Integration

AI503
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Lecture 9

Definite Integral: Review

Recall from last time, the definite integral of a continuous one-variable function f is defined as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$, and $x_k^* \in [x_{k-1}, x_k]$.

Fundamental Theorem of Calculus (Evaluation Theorem)

Theorem

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any antiderivative of f, i.e., $F'(x) = f(x)$.

Proof of FTC

- Divide $[a, b]$ into n subintervals with endpoints

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The distance between subsequent x_k's is $\Delta x = \frac{b-a}{n}$.

- Let F be an antiderivative of f. We see that

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] \\ &= \sum_{k=1}^{n} F'(c_k)\Delta x \quad \text{(Mean Value Theorem)} \\ &= \sum_{k=1}^{n} f(c_k)\Delta x \end{aligned}$$

for some $c_k \in [x_{k-1}, x_k]$.

Proof of FTC

- Since F is differentiable, it is continuous. Taking the limit as $n \to \infty$ gives

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

Problem: Using FTC

Compute

$$\int_0^3 2x \, dx$$

using the fundamental theorem of calculus.

Second Fundamental Theorem of Calculus

Theorem

Let f be continuous on an interval. Then, for x and a in that interval,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof of Second FTC

Suppose $g(x) = \int_a^x f(t)dt$. Then

$$\frac{g(x+h) - g(x)}{h} = \frac{\int_x^{x+h} f(t)dt}{h}.$$

Since f is continuous on $[x, x+h]$, by the Extreme Value Theorem, f attains minimum m and maximum M on $[x, x+h]$, so

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh.$$

Dividing by h gives

$$m \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq M.$$

As $h \to 0$, $m \to f(x)$ and $M \to f(x)$, so by the Squeeze Theorem,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

Problems: Second FTC

Find
$$\frac{d}{dx}\int_2^x \cos t \, dt.$$

Problems: Second FTC

Let $g(x) = \int_1^x \sqrt{1+t^2}\,dt$.

- Find $g'(x)$
- Find $g(x^3)$
- Compute $\frac{d}{dx} g(x^3)$.

Improper Integral

We define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

if this limit exists. In this case, we say that the integral **converges**. If the limit does not exist, we say it **diverges**.

Examples

Compute $\int_0^{\infty} e^{-5x}\, dx$.

Examples

Investigate the convergence of $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx$.

u-Substitution

Recall the **Chain Rule:** If f and g are differentiable functions, then

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

This tells us that

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C.$$

Example: u-Substitution

Compute

$$\int_2^3 2x \sin(x^2)\, dx.$$

Hint: Let $u = x^2$, then $du = 2x\, dx$. After substitution, the integral becomes

$$\int_{u=4}^{u=9} \sin(u)\, du.$$

General Idea

To evaluate integrals of the form

$$\int f'(g(x))\, g'(x)\, dx,$$

let

$$u = g(x), \quad du = g'(x)\, dx.$$

Then

$$\int f'(g(x))\, g'(x)\, dx = \int f'(u)\, du = f(u) + C.$$

Practice

Evaluate:

1. $\displaystyle\int_0^1 x(x^2+1)^4\,dx.$
2. $\displaystyle\int \sin^2 x\,dx.$

Integration by Parts

Recall the **Product Rule:**

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x) + C.$$

Hence,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Integration by Parts Formula

Let $u = f(x)$ and $dv = g'(x)\,dx$. Then

$$\int u\,dv = uv - \int v\,du.$$

For definite integrals:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Examples

Evaluate:

1. $\displaystyle\int_0^1 xe^x \, dx.$
2. $\displaystyle\int_1^2 \ln x \, dx.$

Extending to Two Variables

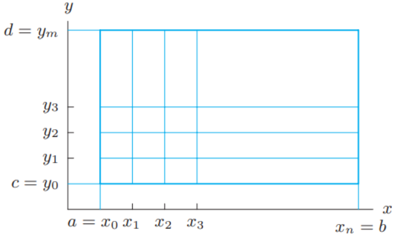
We extend the definition of the integral to a function $f(x, y)$ defined on

$$a \leq x \leq b, \quad c \leq y \leq d.$$

Divide both intervals into n and m equal parts:

$$\Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{m}.$$

- Number of subrectangles:
- Area of each subrectangle:



