Continuity and differentiation

AI503
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Lecture 2

Definition of Limit

Definition

We say

$$\lim_{x \to c} f(x) = L$$

if we can get $f(x)$ as close to L as we want by taking x sufficiently close to c (but $x \neq c$).

Formally, for any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

Note:

- The limit does not require $f(c)$ to be defined.
- The limit does not exist if $f(x)$ behaves differently from left and right.

Definition of Continuity

Definition

f is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

f is continuous on $[a, b]$ if continuous at every point in $[a, b]$.

Intuitively, you can draw the graph of a continuous function over that interval without lifting your pen from the paper!

Example

Show $f(x) = x^3$ is continuous at every $c \in \mathbb{R}$.

Proof: It is equivalent to show that for every c,

$$\lim_{x \to c} f(x) = c^3$$

Notice that

$$|f(x) - f(c)| = |x^3 - c^3| = |(x - c)(x^2 + xc + c^2)|$$

If $|x - c| < 1$, then

$$|x^2 + xc + c^2| \leq |x|^2 + |x||c| + |c|^2 \leq (|c| + 1)^2 + |c|(|c| + 1) + c^2 =: M$$

To guarantee $|x^3 - c^3| < \varepsilon$, choose

$$\delta = \min\{1, \varepsilon/M\}$$

This completes the proof.

Continuity Laws and Theorems

Continuity Laws: If f, g continuous on interval, b constant:

- $bf(x)$ is continuous
- $f(x) + g(x)$ is continuous
- $f(x)g(x)$ is continuous
- $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval
- $f(g(x))$ is continuous, provided that the composite function $f(g(x))$ is defined on the interval

Why are the continuity laws true? How can you prove them?

Intermediate Value Theorem

Theorem

If f continuous on $[a, b]$ and k between $f(a)$ and $f(b)$, there exists c s.t. $f(c) = k$.

Example: Show that $f(x) = x^3 + x - 1$ has a root in $[0, 1]$.

- Compute $f(0) = 0^3 + 0 - 1 = -1$
- Compute $f(1) = 1^3 + 1 - 1 = 1$
- 0 lies between $f(0) = -1$ and $f(1) = 1$
- By the Intermediate Value Theorem, $\exists c \in [0, 1]$ such that $f(c) = 0$

Conclusion: $f(x)$ has at least one root in $[0, 1]$.

Definition of Derivative

Definition

The *derivative* of a function at a point $x = a$ is, rate of change of f at a, defined as
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists, f is differentiable at a.

- Indicates slope of the tangent line.
- Critical point: the x such that $f'(x) = 0$ or undefined.

Example

When we evaluate the derivative of $f(x) = x^2$ at $x = 2$, we carry out the following calculation:

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} \\
&= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} \\
&= \lim_{h \to 0} \frac{4h + h^2}{h} \\
&= \lim_{h \to 0} (4 + h) = 4
\end{aligned}
$$

Derivative Function

Important: $f'(x)$ is a function itself!

Definition

Given a function $f(x)$, its *derivative function* is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Also denoted y', $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$

Derivative Shortcuts

1. Linearity: For constants a, b and functions f, g,

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$$

2. Power Rule: For $n \in \mathbb{R}$,

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

3. Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

4. Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad g(x) \neq 0$$

Derivative Shortcuts

5. Chain Rule: If $y = f(g(x))$,

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

6. Exponential and Logarithm:

$$\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[a^x] = a^x \ln a$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad \frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$$

7. L'Hôpital's Rule: For $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}, \quad \text{if limit exists.}$$

Relation: Differentiable $\implies$ Continuous

Theorem

A differentiable function is always continuous.

Is the converse true?

Counterexample: $f(x) = |x|$ is continuous everywhere but not differentiable at 0.

Tangent Line Approximation

The derivative of a function f at point A can be interpreted as the slope of the tangent line to the graph of the function at A.

