AI511/MM505 Linear Algebra with Applications
Take-Home Exam (Solutions)
Autumn 2024

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- The solutions of the take-home exam have to be submitted as a single PDF file on Digital Eksamen until the deadline of November 6, 2024 at 20:59.

- You are required to solve the exercises on your own and you are not permitted to

  (i) discuss the exercises with anyone else (this includes your fellow students as well as posting questions about the exercises on the internet);
  (ii) use generative models (such as ChatGPT).

  Any suspicious similarities of the solutions or any other concerns will be investigated.

- You are allowed to use everything else (including the lecture material, textbooks, Wikipedia, etc.).

- You are required to demonstrate how you arrive at your solution by providing a suitable amount of explanations and intermediate computations. Exercise 7 is an exception since you are only required to write true or false in Exercise 7.

- You can scan and submit your handwritten solutions as long as the solutions are properly organised and the handwriting is readable.

- You may write the solutions in English or Danish.

The total number of points is 100.

Good luck!

1. Consider the following system of linear equations

$$\begin{cases} x - 3y + 2z = -1; \\ 2x - 5y + 9z = 10; \\ 2x - 5y + 6z = 4. \end{cases} \tag{1}$$

(a) (2 pts) Is $(x, y, z) = (35, 12, 0)$ a solution of (1)?

Solution
It is not. It satisfies the first two equations but not the third one.
We have that

$$\begin{aligned} 35 - 3 \cdot 12 + 2 \cdot 0 &= -1 \\ 2 \cdot 35 - 5 \cdot 12 + 9 \cdot 0 &= 10 \\ 2 \cdot 35 - 5 \cdot 12 + 6 \cdot 0 &= 10 \neq 4. \end{aligned}$$

(b) (2 pts) Give the expression of the augmented matrix corresponding to (1).

Solution
The augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & 9 & 10 \\ 2 & -5 & 6 & 4 \end{array}\right].$$

(c) (8 pts) Transform the augmented matrix into a reduced row echelon form using the Gauss-Jordan elimination to find the solution of (1).

Solution
We have that

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 2 & -5 & 9 & 10 \\ 2 & -5 & 6 & 4 \end{array}\right] &\xrightarrow[R_3 - 2R_1]{R_2 - 2R1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 12 \\ 0 & 1 & 2 & 6 \end{array}\right] \\ &\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 12 \\ 0 & 0 & -3 & -6 \end{array}\right] \\ &\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 12 \\ 0 & 0 & 1 & 2 \end{array}\right] \\ &\xrightarrow[R_1 - 2R_3]{R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{aligned}$$

$$\xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

so that solution of (1) is $(1, 2, 2)$.

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2. Suppose that

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix}.$$

(a) (5 pts) Determine range(A).

Solution
Since $(1, 4)$ and $(2, 3)$ are two linearly independent vectors in $\mathbb{R}^2$, the columns of A span $\mathbb{R}^2$ and hence the range of A is $\mathbb{R}^2$.

(b) (2 pts) What is the largest possible value of rank(A)? Determine the actual value of rank(A).

Solution
Since A is a 2×3 matrix, the largest possible rank of A is 2. The dimension of the range of A is 2 and hence the rank of A is 2.

(c) (5 pts) Give a basis for null(A).

Solution
We solve the homogeneous system of linear equations

$$\begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have that

$$x_1 = -2x_2 - 5x_3$$

and

$$4(-2x_2 - 5x_3) + 3x_2 + 5x_3 = -5x_2 - 15x_3 = 0$$

so that $x_2 = -3x_3$. Hence, the set of solutions is spanned by the vector $(1, -3, 1)$ which is a basis of null(A).

(d) (2 pts) Determine nullity(A).

Solution
Since null(A) is a line, the dimension of null(A) is one and nullity(A) $= 1$.

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3. Suppose that $A, B \in \mathcal{M}_4$ such that $\det(A) = 4$ and $\det(B) = -3$. Compute the following determinants

(a) (4 pts) $\det(A^{-1}BA)$;

Solution
Using the fact that $\det(AB) = \det(A)\det(B)$ for square matrices and the fact that $\det(A^{-1}) = 1/\det(A)$,

$$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A) = \frac{1}{\det(A)}\det(B)\det(A) = -3.$$

(b) (4 pts) $\det(A'BA)$, where A' is the transpose of A;

Solution
Using the fact that $\det(AB) = \det(A)\det(B)$ for square matrices and the fact that $\det(A) = \det(A')$,

$$\det(A'BA) = \det(A')\det(B)\det(A) = \det(A)\det(B)\det(A) = -48.$$

(c) (4 pts) $\det(B^5)$.

