Continuity and differentiation of multivariate functions

AI503
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Lecture 5

Limit (ε-δ Definition)

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} = (a_1, \ldots, a_n)$. We say

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \quad \Longrightarrow \quad |f(\mathbf{x}) - L| < \varepsilon.$$

- As $\mathbf{x}$ gets arbitrarily close to $\mathbf{a}$, $f(\mathbf{x})$ gets arbitrarily close to L.
- Approaching a point $\mathbf{a}$ can occur along infinitely many paths:

Continuity

Definition

f is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

A function is continuous on a region U in $\mathbb{R}^n$ if it is continuous at each point in U. *There is no break in the graph of f!*

Using the limit laws, one can show that for functions of several variables, the following are continues

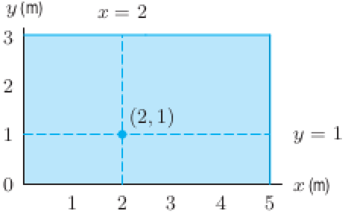
1. sums of continuous functions
2. products of continuous functions
3. compositions of continuous functions
4. quotient of two continuous functions wherever the denominator function is nonzero.

Examples

Is the function continuous in the given region?

1. $\dfrac{1}{x^2+y^2}$ on $-1 \le x, y \le 1$
2. $\dfrac{1}{x^2+y^2}$ on $1 \le x, y \le 2$
3. $\dfrac{y}{x^2+2}$ on $x^2+y^2 \le 1$

Imagine an unevenly heated thin rectangular metal plate lying in the xy-plane. Temperature at (x, y) is $T(x, y)$.



y (m)						
3	85	90	**110**	135	155	180
2	100	110	**120**	145	190	170
1	**125**	**128**	**135**	**160**	**175**	**160**
0	120	135	**155**	160	160	150
x (m)	0	1	2	3	4	5

How does T vary near the point $(2, 1)$?

Partial Derivatives Intuition

Let's make the intuition more precise...

1. Along $y = 1$, consider $u(x) = T(x, 1)$.
2. By definition,

$$u'(2) = \lim_{h \to 0} \frac{T(2+h, 1) - T(2, 1)}{h}.$$

   Then $u'(2)$ measures rate of change of T in x direction at $(2, 1)$.
3. Denote $u'(2)$ as $T_x(2, 1)$. Approximate $T_x(2, 1)$ with $h = 1$.

Estimate rate of change in y direction.

Definition of Partial Derivatives

Partial derivatives of f at (a, b)

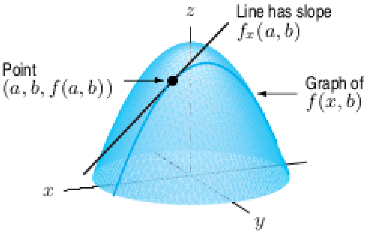
For all points at which the limit exist, we define the partial derivatives at the point (a, b) by

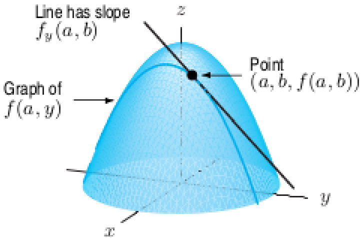
$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

- Functions of (x, y): $f_x(x, y)$, $f_y(x, y)$.
- Alternative notation: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.
- Measures the rate of change in x or y direction.

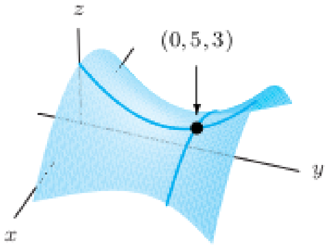
Geometric interpretation





Partial derivatives are the slope of tangent lines of the cross section.

Sign of Partial Derivatives



1. Sign of $f_x(0,5)$?
2. Sign of $f_y(0,5)$?

Computing Partial Derivatives

Since f_x and f_y are derivatives with one variable fixed, we can use ordinary differentiation formulas. Let

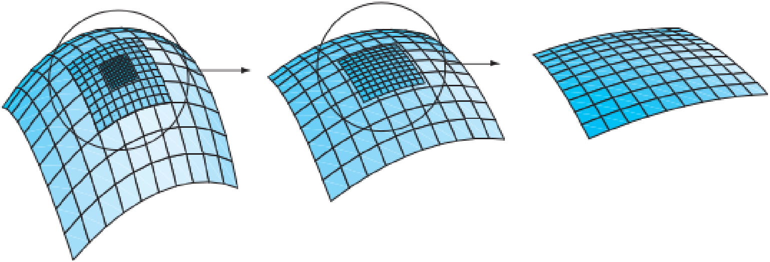
$$f(x,y) = \frac{x^2}{y+1}.$$

Find $f_x(3,2)$.

Tangent Plane

Seeing a plane when we zoom in at a point tells us (provided the plane is not vertical) that $f(x, y)$ is closely approximated near that point by a linear function, $L(x, y)$.

The graph of the function $z = L(x, y)$ is the *tangent plane*.



Tangent Plane

Tangent Plane Equation

At (a, b), if f is differentiable:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example

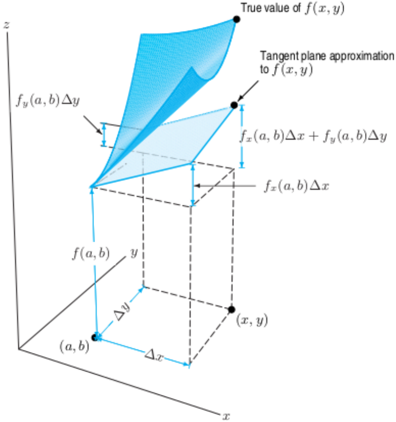
Find tangent plane to $f(x, y) = x^2 + y^2$ at $(3, 4)$.

