Exercises 1

Simon Holm
AI505: Optimization
Teacher: Marco Chiarandini

Contents

1. Exercise 1^+ Python

Show that the function

$$f(x) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$$

only has one stationary point, and that it is neither a maximum or minimum, but a saddle point (or inflection point). Plot the contour lines of f in Python

1.1. Solution

Since

$$\nabla f(x_1, x_2) = (2x_1 + 8, -4x_2 + 12)$$

$$\nabla f(x_1, x_2) = 0 \Longrightarrow \left(-\frac{8}{2}, -\frac{12}{4}\right) = (-4, 3)$$

Then

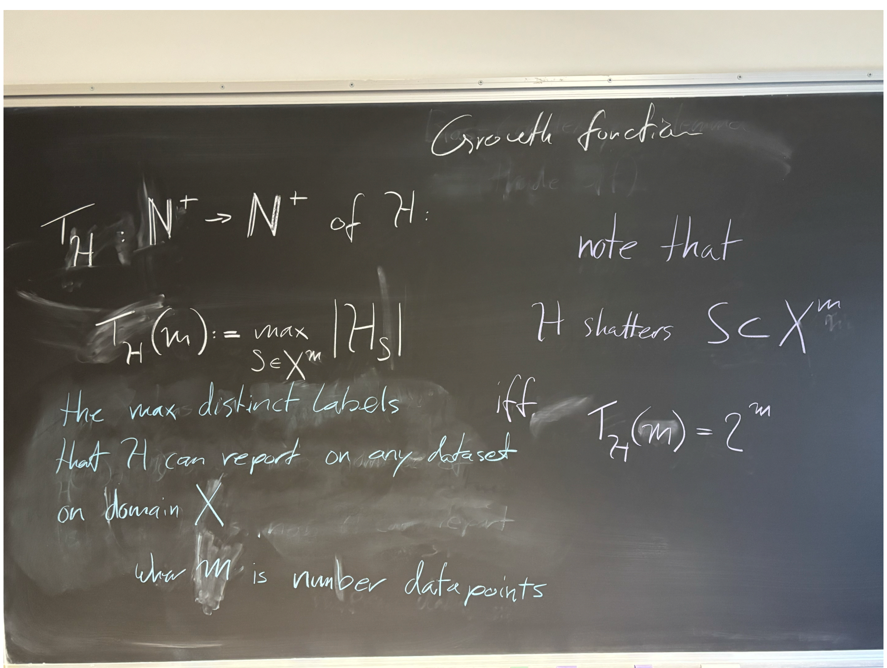
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

For a hessian $\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & -4-\lambda \end{bmatrix}\right) = 0$

So

$$(2 - \lambda)(-4 - \lambda) = 0 \Rightarrow \lambda = 2, -4$$

Since both negative and positive its a saddlepoint



2. Exercise 2^+

Write the second-order Taylor expansion for the function $\cos\left(\frac{1}{x}\right)$ around a nonzero point x, and the third-order Taylor expansion of $\cos(x)$ around any point x. Evaluate the second expansion for the specific case of $x = 1$.

2.1. Solution

- $\cos\left(\frac{1}{x}\right)$

$$\cos\left(\frac{1}{x}\right) \approx \cos\left(\frac{1}{a}\right) + \frac{\frac{1}{a^2}\sin\left(\frac{1}{a}\right)}{1!}(x-a) + \frac{\frac{1}{a^3}2\sin\left(\frac{1}{a}\right) - \frac{1}{a^4}\cos\left(\frac{1}{a}\right)}{2!}(x-a)^2$$

- $\cos(x)$

$$\cos(x) \approx \cos(a) - \frac{\sin(a)}{1!}(x-a) - \frac{\cos(a)}{2!}(x-a)^2 + \frac{\sin(a)}{3!}(x-a)^3$$

- at $x = 1c1$

$$\cos(1) - \frac{\sin(1)}{1!}(x-a) - \frac{\cos(1)}{2!}(x-a)^2$$

$$1.381773291 - 0.8414709848x - 0.2701511530(x-1.0)^2$$

3. Exercise 5^*

Consider the function $f(x_1, x_2) = \left(x_1 + x_2^2\right)^2$. At th epoint $x_0 = [0, 1]$ we consider the search direction $p = [-1, 1]$. Show that p is a descent direction and find all minimizers of the problem $\min_\alpha f(x_0 + \alpha p)$.

