

Calculus written exam 2026

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AI503: Calculus

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1. Problem 1

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by

$$f(x, y, z) = x^2 + y^2 + z^2$$

- (a) Determine the domain and range of f
- (a)
- (b) Describe geometrically the level set formed by $f(x, y, z) = 100$.

1.1. Solution

- (a) Determine the domain and range of f
- $D(f) = \{(x, y, z) : (-\infty, +\infty)\}$
 - $R(f) = [0, +\infty)$
- (b) Describe geometrically the level set formed by $f(x, y, z) = x^2 + y^2 + z^2 = 100$.
- f represents a sphere, generally on the form $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
 - The sphere would have center at $0, 0, 0$
 - And a radius of $\sqrt{100} = 10$

2. Problem 2

(a) Compute the following limit or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x + y^2}{2x + y}$$

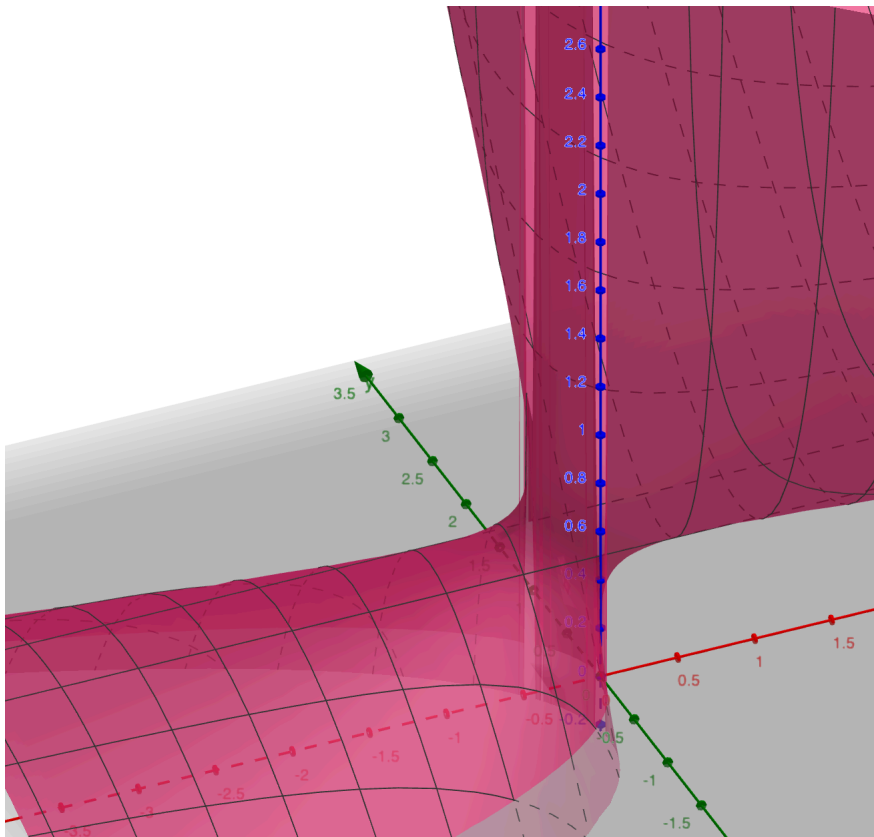
(b) Is the following function continuous along

$$f(x, y) = \begin{cases} x - 1 & y \geq 0, \\ -2 & y < 0 \end{cases}$$

2.1. Solution

(a) Compute the following limit or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x + y^2}{2x + y}$$



$$f(x, y) = \frac{x + y^2}{2x + y} \text{ graphed in GeoGebra}$$

Then i can constant y and check for x -

$$y = 0 \Rightarrow \lim_{x \rightarrow 0} \frac{x + y^2}{2x + y} = \frac{1}{2}$$

$$y = x \Rightarrow \lim_{x \rightarrow 0} \frac{x + y^2}{2x + y} = \frac{1}{3}$$

Since $\frac{1}{2} \neq \frac{1}{3}$

The limit **does not** exist

(b) Is the following function continuous along

$$f(x, y) = \begin{cases} x - 1 & y \geq 0, \\ -2 & y < 0 \end{cases}$$

- **Yes**, regardless of (x, y) the f is defined, therefore it is continuous

3. Problem 3

Let $g(x, y, z) = x^2 - 2xy^2 + az - a$

- (a) Find an equation of the tangent plane to g at the point $(1, 1, 1)$.
 (b) For which value of a does the tangent plane pass through the origin?

3.1. Solution

- (a) Find an equation of the tangent plane to g at the point $(1, 1, 1)$.

