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Question 1: Pole structures of the two-point functions

The two-point functions in momentum space are given by:

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

For a free massless scalar field theory, we set $m = 0$:

$$\begin{aligned} D_F(p) &= \frac{i}{(p^0)^2 - \mathbf{p}^2 + i\epsilon} \\ &= \frac{i}{(p^0 - |\mathbf{p}| + i\epsilon)(p^0 + |\mathbf{p}| - i\epsilon)} \end{aligned}$$

thus we have simple poles at $p^0 = \pm|\mathbf{p}| \mp i\epsilon$. We can compute the residues via the formula:

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

where z_0 are our two poles and $f(z)$ is our two-point function $D_F(p)$. We find:

$$\begin{aligned} \text{Res}(D_F, |\mathbf{p}| - i\epsilon) &= \lim_{p^0 \to |\mathbf{p}| - i\epsilon} (p^0 - |\mathbf{p}| + i\epsilon) \frac{i}{(p^0 - |\mathbf{p}| + i\epsilon)(p^0 + |\mathbf{p}| - i\epsilon)} \\ &= \lim_{p^0 \to |\mathbf{p}| - i\epsilon} \frac{i}{p^0 + |\mathbf{p}| - i\epsilon} = \frac{i}{2(|\mathbf{p}| - i\epsilon)} \\ \text{Res}(D_F, -|\mathbf{p}| + i\epsilon) &= \lim_{p^0 \to -|\mathbf{p}| + i\epsilon} (p^0 + |\mathbf{p}| - i\epsilon) D_F(p) \\ &= \lim_{p^0 \to -|\mathbf{p}| + i\epsilon} \frac{i}{p^0 - |\mathbf{p}| + i\epsilon} = -\frac{i}{2(|\mathbf{p}| + i\epsilon)} \end{aligned}$$

In the non-relativistic limit, we instead have:

$$\begin{aligned} D_F(p) &= \frac{i}{(p^0)^2 - m^2 + i\epsilon} \\ &= \frac{i}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)} \end{aligned}$$

and so in an identical manner, we find simple poles at $p^0 = \pm m \mp i\epsilon$ with residues:

$$\begin{aligned} \text{Res}(D_F, m - i\epsilon) &= \lim_{p^0 \to m - i\epsilon} (p^0 - m + i\epsilon) D_F(p) \\ &= \lim_{p^0 \to m - i\epsilon} \frac{i}{p^0 + m - i\epsilon} = \frac{i}{2(m - i\epsilon)} \\ \text{Res}(D_F, -m + i\epsilon) &= \lim_{p^0 \to -m + i\epsilon} (p^0 + m - i\epsilon) D_F(p) \\ &= \lim_{p^0 \to -m + i\epsilon} \frac{i}{p^0 - m + i\epsilon} = -\frac{i}{2(m + i\epsilon)} \end{aligned}$$

Thus, the Feynmann propagator

$$\begin{aligned}
D_F(x-y) &= \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \delta^3(x-y) \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)}
\end{aligned}$$

Now, let us define

$$\begin{aligned}
I(t) &= \int \frac{dp^0}{2\pi} e^{-ip^0 t} \frac{i}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)} \\
&= \int \frac{dp^0}{2\pi} e^{-ip^0 t} D_F(p)
\end{aligned}$$

where $t = x^0 - y^0$. We can evaluate this integral using the residue theorem. Notice that the inclusion of the $e^{-ip^0 t}$ has the same poles as $D_F(p)$, because it is analytic. We consider two cases: $t > 0$ and $t < 0$. For $t > 0$, we close the contour in the lower half-plane, which encloses the pole at $p^0 = m - i\epsilon$. Thus, we find:

$$\begin{aligned}
I(t) &= \frac{-2\pi i}{2\pi} \cdot \mathrm{Res}(D_F, m - i\epsilon) e^{-i(m-i\epsilon)t} \\
&= -i \cdot \frac{i}{2(m - i\epsilon)} e^{-i(m-i\epsilon)t} \\
&= \frac{1}{2(m - i\epsilon)} e^{-i(m-i\epsilon)t}
\end{aligned}$$