Solution
Using the fact that $\det(AB) = \det(A)\det(B)$ for square matrices,

$$\det(B^5) = \det(B)^5 = -243.$$

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4. (5 pts) Let $S \subset \mathbb{R}^4$ be defined by

$$S = \left\{ \begin{bmatrix} b-5d \\ 2b \\ 2d \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\}.$$

Is S a subspace of $\mathbb{R}^4$?

Solution

Suppose that $v_1, v_2 \in S$ and $c \in \mathbb{R}$. Then

$$v_1 + v_2 = \begin{bmatrix} b_1 - 5d_1 \\ 2b_1 \\ 2d_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} b_2 - 5d_2 \\ 2b_2 \\ 2d_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 - 5(d_1 + d_2) \\ 2(b_1 + b_2) \\ 2(d_1 + d_2) \\ d_1 + d_2 \end{bmatrix} \in S$$

and

$$cv_1 = \begin{bmatrix} cb_1 - 5cd_1 \\ 2cb_1 \\ 2cd_1 \\ cd_1 \end{bmatrix} \in S$$

so S is a subspace of $\mathbb{R}^4$ since S is closed under vector addition and scalar multiplication.

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5. (15 pts) Suppose that

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Does b belong to range(A)?

Solution

We have that

$$\det(A) = 4 \cdot (-1)^{3+1} \cdot (20) + 4 \cdot (-1)^{3+3} \cdot (-20) = 0$$

which implies that the columns of the matrix must be linearly dependent. Hence, the span of any two columns of the matrix is the same as range(A). We look for $k_1, k_2 \in \mathbb{R}$ such that

$$k_1 \begin{bmatrix} -8 \\ 6 \\ 4 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Since $4k_1 = -2$, k_1 must be equal to $-1/2$. Then $-3 + 4k_2 = 1$ and hence $k_2 = 1$. But we also have that $-8k_1 - 2k_2 = 4 - 2 = 2$ so it follows that b belongs to range(A).

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6. Suppose that

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

(a) (4 pts) Compute the eigenvalues of A and their algebraic multiplicities.

Solution
We have that

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = \lambda(\lambda - 4)$$

so that the eigenvalues of A are 0 with algebraic multiplicity of 1 and 4 with algebraic multiplicity of 1.

(b) (8 pts) Compute bases of the eigenspaces of A and geometric multiplicities of the eigenvalues of A.

Solution
We begin with $\lambda = 0$. Then

$$Av = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 4v_2 \\ v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $(-2, 1)$ is a basis for the eigenspace corresponding to $\lambda = 0$ and since the basis has only one vector, the geometric multiplicity of $\lambda = 0$ is equal to 1.
If $\lambda = 4$, then

$$(A - 4I)v = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2v_1 + 4v_2 \\ v_1 - 2v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $(2, 1)$ is a basis for the eigenspace corresponding to $\lambda = 4$ and since the basis has only one vector, the geometric multiplicity of $\lambda = 4$ is equal to 1.

(c) (5 pts) Let E_λ denote the eigenspace corresponding to the eigenvalue λ. For each eigenvalue λ of the matrix A, describe how the matrix A transforms non-zero vectors $v \in E_\lambda$.

Solution
The vectors that belong to the eigenspace corresponding to $\lambda = 0$ are scaled to 0. The vectors that belong to the eigenspace corresponding to $\lambda = 4$ get stretched by a factor of 4.

(d) (15 pts) Find a matrix $A^{1/2}$ such that $A^{1/2}A^{1/2} = A$ and verify your answer by multiplying $A^{1/2}$ with itself to show that the product is indeed equal to A.

Solution
We have that

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

The inverse of P is given by

$$P^{-1} = -\frac{1}{4}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix}.$$

Set

$$D^{1/2} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then $A^{1/2}$ is given by

$$\begin{aligned} A^{1/2} &= PD^{1/2}P^{-1} \\ &= -\frac{1}{4}\begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{4}\begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \\ &= -\frac{1}{4}\begin{bmatrix} -4 & -8 \\ -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}. \end{aligned}$$

We have that

$$A^{1/2}A^{1/2} = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = A$$

so that $A^{1/2}$ is indeed the square root of A.

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7. You are only required to respond true or false to the following statements.

(a) (2 pts) A product of two invertible matrices is an invertible matrix.

Solution
True.

(b) (2 pts) Every subspace of $\mathbb{R}^n$ has a unique basis.

Solution
False.

(c) (2 pts) Every basis of $\mathbb{R}^n$ has n vectors.

Solution
True.

(d) (2 pts) Determinants of triangular matrices are non-negative.

Solution
False.

(e) (2 pts) A sum of two eigenvalues is an eigenvalue.

Solution
False.
