Master's thesis

Simo Tuomisto January 13, 2016

Contents

1 Introduction

2 Theory

2.1 On solar structure and its mathematical treatment

Astronomy is perhaps the oldest human science. For millenia people have looked upon the stars and wondered about their nature. Of these stars the Sun is by far the most interesting as it is the closest to our planet and provides energy that has enabled life to grow on Earth.

In this study the focus will be on the creation of magnetic fields in stars by processes within their convection layers. The focus if further limited to computational simulations of mean field models that are used in order to study the mean large scale magnetic fields of stars.

These models are built on foundation of knowledge regarding the solar structure. In this chapter a small review of this knowledge is undertaken. The goal of this review is to supply a starting point for hydrodynamic simulations of the convection zone, the mean field formulation of the magnetic field and simulations of this magnetic field.

The solar structure and theories that try to explain observed phenomena have been under heavy development during the last century. First problem that remained unsolved was one of energy creation. Calculations of energy that could be provided by chemical or gravitational processes were orders of magnitude lower than the energy Sun provides and this ongoing problem puzzled physicists troughout the 19th century. Finally Einstein's special relativity and advances in nuclear physics provided a method for solar energy production with nuclear fusion.

Nowdays it is known that within Sun's core protons combine into deuterium, hydrogen's heavier isotope, that will fusion with additional proton forming helium-3 and releasing energy through gamma radiation and electron neutrinos. This proton-proton chain releases 0.7 percent of mass as energy. An another contributor to energy production is CNO-chain that uses carbon, nitrogen and oxygen atoms as catalysts for hydrogen - helium fusion. This process is much more temperature dependent $\epsilon_{CNO} \propto T^{16}$, where as proton-proton chain has a much more softer temperature dependency $\epsilon_{pp} \propto T^4$.[4][1] This means that the process is dominant in larger stars, but for stars of Sun's scale proton-proton cycle is the the main energy production mechanism and this will be the focus in this study.

The first widely successful theories of solar structure were formulated based on integrating multiple differential equations based on thermodynamics. [7] [1] A usual set of equations would contain:

(a) Equation of density relating radius and mass.

(b) Equation of hydrostatic force balance relating pressure gradient with gravitational pull of matter.

(c) Equation of state relating pressure and temperature.

(d) Equation of luminocity relating energy generation and transport.

In total a system of equations like these would be used [1]

$$\frac{dM}{dr} = 4\pi\rho r^2 \tag{1}$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \tag{2}$$

$$P = \frac{\rho}{\mu} R_g T \tag{3}$$

$$\frac{dL}{dr} = 4\pi r^2 \rho\epsilon \tag{4}$$

where G is the gravitational constant, μ is the chemical potential and R_g is the specific gas constant for the gas mixture.

These models were complemented by studying what form the matter takes within the sun and what are the main methods of energy transport from Sun's core. The core, where temperature and pressure are high enough for fusion to happen, is contained within $R/R_\odot \leq 0.25$ and material there is completely ionized.

Further away from the core temperature lowers, but the material is still ionized. In this radiative zone the enourmous pressure causes the material to appear completely opaque to the radiation and photons will be absorbed and re-emitted to random directions. For a single photon it takes around 10^7 years of random walking in order to climb towards areas where it can travel more freely. Here energy transport is done by radiation and this zone extends from the core to the bottom of the convection zone at $R/R_\odot \approx 0.713$.[4]

At the bottom of the convection zone the a *convective instability* is reached. Convective instability is defined as

$$\nabla > \nabla_{\mathrm{ad}} \tag{5}$$

where $\nabla = \frac{\partial \ln T}{\partial \ln r}$ is the logarithmic temperature gradient and ∇_{ad} is the adiabatic temperature gradient.[4][3]

In essence this equation tells that if the temperature gradient caused by nuclear energy generation is higher than adiabatic one, radiative diffusion will not be able to transfer all of the generated energy to higher layers. Convection will start as parcels of material hotter than their surroundings will rise from the bottom of the convection zone. These hot packages will transfer heat energy to the higher layers and at a same time cooler and denser packages fall towards the bottom. In the sun the material rises near the equator, then travels towards the poles on the convection zone upper boundary and near the poles it travels back to deeper regions. The photosphere, where light finally escapes the solar interior and starts its journey towards Earth, is situated at the top of the convection zone.

