Page Rank

Linear Algebra and its Applications

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- The chain is ergodic, if after exactly N steps it is possible to go from any state to any other state
- An ergodic Markov chain has a unique stable distribution (also known as the steady-state solution), which we can find from $\lim_{n\to\infty} P^N$ or as the dominating eigenvector (the Perron-Frobenius eigenvector)

Application to web page ranking - the page rank algorithm

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- Where we go next only depends on the current website: what links there are and how many of them

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- Where we go next only depends on the current website: what links there are and how many of them
- This is a Markov process, but is it ergodic?

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The following 4 slides contain important notation for the graded assignment, the matrices A, $\tilde{A}$, $\hat{A}$ and $\bar{A}$.
Given n websites and links between them,

- Define an n by n matrix A as follows:

$$A_{ij} = \begin{cases} 1, & \text{if there is a link from site } j \text{ to site } i \\ 0 & \text{otherwise} \end{cases}$$

- In graph theory, this is known as the *adjacency matrix*, because it encodes the directional links from j to i

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- In graph theory, this is known as the *adjacency matrix*, because it encodes the directional links from j to i
- This is a matrix of zeros and ones
- Note that this is a directed graph:

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- We get the modified adjacency matrix, which we dub $\tilde{A}$, defined as:

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- Expressed in plain language, we obtain $\tilde{A}$ by replacing the zero columns with columns of all ones

Another Matrix notation

- To make this into a stochastic matrix, we want to normalize each column to sum to one. This is encoded in matrix $\hat{A}$, defined as
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- $$\hat{A}_{ij} = \frac{\tilde{A}_{ij}}{\sum_j \tilde{A}_{ij}}$$
- This matrix describes a random walk following links on webpages, with the user resorting to the address bar if they get stuck
- It need not be ergodic, for example if we had 2 pairs of webpages (so 4 pages), with each pair pointing to each other

One more Matrix

- To allow the users to resort to address bar at any state, we define a new matrix $\bar{A}$, which, for an $\alpha \in [0, 1]$ is defined as
- $$\bar{A} = \alpha \hat{A} + (1 - \alpha)\frac{1}{n}\mathbf{1}_n \mathbf{1}_n^T$$

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- The second term is an $n \times n$ matrix of all entires $1/n$, and it models the part where the user quits clicking and uses the address bar instead
- The constant α is called the *dampening factor*, and it can be interpreted as the probability that the user keeps on clicking links
- The matrix $\bar{A}$ is called the *Google matrix*

Page Rank Vector

- The matrix
$$\bar{A} = \alpha\hat{A} + (1-\alpha)\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$$
is an ergodic matrix, because all its entries are positive numbers

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- it was used by google to rank popularity of webpages, which they used in their search engine with high success
- Next we investigate how we can find this page rank vector

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- This is of the form $Ax = b$, i.e. a system of linear equations, and solvable by matrix algebra (numpy)

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- In numpy:

```
import numpy as np
A=np.array(([0.7,0.4],[0.3,0.6]))
q,v=np.linalg.eig(A)
print(v[:,0]/sum(v[:,0]))
[0.57142857 0.42857143]
# c.f. the example from last week
```

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- Run the above for $k = 0, 1, \ldots, N$, or while $||x_{k+1} - x_k||_2 > \epsilon$, whichever comes first

Computational Considerations

- We need to find the maximal eigenvector of matrix

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- We did something similar with the cipher cracking last week, where we didn't explicitly construct the matrix between all the pairs of 26! permutations, but instead just neighbors

Optimized Graph Algorithms

- The python package `networkx` implements many optimized graph algorithms that we can use for more heavy computations
- This represents the matrices as dictionaries (so essentially sparse matrices)

Crash course to `networkx`

import networkx as nx

Task	Command	Notes
Initialization	DG = `nx.DiGraph()`	Initialize Directed graph DG
Add edge	`DG.add_edge('a', 'b')`	Add edge from a to b (nodes added automatically)
Add edge with weight	`DG.add_edge('a', 'c', weight=2)`	Default weight is 1
Compute Page Rank Vector	`nx.pagerank(DG, alpha=0.85)`	α is the dampening factor
Add node	`add_node()`	Add node manually
Remove node	`remove_node()`	Also removes the edges
Remove edge	`remove_edge()`	
Check node	`DG.has_node('a')`	True if node a exist
Check edge	`DG.has_edge('b', 'a')`	True if edge from b to a
List nodes	`list(DG.nodes())`	iterator inside list
List edges	`list(DG.edges())`	iterator inside list
Count nodes	`number_of_nodes()`	similarly for edges

Check the documentation for complete list

We have now covered all the content on this course!

