Question 1: Pole structures of the two-point functions

The two-point functions in momentum space are given by:

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}$$

For a free massless scalar field theory, we set $m = 0$:

$$\begin{aligned} D_F(p) &= \frac{i}{(p^0)^2 - \mathbf{p}^2 + i\epsilon} \\ &= \frac{i}{(p^0 - |\mathbf{p}| + i\epsilon)(p^0 + |\mathbf{p}| - i\epsilon)} \end{aligned}$$

thus we have simple poles at $p^0 = \pm|\mathbf{p}| \mp i\epsilon$. We can compute the residues via the formula:

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

where z_0 are our two poles and $f(z)$ is our two-point function $D_F(p)$. We find:

$$\begin{aligned} \text{Res}(D_F, |\mathbf{p}| - i\epsilon) &= \lim_{p^0 \to |\mathbf{p}| - i\epsilon} (p^0 - |\mathbf{p}| + i\epsilon) \frac{i}{(p^0 - |\mathbf{p}| + i\epsilon)(p^0 + |\mathbf{p}| - i\epsilon)} \\ &= \lim_{p^0 \to |\mathbf{p}| - i\epsilon} \frac{i}{p^0 + |\mathbf{p}| - i\epsilon} = \frac{i}{2(|\mathbf{p}| - i\epsilon)} \\ \text{Res}(D_F, -|\mathbf{p}| + i\epsilon) &= \lim_{p^0 \to -|\mathbf{p}| + i\epsilon} (p^0 + |\mathbf{p}| - i\epsilon) D_F(p) \\ &= \lim_{p^0 \to -|\mathbf{p}| + i\epsilon} \frac{i}{p^0 - |\mathbf{p}| + i\epsilon} = -\frac{i}{2(|\mathbf{p}| + i\epsilon)} \end{aligned}$$

In the non-relativistic limit, we instead have:

$$\begin{aligned} D_F(p) &= \frac{i}{(p^0)^2 - m^2 + i\epsilon} \\ &= \frac{i}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)} \end{aligned}$$

and so in an identical manner, we find simple poles at $p^0 = \pm m \mp i\epsilon$ with residues:

$$\begin{aligned} \text{Res}(D_F, m - i\epsilon) &= \lim_{p^0 \to m - i\epsilon} (p^0 - m + i\epsilon) D_F(p) \\ &= \lim_{p^0 \to m - i\epsilon} \frac{i}{p^0 + m - i\epsilon} = \frac{i}{2(m - i\epsilon)} \\ \text{Res}(D_F, -m + i\epsilon) &= \lim_{p^0 \to -m + i\epsilon} (p^0 + m - i\epsilon) D_F(p) \\ &= \lim_{p^0 \to -m + i\epsilon} \frac{i}{p^0 - m + i\epsilon} = -\frac{i}{2(m + i\epsilon)} \end{aligned}$$

Thus, the Feynmann propagator

$$\begin{aligned}
D_F(x-y) &= \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{(p^0)^2 - m^2 + i\epsilon} \\
&= \delta^3(x-y) \int \frac{dp^0}{2\pi} \frac{ie^{-ip^0(x^0-y^0)}}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)}
\end{aligned}$$

Now, let us define

$$\begin{aligned}
I(t) &= \int \frac{dp^0}{2\pi} e^{-ip^0 t} \frac{i}{(p^0 - m + i\epsilon)(p^0 + m - i\epsilon)} \\
&= \int \frac{dp^0}{2\pi} e^{-ip^0 t} D_F(p)
\end{aligned}$$

where $t = x^0 - y^0$. We can evaluate this integral using the residue theorem. Notice that the inclusion of the $e^{-ip^0 t}$ has the same poles as $D_F(p)$, because it is analytic. We consider two cases: $t > 0$ and $t < 0$. For $t > 0$, we close the contour in the lower half-plane, which encloses the pole at $p^0 = m - i\epsilon$. Thus, we find:

$$\begin{aligned}
I(t) &= \frac{-2\pi i}{2\pi} \cdot \mathrm{Res}(D_F, m - i\epsilon) e^{-i(m-i\epsilon)t} \\
&= -i \cdot \frac{i}{2(m - i\epsilon)} e^{-i(m-i\epsilon)t} \\
&= \frac{1}{2(m - i\epsilon)} e^{-i(m-i\epsilon)t}
\end{aligned}$$

Such that the lower half-plane result is

$$D_{F_l}(x-y) = \delta^3(x-y) \frac{1}{2(m - i\epsilon)} e^{-i(m-i\epsilon)(x^0-y^0)}$$

For $t < 0$, we close the contour in the upper half-plane, and we obtain an equivalent result:

$$D_{F_u}(x-y) = \delta^3(x-y) \frac{1}{2(m + i\epsilon)} e^{i(m+i\epsilon)(x^0-y^0)}$$

We can combine these two results into one expression using the Heaviside step function $\theta(t)$:

$$D_F(x-y) = \delta^3(x-y) \left[\theta(t) \frac{e^{-i(m-i\epsilon)t}}{2(m - i\epsilon)} + \theta(-t) \frac{e^{i(m+i\epsilon)t}}{2(m + i\epsilon)} \right]$$

Taking the limit $\epsilon \to 0$, we find:

$$\begin{aligned} D_F(x-y) &= \delta^3(x-y)\left[\theta(t)\frac{e^{-imt}}{2m} + \theta(-t)\frac{e^{imt}}{2m}\right] \\ &= \delta^3(x-y)\frac{1}{2m}\left[\theta(t)e^{-imt} + \theta(-t)e^{imt}\right] \\ &= \delta^3(x-y)\frac{1}{2m}e^{-im|t|} \end{aligned}$$

$$\boxed{D_F(x-y) = \delta^3(x-y)\frac{1}{2m}e^{-im|x^0-y^0|}} \tag{1}$$

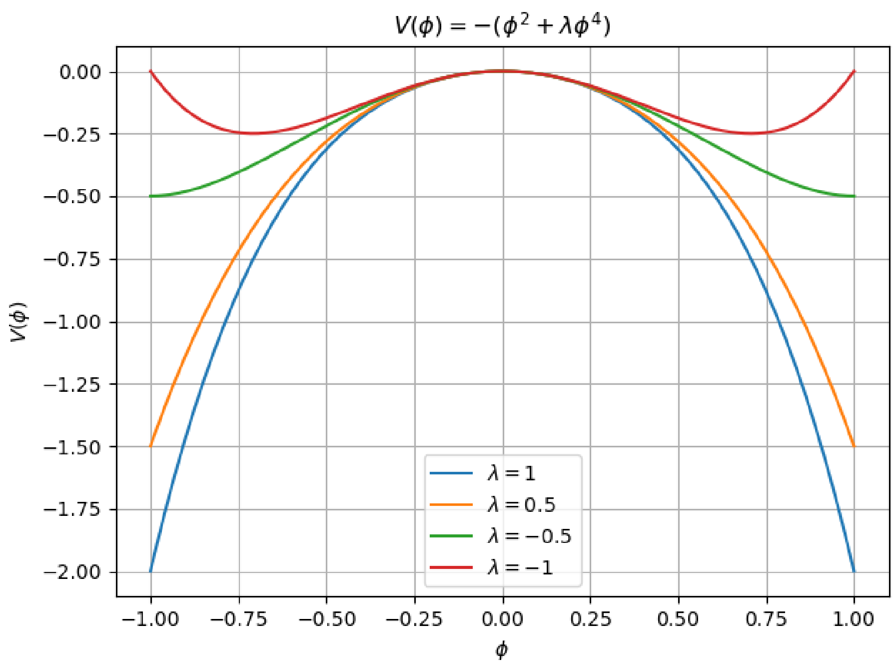
Now, let us consider the scalar field theory with action

$$S = \int d^{D+1}x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + |\lambda_2|\phi^2 + \lambda_3\phi^4\right) \tag{2}$$

We always begin by computing the minima of the potential, which in this case is $|\lambda_2|\phi^2 + \lambda_3\phi^4$. We find:

$$\begin{aligned} &\frac{d}{d\phi}(|\lambda_2|\phi^2 + \lambda_3\phi^4) = 2|\lambda_2|\phi + 4\lambda_3\phi^3 = 0 \\ &\phi(2|\lambda_2| + 4\lambda_3\phi^2) = 0 \end{aligned}$$

Thus, we have three stationary points: $\phi_c = 0$ and $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$. The following plot shows that $\lambda_3 < 0$ is the only case where we have local minima at $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$:



In the case where the potential has $\lambda_2\phi^2$ instead of $|\lambda_2|\phi^2$, then when $\lambda_2 < 0$ we have a single minimum at $\phi = 0$, and when $\lambda_2 > 0$ we have local maxima at $\phi = 0$ and local minima at $\phi = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$.

