Exercises 5

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AI503: Calculus
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Contents

1. Problem 1

1. Evalueate

$$\int_0^3 (3x+7)dx$$

as a limit of Riemann sums.

1.1. Solution

We know that the Riemann sums for a single variable is defined as

$$\int_a^b f\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

We know that for each x: $x_k = a + k\Delta x$ and that $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

So therefore

$$x_k = \frac{3k}{n}$$

$$\begin{aligned}
\int_a^b f\ dx &= \lim_{n\to\infty} \sum_{k=1}^{n} (3x+7) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(3 \cdot \frac{3k}{n} + 7\right) \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{27k}{n^2} + \frac{21}{n}\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{27k}{n^2} + \sum_{k=1}^{n} \frac{21}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{27}{n^2} \sum_{k=1}^{n} k + \frac{21}{n} \sum_{k=1}^{n} 1\right)
\end{aligned}$$

We know the formulas: $\sum_{k=1}^{n} = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} 1) = n$

So...

$$\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{27}{n^2} \frac{n(n+1)}{2} + \frac{21}{n} n\right) \\
&= \lim_{n\to\infty} \left(\frac{27(n+1)}{2} n + 21\right) \\
&= \lim_{n\to\infty} \left(\frac{27(n+1)}{2} n + 21\right) \\
&= \lim_{n\to\infty} \left(\frac{27n}{2n} + \frac{27}{2n} + 21\right) \\
&= \lim_{n\to\infty} \left(\frac{27}{2} + \frac{27}{2n} + 21\right) = \frac{27}{2} + 0 + 21 = \frac{27}{2} + \frac{42}{2} = 34.5
\end{aligned}$$

2. Problem 2

Why does the Fundamental Theorem of Calculus not imply that

$$\int_{-1}^{1} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_{-1}^{1} = \left(-\frac{1}{1} - \left(-\frac{1}{-1} = -2\right)\right)$$

2.1. Solution

Lets use the fundemental theorem of falculus on this.

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

since $\frac{1}{x^2} = x^{-2}$, then $\int \frac{1}{x^2} = \frac{x^{-2+1}}{-2+1} = x^{-1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$

then

$$\int_{-1}^{1} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_{-1}^{1} = \left(-\frac{1}{1} - \left(-\frac{1}{-1} = -2\right)\right)$$

This does not make sense because f has a assymtote.

$$f(x) \to \infty \text{ as } x \to 0^- \text{ or } \ x \to 0^+$$

3. Problem 3

Find a function $g(x)$ such that $g'(x) = \sqrt{1+x^2}$ and $g(2) = 0$.

3.1. Solution

$$g(x) = \int \sqrt{1+x^2}\ dx$$

$$\int \sqrt{1+x(u)^2} \cdot \frac{du}{dx}\ dx$$

$$\int \sqrt{1+\sinh^2(u)} \cdot \cosh(u)\ dx$$

$$\int \sqrt{\cosh^2(u)} \cdot \cosh(u)\ dx$$

$$\int \cosh(u) \cdot \cosh(u)\ du$$

$$\int \cosh^2(u)\ du$$

since $\cos^2(x) = \frac{1}{2}(\cos(u) - 1)$

$$\int \cosh^2(u) = \frac{1}{2} \int (\cosh(u) - 1)\ du$$

$$\frac{1}{2} \int \cosh(u) - \int 1\ du$$

4. Problem 4

Find $\frac{d}{d}x \int_{x^2}^{3} \frac{\sin(t)}{t}\ dt$.

4.1. Solution

we know that

$$\frac{d}{dx} \int_{x^2}^{3} \frac{\sin(t)}{t}\ dt = -\frac{d}{dx} \int_{3}^{x^2} \frac{\sin(t)}{t}\ dt$$

Use u substitution

$$u = x^2 \qquad du = 2x$$

then

$$-\frac{d}{du} \left(\int_{3}^{u} \frac{\sin(t)}{t}\ dt \right) \cdot \frac{du}{dx}$$

since $\frac{d}{dx} \int_{c}^{x} f(t)\ dt = f(u)$

$$\frac{\sin(t)}{t}\ dt \cdot \frac{du}{dx} = \frac{\sin(x^2)}{x^2} 2x = \frac{2\sin(x)}{x}$$

5. Problem 5

True or False? Give reasons for your answer.