- Any slides after this one are just interesting development around Markov theory – you are not expected to master this material
- Markov chains are very useful for both theoretical development and applications: Modeling time series, random walks, sampling, we have only scratched the surface here
- Also: Please remember to fill out the course survey on itslearning

Convergence of Markov Chains

- Recall the weather example from last time:
- $$P = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

  (Note we have flipped this so that P acts on the states by multiplying from the left
- Question: How fast does this chain converge to the stable distribution?

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- If $x = Px$, then necessarily $x = Px = PPx$, and hence $x = P^N x$ for any $N = 1, 2, \ldots$

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- On the other hand, we also have that the sequence $x, P^2x, (P^2)^2x, (P^2)^3x, \ldots$ necessarily converges to the same distribution π, because it is a subsequence of the sequence above

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- Let's try to formalize this idea

Eigendecomposition

- The eigenvalues and eigenvectors of the matrix

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are given as $(\lambda_1, \lambda_2) = (1, 0.3)$ and $v_1 = (4/7, 3/7)$, $v_2 = (1, -1)$.

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- For any vector $x = [x_1, x_2]$, we have the following:

$$(x_1, x_2) = \underbrace{(x_1 + x_2)}_{\alpha_1}\underbrace{(4/7, 3/7)}_{=v_1} + \underbrace{(3/7x_1 - 4/7x_2)}_{=\alpha_2}\underbrace{(1, -1)}_{=v_2},$$

i.e., the vector (x_1, x_2) is expressed as linear combination of the eigenvectors

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- Now for P^N, and $x = (x_1, x_2)$ we have

$$\begin{aligned} P^N x &= P^n(\alpha_1 v_1) + P^n(\alpha_2 v_2) \\ &= \alpha_1 P^n v_1 + \alpha_2 P^n v_2 \\ &= \alpha_1 1^n v_1 + \alpha_2 0.3^n v_2 \\ &= \pi + 0.3^n(3/7x_1 - 4/7x_2)(1, -1). \end{aligned}$$

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- (If we started with the stable distribution, then $\alpha_2 = 0$, and there would be no converge)
- Going back to our example of sequence $(P^k)^i$, if λ is an eigenvalue of P, then λ^k is an eigenvalue of P^k.
- Oftentimes, we usually want to define chains that have fast convergence – a chain that converges to the stable distribution fast is said to be *fast mixing*

Spectral Graph Theory

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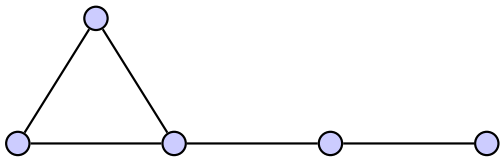
- On the other hand, intuition says that the random walk should converge faster the more connected the graph is
- Question: Can we relate the spectral properties to the geometry of the graph?

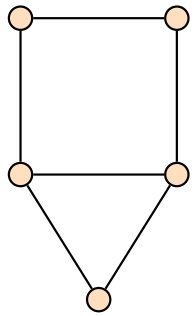
Two Example Graphs

Let's compute the second largest eigenvalue of the transition matrix of the following two graphs (the edges are undirected - we can take it as there being edge both directions)

Graph number 1



Graph number 2



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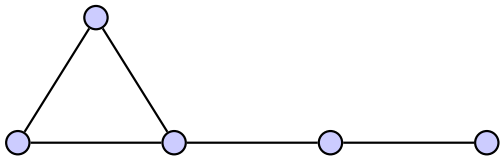
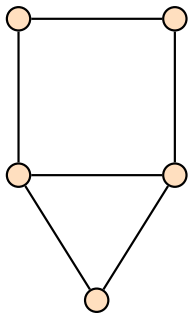


$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Graph number 2



$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Two Example Graphs

$$\hat{A}_1 = \begin{pmatrix} 0 & 1/2 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \end{pmatrix} \quad \hat{A}_2 = \begin{pmatrix} 0 & 1/2 & 1/3 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

$$\lambda_2(\hat{A}_1) \approx -0.86$$

$$\lambda_2(\hat{A}_2) \approx -0.83$$

In python

```
import numpy as np
a=0
A=np.array(([[0,1,1,0,a,],[1,0,1,0,0],[1,1,0,1,0],
[0,0,1,0,1],[a,0,0,1,0]]))
col_sums = A.sum(axis=0)
A = A / col_sums
v,t=np.linalg.eig(A)
np.round(v,2)
>>array([ 1.  ,  0.65, -0.86, -0.3 , -0.5 ])
```

More generally

- The eigenvalues of the transition matrix capture something about the connectivity properties of the graph

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- This type of thinking is also behind the UMAP algorithm for visualizing high dimensional data