Such that the lower half-plane result is

$$D_{F_l}(x-y) = \delta^3(x-y) \frac{1}{2(m - i\epsilon)} e^{-i(m-i\epsilon)(x^0-y^0)}$$

For $t < 0$, we close the contour in the upper half-plane, and we obtain an equivalent result:

$$D_{F_u}(x-y) = \delta^3(x-y) \frac{1}{2(m + i\epsilon)} e^{i(m+i\epsilon)(x^0-y^0)}$$

We can combine these two results into one expression using the Heaviside step function $\theta(t)$:

$$D_F(x-y) = \delta^3(x-y) \left[\theta(t) \frac{e^{-i(m-i\epsilon)t}}{2(m - i\epsilon)} + \theta(-t) \frac{e^{i(m+i\epsilon)t}}{2(m + i\epsilon)} \right]$$

Taking the limit $\epsilon \to 0$, we find:

$$\begin{aligned} D_F(x-y) &= \delta^3(x-y)\left[\theta(t)\frac{e^{-imt}}{2m} + \theta(-t)\frac{e^{imt}}{2m}\right] \\ &= \delta^3(x-y)\frac{1}{2m}\left[\theta(t)e^{-imt} + \theta(-t)e^{imt}\right] \\ &= \delta^3(x-y)\frac{1}{2m}e^{-im|t|} \end{aligned}$$

$$\boxed{D_F(x-y) = \delta^3(x-y)\frac{1}{2m}e^{-im|x^0-y^0|}} \tag{1}$$

Now, let us consider the scalar field theory with action

$$S = \int d^{D+1}x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + |\lambda_2|\phi^2 + \lambda_3\phi^4\right) \tag{2}$$

We always begin by computing the minima of the potential, which in this case is $|\lambda_2|\phi^2+\lambda_3\phi^4$. We find:

$$\begin{aligned} \frac{d}{d\phi}(|\lambda_2|\phi^2 + \lambda_3\phi^4) = 2|\lambda_2|\phi + 4\lambda_3\phi^3 = 0 \\ \phi(2|\lambda_2| + 4\lambda_3\phi^2) = 0 \end{aligned}$$

Thus, we have three stationary points: $\phi_c = 0$ and $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$. The following plot shows that $\lambda_3 < 0$ is the only case where we have local minima at $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$:

In the case where the potential has $\lambda_2\phi^2$ instead of $|\lambda_2|\phi^2$, then when $\lambda_2 < 0$ we have a single minimum at $\phi = 0$, and when $\lambda_2 > 0$ we have local maxima at $\phi = 0$ and local minima at $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$.

Now, we can expand the field around one of the minima $\phi_c = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$ by defining a fluctuation field $\phi = \varphi - \phi_c$. The potential becomes

$$V(\varphi) = -|\lambda_2|\varphi^2 + \lambda_3\varphi^4, \tag{3}$$

and expanding around ϕ_c gives

$$\begin{aligned} V(\phi_c + \phi) &= V(\phi_c) + V'(\phi_c)\phi + \frac{1}{2}V''(\phi_c)\phi^2 + \mathcal{O}(\phi^3) \qquad (4) \\ &= \frac{|\lambda_2|^2}{4\lambda_3} + 2|\lambda_2|\phi^2 + \mathcal{O}(\phi^3), \qquad (5) \end{aligned}$$

where we used $V'(\phi_c) = 0$ and $V''(\phi_c) = 4|\lambda_2|$.