Definition

Suppose f is differentiable at a. Then, for values of x near a, the tangent line approximation of $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

The expression $f(a) + f'(a)(x - a)$ is called the *local linearization* of f near $x = a$.

Error Analysis

The error, $E(x)$, in the approximation is defined by

$$E(x) = f(x) - f(a) + f'(a)(x - a)$$

Then

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0$$

Further, near $x = a$,

$$E(x) \approx \frac{f''(a)}{2}(x - a)^2.$$

Example

Let $f(x) = \sqrt{x}$.

1. Find $f'(4)$.

2. Now give the equation for the tangent line at the point $x = 4$.

3. Draw the graph of $f(x)$ for $0 \leq x \leq 10$. On your drawing, add in the tangent line you just calculated.

4. Use the equation of the tangent line to approximate $\sqrt{4.1}$.

5. Compute the error in 4.

6. More generally, let $E(x)$ be the error in the tangent line approximation to f for x near 4. What does a table of values for $E(x)/(x - 4)$ suggest about $\lim_{x \to 4} E(x)/(x - 4)$?

7. Make another table to see that $E(x) \approx k(x - 4)^2$. Estimate the value of k. Check that a possible value is $k = f''(4)/2$.

A Graphing Exercise

Examine the graph of a function $f(x)$ below. Sketch a graph of $f'(x)$.

Positive and Negative Derivative

Theorem

1. If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.
3. If $f'(x) = 0$ on an interval, then $f(x)$ is constant on that interval.

Mean Value Theorem

Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In other words, $f(b) - f(a) = f'(c)(b - a)$.
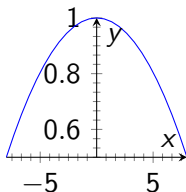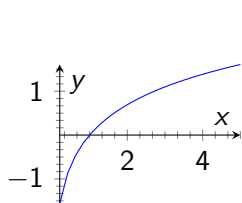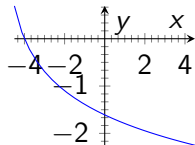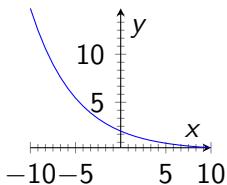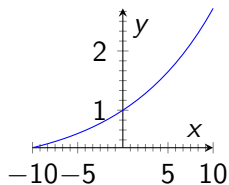
Second Derivative

Definition

The second derivative of a function f is the derivative of the derivative function of f.

$$f''(x) = (f'(x))' = \lim_{h \to 0} \frac{f'(x + h) - f'(x)}{h}$$

- Indicates concavity.
- Inflection point: where concavity changes. Check when $f''(x) = 0$ or undefined. WARNING: Not every point where $f''(x) = 0$ is an inflection point. Example: $f(x) = x^4$.

Example

For each of the following functions, draw tangent lines to the curve on the left-hand side of the graph, in the middle, and on the right-hand side of the graph. What does the trend of your slopes tell you about the sign of the second derivative? Why?

Second Derivative and Concavity

1. If $f'' > 0$ on an interval, then f' is increasing over that interval, so the graph of f is concave up.

2. If $f'' < 0$ on an interval, then f' is decreasing over that, so the graph of f is concave down.

The converse of the above statements are NOT true! Think $f(x) = x^4$.

So, for a general function f twice differentiable:

- If the graph of f is concave up, then $f'' \geq 0$
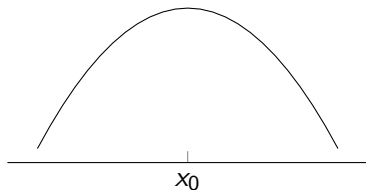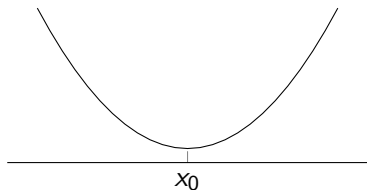- If the graph of f is concave down, then $f'' \leq 0$.

