Exercises from Shan Shan's lectures

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AI503: Calculus
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Contents

1. Lecture 1

1.1. Problem 1

$f(x)$ linear with $f(2) = 4$, $f(4) = -2$.

Find $f(x)$.

1.1.1. Solution

$f(x) = -3x + 10$

1.2. Problem 2

$g(x)$ linear with slope 6, $g(5) = 9$.

Find g (x).

1.2.1. Solution

$9 = 6 \cdot 5 + c$

$c = 9 - 6 \cdot 5 \Rightarrow c = 21$

$g(x) = 6x - 21$

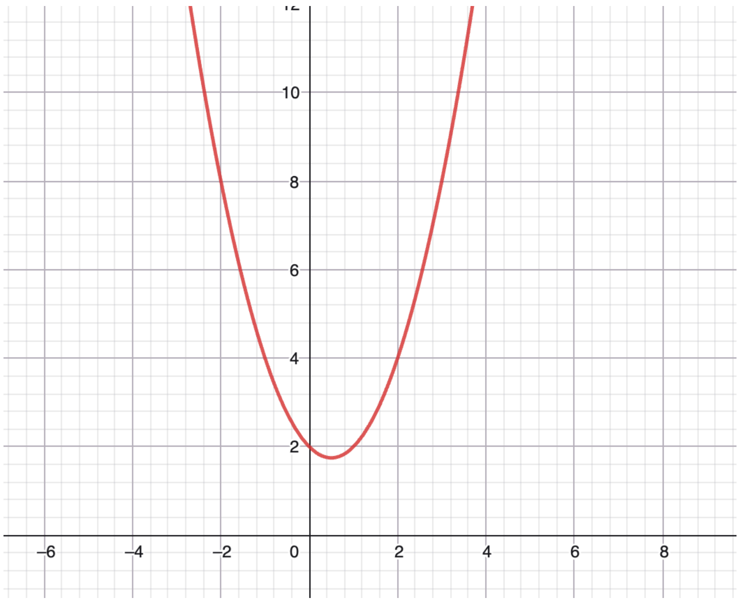
1.3. Problem 3

Draw the following polynomials

(a) $y = x^2 - x + 2$
(b) $y = x^3 + x^2 - 2x - 6$
(c) $y = x^4 - x^3 - 7x^2 - 2x - 6$
(d) $y = x^5 - x^4 - 5x^3 + 2x - 6$

1.3.1. Solution

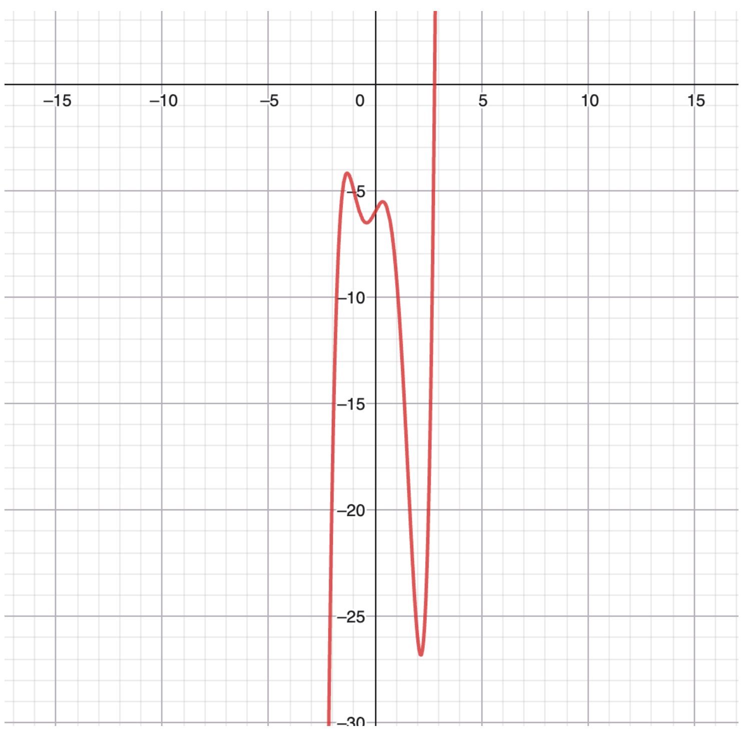
(a) $y = x^2 - x + 2$



(b) $y = x^3 + x^2 - 2x - 6$

(c) $y = x^4 - x^3 - 7x^2 - 2x - 6$

(d) $y = x^5 - x^4 - 5x^3 + 2x - 6$



1.4. Problem 4

Draw the graph of the following functions

(a) $y = (x+1)(x-1)^2$

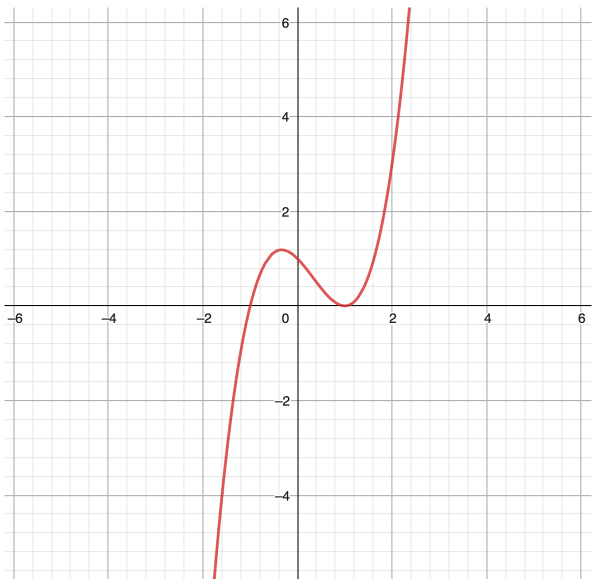
(b) $y = -x^2(x-1)$

Are these functions polynomials? Why?

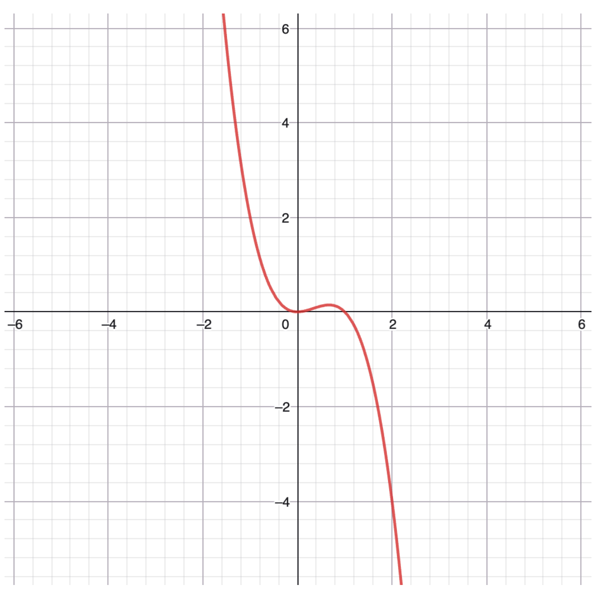
What changes about the behavior of the tails of the graph if its leading coefficient is negative?

1.4.1. Solution

(a) $y = (x+1)(x-1)^2$



(b) $y = -x^2(x-1)$



Both are still polynomials. Any power function is a polynomials

Negaive flips the entire graph

2. Lecture 2

2.1. Problem 1

Show $f(x) = x^3$ is continuous at every $c \in \mathbb{R}$.

2.1.1. Solution

This is the same as:

show that for every c

$$\lim_{x \to c} f(x) = c^3$$

Notice that

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + c^2 + xc)|$$

if $|x - c| < 1$ (which is is when $\lim_{x \to c}$)

Then we can insert $c + 1$ instead of x

$$|x^2 + c^2 + xc| \leq |x^2| + |c^2| + |x|\ |c| \leq |(c + 1)^2| + |c^2| + |c + 1|\ |c| := M$$

We want to show that $|x^3 - c^3| < \varepsilon$, so we choose

$$\delta = \min\left(1, \frac{\varepsilon}{M}\right) \qquad \square$$

2.2. Problem 2

Show that $f(x) = x^3 + x - 1$ has a root in $[0, 1]$

2.2.1. Solution

$f(0) = 0^3 + 0 - 1 = -1 \quad \notin [\mathbf{0}, \mathbf{1}]$

$f(1) = 1^3 + 1 - 1 = 1 \quad \in [\mathbf{0}, \mathbf{1}]$

Since f is continuous and because $f(0) = -1$ and $f(1) = 1$, f has to pass through $y = 0$ at some point between $x = [0, 1]$

Conclusion: f must have 1 or more roots in $[0, 1]$

2.3. Problem 3

Evaluate $f(x) = x^2$ at $x = 2$

2.3.1. Solution

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} \\ &= \lim_{h \to 0} \frac{4 + h^2 + 4h - 4}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 4h}{h} \\ &= \lim_{h \to 0} h + 4 = 4 \end{aligned}$$

2.4. Problem 4

Let $f(x) = \sqrt{x}$.

- Find $f'(4)$

2.4.1. Solution

We know that

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

So

$$f'(4) = \frac{1}{2} \cdot 4^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

The tangent line at $x = 4$ is:

$$y = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$$

The error function (difference between $f(x)$ and its tangent approximation at $x = 4$) is:

$$E(x) = f(x) - [f(4) + f'(4)(x - 4)]$$

As x approaches 4, this error approaches 0. Using the tangent line to approximate $\sqrt{4.1}$:

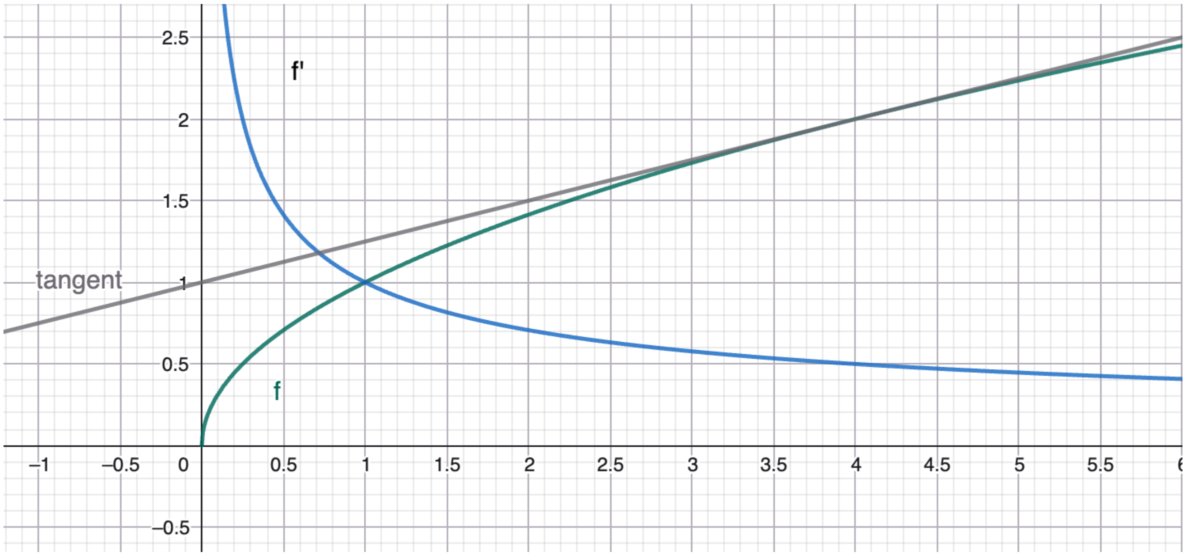
The tangent line at $x = 4$ is $y = 2 + \frac{1}{4}(x - 4)$.

So, for $x = 4.1$:

$$y \approx 2 + \frac{1}{4} \cdot (4.1 - 4) = 2 + \frac{1}{4} \cdot 0.1 = 2 + 0.025 = 2.025$$

The true value is $\sqrt{4.1} \approx 2.0249$, so the tangent line gives a good approximation for values close to $x = 4$.

The image below shows $f(x)$, its derivative $f'(x)$, and the tangent line at $x = 4$:



2.5. Problem 5

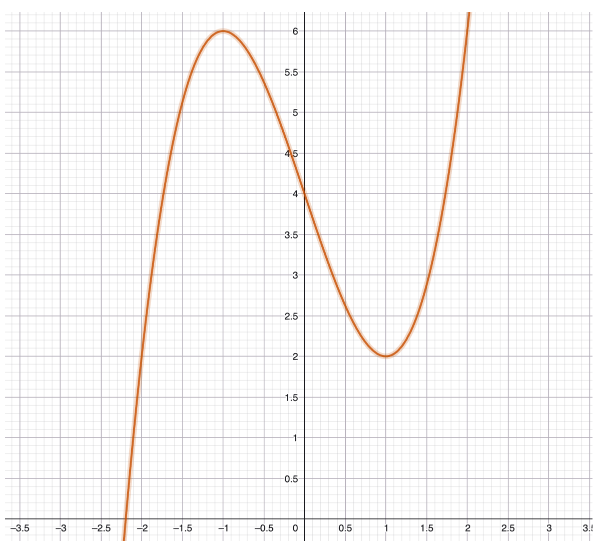
(a) Find and classify all of the local extrema of $f(x) = x^3 - 3x + 4$.

2.5.1. Solution

$$f'(x) = 3x^2 - 3$$

Set to 0

$$\begin{aligned}0 &= 3x^2 - 3\\3 &= 3x^2\\1 &= x^2\\x &= \sqrt{1} = \pm 1\end{aligned}$$



$f'(x) = 0$ gives 2 extremas, -1 and 1, this makes sence given the graph.

2.6. Problem 6

True or False? If $f'(x_0) = 0$, then f has a local max or min at x_0. Explain.

2.6.1. Solution

Yes, this means that the rate of change is 0 (extrema), at point x_0

2.7. Problem 7

Use the second derivative test to classify the critical points of $f(x) = x^3 - 3x + 4$.

2.7.1. Solution

We know that $f'(x) = 3x^2 + 3$ and that $f'(x) = 0 \Rightarrow x = \pm 1$

Then

$$f''(x) = 6x$$

$$f''(1) = 6$$

$$f''(-1) = -6$$

since $f''(1) > 0 \Rightarrow \textbf{Local minimum}$

and $f''(-1) < 0 \Rightarrow \textbf{Local maximum}$

2.8. Problem 8

Consider $f(x) = \left(x^2 - 4\right)^7$. Find and classify all local extrema.

2.8.1. Solution

$$f'(x) = 7\left(x^2 - 4\right)^6 \cdot 2x$$

then

$$0 = 7\left(x^2 - 4\right)^6 \cdot 2x$$

Because of 0-rule either $2x = 0$ or $x^2 - 4 = 0$

so $x = -2, 0, 2$

2.9. Problem 9

Find the global max and min of $f(x) = x(x-1)$ on $0 \leq x \leq 3$.

2.9.1. Solution

$$x^2 - x$$

$$f'(x) = 2x - 1$$

$$0 = 2x - 1 \Rightarrow x = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = -\frac{1}{4}$$

then for the endpoints

$$f(0) = 0$$

$$f(3) = 6$$

Global minimum at $x = \frac{1}{2}$ with $f\left(\frac{1}{2}\right) = -\frac{1}{4}$

Global minimum at $x = 3$ with $f(3) = 6$

2.10. Problem 10

Find the global max and min of $f(x) = x^3 - 7x + 6$ on $-4 \leq x \leq 2$.