Thus, the potential near the minimum is approximately quadratic:

$$V(\phi) \simeq \frac{|\lambda_2|^2}{4\lambda_3} + 2|\lambda_2|\phi^2. \tag{6}$$

The constant term does not affect the dynamics, so the quadratic (free) part of the action becomes

$$S[\phi] = \frac{1}{2}\int d^{D+1}x \, \left(\partial_\mu\phi\,\partial^\mu\phi - 4|\lambda_2|\,\phi^2\right). \tag{7}$$

By comparison with the canonical Klein–Gordon form

$$S = \frac{1}{2}\int d^{D+1}x \, \left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right),$$

we identify the new mass as

$$m = 2\sqrt{|\lambda_2|}. \tag{8}$$

This way, we can write the Feynmann propagator as

$$D_F(p) = \frac{i}{p^2 - 4|\lambda_2| + i\epsilon}.$$

With simple poles at $p^0 = \pm\sqrt{\mathbf{p}^2 + 4|\lambda_2|} \mp i\epsilon$ and residues:

$$\mathrm{Res}(D_F, \sqrt{\mathbf{p}^2 + 4|\lambda_2|} - i\epsilon) = \frac{i}{2(\sqrt{\mathbf{p}^2 + 4|\lambda_2|} - i\epsilon)}$$
$$\mathrm{Res}(D_F, -\sqrt{\mathbf{p}^2 + 4|\lambda_2|} + i\epsilon) = -\frac{i}{2(\sqrt{\mathbf{p}^2 + 4|\lambda_2|} + i\epsilon)}$$

To obtain the propagator in coordinate space, we perform the Fourier transform:

$$D_F(x-y) = \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \frac{i\,e^{-ip\cdot(x-y)}}{p^2 - 4|\lambda_2| + i\epsilon}.$$
$$D_F(x-y) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{dp^0}{2\pi} \frac{i\,e^{-ip^0(x^0-y^0)}}{(p^0)^2 - (\mathbf{p}^2 + 4|\lambda_2|) + i\epsilon}.$$

The integrand has simple poles at

$$p^0 = \pm E_{\mathbf{p}} \mp i\epsilon, \qquad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + 4|\lambda_2|}.$$

Closing the contouts and using the residue theorem in the same manner as before, we find:

$$\boxed{D_F(x-y) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2E_{\mathbf{p}}} \left[\theta(x^0 - y^0)e^{-iE_{\mathbf{p}}(x^0-y^0)} + \theta(y^0 - x^0)e^{+iE_{\mathbf{p}}(x^0-y^0)}\right]} \tag{9}$$

Question 2: Energy-momentum tensor in field theory

For a free scalar field theory, we have the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.\end{aligned}$$

Or, on some arbitrary curved spacetime with metric $g_{\mu\nu}$, we have

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.$$

Running with this, we can compute

$$-\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\right)$$

Using the matrix identity $\ln\det A = \operatorname{tr}\ln A$, and varying both sides, we find

$$\begin{aligned}\delta(\ln\det A) &= \delta(\operatorname{tr}\ln A) \\ &= \operatorname{tr}(\delta\ln A) \\ &= \operatorname{tr}(A^{-1}\delta A)\end{aligned}$$

Therefore, we have

$$\frac{\delta(\det A)}{\det A} = \operatorname{tr}(A^{-1}\delta A).$$

Putting this in the language of metrics, we have

$$\begin{aligned}\frac{\delta g}{g} &= g^{\mu\nu}\delta g_{\mu\nu} \\ \delta g &= g g^{\mu\nu}\delta g_{\mu\nu}\end{aligned}$$

Therefore, we can compute

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= -\frac{1}{2\sqrt{-g}}g g^{\mu\nu}\delta g_{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \\ \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\ g_{\mu\nu}\end{aligned}$$

Putting this back into our expression, we find

$$\begin{aligned}
\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} &= \frac{2}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right)\right) \\
&= -g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \partial_\mu\phi\partial_\nu\phi \\
&= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}
\end{aligned}$$

Or, swapping back to Minkowski space, we have

$$\boxed{T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}.}$$

Just as was obtained using Noether's theorem.

To couple the free field theory Lagrangian to gravity, we replace the Minkowski metric with a general metric $g_{\mu\nu}$, and we include a factor of $\sqrt{-g}$ to ensure that the action is a scalar under general coordinate transformations. Lastly, we would also have to make the derivatives covariant instead of partial, but since ϕ is a scalar field, the covariant derivative reduces to the partial derivative.