Convection zone modelling requires equations that are more complicated than equations (1) - (4) that are used to a simulate static sun. In the convection zone the plasma is in a very turbulent state and it is completely ionized.

Equations that govern the dynamics in the convection zone are: [6]

(a) Mass continuity equation that describes density changes.

(b) Equation of motion that describes plasma movement.

(c) Induction equation that describes magnetic field evolution.

(d) Entropy equation that describes plasma thermodynamics.

These equations can be written in a multiple ways and depending on their complexity, many different phenomena can be incorporated. The equations that are shown next are chosen because they are used by the computational code, Pencil-code, that is used to simulate the convection zone.

Pencil-code is written to solve compressible Navier-Stokes equations with thermodynamics and magnetodynamics. Its equations can be written as [6]

$$\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u} \tag{6}$$

$$\begin{aligned} \frac{D\mathbf{u}}{Dt} &= -c_s^2 \nabla \left(\frac{s}{c_p} + \ln \rho \right) - \nabla \Phi_{\mathrm{grav}} + \frac{\mathbf{j} \times \mathbf{B}}{\rho} \\ &\quad + \nu \left(\nabla^2 \mathbf{u} + \frac{1}{3} \nabla\nabla \cdot \mathbf{u} + 2\mathbf{S} \cdot \nabla \ln \rho \right) + \zeta \left(\nabla\nabla \cdot \mathbf{u} \right) \end{aligned} \tag{7}$$

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \nu \mu_0 \mathbf{j} \tag{8}$$

$$\rho T \frac{Ds}{Dt} = \mathcal{H} - \mathcal{C} + \nabla \cdot K \nabla T + \nu \mu_0 \mathbf{j}^2 + 2\rho\nu \mathbf{S} \otimes \mathbf{S} + \zeta\rho(\nabla \cdot \mathbf{u})^2 \tag{9}$$

As is immeadiately apparent, these equations are three dimensional in space and take multiple phenomena into account.

In equation for the matter continuity (6) the density is calculated in logarithmic scale. In mathematical sense this does not change the results, but for computational methods this is much more suitable as it will protect from rounding errors. It is also very natural for a system like a sun, where density will drop from huge values inside the sun to near zero in outer atmosphere.[6]

The equation of motion (7) has multiple new terms. First term contains forces caused by entropy s and pressure. Here c_s is the speed of sound defined for perfect gas by

$$c_s^2 = \gamma \frac{p}{\rho} \tag{10}$$

Second term is the gradient of the gravitation potential Φ_{grav}. Third term contains Lorentz force in the absense of electrical fields. Fourth and fifth term encompasses viscosity in convection. Here ν is the kinematic viscosity, ζ is the bulk viscosity and $\mathbf{S}$ is the rate-of-shear tensor.[6]

The induction equation (8) is simply a Maxwellian induction equation written in terms of the magnetic vector potential. It is written in Weyl gauge ($\Phi = 0$) and electric fields are assumed to be slowly varying. [6] This assumption is applicable to the solar convection zone where, even though hydrogen is almost totally ionized, charge neutrality exists, electric fields are mainly induced by magnetic fields and turbulence damps these currents. [7] This equation can be, and of often is, written in terms of the magnetic field as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{Pm} \nabla^2 \mathbf{B} \tag{11}$$

where Pm is the magnetic Prandlt number. This form will also used in some chapters that are concerned with mathematical results, but the form (8) is the main equation because of its connection to the computational results.

The entropy equation contains terms for explicit heating $\mathcal{H}$ and cooling $\mathcal{C}$. In case of computational simulations, these are utilized in heating or cooling zones near the simulation borders that simulate solar radiative interior and atmospheric exterior respectively. Also included are terms for heat diffusion, Ohmic heating and friction due to shear and compression. [6]

Advent of efficient algorithms and powerful computers has allowed integration of these equations.