3.1. Solution

First

$$f(x_0 + \alpha p) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix}\right)$$

Then

$$f\left(\begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix}\right) = \left((1-\alpha) + \alpha^2\right)^2$$

Then we can

$$\arg\min_\alpha \left((1-\alpha) + \alpha^2\right)^2$$

$$f' = 2(\alpha^2 - x + 1)(2\alpha - 1) = 0 \Rightarrow \alpha = \frac{1}{2}$$

Since

$$f\left(\begin{bmatrix} 1-\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) = \left(\frac{1}{2} + \left(\frac{1}{2}\right)^2\right)^2 = 0.5625 < f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1$$

p is a descent direction

4. Exercise 6^+

Consider the case of a vector function f : $R^n \to R^m$. The matrix J(x) of first derivatives for this function is defined as follows:

$$J(x) = \left[\frac{\partial f_j}{\partial x_i}\right]_{\substack{j=1..m \\ i=1..n}}$$

write the forward-difference calculations needed to compute J(x) at a given point x.

4.1. Solution

Since f is a vector function, e.g.

$$\frac{\partial f}{\partial x} = \lim_{\varepsilon \to 0} \frac{f_j(\vec{x} + \varepsilon \vec{e}_i) + f(\vec{x})}{\varepsilon}$$

$$J(x) = \left[\frac{f_j(\vec{x}+\varepsilon\vec{e}_i)+f(\vec{x})}{\varepsilon}\right]_{\substack{j=1..m \\ i=1..n}}$$

5. Exercise 7^+

Adopt the forward difference method to approximate the Hessian of $f(x)$ using its gradient, $\nabla f(x)$.

5.1. Solution

Since

$$\nabla f_S(x) = \frac{f(x + hs) - f(x)}{h}$$

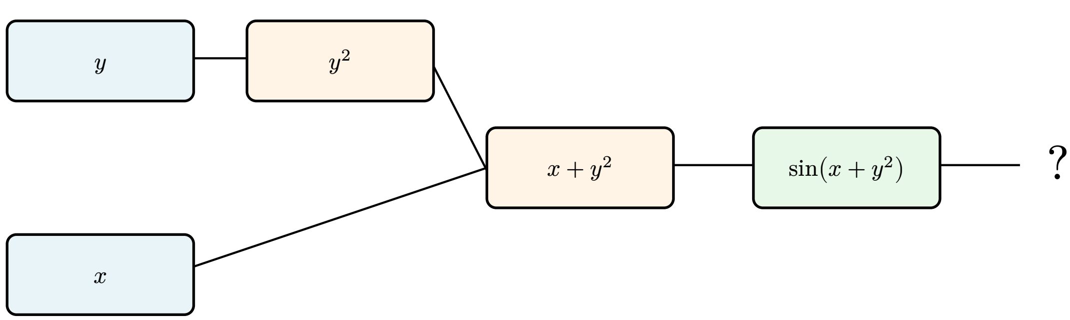
Then

$$H_f(x) = \frac{\nabla f(x + hs) - \nabla f(x)}{h}$$

6. Exercise 10^*

Draw the computational graph for $f(x, y) = \sin(x + y^2)$. Use the computational graph with forward accumulation to compute $\frac{\partial f}{\partial y}$ at $(x, y) = (1, 1)$. Label the intermediate values and partial derivatives as they are propagated through the graph.

6.1. Solution



Now to do forward accumulation on $f(x,y) = \sin(x+y^2)$

First i will compute

$$\frac{\partial}{\partial x} f(1,1) = \cos(2), \quad \frac{\partial}{\partial y} f(1,1) = 2\cos(2)$$

Now for the forward accumulation

$$x = 1, \quad \dot{x} = 0$$

$$y = 1, \quad \dot{y} = 1$$

$$c_1 = x, \quad \dot{c}_1 = \dot{x}$$

$$c_2 = y, \quad \dot{c}_2 = \dot{y}$$

$$c_3 = c_2^2, \quad f(x) = x^2, \quad \dot{c}_3 = \frac{\partial f}{\partial c_2} \frac{\partial c_2}{\partial y} = 2c_2\dot{c}_2$$

$$c_4 = c_1 + c_3, \quad \dot{c}_4 = \dot{c}_1 + \dot{c}_3$$

$$c_5 = \sin(c_4), \quad \dot{c}_5 = \cos(c_4)\dot{c}_4$$