

# Calculus written exam 2026

Simon Holm  
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## Contents

1.	Problem 1 .....	3
1.1.	Solution .....	3
2.	Problem 2 .....	4
2.1.	Solution .....	4
3.	Problem 3 .....	5
3.1.	Solution .....	5
4.	Problem 4 .....	6
4.1.	Solution .....	6
5.	Problem 5 .....	7
5.1.	Solution .....	7
6.	Problem 6 .....	8
6.1.	Solution .....	8
7.	Problem 7 .....	9
7.1.	Solution .....	9
8.	Problem 8 .....	10
8.1.	Solution .....	10
9.	Problem 9 .....	11
9.1.	Solution .....	11
10.	Problem 10 .....	11
10.1.	Solution .....	11

## 1. Problem 1

Let  $f : \mathbb{R}^3 \rightarrow R$  be defined by

$$f(x, y, z) = x^2 + y^2 + z^2$$

- (a) Determine the domain and range of  $f$ 
  - (a)
- (b) Describe geometrically the level set formed by  $f(x, y, z) = 100$ .

### 1.1. Solution

- (a) Determine the domain and range of  $f$

- $D(f) = \{(x, y, z) : (-\infty, +\infty)\}$
- $R(f) = [0, +\infty]$

- (b) Describe geometrically the level set formed by  $f(x, y, z) = x^2 + y^2 + z^2 = 100$ .

- $f$  represents a sphere, generally on the form  $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
- The sphere would have center at  $0, 0, 0$
- And a radius of  $\sqrt{100} = 10$

## 2. Problem 2

(a) Compute the following limit or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x+y^2}{2x+y}$$

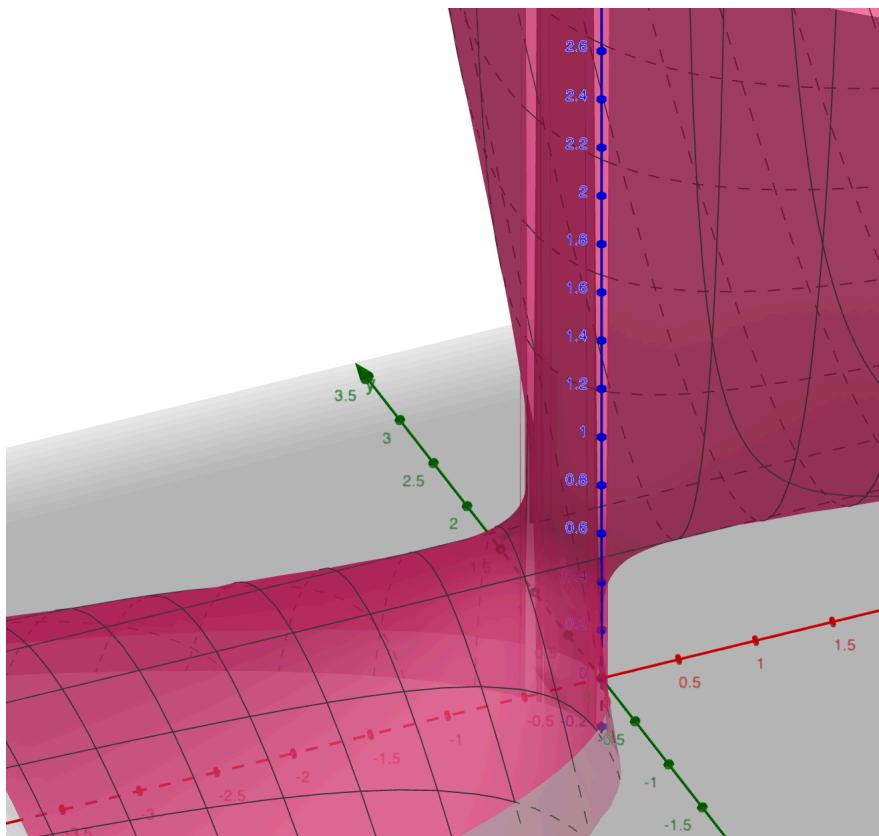
(b) Is the following function continuous along

