Review of the Basics

AI503
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Lecture 1

Functions

Definition

A **function** is a rule that takes numbers from a set A and assigns them a unique number in a set B.

- The set A is called the *domain*.
- The *range* of f is the set of points in B that are hit by f.
- Input is called the *independent variable*, output the *dependent variable*.
- The *graph* of f is the set of points $(x, f(x))$ in the xy-plane.

Examples

1. The following table defines a function with domain $\{15, 16, 17, 18, 19\}$ and range $\{17, 19, 20, 22, 23\}$:

Date in August	15	16	17	18	19
High Temp (C)	22	23	17	20	19

2. The function $f(x) = x^2$ has domain all real numbers, and range all non-negative numbers. Why isn't the range all real numbers?

3. Is $s(x) = \pm\sqrt{x}$ a function? Why?

Properties of functions

Remarks

For a function f

- for every x in the domain, $f(x)$ is uniquely defined
- for every y in the range, there exists x (not necessarily unique) such that $f(x) = y$.

Linear Functions

Definition

A function is *linear* if its slope is constant. Equivalently, any linear function can be written as

$$y = f(x) = mx + c$$

where m is the slope and c is the intercept

Remarks

1. Slope is rise over run, i.e., $\frac{\Delta y}{\Delta x}$
2. Intercept is the value of $f(0)$, i.e., $c = f(0)$.

Examples

1. $f(x)$ linear with $f(2) = 4$, $f(4) = -2$. Find $f(x)$.
2. $g(x)$ linear with slope 6, $g(5) = 9$. Find $g(x)$.

Proportionality

Definitions

- We say that y is *directly proportional* to x if there is a non-zero constant k such that
$$y = kx$$
- We say that y is *inversely proportional* to x if there is a non-zero constant k such that
$$y = \frac{k}{x}$$

In either case, k is called the *constant of proportionality*.

Gravitation Example

Law of gravitation says "The gravitational attraction force F between two point masses is directly proportional to the product of their masses and inversely proportional to the square of their separation distance d." Write this sentence as a formula with masses M_1 and M_2.

Power Functions

Definition

A *power function* is a function of the form

$$f(x) = kx^p,$$

where k and p are constants.

Remarks

1. p can be any real number.
2. f is always well defined on $(0, \infty)$ but not on the entire real line. For example, when $p = -1$ or $p = 0.5$.

Power Functions: Examples

Draw graphs: $f(x) = x,\ x^2,\ x^3,\ x^4,\ x^5$

- What are the behaviors of the tail of the function for odd/even exponent?
- Which functions have the fastest increase/decay?

Draw graphs: $f(x) = x^{-2.5}, x^{-0.3}, x^{0.3}, x^{2.5}$

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Larger $|p|$ implies faster increase/decay.

Polynomials

Definitions

A *polynomial* is a sum of power functions with positive integer (whole number) exponents:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where the a_i are constants, and $a_n \neq 0$. The number n is called the *degree* of the polynomial.

Special names

- $n = 2$ quadratic
- $n = 3$ cubic
- $n = 4$ quartic
- $n = 5$ quintic

Polynomial Exercises

Draw the following polynomials

1. $y = x^2 - x + 2$
2. $y = x^3 + x^2 - 2x - 6$
3. $y = x^4 - x^3 - 7x^2 - 2x - 6$
4. $y = x^5 - x^4 - 5x^3 + 2x - 6.$

Answer the following questions

- What are the behaviors of the tail of the function for odd/even exponent?
- How many times does the function intercept the x axis?

Polynomial Exercises

Draw the graph of the following functions

1. $y = (x+1)(x-1)^2$
2. $y = -x^2(x-1)$.

Are these functions polynomials? Why?

What changes about the behavior of the tails of the graph if its leading coefficient is negative?

Rational Functions

Definition

A *rational function* is a function of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

Examples:

$$\frac{1}{x}, \; \frac{x^2-1}{x^4+x^3}, \; \frac{x^7-8}{x^2+2x-7}.$$

Asymptotes

Definitions

- If $f(x)$ approaches a finite number L as x goes to ∞ (or x goes to $-\infty$), then $y = L$ is called a *horizontal asymptote*.
- If $f(x)$ goes to ∞ (or $-\infty$) as x approaches a finite number K from one side or the other, then $x = K$ is called a *vertical asymptote*.

Rational Function Exercises

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function. Then

1. Suppose $p(a) = 0$ for some number a. What is $f(a)$?
2. Suppose $q(b) = 0$ for some number b. Does f have any asymptote?

Rational Function Exercises

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Remarks

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, then

- f intercept the x-axis where $p(x) = 0$
- f has vertical asymptotes when $q(x) = 0$

Rational Function Exercises

Suppose $r(x) = \frac{x^3-1}{x^2+2x+1}$.

1. Does $r(x)$ intercept the x-axis? Where?
2. Does it have any vertical asymptotes? Where?
3. Where is $r(x)$ positive? Negative?
4. What happens as $x \to \infty$? What about as $x \to -\infty$?

Exponential Growth Equations

Definition

$P(t)$ is an *exponential function* of t if

$$P(t) = P_0 a^t,$$

where P_0 is the initial quantity and a is the factor by which $P(t)$ changes when t increases by 1.