The equation of the tangent plane at the point (x_0, y_0, z_0) is given by:

$$\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

In our case, $(0, -4 + a, a) \cdot (x - 1, y - 1, z - 1) = 0$

This expands to:

$$0(x - 1) + (-4 + a)(y - 1) + a(z - 1) = 0$$

$$(-4 + a)(y - 1) + a(z - 1) = 0$$

$$(a - 4)y - (a - 4) + az - a = 0$$

$$(a - 4)y + az - (2a - 4) = 0$$

- (b) For which value of a does the tangent plane pass through the origin?

For this i plug in $(0, 0, 0)$

$$(a - 4)(0) + a(0) - (2a - 4) = 0$$

$$-(2a - 4) = 0$$

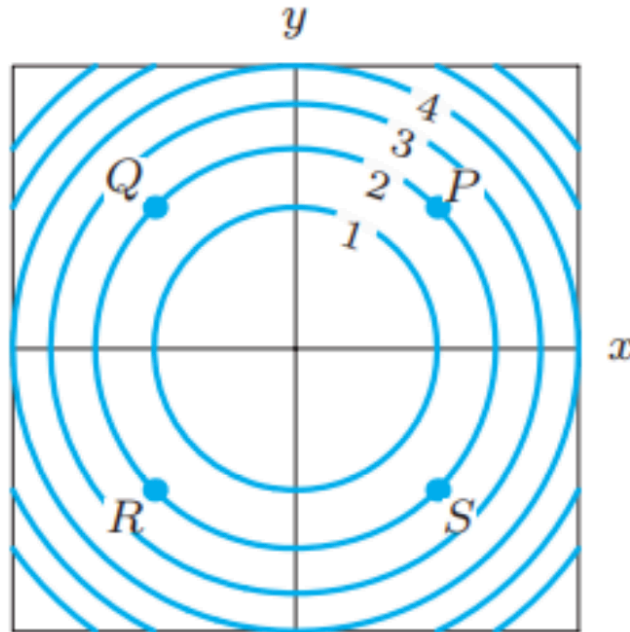
$$2a - 4 = 0$$

$$2a = 4a = 2$$

4. Problem 4

Below is a contour diagram of $f(x, y)$. In each of the following cases, list the marked points in the diagram (there may be none or more than one) at which

- (a) $f_x < 0$ (b) $f_y > 0$ (c) $f_{xx} > 0$ (d) $f_{yy} < 0$



Hint: what kind of function $f(x, y)$ would produce a contour plot like the one shown? Compute its f_x , f_y , f_{xx} and f_{yy} .

4.1. Solution

- (a) For $f_x < 0$ this means that the partial derivative is negative on the x -axis. Since the contour plot is showing an increase $f_x > 0$.

So $f_x < 0$ is false for this contour plot

- (b) As for $f_y > 0$ This means that the partial derivative is positive on the y -axis. Since the contour plot is showing an increase $f_y > 0$.

So $f_y < 0$ is true for this contour plot.

- (c) $f_{xx} > 0$

- (d) $f_{yy} < 0$

5. Problem 5

Let $F(x) = \int_0^x \cos(\pi t) \, dt$.

- (a) Evaluate $F(1)$.
- (b) For what values of x is $F(x)$ positive? negative?

5.1. Solution

- (a) Evaluate $F(1)$.

Since

$$F(x) = \int_0^x \cos(\pi t) \, dt = \int \cos(\pi x) \, dx$$

and since

$$F(x) = \int \cos(\pi x) \, dx = \frac{\sin(\pi x)}{\pi}$$

$$F(1) = \frac{\sin(\pi)}{\pi} = 0$$

- (b) For what values of x is $F(x)$ positive? negative?

Since $\sin(a \cdot \pi) = 0$ for any a

$$F(x) = 0, \quad \text{for any } x$$

6. Problem 6

Let $T(x, y) = e^{-x^2y^2}$ represent the heat of the point (x, y) .

- (a) In which direction should a heat-seeking bug move from the point $(-1, 1)$ to increase its temperature fastest?
- (b) Find the directional derivative of T at the point $P = (-1, 1)$ in the direction of $(1, 1)$.
- (c) Does T have a maximum or a minimum? At where? Hint: You don't need to compute second-order derivatives. Consider that exp is monotone increasing and how that affects the extrema of T .

6.1. Solution

- (a) In which direction should a heat-seeking bug move from the point $(-1, 1)$ to increase its temperature fastest?

In the direction of $\nabla f(-1, 1)$

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = (-2e^{-x^2y^2}xy^2, -2e^{-x^2y^2}x^2y)$$

Then

$$\nabla f(-1, 1) = -2e^{-(-1)^2 \cdot 1^2}xy^2, -2e^{-x^2y^2}x^2y = \frac{2}{e}, -\frac{2}{e}$$

- (b) Find the directional derivative of T at the point $P = (-1, 1)$ in the direction of $(1, 1)$.
first find the unit vector of $(1, 1)$

$$u = \frac{1, 1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$f_u = \nabla f(x, y) \cdot u = \left(\frac{2}{e}, -\frac{2}{e} \right) \cdot (1, 1) = 0$$

- (c) Does T have a maximum or a minimum? At where? Hint: You don't need to compute second-order derivatives. Consider that exp is monotone increasing and how that affects the extrema of T .