$$f(x, y) = \begin{cases} x - 1 & y \geq 0, \\ -2 & y < 0 \end{cases}$$

### 2.1. Solution

(a) Compute the following limit or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x+y^2}{2x+y}$$



$$f(x, y) = \frac{x+y^2}{2x+y} \text{ graphed in GeoGebra}$$

Then i can constant y and check for x-

$$y = 0 \Rightarrow \lim_{x \rightarrow 0} \frac{x+y^2}{2x+y} = \frac{1}{2}$$

$$y = x \Rightarrow \lim_{x \rightarrow 0} \frac{x+y^2}{2x+y} = \frac{1}{3}$$

Since  $\frac{1}{2} \neq \frac{1}{3}$

The limit **does not** exist

- (b) Is the following function continuous along

$$f(x, y) = \begin{cases} x - 1 & y \geq 0, \\ -2 & y < 0 \end{cases}$$

- Yes, regardless of  $(x, y)$  the  $f$  is defined, therefore it is continuous

### 3. Problem 3

Let  $g(x, y, z) = x^2 - 2xy^2 + az - a$

- (a) Find an equation of the tangent plane to  $g$  at the point  $(1, 1, 1)$ .  
(b) For which value of  $a$  does the tangent plane pass through the origin?

#### 3.1. Solution

- (a) Find an equation of the tangent plane to  $g$  at the point  $(1, 1, 1)$ .

The equation of the tangent plane at the point  $(x_0, y_0, z_0)$  is given by:

$$\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

In our case,  $(0, -4 + a, a) \cdot (x - 1, y - 1, z - 1) = 0$

This expands to:

$$\begin{aligned} 0(x - 1) + (-4 + a)(y - 1) + a(z - 1) &= 0 \\ (-4 + a)(y - 1) + a(z - 1) &= 0 \\ (a - 4)y - (a - 4) + az - a &= 0 \\ (a - 4)y + az - (2a - 4) &= 0 \end{aligned}$$

- (b) For which value of  $a$  does the tangent plane pass through the origin?

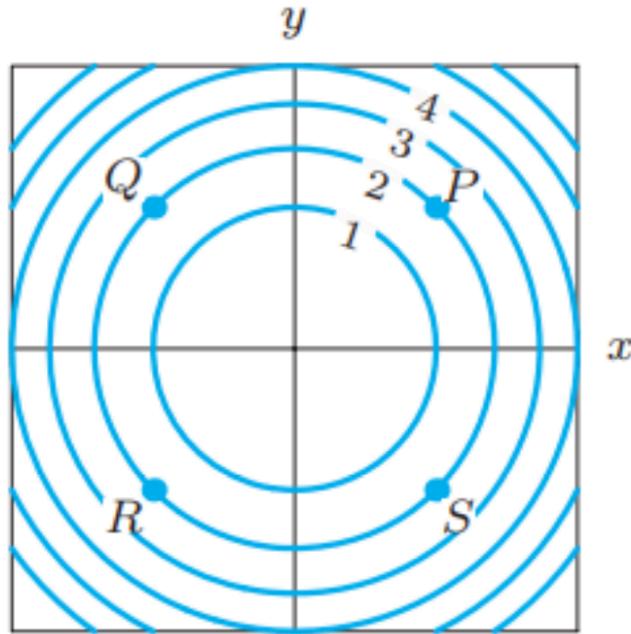
For this i plug in  $(0, 0, 0)$

$$\begin{aligned} (a - 4)(0) + a(0) - (2a - 4) &= 0 \\ -(2a - 4) &= 0 \\ 2a - 4 &= 0 \\ 2a = 4a &= 2 \end{aligned}$$

## 4. Problem 4

Below is a contour diagram of  $f(x, y)$ . In each of the following cases, list the marked points in the diagram (there may be none or more than one) at which

- (a)  $f_x < 0$  (b)  $f_y > 0$  (c)  $f_{xx} > 0$  (d)  $f_{yy} < 0$



Hint: what kind of function  $f(x, y)$  would produce a contour plot like the one shown? Compute its  $f_x, f_y, f_{xx}$  and  $f_{yy}$ .

