AI511/MM505 Linear Algebra with Applications

Lecture 10 – Change of Basis, Fundamental Subspaces, Rank and Nullity

Vaidotas Characiejus

October 7, 2025

Department of Mathematics and Computer Science
University of Southern Denmark

Schedule

Week	Lectures	Exercise sessions	Part
36	AI511/MM505	No exercise sessions	Theory
37	AI511/MM505	AI511/MM505	Theory
38	AI511	AI511	Applications
39	AI511/MM505	AI511/MM505	Theory
40	AI511/MM505	AI511/MM505	Theory
41	**AI511/MM505**	**AI511/MM505**	**Theory**
42	Autumn break		
43	AI511/MM505	AI511/MM505	Theory
44	AI511	AI511/MM505	Applications/theory

Week 44 is split: lectures cover applications, exercises cover theory.

Outline

Recap

Coordinates

- Suppose that $B = \{v_1, \ldots, v_k\}$ is a basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$. Then $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$v = c_1 v_1 + \ldots + c_k v_k$$

  are called the coordinates of $v \in \mathcal{S}$ and $c_1, \ldots, c_k \in \mathbb{R}$ are uniquely determined by $v \in \mathcal{S}$.
- We write

$$[v]_B = (c_1, \ldots, c_n)$$

  to indicate that $c_1, \ldots, c_k \in \mathbb{R}$ are the coordinates of $v \in \mathcal{S}$ in basis B.

Examples

The coordinates of $v = (5, 0)$ with respect to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

of $\mathbb{R}^2$ are $[v]_B = (10, -5)$.

Dimension

- The number of elements in some/any basis is called the dimension of a subspace $\mathcal{S} \subset \mathbb{R}^n$ and denoted by $\mathrm{dim}(\mathcal{S})$.
- We formally define the dimension of $\{0\}$ to be 0.
- $\mathrm{dim}(\mathbb{R}^n) = n$.
- $\mathrm{dim}(\mathrm{span}\{v_1, v_2\}) = 2$, where $v_1, v_2 \in \mathbb{R}^n$ with $n \geq 2$ are two linearly independent vectors.

Bases and dimension

- Suppose that $\mathcal{S} \subset \mathbb{R}^n$ is a subspace and $\mathsf{dim}(\mathcal{S}) = k$. If $|B| = k$, then B is a basis if and only if $\mathsf{span}(B) = \mathcal{S}$ or if B is linearly independent.
- In simple terms, if we know the dimension of the space, it is enough to check if the vectors span the space or if the vectors are linearly independent to determine if the vectors form a basis.

Change of basis

Example on the blackboard

Let

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

be bases of $\mathbb{R}^2$ and suppose that $[v]_B = (5, 1)$. Compute $[v]_C$.

Change of basis

- When we change the basis B that we are working with, the resulting coordinate vectors change as well.
- When we computed $[v]_C$ from $[v]_B$, we computed v itself as an intermediate step.
- This is quite undesirable in general since $[v]_B$ and $[v]_C$ can have a much lower dimension than v (consider a 3-dimensional subspace of $\mathbb{R}^{85}$).

Definition 3.1.2 of Johnston (2021)

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ with bases $B = \{v_1, v_2, \ldots, v_k\}$ and C. The *change-of-basis matrix* from B to C, denoted by $P_{C \leftarrow B}$,is the $k \times k$ matrix whose columns are the coordinate vectors $[v_1]_C$, $[v_2]_C$, $\ldots$, $[v_k]_C$:

$$P_{C \leftarrow B} := \begin{bmatrix} [v_1]_C & [v_2]_C & \ldots & [v_k]_C \end{bmatrix}.$$

Recommended to watch *Change of basis* of 3Blue1Brown.

Change of basis (cont.)

Theorem 3.1.3 of Johnston (2021)

Suppose B and C are bases of a subspace $\mathcal{S}$ of $\mathbb{R}^n$, and let $P_{C \leftarrow B}$ be the change-of-basis matrix from B to C. Then

(a) $P_{C \leftarrow B}[v]_B = [v]_C$ for all $v \in \mathcal{S}$, and

(b) $P_{C \leftarrow B}$ is invertible and $P_{C \leftarrow B}^{-1} = P_{B \leftarrow C}$.

Furthermore, $P_{C \leftarrow B}$ is the unique matrix with property (a).

- In part (a) of the theorem, the middle Bs "cancel out" and leave just the Cs behind (this is why the notation $P_{C \leftarrow B}$ is used instead of $P_{B \rightarrow C}$).
- One of the useful features of change-of-basis matrices is that they can be re-used to change the basis of multiple different vectors.