7. Problem 7

Consider the function

$$f(x, y) = |xy|$$

(a) Is f differentiable at $(1, 0)$? Explain.

(b) Is f differentiable at $(0, 0)$? Explain.

7.1. Solution

f is differentiable on a point (x, y) if $\lim_{(x,y) \rightarrow (a,b)} f(x, y) - \frac{f(a,b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$

(a) Is differentiable at $(1, 0)$? Explain.

$$\lim_{(x,y) \rightarrow (1,0)} f(x, y) - \frac{f(1,0)}{\sqrt{(x-1)^2 + (y)^2}}$$

We check this by setting a constant

$$\text{For } y = 0, \quad \lim_{x \rightarrow 0} = 0$$

$$\text{For } y = x, \quad \lim_{x \rightarrow 0} = 1$$

Since the limit **does not** exist, $f(x, y)$ **is not** differentiable on point $(1, 0)$

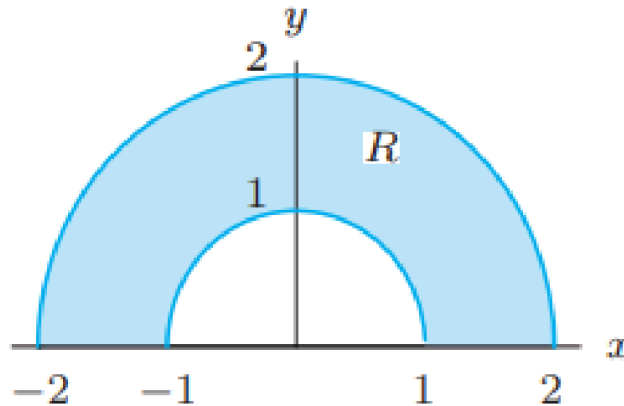
(b) Is f differentiable at $(0, 0)$? Explain.

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) - \frac{f(0,0)}{\sqrt{(x)^2 + (y)^2}} = 0$$

Since the limit **does** exist, $f(x, y)$ **is** differentiable on point $(1, 0)$

8. Problem 8

Use polar coordinate to evaluate $\int_R \sqrt{x^2 + y^2}$ where R is given in the following figure.



8.1. Solution

We want to find

$$\int_R \sqrt{x^2 + y^2} \, dA$$

where $x^2 + y^2 = r^2$ so

and θ must be half a circle: π

$$\int_{\theta=0}^{\pi} \int_{r=0}^1 r \, r \, dr \, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^1 r^2 \, dr \, d\theta$$

Now integrate

$$\int_{\theta=0}^{\pi} \frac{1}{3} \, d\theta = \pi = \frac{\pi}{3}$$

9. Problem 9

Evaluate the following integral by using cylindrical coordinate or spherical coordinate:

$$\iiint_W \frac{z}{(x^2 + y^2)^{\frac{2}{3}}} dV, \quad W = \{(x, y, z) : 0 \leq x^2 + y^2 \leq 4, 0 \leq z \leq 4\}$$

9.1. Solution

in this case this is:

$$0 = \frac{z}{(x^2 + y^2)^{\frac{2}{3}}} \Rightarrow z = (x^2 + y^2)^{\frac{2}{3}}$$

$$\iiint_W z dV$$

Because the of the cone, polar coordinates are easier to work with

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad \underbrace{z = z}_{\text{figure on } z}, \quad dV = r dz dr d\theta$$

In this instance θ is the whole circle around the z axis so $0 \leq \theta \leq 2\pi$

To find r we can use the definition of the circle at $z = 4$

Because of $4 = (x^2 + y^2)^{\frac{2}{3}} \Rightarrow \underbrace{x^2 + y^2 = \sqrt[3]{4}}_{\text{circle at } z=4} \text{ where } \sqrt[3]{4} = r^2, \text{ So } r = 2\sqrt{2}$

This means that $0 \leq r \leq 4$

To find z we look at the cones height from $z = 0$ to $z = 4$.

This means that $\sqrt[3]{x^2 + y^2} \leq z \leq 4 \Rightarrow r \leq z \leq 0$

Finally we can put it all together

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10. Problem 10

Decide whether the following statements are true or false.

(a) If a function is differentiable, it must be continuous.

(b) $\int_0^1 \int_0^x f(x, y) dx dy = \int_0^1 \int_0^y f(x, y) dy dx$

(c) If $\int_R f dA = 0$ then f is zero at all points in R .

(c) (d) If f and g are two functions continuous on a region R , then

$$\int_R f \cdot g dA = \int_R f dA \cdot \int_R g dA$$

(e) A local maximum of f can only occur at the critical points.

10.1. Solution

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