(a) The double integral $\int_R f\ dA$ is always positive
(b) If $f(x, y) = k$ for all points in a region R, then $\int_R f\ dA = k \cdot \text{Area}R$
(c) If $\int_R f\ dA = 0$ then $f(x, y) = 0$ at all points of R

5.1. Solution

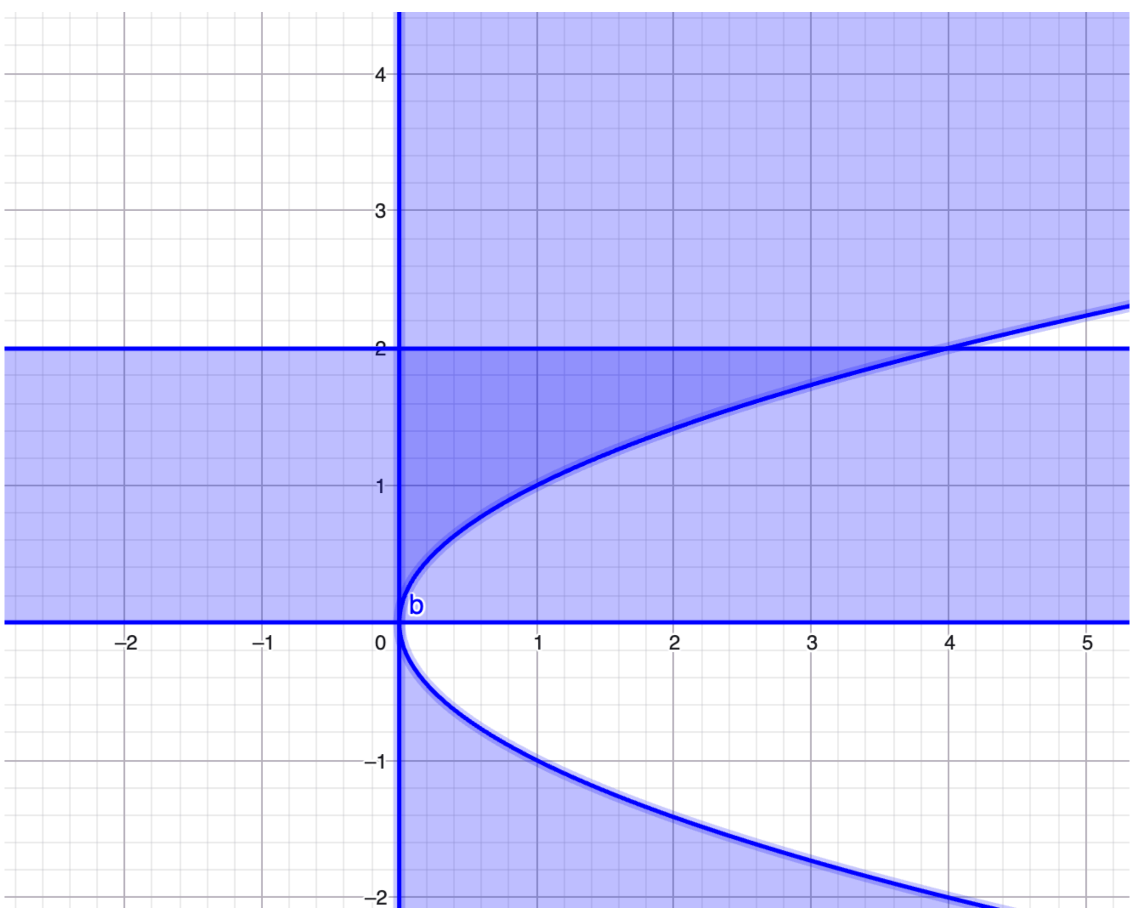
(a) Yes, with a function f inside the $\int$ sign anything is possible
(b) Yes, since f is a constant k, then $k \int_R f\ dA$
(c) No, it can be 0 but doesnt have to.

