Problem Set 1

Note: *This problem set is due September 9 by midnight. Please make sure that the submission reflects your own work. Note also that the chatGPT answers are mostly imprecise and in some case incorrect, so my suggestion would be to figure out the answers by yourself.*

1. Give short answers to the following questions. You only require undergraduate level physics to answer these questions.

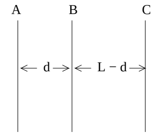
(a) Consider an electron moving with a momentum $\mathbf{p}$. At a given instant it emits a photon of momentum $\mathbf{q}$. Given that momentum is conserved in the process, show that energy can never be conserved.

*(b) What do you think would happen to the hydrogen atom ionisation energy if the photon picks up a small mass μ such that $\mu \to 0$? Give only qualitative discussions.

(c) Suppose a particle of mass m decays at rest into an e^+ and an e^- where $m > 2m_e$. If this decay occurs in a magnetic field B perpendicular to the plane of decay, estimate the radius of the orbits these charged particles travel in.

(d) Suppose a particle z is expected to be produced in the reaction $x + y \to z$. Imagine we device our simplest experimental setup by shooting the particle x of mass m_x to a stationary particle y of mass m_y to create the particle z of mass m_z. Show that the *total* energy of the z particle in our setup will be $E = \frac{(m_z^2 - m_x^2 + m_y^2)c^2}{2m_y}$. How does this energy change if I create the z particle by a head-on collision of x and y particles with equal and opposite momenta?

2. Consider a 1+1 dimensional non-compact subspace divided by two parallel and infinite plates A and C. These two plates are kept at a distance L from each other where L could be any number. Our aim here is to compute the vacuum energy of a quantum field using this apparatus. As is well known, the actual vacuum energy is of course not observable but a shift in the vacuum energy should be detectable. To determine the shift in vacuum energy and in turn calculate the force, we have to vary the distance between the plates. However this is not a very efficient technique because we also have to worry about how the energy density outside the two plates vary. To avoid this problem, we can use the following trick: insert another parallel plate between the original two plates. This plate is kept at a distance d from the first plate marked A (see figure below):



One immediate usefulness of inserting a third plate is that we don't have to worry about the world outside the plates. Assume now that the space between the plates is filled with a massless scalar field (i.e massless spin zero field in 1+1 dimensions)[1]. Show that the vacuum energy of the scalar field trapped between the plates is given by

$$E = \frac{\pi c \hbar}{2} \left(\sum_{n=1}^{\infty} \frac{n}{d} + \sum_{m=1}^{\infty} \frac{m}{L-d} \right) \tag{1}$$

The above formula, as can be easily seen, doesn't take into account the fact that high frequency modes can leak out of the plates to outside world. One way to implement this subtlety is to transform all integers n, m to:

$$n \to n\, e^{-\frac{an\pi}{d}}, \quad m \to m\, e^{-\frac{am\pi}{L-d}} \tag{2}$$

The above criteria, when substituted into (1) will guarantee that frequencies much higher than a^{-1} are automatically removed from the summation. This process is called *regularization* in quantum field theory. Using this regularization scheme, calculate the total vacuum energy E of the system.

Once you know the total energy it is easy to calculate the force between the plates due to vacuum fluctuation of the scalar field. Show that the force between the plates is given by

$$F \equiv -\frac{\partial E}{\partial d} \approx -\frac{\pi c \hbar}{24 d^2} \tag{3}$$

in the limit where $L >> d$. Due to the minus sign the force is in fact attractive, and because of the presence of $\hbar$ its a small quantum mechanical effect. This force has actually been observed, confirming the existence of vacuum energy due to quantum fields!

