AI511/MM505 Linear Algebra with Applications

Lecture 9 – Coordinates, Dimension

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Outline

Recap

Definition 2.3.1 of Johnston (2021)

A *subspace* of $\mathbb{R}^n$ is a non-empty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ with the properties that

(a) if $v, w \in \mathcal{S}$, then $v + w \in \mathcal{S}$;

(b) if $v \in \mathcal{S}$ and $c \in \mathbb{R}$, then $cv \in \mathcal{S}$.

Intuitively, the definition means that we cannot leave the subspace by adding or scaling its vectors.

Subspaces of $\mathbb{R}^2$

- How do the subspaces look like in $\mathbb{R}^2$?
- The zero subspace
$$\{0\}$$
with $0 = (0, 0)$.
- The lines that go through the origin
$$\mathrm{span}\{v\} = \{tv : t \in \mathbb{R}\}$$
with $v \in \mathbb{R}^2$.
- The whole space $\mathbb{R}^2$
$$\mathrm{span}\{v_1, v_2\} = \{t_1 v_1 + t_2 v_2 : t_1, t_2 \in \mathbb{R}\},$$
where $v_1, v_2 \in \mathbb{R}^2$ are any two linearly independent vectors.

Linear combination and span

Definition 1.1.3 of Johnston (2021)

A *linear combination* of the vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is any vector of the form

$$c_1v_1 + c_2v_2 + \ldots + c_kv_k,$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Definition 2.3.3 of Johnston (2021)

If $B = \{v_1, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of those vectors is called their *span*, and it is denoted by $\mathsf{span}(B)$ and $\mathsf{span}(v_1, \ldots, v_k)$.

Definition 2.3.4 of Johnston (2021)

A set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ is *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, at least one of which is not zero, such that

$$c_1 v_1 + \ldots + c_k v_k = 0.$$

If S is not linearly dependent then it is called *linearly independent*.

Basis

- A set $B = \{v_1, \ldots, v_k\}$ is called a basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ if it spans $\mathcal{S}$ and if it is linearly independent.
- If B is a basis and if we insert one more vector into B, $\mathcal{S}$ becomes linearly dependent.
- If B is a basis and if we remove one vector from B, B does not span $\mathcal{S}$ anymore.
- A basis is not too big and not too small–it is just right.

Euclidean vector spaces (cont.)

Euclidean vector spaces (cont.)

Coordinates

Definition 3.1.1 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, $B = \{v_1, v_2, \ldots, v_k\}$ is a basis for $\mathcal{S}$, and $v \in \mathcal{S}$. Then the unique scalars $c_1, c_2, \ldots, c_k$ for which

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

are called the *coordinates* of v with respect to B, and the vector

$$[v]_B := (c_1, c_2, \ldots, c_k)$$

is called the *coordinate vector* of v with respect to B.

When we work with coordinates or coordinate vectors, we simply use the basis vectors in the order written. If $B = \{v_1, v_2, \ldots, v_k\}$, then the "first" basis vector is v_1, the "second" basis vector is v_2, and so on.

Examples of coordinates

- For $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we have

$$[v]_B = (v_1, \dots, v_n)$$

  with respect to the standard basis $B = \{e_1, \dots, e_n\}$.
- The coordinates of $v = (5, 0)$ with respect to the basis

$$B' := \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

  of $\mathbb{R}^2$ are $[v]_{B'} = (10, -5)$.

Euclidean vector spaces (cont.)

Dimension

Number of vectors in a basis

Theorem 2.4.1 of Johnston (2021)

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ and suppose that $B, C \subset \mathcal{S}$ are finite sets with the properties that B is linearly independent and $\mathsf{span}(C) = \mathcal{S}$. Then

$$|B| \leq |C|.$$

Linearly independent sets are "small", while spanning sets are "big".

Dimension

Corollary 2.4.2 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$. Every basis of $\mathcal{S}$ has the same number of vectors.

Proof.

Suppose that B and C are bases of $\mathcal{S}$. Then $|B| \leq |C|$ since B is linearly independent and $\mathrm{span}(C) = \mathcal{S}$. On the other hand, $|C| \leq |B|$ since C is linearly independent and $\mathrm{span}(B) = \mathcal{S}$. Hence, $|B| = |C|$. $\square$

Definition 2.4.2 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$. The number of vectors in a basis of $\mathcal{S}$ is called the *dimension* of $\mathcal{S}$ and is denoted by $\mathrm{dim}(\mathcal{S})$.

Examples of dimensions

- We formally define the dimension of $\{0\}$ to be 0.
- $\dim(\mathbb{R}^n) = n$.
- Lines are 1-dimensional since a single vector acts as a basis of a line.
- Planes in $\mathbb{R}^3$ are 2-dimensional since two non-parallel vectors form a basis of a plane.

Creating a basis

Theorem 2.4.3 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ and $B \subset \mathcal{S}$ is a finite set of vectors.

(a) If B is linearly independent, then there is a basis C of $\mathcal{S}$ with $B \subset C$.

(b) If B spans $\mathcal{S}$, then there is a basis C of $\mathcal{S}$ with $C \subset B$.

We can always toss away vectors from a spanning set until it is small enough to be a basis, and we can always add new vectors to a linearly independent set until it is big enough to be a basis.

Basis

If one already knows $\dim(\mathcal{S})$, then checking if something is a basis becomes easier.

