Problem 1:

$$\begin{cases} x - 3y + 2z = -1 \\ 2x - 5y + 9z = 10 \\ 2x - 5y + 6z = 4 \end{cases} \qquad (1)$$

(α) Is $(x, y, z) = (35, 12, 0)$ a solution of (1)?

We can either check directly on (1) or put it first in RREF form since this is asked later.

We will check this directly:

In the third equation:

$$2 \cdot 35 - 5 \cdot 12 + 6 \cdot 0 \neq 4$$

so $(35, 12, 0)$ is not a solution of (1).

(β) Give the expression of the augmented matrix corresponding to (1)

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & 9 & 10 \\ 2 & -5 & 6 & 4 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & 9 & 10 \\ 2 & -5 & 6 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 12 \\ 0 & 1 & 2 & 6 \end{array}\right]$$

$$\xrightarrow[R_3-R_2]{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 17 & 35 \\ 0 & 1 & 5 & 12 \\ 0 & 0 & -3 & -6 \end{array}\right] \xrightarrow{R_3/-3} \left[\begin{array}{ccc|c} 1 & 0 & 17 & 35 \\ 0 & 1 & 5 & 12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_1-17R_3]{R_2-5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\Rightarrow$ $x = 1$, $y = 2$, $z = 2$ $\Rightarrow$ $(1,2,2)$ is the unique solution.

Problem 2: $A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix}$

(a) Determine the range of A.

The range is the column space $\text{span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}$

We need to check linear independence (which we can already see since we have 3 vectors in $\mathbb{R}^2$).

We will put A is RREF

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_2/-5} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

linear independent columns so we need the corresponding columns of the original matrix

$$\Rightarrow \text{range}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} = \mathbb{R}^2$$

(B) Recall that the rank is the maximal number of non-zero linear independent rows or columns.

Since we only have two rows. $\text{rank}(A) \leq 2$.

By putting the matrix in RREF we already proved that it has exactly two non-zero linearly independent columns. This means that $\text{rank}(A) = 2$.

(c) Give a basis for null(A)

We solve $A\bar{x} = 0$ for $\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{matrix} x + 2y + 5z = 0 \\ 4x + 3y + 5z = 0 \end{matrix}$$ (solve for any variable !)

$$\Leftrightarrow \begin{matrix} y = -3x \\ z = x \end{matrix}$$

$\Leftrightarrow x + 2y = 4x + 3y$

$\Leftrightarrow 3x + y = 0 \Leftrightarrow y = -3x$

$$\text{null}(A) = \text{span}\left(\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{matrix} x + 2y + 5z = 0 \\ 4x + 3y + 5z = 0 \end{matrix} \right\} \right)$$

$$= \text{span}\left(\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{matrix} y = -3x \\ y = z \end{matrix} \right\} \right)$$

$$= \text{span}\left(\left\{ \begin{bmatrix} x \\ -3x \\ x \end{bmatrix} \in \mathbb{R}^3 \right\} \right)$$

$$= \text{span}\left(\left\{ x \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \right\} \right)$$

has a basis $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$

(d) Determine nullity(A).

nullity(A) = dim null(A) = 1 as seen above.

Problem 3:

(a) $\det(A^{-1} B A) = \det(A^{-1}) \cdot \det(B) \cdot \det(A)$

$$= \frac{1}{\det(A)} \cdot \det(B) \cdot \det(A)$$

$$= \det(B)$$

$$= -3$$

(B) $\det(A' B A) = \det(A') \cdot \det(B) \cdot \det(A)$

$$= \det(A) \cdot \det(B) \cdot \det(A)$$

$$= 4 \cdot (-3) \cdot 4$$

$$= -48$$

(c) $\det(B^5) = \det(B)^5 = (-3)^5$

Problem 4:

- For $b = d = 0$ it contains the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- Let $\begin{bmatrix} b_1 - 5d_1 \\ 2b_1 \\ 2d_1 \\ d_1 \end{bmatrix}, \begin{bmatrix} b_2 - 5d_2 \\ 2b_2 \\ 2d_2 \\ d_2 \end{bmatrix} \in S$

then

$$\begin{bmatrix} b_1 - 5d_1 \\ 2b_1 \\ 2d_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} b_2 - 5d_2 \\ 2b_2 \\ 2d_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} (b_1+b_2) - 5(d_1+d_2) \\ 2(b_1+b_2) \\ 2(d_1+d_2) \\ (d_1+d_2) \end{bmatrix} \in S$$