6. Problem 6

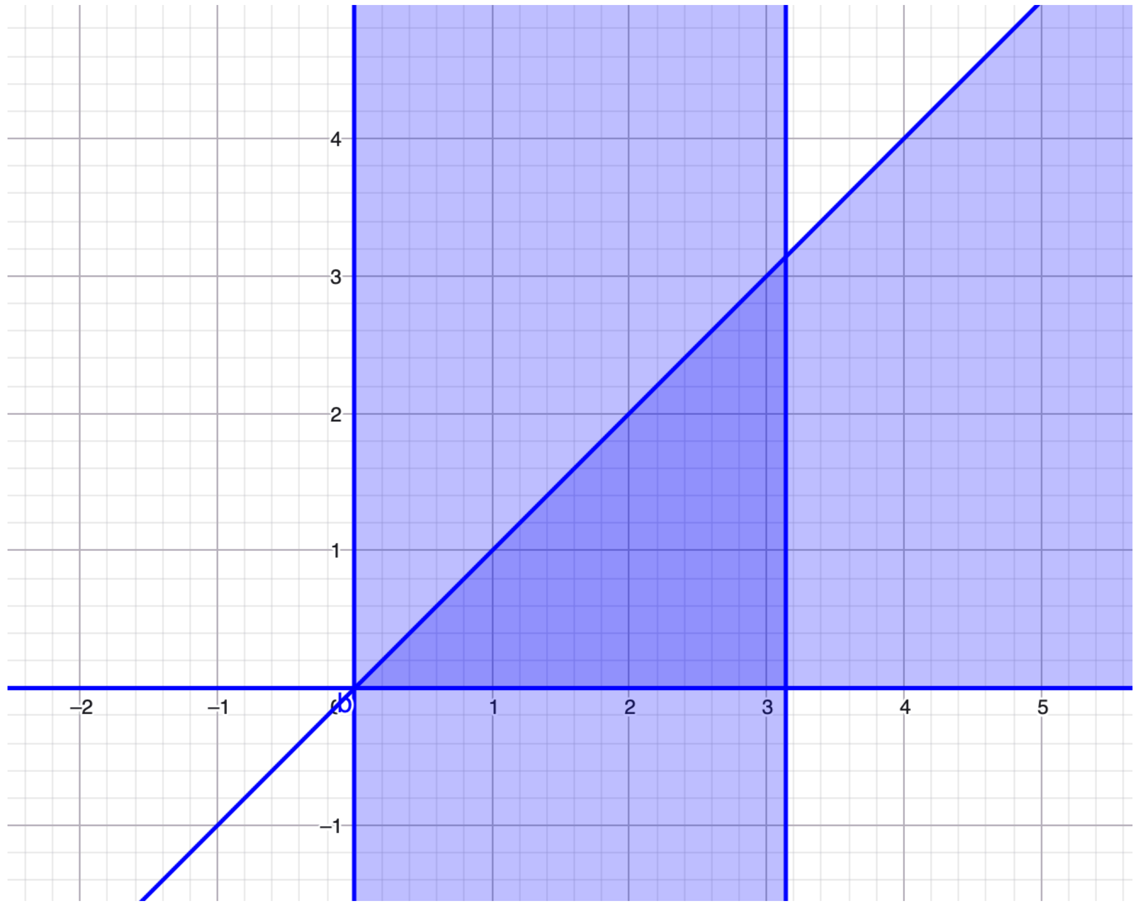
Sketch the region of integration
(a) $\int_0^2 \int_0^{y^2} y^2 x dy dx$
(b) $\int_0^{\pi} \int_0^{x} y \sin(x) dy dx$

6.1. Solution

(a) We know that $0 \leq x \leq y^2$ and that $0 \leq y \leq 2$



(b) We know that $0 \leq y \leq x$ and that $0 \leq x \leq \pi$



7. Problem 7

A city occupies a semicircular region of radius 3 km bordering on the ocean. Find the average distance from points in the city to the ocean.