2.10.1. Solution

$$f'(x) = 3x^2 - 7$$

$$0 = 3x^2 - 7 \Rightarrow x^2 = \frac{7}{3} \Rightarrow x = \pm\sqrt{\frac{7}{3}}$$

$$f\left(\sqrt{\frac{7}{3}}\right) = -1.13$$

$$f\left(-\sqrt{\frac{7}{3}}\right) = 13.13$$

then for endpoints

$$f(-4) = -30$$

$$f(2) = 0$$

Global minimum at $x = -4$ with $f(-4) = -30$

Global minimum at $x = -\sqrt{\frac{7}{3}}$ with $f\left(-\sqrt{\frac{7}{3}}\right) = 13.13$

2.11. Problem 11

Find the global max and min of $g(x) = \ln(1 + x^2)$ on $-1 \leq x \leq 2$.

2.11.1. Solution

$$f'(x) = \frac{1}{1 + x^2} \cdot 2x$$

When $f'(x) = 0 \Rightarrow x = 0$

$f(0) = 0$

now for endpoints

$f(-1) = 0.69$

$f(2) = 1.61$

Global minimum at $x = 0$ with $f(0) = 0$

Global minimum at $x = 2$ with $f(2) = 1.13$

2.12. Problem 12

Find the global max and min of $f(t) = te^{-t}$ for $t \geq 0$. (Hint: careful here, as the domain is neither closed nor finite)

2.12.1. Solution

$$f'(x) = e^{-t} - te^{-t}$$

$$0 = e^{-t} - te^{-t} \Rightarrow t = 1$$

$$f(1) = \frac{1}{e} \approx 0.37$$

Then for endpoints

$$f(0) = 0$$

Since i cant do $f(\infty)$, i can use L'Hopitals rule to show that

$$\lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0$$

so.. Global minimum at $x = 0$ with $f(0) = 0$ and $x = \infty$ with $f(\infty) = 0$

Global minimum at $x = 1$ with $f(1) = 0.37$

3. Lecture 3

3.1. Problem 1

Write the definition of $\lim_{x \to c} f(x) = L$

3.1.1. Solution

$\lim_{x \to c} f(x)$ is when we get $f(x)$ close enogh to L so that

For any small $\varepsilon >$, there is $\delta > 0$ such that if

$$|x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon$$

note that $x \neq c$

3.2. Problem 2

Write the definition of a function f continuous at $x = c$

3.2.1. Solution

f is continous at point $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

Note that, if f is continous in $[a, b]$, then f is continous on all points within $[a, b]$

3.3. Problem 3

Write the definition of the derivative of f at a point $x = a$

3.3.1. Solution

for a point a, then

$$f'(a) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

If this limit exists, f is differentiable at a.

3.4. Problem 4

Write the definition of the second derivative

3.4.1. Solution

Since $f''(x) = f'(f'(x))$

$$f''(a) = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

3.5. Problem 5

Use the second derivative test to classify the critical points of $f(x) = x^3 - 3x + 4$

We know that $f'(x) = 3x^2 + 3$ and that $f'(x) = 0 \Rightarrow x = \pm 1$

Then

$$f''(x) = 6x$$

$$f''(1) = 6$$

$$f''(-1) = -6$$

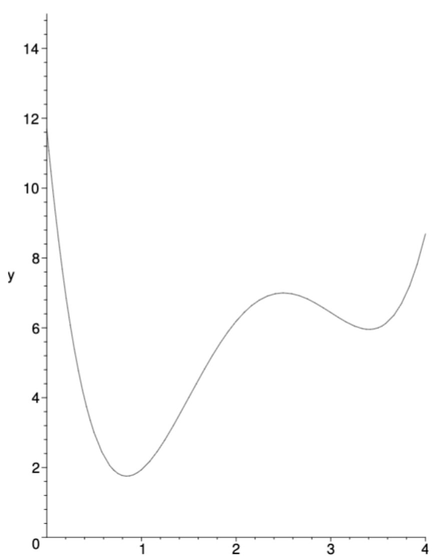
$f''(1) > 0 \Rightarrow \textbf{Local minimum}$

and $f''(-1) < 0 \Rightarrow \textbf{Local maximum}$

3.5.1. Solution

3.6. Problem 6

Consider the function below:
- Identify the local minima and maxima.
- Identify the global minimum and maximum.



3.6.1. Solution

Local minima at $x \approx 0.8$ with $f(0.8) \approx 2$

Local maxima at $x \approx 2.5$ with $f(2.5) \approx 7$

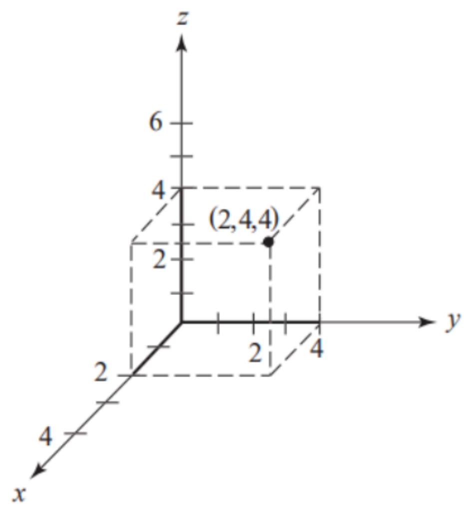
Local minima at $x \approx 3.5$ with $f(3.5) \approx 6$

Global minimum at $x \approx 0.8$ with $f(0.8) \approx 1.9$

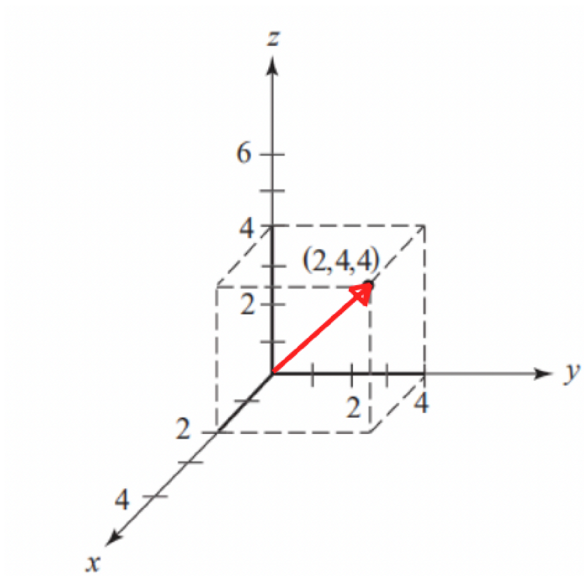
Global maximum at $x \approx 0$ with $f(0) \approx 10.8$

3.7. Problem 7

Draw the vector $(2, 4, 4)$ on the following graph and compute its length.



3.7.1. Solution



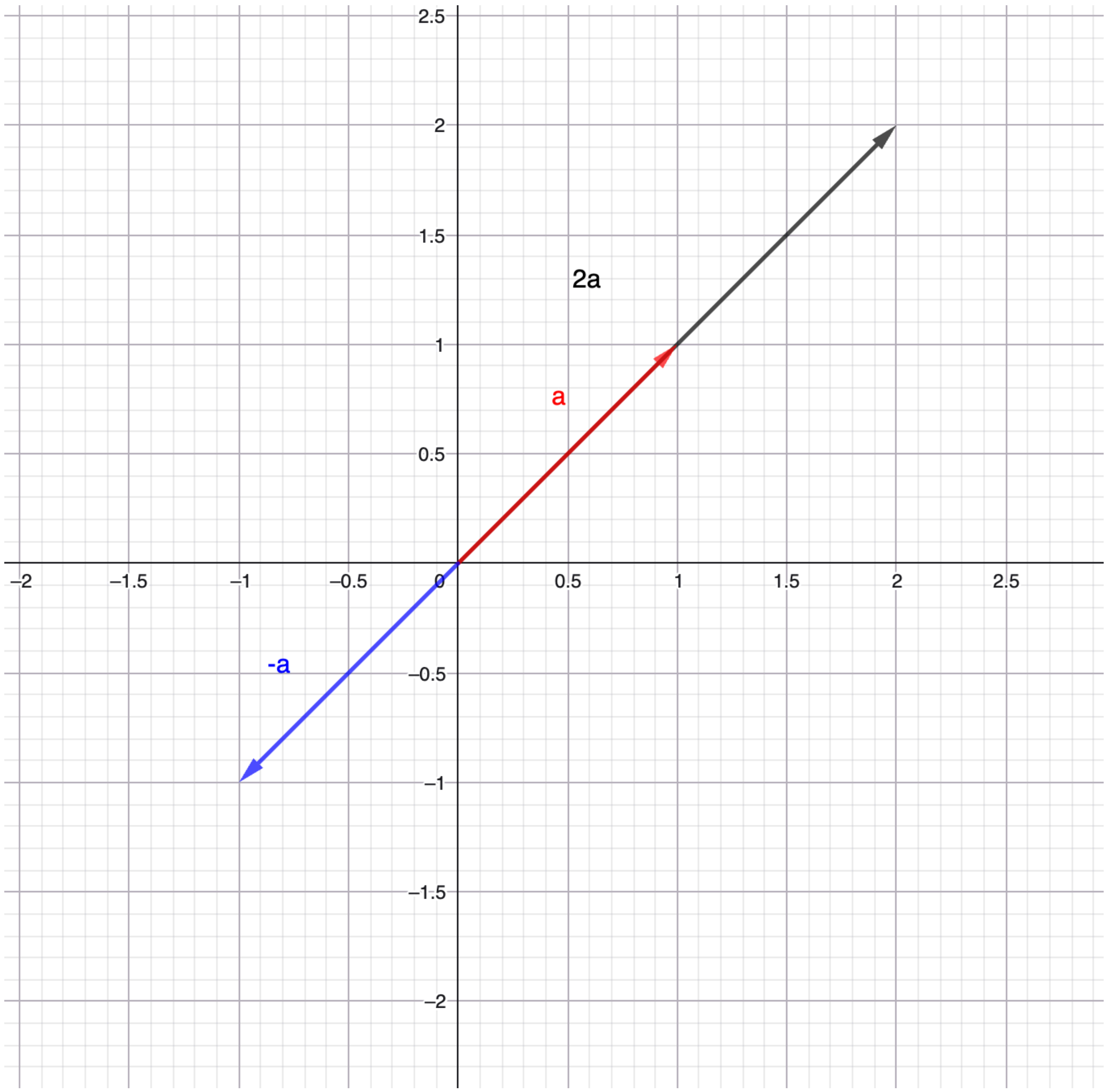
$$\|(2, 4, 4)\| = \sqrt{2^2 + 4^2 + 4^2} = \sqrt{68}$$

3.8. Problem 8

Let $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- Draw a, $2a$, and $-a$.
- How can $-a$ be normalized to a unit vector?

3.8.1. Solution

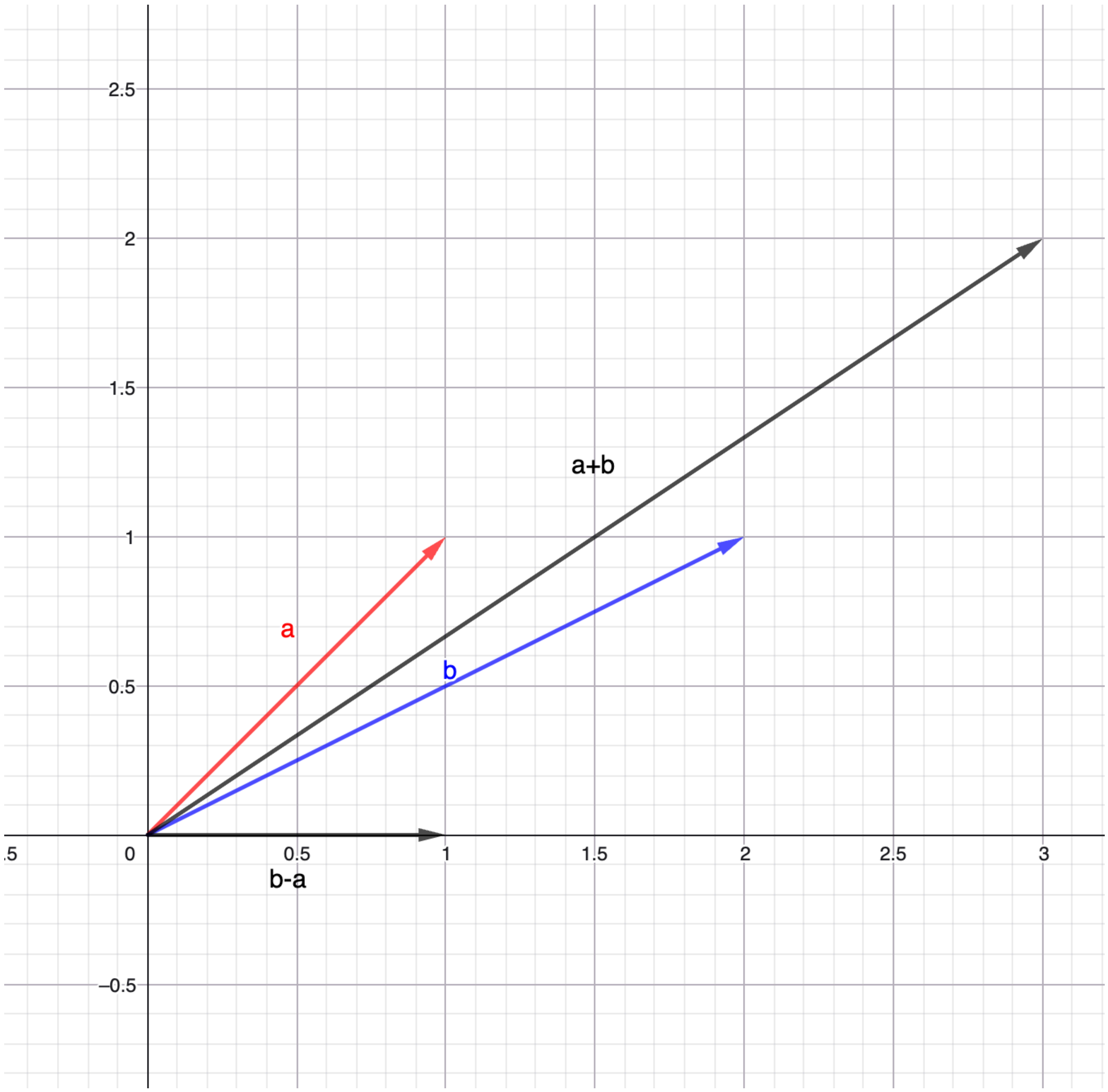
- 

- Normalized to a unit vector $(-a)_u = \frac{-a}{\|-a\|} = \frac{-1,-1}{\sqrt{(-1)^2+(-1)^2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