- Let $c \in \mathbb{R}$ and $\begin{bmatrix} b - 5d \\ 2b \\ 2d \\ d \end{bmatrix} \in S.$

Then $c \cdot \begin{bmatrix} 6 - 5d \\ 2b \\ 2d \\ d \end{bmatrix} = \begin{bmatrix} (cb) - 5(c d) \\ 2(cb) \\ 2(cd) \\ (cd) \end{bmatrix} \in S$

$\Rightarrow$ S is a subspace of $\mathbb{R}^4$

Problem 5:

I first put A in RREF

A has RREF $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

so a basis for the range are the first two columns of the <u>original</u> A which is $\left\{ \begin{bmatrix} -8 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} \right\}$

Then let $d_1, d_2 \in \mathbb{R}$:

$$d_1 \begin{bmatrix} -8 \\ 6 \\ 4 \end{bmatrix} + d_2 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -8\lambda_1 - 2\lambda_2 \\ 6\lambda_1 + 4\lambda_2 \\ 4\lambda_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \quad \lambda_1 = -\frac{1}{2}, \quad \lambda_2 = 1$$

So b is in the range of A.

Problem 6:

(a) $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

$$= (2-\lambda)^2 - 4$$
$$= 4 - 4\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 4\lambda$$
$$= \lambda(\lambda - 4)$$

$\Rightarrow \det(A - \lambda I) = 0 \Rightarrow \lambda = 0$ or $\lambda = 4$ (simple roots)

$\Rightarrow$ $\lambda = 0$ is an eigenvalue with algebraic multiplicity 1
$\lambda = 4$ is an eigenvalue with algebraic multiplicity 1

(B) <u>For $\lambda = 0$</u> :

$$(A - \lambda I_2)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (\text{null}(A))$$

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 2x + 4y = 0 \\ x + 2y = 0 \end{matrix} \iff x = -2y.$$

$$E_0 = \text{span}\left(\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = -2y \right\} \right)$$

$$= \text{span}\left(\left\{ \begin{bmatrix} -2y \\ y \end{bmatrix} \in \mathbb{R}^2 \right\} \right)$$

$$= \text{span}\left(\left\{ y\begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \right\} \right.$$

$$= \text{span}\left(\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \right) \Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \text{ is a basis}$$

and $\lambda = 0$ has geometric multiplicity 1 as the dimension of the eigenspace is 1.

<u>For $\lambda = 4$</u>

$$(A - 4I_2)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} -2x + 4y \\ x - 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{aligned} -2x + 4y &= 0 \\ x - 2y &= 0 \end{aligned} \quad \Leftrightarrow \quad x = 2y$$

$$E_4 = \text{span}\left(\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = 2y \right\} \right)$$

$$= \text{span}\left(\left\{ y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \right\} \right.$$

$\Rightarrow$ a basis is $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ with geometric multiplicity 1.

(c) For $0 \neq u \in E_0$ then $Au = 0 \cdot u = 0$
this is null(A).

For $0 \neq u \in E_4$ then $Au = 4 \cdot u$

(d) We set $P := \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}$ a matrix with

the eigenvectors. Then we know that

$$P^{-1} A P = D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow \quad A = P D P^{-1}$$

Consider $D^{1/2} := \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{4} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

Then $A^{1/2} := P D^{1/2} P^{-1}$

$$A^{1/2} \cdot A^{1/2} = P D^{1/2} P^{-1} \cdot P D^{1/2} P^{-1} = P D^{1/2} \cdot D^{1/2} P^{-1}$$

$$= A.$$

Problem 7:

(a) True: $(AB)^{-1} = B^{-1} \cdot A^{-1}$

(B) False: a basis is not unique, dimension is

(C) True: $\dim \mathbb{R}^n = n$

(d) False: $A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$

(e) False: Counterexample: $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

with eigenvalues $1, 2$ but $1+2=3$ is not an eigenvalue.

Extra exercise with 2-dimensional eigenspace:

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \iff (\lambda-3)^2(\lambda-2) = 0$$

$$\Rightarrow \lambda = 3 \text{ or } \lambda = 2.$$

We will do the $\lambda = 3$ case because it has a 2-dimensional eigenspace.

$$(A - 3I_3)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\iff \quad -x - 2z = 0.$

$\iff \quad x = -2z$

$$E_3 = \mathrm{span}\left(\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = -2z \right\} \right.$$

$$= \mathrm{span}\left(\left\{ \begin{bmatrix} -2z \\ 0 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 \right\} \right.$$

$$= \mathrm{span}\left(\left\{ z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \right\} \right.$$

$\Rightarrow$ a basis is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$