Problem Set 02

Note: The problem set is due September 26 before midnight. Please make sure your submission reflects your own work. For some more details on basic quantization of fields, you may refer to Peskin and Schroeder's "An introduction to quantum field theory".

1. **A field with an unusual action.** Let us consider a scalar field $\varphi(x) \equiv \varphi(x,t)$ in $3+1$ dimensions with the following action, that may appear a bit unusual to you:

$$S \;=\; \frac{1}{2}\int d^4x \left[\partial_\mu\varphi(x)\,\partial^\mu\varphi(x) \;-\; \varphi^2(x)\,\exp\left(\int d^4y\left(F_1(x-y)\,\varphi(y) + F_2(x-y)\,\varphi^3(y)\right)\right)\right]$$

   where $F_i(x-y)$, for $i = 1, 2$ are two given functions that depend on two points (x, y) in the 4d space-time. Due to the presence of this function, we have to add an integral on the exponential term. Determine the EOM for the scalar field and also the boundary term. Note: you might have to be extra careful while dealing with the integrand in the exponent. Determine, how your EOM and the boundary term change if I make the following replacement:

$$\varphi(y) \;\to\; \varphi(y) + \lambda\,\partial'_\mu\partial'^\mu\varphi(y)$$

   where λ is another parameter and ∂'_μ is defined with respect to the y coordinates. Provide quantitative answers.

2. **Simple QFT on a toroidal universe.** In the class we discussed how one should go about studying simple quantum field theory in $1+1$ dimensional space-time. We also elaborated on how the quantization procedure should be performed.

   (a) As a starter, please verify equations (2.25) till (2.31) of Peskin and Schroeder. The discussion is self-contained, so you should not have any issues.

   (b) Let us now assume that we have defined a real scalar field $\varphi(x,t) \equiv \varphi(x_1, x_2, t)$ on a two-dimensional torus, which we will refer to as our toroidal universe. The action for such a scalar field is defined in the standard way:

$$S \;=\; \int d^3x\,\mathcal{L} \;=\; \frac{1}{2}\int d^3x\left(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2\right)$$

   where m is the mass of the scalar field, $\mathcal{L}$ is the Lagrangian and the repeated indices are summed over. One could also determine the equations of motion (EOMs) from the Lagrangian. Since you may have just learnt to do this, let me just say that, given any field theory Lagrangian $\mathcal{L}$, the EOM may be derived from:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) \;-\; \frac{\partial\mathcal{L}}{\partial\varphi} \;=\; 0$$

   where as before, the repeated indices are summed over. Using the above formula, determine the EOMs for the scalar field on our toroidal universe. How is your result different from a flat $2+1$ dimensional universe?

   (c) Once you have the EOMs, the quantization procedure should follow the technique laid out in the class. In other words, you should be able to write the field φ now as an operator using the creation and the annihilation operators. Is it true that the quantization procedure will yield an infinite number of harmonic oscillators for this case? Provide quantitative details.

(d) How is the expression of the operator different from what we have for a flat $2+1$ dimensional universe? Provide quantitative details.

(e) We can simplify the above story by reducing one of the spatial dimension. For example, let us study QFT on a circle, i.e $1+1$ dimensional universe with circular topology and non-compact temporal direction. We will call this the circular universe. How does the EOM and the quantization procedure works now? Do you still get an infinite set of simple harmonic oscillators? If yes, can you express the field operator in terms of creation and annihilation operators? Again, provide quantitative details. Note that you will not require any resources beyond what we discussed in the class to solve both the problems. We could even modify the story by introducing complex scalar fields, but that's for another day.

3. **Simple QFT in $0+1$ dimensions.** Let us now study a slight variation of the above set of problems by reducing the spatial dimensions to nothing! In other words, let us define our QFT in zero space and one time dimensions. Consider again a real scalar field $\varphi(t)$ with a mass m. Using similar approach to above, determine the EOMs for the scalar field.

   (a) Let us now go to the quantum aspects of the theory. Using the procedure laid out in the class, do you think we can again express the QFT in terms of an infinite number of harmonic oscillators? If not, then how do you express the field operator φ in this case? Provide quantitative details to justify your answer.

   (b) One could also compute the Hamiltonian of the system for both the classical and the quantum fields. Again, you may not know how to derive the Hamiltonian from the action, so let me write the expression. Just like in classical mechanics of point particles, Hamiltonian requires us to define the so-called conjugate momentum $\Pi(x)$. This we can define as $\Pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)}$ which in turn is defined at a fixed time[1]. Using this the Hamiltonian is expressed in $d+1$ dimensions by the following expression[2]:

$$H \;=\; \int d^d x \left(\frac{1}{2}\Pi^2 \;+\; \frac{1}{2}(\nabla\varphi)^2 \;+\; \frac{1}{2}m^2\varphi^2\right)$$

   where $\nabla \equiv \partial_i \partial^i$ is the laplacian, with i being the spatial directions. Using this Hamiltonian, compute the lowest energy of the system and compare your answer with the two cases discussed before, i.e the toroidal and the circular universes. What differences do you see? Discuss quantitatively.