Now, we can expand the field around one of the minima $\phi_c = \pm\sqrt{\frac{-|\lambda_2|}{2\lambda_3}}$ by defining a fluctuation field $\phi = \varphi - \phi_c$. The potential becomes

$$V(\varphi) = -|\lambda_2|\varphi^2 + \lambda_3\varphi^4, \tag{3}$$

and expanding around ϕ_c gives

$$\begin{aligned} V(\phi_c + \phi) &= V(\phi_c) + V'(\phi_c)\phi + \frac{1}{2}V''(\phi_c)\phi^2 + \mathcal{O}(\phi^3) && (4)\\ &= \frac{|\lambda_2|^2}{4\lambda_3} + 2|\lambda_2|\phi^2 + \mathcal{O}(\phi^3), && (5) \end{aligned}$$

where we used $V'(\phi_c) = 0$ and $V''(\phi_c) = 4|\lambda_2|$.

Thus, the potential near the minimum is approximately quadratic:

$$V(\phi) \simeq \frac{|\lambda_2|^2}{4\lambda_3} + 2|\lambda_2|\phi^2. \tag{6}$$

The constant term does not affect the dynamics, so the quadratic (free) part of the action becomes

$$S[\phi] = \frac{1}{2}\int d^{D+1}x \left(\partial_\mu\phi\,\partial^\mu\phi - 4|\lambda_2|\,\phi^2\right). \tag{7}$$

By comparison with the canonical Klein–Gordon form

$$S = \frac{1}{2}\int d^{D+1}x \left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right),$$

we identify the new mass as

$$m = 2\sqrt{|\lambda_2|}. \tag{8}$$

This way, we can write the Feynmann propagator as

$$D_F(p) = \frac{i}{p^2 - 4|\lambda_2| + i\epsilon}.$$

With simple poles at $p^0 = \pm\sqrt{\mathbf{p}^2 + 4|\lambda_2|} \mp i\epsilon$ and residues:

$$\begin{aligned} \mathrm{Res}(D_F, \sqrt{\mathbf{p}^2 + 4|\lambda_2|} - i\epsilon) &= \frac{i}{2(\sqrt{\mathbf{p}^2 + 4|\lambda_2|} - i\epsilon)}\\ \mathrm{Res}(D_F, -\sqrt{\mathbf{p}^2 + 4|\lambda_2|} + i\epsilon) &= -\frac{i}{2(\sqrt{\mathbf{p}^2 + 4|\lambda_2|} + i\epsilon)} \end{aligned}$$

To obtain the propagator in coordinate space, we perform the Fourier transform:

$$D_F(x-y) = \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \frac{i\,e^{-ip\cdot(x-y)}}{p^2 - 4|\lambda_2| + i\epsilon}.$$

$$D_F(x-y) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \int \frac{dp^0}{2\pi} \frac{i\,e^{-ip^0(x^0-y^0)}}{(p^0)^2 - (\mathbf{p}^2 + 4|\lambda_2|) + i\epsilon}.$$

The integrand has simple poles at

$$p^0 = \pm E_{\mathbf{p}} \mp i\epsilon, \qquad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + 4|\lambda_2|}.$$

Closing the contouts and using the residue theorem in the same manner as before, we find:

$$\boxed{D_F(x-y) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2E_{\mathbf{p}}} \left[\theta(x^0-y^0)e^{-iE_{\mathbf{p}}(x^0-y^0)} + \theta(y^0-x^0)e^{+iE_{\mathbf{p}}(x^0-y^0)}\right]} \tag{9}$$

Question 2: Energy-momentum tensor in field theory

For a free scalar field theory, we have the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \\ &= \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.\end{aligned}$$

Or, on some arbitrary curved spacetime with metric $g_{\mu\nu}$, we have

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.$$

Running with this, we can compute

$$-\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left(\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\right)$$

Using the matrix identity $\ln\det A = \operatorname{tr}\ln A$, and varying both sides, we find

$$\begin{aligned}\delta(\ln\det A) &= \delta(\operatorname{tr}\ln A) \\ &= \operatorname{tr}(\delta\ln A) \\ &= \operatorname{tr}(A^{-1}\delta A)\end{aligned}$$

Therefore, we have

$$\frac{\delta(\det A)}{\det A} = \operatorname{tr}(A^{-1}\delta A).$$

Putting this in the language of metrics, we have

$$\begin{aligned}\frac{\delta g}{g} &= g^{\mu\nu}\delta g_{\mu\nu} \\ \delta g &= g g^{\mu\nu}\delta g_{\mu\nu}\end{aligned}$$

Therefore, we can compute

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu} \\ \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} &= -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\end{aligned}$$

Putting this back into our expression, we find

$$\begin{aligned}\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} &= \frac{2}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right)\right) \\ &= -g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \partial_\mu\phi\partial_\nu\phi \\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}\end{aligned}$$

Or, swapping back to Minkowski space, we have

$$\boxed{T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}.}$$

Just as was obtained using Noether's theorem.

To couple the free field theory Lagrangian to gravity, we replace the Minkowski metric with a general metric $g_{\mu\nu}$, and we include a factor of $\sqrt{-g}$ to ensure that the action is a scalar under general coordinate transformations. Lastly, we would also have to make the derivatives covariant instead of partial, but since ϕ is a scalar field, the covariant derivative reduces to the partial derivative.

Question 3

We can imagine ϕ and $\phi^\dagger$ as a sum of two real fields ϕ_1 and ϕ_2, such that $\phi = \phi_1 + i\phi_2$ and $\phi^\dagger = \phi_1 - i\phi_2$. Plugging this presecription into our Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^\dagger \\ &= \frac{1}{2}\partial_\mu(\phi_1 + i\phi_2)\partial^\mu(\phi_1 - i\phi_2) \\ &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{i}{2}\partial_\mu\phi_2\partial^\mu\phi_1 - \frac{i}{2}\partial_\mu\phi_1\partial^\mu\phi_2 \\ &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{i}{2}\partial_\mu\partial^\mu\phi_2\phi_1 - \frac{i}{2}\partial_\mu\partial^\mu\phi_2\partial^\mu\phi_1 \\ &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2\end{aligned}$$

because two independent scalar fields ϕ_1, ϕ_2 commute. Thus, we can see this Lagrangian describes two independent scalar field theories. Now, let us quantize both ϕ_1 and ϕ_2 independently, giving them each their own creation and annihilation operators, which we shall label α and β.

$$\phi_1(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\alpha_k e^{-ik\cdot x} + \alpha_k^\dagger e^{ik\cdot x})$$
$$\phi_2(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\beta e^{-ik\cdot x} + \beta^\dagger e^{ik\cdot x})$$