3.9. Problem 9

Draw $\boldsymbol{b} - \boldsymbol{a}$

3.9.1. Solution



3.10. Problem 10

Let

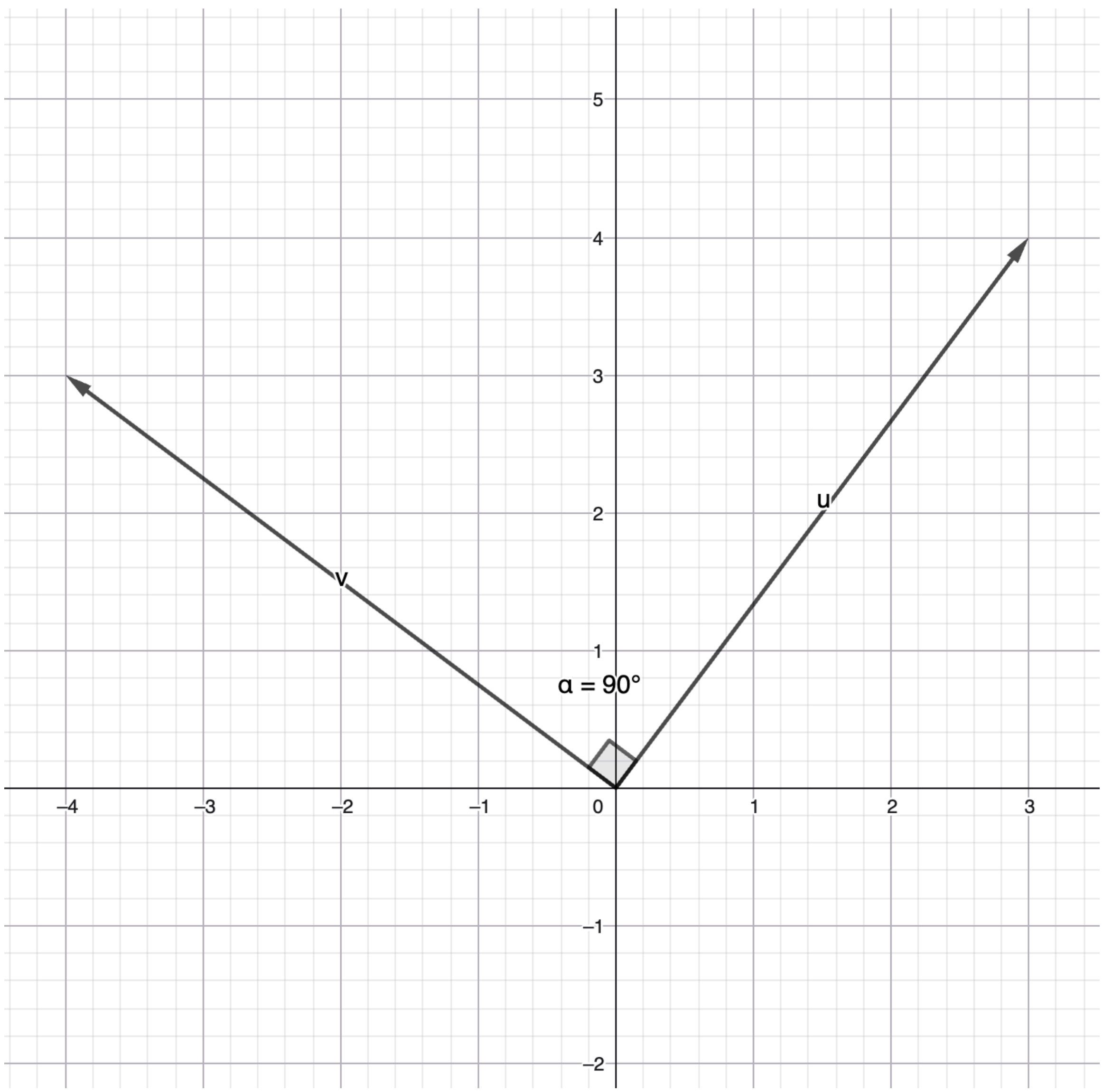
$$\boldsymbol{v} = (3, 4) \quad \boldsymbol{w} = (-4, 3)$$

- (1) Compute $\boldsymbol{v} \cdot \boldsymbol{w}$ ($\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{w}$)
- (2) Use (1) to deduce that the vectors form a right angle
- Draw the two vectors on the plane to confirm (2)

3.10.1. Solution

- $\boldsymbol{v} \cdot \boldsymbol{w} = 3 \cdot (-4) + 4 \cdot 3 = -12 + 12 = 0$
- When dot product $= 0$, angle is between them is 90

$$\cos(90)\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 0 = \boldsymbol{v} \cdot \boldsymbol{w}$$

  Since neither $\|\boldsymbol{v}\|$ or $\|\boldsymbol{w}\|$ is 0 $\cos(\theta) = 0$, so $\theta = 90$
- 

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3.11. Problem 11

Can you give the standard basis vectors in $\mathbb{R}^n$?

3.11.1. Solution

for any vector a in $\mathbb{R}^n$

$$a = a_1 e_1 + a_2 e_2 + ... + a_n e_n$$

Then the standart basis vectors are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad ... \quad , e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

3.12. Problem 12

What are the domain and range for each of these functions?

- $f(x, y) = x^2 + y^2$
- $f(x, y) = (x^2 + y^2) \log(xy)$
- $f(x, y, z) = \log(z) x^2 y^2$

3.12.1. Solution

- $f(x, y) = x^2 + y^2$
  - $D(f) = \mathbb{R}^2$
  - $R(f) = [0, \infty)$
- $f(x, y) = (x^2 + y^2) \log(xy)$
  - $D(f) = \{(x, y) \in \mathbb{R}^2 \mid xy > 0\}$
  - $R(f) = \mathbb{R}$
- $f(x, y, z) = \log(z) x^2 y^2$
  - $D(f) = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$
  - $R(f) = \mathbb{R}$

4. Lecture 4

4.1. Problem 1

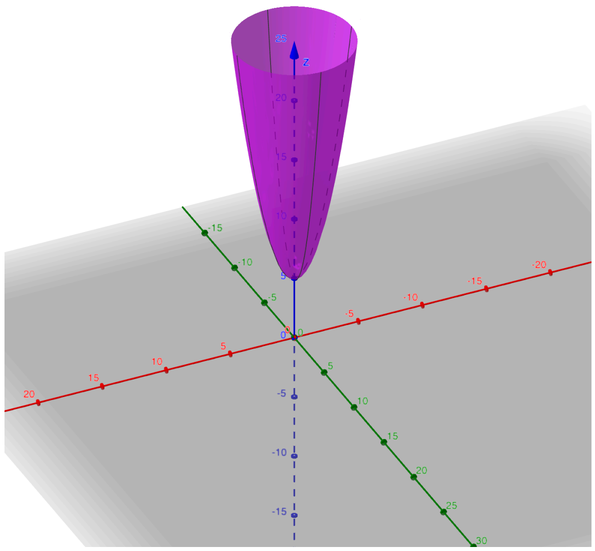
Can you describe in words the graphs of the following functions?

- $g(x, y) = x^2 + y^2 + 5$
- $h(x, y) = -x^2 - y^2$
- $q(x, z) = x^2 + (z - 1)^2$

4.1.1. Solution

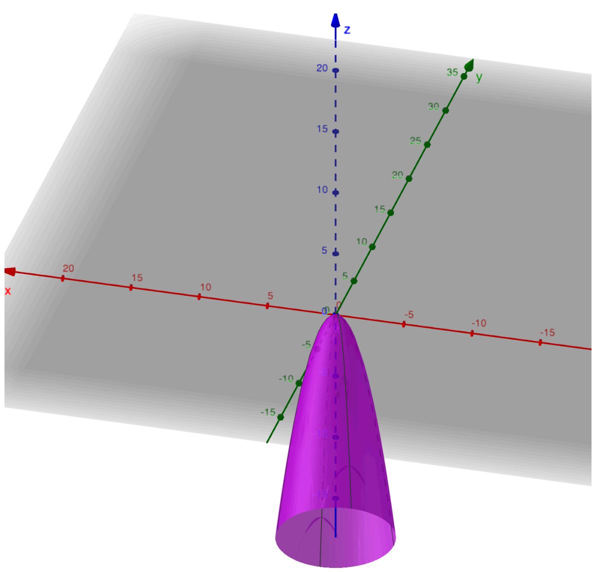
- $g(x, y) = x^2 + y^2 + 5$

  upward circular paraboloid (bowl) that is shifted up by 5



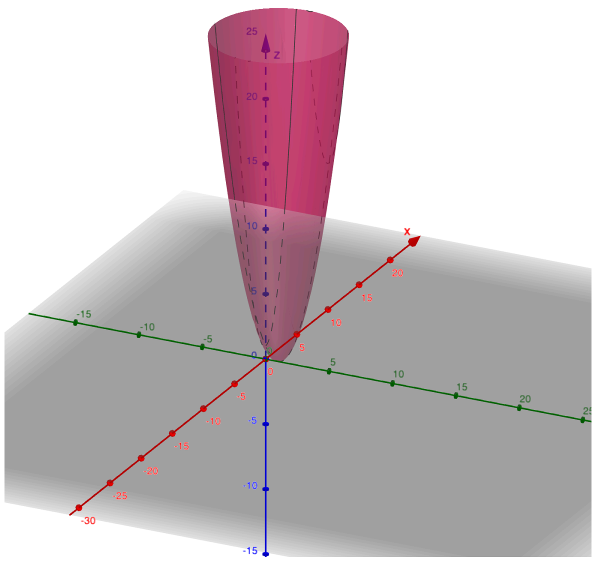
- $h(x, y) = -x^2 - y^2$

  downward circular paraboloid (bowl)



- $q(x, z) = x^2 + (z - 1)^2$

  upward circular paraboloid (bowl) that is shifted + 1 in the direction of z



4.2. Problem 2

Describe the cross-sections of the function $g(x, y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g.

4.2.1. Solution

- with $x^2 - y^2$ and $y = c$ is an upwards parabola
- with $x^2 - y^2$ and $x = c$ is an downwards parabola
- together = saddle surface

4.3. Problem 3

Describe the level sets of the functions

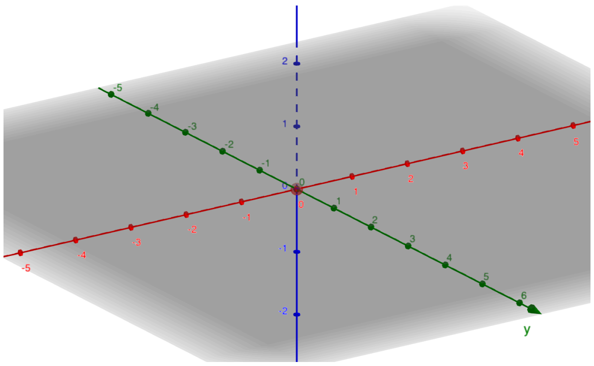
- $f(x, y, z) = x^2 + y^2 + z^2$
- $f(x, y, z) = x^2 + y^2 - z^2$

4.3.1. Solution

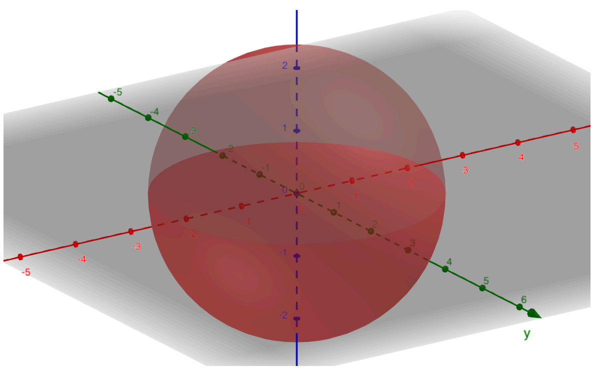
- $f(x, y, z) = x^2 + y^2 + z^2$

  Level sets $x^2 + y^2 + z^2 = c$

  For $c = 0$ a single point at $(0, 0, 0)$

  For $c > 0$ a 3D-sphere with $r = c$

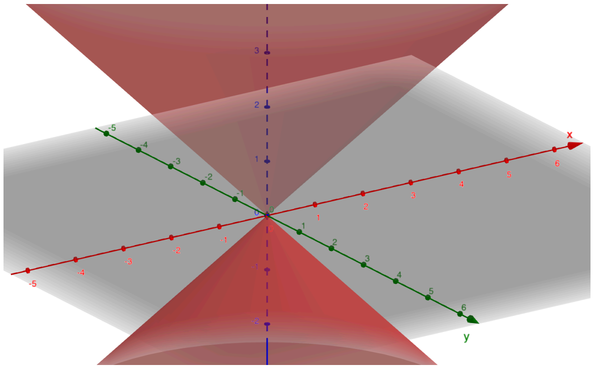
  

  For $c > 0$ a single point at $(0, 0, 0)$

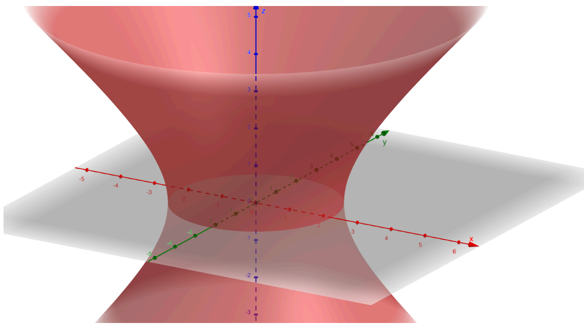
  Undefindes as $(-1)^2 = 1$

- $f(x, y, z) = x^2 + y^2 - z^2$

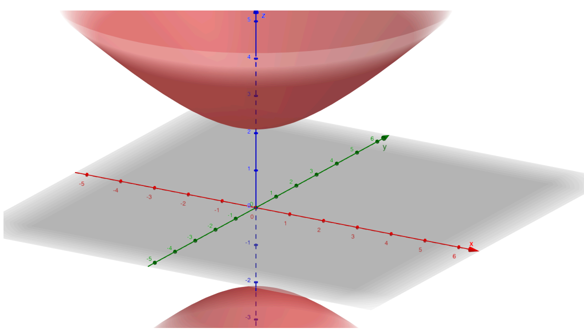
  For $c = 0$ a double cone with center at $(0, 0, 0)$



  For $c < 0$ a hyperboloid of one sheet



  For $c > 0$ a hyperboloid of one sheets



4.4. Problem 4

Find the equation of the plane passing through the points $(1, 0, 1), (1, -1, 3)$, and $(3, 0, -1)$.

4.4.1. Solution

The equation for the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

with

$$\text{point } P(x_0, y_0, z_0) \text{ and normal vector } N(a, b, c)$$

i will use the tree points to make 2 vectors in the plane

$$v = (1, 0, 1) - (1, -1, 3) = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

and

$$w = (3, 0, -1) - (1, -1, 3) = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$$

Now by using cross product $N = v \times w$

$$N = v \times w = \begin{pmatrix} -2 \\ -4 \\ -2 \end{pmatrix}$$

with point $P(1, 0, 1)$ the plane becomes

$$-2(x - 1) - 4(y - 0) - 2(z - 1) = 0$$

$$-2x + 2 - 4y - 2z + 2 = 0$$

$$-2x - 4y - 2z + 4 = 0$$

4.5. Problem 5

What is the minimum number of points required to uniquely determine a plane?