Question 3

We can imagine ϕ and $\phi^\dagger$ as a sum of two real fields ϕ_1 and ϕ_2, such that $\phi = \phi_1 + i\phi_2$ and $\phi^\dagger = \phi_1 - i\phi_2$. Plugging this presecription into our Lagrangian

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^\dagger \\
&= \frac{1}{2}\partial_\mu(\phi_1 + i\phi_2)\partial^\mu(\phi_1 - i\phi_2) \\
&= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{i}{2}\partial_\mu\phi_2\partial^\mu\phi_1 - \frac{i}{2}\partial_\mu\phi_1\partial^\mu\phi_2 \\
&= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{i}{2}\partial_\mu\partial^\mu\phi_2\phi_1 - \frac{i}{2}\partial_\mu\partial^\mu\phi_2\partial^\mu\phi_1 \\
&= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2
\end{aligned}$$

because two independent scalar fields ϕ_1, ϕ_2 commute. Thus, we can see this Lagrangian describes two independent scalar field theories. Now, let us quantize both ϕ_1 and ϕ_2 independently, giving them each their own creation and annihilation operators, which we shall label α and β.

$$\begin{aligned}
\phi_1(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\alpha_k e^{-ik\cdot x} + \alpha_k^\dagger e^{ik\cdot x}) \\
\phi_2(x) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\beta e^{-ik\cdot x} + \beta^\dagger e^{ik\cdot x})
\end{aligned}$$

Using these definitions, we write ϕ and $\phi^\dagger$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left((\alpha_k + i\beta_k)e^{-ik\cdot x} + (\alpha_k^\dagger + i\beta_k^\dagger)e^{ik\cdot x}\right)$$
$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left((\alpha_k - i\beta_k)e^{ik\cdot x} + (\alpha_k^\dagger - i\beta_k^\dagger)e^{-ik\cdot x}\right)$$

let us now define $a_k = \alpha_k + i\beta_k$ and $b_k = \alpha_k - i\beta_k$. Then, immediately we see $a_k^\dagger = \alpha_k^\dagger - i\beta_k^\dagger$ and $b_k^\dagger = \alpha_k^\dagger + i\beta_k^\dagger$ such that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right)$$
$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k^\dagger e^{ik\cdot x} + b_k e^{-ik\cdot x}\right)$$

Now, let's compute $[\phi(x), \phi^\dagger(y)]$

$$\begin{aligned}
[\phi(x), \phi^\dagger(y)] &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\left[\left(a_\mathbf{k}e^{-ik\cdot x} + b_\mathbf{k}^\dagger e^{+ik\cdot x}\right), \left(a_{\mathbf{k}'}^\dagger e^{+ik'\cdot y} + b_{\mathbf{k}'}e^{-ik'\cdot y}\right)\right] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\Big(a_\mathbf{k}a_{\mathbf{k}'}^\dagger e^{-ik\cdot x}e^{+ik'\cdot y} + a_\mathbf{k}b_{\mathbf{k}'}e^{-ik\cdot x}e^{-ik'\cdot y} \\
&\qquad + b_\mathbf{k}^\dagger a_{\mathbf{k}'}^\dagger e^{+ik\cdot x}e^{+ik'\cdot y} + b_\mathbf{k}^\dagger b_{\mathbf{k}'}e^{+ik\cdot x}e^{-ik'\cdot y} \\
&\qquad - a_{\mathbf{k}'}^\dagger a_\mathbf{k}e^{+ik'\cdot y}e^{-ik\cdot x} - b_{\mathbf{k}'}a_\mathbf{k}e^{-ik'\cdot y}e^{-ik\cdot x} \\
&\qquad - a_{\mathbf{k}'}^\dagger b_\mathbf{k}^\dagger e^{+ik'\cdot y}e^{+ik\cdot x} - b_{\mathbf{k}'}b_\mathbf{k}^\dagger e^{-ik'\cdot y}e^{+ik\cdot x}\Big) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\Big([a_\mathbf{k}, a_{\mathbf{k}'}^\dagger]\,e^{-ik\cdot x}e^{+ik'\cdot y} + [a_\mathbf{k}, b_{\mathbf{k}'}]\,e^{-ik\cdot x}e^{-ik'\cdot y} \\
&\qquad + [b_\mathbf{k}^\dagger, a_{\mathbf{k}'}^\dagger]\,e^{+ik\cdot x}e^{+ik'\cdot y} + [b_\mathbf{k}^\dagger, b_{\mathbf{k}'}]\,e^{+ik\cdot x}e^{-ik'\cdot y}\Big) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_\mathbf{k}}\sqrt{2\omega_{\mathbf{k}'}}}\left((2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,e^{-ik\cdot x}e^{+ik'\cdot y} - (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,e^{+ik\cdot x}e^{-ik'\cdot y}\right) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left(e^{-ik\cdot(x-y)} - e^{+ik\cdot(x-y)}\right). \\
&= \int \frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\frac{1}{2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=\omega_k} + \frac{1}{-2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=-\omega_k}\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ik\cdot(x-y)} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ik\cdot(x-y)}
\end{aligned}$$