Using these definitions, we write ϕ and $\phi^\dagger$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left((\alpha_k + i\beta_k)e^{-ik\cdot x} + (\alpha_k^\dagger + i\beta_k^\dagger)e^{ik\cdot x}\right)$$
$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left((\alpha_k - i\beta_k)e^{ik\cdot x} + (\alpha_k^\dagger - i\beta_k^\dagger)e^{-ik\cdot x}\right)$$

let us now define $a_k = \alpha_k + i\beta_k$ and $b_k = \alpha_k - i\beta_k$. Then, immediately we see $a_k^\dagger = \alpha_k^\dagger - i\beta_k^\dagger$ and $b_k^\dagger = \alpha_k^\dagger + i\beta_k^\dagger$ such that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right)$$
$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k^\dagger e^{ik\cdot x} + b_k e^{-ik\cdot x}\right)$$

Now, let's compute $[\phi(x), \phi^\dagger(y)]$

$$
\begin{aligned}
[\phi(x), \phi^\dagger(y)] &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\left[\left(a_{\mathbf{k}}e^{-ik\cdot x} + b^\dagger_{\mathbf{k}}e^{+ik\cdot x}\right), \left(a^\dagger_{\mathbf{k}'}e^{+ik'\cdot y} + b_{\mathbf{k}'}e^{-ik'\cdot y}\right)\right] \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\Big(a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{-ik\cdot x}e^{+ik'\cdot y} + a_{\mathbf{k}}b_{\mathbf{k}'}e^{-ik\cdot x}e^{-ik'\cdot y} \\
&\qquad + b^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}e^{+ik\cdot x}e^{+ik'\cdot y} + b^\dagger_{\mathbf{k}}b_{\mathbf{k}'}e^{+ik\cdot x}e^{-ik'\cdot y} \\
&\qquad - a^\dagger_{\mathbf{k}'}a_{\mathbf{k}}e^{+ik'\cdot y}e^{-ik\cdot x} - b_{\mathbf{k}'}a_{\mathbf{k}}e^{-ik'\cdot y}e^{-ik\cdot x} \\
&\qquad - a^\dagger_{\mathbf{k}'}b^\dagger_{\mathbf{k}}e^{+ik'\cdot y}e^{+ik\cdot x} - b_{\mathbf{k}'}b^\dagger_{\mathbf{k}}e^{-ik'\cdot y}e^{+ik\cdot x}\Big) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\Big([a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]\,e^{-ik\cdot x}e^{+ik'\cdot y} + [a_{\mathbf{k}}, b_{\mathbf{k}'}]\,e^{-ik\cdot x}e^{-ik'\cdot y} \\
&\qquad + [b^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]\,e^{+ik\cdot x}e^{+ik'\cdot y} + [b^\dagger_{\mathbf{k}}, b_{\mathbf{k}'}]\,e^{+ik\cdot x}e^{-ik'\cdot y}\Big) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{k}'}}}\Big((2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,e^{-ik\cdot x}e^{+ik'\cdot y} - (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,e^{+ik\cdot x}e^{-ik'\cdot y}\Big) \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{-ik\cdot(x-y)} - e^{+ik\cdot(x-y)}\right). \\
&= \int \frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\frac{1}{2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=\omega_k} + \frac{1}{-2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=-\omega_k}\right) \\
&= \int \frac{d^3k}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ik\cdot(x-y)} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ik\cdot(x-y)}
\end{aligned}
$$

which we immediately recognize as $D_R(x-y)$, the retarded propagator. It makes sense that the commutator would yield the retarded propagator because the commutator measures how much the quantum field ϕ at one point in spacetime x interferes with $\phi^\dagger$ at a later point y, and $D_R(x-y)$ describes what this causal effect is.

Next, to find the Hamiltonian, we will need to compute the conjugate momenta

$$
\begin{aligned}
\pi(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \tfrac{1}{2}\partial^0\phi^\dagger \\
\pi^\dagger(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)} = \tfrac{1}{2}\partial^0\phi
\end{aligned}
$$

then, the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L} \\ &= \partial^0\phi^\dagger\partial_0\phi + \partial^0\phi\partial_0\phi^\dagger - \left(\partial_\mu\phi^\dagger\partial^\mu\phi\right) \\ &= \partial^0\phi^\dagger\partial_0\phi + \partial^0\phi\partial_0\phi^\dagger - \left(\partial^0\phi^\dagger\partial_0\phi - \nabla\phi^\dagger\cdot\nabla\phi\right) \\ &= \nabla\phi^\dagger\cdot\nabla\phi + \partial^0\phi^\dagger\partial_0\phi\end{aligned}$$

Now, considering the massive case, we have the Lagrangian

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \tfrac{1}{2}m^2\phi^\dagger\phi$$

If we consider a transformation

$$\phi \;\to\; \phi' \;=\; e^{-i\theta}\phi, \qquad \phi^\dagger \;\to\; \phi'^\dagger \;=\; e^{+i\theta}\phi^\dagger.$$

then we can tell that if we were to plug this into the Lagrangian, we would find no change since in both terms, the exponential factors cancel. Further, expanding the rotated fields in terms of real components, we find

$$\phi' \;=\; \phi_1' + i\phi_2' \;=\; e^{-i\theta}\,(\phi_1 + i\phi_2) \;=\; (\cos\theta\,\phi_1 + \sin\theta\,\phi_2) \;+\; i(\cos\theta\,\phi_2 - \sin\theta\,\phi_1)$$

so

$$\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} \;=\; \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.$$

Thus the global phase $\phi \to e^{-i\theta}\phi$ acts as a rotation by angle θ in the (ϕ_1, ϕ_2) plane. Now, considering a small θ, we can expand the exponentials to first order in θ:

$$\begin{aligned}e^{-i\theta} &\simeq 1 - i\theta, \\ e^{+i\theta} &\simeq 1 + i\theta,\end{aligned}$$

so

$$\begin{aligned}\phi' &= (1 - i\theta)\phi = \phi - i\theta\phi, \\ \therefore\, \delta\phi &= -i\theta\,\phi \\ \phi'^\dagger &= (1 + i\theta)\phi^\dagger = \phi^\dagger + i\theta\phi^\dagger, \\ \therefore\, \delta\phi^\dagger &= +i\theta\,\phi^\dagger\end{aligned}$$

Now, we can find the conserved current using Noether's theorem:

$$\begin{aligned}J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\,\delta\phi^\dagger \\ &= \left(\partial^\mu\phi^\dagger\right)(-i\theta\,\phi) \;+\; \left(\partial^\mu\phi\right)(+i\theta\,\phi^\dagger) \\ &= \frac{i\theta}{2}\Big(\phi^\dagger\partial^\mu\phi \;-\; (\partial^\mu\phi^\dagger)\,\phi\Big).\end{aligned}$$

Then, the conserved charge Q can be found by integrating the charge density over space

$$Q = \int d^3x J^0 = \int d^3x \, \frac{i\theta}{2}\Big(\phi^\dagger \partial^0 \phi - \partial^0 \phi^\dagger \phi\Big).$$
$$= \int d^3x \frac{i\theta}{2}\Big(\phi^\dagger \pi^\dagger - \pi\phi\Big)$$

Knowing that all of ϕ, π (and their daggered counterparts) are quantized, then we can express the charge in terms of their mode expansions. For this we can compute $\partial^0\phi$ and $\partial^0\phi^\dagger$. Using

$$\begin{aligned}\partial^0 c_k e^{-ik\cdot x} &= c_k \partial^0 e^{-ik^0x^0 + i\mathbf{k}\cdot\mathbf{x}} \\ &= -ik^0 c_k e^{-ik\cdot x} \\ &= -i\omega_k c_k e^{-ik\cdot x}\end{aligned}$$