Change of basis (cont.)

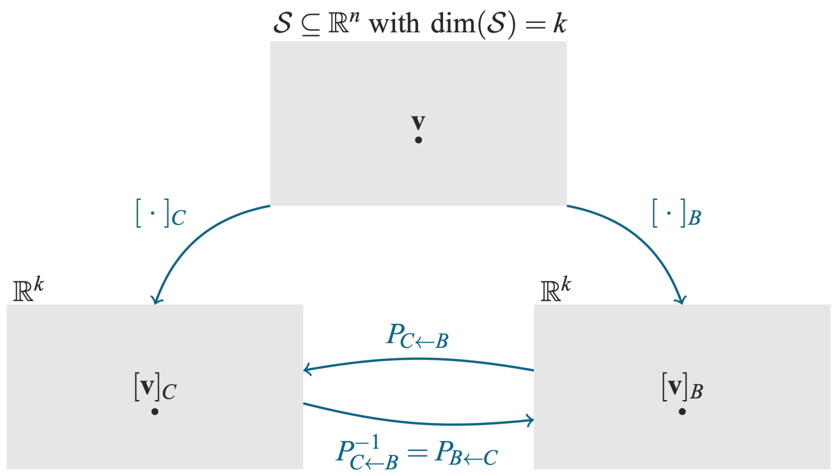

Figure 3.5 of Johnston (2021)

Example on the blackboard (cont.)

Let

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad C = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

be bases of $\mathbb{R}^2$ and suppose that $[v]_B = (5, 1)$. Compute the change-of-basis matrix $P_{C \leftarrow B}$ and calculate $P_{C \leftarrow B}[v]_B$.

Recall that $[v]_C = (2, 2)$.

Fundamental subspaces of a matrix

Range and null space revisited

Definition 2.3.2 of Johnston (2021)

Suppose $A \in \mathbb{R}^{m \times n}$.

(a) The *range* of A is the subspace of $\mathbb{R}^m$, denoted by $\mathsf{range}(A)$, that consists of all vectors of the form Ax.

(b) The *null space* of A is the subspace of $\mathbb{R}^n$, denoted by $\mathsf{null}(A)$, that consists of all solutions x of the system of linear equations $Ax = 0$.

Theorem 2.3.2 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{m \times n}$ has columns $a_1, \ldots, a_n$. Then

$$\mathsf{range}(A) = \mathsf{span}(a_1, \ldots, a_n).$$

Do the columns of a matrix form a basis for its range?

Example: finding bases of $\mathrm{range}(A)$ and $\mathrm{null}(A)$

- Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{1}$$

- We can see that a basis of $\mathrm{range}(A)$ is (think why this is the case)

$$\{(1, 2, 1), (1, -1, 0)\}.$$

- How do we construct a basis for $\mathrm{null}(A)$? We need to solve the system of linear equations $Ax = 0$.

Example: finding bases for $\mathrm{range}(A)$ and $\mathrm{null}(A)$ (cont.)

We have that

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{row-reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and hence the set of solutions (which is a subspace) can be written as

$$\{(-t, -t, t) : t \in \mathbb{R}\}$$

which means that $(-1, -1, 1)$ is a basis of $\mathrm{null}(A)$.

Finding a basis of $\mathrm{range}(A)$ from a row echelon form

- The columns of a matrix A that have a leading entry in one of its row echelon forms make up a basis of $\mathrm{range}(A)$.
- Recall that if a matrix A has row echcelon form R, then x solves the linear system $Ax = 0$ if and only if it solves the linear system $Rx = 0$.
- In other words, a linear combination of the columns of A equals 0 if and only if the same linear combination of the columns of R equals 0.
- This means that the linear dependence structure of the columns of A is the same as that of the columns of R.
- It follows that a particular subset of the columns of A is a basis of $\mathrm{range}(A)$ if and only if that same subset of columns of R is a basis of $\mathrm{range}(R)$.

Four fundamental subspaces

Definition 2.4.3 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{m \times n}$. The four subspaces

$$\mathsf{range}(A), \quad \mathsf{null}(A), \quad \mathsf{range}(A^T), \quad \text{and} \quad \mathsf{null}(A^T)$$

are called the *four fundamental subspaces* associated with A.

The bases of $\mathsf{range}(A^T)$ and $\mathsf{null}(A^T)$ can be found by finding the reduced REF of A^T and using the method for finding bases of the range and null space discussed previously.