7.1. Solution

This was done in class

8. Problem 8

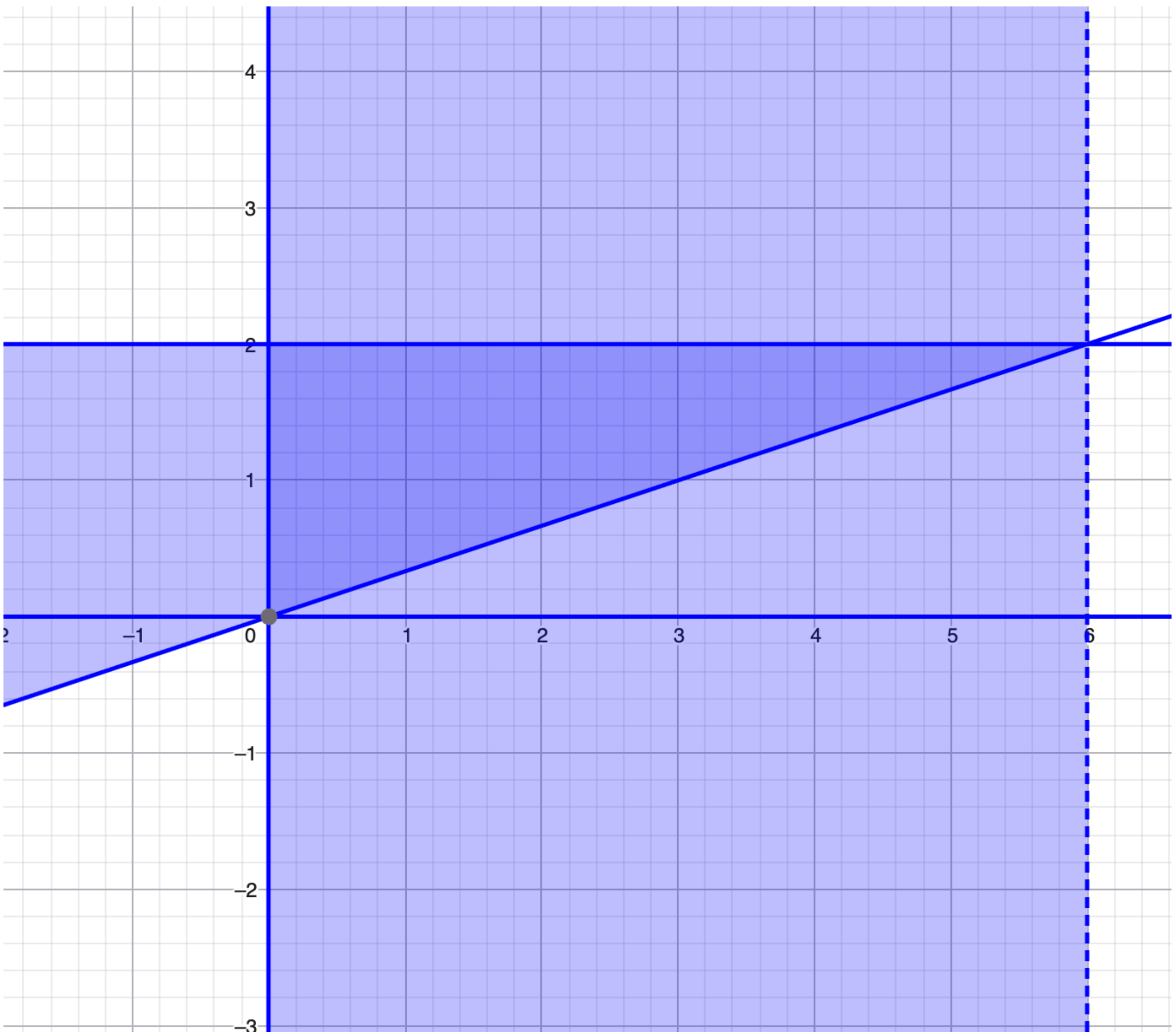
Sketch the region of integration for the iterated integral

$$\int_0^6 \int_{\frac{\pi}{3}}^2 x\sqrt{y^3+1}dydx$$

Then reverse the order of integration.