Remarks

- If $a > 1$, we have *exponential growth*. If $0 < a < 1$, we have *exponential decay*.
- Why does it not make sense to have $a = 1$ or $a \leq 0$?

Examples

1. On the computer, graph the function $f(x) = a^x$ for $a = 1, 2, 3, 5, 10$.
2. Do the same for $a = 0.95, 0.9, 0.8, 0.5, 0.1$
3. All of the graphs you drew are concave up or down?
4. Which functions have the fastest increase/decrease?

The special number e

The number e is defined as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

The standard exponential function is

$$\exp(x) = e^x .$$

Notice that e is defined as the *limit* of a function of n as n approaching infinity. We will discuss in details what does this mean in the next lecture. For now, let us simply take e as a real number which is roughly 2.7169.

Inverse functions

Definitions

For a function $f: X \to Y$, its inverse $f^{-1}: Y \to X$ sends each element $y \in Y$ to the unique element $x \in X$ such that $f(x) = y$.

- The inverse of f, denoted $f^{-1}(x)$, is a function that undoes the operation of f.
- f^{-1} exists if and only if f is one-to-one.
- Two functions f and g are inverses if $f(g(x)) = x$ and $g(f(x)) = x$
- Graphically: reflection over $y = x$

Examples

1. Find $f^{-1}(x)$ for $f(x) = 2x + 3$, $f(x) = x^2$ $(x \geq 0)$
2. Sketch $y = f(x)$ and $y = f^{-1}(x)$
3. Is exponential function invertible?

Logarithms

Definition

If $f(x) = a^x$, we define the logarithm function to be the inverse of f. In other words

$$\log_a x = x \Leftrightarrow a^c = x.$$

- Domain: $x > 0$, Range: all real numbers
- Graph crosses $(1, 0)$, passes through $(a, 1)$

Examples

$\log(x)$

If $f(x) = 10^x$, then we define the *logarithm base 10* of x to be $f^{-1}(x)$. In other words

$$\log_{10} x = c \Leftrightarrow 10^c = x.$$

Often, we leave off the 10 and just write $\log x$.

$\ln(x)$

If $f(x) = e^x$, then we define the *natural logarithm* of x to be $f^{-1}(x)$. In other words

$$\log_e x = c \Leftrightarrow e^c = x.$$

It is most often written $\log_e x = \ln x$.

Angles and Radians

- Angle measured in degrees (0°–360°) or radians (0–2π)
- Conversion: $180^\circ = \pi$ radians
- Common angles:

$$30^\circ = \pi/6, \quad 45^\circ = \pi/4, \quad 60^\circ = \pi/3, \quad 90^\circ = \pi/2$$

Trigonometric Functions

Definition in Right Triangle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- Reciprocal functions:

$$\csc\theta = 1/\sin\theta,\ \sec\theta = 1/\cos\theta,\ \cot\theta = 1/\tan\theta$$

- Can be used to parametrize the unit circle (circle of radius 1 centered at the origin)

$$\text{Coordinates: } (\cos\theta, \sin\theta)$$

Commonly used formulas

Basic trig identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Angle sum and difference

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Transforming Trig Functions

- Graph: $y = \sin x$, $y = \cos x$
- Vertical shift: $y = \sin x + D$
- Vertical stretch: $y = A \sin x$, amplitude A
- Horizontal stretch: $y = \sin(Bx)$, period $T = 2\pi/B$
- Phase shift: $y = \sin(x - C)$

Definition of a Limit

$$\lim_{x \to c} f(x) = L$$

(In words: "the limit as x goes to c of $f(x)$ equals L.")

- $x \to c$ means x approaches c but is not equal to c.
- $\lim_{x \to c} f(x)$ *does not exist* if no such L exists.

Intuitive Definition

if $f(x)$ gets as close to L as we want by taking x *sufficiently close* to c

Formal Definition

For any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

Left and Right Limits

Definition

$$\lim_{x \to c^+} f(x) \quad \text{approaches } c \text{ from the right (positive side)}$$

$$\lim_{x \to c^-} f(x) \quad \text{approaches } c \text{ from the left (negative side)}$$

Important: $\lim_{x \to c} f(x)$ exists and equals L if and only if

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$$

Exercises: Investigate Limits

By looking at left and right limits, investigate $\lim_{x \to 0} f(x)$:

1. $f(x) = \frac{|x|}{x}$
2. $f(x) = \frac{1}{x}$

Limits at Infinity

$$\lim_{x \to \infty} f(x) = L$$

Intuitive Definition

We can get $f(x)$ as close to L as we please by taking x sufficiently large.

Formal Definition

If for every $\varepsilon > 0$, there exists a number $N > 0$ such that for all $x > N$,

$$|f(x) - L| < \varepsilon.$$

Limit Laws

Assume $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then:

- $\lim_{x \to c} k = k$ for constant k
- $\lim_{x \to c} [kf(x)] = k \lim_{x \to c} f(x)$
- $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
- $\lim_{x \to c} [f(x)g(x)] = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x))$
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim g(x) \neq 0$

Exercises: Compute Limits

1. $\lim_{x\to 5} \frac{x^3-x}{2x+3}$

2. $$\lim_{x\to\infty} \frac{3x^2+17x-7}{2x^2-1}$$

3. $\lim_{h\to 0} \frac{2(h+2)^2-2h^2}{h}$ (Hint: simplify first)