Therefore

$$\begin{aligned}\tfrac{1}{2}\partial^0\phi = \pi^\dagger(x) &= \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\Big((-i\omega_k)a_k e^{-ik\cdot x} + (i\omega_k)b_k^\dagger e^{ik\cdot x}\Big) \\ &= \frac{-i}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\Big(a_k e^{-ik\cdot x} - b_k^\dagger e^{ik\cdot x}\Big)\end{aligned}$$

and similarly

$$\pi(x) = \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\Big(a_k^\dagger e^{ik\cdot x} - b_k e^{-ik\cdot x}\Big)$$

And so the conserved charge is given by

$$\begin{aligned}\hat{Q} = \frac{i}{4}\int \frac{d^3x\, d^3k\, d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\Big[&\big(a_{k'}^\dagger e^{ik'\cdot x} + b_{k'} e^{-ik'\cdot x}\big)(-i)\big(a_k e^{-ik\cdot x} - b_k^\dagger e^{ik\cdot x}\big) \\ &- \big(a_{k'}^\dagger e^{ik'\cdot x} - b_{k'} e^{-ik'\cdot x}\big)(i)\big(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\big)\Big] \\ = \frac{1}{4}\int \frac{d^3x\, d^3k\, d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\Big[&\big(a_{k'}^\dagger a_k e^{i(k'-k)\cdot x} - a_{k'}^\dagger b_k^\dagger e^{i(k'+k)\cdot x} + b_{k'} a_k e^{i(k+k')\cdot x} - b_{k'} b_k^\dagger e^{-i(k-k')\cdot x}\big) \\ &+ \big(a_{k'}^\dagger a_k e^{-i(k'-k)\cdot x} + a_{k'}^\dagger b_k^\dagger e^{-i(k'+k)\cdot x} - b_{k'} a_k e^{i(k'+k)\cdot x} - b_{k'} b_k^\dagger e^{-i(k-k')\cdot x}\big)\Big]\end{aligned}$$

We now carry out the spatial integral. Each exponential gives either $e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}$ or $e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}}$, so

$$\int d^3x\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}'-\mathbf{k}), \qquad \int d^3x\, e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}} = (2\pi)^3\delta^{(3)}(\mathbf{k}'+\mathbf{k}).$$

$$\begin{aligned}\hat{Q} = \frac{1}{4}\int \frac{d^3x\, d^3k\, d^3k'}{(2\pi)^6}\sqrt{\frac{\omega_{k'}}{\omega_k}}\,\Big[\big(a_k^\dagger a_k - a_{-k}^\dagger b_k^\dagger + b_{-k}a_k - b_k b_k^\dagger\big) \\ + \big(a_k^\dagger a_k + a_{-k}^\dagger b_k^\dagger - b_{-k}a_k - b_k b_k^\dagger\big)\Big] \\ = \frac{1}{4}\int \frac{d^3k}{(2\pi)^3}\Big[2a_k^\dagger a_k - 2b_k b_k^\dagger\Big] \\ = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Big(a_k^\dagger a_k - b_k b_k^\dagger\Big).\end{aligned}$$

Using $b_k b_k^\dagger = b_k^\dagger b_k + (2\pi)^3\delta^{(3)}(0)$, we normal order and drop the infinite constant.

$$:\hat{Q}:= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\big(a_k^\dagger a_k - b_k^\dagger b_k\big).$$

Thus the conserved charge counts particles minus antiparticles.

Now, if the rotational angle is space dependent $\theta = \theta(x)$, we see that the Lagrangian is no longer invariant under $\phi \to e^{-i\theta(x)}\phi$ because the first term

$$\begin{aligned}\partial_\mu\phi\partial^\mu\phi^\dagger &\to \partial_\mu(e^{-i\theta(x)}\phi)\partial^\mu(e^{i\theta(x)}\phi) \\ &= \big(-i\partial_\mu\theta(x)e^{-i\theta(x)}\phi + e^{-i\theta(x)}\partial_\mu\phi\big)\big(i\partial_\mu\theta(x)e^{i\theta(x)}\phi^\dagger + e^{i\theta(x)}\partial_\mu\phi^\dagger\big) \\ &= \big((\partial_\mu\theta)^2\phi\phi^\dagger - i\partial_\mu\theta\phi\partial_\mu\phi^\dagger + i\partial_\mu\phi\partial_\mu\theta\phi^\dagger + \partial_\mu\phi\partial_\mu\phi^\dagger\big)\end{aligned}$$

For a small angle change, we can ignore the $(\partial_\mu\theta)^2$ term, leaving us with our new Lagrangian:

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi^\dagger - \frac{i}{2}\partial_\mu\theta(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger) - \frac{1}{2}m^2\phi\phi^\dagger$$

Which is obviously different than prior to the transformation. Recalling that the conserved current for the global gauge transformation was $J^\mu = \frac{i}{2}\big(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi\big)$, we can read off that the variation in the Lagrangian is

$$\begin{aligned}\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} &= -\frac{i}{2}\partial_\mu\theta(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger) \\ &= \partial_\mu\theta J^\mu\end{aligned}$$

as desired.

To show how the current is not invariant, take:

$$\begin{aligned}J'^\mu &\to \frac{i}{2}\big(e^{-i\theta(x)}\phi\partial_\mu(e^{i\theta(x)}\phi^\dagger) - \partial_\mu(e^{-i\theta(x)}\phi)e^{i\theta(x)}\phi^\dagger\big) \\ &= \frac{i}{2}\big(e^{-i\theta(x)}\phi(i\partial_\mu\theta(x)e^{i\theta(x)}\phi^\dagger + e^{i\theta(x)}\partial_\mu\phi^\dagger) - (-i\partial_\mu\theta(x)e^{-i\theta(x)}\phi + e^{-i\theta(x)}\partial_\mu\phi)e^{i\theta(x)}\phi^\dagger\big) \\ &= \frac{i}{2}\big(i\partial_\mu\theta\phi\phi^\dagger + \phi\partial_\mu\phi^\dagger + i\partial_\mu\theta\phi\phi^\dagger - \partial_\mu\phi\phi^\dagger\big) \\ &= \frac{i}{2}\big(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger\big) + |\phi|^2\partial_\mu\theta\end{aligned}$$

Such that

$$\delta J^\mu = |\phi|^2 \partial_\mu \theta$$

Now, let us consider the same transformation $\phi \to e^{-i\theta(x)}\phi$ but with our modified Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^\dagger \partial^\mu \phi - eJ^\mu A_\mu + \frac{e^2}{2}|\phi|^2 A_\mu A^\mu - \frac{1}{2}m^2\phi^\dagger\phi$$

Then the variation in the Lagrangian is

$$\begin{aligned}
\delta\mathcal{L} &= \delta\mathcal{L}_{\text{old}} + \delta\mathcal{L}_{\text{new}} \\
&= \partial_\mu\theta J^\mu - e\delta J^\mu A_\mu - eJ^\mu \delta A_\mu + \frac{e^2}{2}|\phi|^2\delta A_\mu A^\mu + \frac{e^2}{2}|\phi|^2 A_\mu \delta A^\mu \\
&= \partial_\mu\theta J^\mu - e(|\phi|^2\partial_\mu\theta)A_\mu - eJ^\mu\delta A_\mu + e^2|\phi|^2\delta A_\mu A^\mu \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \big(eJ^\mu - e^2|\phi|^2A^\mu\big)\delta A_\mu \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \big(eJ^\mu - e^2|\phi|^2A^\mu\big)\left(\frac{1}{e}\partial_\mu\theta\right) \\
&= \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) - \partial_\mu\theta\big(J^\mu - e|\phi|^2A^\mu\big) \\
&= 0
\end{aligned}$$

and so we see that if we choose $\delta A_\mu = \frac{1}{e}\partial_\mu\theta$, then the variation in the Lagrangian is zero, and thus the Lagrangian is invariant under the local gauge transformation.