8.1. Solution

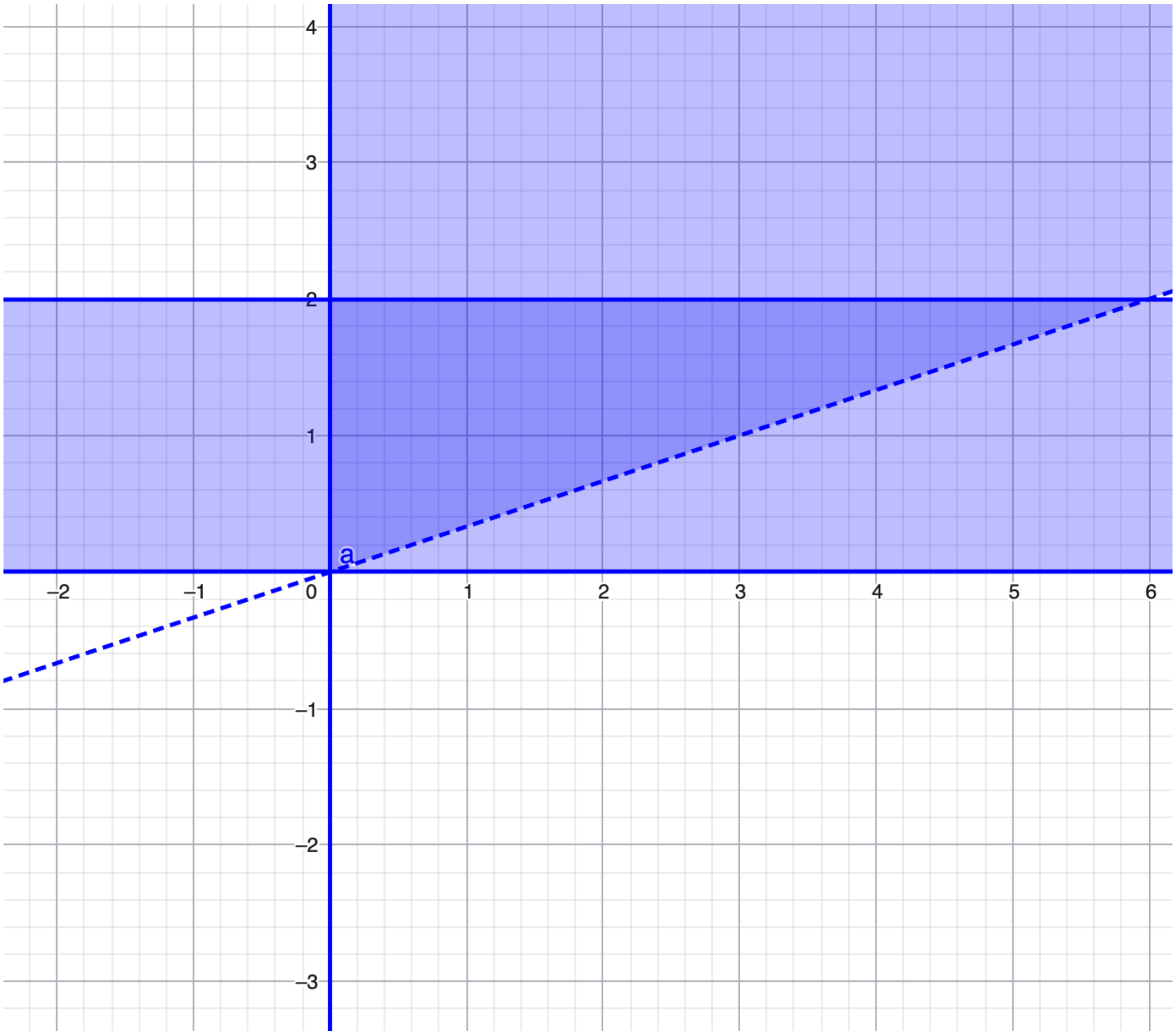
We know that $\frac{x}{3} \leq y \leq 2$ and that $0 \leq x \leq 6$



To reverse the orde, first swich the boundarie $\frac{x}{3}$ beacuse it depends on x. so...

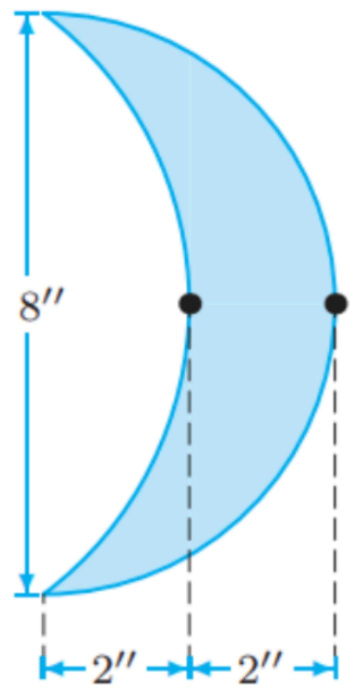
$$y = \frac{x}{3} \Leftrightarrow x = 3y$$

Now we know that $0 \leq y \leq 2$ and that $0 \leq x \leq 3y$



9. Problem 9

Find the area of the crescent-moon shape with circular arcs as edges and the dimensions shown in figure below.



9.1. Solution

We know that

$$A = A_{\text{small}} - A_{\text{segm}}$$

$$A = \frac{1}{2}\pi 4^2 - A_{\text{segm}}$$

So we have to find find the segment of the bigger circle

We know that $A_{\text{seg}} = \frac{1}{2}\theta R^2 - \frac{1}{2}ad$. (from)

and since that $R^2 = d^2 + \left(\frac{a}{2}\right)^2 = (R-h)^2 + \left(\frac{a}{2}\right)^2$

$$R^2 = R^2 - 2hR + h^2 + \frac{a^2}{4} \implies R = \frac{1}{2h}\left(h^2 + \frac{a^2}{4}\right) = \frac{1}{2}\left(h + \frac{a^2}{4h}\right)$$

since $\sin\left(\frac{\theta}{2}\right) = \frac{\frac{a}{2}}{R}$

$$\theta = 2\sin^{-1}\left(\frac{a}{2R}\right)$$

$$A_{\text{seg}} = \sin^{-1}\left(\frac{ah}{h^2 + \frac{a^2}{4}}\right) \cdot \left(\frac{h}{2} + \frac{a^2}{8}h\right) - \frac{1}{2}a\left(\frac{a^2}{8h} - \frac{h}{2}\right)$$

$$A = \frac{1}{2}\pi 4^2 - 12.68$$