AI511/MM505 Linear Algebra with Applications
Take-Home Exam
Autumn 2024

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- The solutions of the take-home exam have to be submitted as a single PDF file on Digital Eksamen until the deadline of November 6, 2024 at 20:59.

- You are required to solve the exercises on your own and you are not permitted to

  (i) discuss the exercises with anyone else (this includes your fellow students as well as posting questions about the exercises on the internet);
  (ii) use generative models (such as ChatGPT).

  Any suspicious similarities of the solutions or any other concerns will be investigated.

- You are allowed to use everything else (including the lecture material, textbooks, Wikipedia, etc.).

- You are required to demonstrate how you arrive at your solution by providing a suitable amount of explanations and intermediate computations. Exercise 7 is an exception since you are only required to write true or false in Exercise 7.

- You can scan and submit your handwritten solutions as long as the solutions are properly organised and the handwriting is readable.

- You may write the solutions in English or Danish.

The total number of points is 100.

Good luck!

1. Consider the following system of linear equations

$$\begin{cases} x - 3y + 2z = -1; \\ 2x - 5y + 9z = 10; \\ 2x - 5y + 6z = 4. \end{cases} \tag{1}$$

   (a) (2 pts) Is $(x, y, z) = (35, 12, 0)$ a solution of (1)?
   (b) (2 pts) Give the expression of the augmented matrix corresponding to (1).
   (c) (8 pts) Transform the augmented matrix into a reduced row echelon form using the Gauss-Jordan elimination to find the solution of (1).

2. Suppose that

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \end{bmatrix}.$$

   (a) (5 pts) Determine range(A).
   (b) (2 pts) What is the largest possible value of rank(A)? Determine the actual value of rank(A).
   (c) (5 pts) Give a basis for null(A).
   (d) (2 pts) Determine nullity(A).

3. Suppose that $A, B \in \mathcal{M}_4$ such that $\det(A) = 4$ and $\det(B) = -3$. Compute the following determinants

   (a) (4 pts) $\det(A^{-1}BA)$;
   (b) (4 pts) $\det(A'BA)$, where A' is the transpose of A;
   (c) (4 pts) $\det(B^5)$.

4. (5 pts) Let $S \subset \mathbb{R}^4$ be defined by

$$S = \left\{ \begin{bmatrix} b - 5d \\ 2b \\ 2d \\ d \end{bmatrix} : b, d \in \mathbb{R} \right\}.$$

   Is S a subspace of $\mathbb{R}^4$?

5. (15 pts) Suppose that

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

   Does b belong to range(A)?

6. Suppose that
$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

(a) (4 pts) Compute the eigenvalues of A and their algebraic multiplicities.
(b) (8 pts) Compute bases of the eigenspaces of A and geometric multiplicities of the eigenvalues of A.
(c) (5 pts) Let E_λ denote the eigenspace corresponding to the eigenvalue λ. For each eigenvalue λ of the matrix A, describe how the matrix A transforms non-zero vectors $v \in E_\lambda$.
(d) (15 pts) Find a matrix $A^{1/2}$ such that $A^{1/2}A^{1/2} = A$ and verify your answer by multiplying $A^{1/2}$ with itself to show that the product is indeed equal to A.

7. You are only required to respond true or false to the following statements.

(a) (2 pts) A product of two invertible matrices is an invertible matrix.
(b) (2 pts) Every subspace of $\mathbb{R}^n$ has a unique basis.
(c) (2 pts) Every basis of $\mathbb{R}^n$ has n vectors.
(d) (2 pts) Determinants of triangular matrices are non-negative.
(e) (2 pts) A sum of two eigenvalues is an eigenvalue.