To verify the kinetic term for the gauge field is also invariant, we note that, using the result above, A_μ transforms as $A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. Therefore:

$$\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu &\to \partial_\mu(A_\nu + \frac{1}{e}\partial_\nu\theta) - \partial_\nu(A_\mu + \frac{1}{e}\partial_\mu\theta) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{e}(\partial_\mu\partial_\nu\theta - \partial_\nu\partial_\mu\theta) \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

since partial derivatives commute. Thus the kinetic term for the gauge field is invariant. Specifically, if $(\partial_\mu A_\nu - \partial_\nu A_\mu)$ is invariant, then so is $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$.

Meanwhile, if we have some mass term such as $\frac{1}{2}m^2 A_\mu A^\mu$, then under the gauge transformation

$$\begin{aligned}
A_\mu A^\mu &\to (A_\mu + \frac{1}{e}\partial_\mu\theta)(A^\mu + \frac{1}{e}\partial_\mu\theta) \\
&= A_\mu A^\mu + \frac{2}{e}A^\mu\partial_\mu\theta + \frac{1}{e^2}(\partial_\mu\theta)^2
\end{aligned}$$

which is not equal to the original term, and so the mass term is not gauge invariant.

Now, let us introduce the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$. Then, under the gauge transformation, we have

$$\begin{aligned}
D_\mu\phi &= (\partial_\mu - ieA_\mu)\phi \\
&\to (\partial_\mu - ie(A_\mu + e^{-1}\partial_\mu\theta))e^{-i\theta}\phi \\
&= \partial_\mu(e^{-i\theta}\phi) - ieA_\mu e^{-i\theta}\phi - i\partial_\mu\theta e^{-i\theta}\phi \\
&= i\partial_\mu\theta e^{-i\theta}\phi + e^{-i\theta}\partial_\mu\phi - ieA_\mu e^{-i\theta}\phi - i\partial_\mu\theta e^{-i\theta}\phi \\
&= e^{-i\theta}\partial_\mu\phi - ieA_\mu e^{-i\theta}\phi \\
&= \big(\partial_\mu\phi - ieA_\mu\phi\big)e^{-i\theta}\phi \\
&= e^{-i\theta}D_\mu\phi
\end{aligned}$$

Thus, the covariant derivative of ϕ transforms in the same way as ϕ and not as its derivative.

Finally, to write the QED Lagrangian using the covariant derivative, we start with a well known definition for the electromagnetic field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Then, the QED Lagrangian can be written as

$$\mathcal{L}_{\text{QED}} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi^\dagger) - \frac{1}{2}m^2\phi\phi^\dagger - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Question 4: Quantum field theory in 2 + 1 dimensions

Computing the mode expansions

We start from the free real scalar field action in $(2+1)$ dimensions:

$$S[\phi] = \frac{1}{2}\int d^3x\,\big(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\big). \tag{10}$$

The corresponding Euler–Lagrange equation is the Klein-Gordon equation:

$$(\partial_t^2 - \nabla^2 + m^2)\,\phi = 0. \tag{11}$$

Plane

For the two dimensional square sheet (Surface A), which I will define to have the bottom left corner at $(0,0)$ and length L, we have boundary conditions $\phi(x=0) = \phi(x=L) = \phi(y=0) = \phi(y=L) = 0$. The Laplacian eigenfunctions satisfying these boundary conditions are separable:

$$\phi(t,x,y) = f(t)\,\sin(p_x x)\,\sin(p_y y),$$

where p_x and p_y are quantized by the boundary conditions. Imposing $\sin(p_x L) = 0$ and $\sin(p_y L) = 0$ gives

$$p_x = \frac{m\pi}{L}, \qquad p_y = \frac{n\pi}{L}, \qquad m,n \in \mathbb{Z}. \tag{12}$$

Substituting the separable ansatz

$$\phi(t,x,y) = f(t)\,\sin(p_x x)\,\sin(p_y y)$$

into the Klein–Gordon equation (11),

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0,$$

we evaluate each term separately.

$$\partial_t^2 \phi = \ddot{f}(t)\,\sin(p_x x)\,\sin(p_y y).$$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[f(t)\sin(p_x x)\sin(p_y y)\right] = -f(t)\,(p_x^2 + p_y^2)\,\sin(p_x x)\,\sin(p_y y),$$

Putting everything together:

$$(\partial_t^2 - \nabla^2 + m^2)\phi = \left[\ddot{f}(t) + (p_x^2 + p_y^2 + m^2)f(t)\right]\sin(p_x x)\sin(p_y y).$$

For the equation to hold for all x, y, the prefactor of the sine functions must vanish, giving the ordinary differential equation

$$\ddot{f}(t) + \omega_p^2 f(t) = 0,$$

with

$$\omega_p^2 = p_x^2 + p_y^2 + m^2. \tag{13}$$

The general solution of this ordinary differential equation is

$$f(t) = a_p e^{-i\omega_p t} + b_p e^{+i\omega_p t}. \tag{14}$$

Putting the spatial and temporal parts together, the complete field can be written as a double sum over discrete momentum modes:

$$\phi(t,x,y) = \sum_{m,n} \sin(p_x x)\sin(p_y y)\big(a_p e^{-i\omega_p t} + b_p e^{+i\omega_p t}\big), \tag{15}$$

Since we want $\phi = \phi^\dagger$, then we find that $b_p = a_p^\dagger$

Next, we require that $[\phi(x), \pi(y)] = i\delta^2(x-y)$ and so we must compute $\pi(y)$. This is given by

$$\begin{aligned}
\pi(y) &= \frac{\mathcal{L}}{\partial_0\phi} = \partial_0\phi \\
&= \sum_{m,n} \sin(p_x x)\sin(p_y y)\big((-i\omega_p)a_p e^{-i\omega_p t} + (i\omega_p)a_p^\dagger e^{+i\omega_p t}\big) \\
&= -i\sum_{m,n} \omega_p \sin(p_x x)\sin(p_y y)\big(a_p e^{-i\omega_p t} - a_p^\dagger e^{+i\omega_p t}\big)
\end{aligned}$$

Now, we compute the commutator. Let us first define $u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$. Then:

$$
\begin{aligned}
[\phi(x), \pi(y)] = &-i \sum_{m,n} \sum_{j,k} \omega_q u_n(x) u_m(y) u_j(x') u_k(y') \Big((a_p e^{-i\omega_p t} + a_p^\dagger e^{i\omega_p t})(a_q e^{-i\omega_q t} - a_q^\dagger e^{i\omega_q t}) \Big) \\
&+ i \sum_{m,n} \sum_{j,k} \omega_q u_n(x) u_m(y) u_j(x') u_k(y') \Big((a_q e^{-i\omega_q t} - a_q^\dagger e^{i\omega_q t})(a_p e^{-i\omega_p t} + a_p^\dagger e^{i\omega_p t}) \Big) \\
= &\, i \sum_{m,n} \sum_{j,k} \omega_q u_n(x) u_m(y) u_j(x') u_k(y') \times \Big((a_q e^{-i\omega_q t} - a_q^\dagger e^{i\omega_q t})(a_p e^{-i\omega_p t} + a_p^\dagger e^{i\omega_p t}) - \\
&(a_p e^{-i\omega_p t} + a_p^\dagger e^{i\omega_p t})(a_q e^{-i\omega_q t} - a_q^\dagger e^{i\omega_q t}) \Big) \\
= &\, i \sum_{m,n} \sum_{j,k} \omega_q u_n(x) u_m(y) u_j(x') u_k(y') \Big((a_q a_p e^{-i\omega_q t} e^{-i\omega_p t} - a_p a_q e^{-i\omega_p t} e^{-i\omega_q t}) + \\
&(a_q a_p^\dagger e^{-i\omega_q t} e^{i\omega_p t} - a_p^\dagger a_q e^{i\omega_p t} e^{-i\omega_q t}) - (a_p a_q^\dagger e^{-i\omega_p t} e^{i\omega_q t} - a_q^\dagger a_p e^{i\omega_q t} e^{-i\omega_p t}) + \\
&(a_q^\dagger a_p^\dagger e^{i\omega_q t} e^{-i\omega_p t} - a_q^\dagger a_p^\dagger e^{i\omega_q t} e^{-i\omega_p t}) \Big) \\
= &\, i \sum_{m,n} \sum_{j,k} \omega_q u_n(x) u_m(y) u_j(x') u_k(y') \Big([a_q, a_p] e^{-it(\omega_q + \omega_p)} + [a_q, a_p^\dagger] e^{-it(\omega_q - \omega_p)} - \\
&[a_p, a_q^\dagger] e^{-it(\omega_p - \omega_q)} + [a_q^\dagger, a_p^\dagger] e^{-it(\omega_p - \omega_q)} \Big)
\end{aligned}
$$