4.5.1. Solution

3

4.6. Problem 6

In one-variable calculus, a limit exists at x = a if the left-hand and right-hand limits match:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

Does this make sense for multivariate functions?

4.6.1. Solution

Yes, but this must be true for all paths. (impossible task in practice)

So to show that a multivariate limit does not exist, you look for counterexamples:

$$\frac{2xy}{x^2 - y^2}$$

$$y = 0 \Rightarrow \lim_{x \to 0} \frac{2xy}{x^2 - y^2} = 0$$

$$y = x \Rightarrow \lim_{x \to 0} \frac{2x^2}{2x^2} = 1$$

Limit does not exits since $0 \neq 1$

4.7. Problem 7

Does the following limit exists?

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$$

4.7.1. Solution

$$y = 0 \Rightarrow \lim_{x \to 0} \frac{xy}{x^2 + y^2} = 0$$

$$y = x \Rightarrow \lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

DNE

4.8. Problem 8

Does the following limits exists?

- $\lim_{(x,y)\to(0,0)} e^{-x-y}$
- $\lim_{(x,y)\to(0,0)} x^2 + y^2$
- $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$

4.8.1. Solution

- $\lim_{(x,y)\to(0,0)} e^{-x-y}$
  - $y = 0 \Rightarrow \lim_{x\to 0} e^{-x-y} = 0$
  - $y = x \Rightarrow \lim_{x\to 0} e^{-x-y} = \frac{1}{2}$
- $\lim_{(x,y)\to(0,0)} x^2 + y^2$
  - this is just 0
- $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$

  We want to prove that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$

  By definition

$$\forall \varepsilon > 0, \exists \delta > 0 \mid \underbrace{\left(\sqrt{x^2+y^2}\right)}_{\text{(x,y) to (0,0)}} < \delta \Longrightarrow \left|\frac{x^2 y}{x^2} + y^2\right| < \varepsilon$$

Since

$$\frac{x^2|y|}{x^2+y^2} \leq \frac{x^2+y^2}{x^2+y^2}|y|$$

then

$$\frac{x^2|y|}{x^2+y^2} \leq |y| \leq \sqrt{x^2+y^2} \leq \delta$$

choose $\varepsilon = \delta$ then

$$\frac{x^2|y|}{x^2+y^2} \leq \varepsilon$$

So

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0 \quad \square$$

4.9. Problem 9

Is the function continuous in the given region?

- $\frac{1}{x^2+y^2}$ on $-1 \leq x, y \leq 1$
- $\frac{1}{x^2+y^2}$ on $1 \leq x, y \leq 2$
- $\frac{y}{x^2+2}$ on $x^2 + y^2 \leq 1$

4.9.1. Solution

- $\frac{1}{x^2+y^2}$ on $-1 \leq x, y \leq 1$

  at $x = y = 0$, $\frac{1}{x^2+y^2}$ is undefined

- $\frac{1}{x^2+y^2}$ on $1 \leq x, y \leq 2$

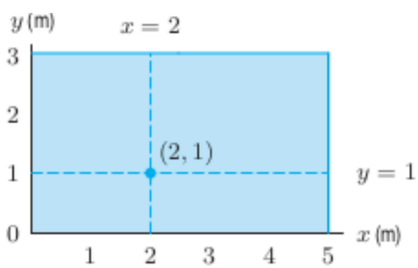
  Since $1 \leq x$, $x^2 + y^2$ can never be 0, so yes it is continuous

- $\frac{y}{x^2+2}$ on $x^2 + y^2 \leq 1$

  Because of x^2, $x^2 + 2$ can not become 0, so yes it is continuous

4.10. Problem 10

Imagine an unevenly heated thin rectangular metal plate lying in the xy-plane. Temperature at (x, y) is $T(x, y)$



y (m) \ x (m)	0	1	2	3	4	5
3	85	90	**110**	135	155	180
2	100	110	**120**	145	190	170
1	**125**	**128**	**135**	**160**	**175**	**160**
0	120	135	**155**	160	160	150

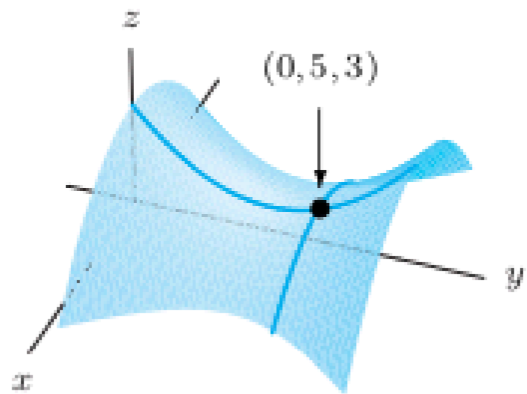
How does T vary near the point (2, 1)?

4.10.1. Solution

Its increasing towards $x = 3$ and decreasing towards $x = 1$

Also its increasing towards $y = 0$ and decreasing towards $y = 2$

4.11. Problem 11



- What is the sign of $f_x(0,5)$?
- What is the sign of $f_x(0,5)$?

4.11.1. Solution

I am assuming that the direction of the derivative is towards the x and y axis (which are the positive ends)

- What is the sign of $f_x(0,5)$?

  negative

- What is the sign of $f_x(0,5)$?

  positive

4.12. Problem 12

Let

$$f(x,y) = \frac{x^2}{y+1}$$

Find $f_{x(3,2)}$

4.12.1. Solution

$$f_{x(x,y)} = \frac{\partial}{\partial x}\frac{x^2}{y+1} = \frac{2x}{y-1}$$

$$f_{x(3,2)} = \frac{2 \cdot 3}{2-1} = \frac{6}{1} = 6$$

4.13. Problem 13

Find tangent plane to $f(x, y) = x^2 + y^2$ at $(3, 4)$

4.13.1. Solution

If f is differentiable, then

$$z = f(a, b) + f_{x(a,b)}(x - a) + f_{y(a,b)}(y - b)$$

Find the derivatives

$$f_{x(x,y)} = 2x \quad f_{y(x,y)} = 2y$$

Now plug it into the tangent equation

$$z = 3^2 + 4^2 + (2 \cdot 3)(x - 3) + (2 \cdot 4)(y - 4)$$

$$z = 25 + 6(x - 3) + 8(y - 4)$$

$$z = 25 + 6x - 18 + 8y - 32$$

$$z = 6x + 8y - 25$$

4.14. Problem 14

Let $f(x,y) = x^2 + y^2$ and call the tangent plane you computed at $(3,4)$ $L(x,y)$. We define the error function

$$E(x,y) = f(x,y) - L(x,y)$$

The error near $(3,4)$ is given by $E(3+h, 4+k)$ where h, k are small.

What is the distance between $(3+h, 4+k)$ and $(3,4)$?

Compare $|E(3+h, 4+k)|$ with $h^2 + k^2$.

Observe that:

$$\frac{|E(x,y)|}{h^2 + k^2} \approx c$$

What can you say about the error near $(3,4)$?

4.14.1. Solution

- What is the distance between $(3+h, 4+k)$ and $(3,4)$?

$$\text{dist} = \sqrt{h^2 + k^2}$$

- Compare $|E(3+h, 4+k)|$ with $h^2 + k^2$.

$$E(3+h, 4+k) = f(3+h, 4+h) - L(3+h, 4+h)$$

$$25 + 6h + 8k + h^2 + k^2 - 25 + 6h + 8k = h^2 + k^2$$

  So actually we see that

$$|E(3+h, 4+k)| = h^2 + k^2$$

Now

$$\frac{|E(x,y)|}{h^2 + k^2} = 1$$

- What can you say about the error near $(3,4)$? The error is quadratic in the distance from $(3,4)$

  When $(h,k) \to (0,0)$, $\sqrt{h^2 + k^2} \to 0$ faster than $h^2 + k^2$

$$\frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2} \to 0$$

  This means that the error goes to 0 faster than the distance itself

  For the tangent, this means that even if you move a little bit away (small h, k), the value of the tangent plane is almost the same as the function itself.

5. Lecture 5

5.1. Problem 1

Compute partial derivatives by treating all other variables as constants

$$f(x, y, z) = x^2 yz$$

5.1.1. Solution

Using

$$\frac{\partial f}{\partial x_i}(x_1, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{h}$$

$$\frac{\partial f}{\partial x} = 2xyz \quad \frac{\partial f}{\partial y} = x^2 z \quad \frac{\partial f}{\partial z} = x^2 y$$

5.2. Problem 2

Compute the partial derivatives of

$$f(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^2$$

5.2.1. Solution

$$\frac{\partial}{\partial x_1} \sum_{i=1}^{n} x_i^2, \frac{\partial}{\partial x_2} \sum_{i=1}^{n} x_i^2, ..., \frac{\partial}{\partial x_n} \sum_{i=1}^{n} x_i^2$$

$$2x_1, 2x_2, ..., 2x_n$$

5.3. Problem 3

When $u = i$ or $u = j$, what is $f_u(a, b)$?

5.3.1. Solution

This means that either $u_1 = 1$ and $u_2 = 0$ or $u_1 = 0$ and $u_2 = 1$

$$u = 1i + 0j = i \quad \vee u = 0i + 1j = j$$

Because of this we only stepping either $a + h$ or $b + h$

This means that

$$f_{\boldsymbol{u}} = \lim_{h \to 0} \frac{(a, b) = f(a + h, b) - f(a, b)}{h}$$

or

$$f_{\boldsymbol{u}} = \lim_{h \to 0} \frac{(a, b) = f(a, b + h) - f(a, b)}{h}$$

This is exactly the same as taking the partial derivative with respect to the variable in that direction (partial derivatives) so

$$f_i(x, y) = f_x(x, y)$$

$$f_j(x, y) = f_y(x, y)$$

5.4. Problem 4

For $f_u(x, y)$ What happens when u is not a unit vector?

5.4.1. Solution

We take larger steps and therefore the $f_u(x, y)$ is scaled by some number

Let u not be a unitvector, then

$$f_u(x, y) = k \cdot f_{\hat{u}}(x, y) \quad \text{where } \hat{u} = \frac{u}{\|u\|}$$

Where $k = \|u\|$, when $u \neq 0$

To get the true rate of change per unit distance, you divide by $\|u\|$ (i.e., use a unit vector).

5.5. Problem 5

Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of $i + j$.

5.5.1. Solution

$$u = i + j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{u} = \frac{1, 1}{\|u\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\begin{aligned} f_{\boldsymbol{u}}(x, y) &= \lim_{h \to 0} \frac{f\left(a + \frac{h}{\sqrt{2}}, b + \frac{h}{\sqrt{2}}\right) - f(a, b)}{h} \\ f_{\boldsymbol{u}}(1, 0) &= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(1, 0)}{h} \\ &= \lim_{h \to 0} \frac{\left(1 + \frac{h}{\sqrt{2}}\right)^2 + \left(\frac{h}{\sqrt{2}}\right)^2 - 1}{h} = \sqrt{2} \end{aligned}$$

5.6. Problem 6

- Approximate $f(a_1 + hu_1, ..., a_n + hu_n)$ with the first-order approximation at $(hu_1, ..., hu_n)$
- What happens when you simplify the limit in the previous definition?

5.6.1. Solution

- Approximate $f(a_1 + hu_1, ..., a_n + hu_n)$ with the first-order approximation at $(hu_1, ..., hu_n)$

For $x = (x_1, ..., x_n)$ near $a = (a_1, ..., a_n)$

$$f(x) \approx L(x) = f(a) + \sum_{x=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

For $x = (a_1 + hu_1, ..., a_n + hu_a)$ near $h = (hu_1, ..., hu_n)$

$$f(x) \approx L(x) = f(h) + \sum_{x=1}^{n} \frac{\partial f}{\partial x_i}(h)a_i$$

or

$$f(a_1 + hu_1, ..., a_n + hu_a) \approx L(a_1 + hu_1, ..., a_n + hu_a) =$$
$$f(hu_1, ..., hu_n) + \sum_{x=1}^{n} \frac{\partial f}{\partial x_i}(hu_1, ..., hu_n)a_i$$

- What happens when you simplify the limit in the previous definition?

  For the previous definition (in the notes)

$$f_{\boldsymbol{u}}(a) = \lim_{h \to 0} \frac{f(a_1 + hu_1, ..., a_n + hu_n) - f(a_1, ..., a_n)}{h}$$

$$f_{\boldsymbol{u}}(a) \approx \lim_{h \to 0} \frac{f(h) + \sum_{x=1}^{n} \frac{\partial}{\partial x_i} f(h)a_i - f(a)}{h}$$

$$f_{\boldsymbol{u}}(a) \approx \sum_{x=1}^{n} \frac{\partial}{\partial x_i} f(h) \frac{a_i}{h}$$

  since $\frac{a_i}{h} = u_i$

$$f_{\boldsymbol{u}}(a)a = \sum_{x=1}^{n} \frac{\partial}{\partial x_i} f(h)u_i = \nabla f(a) \cdot u$$

5.7. Problem 7

Use the preceding formula ($\nabla f(a) \cdot u$) to calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of $i + j$.

5.7.1. Solution

Now that we know that $f_u(x, y) = \nabla f(a) \cdot u$

$$u = i + j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{u} = \frac{1, 1}{\|u\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$f_{\hat{u}}(x, y) = \nabla f(x, y) \cdot u = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{2x}{\sqrt{2}} + \frac{2y}{\sqrt{2}}$$

$$f_{\hat{u}}(1, 0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{2}{\sqrt{2}} + \frac{0}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

5.8. Problem 8

Find the gradient of $f(x, y) = x + e^y$ at $(1, 1)$. Use it to compute the directional derivative in the direction of $i + j$.

5.8.1. Solution

$$u = i + j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{u} = \frac{1, 1}{\|u\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$f_{\hat{u}}(x, y) = \nabla f(x, y) \cdot u = \begin{pmatrix} 1 \\ e^y \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} + \frac{e^y}{\sqrt{2}}$$

$$f_{\hat{u}}(1, 1) = \frac{1}{\sqrt{2}} + \frac{e^1}{\sqrt{2}} = \frac{1 + e}{\sqrt{2}}$$

6. Lecture 6

6.1. Problem 1

We computed the directional derivative by the inner product with the gradient vector

$$f_u(a) = \nabla f(a) \cdot u$$

Recall that the inner product measures the alignment of two vectors:

$$\nabla f(a) \cdot u = \|\nabla f(a)\| \cdot \underbrace{\|u\|}_{1} \cdot \cos(\theta) = \|\nabla f(a)\| \cdot \cos(\theta)$$

Think, what does this tell us about the gradient vector?

