AI511/MM505 Linear Algebra with Applications
Take-Home Exam
Autumn 2025

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- The solutions of the take-home exam have to be submitted as a single PDF file on Digital Eksamen until the deadline of November 6, 2025 at 9:59.

- You are required to solve the exercises on your own and you are not permitted to

  (i) discuss the exercises with anyone else (this includes your fellow students as well as posting questions about the exercises on the internet);
  (ii) use generative models (such as ChatGPT).

  Any suspicious similarities of the solutions or any other concerns will be investigated.

- You are allowed to use everything else (including the lecture material, textbooks, Wikipedia, etc.).

- You are required to demonstrate how you arrive at your solution by providing a suitable amount of explanations and intermediate computations. Exercise 6 is an exception since you are only required to write true or false in Exercise 6.

- You can scan and submit your handwritten solutions as long as the solutions are properly organised and the handwriting is readable.

- You may write the solutions in English or Danish.

The total number of points is 100.

Good luck!

1. Consider the following system of linear equations

$$\begin{cases} 5x + 2y - 3z = 9; \\ 6x + 3y - 3z = 12; \\ 4x + 2y - 2z = 8. \end{cases} \tag{1}$$

   (a) (2 pts) Is $(x, y, z) = (1, 2, 0)$ a solution of (1)?
   (b) (2 pts) Give the expression of the augmented matrix corresponding to (1).
   (c) (10 pts) Transform the augmented matrix into a reduced row echelon form to find the set of solutions of (1).

2. (8 pts) Suppose that

$$A = \begin{bmatrix} 3 & 8 & 1 & 7 \\ 2 & 9 & 10 & 2 \\ 1 & 1 & 8 & 8 \\ 2 & 2 & 5 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 2 & 4 & 8 \\ 3 & 4 & 2 & 4 \\ 6 & 10 & 3 & 6 \\ 10 & 9 & 2 & 4 \end{bmatrix}.$$

   Calculate the determinant of C, where $C = AB$.

3. Consider the subsets of $\mathbb{R}^3$

$$A = \{(8, 7, 6)\}, \quad B = \{(4, 9, 8), (-3, 5, 7)\},$$

   and

$$C = \{(3, 7, 1), (8, 10, 9), (-4, 8, -14)\}.$$

   (a) (10 pts) Determine which of the subsets A, B, and C are bases of some subspace of $\mathbb{R}^3$ and which are not bases of any subspace of $\mathbb{R}^3$.
   (b) (4 pts) Determine the dimensions of the subspaces that A, B, and C span.

4. Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   (a) (2 pts) Show that

$$A = \frac{ww^T}{w^T w},$$

   where $w = (1, 1)$.
   (b) (8 pts) Explain how $\mathbb{R}^2$ is transformed by the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_A(x) = Ax$ for $x \in \mathbb{R}^2$.
   (c) (8 pts) The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$ with their corresponding eigenvectors $2^{-1/2}(1, 1)$ and $2^{-1/2}(-1, 1)$ (they are given – you do not need to compute them). Let E_λ denote the eigenspace corresponding to the eigenvalue λ. Describe how the matrix A transforms non-zero vectors in E_{λ_1} and in E_{λ_2}. Make a connection with your response in part (b).

5. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and define the function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by setting

$$f(x) = \frac{x^T A x}{x^T x}$$

for nonzero $x \in \mathbb{R}^n$.

(a) (2 pts) What does the assumption that $A \in \mathbb{R}^{n\times n}$ is symmetric tell us about its eigenvalues?
(b) (8 pts) Suppose that v is an eigenvector of A with its corresponding eigenvalue λ. Show that $f(v) = \lambda$, i.e., the value of f at an eigenvector is the corresponding eigenvalue.
(c) (2 pts) Suppose that $X \in \mathbb{R}^{n\times p}$ with $n \geq p$. What is the size of the matrix $X^T X$? Is the matrix $X^T X$ symmetric?
(d) (12 pts) Show that the eigenvalues of $X^T X$ are non-negative.
(e) (12 pts) Suppose additionally that $X \in \mathbb{R}^{n\times p}$ is of full rank. Show that the eigenvalues of $X^T X$ are positive.

6. You are only required to respond true or false to the following statements.

(a) (2 pts) A product of an invertible matrix and a non-invertible matrix is an invertible matrix.
(b) (2 pts) Every subspace of $\mathbb{R}^n$ has infinitely many vectors.
(c) (2 pts) If $A \in \mathbb{R}^{3\times 3}$ with $\det(A) = 3$, then $\operatorname{rank}(A) = 3$.
(d) (2 pts) 0 is an eigenvalue of every square matrix.
(e) (2 pts) Every diagonalisable matrix is invertible.