Theorem 2.4.4 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ and $B \subset \mathcal{S}$ is a set containing k vectors.

(a) If $k \neq \dim(\mathcal{S})$ then B is not a basis of $\mathcal{S}$.

(b) If $k = \dim(\mathcal{S})$, then the following are equivalent

- (i) B spans $\mathcal{S}$;
- (ii) B is linearly independent;
- (iii) B is a basis of $\mathcal{S}$.

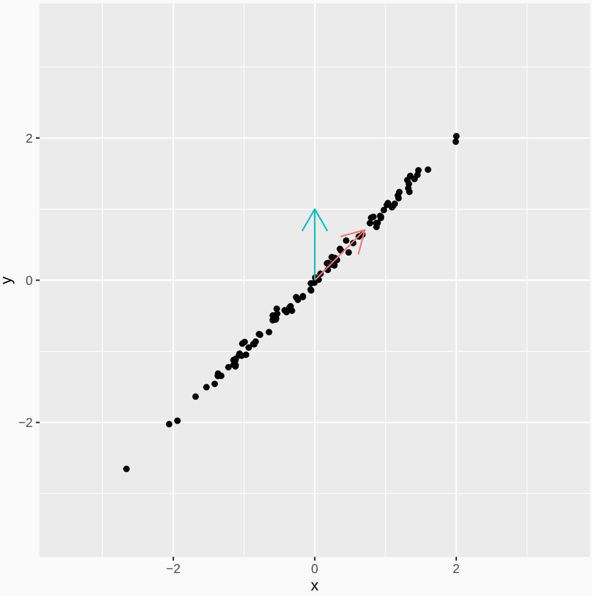
For instance, any two non-proportional vectors in $\mathbb{R}^2$ are a basis.

Proof.

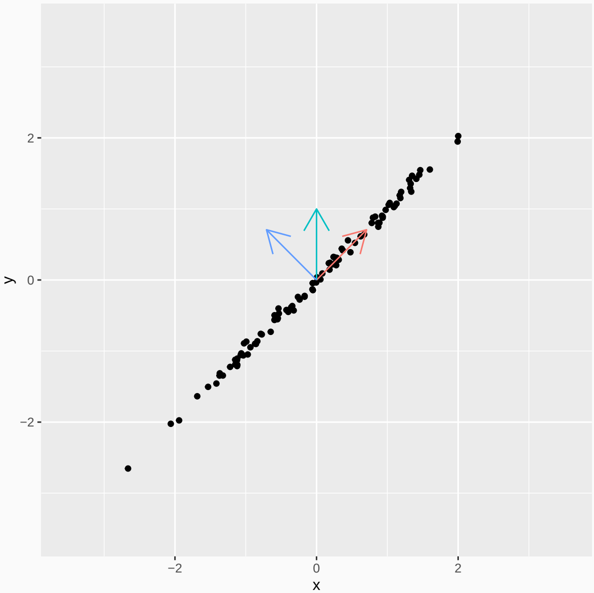
B spans $\mathcal{S} \implies B$ is linearly independent. Suppose otherwise, i.e., $\exists v \in B$ such that $v \in \mathsf{span}(B \setminus \{v\})$. Then $\mathsf{span}(B \setminus \{v\}) = \mathsf{span}(B) = \mathcal{S}$. But this is a contradiction since no set with $k-1$ elements can span a k-dimensional subspace.

B is linearly independent $\implies \mathsf{span}(B) = \mathcal{S}$. Suppose otherwise, i.e., $\exists v \in \mathcal{S}$ such that $v \notin \mathsf{span}(B)$. Then the set $B \cup \{v\}$ has $k+1$ vectors and still is linearly independent. But this is a contradiction since a set with $k+1$ vectors cannot be linearly independent in an k dimensional subspace (see Theorem 2.4.1 of Johnston (2021)). $\square$

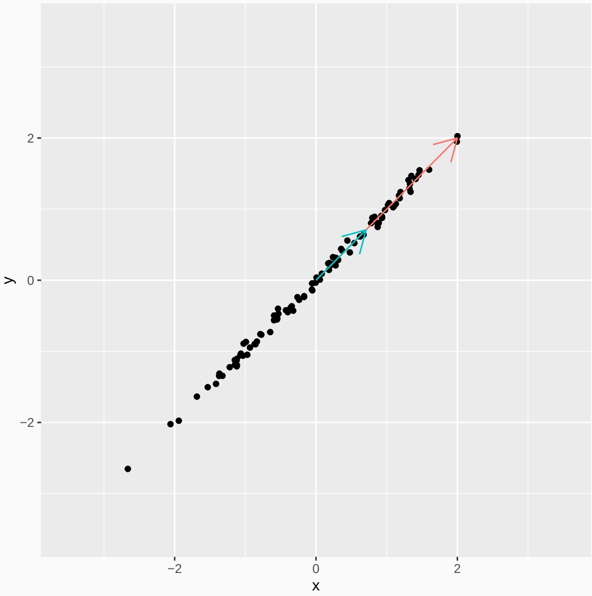
Are the two vectors a basis of $\mathbb{R}^2$?



Are the three vectors a basis of $\mathbb{R}^2$?



Are the two vectors a basis of $\mathbb{R}^2$ or of some subspace of $\mathbb{R}^2$?



References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.