Differentiation (continued)

AI503
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Lecture 3

Warm up exercises

Without looking at your notes, write out the following

1. Definition of $\lim_{x \to c} f(x) = L$
2. Definition of a function f continuous at $x = c$
3. Definition of the derivative of f at a point $x = a$
4. Definition of the second derivative

Review

Definition

We say $\lim_{x \to c} f(x) = L$ if we can get $f(x)$ as close to L as we want by taking x sufficiently close to c (but $x \neq c$).

Formally, for any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

Definition

f is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

f is continuous on $[a, b]$ if continuous at every point in $[a, b]$.

Review

Definition

The *derivative* of a function at a point $x = a$ is, rate of change of f at a, defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists, f is differentiable at a.

Definition

The second derivative of a function f is the derivative of the derivative function of f.

$$f''(x) = (f'(x))' = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Review

Intermediate Value Theorem

If f continuous on $[a, b]$ and k between $f(a)$ and $f(b)$, there exists c s.t. $f(c) = k$.

Mean Value Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, $f(b) - f(a) = f'(c)(b - a)$.

Why is f not assumed to be differentiable at $x = a$ or $x = b$?

Review

Theorem

1. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.
3. If $f'(x) = 0$ on an interval, then $f(x)$ is constant on that interval.

Theorem

1. If $f'' > 0$ on an interval, then f' is increasing over that interval, so the graph of f is concave up.
2. If $f'' < 0$ on an interval, then f' is decreasing over that, so the graph of f is concave down.
3. If $f'' = 0$ on an interval, then f' is constant on that interval, so the graph of f is a straight line (linear).

Careful with the converse statements: $\geq, \leq$ instead of $>, <$.

Review

Tangent Line Approximation

Suppose f is differentiable at a. Then, for values of x near a, the tangent line approximation of $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

The expression $f(a) + f'(a)(x - a)$ is called the *local linearization* of f near $x = a$.
The error, $E(x)$, defined by $E(x) = f(x) - f(a) - f'(a)(x - a)$, can be approximated near $x = a$ by

$$E(x) \approx \frac{f''(a)}{2}(x - a)^2.$$

Optimization

The First Derivative Test

Let x_0 be a critical point of a continuous function f.

- If f' changes from positive to negative at x_0, then f has a local **maximum**.
- If f' changes from negative to positive at x_0, then f has a local **minimum**.

True or False? If $f'(x_0) = 0$, then f has a local max or min at x_0. Explain.

Optimization

The First Derivative Test

Let x_0 be a critical point of a continuous function f.

- If f' changes from positive to negative at x_0, then f has a local **maximum**.
- If f' changes from negative to positive at x_0, then f has a local **minimum**.

True or False? If $f'(x_0) = 0$, then f has a local max or min at x_0. Explain.

Important: All local extrema occur either at the end points of the domain or at the critical points. But not all critical points are local extrema.

The Second Derivative Test:

Let x_0 be a critical point of a continuous function f.

- If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local **minimum** at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a local **maximum** at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test is inconclusive.

Examples

1. Use the second derivative test to classify the critical points of $f(x) = x^3 - 3x + 4$.

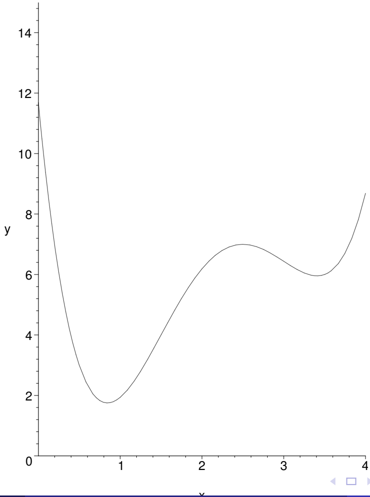
Local vs. Global Extrema

- A **global maximum** of f is the greatest value of f over its domain (or a specified interval).
- A **global minimum** is the least value of f over its domain (or a specified interval).
- Local extrema only describe behavior *near a point*, while global extrema compare values over the entire interval/domain.

Example: Global vs. Local Extrema

Consider the function below:

- Identify the local minima and maxima.
- Identify the global minimum and maximum.



Procedure for Finding Global Extrema

Use the previous graph to guide you, what are the possible locations for the global extrema?

To find the global max and min of f on $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at the critical points and endpoints a, b.
3. Compare values:
   - Largest = global maximum
   - Smallest = global minimum

Practice: Global Extrema

1. Find the global max and min of $g(x) = \ln(1 + x^2)$ on $-1 \leq x \leq 2$.

Extreme Value Theorem

Theorem

If f is continuous on a closed finite interval $[a, b]$, then f is guaranteed to have both a global maximum and a global minimum on that interval.