which we immediately recognize as $D_R(x-y)$, the retarded propagator. It makes sense that the commutator would yield the retarded propagator because the commutator measures how much the quantum field ϕ at one point in spacetime x interferes with $\phi^\dagger$ at a later point y, and $D_R(x-y)$ describes what this causal effect is.

Next, to find the Hamiltonian, we will need to compute the conjugate momenta

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \tfrac{1}{2}\partial^0 \phi^\dagger$$
$$\pi^\dagger(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\dagger)} = \tfrac{1}{2}\partial^0 \phi$$

then, the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L}\\ &= \partial^0\phi^\dagger\partial_0\phi + \partial^0\phi\partial_0\phi^\dagger - \big(\partial_\mu\phi^\dagger\partial^\mu\phi\big)\\ &= \partial^0\phi^\dagger\partial_0\phi + \partial^0\phi\partial_0\phi^\dagger - \big(\partial^0\phi^\dagger\partial_0\phi - \nabla\phi^\dagger\cdot\nabla\phi\big)\\ &= \nabla\phi^\dagger\cdot\nabla\phi + \partial^0\phi^\dagger\partial_0\phi\end{aligned}$$

Now, considering the massive case, we have the Lagrangian

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \tfrac{1}{2}m^2\phi^\dagger\phi$$

If we consider a transformation

$$\phi \;\rightarrow\; \phi' \;=\; e^{-i\theta}\phi, \qquad \phi^\dagger \;\rightarrow\; \phi'^\dagger \;=\; e^{+i\theta}\phi^\dagger.$$

then we can tell that if we were to plug this into the Lagrangian, we would find no change since in both terms, the exponential factors cancel. Further, expanding the rotated fields in terms of real components, we find

$$\phi' \;=\; \phi_1' + i\phi_2' \;=\; e^{-i\theta}\,(\phi_1 + i\phi_2) \;=\; (\cos\theta\,\phi_1 + \sin\theta\,\phi_2) \;+\; i(\cos\theta\,\phi_2 - \sin\theta\,\phi_1)$$

so

$$\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} \;=\; \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.$$

Thus the global phase $\phi \rightarrow e^{-i\theta}\phi$ acts as a rotation by angle θ in the (ϕ_1, ϕ_2) plane. Now, considering a small θ, we can expand the exponentials to first order in θ:

$$e^{-i\theta} \simeq 1 - i\theta,$$
$$e^{+i\theta} \simeq 1 + i\theta,$$

so

$$\begin{aligned}
\phi' &= (1-i\theta)\phi = \phi - i\theta\phi, \\
\therefore \delta\phi &= -i\theta\,\phi \\
\phi'^{\dagger} &= (1+i\theta)\phi^{\dagger} = \phi^{\dagger} + i\theta\phi^{\dagger}, \\
\therefore \delta\phi^{\dagger} &= +i\theta\,\phi^{\dagger}
\end{aligned}$$