Optimization: Local Extrema

Let a be a point in the domain of a differentiable function f. Then

- f has a **local maximum** at a if $f(a) \geq f(x)$ for all x near a.
- f has a **local minimum** at a if $f(a) \leq f(x)$ for all x near a.

We use the term extrema to refer to both maxima and minima.

Critical Points and Graphs

What happens to f' around x_0 in each of the graphs?

The First Derivative Test

All local extrema occur either at the end points of the domain or at the critical points.

The First Derivative Test

Let x_0 be a critical point of a continuous function f.

- If f' changes from positive to negative at x_0, then f has a local **maximum**.
- If f' changes from negative to positive at x_0, then f has a local **minimum**.

Examples

1. Find and classify all of the local extrema of $f(x) = x^3 - 3x + 4$.
2. True or False? If $f'(x_0) = 0$, then f has a local max or min at x_0. Explain.

The Second Derivative Test

All local extrema occur either at the end points of the domain or at the critical points.

The Second Derivative Test:

Let x_0 be a critical point of a continuous function f.

- If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local **minimum** at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a local **maximum** at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test is inconclusive.

Examples

1. Use the second derivative test to classify the critical points of $f(x) = x^3 - 3x + 4$.
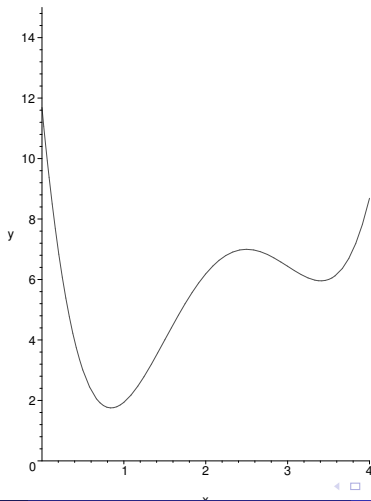2. Consider $f(x) = (x^2 - 4)^7$. Find and classify all local extrema.

Local vs. Global Extrema

- A **global maximum** of f is the greatest value of f over its domain (or a specified interval).
- A **global minimum** is the least value of f over its domain (or a specified interval).
- Local extrema only describe behavior *near a point*, while global extrema compare values over the entire interval/domain.

Example: Global vs. Local Extrema

Consider the function below:

- Identify the local minima and maxima.
- Identify the global minimum and maximum.

Procedure for Finding Global Extrema

Use the previous graph to guide you, what are the possible locations for the global extrema? They appear at local extrame!

To find the global max and min of f on $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at the critical points and endpoints a, b.
3. Compare values:
   - Largest = global maximum
   - Smallest = global minimum

1. Find the global max and min of $f(x) = x(x-1)$ on $0 \le x \le 3$.
2. Find the global max and min of $f(x) = x^3 - 7x + 6$ on $-4 \le x \le 2$.
3. Find the global max and min of $g(x) = \ln(1 + x^2)$ on $-1 \le x \le 2$.

Extreme Value Theorem

Theorem

If f is continuous on a closed finite interval $[a, b]$, then f is guaranteed to have both a global maximum and a global minimum on that interval.

- Endpoints a and b are possible locations for global extrema.
- Critical points in (a, b) are also candidates.

- Let us draw the graph of $g(x) = \frac{1}{x}$. Does g have a global maximum over the interval $0 < x \leq 1$?
- Let us draw the graph of $h(x) = x^2$. Does h have a global maximum over the interval $0 \leq x < \infty$?

Why Hypotheses Matter

- $g(x) = \frac{1}{x}$ on $0 < x \leq 1$ does not have a global maximum (domain not closed).
- $h(x) = x^2$ on $0 \leq x < \infty$ does not have a global maximum (domain not bounded).
- Both fail the Extreme Value Theorem assumptions.

Find the global max and min of $f(t) = te^{-t}$ for $t \geq 0$. (Hint: careful here, as the domain is neither closed nor finite)