### 4.1. Solution

- (a) For  $f_x < 0$  this means that the partial derivative is negative on the  $x$ -axis. Since the contour plot is showing an increase  $f_x > 0$ .

So  $f_x < 0$  is false for this contour plot

- (b) As for  $f_y > 0$  This means that the partial derivative is positive on the  $y$ -axis. Since the contour plot is showing an increase  $f_y > 0$ .

So  $f_y < 0$  is true for this contour plot.

- (c)  $f_{xx} > 0$

- (d)  $f_{yy} < 0$

## 5. Problem 5

Let  $F(x) = \int_0^x \cos(\pi t) dt$ .

- (a) Evaluate  $F(1)$ .
- (b) For what values of  $x$  is  $F(x)$  positive? negative?

### 5.1. Solution

- (a) Evaluate  $F(1)$ .

Since

$$F(x) = \int_0^x \cos(\pi t) dt = \int \cos(\pi x) dx$$

and since

$$F(x) = \int \cos(\pi x) dx = \frac{\sin(\pi x)}{\pi}$$

$$F(1) = \frac{\sin(\pi)}{\pi} = 0$$

- (b) For what values of  $x$  is  $F(x)$  positive? negative?

Since  $\sin(a \cdot \pi) = 0$  for any  $a$

$$F(x) = 0, \quad \text{for any } x$$

## 6. Problem 6

Let  $T(x, y) = e^{-x^2 y^2}$  represent the heat of the point  $(x, y)$ .

- In which direction should a heat-seeking bug move from the point  $(-1, 1)$  to increase its temperature fastest?
- Find the directional derivative of  $T$  at the point  $P = (-1, 1)$  in the direction of  $(1, 1)$ .
- Does  $T$  have a maximum or a minimum? At where? Hint: You don't need to compute second-order derivatives. Consider that  $\exp$  is monotone increasing and how that affects the extrema of  $T$ .

### 6.1. Solution

- In which direction should a heat-seeking bug move from the point  $(-1, 1)$  to increase its temperature fastest?

In the direction of  $\nabla f(-1, 1)$

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = \left( -2e^{-x^2 y^2} xy^2, -2e^{-x^2 y^2} x^2 y \right)$$

Then

$$\nabla f(-1, 1) = -2e^{-(-1)^2 y^2} xy^2, -2e^{-x^2 y^2} x^2 y = \frac{2}{e}, -\frac{2}{e}$$

- Find the directional derivative of  $T$  at the point  $P = (-1, 1)$  in the direction of  $(1, 1)$ .

first find the unit vector of  $(1, 1)$

$$u = \frac{1, 1}{\sqrt{2}} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$f_u = \nabla f(x, y) \cdot u = \left( \frac{2}{e}, -\frac{2}{e} \right) \cdot (1, 1) = 0$$

- Does  $T$  have a maximum or a minimum? At where? Hint: You don't need to compute second-order derivatives. Consider that  $\exp$  is monotone increasing and how that affects the extrema of  $T$ .

## 7. Problem 7

Consider the function

$$f(x, y) = |xy|$$

- (a) Is f differentiable at (1, 0)? Explain.
- (b) Is f differentiable at (0, 0)? Explain.

### 7.1. Solution

F is differentiable on a point  $(x, y)$  if  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) - \frac{f(a,b)}{\sqrt{(x-a)^2 + (y-b)^2}}$

- (a) Is differentiable at (1, 0)? Explain.

$$\lim_{(x,y) \rightarrow (1,0)} f(x, y) - \frac{f(1,0)}{\sqrt{(x-1)^2 + (y)^2}}$$

We check this by setting a constant

$$\text{For } y = 0, \quad \lim_{x \rightarrow 0} = 0$$

$$\text{For } y = x, \quad \lim_{x \rightarrow 0} = 1$$

Since the limit **does not** existst,  $f(x, y)$  **is not** differentiable on poin (1, 0)

