Homework

1. A drug is injected into a patient's blood vessel. The function $c = f(x, t)$ represents the concentration of the drug at a distance x mm in the direction of the blood flow measured from the point of injection and at time t seconds since the injection. What are the units of the following partial derivatives? What are their practical interpretations? What do you expect their signs to be?

   (a) $\partial c/\partial x$

   (b) $\partial c/\partial t$

2. Is there a function f which has the following partial derivatives? If so what is it? Are there any others?

$$f_x(x, y) = 4x^3y^2 - 2y^4$$
$$f_y(x, y) = 2x^4y - 12xy^3$$

3. Find the rate of change of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$ in the direction of the vector $\mathbf{u} = (0.6, 0.8)$.

4. Let $f(x, y) = x^2y^3$. At the point $(-1, 2)$, find a vector that is

   (a) in the direction of maximum rate of change

   (b) in the direction of minimum rate of change

   (c) in a direction in which the rate of change is zero

5. Find a unit vector normal to the surface S given by $z = x^2y^2 + y + 1$ at the point (0, 0, 1).

6. A student was asked to find the directional derivative of $f(x, y) = x^2e^y$ at the point $(1, 0)$ in the direction of $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$. The student's answer was

$$f_{\mathbf{v}}(1, 0) = \text{grad} f(1, 0) \cdot \mathbf{v} = \frac{8}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

   At a glance, how do you know this is wrong?

7. Find the equation of the tangent plane at the given point.

$$f(x, y) = \ln(x^1 + 1) + y^2$$

   at the point $(0, 3, 9)$.

8. Are the following statements true or false?

   (a) If $f(x, y)$ has $f_y(x, y) = 0$ then f must be a constant.

   (b) If f is a symmetric two-variable function, that is $f(x, y) = f(y, x)$ then $f_x(x, y) = f_y(x, y)$.

   (c) For $f(x, y)$, if $\frac{f(0.01,0)-f(0,0)}{0.01} > 0$, then $f_x(0, 0) > 0$.

9. (Challenge yourself!) Given a function defined as following.

$$L(\theta_1, \theta_2, b) = -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)}\log(\hat{y}^{(i)}) + (1-y^{(i)})\log(1-\hat{y}^{(i)})\right]$$

where

$$\hat{y}^{(i)} = f(x_1^{(i)}, x_2^{(i)}).$$

and

$$f(x_1, x_2) = \frac{1}{1+e^{-(\theta_1 x_1 + \theta_2 x_2 + b)}}$$

Suppose $x_1^{(i)}, x_2^{(i)}, y^{(i)} \in \mathbb{R}, i = 1..., m$ are given.
Use the fact that

$$\sigma(x) = \frac{1}{1+e^{-x}}, \qquad \sigma'(x) = \sigma(x)\big(1-\sigma(x)\big)$$

to prove the following is true.

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= \frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})x_1^{(i)} \\
\frac{\partial L}{\partial \theta_2} &= \frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})x_2^{(i)} \\
\frac{\partial L}{\partial b} &= \frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)}).
\end{aligned}$$