And using $[a_q, a_q] = [a_q^\dagger, a_p^\dagger] = 0$

$$
\begin{aligned}
[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] &= i \sum_{m,n} \sum_{j,k} \omega_{jk}\, u_n(x) u_m(y) u_j(x') u_k(y') \\
&\quad \times \Big([a_{jk}, a_{mn}^\dagger]\, e^{-i(\omega_{jk} - \omega_{mn})t} - [a_{mn}, a_{jk}^\dagger]\, e^{-i(\omega_{mn} - \omega_{jk})t} \Big) \\
&= i \sum_{m,n} 2\, \omega_{mn}\, C_{mn}\, u_n(x) u_n(x')\, u_m(y) u_m(y')
\end{aligned}
$$

where we have set

$$[a_{mn}, a_{m'n'}^\dagger] = C_{mn}\, \delta_{mm'} \delta_{nn'}$$

so that the phases cancel when $m = m'$, $n = n'$ and the two mixed terms contribute equally.

Using completeness twice,

$$\sum_{n=1}^{\infty} u_n(x) u_n(x') = \frac{L}{2} \delta(x - x'), \qquad \sum_{m=1}^{\infty} u_m(y) u_m(y') = \frac{L}{2} \delta(y - y'),$$

we obtain

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\, (L/2)^2 \left(\sum_{m,n} 2\, \omega_{mn}\, C_{mn} \right) \delta(x - x') \delta(y - y').$$

Imposing the canonical equal-time commutator on the box,

$$[\phi(t,\mathbf{x}),\pi(t,\mathbf{x}')] = i\,\delta(x-x')\delta(y-y'),$$

forces the coefficient of $\delta(x-x')\delta(y-y')$ to be 1 mode by mode. Hence

$$2\,\omega_{mn}\,C_{mn} = \frac{1}{(L/2)^2} \quad\Longrightarrow\quad C_{mn} = \frac{1}{(L/2)^2\,2\,\omega_{mn}}.$$

Equivalently, if we define rescaled mode operators

$$\tilde{a}_{mn} \;\equiv\; (L/2)\,\sqrt{2\,\omega_{mn}}\,a_{mn},$$

then their algebra is

$$[\tilde{a}_{mn},\tilde{a}^\dagger_{m'n'}] = (L/2)^2\,(2\,\omega_{mn})\,\delta_{mm'}\delta_{nn'},$$

and the normalized mode expansion is

$$\phi(t,x,y) = \sum_{m,n}\frac{1}{(L/2)\sqrt{2\,\omega_{mn}}}\,\sin(p_x x)\sin(p_y y)\Big(\tilde{a}_{mn}\,e^{-i\omega_{mn}t} + \tilde{a}^\dagger_{mn}\,e^{+i\omega_{mn}t}\Big).$$

Cylinder

Now, for the cylinder, the same boundary exist for for $\phi(y=0)=\phi(y=L)=0$ but now we impose instead $\phi(x=0)=\phi(x=2\pi R)$

A convenient separable basis is

$$u_m(x) = e^{ip_x x}, \qquad p_x = \frac{m}{R}, \quad m\in\mathbb{Z}, \qquad v_n(y) = \sin\Big(\frac{n\pi y}{L}\Big), \quad n\in\mathbb{Z}_+.$$

These obey the orthogonality/completeness relations

$$\int_0^{2\pi R} dx\,e^{i(p_x-p_x')x} = 2\pi R\,\delta_{m,m'},$$

$$\int_0^L dy\,\sin\Big(\frac{n\pi y}{L}\Big)\sin\Big(\frac{n'\pi y}{L}\Big) = \frac{L}{2}\,\delta_{n,n'},$$

$$\sum_{m\in\mathbb{Z}}\frac{e^{im(x-x')/R}}{2\pi R} = \delta(x-x'), \qquad \sum_{n=1}^{\infty}\frac{2}{L}\,\sin\Big(\frac{n\pi y}{L}\Big)\sin\Big(\frac{n\pi y'}{L}\Big) = \delta(y-y').$$

As we saw for the sheet, plug a separable ansatz into the Klein-Gordon equation. With

$$\phi(t,x,y) = f_{mn}(t)\,u_m(x)\,v_n(y),$$

one finds the same oscillator equation

$$\ddot{f}_{mn}(t) + \omega_{mn}^2 f_{mn}(t) = 0, \qquad \omega_{mn}^2 = p_x^2 + p_y^2 + m^2 = \Big(\frac{m}{R}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2 + m^2,$$

so

$$f_{mn}(t) = \beta_{mn}\,e^{-i\omega_{mn}t} + \beta^\dagger_{mn}\,e^{+i\omega_{mn}t}.$$

Reality of the field enforces the usual relation between opposite momenta. With the current basis it is convenient to keep β_{mn} and $\beta^\dagger_{mn}$ and let the commutator fix the normalization.

Thus the mode expansion is

$$\phi(t,x,y) = \sum_{m\in\mathbb{Z}}\sum_{n=1}^{\infty} v_n(y)\Big(\beta_{mn}\, e^{-i(\omega_{mn}t - p_x x)} + \beta^\dagger_{mn}\, e^{+i(\omega_{mn}t-p_x x)}\Big). \tag{16}$$

The canonical momentum is $\pi = \partial_t\phi$, so

$$\pi(t,x,y) = \sum_{m,n} v_n(y)\Big(-i\omega_{mn}\,\beta_{mn}\, e^{-i(\omega_{mn}t-p_x x)} + i\omega_{mn}\,\beta^\dagger_{mn}\, e^{+i(\omega_{mn}t-p_x x)}\Big).$$

Again, impose

$$[\phi(t,\mathbf{x}),\pi(t,\mathbf{x}')] = i\,\delta(x-x')\delta(y-y').$$

$$\begin{aligned}[\phi(t,x,y),\pi(t,x',y')] = \sum_{m,n}\sum_{r,s} &\sin\Big(\frac{n\pi y}{L}\Big)\sin\Big(\frac{s\pi y'}{L}\Big)\\ &\times\Big(\ (+i\omega_{rs})\, e^{-i(p_m x - p_r x')}\,[\beta_{mn},\beta^\dagger_{rs}] \ +\ (-i\omega_{rs})\, e^{+i(p_m x-p_r x')}\,[\beta^\dagger_{mn},\beta_{rs}]\Big).\end{aligned}$$