6.1.1. Solution

Because of

$$f_u(a) = \|\nabla f(a)\| \cdot \cos(\theta)$$

As long as $\|f\| \neq 0$

- When looking for the **maximum** increase. Assume that $\theta = 0$

  Then $f_u(a)$ is maximized, when the direction u points along of $\nabla f(a)$ (i.e. when $\theta = 0$)

  $$f_u(a) = \|\nabla f(a)\|$$

- We can also **minimize** by $\theta = 180$

  Then $f_u(a)$ is minimized, when the direction u points opposite of $\nabla f(a)$ (i.e. when $\theta = 180$)

  $$f_u(a) = -\|\nabla f(a)\|$$

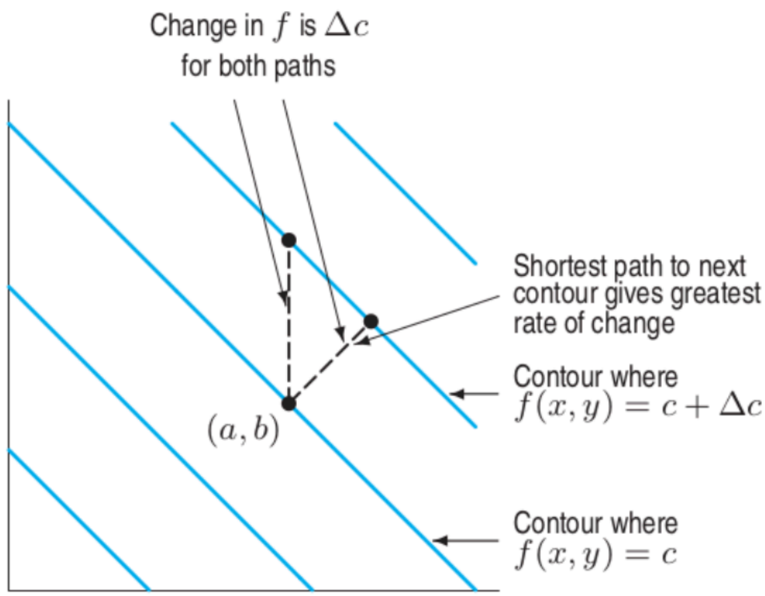
- What about **no change**?

  Then, when the direction u is perpendiculat ($\perp$) to $\nabla f(a)$ (i.e. when $\theta = 90$)

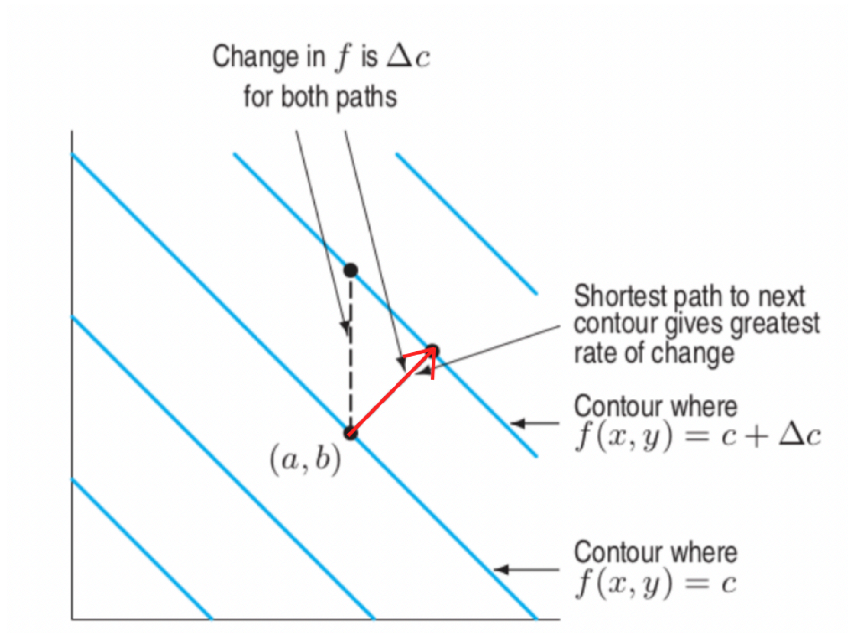
  $$f_u(a) = 0$$

6.2. Problem 2

In the following picture, mark the gradient vector of f at (a, b).



6.2.1. Solution



6.3. Problem 3

Let's consider a simple classification example where we need to classify whether a student passes an exam based on two features:

- $x_1 =$ hours studied
- $x_2 =$ hours slept

The goal is to make a binary classification such that

- $y = 1$ if the student passes (positive class).
- $y = 0$ if the student passes (positive class).

Suppose a data set $\left\{x_1^{(i)}, x_2^{(i)}, y^{(i)}\right\}$ with $i = 1, ..., m$ is given.

The logistic regression model is defined by

$$f(x_1, x_2) = \frac{1}{1 + e^{-(\theta_1 x_1 + \theta_2 x_2 + b)}}$$

where θ_1 and θ_2 are parameters (weights) for the features. and b is some bias term.

The predicted outcome using the logistic model is

$$y^{(i)} = f\left(x_1^{(i)}, x_2^{(i)}\right)$$

Think: What is the range of f?

The amount of error we made is characterized by the following function. Let us define

$$L(\theta_1, \theta_2, b) = \sum_{i=1}^{m} [y^i \log(\hat{y}^i) + (1 - y^i) \log(1 - \hat{y}^i)]$$

What is the gradient vector?

6.3.1. Solution

- What is the range of f?

  The range of the general logistic function $\sigma(z) = \left(\frac{1}{1+e^{-z}}\right)$ is:

  - As $z \to +\infty, \sigma(z) \to 0$
  - As $z \to -\infty, \sigma(z) \to 1$

- What is the gradient vector?

$$\nabla f(\theta_1, \theta_2, b) = \left(\frac{\partial L}{\partial \theta_1}, \frac{\partial L}{\partial \theta_2}, \frac{\partial L}{\partial b}\right)$$

To simplify $L = \sum_{i=1}^{m} l^i$ where $l^i = [y^i \log(\hat{y}^i) + (1 - y^i)\log(1 - \hat{y}^i)]$

(a) $\dfrac{\partial L}{\partial \theta_1} = \dfrac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})x_1^{(i)}$

So we need to find $\frac{\partial l}{\partial \theta_1}$

$$\begin{aligned}\frac{\partial l}{\partial \theta_1} &= -\left[\frac{\partial}{\partial \theta_1} y \log(\hat{y}) + \frac{\partial}{\partial \theta_1}(1 - y)\log(1 - \hat{y})\right] \\ &= -\left[y \frac{1}{\hat{y}}\frac{\partial \hat{y}}{\partial \theta_1} + (1 - y)\frac{1}{1 - \hat{y}}\frac{\partial (1 - \hat{y})}{\partial \theta_1}\right] \qquad \text{(applying chainrule)}\end{aligned}$$

since $\frac{\partial(1-\hat{y})}{\partial \theta_1} = \frac{\partial 1}{\partial \theta_1} - \frac{\partial \hat{y}}{\partial \theta_1} = 0 - \frac{\partial \hat{y}}{\partial \theta_1}$

$$\begin{aligned}&= -\left[y \frac{1}{\hat{y}}\frac{\partial \hat{y}}{\partial \theta_1} + (1 - y)\frac{1}{1 - \hat{y}}\frac{\partial \hat{y}}{\partial \theta_1}\right] \\ &= -\left[y \frac{1}{\hat{y}}\frac{\partial \hat{y}}{\partial \theta_1} + \frac{1 - y}{1 - \hat{y}}\frac{\partial \hat{y}}{\partial \theta_1}\right]\end{aligned}$$

Then factor out $\frac{\partial \hat{y}}{\partial \theta_1}$

$$= -\left(\frac{y}{\hat{y}} + \frac{1 - y}{1 - \hat{y}}\right) \cdot \frac{\partial \hat{y}}{\partial \theta_1}$$

This can be simplified by:

$$\frac{y}{\hat{y}} - \frac{1 - y}{1 - \hat{y}} = \frac{y(1 - \hat{y})}{\hat{y}(1 - \hat{y})} - \frac{\hat{y}(1 - y)}{\hat{y}(1 - y)} = \frac{y(1 - \hat{y}) - (\hat{y}(1 - y))}{\hat{y}(1 - y)}$$

$$\frac{y - y\hat{y} - (\hat{y} - \hat{y}y)}{\hat{y}(1 - y)} = \frac{y - y\hat{y} - \hat{y} + \hat{y}y}{\hat{y}(1 - y)} = \frac{y - \hat{y}}{\hat{y}(1 - y)}$$

So now

$$\frac{\partial l}{\partial \theta_1} = -\frac{y - \hat{y}}{\hat{y}(1 - y)}\frac{\partial \hat{y}}{\partial \theta_1}.$$

Since $\sigma(x) = \frac{1}{1+e^{-x}}, \sigma'(x) = \sigma(x)(1 - \sigma(x))$ and by chain-rule

Then for $f(z) = \frac{1}{1+e^{z}}$

$$\frac{\partial \hat{y}}{\partial \theta_1} \Rightarrow \frac{\partial \hat{y}}{\partial z}\frac{\partial z}{\partial \theta_1} = \hat{y}(1 - \hat{y}) \cdot x_1$$

So now

$$\frac{\partial l}{\partial \theta_1} = -\frac{y - \hat{y}}{\hat{y}(1 - y)}\frac{\partial \hat{y}}{\partial \theta_1} = \frac{y - \hat{y}}{\hat{y}(1 - y)}\hat{y}(1 - \hat{y}) \cdot x_1 =$$

$$\frac{\partial l}{\partial \theta_1}(\hat{y} - y)x_1 \Longrightarrow \boldsymbol{\frac{\partial L}{\partial \theta_1} = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)x_1^{(i)}}$$

(b) $\frac{\partial L}{\partial \theta_2} = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)x_2^{(i)}$

Following the same procedure as (a) but differentiating w.r.t. θ_2 instead of θ_1

So.

$$\frac{\partial \hat{y}}{\partial \theta_2} \Rightarrow \frac{\partial \hat{y}}{\partial z}\frac{\partial z}{\partial \theta_2} = \hat{y}(1 - \hat{y}) \cdot x_2$$

So now

$$\frac{\partial l}{\partial \theta_2} = -\frac{y - \hat{y}}{\hat{y}(1 - y)}\frac{\partial \hat{y}}{\partial \theta_2} = \frac{y - \hat{y}}{\hat{y}(1 - y)}\hat{y}(1 - \hat{y}) \cdot x_2 =$$

$$\frac{\partial l}{\partial \theta_2}(\hat{y} - y)x_2 \Longrightarrow \boldsymbol{\frac{\partial L}{\partial \theta_2} = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)x_2^{(i)}}$$

(c) $\frac{\partial L}{\partial b} = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)$

Following the same procedure as (a) but differentiating w.r.t. b instead of θ_1

So.

$$\frac{\partial \hat{y}}{\partial b} \Rightarrow \frac{\partial \hat{y}}{\partial z}\frac{\partial z}{\partial b} = \hat{y}(1 - \hat{y})$$

So now

$$\frac{\partial l}{\partial b} = -\frac{y - \hat{y}}{\hat{y}(1 - y)}\frac{\partial \hat{y}}{\partial b} = \frac{y - \hat{y}}{\hat{y}(1 - y)}\hat{y}(1 - \hat{y}) =$$

$$\frac{\partial l}{\partial b}(\hat{y} - y)x_2 \Longrightarrow \boldsymbol{\frac{\partial L}{\partial b} = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)}$$

$$\nabla f(\theta_1, \theta_2, b) = \left(\frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)x_1^{(i)}, \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)x_2^{(i)}, \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)\right)$$

$$\nabla f(\theta_1, \theta_2, b) = \frac{1}{m}\sum_{i=1}^{m}\big(\hat{y}^{(i)} - y^{(i)}\big)\left(x_1^{(i)}, x_2^{(i)}, 1\right)$$

6.4. Problem 4

A function $f(x_1, ..., x_n)$ has n first-order partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad i = 1, ..., n$$

How many second-order partial derivatives does it have?

6.4.1. Solution

Since

$$\frac{\partial^2 f}{\partial x_i^2}, \frac{\partial^2 f}{\partial x_i \partial x_j} \quad i, j = 1, ..., n, \quad i \neq j$$

n^2 total mixed derivatives. So

$$n + n(n-1) = n^2$$

6.5. Problem 5

Compute the second-order partial derivatives of

$$f(x_1, x_2) = x_1 x_2^2 + 3x_1^2 e^{x_2}$$

6.5.1. Solution

There should be $n^2 = 2^2 = 4$ possible second-order partial derivatives of $f(x_1, x_2)$

These are

$$\frac{\partial^2 f}{\partial x_1^2}, \quad \frac{\partial^2 f}{\partial x_2^2}, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2}, \quad \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

Now compute them all

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}(x_2^2 + 6x_1 e^{x_2}) = 6e^{x_2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(2x_1 x_2 + 3x_1^2 e^{x_2}) = 2x_1 + 3x_1^2 e^{x_2}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}(2x_1 x_2 + 3x_1^2 e^{x_2}) = 2x_2 + 6xe^{x_2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(x_2^2 + 6x_1 e^{x_2}) = 2x_2 + 6xe^{x_2}$$

6.6. Problem 6

Find and analyze the critical points of

$$f(x,y) = x^2 - 2x + y^2 - 4y + 5$$

6.6.1. Solution

Then

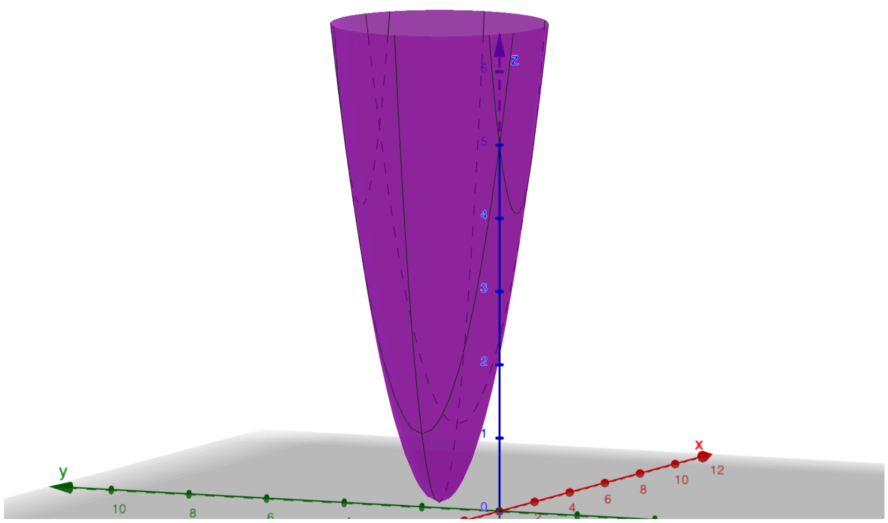
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For a hessian $\det\left(\left[\begin{smallmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{smallmatrix}\right]\right) = 0$