2.2 The origin of the solar magnetic field and the solar dynamo theory

The origin of solar magnetic field was an open question at the start of the 20th century. In 1908 Hale showed that there was a connection between sunspots, darker regions on the solar disk, and the solar magnetic field. Sunspots numbers had been counted since Galileo re-discovered them in the 1610 and statistical discoveries regarding them had been made in the 19th century. It was known that they usually appear in pairs of positive and negative polarity and they vary cyclically in numbers. In 1843 Schwabe formulated an empirical law that states that solar sunspot numbers vary cyclically with a period of 11 years.[7]

In 1858 Carrington discovered that during this cycle there exists a latitudial drift. Sunspots are born in the higher latitudes and from there they will migrate towards the equator. After measurements on solar magnetic field begun, it was also shown that the mean radial component of the solar magnetic field behaves in a similar fashion.[7] It was also found out that polarity of the magnetic field will alternate between cycles and as a result the radial component of the solar magnetic has a composite period of 22 years. This cyclical migration is usually shown as butterfly diagrams, where the number of sunspots on a latitude is plotted against time. The name of the diagram comes from the characteristic butterfly shape. [7][2]

Figure 1: Butterfly diagram

It took some time before theories started to explain the measurements. In the 19th century it was postulated that solar magnetic field could be a remnant from initial magnetic field that sun might have had before its formation. However calculations showed that this would be highly unlikely as Ohmic decay would destroy the field in the convection zone in a much shorter timescale than Sun's current lifetime. This meant that new methods needed to be invented. [1] [7]

Dynamo theories were suggested as a method of magnetic field generation. In these theories magnetic field generation was due to moving plasma in the convection zone. Motion of the plasma across pre-existing magnetic field lines would result in a current induction that would drive a new magnetic field. This interplay of magnetic field and plasma motion would then grow an initial magnetic field in an exponential fashion to a macroscopic magnetic field. These macroscopic fields would then cause phenomena like the sunspots and cyclical variations. [7]

Nowadays these models are accepted, but in their early years there was much resistance to the models. One of the reasons for this resistance was Cowling's theory in 1934 regarding steady fields. It showed that steady axisymmetric magnetic field could not be maintained by dynamo action. This caused a lot of problems for simple models as it showed that the solar dynamo problem is dynamic in nature and needed constant field generation. Only when hydrodynamic specialities and small scale phenomena were incorporated, the dynamo theory started to become the favored one.[5][7]

Firstly measurements showed that Sun's convection zone undergoes differential rotation, where the equatorial region has a faster rotation velocity than its polar regions. This added information could be used to show that toroidal field could be generated from the poloidal field as the faster rotation of the equatorial regions would cause poloidal field lines carried by the plasma to undergo elongation and turning. The net result would be a transformation of poloidal field into toroidal one.[7]

Secondly in 1955 Parker showed that rising and expanding plasma blobs containing flux tubes could be twisted by Coriolis force. The net result would be that turbulent eddies would twist the toroidal component of the magnetic field into a poloidal field. This α-effect would often be written as

$$\mathbf{E}_{\varphi} = \alpha \mathbf{B}_{\varphi} \tag{12}$$

Its name comes from the constant of proportionality α that describes the mean eddy velocity.[7]

This starting point lead to the formulation of *mean field theories*, that would try to explain the properies of mean magnetic fields in terms of the turbulent processes in the plasma.

2.3 Mean field theory

For a long time turbulent motions were regarded as destructive to large scale phenomena of physical system, but after it was found out that asymmetries in the turbulence could result in an emergent mean behaviour a well defined mathematical treatment was required. *Mean field electrodynamics* is a general theory that tries to predict large scale mean electric- and magnetic fields that are created by small scale turbulent motion. This theory gave answers to questions regarding solar magnetic field at a time when computational methods were not powerful enough to simulate turbulance in a system like the sun. [5]