Now use the standard completeness relations on the cylinder:

$$\sum_{m\in\mathbb{Z}} e^{+ip_m\Delta x} = 2\pi R\,\delta(\Delta x), \qquad \sum_{n=1}^{\infty}\sin\Big(\frac{n\pi y}{L}\Big)\sin\Big(\frac{n\pi y'}{L}\Big) = \frac{L}{2}\,\delta(y-y').$$

thus,

$$[\phi(t,x,y),\pi(t,x',y')] = i\,(2\pi R)\,(L/2)\sum_{m,n}\omega_{mn}\,[\beta_{mn},\beta^\dagger_{mn}]\,\delta(x-x')\delta(y-y').$$

This alone only constrains the sum over modes. To get each mode separately, Multiply both sides of the above by

$$\frac{1}{(2\pi R)}\int_0^{2\pi R} dx\, e^{-ip_\ell x}\,\frac{2}{L}\int_0^L dy\,\sin\Big(\frac{n'\pi y}{L}\Big)\,\frac{1}{(2\pi R)}\int_0^{2\pi R} dx'\, e^{+ip_\ell x'}\,\frac{2}{L}\int_0^L dy'\,\sin\Big(\frac{n'\pi y'}{L}\Big)$$

And use completeness/orthogonality on the cylinder again. The left-hand side (from the canonical commutator $i\,\delta(x-x')\delta(y-y')$) cancels with the deltas on the RHS and leaves just. i. The right-hand side collapses the double sum to the single (ℓ, n') mode and produces the extra factor of 2 coming from ∂_t (i.e. the $\beta\beta^\dagger - \beta^\dagger\beta$ pair):

$$i \ = \ i\,(2\pi R)\Big(\frac{L}{2}\Big)\,\big(2\,\omega_{\ell n'}\big)\,[\beta_{\ell n'},\beta^\dagger_{\ell n'}]\,.$$

Therefore,

$$[\beta_{mn},\beta^\dagger_{m'n'}] = \frac{1}{(2\pi R)\,(L/2)\,(2\,\omega_{mn})}\,\delta_{mm'}\delta_{nn'} = \frac{1}{2\pi R\,L\,\omega_{mn}}\,\delta_{mm'}\delta_{nn'}\,,$$

and

$$[\beta_{mn}, \beta_{m'n'}] = [\beta^\dagger_{mn}, \beta^\dagger_{m'n'}] = 0.$$

Defining

$$b_{mn} \;\equiv\; \frac{\beta_{mn}}{\sqrt{2\pi R\, L\,\omega_{mn}}}\,, \qquad b^\dagger_{mn} \;\equiv\; \frac{\beta^\dagger_{mn}}{\sqrt{2\pi R\, L\,\omega_{mn}}}\,,$$

one immediately finds

$$[b_{mn}, b^\dagger_{m'n'}] = \delta_{mm'}\delta_{nn'}\,,$$

and the standard normalized mode expansion for ϕ.

$$\phi(t,x,y) = \frac{1}{\sqrt{\pi R L}} \sum_{m\in\mathbb{Z}} \sum_{n=1}^{\infty} \frac{\sin\!\left(\frac{n\pi y}{L}\right)}{\sqrt{2\omega_{mn}}} \Big(b_{mn}\, e^{-i(\omega_{mn}t - p_x x)} + b^\dagger_{mn}\, e^{+i(\omega_{mn}t - p_x x)} \Big), \tag{17}$$

Torus

Finally, for the torus, we impose periodic boundary conditions in both directions.

$$\phi(t, x + 2\pi R_x, y) = \phi(t,x,y), \qquad \phi(t,x,y+2\pi R_y) = \phi(t,x,y).$$

Use the separable plane-wave basis

$$u_m(x) = e^{ip_x x}, \quad p_x = \frac{m}{R_x}, \quad m \in \mathbb{Z}, \qquad v_n(y) = e^{ip_y y}, \quad p_y = \frac{n}{R_y}, \quad n \in \mathbb{Z}.$$

Orthogonality/completeness:

$$\int_0^{2\pi R_x} dx\, e^{i(p_x - p'_x)x} = 2\pi R_x\, \delta_{m,m'}, \qquad \sum_{m\in\mathbb{Z}} \frac{e^{im(x-x')/R_x}}{2\pi R_x} = \delta(x-x'),$$
$$\int_0^{2\pi R_y} dy\, e^{i(p_y - p'_y)y} = 2\pi R_y\, \delta_{n,n'}, \qquad \sum_{n\in\mathbb{Z}} \frac{e^{in(y-y')/R_y}}{2\pi R_y} = \delta(y-y').$$

As before, with the ansatz $\phi(t,x,y) = f_{mn}(t)\, u_m(x)\, v_n(y)$, the Klein–Gordon equation

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0$$

yields

$$\ddot{f}_{mn}(t) + \omega_{mn}^2 f_{mn}(t) = 0, \qquad \omega_{mn}^2 = p_x^2 + p_y^2 + m^2 = \left(\frac{m}{R_x}\right)^2 + \left(\frac{n}{R_y}\right)^2 + m^2,$$

so

$$f_{mn}(t) = \beta_{mn}\, e^{-i\omega_{mn}t} + \beta^\dagger_{mn}\, e^{+i\omega_{mn}t}.$$

(Reality implies $\beta^\dagger_{mn} = \beta_{-m,-n}$; we keep $\beta, \beta^\dagger$ and let the commutator fix the normalization.)

Thus the mode expansion is

$$\phi(t,x,y) = \sum_{m,n\in\mathbb{Z}} \Big(\beta_{mn}\, e^{-i(\omega_{mn}t - p_x x - p_y y)} + \beta^\dagger_{mn}\, e^{+i(\omega_{mn}t - p_x x + p_y y)} \Big), \tag{18}$$

and $\pi = \partial_t \phi$,

$$\pi(t,x,y) = \sum_{m,n} \Big(-i\omega_{mn}\beta_{mn}\, e^{-i(\omega_{mn}t - p_x x - p_y y)} + i\omega_{mn}\beta^\dagger_{mn}\, e^{+i(\omega_{mn}t - p_x x + p_y y)} \Big).$$

Again, impose

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\,\delta(x-x')\delta(y-y').$$

A direct computation gives, using the periodic completeness,

$$[\phi(t,x,y), \pi(t,x',y')] = i\,(2\pi R_x)\,(2\pi R_y) \sum_{m,n} \omega_{mn}\, [\beta_{mn}, \beta^\dagger_{mn}]\, \delta(x-x')\delta(y-y').$$

To isolate a single mode (ℓ, k), multiply both sides by

$$\frac{1}{2\pi R_x}\int_0^{2\pi R_x} dx\, e^{-ip_\ell x}\ \frac{1}{2\pi R_y}\int_0^{2\pi R_y} dy\, e^{-iq_k y}\ \frac{1}{2\pi R_x}\int_0^{2\pi R_x} dx'\, e^{+ip_\ell x'}\ \frac{1}{2\pi R_y}\int_0^{2\pi R_y} dy'\, e^{+iq_k y'},$$

with $q_k = k/R_y$. The LHS evaluates to i, while the RHS collapses the double sum to (ℓ, k) and picks up the usual extra factor 2 from the $\beta\beta^\dagger - \beta^\dagger\beta$ pair, yielding

$$i \;=\; i\,(2\pi R_x)(2\pi R_y)\,(2\,\omega_{\ell k})\,[\beta_{\ell k}, \beta^\dagger_{\ell k}].$$

therefore

$$[\beta_{mn}, \beta^\dagger_{m'n'}] = \frac{1}{(2\pi R_x)(2\pi R_y)\,(2\,\omega_{mn})}\,\delta_{mm'}\delta_{nn'} = \frac{1}{4\pi^2 R_x R_y\,\omega_{mn}}\,\delta_{mm'}\delta_{nn'}, \tag{19}$$

and $[\beta, \beta] = [\beta^\dagger, \beta^\dagger] = 0$.