So

$$(2-\lambda)(2-\lambda) - 0 \cdot 0 = 0 \Rightarrow \lambda = 2$$

So local mimimum



As shown here, the critical point is also the global min

$$0 = \frac{\partial f}{\partial x} \Rightarrow x = 1$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow y = 2$$

gobal min is at $x = 1, y = 2 \rightarrow f(1,2) = 0$

6.7. Problem 7

Find and analyze any critical points of

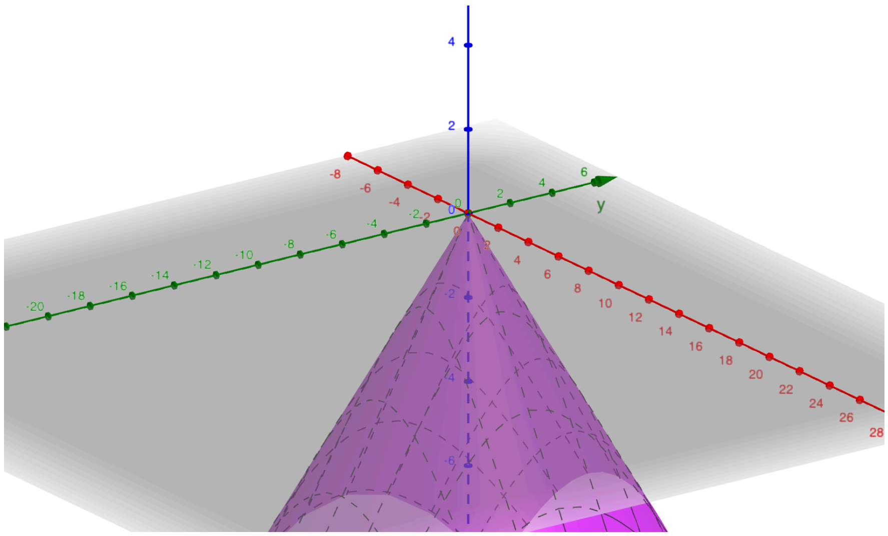
$$f(x,y) = -\sqrt{x^2 + y^2}$$

6.7.1. Solution

$$f_x(x,y) = -\frac{x}{\sqrt{x^2+y^2}}, \quad f_y(x,y) = -\frac{y}{\sqrt{x^2+y^2}}$$

$$f_x(0,0) = \frac{0}{0} = \text{DNE}, \quad f_y(0,0) = \frac{0}{0} = \text{DNE}$$

f is a cone though, where its global maximum is $f(0,0) = 0$



6.8. Problem 8

Find and analyze any critical points of

$$f(x, y) = x^2 + y^2$$

6.8.1. Solution

Then

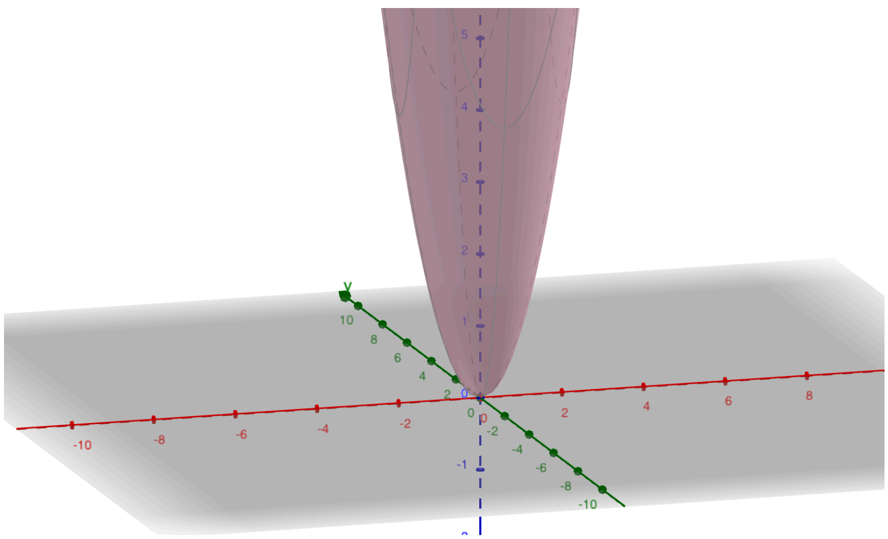
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For a hessian $\det\left(\left[\begin{smallmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{smallmatrix}\right]\right) = 0$

So

$$(2 - \lambda)(2 - \lambda) - 0 \cdot 0 = 0 \Rightarrow \lambda = 2$$

So local mimimum



The minimum must also be the global min

$$0 = \frac{\partial f}{\partial x} \Rightarrow x = 0$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow y = 0$$

gobal min is at $0, 0 \to f(0, 0) = 0$

6.9. Problem 7

Classify the critical points of

$$f(x, y) = x^4 + y^4$$

6.9.1. Solution

Then

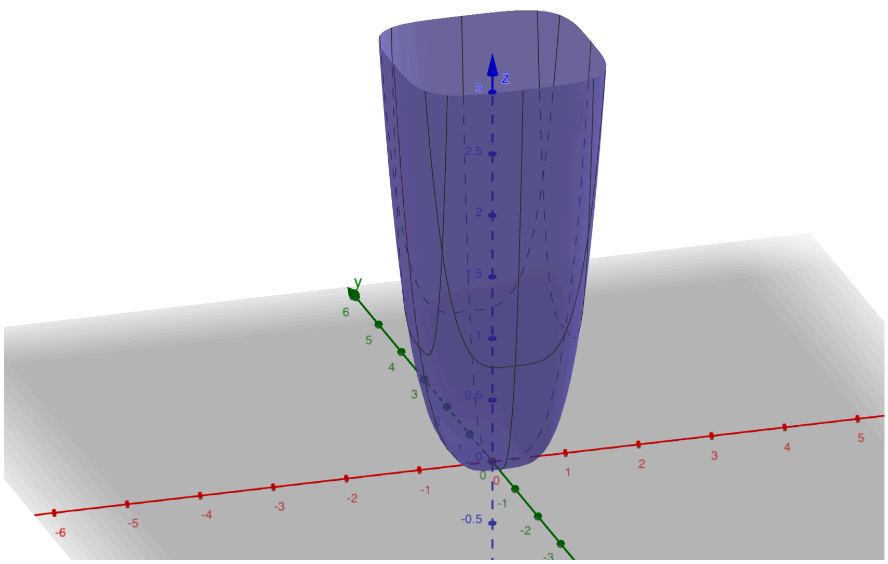
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

For a hessian $\det\left(\begin{bmatrix} 12x^2-\lambda & 0 \\ 0 & 12y^2-\lambda \end{bmatrix}\right) = 0$

So

$$(12x^2 - \lambda)(12y^2 - \lambda) = 0 \Rightarrow \lambda = 2$$

Because of $\lambda_1, \lambda_2 < 0$, this indicates local mimimum



The local minimum must also be the global min

$$0 = \frac{\partial f}{\partial x} \Rightarrow x = 0$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow y = 0$$

gobal min is at $0, 0 \rightarrow f(0, 0) = 0$

6.10. Problem 8

Classify the critical points of

$$f(x, y) = -x^4 - y^4$$

6.10.1. Solution

Then

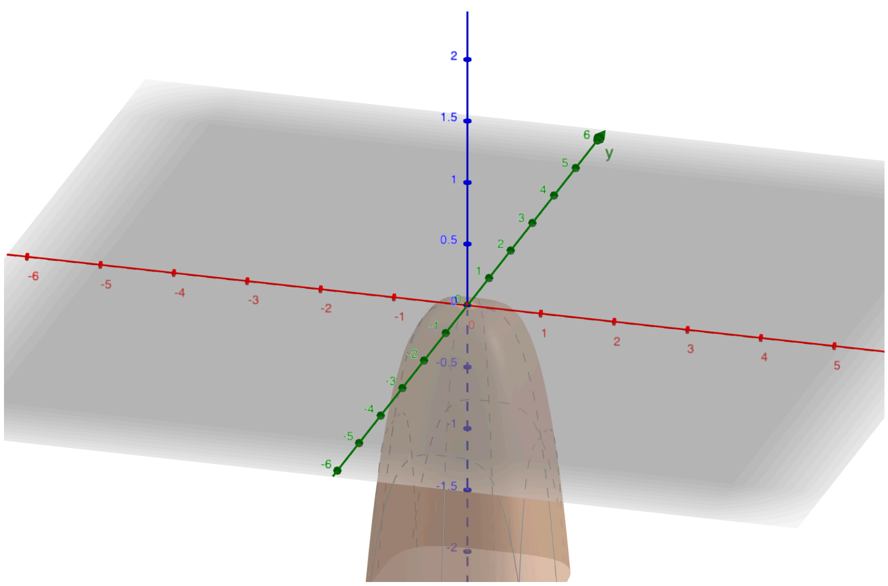
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

For a hessian $\det\left(\begin{bmatrix} -12x^2-\lambda & 0 \\ 0 & -12y^2-\lambda \end{bmatrix}\right) = 0$

So

$$(12x^2 - \lambda)(12y^2 - \lambda) = 0 \Rightarrow \lambda = -12 * x^2$$

Since $\lambda_1, \lambda_2 \leq 0$, I cannot directly conclude anything, only based on the eigenvalues from the hessian matrix. But since for $x, y \neq 0$, all eigenvaleus are strictly negative, this is most likely a local maximum



The local max is clearly shown above. The local max must also be the global maxima

$$0 = \frac{\partial f}{\partial x} \Rightarrow x = 0$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow y = 0$$

gobal min is at $0, 0 \to f(0, 0) = 0$

6.11. Problem 9

Classify the critical points of

$$f(x, y) = x^4 - y^4$$

6.11.1. Solution

Then

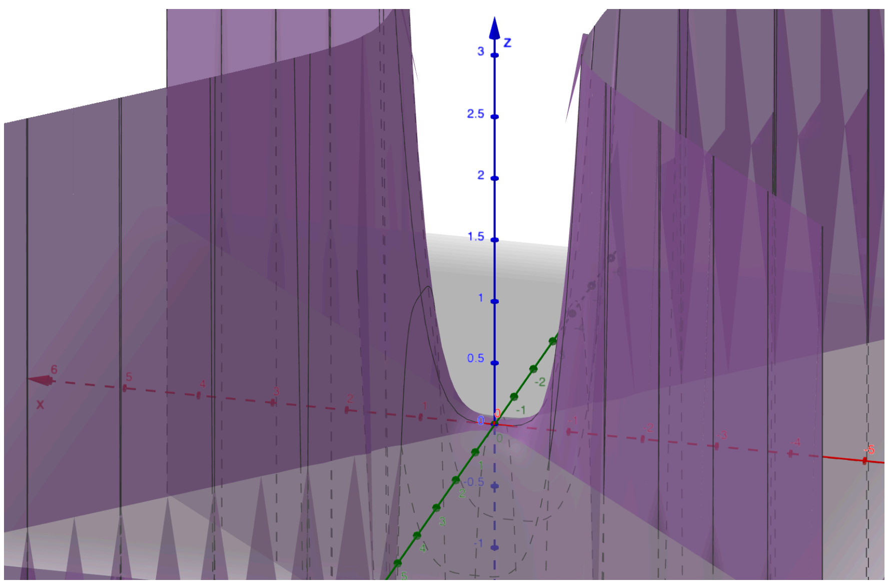
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x \partial y} \\ \frac{\partial f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

For a hessian $\det\left(\begin{bmatrix} 12x^2-\lambda & 0 \\ 0 & -12y^2-\lambda \end{bmatrix}\right) = 0$

So

$$(12x^2 - \lambda)(12y^2 - \lambda) = 0 \Rightarrow \lambda = -12x^2, 12y^2$$

Since $\lambda \leq 0 \geq \lambda$ and , I cannot directly conclude anything, only based on the eigenvalues from the hessian matrix. But since for $x, y \neq 0$, the eigenvaleus have mixed signs, this is most likely a sadle point



The saddlepoint can be seen above.

$$0 = \frac{\partial f}{\partial x} \Rightarrow x = 0$$

$$0 = \frac{\partial f}{\partial y} \Rightarrow y = 0$$

The saddle point is at $0, 0 \to f(0, 0) = 0$

7. Lecture 7

7.1. Problem 1

Do all functions have global extrema? Can you think of a counterexample?

7.1.1. Solution

The global max is the the largest value $f(x)$ takes.

But if $f(x) \to +\infty$, then $f(x)$ does not have a global extrema

7.2. Problem 2

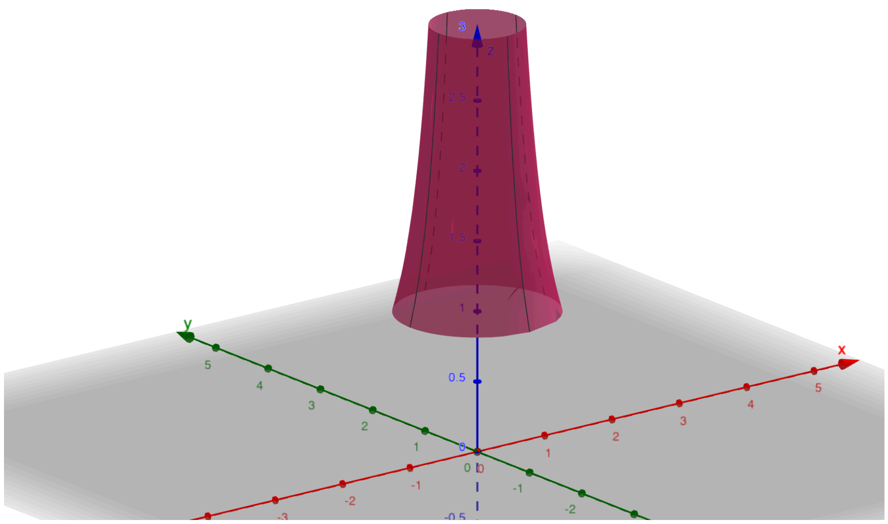
Does the function

$$f(x, y) = \frac{1}{x^2 + y^2}$$

have global maxima and minima on the region R given by $0 < x^2 + y^2 \leq 1$?

7.2.1. Solution

First inspect f



Since R is $(0, 1]$, R is not closed, therfore f does not have a global max

It does however has a global min at $\frac{1}{x^2+y^2} = 1)$

7.3. Problem 3

Does the function

$$f(x,y) = x^2y^2$$

have global maxima and minima in the xy -plane?

7.3.1. Solution

The, range for f is $[0, \infty)$ as $|x|$ and $|y|$ grows arbitrarily large, and $f(x,y) \to \infty$

So global minima at $f(x,y) = 0$

And no global maxima

7.4. Problem 4

Suppose that $z = f(x,y) = x\sin(y)$, where $x = t^2$ and $y = 2t + 1$

Compute $f'(x,y)$ directly and using the chain rule.

7.4.1. Solution

$$\frac{\partial}{\partial t}f(x,y) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Then

$$\frac{\partial}{\partial t}f(x,y) = \sin(y) \cdot 2t + x\cos(y) \cdot 2$$

Then substitute x and y

$$\frac{\partial}{\partial t}f(x,y) = 2\sin(2t+1) \cdot t + 2t^2\cos(2t+1)$$

7.5. Problem 5

If f, g, h are differentiable and if $z = f(x,y)$, with $x = g(u,v)$ and $y = h(u,v)$

What is $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$?

7.5.1. Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

8. Lecture 8

8.1. Problem 1

Show that if f is differentiable with the local linearization

8.1.1. Solution

$$L(x) = f(a) - \sum_{i=1}^{n} m_i(x_i - a_i)$$

Then

$$m_i = \frac{\partial f}{\partial x_i}(a)$$

since

$$f(x) \approx L(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i - a_i)$$

8.2. Problem 2

Consider the function

$$f(x, y) = \sqrt{x^2 + y^2}$$

is f differentiable at the origin?