Local Linearity

Tangent Plane Approximation

For (x, y) near (a, b):

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$



Example

Let $f(x,y) = x^2 + y^2$ and call the tangent plane you computed at (3,4) $L(x,y)$. We define the error function

$$E(x,y) = f(x,y) - L(x,y).$$

1. The error near (3,4) is given by $E(3+h, 4+k)$ where h, k are small.
2. What is the distance between $(3+h, 4+k)$ and $(3,4)$?
3. Compare $|E(3+h, 4+k)|$ with $h^2 + k^2$.
4. Observe:
$$\frac{|E(x,y)|}{h^2 + k^2} \approx \text{constant}.$$
5. What can you say about the error near $(3,4)$?

Local linearity

Tangent Plane Approximation

For (x, y) near (a, b):

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Error behavior: If f has continuous second derivatives, then

$$f(x,y) - L(x,y) = O\left(\|(x,y) - (a,b)\|^2\right).$$

That is, the error is proportional to the *square of the distance* from (x, y) to (a, b).

Partial Derivatives in n Variables

Partial derivatives of f at $(x_1, \ldots, x_n)$

For all points at which the limit exist, we define the partial derivatives at the point $(x_1, \ldots, x_n)$ by

$$\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

- Compute by treating all other variables as constants.
- Example:

$$f(x, y, z) = x^2 yz$$

$$\frac{\partial f}{\partial x} = 2xyz, \quad \frac{\partial f}{\partial y} = x^2 z, \quad \frac{\partial f}{\partial z} = x^2 y$$

Example

Compute the partial derivatives of

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2.$$

Local linearity

Tangent hyperplane approximation/ First order approximation

For $\mathbf{x} = (x_1, \ldots, x_n)$ near $\mathbf{a} = (a_1, \ldots, a_n)$:

$$f(\mathbf{x}) \approx L(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)$$

Error behavior: If f has continuous second derivatives, then

$$f(\mathbf{x}) - L(\mathbf{x}) = O\left(\|\mathbf{x} - \mathbf{a}\|^2\right).$$

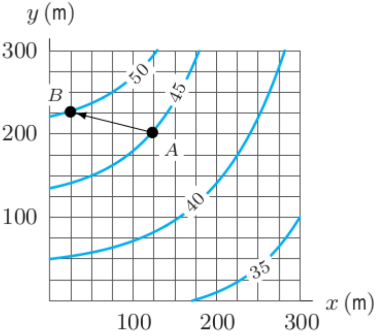
Directional Derivatives

The partial derivatives of a function f tell us the rate of change of f in the directions parallel to the coordinate axes.

How to compute the rate of change in an arbitrary direction?

Example

The figure below shows the temperature at the point (x, y). Estimate the average rate of change from point A to point B.



Question

Suppose $f(x, y)$ is an arbitrary function. How would you compute its rate of change at $P = (a, b)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$?

- For $h > 0$, consider the point $Q = (a + hu_1, b + hu_2)$
- The displacement from P is $h\mathbf{u}$.
- Since $\|\mathbf{u}\| = 1$, the distance between P, Q is h.
- The average rate of change at P is thus

$$\frac{\text{Change in } f}{\text{Distance between } P, Q} = \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

- Taking the limit as $h \to 0$ gives the instantaneous rate of change.

Definition: Directional Derivative

Directional Derivative

The **directional derivative** of f at (a, b) in the direction of a *unit vector* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$f_{\mathbf{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

Questions

When $u = \mathbf{i}$ or $u = \mathbf{j}$, what is $f_{\mathbf{u}}(a, b)$?

Questions

What happens when $\mathbf{u}$ is not a unit vector?

Questions

Calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of $\mathbf{i} + \mathbf{j}$.

Definition: Directional Derivative in n Variables

Directional Derivative

Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\mathbf{a} = (a_1, \ldots, a_n)$. The **directional derivative** of f at $\mathbf{a}$ in the direction of a *unit vector* $\mathbf{u} = (u_1, \ldots, u_n)$ is

$$f_{\mathbf{u}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1 + hu_1, \ldots, a_n + hu_n) - f(a_1, \ldots, a_n)}{h},$$

provided the limit exists.

Computing Directional Derivatives from Partial Derivatives

- Approximate $f(a_1 + hu_1, \ldots, a_n + hu_n)$ with the first-order approximation at $(hu_1, \ldots, hu_n)$
- What happens when you simplify the limit in the previous definition?

Questions

Use the preceding formula to calculate the directional derivative of $f(x, y) = x^2 + y^2$ at $(1, 0)$ in the direction of $\mathbf{i} + \mathbf{j}$.

Gradient Vector

Gradient

If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable, the gradient at (a, b) is

$$\nabla f(a, b) = (f_x(a, b), f_y(a, b))$$

Compute using Gradient

If f is differentiable at (a, b) and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector:

$$f_{\mathbf{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \nabla f(a, b) \cdot \mathbf{u}.$$

Example

Find the gradient of $f(x, y) = x + e^y$ at $(1, 1)$. Use it to compute the directional derivative in the direction of $\mathbf{i} + \mathbf{j}$.

Gradient Vector in n Variables

Gradient

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} = (a_1, \ldots, a_n)$. The **gradient** of f at $\mathbf{a}$ is

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}), \frac{\partial f}{\partial x_2}(\mathbf{a}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{a}) \right).$$

Directional Derivative via Gradient

If $\mathbf{u} = (u_1, \ldots, u_n)$ is a unit vector, the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{u}$ is

$$f_{\mathbf{u}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a}) u_i.$$

Alternative notation

$$\nabla f = \mathrm{grad} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$