Define

$$b_{mn} \equiv \frac{\beta_{mn}}{\sqrt{(2\pi R_x)(2\pi R_y)\,\omega_{mn}}}, \qquad [b_{mn}, b^\dagger_{m'n'}] = \delta_{mm'}\delta_{nn'}.$$

Then

$$\phi(t,x,y) = \frac{1}{\sqrt{(2\pi R_x)(2\pi R_y)}} \sum_{m,n\in\mathbb{Z}} \frac{1}{\sqrt{2\omega_{mn}}} \Big(b_{mn}\, e^{-i(\omega_{mn}t - p_x x - p_y y)} + b^\dagger_{mn}\, e^{+i(\omega_{mn}t - p_x x + p_y y)} \Big), \tag{20}$$

Propagators (From mode expansions)

Plane

Using the mode expansion for the plane, we have:

$$\langle 0|\, \phi(t,x,y)\phi(t',x',y')\,|0\rangle = \frac{4}{L^2}\,\langle 0|\, \Big[\sum_{m,n} \frac{\sin(p_x x)\sin(p_y y)}{\sqrt{2\omega_p}} \big(a_p e^{-i\omega_p t} + a^\dagger_p e^{i\omega_p t} \big) \Big]$$

$$\Big[\sum_{m',n'} \frac{\sin(p_{x'} x')\sin(p_{y'} y')}{\sqrt{2\omega_{p'}}} \big(a_{p'} e^{-i\omega_{p'} t'} + a^\dagger_{p'} e^{i\omega_{p'} t'} \big) \Big]\, |0\rangle$$

where we assume $t > t'$. Expanding the products, we see that only the term with $a_p a_{p'}^\dagger$ will contribute, since $a_p |0\rangle = 0$ and $\langle 0| a_{p'}^\dagger = 0$. Thus:

$$\begin{aligned}\langle 0| \phi(t,x,y)\phi(t',x',y') |0\rangle &= \frac{4}{L^2}\sum_{m,n}\sum_{m',n'}\frac{\sin(p_x x)\sin(p_y y)\sin(p_{x'}x')\sin(p_{y'}y')}{\sqrt{2\omega_p}\sqrt{2\omega_{p'}}} \\ &\quad \langle 0| a_p a_{p'}^\dagger |0\rangle e^{-i\omega_p t}e^{i\omega_{p'}t'} \\ &= \frac{4}{L^2}\sum_{m,n}\sum_{m',n'}\frac{\sin(p_x x)\sin(p_y y)\sin(p_{x'}x')\sin(p_{y'}y')}{\sqrt{2\omega_p}\sqrt{2\omega_{p'}}} \\ &\quad \delta_{mm'}\delta_{nn'}e^{-i\omega_p(t-t')} \\ &= \frac{4}{L^2}\sum_{m,n}\frac{\sin(p_x x)\sin(p_y y)\sin(p_x x')\sin(p_y y')}{2\omega_p}e^{-i\omega_p(t-t')}\end{aligned}$$

Cylinder

Using the mode expansion for the cylinder, we have:

$$\begin{aligned}\langle 0| \phi(t,x,y)\phi(t',x',y') |0\rangle = \frac{1}{\pi RL}\langle 0| &\Big[\sum_{m,n}\frac{\sin\left(\frac{n\pi y}{L}\right)}{\sqrt{2\omega_{mn}}}\left(b_{mn}e^{-i(\omega_{mn}t-p_x x)} + b_{mn}^\dagger e^{i(\omega_{mn}t-p_x x)}\right)\Big] \\ &\Big[\sum_{m',n'}\frac{\sin\left(\frac{n'\pi y'}{L}\right)}{\sqrt{2\omega_{m'n'}}}\left(b_{m'n'}e^{-i(\omega_{m'n'}t'-p_{x'}x')} + b_{m'n'}^\dagger e^{i(\omega_{m'n'}t'-p_{x'}x')}\right)\Big] |0\rangle\end{aligned}$$

where we assume $t > t'$. Expanding the products, we see that only the term with $b_{mn}b_{m'n'}^\dagger$ will contribute, since $b_{mn}|0\rangle = 0$ and $\langle 0| b_{m'n'}^\dagger = 0$. Thus:

$$\begin{aligned}\langle 0| \phi(t,x,y)\phi(t',x',y') |0\rangle &= \frac{1}{\pi RL}\sum_{m,n}\sum_{m',n'}\frac{\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n'\pi y'}{L}\right)}{\sqrt{2\omega_{mn}}\sqrt{2\omega_{m'n'}}} \\ &\quad \langle 0| b_{mn}b_{m'n'}^\dagger |0\rangle e^{-i(\omega_{mn}t-p_x x)}e^{i(\omega_{m'n'}t'-p_{x'}x')} \\ &= \frac{1}{\pi RL}\sum_{m,n}\sum_{m',n'}\frac{\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n'\pi y'}{L}\right)}{\sqrt{2\omega_{mn}}\sqrt{2\omega_{m'n'}}} \\ &\quad \delta_{mm'}\delta_{nn'}e^{-i(\omega_{mn}(t-t')-p_x(x-x'))} \\ &= \frac{1}{\pi RL}\sum_{m,n}\frac{\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi y'}{L}\right)}{2\omega_{mn}}e^{-i(\omega_{mn}(t-t')-p_x(x-x'))}\end{aligned}$$

Torus

Using the mode expansion for the torus, we have:

$$\langle 0|\,\phi(t,x,y)\phi(t',x',y')\,|0\rangle = \frac{1}{(2\pi)^2 R_x R_y}\langle 0|\Big[\sum_{m,n}\frac{1}{\sqrt{2\omega_{mn}}}\big(b_{mn}e^{-i(\omega_{mn}t-p_x x-p_y y)}+b^\dagger_{mn}e^{i(\omega_{mn}t-p_x x+p_y y)}\big)\Big]$$

$$\Big[\sum_{m',n'}\frac{1}{\sqrt{2\omega_{m'n'}}}\big(b_{m'n'}e^{-i(\omega_{m'n'}t'-p_{x'}x'-p_{y'}y')}+b^\dagger_{m'n'}e^{i(\omega_{m'n'}t'-p_{x'}x'+p_{y'}y')}\big)\Big]\,|0\rangle$$

where we assume $t > t'$. Expanding the products, we see that only the term with $b_{mn}b^\dagger_{m'n'}$ will contribute, since $b_{mn}\,|0\rangle = 0$ and $\langle 0|\,b^\dagger_{m'n'} = 0$. Thus:

$$\langle 0|\,\phi(t,x,y)\phi(t',x',y')\,|0\rangle = \frac{1}{(2\pi)^2 R_x R_y}\sum_{m,n}\sum_{m',n'}\frac{1}{\sqrt{2\omega_{mn}}\sqrt{2\omega_{m'n'}}}$$

$$\langle 0|\,b_{mn}b^\dagger_{m'n'}\,|0\rangle\,e^{-i(\omega_{mn}t-p_x x-p_y y)}e^{i(\omega_{m'n'}t'-p_{x'}x'+p_{y'}y')}$$

$$= \frac{1}{(2\pi)^2 R_x R_y}\sum_{m,n}\sum_{m',n'}\frac{1}{\sqrt{2\omega_{mn}}\sqrt{2\omega_{m'n'}}}$$

$$\delta_{mm'}\delta_{nn'}e^{-i(\omega_{mn}(t-t')-p_x(x-x')-p_y(y-y'))}$$

$$= \frac{1}{(2\pi)^2 R_x R_y}\sum_{m,n}\frac{1}{2\omega_{mn}}e^{-i(\omega_{mn}(t-t')-p_x(x-x')-p_y(y-y'))}$$

Propagators (From the Path Integral)