In mean field theories relevant fields are separated into their mean and fluctuating parts. The mean part is defined as the ensemble mean of the field and it is denoted as $\overline{\mathbf{F}}$ for field F. This averaging operator is defined in a way that the following Reynold's relations, hold:

$$\begin{aligned} \mathbf{F} &= \overline{\mathbf{F}} + \mathbf{f} \\ \overline{\overline{\mathbf{F}}} &= \overline{\mathbf{F}} \\ \overline{\mathbf{f}} &= 0 \\ \overline{\mathbf{F} + \mathbf{G}} &= \overline{\mathbf{F}} + \overline{\mathbf{G}} \\ \overline{\overline{\mathbf{F}}\mathbf{G}} &= \overline{\mathbf{F}}\overline{\mathbf{G}} \\ \overline{\overline{\mathbf{F}}\mathbf{g}} &= 0 \end{aligned} \tag{13}$$

Here G is an another field and f and g are the fluctuating parts of the fields F and G respectively. The averaging operation is also interchangable with integration and differentation.[5]

Taking the induction equation (11) as the starting point and writing magnetic field $\mathbf{B}$ and velocity field $\mathbf{u}$ in terms of its mean and fluctuating parts results in a following equation: [5]

$$\frac{\partial(\overline{\mathbf{B}} + \mathbf{B}')}{\partial t} = \nabla \times ((\overline{\mathbf{u}} + \mathbf{u}') \times (\overline{\mathbf{B}} + \mathbf{B}')) + \frac{1}{Pm}\nabla^2(\overline{\mathbf{B}} + \mathbf{B}') \tag{14}$$

or

$$\begin{aligned} &\frac{\partial \mathbf{B}'}{\partial t} - \nabla \times (\overline{\mathbf{u}} \times \mathbf{B}') - \nabla \times (\mathbf{u}' \times \mathbf{B}') - \frac{1}{Pm}\nabla^2\mathbf{B}' \\ &= -\frac{\partial \overline{\mathbf{B}}}{\partial t} + \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times (\mathbf{u}' \times \overline{\mathbf{B}}) + \frac{1}{Pm}\nabla^2\overline{\mathbf{B}} \\ &= f(\overline{\mathbf{u}}, \mathbf{u}', \overline{\mathbf{B}}) \end{aligned} \tag{15}$$

where f is some functional of $\overline{\mathbf{u}}$, $\mathbf{u}'$ and $\overline{\mathbf{B}}$. Thus knowledge of velocity fields and the mean magnetic field will determine the fluctuations of the magnetic fields. [5] A corollary for this is that if the velocity fields and the magnetic field fluctuation $\mathbf{B}'$ are known, mean magnetic field $\overline{\mathbf{B}}$ can be solved from

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \frac{1}{Pm} \nabla^2 \overline{\mathbf{B}} + g(\overline{\mathbf{u}}, \mathbf{u}', \overline{\mathbf{B}}) \tag{16}$$

where g is some functional of $\overline{\mathbf{u}}$, $\mathbf{u}'$ and $\mathbf{B}'$. This is usually written in terms of turbulent electromotive force $\boldsymbol{\mathcal{E}}$ as $\nabla \times \boldsymbol{\mathcal{E}}$. Reason for this is that when the mean field averaging is applied to the Ohm's law

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \tag{17}$$

it will appear as the result. So

$$\begin{aligned}
\bar{\mathbf{j}} &= \sigma \overline{(\mathbf{E} + \mathbf{u} \times \mathbf{B})} \\
&= \sigma \left(\overline{\mathbf{E}} + \overline{\mathbf{u} \times \mathbf{B}} \right) \\
&= \sigma \left(\overline{\mathbf{E}} + \overline{(\overline{\mathbf{u}} + \mathbf{u}') \times (\overline{\mathbf{B}} + \mathbf{B}')} \right) \\
&= \sigma \left(\overline{\mathbf{E}} + \overline{\overline{\mathbf{u}} \times \overline{\mathbf{B}}} + \overline{\mathbf{u}' \times \overline{\mathbf{B}}} + \overline{\overline{\mathbf{u}} \times \mathbf{B}'} + \overline{\mathbf{u}' \times \mathbf{B}'} \right) \\
&= \sigma \left(\overline{\mathbf{E}} + \overline{\mathbf{u}} \times \overline{\mathbf{B}} + \overline{\mathbf{u}' \times \mathbf{B}'} \right) \\
&= \sigma \left(\overline{\mathbf{E}} + \overline{\mathbf{u}} \times \overline{\mathbf{B}} + \boldsymbol{\mathcal{E}} \right)
\end{aligned} \tag{18}$$

