AI511/MM505 Linear Algebra with Applications
Exercises 5
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The exercises are from Sections 2.4 and 3.1 of Johnston (2021) and Sections 4.5, 4.6, and 4.7 of Anton, Rorres, and Kaul (2019).

1. Find the coordinate vector of $v = (2, -1, 3)$ relative to the basis $S = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$, where
$$v_1 = (1, 0, 0), \quad v_2 = (2, 2, 0), \quad \text{and} \quad v_3 = (3, 3, 3).$$
2. The vectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{v_1, v_2\}$ to a basis for $\mathbb{R}^4$.
3. Consider the bases $B = \{u_1, u_2\}$ and $B' = \{u'_1, u'_2\}$ for $\mathbb{R}^2$, where
$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ u'_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad u'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$
   (a) Find the change-of-basis matrix B' to B.
   (b) Find the change-of-basis matrix B to B'.
   (c) Compute the coordinate vector $[w]_B$, where
   $$w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$
   and use the change-of-basis matrix $P_{B' \leftarrow B}$ to compute $[w]_{B'}$.
   (d) Check your work by computing $[w]_{B'}$ directly.
4. Compute the change-of-basis matrix $P_{C \leftarrow B}$ for each of the following pairs of bases B and C.
   (a) $B = \{(1, 2), (3, 4)\}$, $C = \{(1, 0), (0, 1)\}$;
   (b) $B = \{(1, 0), (0, 1)\}$, $C = \{(2, 1), (-4, 3)\}$;
   (c) $B = \{(1, 2), (3, 4)\}$, $C = \{(2, 1), (-4, 3)\}$.

5. Determine which of the following statements are true and which are false.
   (a) If B is a basis of a subspace $\mathcal{S}$ of $\mathbb{R}^n$, then $P_{B \leftarrow B} = I$.
   (b) Every basis of $\mathbb{R}^3$ contains exactly 3 vectors.
   (c) Every set of 3 vectors in $\mathbb{R}^3$ forms a basis of $\mathbb{R}^3$.
   (d) If $\mathcal{S} = \operatorname{span}(v_1, \ldots, v_k)$, $\{v_1, \ldots, v_k\}$ is a basis of $\mathcal{S}$.
   (e) The rank of the zero matrix is 0.
   (f) The $n \times n$ identity matrix has rank n.
   (g) If $A \in \mathbb{R}^{3\times 7}$, then $\operatorname{rank}(A) \leq 3$.
   (h) If A and B are matrices of the same size, then $\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B)$.
   (i) If A and B are square matrices of the same size, then $\operatorname{rank}(AB) = \operatorname{rank}(A) \cdot \operatorname{rank}(B)$.
   (j) The rank of a matrix equals the number of non-zero rows that it has.
   (k) If $A, B \in \mathbb{R}^{m\times n}$ are row equivalent, then $\operatorname{null}(A) = \operatorname{null}(B)$.
6. For each of the following matrices A, find bases for each of $\operatorname{range}(A)$, $\operatorname{null}(A)$, $\operatorname{range}(A^T)$, and $\operatorname{null}(A^T)$.

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$.
7. Compute the rank and nullity of the following matrices.

   (a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$; (d) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$.
8. In this exercise, we explore a relationship between the range and null space of a matrix.
   (a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ whose range is the same as its null space. [Hint: use the rank–nullity theorem as a starting point.]
   (b) Show that there does not exist a matrix $A \in \mathbb{R}^{3\times 3}$ whose range is the same as its null space.
9. Let J_n be the $n \times n$ matrix, all of whose entries are 1. Compute $\operatorname{rank}(J_n)$.
10. Suppose A and B are 4×4 matrices with the property that $Av \neq Bv$ for all $v \neq 0$. What is the rank of the matrix $A - B$?

References

Johnston, Nathaniel (2021). *Introduction to Linear and Matrix Algebra.* Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). *Elementary Linear Algebra: Applications Version.* 12th edition. John Wiley & Sons, Inc.