4. **QFT with massless particles.** All the three different types of QFT that we discussed above have massive particles. Let us now consider these cases with massless particles, i.e massless particles on toroidal, circular and $0+1$ dimensional universes. Using the quantization techniques developed above, do you notice any differences? For example do you still expect the fields φ in the three cases to be expressible in terms of infinite number of harmonic oscillators? Provide quantitative arguments.

5. **Schrödinger equation from quantum field theory.** One of the important equation for non-relativistic QM is the Schrödinger equation, so we should be able to recover it from

[1] Recall in classical mechanics (CM) $p = \frac{\partial L}{\partial \dot{q}}$ where $\dot{q} = \frac{\partial q}{\partial t}$ and q is the dynamical variable that parametrizes the position.

[2] Once you know the Lagrangian and the conjugate momentum, you should be able to derive this using standard formula for the Hamiltonian in classical mechanics!

QFT. We mentioned this briefly in the class so it's time now to recover Schrödinger equation quantitatively. First however let us clear up one subtlety that sometimes make us confuse between the classical and the quantum Schrödinger equations. To this end, let us take a real scalar field of mass m, and express the solution of the Klein-Gordon equation as $e^{-imt}\,\tilde{\varphi}(x,t)$, where $\tilde{\varphi}$ satisfies:

$$\ddot{\tilde{\varphi}} \;-\; 2im\,\dot{\tilde{\varphi}} \;-\; \nabla^2\tilde{\varphi} \;=\; 0$$

Derive the above equation. The above equation is purely relativistic and we can go to the non-relativistic limit by taking the momentum $|p| \ll m$. Argue that, in this limit $\tilde{\varphi}$ satisfies the following equation:

$$i\,\frac{\partial\tilde{\varphi}(x,t)}{\partial t} \;=\; -\,\frac{1}{2m}\,\nabla^2\tilde{\varphi}(x,t)$$

which is the classical Schrödinger equation (which means $\tilde{\varphi}$ is not a wave-function). To get the quantum Schrödinger equation we start by defining an operator $\varphi(x)$ as:

$$\varphi(x) \;=\; \int \frac{d^3k}{(2\pi)^3}\, a_{\boldsymbol{k}}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$

where $\boldsymbol{k} = (k_x, k_y, k_z)$ and $\boldsymbol{x} = (x, y, z)$ with $a_{\boldsymbol{k}}$ being the annihilation operator. Note the absence of $(2\omega_{\boldsymbol{k}})^{-1/2}$ factor in the non-relativistic limit. Is this consistent? Justify quantitatively. We can use the above definition to write the state $|x\rangle = \varphi^\dagger(x)|0\rangle$. Justify that the operator

$$X \;=\; \int d^3x\; \boldsymbol{x}\, \varphi^\dagger(x)\, \varphi(x)$$

can be viewed as a natural position operator from QFT. In a similar vein we can define a momentum operator P in a similar way by replacing x by p in the definition of X above[3]. Let us also define a state $|\psi\rangle \equiv \int d^3x\, \psi(x)\, |x\rangle$, where $\psi(x)$ is a function that will eventually be identified with Schrödinger wave-function. Show that the position and the momentum operators act on the state $|\psi\rangle$ as:

$$X|\psi\rangle \;=\; \int d^3x\; \boldsymbol{x}\, \psi(x)\, |x\rangle, \qquad P|\psi\rangle \;=\; \int d^3x\, \big(-\,i\nabla\psi(x)\big)\, |x\rangle$$

and from there justify the commutation relation: $[X_i, P_j]|\psi\rangle \;=\; i\delta_{ij}\,|\psi\rangle$ which is the key commutation relation between position and momentum operator in QM. Finally show that $\psi(x,t)$ satisfies:

$$i\,\frac{\partial\psi(x,t)}{\partial t} \;=\; -\,\frac{1}{2m}\,\nabla^2\psi(x,t)$$

which should tell you why $\psi(x,t)$ may indeed be regarded as the one-particle wave-function of QM. The derivation of the non-relativistic Schrödinger equation from QFT should tell you how the QFT formalism is powerful enough to reproduce all the known aspects of QM; and more.

[3] Can you derive them from Noether's theorem that we derived in the class?