where mean field averages of quantities like $\overline{\mathbf{u}' \times \overline{\mathbf{B}}}$ are zero because $\overline{\overline{\mathbf{F}}\mathbf{g}} = 0$ and $\overline{\overline{\mathbf{u}} \times \overline{\mathbf{B}}} = \overline{\mathbf{u}} \times \overline{\mathbf{B}}$ because $\overline{\overline{\mathbf{F}}\overline{\mathbf{G}}} = \overline{\mathbf{F}}\overline{\mathbf{G}}$.

In the view of this result the effect of the turbulence on the mean magnetic field appears as an additional turbulent electromotive force field that will affect the mean field dynamics. [5]

Different analyzes of this turbulent electromotive force have been done. By writing a correlation tensor

$$P_{ik}(x, \xi, t, \tau) = \overline{\mathbf{u}'(x,t)\mathbf{B}'(x+\xi, t+\tau)} \tag{19}$$

that describes the interaction between velocity fluctuations and magnetic fluctuations in points of different times and locations a form

$$\mathcal{E}_i = \varepsilon_{ijk} P_{jk}(x, 0, t, 0) \tag{20}$$

for the turbulent electromotive force can be created. [5]

These interactions can be extended as a series of higher order correlations, but quite often these interactions are limited to few orders of fluctuating quantities. One of the most popular approximations is the second order correlation approximation (SOCA) that is sometimes also called the first order smoothing approximation. In this approximation higher than second order terms of interactions are not used and this gives closure to the feedback loop between interaction. Nevertheless, solving these equations is not in any means easy. [5] [**Schrinner**]

Other methodes include two-scale approach that expands the components of $\boldsymbol{\mathcal{E}}$ as a function of $\overline{\mathbf{B}}$ and its gradients. [2] In this model the components take form

$$\mathcal{E}_i = \alpha_{ij}(\hat{g}, \hat{\Omega}, \overline{\mathbf{B}}, ...)\overline{B}_j + \eta_{ijk}(\hat{g}, \hat{\Omega}, \overline{\mathbf{B}}, ...)\frac{\partial \overline{B}_j}{\partial x_k} \tag{21}$$

where $\hat{g}$ is the gravitational vector, $\hat{\Omega}$ is the rotational vector, ellipses describe other relevant quantities and α and η are called turbulent transport coefficients.

A famous covariant form for $\boldsymbol{\mathcal{E}}$ is [**Schrinner**]

$$\boldsymbol{\mathcal{E}} = -\alpha\overline{\mathbf{B}} - \gamma \times \overline{\mathbf{B}} - \beta(\nabla \times \overline{\mathbf{B}}) - \delta \times (\nabla \times \overline{\mathbf{B}}) - \kappa(\nabla\overline{\mathbf{B}})^{(\mathrm{sy})} \tag{22}$$

This is the form that will be used throughout this work.

Here α and β are symmetric tensors of second rank, γ and δ are vectors and κ is a tensor of third rank. α contributes to α-effect that twists magnetic field lines in rising flux tubes. γ is widely associated with magnetic pumping that transports mean magnetic flux. [**Schrinner**]

---

stuff about rotational turbulance, alpha-effect, gamma pumping etc

2.4 Test field method

Test field method, that aims to solve the turbulent electromotive force coefficients by using magnetohydrodynamic simulations as a reference, was first described by Martin Schrinner in in his doctoral dissertation.[**Schrinner**] The method goes as follows:

(a) Calculate azimuthal averages of turbulent electromotive forces from proper hydrodynamic simulations.