Now, we can find the conserved current using Noether's theorem:

$$\begin{aligned}
J^{\mu} &= \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\,\delta\phi \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{\dagger})}\,\delta\phi^{\dagger} \\
&= \left(\partial^{\mu}\phi^{\dagger}\right)(-i\theta\,\phi) \;+\; \left(\partial^{\mu}\phi\right)(+i\theta\,\phi^{\dagger}) \\
&= \frac{i\theta}{2}\Big(\phi^{\dagger}\partial^{\mu}\phi \;-\; (\partial^{\mu}\phi^{\dagger})\,\phi\Big).
\end{aligned}$$

Then, the conserved charge Q can be found by integrating the charge density over space

$$\begin{aligned}
Q = \int d^3x J^0 &= \int d^3x\,\frac{i\theta}{2}\Big(\phi^{\dagger}\partial^0\phi - \partial^0\phi^{\dagger}\phi\Big). \\
&= \int d^3x\frac{i\theta}{2}\Big(\phi^{\dagger}\pi^{\dagger} - \pi\phi\Big)
\end{aligned}$$

Knowing that all of ϕ, π (and their daggered counterparts) are quantized, then we can express the charge in terms of their mode expansions. For this we can compute $\partial^0\phi$ and $\partial^0\phi^{\dagger}$. Using

$$\begin{aligned}
\partial^0 c_k e^{-ik\cdot x} &= c_k\partial^0 e^{-ik^0x^0 + i\mathbf{k}\cdot\mathbf{x}} \\
&= -ik^0 c_k e^{-ik\cdot x} \\
&= -i\omega_k c_k e^{-ik\cdot x}
\end{aligned}$$

Therefore

$$\begin{aligned}
\tfrac{1}{2}\partial^0\phi = \pi^{\dagger}(x) &= \tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\Big((-i\omega_k)a_k e^{-ik\cdot x} + (i\omega_k)b_k^{\dagger}e^{ik\cdot x}\Big) \\
&= \frac{-i}{2}\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\Big(a_k e^{-ik\cdot x} - b_k^{\dagger}e^{ik\cdot x}\Big)
\end{aligned}$$

and similarly

$$\pi(x) = \frac{i}{2}\int\frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\Big(a_k^{\dagger}e^{ik\cdot x} - b_k e^{-ik\cdot x}\Big)$$

And so the conserved charge is given by

$$\begin{aligned}
\hat{Q} = \frac{i}{4}\int&\frac{d^3x\,d^3k\,d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\Big[\big(a_{k'}^{\dagger}e^{ik'\cdot x} + b_{k'}e^{-ik'\cdot x}\big)(-i)\big(a_k e^{-ik\cdot x} - b_k^{\dagger}e^{ik\cdot x}\big) \\
&- \big(a_{k'}^{\dagger}e^{ik'\cdot x} - b_{k'}e^{-ik'\cdot x}\big)(i)\big(a_k e^{-ik\cdot x} + b_k^{\dagger}e^{ik\cdot x}\big)\Big] \\
= \frac{1}{4}\int&\frac{d^3x\,d^3k\,d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\Big[\big(a_{k'}^{\dagger}a_k e^{i(k'-k)\cdot x} - a_{k'}^{\dagger}b_k^{\dagger}e^{i(k'+k)\cdot x} + b_{k'}a_k e^{i(k+k')\cdot x} - b_{k'}b_k^{\dagger}e^{-i(k-k')\cdot x}\big) \\
&+ \big(a_{k'}^{\dagger}a_k e^{-i(k'-k)\cdot x} + a_{k'}^{\dagger}b_k^{\dagger}e^{-i(k'+k)\cdot x} - b_{k'}a_k e^{i(k'+k)\cdot x} - b_{k'}b_k^{\dagger}e^{-i(k-k')\cdot x}\big)\Big]
\end{aligned}$$

