AI511/MM505 Linear Algebra with Applications
Exercises 1
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The exercises are from Sections 1.1, 1.2, 1.3, and 1.4 of Anton, Rorres, and Kaul (2019).

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

   (a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$.

   (b) $x_1 + 3x_2 + x_1x_3 = 2$.

   (c) $x_1 = -7x_2 + 3x_3$.

   (d) $x_1^{-2} + x_2 + 8x_3 = 5$.

2. Find a system of linear equations in the unknowns $x_1, x_2, x_3, \ldots$, that corresponds to the given augmented matrix.

   (a) $\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{array}\right]$.

   (b) $\left[\begin{array}{cccc|c} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{array}\right]$.

3. Find the augmented matrix for the linear system.

   (a) $\begin{cases} -2x_1 = 6; \\ 3x_1 = 8; \\ 9x_1 = -3. \end{cases}$

   (b) $\begin{cases} 6x_1 - x_2 + 3x_3 = 4; \\ \phantom{6x_1 - {}} 5x_2 - x_3 = 1. \end{cases}$

4. Determine whether the 3-tuple $(5, 8, 1)$ is a solution of the linear system

$$\begin{cases} x + 2y - 2z = 3; \\ 3x - y + z = 1; \\ -x + 5y - 5z = 5. \end{cases}$$

5. Consider a system of linear equations

$$\begin{cases} x - y \;\; + 2z = 5; \\ 2x - 2y + 4z = 10; \\ 3x - 3y + 6z = 15. \end{cases} \tag{1}$$

Does (1) have a solution? If yes, how many solutions does (1) have? Explain why this happens and describe how the solution set looks like geometrically.

Row-reduce the augmented matrix of (1) to obtain the following matrix in reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{2}$$

Do the second and the third lines of (2) impose any constraints on the solution? In such situations when we have an infinite number of solutions, we describe a *general solution* of the system of linear equations using *parametric equations*. We have that $x - y + 2z = 5$ and assign an arbitrary value r to y and an arbitrary value s to z. Once we have the values of y and z, $x = 5 + r - 2s$. The parametric equations that describe a general solution in this example are given by

$$x = 5 + r - 2s, \quad y = r, \quad \text{and} \quad z = s. \tag{3}$$

The solution set can then be written as

$$\{(5 + r - 2s, r, s) : r, s \in \mathbb{R}\}. \tag{4}$$

y and z are called *free variables* of (1).

Consider a simpler situation when we only have one linear equation

$$7x - 5y = 3 \tag{5}$$

and use parametric equations to describe the solution set of linear equation (5).

6. Determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$.

7. Suppose that the augmented matrix for a linear system has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$.　　(b) $\left[\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$.

8. Solve the systems by Gauss-Jordan elimination.

(a) $\begin{cases} x_1 + x_2 + 2x_3 = 8; \\ -x_1 - 2x_2 + 3x_3 = 1; \\ 3x_1 - 7x_2 + 4x_3 = 10. \end{cases}$　　(b) $\begin{cases} 2x_1 + 2x_2 + 2x_3 = 0; \\ -2x_1 - 5x_2 + 2x_3 = 1; \\ 8x_1 + x_2 + 4x_3 = -1. \end{cases}$

9. Use the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad \text{and} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

to compute the following expressions if they are defined.

(a) $D + E$.　　(b) $\mathrm{tr}(D)$.

(c) AB.　　(d) BA.

(e) $(3E)D$.

10. Find matrices A, x, and b that express the given linear system as a single matrix equation $Ax = b$, and write out this matrix equation.

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 = 7; \\ 9x_1 - x_2 + x_3 = -1; \\ x_1 + 5x_2 + 4x_3 = 0. \end{cases}$$

11. Use the formula for the inverse of a 2×2-matrix to compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}.$$

12. If $A \in \mathbb{R}^{n\times n}$, then it can be multiplied with itself and $A^2 := AA$ which is also in $\mathbb{R}^{n\times n}$. Hence, $A^3 := A^2 A$ is a well-defined operation as well. Inductively, we define

A^k for any $k = 1, 2, \ldots$ and set $A^0 \coloneqq I_n$. If $p(x) = a_0 + a_1 x + \ldots + a_k x^k$ is a degree k polynomial with real coefficients,

$$p(A) \coloneqq a_0 I_n + a_1 A + a_2 A^2 + \ldots + a_k A^k \in \mathbb{R}^{n \times n}.$$

Compute $p(A)$ for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

and the polynomial

$$p(x) = 2x^2 - x + 1.$$

References

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). *Elementary Linear Algebra: Applications Version*. 12th edition. John Wiley & Sons, Inc.