(b) Apply a magnetic test field and the mean velocity field to a two dimensional induction equation and solve resulting magnetic field and its derivative.

(c) Use multiple linearly independent test fields in order to describe a linear relation between magnetic fields and the turbulent electromotive force.

First step consists of calculating turbulent electromotive force from the full induction equation calculated during hydrodynamic simulations. The equations of Schrinner are similar to equations (6) - (9) described in chapter 2.1.

The form of induction equation he used is the one of equation (16)

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \boldsymbol{\mathcal{E}} + \frac{1}{Pm}\nabla^2 \overline{\mathbf{B}} \tag{23}$$

where Pm is the magnetic Prandtl number and the electromotive force is

$$\boldsymbol{\mathcal{E}} = \overline{\mathbf{u}' \times \mathbf{B}'} \tag{24}$$

This equation is been written in terms of the mean fields and magnetic field and velocity field have alredy been decomposed into their mean and residual fields

$$\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}' \qquad \mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \tag{25}$$

As Schrinner describes in his thesis, if $\overline{\mathbf{u}}$ has been used as a starting point for magnetohydrodynamic simulations and the $\mathbf{u}$ has been recorded, the only information required to calculate $\boldsymbol{\mathcal{E}}$ is contained in the residual magnetic fields, that can be calculated from its own induction equation

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \mathbf{B}') + \nabla \times (\mathbf{u}' \times \overline{\mathbf{B}}) + \nabla \times \boldsymbol{G} + \frac{1}{Pm}\nabla^2 \mathbf{B}' \tag{26}$$

where $\boldsymbol{G} = \mathbf{u}' \times \mathbf{B}' - \overline{\mathbf{u}' \times \mathbf{B}'}$.

Now using this equation and the knowledge that $\boldsymbol{\mathcal{E}} = \overline{\mathbf{u}' \times \mathbf{B}'}$ is a functional of $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}'$ Schrinner shows that there is a integral representation for $\boldsymbol{\mathcal{E}}$ and by assuming that $\boldsymbol{\mathcal{E}}$ depends instantaneously on $\overline{\mathbf{B}}$ a Taylor series expansion form for $\boldsymbol{\mathcal{E}}$ can be found

$$\mathcal{E}_i = a_{ij}\overline{\mathbf{B}}_j + b_{ijk}\frac{\partial \overline{\mathbf{B}}_j}{\partial x_k} \tag{27}$$

Here the coefficients a_{ij} and b_{ijk} can be written as integrals of kernel K over a small range of x and t.

In spherical coordinates the coefficients have to be written in covariant form and equation ($\boldsymbol{\mathcal{E}}$ expansion) is written as

$$\mathcal{E}^{\kappa} = \hat{a}^{\kappa}_{\lambda}\overline{\mathbf{B}}^{\lambda} + \hat{b}^{\kappa}{}_{\lambda}{}^{\mu}\overline{\mathbf{B}}^{\lambda}{}_{;\mu} \tag{28}$$

This can be rewritten as

$$\boldsymbol{\mathcal{E}} = \boldsymbol{a}\overline{\mathbf{B}} + \boldsymbol{b}\nabla\overline{\mathbf{B}} \tag{29}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are second rank tensors. They can be split into their symmetric and antisymmetric parts and that results in a form

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\alpha}\overline{\mathbf{B}} - \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}(\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\delta} \times (\nabla \times \overline{\mathbf{B}}) - \boldsymbol{\kappa}(\nabla\overline{\mathbf{B}})^{(\mathrm{sy})} \tag{30}$$

The tensors and vectors in this equation contribute to different dynamo effects. Schrinner takes into account expected symmetry properties of these coefficients when they have been written in spherical coordinate system. This results in a expansion

$$\mathcal{E}_{\kappa} = \tilde{a}_{\kappa\lambda}\overline{\mathbf{B}}_{\lambda} + \tilde{b}_{\kappa\lambda r}\frac{\partial \overline{\mathbf{B}}_{\lambda}}{\partial r} + \tilde{b}_{\kappa\lambda\theta}\frac{1}{r}\frac{\partial \overline{\mathbf{B}}_{\lambda}}{\partial \theta} \tag{31}$$

There are 27 independent coefficients and the aim of the test field process is to find these coefficients.