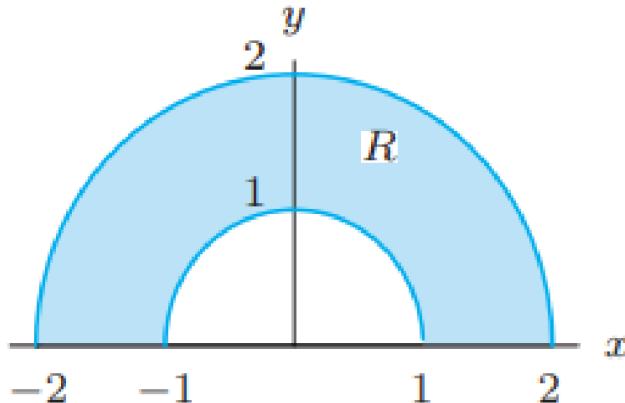
- (b) Is f differentiable at (0, 0)? Explain.

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) - \frac{f(0,0)}{\sqrt{(x)^2 + (y)^2}} = 0$$

Since the limit **does** existst,  $f(x, y)$  **is** differentiable on poin (1, 0)

## 8. Problem 8

Use polar coordinate to evaluate  $\int_R \sqrt{x^2 + y^2}$  where  $R$  is given in the following figure.



### 8.1. Solution

We want to find

$$\int_R \sqrt{x^2 + y^2} \, dA$$

where  $x^2 + y^2 = r^2$  so

and  $\theta$  must be half a circle:  $\pi$

$$\int_{\theta=0}^{\pi} \int_{r=0}^1 r \, r \, dr \, d\theta = \int_{\theta=0}^{\pi} \int_{r=0}^1 r^2 \, dr \, d\theta$$

Now integrate

$$\int_{\theta=0}^{\pi} \frac{1}{3} \, d\theta = \pi = \frac{\pi}{3}$$

## 9. Problem 9

Evaluate the following integral by using cylindrical coordinate or spherical coordinate:

$$\iiint_W \frac{z}{(x^2 + y^2)^{\frac{2}{3}}} dV, \quad W = \{(x, y, z) : 0 \leq x^2 + y^2 \leq 4, 0 \leq z \leq 4\}$$

### 9.1. Solution

in this case this is:

$$0 = \frac{z}{(x^2 + y^2)^{\frac{2}{3}}} \Rightarrow z = (x^2 + y^2)^{\frac{2}{3}}$$

$$\iiint_W z dV$$

Because the of the cone, polar coordinates are easier to work with

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad \underbrace{z = z}_{\text{figure on } z}, \quad dV = r dz dr d\theta$$

In this instance  $\theta$  is the whole circle around the  $z$  axis so  $0 \leq \theta \leq 2\pi$

To find  $r$  we can use the definition of the circle at  $z = 4$

$$\text{Because of } 4 = (x^2 + y^2)^{\frac{2}{3}} \Rightarrow \underbrace{x^2 + y^2}_{\text{circle at } z=4} = \sqrt[3]{4} \text{ where } \sqrt[3]{4} = r^2, \text{ So } r = 2\sqrt{2}$$

This means that  $0 \leq r \leq 4$

To find  $z$  we look at the cones height from  $z = 0$  to  $z = 4$ .

$$\text{This means that } \sqrt[3]{x^2 + y^2} \leq z \leq 4 \implies r \leq z \leq 0$$

Finally we can put it all together

.. didnt finish

## 10. Problem 10

Decide whether the following statements are true or false.

- (a) If a function is differentiable, it must be continuous.
- (b)  $\int_0^1 \int_0^x f(x, y) dx dy = \int_0^1 \int_0^y f(x, y) dy dx$
- (c) If  $\int_R f dA = 0$  then  $f$  is zero at all points in  $R$ .
- (d) If  $f$  and  $g$  are two functions continuous on a region  $R$ , then

$$\int_R f \cdot g dA = \int_R f dA \cdot \int_R g dA$$

- (e) A local maximum of  $f$  can only occur at the critical points.

### 10.1. Solution

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