Exercises 2

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AI503: Calculus
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Contents

1. Problem 1

A company's profit function is given by

$$P(x) = -2x^2 + 12x - 5 \tag{1}$$

where x is the number of units produced. Find the production level that maximizes profit.

1.1. Solution

Find $f'(x)$

$$f'(x) = -4x + 12 \tag{2}$$

Then

$$0 = -4x + 12 \Rightarrow x = \frac{12}{4} = 3 \tag{3}$$

Since $f''(x) = -4 < 0$ graph is a concave down everywhere

Because of this $x = 3$ must be the maximum

2. Problem 2

Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$. Compute:

(a) $\vec{u} + \vec{v}$

(b) $2\vec{u} - 3\vec{v}$

(c) $\vec{u} \bullet \vec{v}$

2.1. Solution

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \mathbf{5} \\ \mathbf{1} \\ \mathbf{5} \end{pmatrix}$

(b) $2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 12 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} \mathbf{-10} \\ \mathbf{7} \\ \mathbf{0} \end{pmatrix}$

(c) $\vec{u} \bullet \vec{v} = 1 \cdot 4 + 2 \cdot (-1) + 3 \cdot 2 = \mathbf{8}$

3. Problem 3

Find the length (norm) of the vector $\vec{w} = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix}$. And normalize it to a unit vector.

3.1. Solution

The length of vector is definde as

$$\|\vec{w}\| = \sqrt{2^2 + (-1^2) + 2^2 + 2^2} = \sqrt{13} \tag{4}$$

To normalize $\vec{w}$, set the length to 1 by: $\vec{w}_u = \frac{1}{\sqrt{13}} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} \\ -\frac{1}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}$

Now $\|\vec{w}\| = \sqrt{\left(\frac{2}{\sqrt{13}}\right)^2 + \left(-\frac{1}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2} = 1$

4. Problem 4

Let $a = (1, 0, -1)$ and $b = (2, 3, 1)$. Compute the angle between a and b.

4.1. Solution

Dot product is defined as

$$a \bullet b = \|a\| \|b\| \cdot \underbrace{\cos(\theta)}_{\angle(a,b)} \tag{5}$$

So.

$$\theta = \cos^{-1}\left(\frac{a \bullet b}{\|a\| \|b\|}\right) \tag{6}$$

Now to find θ between a and b

$$\theta = \cos^{-1}\left(\frac{2-1}{\sqrt{2} \cdot \sqrt{14}}\right) \tag{7}$$

5. Problem 5

Find a linear function whose graph is the plane intersects the xy-plane along the line $y = 2x + 2$ and contains the point $(1, 2, 2)$

5.1. Solution

The plane equation in 3d:

$$ax + by + cx = d \tag{8}$$

Where $y = 2x + 2$ and z is fixed to $z = 0$ at

So then

$$ax + b(2x + 2) + c(0) = ax + 2bx + b2) = d \tag{9}$$

This can be rewritten to

$$(a + 2b)x + b2 = d \tag{10}$$

Since d is fixed: $a + 2b = 0$, $a = -2b$ and $d = 2b$

Now to choose some arbitrary b-value $b = 1$ (for simplicity)

Then

$$-2x + y + cz = 2 \tag{11}$$

This plane is not fixed in the point yet, lets do that and find c to fix the plane to the point

$$(1, 2, 2) \Rightarrow -2(1) + 2 + c2 = 2 \Rightarrow c2 = 2 \Rightarrow c = 1 \tag{12}$$

Finally

$$\boldsymbol{-2x + y + z = 2} \tag{13}$$

6. Problem 6

Determine if z is a function of x and y:

(a) $6x - 4y + 2z = 10$

(b) $x^2 + y^2 + z^2 = 100$

(c) $3x^2 - 5y^2 + 5x = 10 + x + y$

6.1. Solution

(a)

$$\begin{aligned} 6x - 4y + 2z &= 10 \\ 2z &= -6x + 4y + 10 \\ \boldsymbol{z} &= \boldsymbol{-3x + 2y + 5} \quad \checkmark \end{aligned} \tag{14}$$

(b)

$$\begin{aligned} x^2 + y^2 + z^2 &= 100 \\ z^2 &= 100 - x^2 - y^2 \\ \boldsymbol{z} &= \boldsymbol{\sqrt{100 - x^2 - y^2}} \quad \checkmark \end{aligned} \tag{15}$$

(c)

$$\begin{aligned} 3x^2 - 5y^2 + 5x &= 10 + x + y \\ 5z &= -3x^2 - 5y^2 + 10 + x + y \\ \boldsymbol{z} &= \boldsymbol{\frac{-3x^2 - 5y^2 + 10 + x + y}{5}} \quad \checkmark \end{aligned} \tag{16}$$

7. Problem 7

Show that the following limit does not exits

$$\lim_{(x,y)\to(0,0)} \frac{x + y^2}{2x + y} \tag{17}$$

7.1. Solution

Try different paths $x = 0$ and $y = 0$

First $x = 0$

$$\lim_{x\to 0} \frac{x}{2x} = \frac{1}{2} \tag{18}$$

Then $y = 0$

$$\lim_{y\to 0} \frac{y^2}{y} = 0 \tag{19}$$

Since $\frac{1}{2} \neq 0$ The limit does not exist

8. Problem 8

Find the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x + y}{\sin(y) + 2} \tag{20}$$

8.1. Solution

Try different paths $x = 0$ and $y = 0$

First $x = 0$

$$\lim_{x \to 0} \frac{y}{2} = 0 \tag{21}$$

Then $y = 0$

$$\lim_{y \to 0} \frac{x}{2} = 0 \tag{22}$$