Next step of the process is writing the equation for the residual magnetic field

$$\frac{\partial \mathbf{B}'}{\partial t} - \nabla \times (\overline{\mathbf{u}} \times \mathbf{B}') - \nabla \times \boldsymbol{G} - \frac{1}{Pm}\nabla^2\mathbf{B}' = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}_T) \tag{32}$$

with a predefined magnetic test field $\overline{\mathbf{B}}_T$. In the words of Schrinner: "$\boldsymbol{\mathcal{E}}$ is 'measured' due to the action of the velocity field on a prescribed mean test field $\overline{\mathbf{B}}_T$."

This equation can be written as

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{u}' \times (\overline{\mathbf{B}}_T + \mathbf{B}'))' + \frac{1}{Pm} \nabla^2 \mathbf{B}' \tag{33}$$

and using a magnetohydrodynamic solver similar to the one used to solve the initial induction equation, this equation can be solved. Given the velocity field from a complex simulation and three orthogonal test fields $\overline{\mathbf{B}}_{T\lambda}$ three linear equations can be solved at a time. Using nine different test fields a linear relation

$$\mathcal{E}_\kappa^{(i)} = \left(\overline{\mathbf{B}}_{T\lambda}^{(i)}, \frac{\partial \overline{\mathbf{B}}_{T\lambda}^{(i)}}{\partial x_\mu} \right) \begin{pmatrix} \tilde{a}_{\kappa\lambda} \\ \tilde{b}_{\kappa\lambda\theta} \end{pmatrix} \quad , i = 1, ..., 9 \tag{34}$$

can be created and as the system is fully determined, the coefficients can be solved.

3 Mean magnetic field simulations

3.1 Implementation

3.1.1 Pencil code

The implementation of the mean field induction equation was written into Pencil Code-program. The Pencil Code is a high order finite-difference hydrodynamics and magnetohydrodynamics solver. It is a moduler program that can be easily extended. The existing modules include hydrodynamic, magnetic and test field modules. [6]

The magnetic mean field module was created using a so-called "special" physics module as a starting point. Special modules are modules that modify the equations to be solved with minimal changes required to the existing code. Hooks set in the main modules will call subroutines in the special modules that can modify variables in order to change the physics in question. [6]

In case of the mean field module the focus is in the magnetic module that handles the evolution of the induction equation. Induction equation that the code solves is

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{j} \tag{35}$$

This equation supposes the Weyl gauge ($\Phi = 0$) and slowly varying electrical fields. This equation is not a mean field equation, but it will be presented in the mean field fashion in the following section.

The name of the program comes from its utilization of "pencils", x dimension sized vectors that contain relevant physical quantities. These are calculated for each gridpoint in y- and z-directions in order to reduce cache misses. New state for the system is then calculated using .

The Pencil Code uses sixth-order derivatives in it's finite difference model. Time stepping is done by a 2N-scheme. [6]

Cite some Runge-Kutta book

3.1.2 The induction equation

As said in the previous section, the non-mean field induction equation (eq. (45)) in the code is implemented in Weyl gauge ($\Phi = 0$) and it is written as

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{j} \tag{36}$$

Scrinner's mean field induction equation was in turn written in terms of mean magnetic fields:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \boldsymbol{\mathcal{E}} + \frac{1}{Pm} \nabla^2 \overline{\mathbf{B}} \tag{37}$$

Thus in order to utilize the existing solver, this mean field induction equation has to be written in terms of mean vector potential.

