Exercises 1

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AI503: Calculus
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Contents

1. Problem 1

What are the domain and range of the following functions?

(a) $g(x) = \frac{1}{x}$
(b) $f(x) = \sqrt{x-3}$

1.1. Solution

(a) since $g(x) = \text{DNE}$ when $x = 0$

Therefore the domain is $x \in \mathbb{R} \setminus \{0\}$

(b) I assume that $f(x)$ is only defined in non-complex numbers (not i)

The domain is therefore: $x \in [3, \infty)$

2. Problem 2

Let

$$f(x) = \begin{cases} x - 5 & x \geq 1, \\ -3x & x < 1 \end{cases} \tag{1}$$

Evaluate $\lim_{x \to 1} f(x)$

2.1. Solution

To evaluate the limit check right hand limit $x \to 1^+$ and left hand limit $x \to 1^-$

$$\lim_{x \to 1^+} f(x) \to 1 - 5 = -4 \tag{2}$$

$$\lim_{x \to 1^-} f(x) \to -3 \cdot 1 = -3 \tag{3}$$

And since $-4 \neq -3$ the limit $\lim_{x \to 1} f(x)$ **does not exits**

3. Problem 3

Compute:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} \tag{4}$$

3.1. Solution

On the surface, this looks complicated but since

$$\frac{x^2 - 4}{x - 2} = \frac{x^2 + 2x - 2x - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2 \tag{5}$$

So actually

$$\lim_{x \to 2} x + 2 = 4 \tag{6}$$

4. Problem 4

Evaluate

$$\lim_{x \to 0} \frac{\sin(5x)}{x} \tag{7}$$

4.1. Solution

Since $\frac{\sin(5 \cdot 0)}{0}$ is undefined. Use L'Hôpital's rule that is

> **Theorem: L'Hôpital's Rule**
>
> Let f and g be functions that are differentiable on an open interval I containing a, except possibly at a itself
>
> Assume that
>
> $$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0 \tag{8}$$
>
> or that
>
> $$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty \tag{9}$$
>
> Then also assume that $g'(x) \neq 0, \forall x \neq a$
>
> The the limit is
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{10}$$

by theorem

$$\lim_{x \to 0} \frac{\sin(5x)}{x} = \lim_{x \to 0} \frac{\cos(5x)}{1} = \frac{5 \cdot \cos(0)}{1} = 5 \tag{11}$$

5. Problem 5

Determine whether

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0 \end{cases} \tag{12}$$

is continuos at $x = 0$

5.1. Solution

Yes it is, since $f(0) = 1$

6. Problem 6

Find all points where $g(x) = \frac{x^2-1}{x-1}$ is continuos.

6.1. Solution

Since $g(0)$ is undefined

This is continuos on all points exept 1, so $(-\infty, 1) \cup (1, \infty)$

7. Problem 7

Differentiate

(a) $f(x) = x^3 \sin(x)$
(b) $g(x) = \ln\left(\sqrt{1+x^2}\right)$

7.1. Solution

$\frac{d}{dx} ax^n + b$

(a) Use product rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \tag{13}$$

so..

$$\frac{d}{dx}f(x) = 3x^2 \cdot \sin(x) + x^3 \cdot \cos(x) \tag{14}$$

(b) Use chain rule

$$\frac{d}{dx}(f(g(h(x)))) = f'(g(x)) \cdot (g'(h(x)) \cdot h'(x)) \tag{15}$$

so..

$$\frac{d}{dx}g(x) = \frac{1}{\sqrt{1+x^2}} \cdot \left(\frac{1}{2}(1+x^2) \cdot 2x\right) \tag{16}$$

8. Problem 8

Find the equation of the tangent line to

$$y = e^{2x} \tag{17}$$

at $x = 0$

8.1. Solution

Use chainrule again

$$\frac{d}{dx}e^{2x} = e^{2x} \cdot 2 \tag{18}$$

at $x = 0$

$$a = 2 \tag{19}$$

This is the slope of the tangent

Now find the tangent intersect at $x = 0$

$$y(0) = e^{2 \cdot 0} = 1 \tag{20}$$

So point $(0, 1)$ is the intersect

Now for the tagentline $y - y_0 = a(x - x_0)$

This equals to

$$y - 1 = 2(x - 0) \implies \boldsymbol{y = 2x + 1} \tag{21}$$

9. Problem 9

Consider $f(x) = \left(x^2 - 4\right)^7$. Find and classify all local extrema

9.1. Solution

Use chainrule Find $f'(x) = 0$

$$0 = 7\left(x^2 - 4\right)^6 \cdot 2x = 14x\left(x^2 - 4\right)^6 \tag{22}$$

Use 0 rule. either $14x$ or $\left(x^2 - 4\right)^6$ equals 0 for the result to equal 0

So..

$$14x\left(x^2 - 4\right)^6 \Rightarrow x = 0 \quad \text{or} \quad x^2 - 4 \Rightarrow x = \pm 2 \tag{23}$$

So critical points are $= -2, 0, 2$.

10. Problem 10

Find the global max and min of $f(x) = x^3 - 7x + 6$ on $-4 \leq x \leq 2$.

10.1. Solution

For this find $f'(x)$ and choose max and min.

$$f'(x) = 3x^2 - 7 \tag{24}$$

Now solve for $f'(x) = 0$

$$x = \pm\sqrt{\frac{7}{3}} \tag{25}$$

So see which of these are the max and min

$$f''(x) = 6x \Rightarrow 6 \cdot \sqrt{\frac{7}{3}} > 0 \quad \text{and} \quad 6 \cdot \left(-\sqrt{\frac{7}{3}}\right) < 0 \tag{26}$$