- Endpoints a and b are possible locations for global extrema.
- Critical points in (a, b) are also candidates.

Why Hypotheses Matter

- Let us draw the graph of $g(x) = \frac{1}{x}$. Does g have a global maximum over the interval $0 < x \leq 1$?
- Let us draw the graph of $h(x) = x^2$. Does h have a global maximum over the interval $0 \leq x < \infty$?

What happens when the domain is neither closed nor finite

Find the global max and min of $f(t) = te^{-t}$ for $t \geq 0$.

From $\mathbb{R}$ to $\mathbb{R}^n$

- Single variable: number line $\mathbb{R}$
- Multivariable: vector space

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$$

- Visualization:
  - $\mathbb{R}^2$: plane
  - $\mathbb{R}^3$: space
  - $\mathbb{R}^n$: abstract

Vectors in $\mathbb{R}^n$

Points in $\mathbb{R}^n$ are **vectors**.

Definition

A **vector** $\mathbf{a} \in \mathbb{R}^n$ is an ordered n-tuple of real numbers

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)$$

Geometrically, vectors can be thought of as arrows emanating from the origin.

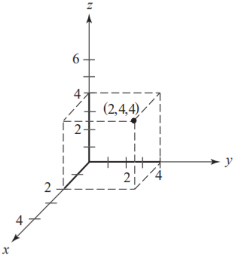
- Magnitude/length of **a**:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$$

- A unit vector has magnitude 1

Example

Draw the vector $(2, 4, 4)$ on the following graph and compute its length.



Scalar Multiplication

Definition

Let $\mathbf{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$c\mathbf{a} = (ca_1, ca_2, \ldots, ca_n)$$

Geometrically, $c\mathbf{a}$ has length $|c|\|\mathbf{a}\|$, same direction if $c > 0$, opposite if $c < 0$.

Example

Let $\mathbf{a} = (1, 1)$.

1. Draw $\mathbf{a}, 2\mathbf{a}$, and $-\mathbf{a}$.
2. How can $-\mathbf{a}$ be normalized to a unit vector?

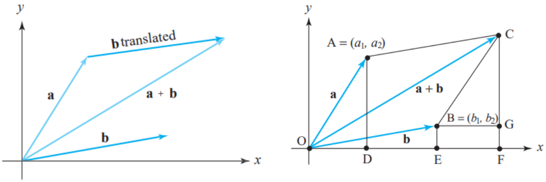
Vector Addition

Definition

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_n + b_n)$$

Geometrically: head-to-tail or parallelogram rule.



Exercise

Draw $\mathbf{b} - \mathbf{a}$.

Displacement Vector

Definition

The displacement vector from $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, \ldots, x_n - y_n).$$

The distance between $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Dot Product in $\mathbb{R}^n$

Definition

Let $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$.

- **Geometric:**
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta,$$
where θ is the angle between the vectors (generalized from 3D).
- **Algebraic:**
$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

1. Is the dot product a number or a vector? (Answer: a scalar)
2. The dot product measures alignment of vectors.

Example

Let

$$\mathbf{v} = (3, 4), \quad \mathbf{w} = (-4, 3)$$

1. Compute $\mathbf{v} \cdot \mathbf{w}$
2. Use (1) to deduce that the vectors form a right angle
3. Draw the two vector on the plane to confirm (2)

Example

Let

$$\mathbf{v} = (3, 4), \quad \mathbf{w} = (-4, 3)$$

1. Compute $\mathbf{v} \cdot \mathbf{w}$
2. Use (1) to deduce that the vectors form a right angle
3. Draw the two vector on the plane to confirm (2)

Dot product zero $\implies$ vectors are perpendicular

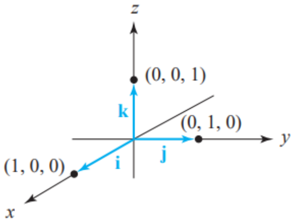
Standard Basis Vectors in $\mathbb{R}^3$

Let

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

Check that any $\mathbf{a} = (a_1, a_2, a_3)$ can be written as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$



Exercise

Can you give the standard basis vectors in $\mathbb{R}^n$?

Multivariate functions

Definition

A **function of n variables** is a function whose domain is a subset A of $\mathbb{R}^n$ and whose range is a subset B of $\mathbb{R}$.

The reason we say "n variables" is simply that we regard the coordinates of a point $\mathbf{x} = (x_1, \ldots, x_n) \in A$ as n variables, and $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ depends on these variables.

Examples

What are the domain and range for each of these functions?

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = (x^2 + y^2) \log(xy)$
3. $f(x, y, z) = \log(z) x^2 y^2$