The last term in the equation can be rewritten by using the vector identity

$$\nabla \times \nabla \times \mathbf{x} = \nabla(\nabla \cdot \mathbf{x}) - \nabla^2 \mathbf{x} \tag{38}$$

and the Gauss's law for magnetism

$$\nabla \cdot \overline{\mathbf{B}} = 0 \tag{39}$$

to get

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \boldsymbol{\mathcal{E}} - \frac{1}{Pm} \nabla \times \nabla \times \overline{\mathbf{B}} \tag{40}$$

Now

$$\overline{\mathbf{B}} = \nabla \times \overline{\mathbf{A}} \tag{41}$$

and applying this to the left side of the equation (40) results in

$$\nabla \times \left(\frac{\partial \overline{\mathbf{A}}}{\partial t} - \overline{\mathbf{u}} \times \overline{\mathbf{B}} - \boldsymbol{\mathcal{E}} + Pm^{-1} \nabla \times \overline{\mathbf{B}} \right) = 0 \tag{42}$$

Uncurling this equation results in

$$\frac{\partial \overline{\mathbf{A}}}{\partial t} = \overline{\mathbf{u}} \times \overline{\mathbf{B}} + \boldsymbol{\mathcal{E}} - Pm^{-1} \nabla \times \overline{\mathbf{B}} + \nabla \varphi \tag{43}$$

where φ is the electric potential. Taking Weyl's gauge sets this term to zero, as $\varphi = 0$.

Now the last term can be rewritten by starting from Ampere's law

$$\nabla \times \overline{\mathbf{B}} = \mu_0 \left(\bar{\mathbf{j}} + \epsilon_0 \frac{\partial \overline{\mathbf{E}}}{\partial t} \right) \tag{44}$$

and assuming that the overall system is an approximation with slowly varying electric fields $\left(\frac{\partial \overline{\mathbf{E}}}{\partial t} = 0 \right)$. In addition $Pm = \nu \eta^{-1}$ and the end result is

$$\frac{\partial \overline{\mathbf{A}}}{\partial t} = \overline{\mathbf{u}} \times \overline{\mathbf{B}} + \boldsymbol{\mathcal{E}} - \frac{\mu_0 \eta}{\nu} \bar{\mathbf{j}} \tag{45}$$

The end result is a similar equation to the typical induction equation (eq. 45), but this time an additional term is included. This turbulent electromotive force contains the contributions from smaller scale dynamic processes.

3.1.3 Technical implementation

Information regarding parallel IO, HDF5 and Python

3.2 Magnetohydrodynamic simulations

3.2.1 Physical parameters of the simulations

Information regarding Millenium-simulation Plots of physical fields

---

3.2.2 Parameters of the turbulent electromotive force

Information regarding the parameters of the emf Plots of these

---

4 Results

4.0.1 Parameters of the turbulent electromotive force

Parameter evolution during different epochs

4.1 Simulated mean magnetic field

Information obtained from simulating the mean magnetic field

4.2 Comparison of mean field- and magnetohydrodynamic simulations

Self-evident

5 Conclusions

6 Sources

References

[1] Erika Böhm-Vitense. *Introduction to stellar astrophysics. Vol. 3, Stellar structure and evolution*. Cambridge: Cambridge University Press, 1992. ISBN: 0-521-34404-2.

[2] Axel Brandenburg and Kandaswamy Subramanian. 'Astrophysical magnetic fields and nonlinear dynamo theory'. In: *Physics Reports* 417.1?4 (2005). ISSN: 0370-1573. DOI: `http://dx.doi.org/10.1016/j.physrep.2005.06.005`.

[3] Gary Glatzmaier. *Lecture notes on Physics of Stars*. 2013. URL: `http://es.ucsc.edu/~glatz/astr_112/lectures/notes12.pdf`.

[4] Petri Käpylä. 'Local numerical modelling of magnetoconvection and turbulence - implications for mean-field theories'. PhD thesis. University of Helsinki, 2006. ISBN: 952-10-3396-7.

[5] 'Mean-Field Magnetohydrodynamics and Dynamo Theory'. In: *Mean-Field Magnetohydrodynamics and Dynamo Theory*. Ed. by F. Krause and K.-H. Rädler. Pergamon, 1980. ISBN: 978-0-08-025041-0.

[6] *Pencil Code manual*. 2015.

[7] Eric Ronald Priest. *Solar magnetohydrodynamics*. Dordrecht: D. Reidel, 1982. ISBN: 90-277-1374-X.