AI511/MM505 Linear Algebra with Applications
Exercises 6
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The exercises are from Sections 3.3 and 3.4 of Johnston (2021) and Sections 5.1 and 5.2 of Anton, Rorres, and Kaul (2019).

1. Confirm by multiplication that v is an eigenvector of A and find the corresponding eigenvalue, where
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
2. Find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrices

   (a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
3. Find a matrix P that diagonalises A, and check your work by computing $P^{-1}AP$, where

   (a) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$; (b) $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
4. Compute the matrix A^{10} by diagonalising the matrix A, where
$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}.$$
5. Let
$$A = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix}.$$
   Find the square root of A, i.e., the matrix $A^{1/2}$ such that $A^{1/2}A^{1/2} = A$.
6. Determine which of the following statements are true and which are false.

(a) Every matrix has at least one real eigenvalue.

(b) A set of two eigenvectors corresponding to the same eigenvalue of a matrix must be linearly dependent.

(c) If two matrices are row equivalent then they must have the same eigenvalues.

(d) If v and w are eigenvectors of a matrix $A \in \mathbb{R}^{n\times n}$ corresponding to an eigenvalue λ then so is any non-zero linear combination of v and w.

(e) If $A \in \mathbb{R}^{n\times n}$, then $\det(A - \lambda I) = \det(\lambda I - A)$.

(f) Every diagonal matrix $A \in \mathbb{R}^{n\times n}$ is symmetric.

(g) The geometric multiplicity of an eigenvalue λ of $A \in \mathbb{R}^{n\times n}$ equals $\text{nullity}(A - \lambda I)$.

(h) It is possible for an eigenvalue to have geometric multiplicity equal to 0.

(i) If an eigenvalue has algebraic multiplicity 1, then its geometric multiplicity must also equal 1.

(j) If an eigenvalue has geometric multiplicity 1, then its algebraic multiplicity must also equal 1.

7. Let $k \in \mathbb{R}$ and consider the following matrix:

$$A = \begin{bmatrix} k & 1 \\ -1 & 1 \end{bmatrix}.$$

For which values of k does A have (a) two distinct real eigenvalues; (b) only one distinct real eigenvalue; (c) no real eigenvalues?

8. Suppose that $A \in \mathbb{R}^{n\times n}$ is a matrix such that $A^2 = 0$ (the square of A is equal to the zero matrix). Such matrices are called nilpotent. Find all possible eigenvalues of A.

References

Johnston, Nathaniel (2021). *Introduction to Linear and Matrix Algebra*. Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). *Elementary Linear Algebra: Applications Version*. 12th edition. John Wiley & Sons, Inc.