We now carry out the spatial integral. Each exponential gives either $e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}$ or $e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}$, so

$$\int d^3x\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k}), \qquad \int d^3x\, e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}'+\mathbf{k}).$$

$$\begin{aligned}\hat{Q} &= \frac{1}{4}\int \frac{d^3x\, d^3k\, d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\Big[\big(a_k^\dagger a_k - a_{-k}^\dagger b_k^\dagger + b_{-k}a_k - b_k b_k^\dagger\big) \\ &\qquad + \big(a_k^\dagger a_k + a_{-k}^\dagger b_k^\dagger - b_{-k}a_k - b_k b_k^\dagger\big)\Big] \\ &= \frac{1}{4}\int \frac{d^3k}{(2\pi)^3}\Big[2a_k^\dagger a_k - 2b_k b_k^\dagger\Big] \\ &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big(a_k^\dagger a_k - b_k b_k^\dagger\Big).\end{aligned}$$

Using $b_k b_k^\dagger = b_k^\dagger b_k + (2\pi)^3\delta^{(3)}(0)$, we normal order and drop the infinite constant.

$$:\hat{Q}:= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\big(a_k^\dagger a_k - b_k^\dagger b_k\big).$$

Thus the conserved charge counts particles minus antiparticles.

Now, if the rotational angle is space dependent $\theta = \theta(x)$, we see that the Lagrangian is no longer invariant under $\phi \to e^{-i\theta(x)}\phi$ because the first term

$$\begin{aligned}\partial_\mu\phi\partial^\mu\phi^\dagger &\to \partial_\mu(e^{-i\theta(x)}\phi)\partial^\mu(e^{i\theta(x)}\phi) \\ &= \big(-i\partial_\mu\theta(x)e^{-i\theta(x)}\phi + e^{-i\theta(x)}\partial_\mu\phi\big)\big(i\partial_\mu\theta(x)e^{i\theta(x)}\phi^\dagger + e^{i\theta(x)}\partial_\mu\phi^\dagger\big) \\ &= \big((\partial_\mu\theta)^2\phi\phi^\dagger - i\partial_\mu\theta\phi\partial_\mu\phi^\dagger + i\partial_\mu\phi\partial_\mu\theta\phi^\dagger + \partial_\mu\phi\partial_\mu\phi^\dagger\big)\end{aligned}$$

For a small angle change, we can ignore the $(\partial_\mu\theta)^2$ term, leaving us with our new Lagrangian:

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi^\dagger - \frac{i}{2}\partial_\mu\theta(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger) - \frac{1}{2}m^2\phi\phi^\dagger$$

Which is obviously different than prior to the transformation. Recalling that the conserved current for the global gauge transformation was $J^\mu = \frac{i}{2}\big(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi\big)$, we can read off that the variation in the Lagrangian is

$$\begin{aligned}\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} &= -\frac{i}{2}\partial_\mu\theta(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger) \\ &= \partial_\mu\theta J^\mu\end{aligned}$$

as desired.

To show how the current is not invariant, take:

$$
\begin{aligned}
J'^{\mu} &\to \frac{i}{2}\big(e^{-i\theta(x)}\phi\partial_\mu(e^{i\theta(x)}\phi^\dagger) - \partial_\mu(e^{-i\theta(x)}\phi)e^{i\theta(x)}\phi^\dagger\big) \\
&= \frac{i}{2}\big(e^{-i\theta(x)}\phi(i\partial_\mu\theta(x)e^{i\theta(x)}\phi^\dagger + e^{i\theta(x)}\partial_\mu\phi^\dagger) - (-i\partial_\mu\theta(x)e^{-i\theta(x)}\phi + e^{-i\theta(x)}\partial_\mu\phi)e^{i\theta(x)}\phi^\dagger\big) \\
&= \frac{i}{2}\big(i\partial_\mu\theta\phi\phi^\dagger + \phi\partial_\mu\phi^\dagger + i\partial_\mu\theta\phi\phi^\dagger - \partial_\mu\phi\phi^\dagger\big) \\
&= \frac{i}{2}\big(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger\big) - |\phi|^2\partial_\mu\theta
\end{aligned}
$$

Such that

$$
\delta J^\mu = |\phi|^2\partial_\mu\theta
$$

Now, let us consider the same transformation $\phi \to e^{-i\theta(x)}\phi$ but with our modified Lagrangian