3. Imagine we want to measure the propagation of a point particle from x to y in say a 1+1 dimensional spacetime. The amplitude is given by $\mathcal{A} = \langle y | e^{-iHT} | x \rangle$, where H is the

[1] You should view this system as though the fields are all constrained to remain in 1+1 dimensions. Therefore the plates are two dimensional and all the interesting dynamics are happening in the 1+1 dimensional subspace, as mentioned at the beginning of the problem.

hamiltonian and T is the total time of propagation. Let us divide the distance between x and y into N points labelled by $(x, x_1, x_2, ..., x_{N-1}, y)$, and the time interval T into N equal intervals $\delta t \equiv \frac{T}{N}$. Show that the amplitude $\mathcal{A}$ can be divided as:

$$\mathcal{A} = \left(\prod_{j=1}^{N} \int dx_j\right) \langle y|e^{-iH\delta t}|x_{N-1}\rangle\langle x_{N-1}|e^{-iH\delta t}|x_{N-2}\rangle...\langle x_2|e^{-iH\delta t}|x_1\rangle\langle x_1|e^{-iH\delta t}|x\rangle \tag{4}$$

To analyse this we will focus on an individual factor $\langle x_{j+1}|e^{-iH\delta t}|x_j\rangle$, and consider the hamiltonian to be $H = \frac{\mathbf{p}^2}{2m}$ where $\mathbf{p}$ tells us that its an operator. Using this arrangement, show that:

$$\langle x_{j+1}|e^{-iH\delta t}|x_j\rangle \;=\; \sqrt{\frac{-2\pi i m}{\delta t}}\,\exp\left[\frac{im(x_{j+1}-x_j)^2}{2\delta t}\right] \tag{5}$$

From here show that once we plug in (5) in (4), we get:

$$\langle y|e^{-iHT}|x\rangle = \int \mathcal{D}x\; e^{i\int_0^T dt\frac{1}{2}m\dot{x}^2} \tag{6}$$

where $\int \mathcal{D}x = \lim_{N\to\infty}(-2\pi i m/\delta t)^{1/2}\prod_{j=0}^{N-1}\int dx_j$ and we have used the following replacement in the *continuum* limit $\delta t \to 0$:

$$\frac{x_{j+1}-x_j}{\delta t} \;\to\; \dot{x}, \qquad \delta t\sum_{j=0}^{N-1} \;\to\; \int_0^T dt \tag{7}$$

4. Give short answers to the following questions:

(a) Consider the example of a scalar field $\varphi(x,t)$ in (say) 1+1 dimensions with a lagrangian that is slightly different from the standard one in the following way:

$$\mathcal{L} \;=\; \int dx\left[\frac{1}{2}(\partial_0\varphi)^2 \;-\; \frac{1}{2}(\partial_x\varphi)^2 - \frac{\lambda}{4}\left(\varphi^2 - \frac{m^2}{\lambda}\right)^2\right] \tag{8}$$

where λ and m are the parameters of the theory. Now using (8) show that the system has two kinds of *classical* solutions: a trivial vacuum solution φ_0 and a non-trivial space dependent background solution $\varphi_{\rm cl}(x)$.

(b) Imagine we define a quantum field theory in five dimensions with coordinates (r, x, y, z, t) where t is the time coordinate and (x, y, z) are the standard three space dimensions. The lagrangian for the theory, written in a slightly unconventional way, is

$$S = \int \frac{d\omega d^3q}{(2\pi)^4} \int dr \Big[A(r)\phi(r,-\omega)\phi''(r,\omega) + B(r)\phi'(r,-\omega)\phi'(r,\omega) \\ + C(r)\phi(r,-\omega)\phi'(r,\omega) + D(r)\phi(r,-\omega)\phi(r,\omega) \Big] \tag{9}$$

where we have converted the 3+1 dimensional spacetime to its Fourier space with coordinates (q_x, q_y, q_z, ω). The field ϕ is a function of ω and r only (for simplicity), and $A(r), B(r), C(r), D(r)$ are some non-trivial functions of r whose details are not required for us. The derivatives appearing above are all wrt the fifth coordinate r. In addition to that, let us also assume that the five dimensional space has a *boundary* so that the boundary is a 3+1 dimensional slice. This means that we cannot eliminate the boundary term when we vary the action to determine the equation of motion. Keeping this in mind, determine the equation of motion of $\phi(r, \omega)$ and the boundary term.

This problem is not as difficult as it sounds, if you follow the steps carefully.