8.2.1. Solution

To check this

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2}}{h} = \lim_{h \to 0} \frac{|h|}{h} = \text{DNE}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2}}{h} = \lim_{h \to 0} \frac{|h|}{h} = \text{DNE}$$

f is **not** differentiable at the origin

8.3. Problem 3

Consider the function

$$f(x, y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$$

Compute the partial derivatives at $(x, y) = (0, 0)$.

Is f differentiable at $(0, 0)$?

8.3.1. Solution

To check this

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}} \cdot 0^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0^{\frac{1}{3}} \cdot h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Note that to prove differentiability f must also be continuous on a small disk centered at the point a (local linearization)

For differentiability:

$$\lim_{h \to 0} \frac{E(x)}{\|h\|} = 0$$

Lets do that

$$\lim_{(h_1, h_2) \to (0,0)} \frac{f(x) - L(x)}{\sqrt{h_1^2 + h_1^2}}$$

$$\lim_{(x,y) \to (a,b)} \frac{f(x) - L(x)}{\sqrt{h_1^2 + h_1^2}}$$

Since

$$L(0, 0) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 0 + 0x + 0y = 0$$

then

$$\lim_{(x,y) \to (0,0)} \frac{x^{\frac{1}{3}} y^{\frac{1}{3}}}{\sqrt{(x)^2 + (y)^2}} = \text{DNE}$$

f is **not** differentiable at $(0, 0)$

8.4. Problem 4

Show that the function $f(x,y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

8.4.1. Solution

The domain of f is $(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)$

If the partial derivatives, f_x and f_y , of a function f exist and are continuous on a small disk centered at the point a, then f is differentiable at a.

Because of this, we want to show that f_x and f_y both exist and are continous on the whole domain

$$f_x(x,y) = \frac{2x}{x^2 + y^2}, \quad f_y(x,y) = \frac{2y}{x^2 + y^2}$$

These are continuous on th whole domain, since $(x,y) \neq (0,0)$

8.5. Problem 5

Use the definition of the definite integral to compute

$$\int_a^b c \, \mathrm{d}x$$

8.5.1. Solution

The definition of the definite integral is:

$$\int_a^b f(x) \, \mathrm{d}x = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

Where $x_k = a + k \cdot \Delta x, \Delta x = \frac{b-a}{n}$ and $x_k^* \in [x_{k-1}, x_k]$

So then

$$\int_a^b c \, \mathrm{d}x = \lim_{n \to \infty} \sum_{k=1}^{n} c \frac{b-a}{n} = \lim_{n \to \infty} n \cdot c \cdot \frac{b-a}{n} = c(b-a)$$

8.6. Problem 6

Use the definition of definite integral to simplify

$$\int_a^b cf(x)\,\mathrm{d}x \quad \text{for a constant c.}$$

8.6.1. Solution

The definition of the definite integral is:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Where $x_k = a + k\cdot\Delta x$, $\Delta x = \frac{b-a}{n}$ and $x_k^* \in [x_{k-1}, x_k]$

$$\int_a^b cf(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n} cf(x_k^*)\Delta x$$

$$\lim_{n\to\infty} c\cdot\sum_{k=1}^{n} f(x_k^*)\Delta x = c\cdot\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = c\cdot\int_a^b f(x)\ dx$$

8.7. Problem 7

Use the definition of definite integral to show that

$$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

8.7.1. Solution

$$\begin{aligned}\int_a^b f(x) + g(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*) + g(x_k^*)\Delta x \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x + \lim_{n\to\infty}\sum_{k=1}^{n} g(x_k^*)\Delta x \\ &= \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\end{aligned}$$

8.8. Problem 8

Evaluate

$$\int_0^3 2x \,\mathrm{d}x$$

as a limit of Riemann sums.

8.8.1. Solution

$$\begin{aligned}\int_0^3 2x \,\mathrm{d}x &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} 2x \cdot \frac{3}{n}\end{aligned}$$

Since $x = k\Delta x = \frac{3k}{n}$

$$\begin{aligned}&= \lim_{n\to\infty} \textstyle\sum_{k=1}^{n} \frac{6k}{n} \cdot \frac{3}{n} \\ &= \lim_{n\to\infty} \textstyle\sum_{k=1}^{n} \frac{18k}{n^2} \\ &= \lim_{n\to\infty} \textstyle\frac{18}{n^2} \cdot \sum_{k=1}^{n} k \\ &= \lim_{n\to\infty} \textstyle\frac{18}{n^2} \cdot n(n+1) \\ &= \lim_{n\to\infty} \textstyle\frac{18(n+1)}{2n} \\ &= \lim_{n\to\infty} \textstyle\frac{9(n+1)}{n} \\ &= \lim_{n\to\infty} 9 + \textstyle\frac{1}{n} = 9\end{aligned}$$

8.9. Problem 9

Compute

$$\int_0^3 2x \ dx$$

Using the fundamental theorem of calculus.

8.9.1. Solution

$$\int_a^b f(x) \,\mathrm{d}xx = F(b) - F(a)$$

$$\int_0^3 2x \,\mathrm{d}xx = F(3) - F(0)$$

Since

$$\int 2x = x^2$$

$$F(3) - F(0) = 9$$

8.10. Problem 10

Find

$$\frac{d}{dx}\left[\int_2^x \cos(t)\,\mathrm{d}x\right]$$

8.10.1. Solution

Since

$$\frac{d}{dx}\left[\int_a^x g(t)\,\mathrm{d}x\right] = g(x)$$

by second FTC

then

$$\frac{d}{dx}\left[\int_2^x \cos(t)\,\mathrm{d}x\right] = \cos(x)$$

8.11. Problem 11

Let

$$g(x) = \int_1^x \sqrt{1+t^2}\,\mathrm{d}x$$

- Find $g'(x)$
- Find $g(x^3)$
- Compute $\frac{d}{dx}g(x^3)$.

8.11.1. Solution

- Find $g'(x)$ Since

  $$\frac{d}{dx}\left[\int_a^x g(t)\,\mathrm{d}x\right] = g(x)$$

  by second FTC

  then

  $$\frac{d}{dx}\left[\int_1^x \sqrt{1+t^2}\,\mathrm{d}x\right] = \sqrt{1+x^2}$$

- Find $g(x^3)$

I want to express

$$g(x^3) = \int_1^{x^3} \sqrt{1+t^2}\,\mathrm{d}x$$

- Compute $\frac{d}{dx} g(x^3)$.

  Since

$$\frac{d}{dx}\left[\int_a^x g(t)\,\mathrm{d}x\right] = g(x)$$

by second FTC

and

$$g(x^3) = \int_1^{x^3} \sqrt{1+t^2}\,\mathrm{d}x$$

then

$$\frac{d}{dx} g(x^2) = \frac{d}{dx}\left[\int_1^{x^3} \sqrt{1+t^2}\,\mathrm{d}x\right]$$

This is done by chainrule

$$\frac{d}{dt} f(x) = \frac{df}{dx} \cdot \frac{dx}{dt}$$

So

$$\frac{d}{dx}\left[\int_1^{x^3} \sqrt{1+t^2}\,\mathrm{d}x\right]$$

$$\frac{d}{dx}\left[\int_1^{x^3} \sqrt{1+t^2}\,\mathrm{d}x\right] = \sqrt{1+x^6} \cdot 3x^2$$

9. Lecture 9

9.1. Problem 1

Compute

$$\int_0^\infty e^{-5x}\,\mathrm{d}x$$

9.1.1. Solution

Since

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

So

$$\int_0^\infty e^{-5x}\,\mathrm{d}x = \lim_{b\to\infty}\int_0^b e^{-5x}\,\mathrm{d}x$$

By first FTC

$$\lim_{b\to\infty} F(b) - F(0) = \lim_{b\to\infty} -\frac{e^{-5b}}{5} - \left(-\frac{e^{-5(0)}}{5}\right) = 0 - \left(-\frac{1}{5}\right) = \frac{1}{5}$$

9.2. Problem 2

Investigate the convergence of

$$\int_0^2 \frac{1}{(x-2)^2}\,\mathrm{d}x.$$

9.2.1. Solution

By looking at the graph i can see that has assymtotic behvaior close to $x = 2$

This makes sense, because f is undefined at $x = 2$. Therfore

$$\lim_{b\to 2}\int_0^b \frac{1}{(x-2)^2}\,\mathrm{d}x$$

Then using first FTC

$$\lim_{b\to 2}\int_0^b \frac{1}{(x-2)^2}\,\mathrm{d}x = \lim_{b\to 2} F(b) - F(0)$$

Since $F(x) = -\frac{1}{x-2}$

$$\lim_{b\to 2} F(b) - F(0) = \lim_{b\to 2}\left(-\frac{1}{b-2} - \left(-\frac{1}{-2}\right)\right) = \lim_{b\to 2}\left(-\frac{1}{b-2} - \frac{1}{2}\right)$$

Since f diverges from 2^-

$$\lim_{b\to 2^-}\left(-\frac{1}{b-2} - \frac{1}{2}\right) = +\infty$$

9.3. Problem 3

Compute

$$\int_2^3 2x \sin(x^2)\, \mathrm{d}x$$

Hint: Let $u = x^2$, then $du = 2x\, \mathrm{d}x$. After substitution, the integral becomes

$$\int_{u(a)}^{u(b)} \sin(u)\, \mathrm{d}u = \int_{2^2}^{3^2} \sin(u)\, \mathrm{d}u$$

9.3.1. Solution

$$\int_3^9 \sin(u)\, \mathrm{d}u = F(9) - F(3) = \cos(9) - \cos(3) \approx 0.26$$

9.4. Problem 4

Evaluate

$$\int_0^1 x(x-1)^4\, \mathrm{d}x$$

9.4.1. Solution

Let $u = x$, then $du = x\, \mathrm{d}x$. After substitution, the integral becomes

$$\int_{u(a)}^{u(b)} u^4\, \mathrm{d}u = \int_{0-1}^{1-1} u^4\, \mathrm{d}u$$

Then

$$\int_{-1}^{0} u^4\, \mathrm{d}u = F(0) - F(-1)$$

Since

$$F(x) = \frac{x^5}{5}$$

$$F(0) - F(-1) = \frac{0^5}{5} - \frac{(-1)^5}{5} = \frac{1}{5}$$

9.5. Problem 5

Evaluate

$$\int \sin^2(x)\,\mathrm{d}x$$

9.5.1. Solution

$$\int \sin^2(x)\,\mathrm{d}x = -\frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + C$$

9.6. Problem 6

Evaluate

$$\int_0^1 xe^x\,\mathrm{d}x$$

9.6.1. Solution

For definite integrals

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = (f(b)g(b) - f(a)g(a)) - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

For this.

$$\int_0^1 xe^x\,\mathrm{d}x = (1 \cdot e^1 - 0 \cdot e^0 - \int_0^1 1 \cdot e^x\,\mathrm{d}x$$

$$\int_0^1 xe^x\,\mathrm{d}x = e - \int_0^1 e^x\,\mathrm{d}x = e - (e^1 - e^0) = 1$$

9.7. Problem 7

Evaluate

$$\int_1^2 \ln(x)\,\mathrm{d}x$$

9.7.1. Solution

Since doint $\int \ln(x)\,\mathrm{d}x$ is not very intuative

Let

$$\int_1^2 \ln(x)\,\mathrm{d}x = \int_1^2 \ln(x) \cdot 1\,\mathrm{d}x$$

Now for definite integrals

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = (f(b)g(b) - f(a)g(a)) - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

For this.

$$\int_1^2 \ln(x) \cdot 1\,\mathrm{d}x = (\ln(2) \cdot 2 - \ln(0) \cdot 0) - \int_1^2 \frac{1}{x} \cdot x\,\mathrm{d}x$$

Since

$$\int_1^2 \frac{1}{x} \cdot x\,\mathrm{d}x = \int_1^2 1\,\mathrm{d}x = 0$$

$$\int_1^2 \ln(x) \cdot 1\,\mathrm{d}x = 2\ln(2) \approx 1.39$$

9.8. Problem 8

Let M_{ij} and L_{ij} denote the max and min of f on the ij-th rectangle. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle:

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \Delta x \Delta y$$

- How does $\sum_{(i,j)} f(u_{ij}, v_{ij}) \Delta x \Delta y$ look geometrically?
- What happens as $n, m \to \infty$?

9.8.1. Solution

- How does $\sum_{(i,j)} f(u_{ij}, v_{ij}) \Delta x \Delta y$ look geometrically?

  An area by f aproximated by rectangles size $\Delta x \cdot \Delta y$

- What happens as $n, m \to \infty$?

  As$n, m \to \infty$ each rectangle gets increasingly smaller, and the numeber of rectangles because larger. So large that the approximation becomes the area

9.9. Problem 9

If $g(x, y) = 1$ what is $\int_R g \, \mathrm{d}A$

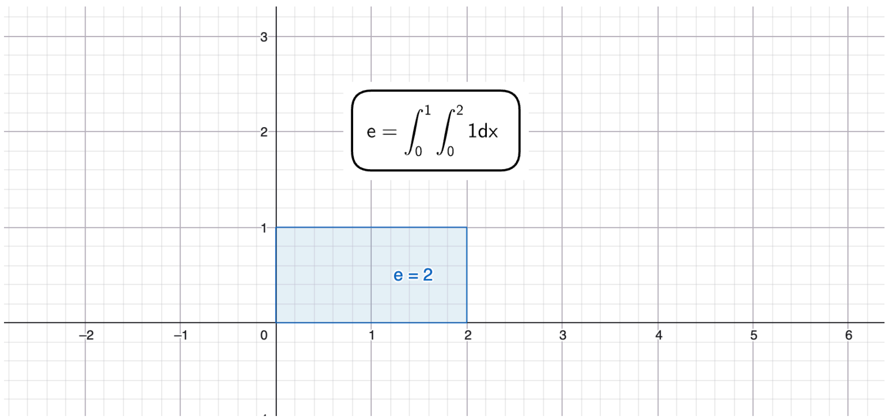
9.9.1. Solution

Since

$$\int_R 1 \, \mathrm{d}A = \int_a^b \int_c^d 1 \, \mathrm{d}y \, \mathrm{d}x = (b - a) \cdot (c - d)$$

Then

$$\int_R g \, \mathrm{d}A = \text{Areal of } R$$



9.10. Problem 10

How do we compute the average value of f over R?

9.10.1. Solution

As for single variable intergration

$$f_{\text{avg}} = \frac{1}{b - a} \cdot \int_a^b f(x) \, \mathrm{d}x \Longrightarrow \frac{\text{Area}}{\text{Length of interval}}$$

Then for two variables

$$f_{\text{avg}} = \frac{1}{\iint_R 1 \, \mathrm{d}} \cdot \iint_R f(x) \, \mathrm{d}A \Longrightarrow \frac{\text{Volume under surface } f(x, y)}{\text{Area of R}}$$

9.11. Problem 11

Let $f(x, y) = x^2 y, R = [0, 1] \times [0, 1]$.