Riemann Sums for Two Variables

Let M_{ij} and L_{ij} denote the max and min of f on the ij-th rectangle. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle:

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \Delta x \Delta y \leq$$

Question:

- How does $\sum_{ij} f(u_{ij}, v_{ij}) \Delta x \Delta y$ look geometrically?
- What happens as $n, m \to \infty$?

Double Integral: Definition

Let $f(x, y)$ be a function on $R = [a, b] \times [c, d]$. The *definite integral* of f over R is

$$\int_R f \, dA = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \, \Delta x \Delta y,$$

if the limit exists. Then f is said to be **integrable** on R.

Interpretation and Notation

- $dA = dx\,dy$ represents the area of an infinitesimal rectangle.

$$\int_R f\,dA = \int_R f(x,y)\,dx\,dy$$

- $\int_R f\,dA$ gives the volume under the surface $z = f(x,y)$ over R.
- If f is continuous (or has finitely many jump discontinuities), it is integrable on R.

Questions

1. If $g(x, y) = 1$, what is $\int_R g \, dA$?
2. How do we compute the average value of f over R?

Iterated Integral

If f is integrable on $R = [a, b] \times [c, d]$, then

$$\boxed{\int_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy.}$$

To evaluate:

- Integrate with respect to x first (treat y constant).
- Then integrate the result with respect to y.

Example: Iterated Integral

Let $f(x, y) = x^2 y$, $R = [0, 1] \times [0, 1]$. Compute $\int_R f \, dA$.

$$\int_0^1 \int_0^1 x^2 y \, dx \, dy$$

Example: Iterated Integral

What if we switch the order of integration?

$$\int_0^1 \int_0^1 x^2 y \, dy \, dx$$

Why does the result stay the same?

Example: Volume Problem

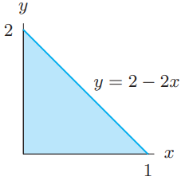
A building is 8 m wide and 16 m long. It has a flat roof 12 m high at one corner, and 10 m high at each adjacent corner. Find the volume of the building.

Integrals over Non-Rectangular Regions

We can define $\int_R f \, dA$ for other shapes (triangles, circles, etc.) by enclosing R in a rectangle and summing over subrectangles inside R.

Example

The density at (x, y) of a triangular metal plate is $\delta(x, y)$. Express its mass as an iterated integral.



Limits on Iterated Integrals

Rules:

1. The limits of the outer integral must be constants.
2. The limits of the inner integral can depend on the outer variable.

Example

A city occupies a semicircular region of radius 3 km bordering the ocean. Find the average distance from points in the city to the ocean.

Example

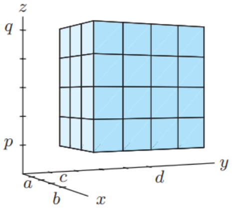
Sketch the region for

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx,$$

then reverse the order of integration.

Triple Integrals: Definition

We can define integrals over a solid region W in $\mathbb{R}^3$ by subdividing into boxes.



Using similar reasoning, express $\int_W f \, dV$ as a limit of Riemann sums.

Triple Integrals as Iterated Integrals

If f is integrable on $W = [a, b] \times [c, d] \times [p, q]$, then

$$\int_W f \, dV = \int_p^q \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz.$$

Other integration orders are also possible.

Example

Compute $\int_W (1 + xyz)\, dV$ over the cube $0 \leq x, y, z \leq 4$.

Example

Set up an iterated integral for the mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density $\delta(x, y, z) = z$.