AI511/MM505 Linear Algebra with Applications
Exercises 3
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Some of the exercises are from Section 1.4 of Johnston (2021) and Sections 1.8, 2.2, and 2.3 of Anton, Rorres, and Kaul (2019).

1. Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{bmatrix}.$$

2. Show that the determinant is equal to 0 without directly evaluating the determinant.

$$\begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}.$$

3. Use the determinant to decide whether the given matrix is invertible.

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}.$$

4. Find the domain and codomain of the transformation defined by the equations.

$$\begin{aligned} w_1 &= 4x_1 + 5x_2; \\ w_2 &= x_1 - 8x_2. \end{aligned}$$

5. The images of the standard basis vectors for $\mathbb{R}^3$ are given for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Find the standard matrix for the transformation, and find $T(x)$.

$$T(e_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad T(e_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}; \quad x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

6. A function of the form $f(x) = mx + b$ is commonly called a "linear function" because the graph of $y = mx + b$ is a line. Is f a matrix transformation on $\mathbb{R}$?
7. Find the standard matrix A for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ for which

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

8. Counter-clockwise rotation of vectors in $\mathbb{R}^2$ by an angle θ in radians is a linear transformation, which is denoted by R^θ and its standard matrix is given by

$$[R^\theta] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for $\theta \in \mathbb{R}$.

   (a) Find the standard matrix of the linear transformation that rotates $\mathbb{R}^2$ by $\pi/4$ radians counterclockwise and explain where e_1 and e_2 are mapped.

   (b) Find the standard matrix of the linear transformation that rotates $\mathbb{R}^2$ by $\pi/4$ radians counterclockwise and rotate $v = (1, 3)$ by $\pi/4$ radians counter-clockwise.

   (c) Find the standard matrix of the linear transformation that rotates $\mathbb{R}^2$ by $\pi/6$ radians clockwise and rotate $w = (\sqrt{3}, 3)$ by $\pi/6$ radians clockwise.

   (d) Show that $[R^\theta]$ is invertible and find its inverse. If $[R^\theta]$ rotates vectors counterclockwise, what does $[R^\theta]^{-1}$ do?

   (e) Does R^θ change the length of a vector? Show that it does not change the length or find an example where the length changes. [Hint: How does the matrix $[R^\theta]^T[R^\theta]$ look like?]

   (f) Does R^θ change the angle between two unit vectors? Show that it does not change the angle or find an example where the angle changes.

9. Suppose that

$$A = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}.$$

Computer the matrix A^{160}. [Hint: Think about this problem geometrically.]

10. Suppose that $u \in \mathbb{R}^n$ is a unit vector and $A = uu^T \in \mathbb{R}^{n \times n}$.

   (a) Show that $A^2 = A$. Provide a geometric interpretation of this fact.

   (b) Is A invertible? If it is, find its inverse.

   (c) Describe the range of the matrix transformation T_A.

11. Consider the Euclidean vector space $\mathbb{R}^2$. Show that straight lines remain straight lines under a linear transformation that maps $\mathbb{R}^2$ to $\mathbb{R}^2$. [Hints: (a) think about how we can express a straight line in a plane; (b) any linear transformation is a matrix transformation; (c) take an arbitrary point on the straight line and map it; (d) consider what happens when we map different points of the straight line and whether the images of these points lie on a straight line.]

12. Suppose that we want to predict the value of some variable y using the values of $x_1, \ldots, x_p$. The predicted value of y is given by the model

$$\hat{y} = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p = x^T \beta,$$

where

$$x = \begin{bmatrix} 1 & x_1 & x_2 & \ldots & x_p \end{bmatrix}^T \in \mathbb{R}^{p+1} \quad \text{and} \quad \beta = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_p \end{bmatrix}^T \in \mathbb{R}^{p+1}.$$

Suppose that we observe $(y_1, x_{11}, x_{12}, \ldots, x_{1p}), \ldots, (y_n, x_{n1}, x_{n2}, \ldots, x_{np})$ and our goal is to choose the value of $\beta \in \mathbb{R}^{p+1}$ that leads to the most accurate predictions. We do this by minimising the quadratic loss function $f(\beta) = \|y - X\beta\|^2$, i.e., the optimal value of β is given by

$$\hat{\beta} = \underset{\beta \in \mathbb{R}^{p+1}}{\arg\min} f(\beta) = \underset{\beta \in \mathbb{R}^{p+1}}{\arg\min} \|y - X\beta\|^2,$$

where

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1p} \\ 1 & x_{21} & x_{22} & \ldots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \ldots & x_{np} \end{bmatrix} \in \mathbb{R}^{n \times (p+1)}.$$

(a) Show that the quadratic loss function can be expressed as

$$f(\beta) = \beta^T A \beta + b^T \beta + c \tag{1}$$

by deriving the expressions of $A \in \mathbb{R}^{(p+1)\times(p+1)}$, $b \in \mathbb{R}^{p+1}$, and $c \in \mathbb{R}$. The quadratic function in (1) is discussed in the first lecture of the course.

(b) The stationary point (i.e., the value of β, where all partial derivatives of f are simultaneously equal to 0) of the function in (1) is a solution of the system of linear equations

$$(A + A^T)\beta = -b.$$

Use the expressions of A and b derived in part (a), assume that $X^T X$ is invertible and obtain the expression of the stationary point of the quadratic loss function.

References

Johnston, Nathaniel (2021). *Introduction to Linear and Matrix Algebra.* Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). *Elementary Linear Algebra: Applications Version.* 12th edition. John Wiley & Sons, Inc.