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^\dagger\partial^\mu\phi - eJ^\mu A_\mu + \frac{e^2}{2}|\phi|^2 A_\mu A^\mu - \frac{1}{2}m^2\phi^\dagger\phi
$$

Then the variation in the Lagrangian is

$$
\begin{aligned}
\delta\mathcal{L} &= \delta\mathcal{L}_{\text{old}} + \delta\mathcal{L}_{\text{new}} \\
&= \partial_\mu\theta J^\mu - e\delta J^\mu A_\mu - eJ^\mu\delta A_\mu + \frac{e^2}{2}|\phi|^2\delta A_\mu A^\mu + \frac{e^2}{2}|\phi|^2 A_\mu\delta A^\mu \\
&= \partial_\mu\theta J^\mu - e(|\phi|^2\partial_\mu\theta)A_\mu - eJ^\mu\delta A_\mu + e^2|\phi|^2\delta A_\mu A^\mu \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \big(eJ^\mu - e^2|\phi|^2A^\mu\big)\delta A_\mu \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \big(eJ^\mu - e^2|\phi|^2A^\mu\big)\left(\frac{1}{e}\partial_\mu\theta\right) \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) \\
&= 0
\end{aligned}
$$

and so we see that if we choose $\delta A_\mu = \frac{1}{e}\partial_\mu\theta$, then the variation in the Lagrangian is zero, and thus the Lagrangian is invariant under the local gauge transformation.

To verify the kinetic term for the gauge field is also invariant, we note that, using the result above, A_μ transforms as $A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. Therefore:

$$
\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu &\to \partial_\mu(A_\nu + \frac{1}{e}\partial_\nu\theta) - \partial_\nu(A_\mu + \frac{1}{e}\partial_\mu\theta) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{e}(\partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}
$$

since partial derivatives commute. Thus the kinetic term for the gauge field is invariant. Specifically, if $(\partial_\mu A_\nu - \partial_\nu A_\mu)$ is invariant, then so is $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$.

Meanwhile, if we have some mass term such as $\frac{1}{2}m^2 A_\mu A^\mu$, then under the gauge transformation

$$\begin{aligned} A_\mu A^\mu &\to (A_\mu + \frac{1}{e}\partial_\mu\theta)(A^\mu + \frac{1}{e}\partial_\mu\theta) \\ &= A_\mu A^\mu + \frac{2}{e}A^\mu\partial_\mu\theta + \frac{1}{e^2}(\partial_\mu\theta)^2 \end{aligned}$$

which is not equal to the original term, and so the mass term is not gauge invariant.

Now, let us introduce the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$. Then, under the gauge transformation, we have

$$\begin{aligned} D_\mu\phi &= (\partial_\mu - ieA_\mu)\phi \\ &\to (\partial_\mu - ie(A_\mu + e^{-1}\partial_\mu\theta))e^{-i\theta}\phi \\ &= \partial_\mu(e^{-i\theta}\phi) - ieA_\mu e^{-i\theta}\phi - i\partial_\mu\theta e^{-i\theta}\phi \\ &= i\partial_\mu\theta e^{-i\theta}\phi + e^{-i\theta}\partial_\mu\phi - ieA_\mu e^{-i\theta}\phi - i\partial_\mu\theta e^{-i\theta}\phi \\ &= e^{-i\theta}\partial_\mu\phi - ieA_\mu e^{-i\theta}\phi \\ &= \big(\partial_\mu\phi - ieA_\mu\phi\big)e^{-i\theta}\phi \\ &= e^{-i\theta}D_\mu\phi \end{aligned}$$

Thus, the covariant derivative of ϕ transforms in the same way as ϕ and not as its derivative.

Finally, to write the QED Lagrangian using the covariant derivative, we start with a well known definition for the electromagnetic field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Then, the QED Lagrangian can be written as

$$\mathcal{L}_{\text{QED}} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi^\dagger) - \frac{1}{2}m^2\phi\phi^\dagger - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$