Compute $\int_R f \, \mathrm{d}A$

$$\int_0^1 \int_0^1 x^2 y \, \mathrm{d}x \, \mathrm{d}y$$

9.11.1. Solution

First

$$\int_0^1 x^2 y \, \mathrm{d}x = \frac{1^3 \cdot y}{3} = \frac{y}{3}$$

Then

$$\int_0^1 \frac{y}{3} \, \mathrm{d}y = \frac{1^2}{6} = \frac{1}{6}$$

9.12. Problem 12

What if we switch the order of integration?

$$\int_0^1 \int_0^1 x^2 y \, \mathrm{d}y \, \mathrm{d}x$$

9.12.1. Solution

First

$$\int_0^1 x^2 y \, \mathrm{d}y = \frac{x^2 y^2}{2} = \frac{x^2}{2}$$

Then

$$\int_0^1 \frac{x^2}{2} \, \mathrm{d}y = \frac{1^3}{6} = \frac{1}{6}$$

Same answer

9.13. Problem 13

A building is 8m wide and 16m long. It has a flat roof 12m high at one corner, and 10m high at each adjacent corner.

Find the volume of the building.

9.13.1. Solution

I assume that the highest corner of the roof is on point $P(0, 0, 12)$

Based on the height of each corner i can know that the surface of the roof, has vector $a = (8, 0, -2)$ and $b = (0, 16, -2)$

i can the find $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ with a normal vector N.

Now by using cross product $N = v \times w$

$$N = v \times w = \begin{pmatrix} 32 \\ 16 \\ 128 \end{pmatrix}$$

with point $P(0, 0, 12)$ the plane becomes

$$32(x - 0) + 16(y - 0) + 128(z - 12) = 0$$

$$32x + 16y + 128z - 1536 = 0$$

$$z = -\frac{x}{4} - \frac{y}{8} + 12$$

Then i can

$$\int_0^8 \int_0^{16} -\frac{x}{4} - \frac{y}{8} + 12 \, \mathrm{d}y \, \mathrm{d}x$$

Since

$$\int -\frac{x}{4} - \frac{y}{8} + 12 \, \mathrm{d}y = -\frac{1}{4}xy - \frac{1}{16}y^2 + 12y$$

Then

$$F(16) - F(0) = -4x + 176$$

Now since

$$\int -2x^2 + 176x \, \mathrm{d}x = -2x^2 + 176x$$

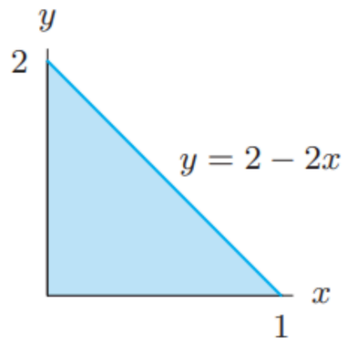
Then

$$F(8) - F(0) = 1280$$

Volume of the building is $\mathbf{1280 \ m^3}$

9.14. Problem 14

The density at (x, y) of a triangular metal plate is $\delta(x, y)$. Express its mass as an iterated integral.



9.14.1. Solution

We wish to express

$$m = \iint_R \delta(x, y)\,\mathrm{d}A$$

Where $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}$

Hence

$$m = \int_0^1 \int_0^{2-2x} \delta(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

9.15. Problem 15

A city occupies a semicircular region of radius 3 km bordering the ocean. Find the average distance from points in the city to the ocean.

9.15.1. Solution

We can use polar coordinates where $0 \leq 3$ and $0 \leq \theta \leq \pi$

Since it is a semicircle of radius R, the area is $\left(\frac{1}{2}\right)\pi R^2 = \left(\frac{1}{2}\right)\pi(3^2) = \left(\frac{9}{2}\right)\pi$.

We know that $f_{\text{avg}} = \frac{\text{Distance}}{\text{Area of R}}$

Hence

$$f_{\text{avg}} = \frac{\iint \text{distance}\,\mathrm{d}A}{\text{Area of R}} = \frac{\int_0^\pi \int_0^R r^2\,\mathrm{d}r\,\mathrm{d}\theta}{\left(\frac{1}{2}\right)\pi R^2}$$

Since $\int_0^R r^2\,\mathrm{d}r = \frac{R^3}{3}$

$$f_{\text{avg}} = \frac{\int_0^\pi \frac{R^3}{3}\,\mathrm{d}\theta}{\left(\frac{1}{2}\right)\pi R^2}$$

Since $\int_0^\pi \frac{R^3}{3}\,\mathrm{d}\theta = R^3\frac{\pi}{3}$

Then

$$f_{\text{avg}} = \frac{R^3\frac{\pi}{3}}{\left(\frac{1}{2}\right)\pi R^2}$$

Now substitute R for 3.

$$f_{\text{avg}} = \frac{\frac{3^3\pi}{3}}{\frac{3^2\pi}{2}} = \frac{9\pi}{\frac{9}{2}\pi} = 2$$

9.16. Problem 16

Compute

$$\int_W (1 + xyz)\,\mathrm{d}V$$

over the cube $0 \leq x, y, z \leq 4$.

9.16.1. Solution

We know that $W = \{(x, y, z) : 0 \leq x, y, z \leq 4\}$

This means that $W = [0, 4] \times [0, 4] \times [0, 4]$

And therefore

$$\int_W (1 + xyz)\,\mathrm{d}V = \iiint_W (1 + xyz)\,\mathrm{d}V = \int_0^4 \int_0^4 \int_0^4 (1 + xyz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Since $\int (1 + xyz)\,\mathrm{d}x = \frac{1}{2}x^2yz + x$

Then

$$F(4) - F(0) = 8yz + 4$$

And

$$\int_W (1 + xyz)\,\mathrm{d}V = \int_0^4 \int_0^4 (8yz + 4)\,\mathrm{d}y\,\mathrm{d}z$$

Since $\int (8yz + 4)\,\mathrm{d}y = 4y^2z + 4y$

Then

$$F(4) - F(0) = 64 * z + 16$$

And

$$\int_W (1 + xyz)\,\mathrm{d}V = \int_0^4 64z + 16\,\mathrm{d}z$$

Since $\int (64z + 16)\,\mathrm{d}z = 32z^2 + 16z$

Then

$$F(4) - F(0) = 576$$

And

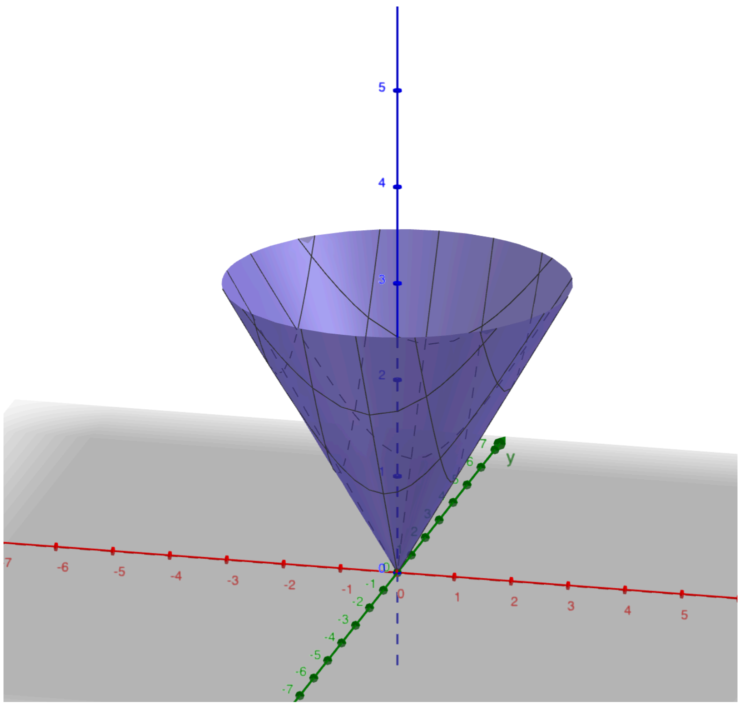
$$\int_W (1 + xyz)\,\mathrm{d}V = \mathbf{576}$$

9.17. Problem 17

Set up an iterated integral for the mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density $\delta(x, y, z) = z$.

9.17.1. Solution

The cone is given by $z = \sqrt{x^2 + y^2}$ and is cut of by the plane $z = 3$ meaning $z \leq 3$



$$z = \sqrt{x^2 + y^2}, \quad \text{where } z \leq 3$$

Then

$$m = \iiint_W \delta(x, y, z) \, \mathrm{d}V$$

in this case this is:

$$\iiint_W z \, \mathrm{d}V$$

Because the of the cone, polar coordinates are easier to work with

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad \underbrace{z = z}_{\text{cone is on } z}, \quad \mathrm{d}V = r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

In this instance θ is the whole circle around the z axis so $0 \leq \theta \leq 2\pi$

To find r we can use the definition of the circle at $z = 3$

Because of $3 = \sqrt{x^2 + y^2} \Rightarrow \underbrace{x^2 + y^2 = 9}_{\text{circle at } z=3}$ where $9 = r^2$, So $r = 3$

This means that $0 \leq r \leq 3$

To find z we look at the cones height from $z = 0$ to $z = 3$.

For any (r, θ) we know that $z = \sqrt{x^2 + y^2} = r$

This means that $\sqrt{x^2 + y^2} \leq z \leq 3 \quad \Longrightarrow r \leq z \leq 0$

Finally we can put it all together

$$\int_0^{2\pi} \int_0^3 \int_r^3 \delta(x, y, z) r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{2\pi} \int_0^3 \int_r^3 \sqrt{x^2 + y^2} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$$

10. Lecture 10

No new problems

11. Lecture 11

11.1. Problem 1

Evaluate

$$\int_0^1 xe^{x^2}\,\mathrm{d}x$$

(Recall: set $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$.)

11.1.1. Solution

When $u = x^2$ and $\mathrm{d}x = 2x$

Then

$$\int_0^1 xe^{x^2}\,\mathrm{d}x = \int_0^1 \frac{1}{2} \cdot e^u\,\mathrm{d}u$$

Now

$$\int_0^1 \frac{1}{2} \cdot e^u\,\mathrm{d}u = F(1) - F(0)$$

Since $\int \frac{1}{2} \cdot x^2\,\mathrm{d}x = \frac{x^3}{6}$

$$F(1) - F(0) = \frac{1^3}{6} = \frac{1}{6}$$

Making

$$\int_0^1 xe^{x^2}\,\mathrm{d}x = \frac{1}{6}$$

11.2. Problem 2

Let $D_* = [0,1] \times [0, 2\pi]$ with coordinates (r, θ) and

$$T(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

What is $T(D_*)$?

11.2.1. Solution

D_* describe all points with distance to the center $(0,0)$ of $0 \leq \text{distance} \leq 1$

As a transformation T this can be described as a disk with center $(0,0)$

This means that $\sqrt{x^2 + y^2} < 1$

$T(D_*) = \left\{(x, y) : \sqrt{x^2 + y^2} \leq 1\right\}$

11.3. Problem 3

$T(u, v) = (\frac{u+v}{2}, \frac{u-v}{2})$ on $D_* = [-1, 1]^2$.

Find $T(D_*)$

11.3.1. Solution

D_* describes all points within a 2×2 square with center in $(0,0)$

The transformation $T(D_*)$ transforms it to a stilted square with a digonal of 2

$T(D_*) = \{(x, y) :\}$

11.4. Problem 4

Use polar coordinates to compute

$$\int_\infty^\infty e^{-x^2} \,\mathrm{d}x$$

Standard trick: square the integral and use polar coordinates to obtain $\sqrt{\pi}$

11.4.1. Solution

By squaring $\int_\infty^\infty e^{-x^2} \,\mathrm{d}x$

I can fit this equation to use polar coordinates

$$\left(\int_\infty^\infty e^{-x^2} \,\mathrm{d}x\right)^2 = \int_\infty^\infty \int_\infty^\infty e^{-x^2} e^{-y^2} \,\mathrm{d}x \,\mathrm{d}y, \quad \text{where } x = y$$

Now $x = r\cos(\theta)$ and $\mathrm{d}x \,\mathrm{d}y = r \,\mathrm{d}r \,\mathrm{d}\theta$

$$\int_0^{2\pi} \int_0^\infty r e^{-r^2(\cos(\theta) + \sin(\theta))^2} \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^\infty r e^{-r^2} \,\mathrm{d}r \,\mathrm{d}\theta$$

Then use u-subsitution where $u = r^2$ and $\mathrm{d}u = 2r\,\mathrm{d}r \Rightarrow \frac{1}{2}\,\mathrm{d}u = r\,\mathrm{d}r$

$$\int_0^{2\pi} \int_0^{\infty} e^{-u}\,\mathrm{d}u\,\mathrm{d}\theta$$

since $\int e^{-u}\,\mathrm{d}u = -e^{-u}$

$$F(\infty) - F(0) = -e^{-\infty} - (-e^0) = \frac{1}{2}(0 - (-1)) = \frac{1}{2}$$

Then

$$\int_0^{2\pi} \frac{1}{2}\,\mathrm{d}\theta = \frac{1}{2}2\pi = \pi$$

Then

$$\left(\int_{\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \pi \quad \Rightarrow \quad \int_{\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

11.5. Problem 5

Let $x = r\cos\theta, y = r\sin\theta$ and $z = z$. Let W_* consists of points in the form (r, θ, z) where $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1$.

Find the image set and compute its Jacobian.

11.5.1. Solution

This transformation

$$T(x, y, z) = (r\cos(\theta), r\sin(\theta), z))$$

Transforms $W_* = \{(r, \theta, z) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1\}$ (a cylinder, $r = 1, h = 1$)

To $V = \{(x, y, z) : -1 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq z \leq 1\}$ (a cuboid, $w = 2, d = 2, h = 1$)

The Jacobian for 3D is:

$$J(r, \theta, z) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11.6. Problem 6

Let $x = \rho \sin(\varphi) \cos(\theta), y = \rho \sin(\varphi) \sin(\theta)$ and $z = \rho \cos(\varphi)$. Compute its Jacobian and use this to derive the change of variable spherical cooridnate formula.

11.6.1. Solution

The Jacobian for 3D is:

$$J(\rho, \varphi, \theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin(\varphi) \cos(\theta) & \rho \cos(\varphi) \cos(\theta) & -\rho \sin(\varphi) \sin(\theta) \\ \sin(\varphi) \sin(\theta) & \rho \cos(\varphi) \sin(\theta) & -\rho \sin(\varphi) \cos(\theta) \\ \cos(\varphi) & -\rho \sin(\varphi) & 0 \end{bmatrix}$$

Now i can take the determinant

$$|J(\rho, \varphi, \theta)| = \rho^2 \sin(\varphi)$$

This means that any shape that is transformed by

$$T(x, y, z) = (\rho \sin(\varphi) \cos(\theta), \rho \sin(\varphi) \sin(\theta), \rho \cos(\varphi))$$

Is then also scaled by $\rho^2 \sin(\varphi)$

12. Lecture 12

Nothing new, besides Taylor series **[NOT CURRICULUM]**