Rank and nullity of a matrix

Inverse of a matrix revisited

Theorem 2.2.4 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{n \times n}$. The following are euquivalent:

(a) A is invertible.

(b) The system of linear equations $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.

(c) The system of linear equations $Ax = 0$ has a unique solution.

- Hence, the range of an invertible matrix is all of $\mathbb{R}^n$ (think why this is the case).
- Geometrically speaking, invertible matrices do not "squash" space down into a smaller-dimensional subspace.

Non-invertible matrices

- If $A \in \mathbb{R}^{n\times n}$ is non-invertible, it means that there is at least one $b \in \mathbb{R}^n$ such that $Ax \neq b$ for any choice of $x \in \mathbb{R}^n$.
- It follows that $\mathsf{range}(A)$ is strictly smaller than $\mathbb{R}^n$, i.e., there is at least one $b \in \mathbb{R}^n$ such that $b \notin \mathsf{range}(A)$.
- The range of some non-invertible matrices is smaller while the range of some other non-invertible matrices is bigger.
- The size of $\mathsf{range}(A)$ can be used to describe how non-invertible the matrix is (the smaller the range of a matrix, the further the matrix is away from being invertible).
- How can we describe the size of $\mathsf{range}(A)$?

Rank of a matrix

Definition 2.4.4 of Johnston (2021)

The *rank* of $A \in \mathbb{R}^{m \times n}$ is denoted by $\mathsf{rank}(A)$ and defined by

$$\mathsf{rank}(A) = \mathsf{dim}(\mathsf{range}(A)).$$

Observe that the rank is defined for all matrices (not necessarily square) but our intuition is based on invertibility of matrices which is only defined for square matrices.

Examples

- Suppose that $u \in \mathbb{R}^n$ is a unit vector and recall that uu^T is the standard matrix of the projection onto the line in the direction of u.
- The range of uu^T is 1-dimensional, so $\mathrm{rank}(uu^T) = 1$.
- The rank of A in (1) is 2.

Nullity of a matrix

Somewhat complementary to the rank of a matrix is its nullity, which is the dimension of its null space.

Definition

The *nullity* of $A \in \mathbb{R}^{m \times n}$ is denoted by $\mathsf{nullity}(A)$ and defined by

$$\mathsf{nullity}(A) = \mathsf{dim}(\mathsf{null}(A)).$$

$A \in \mathbb{R}^{n \times n}$ is invertible if and only if the system of linear equations $Ax = 0$ has a unique solution (i.e., its null space is $\{0\}$), which is equivalent to it having nullity 0.

Invertibility of a matrix revisited

Theorem 2.4.9 of Johnston (2021)

Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

(a) A is invertible.

(b) $\operatorname{rank}(A) = n$.

(c) $\operatorname{nullity}(A) = 0$.

Relationship between $\mathsf{rank}(A)$ and $\mathsf{nullity}(A)$

- Recall that multiplying A by the vector x is equivalent to forming a linear combination of the columns of A, using the entries of x as the weights.
- If there are more ways to linearly combine the columns of A into the 0 vector, $\mathsf{null}(A)$ and $\mathsf{nullity}(A)$ is larger.
- Intuitively, the more ways you can combine the columns of A to get zero, the fewer ways there should be to combine them to reach any other non-zero vector.
- There should be a relationship between $\mathsf{rank}(A)$ and $\mathsf{nullity}(A)$. How does it look like?

Rank–nullity theorem

Theorem 2.4.10 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{m \times n}$. Then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Rank–nullity theorem (cont.)

$$\operatorname{rank}(A) = 3$$

$$\operatorname{rank}(A) = 3 \quad \begin{bmatrix} 0 & 1 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\operatorname{nullity}(A) = 6$$

- The rank of a matrix equals the number of leading columns that its reduced REF has, which equals the number of non-zero rows.
- Its nullity is the number of non-leading columns in its reduced REF, which equals the number of free variables.

Figure 2.24 of Johnston (2021)

Rank inequality and full rank

- We have that

$$\mathrm{rank}(A) \leq \min\{m, n\}$$

for $A \in \mathbb{R}^{m \times n}$ (think why this inequality is true).

- We say that A has *full rank* if $\mathrm{rank}(A) = \min\{m, n\}$.

Rank and matrix addition and multiplication

Theorem 2.4.11 of Johnston (2021)

Let A and B be matrices (with sizes such that the operations below make sense). Then

(i) $\mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$;

(ii) $\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.