AI511/MM505 Linear Algebra with Applications
Take-Home Exam Autumn 2025

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AI503: Linear Algebra
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Contents

Problem 1

Consider the following system of linear equations

$$\begin{cases} 5x + 2y - 3z = 9 \\ 6x + 3y - 3z = 12 \\ 4x + 2y - 2z = 8 \end{cases}$$

a) (2 pts) Is $(x, y, z) = (1, 2, 0)$ a solution of the system?
b) (2 pts) Give the expression of the augmented matrix corresponding to the system.
c) (10 pts) Transform the augmented matrix into a reduced row echelon form to find the set of solutions of the system.

Answer

a)

$$(1, 2, 0) = \begin{cases} 5 + 2 \cdot 2 = 9 \\ 6 + 3 \cdot 2 = 12 \\ 4 + 2 \cdot 2 = 8 \end{cases}$$

This is **true**

b)

$$(x, y, z) = \left[\begin{array}{ccc|c} 5 & 2 & -3 & 9 \\ 6 & 3 & -3 & 12 \\ 4 & 2 & -2 & 8 \end{array}\right]$$

c) Now do row operations to transform this matrix intro RREF

$$\left[\begin{array}{ccc|c} 5 & 2 & -3 & 9 \\ 6 & 3 & -3 & 12 \\ 4 & 2 & -2 & 8 \end{array}\right] \xRightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 6 & 3 & -3 & 12 \\ 4 & 2 & -2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 6 & 3 & -3 & 12 \\ 4 & 2 & -2 & 8 \end{array}\right] \xRightarrow{R_2 - \frac{2}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & -2 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & -2 & 8 \end{array}\right] \xRightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{array}\right] \xRightarrow{R_3 \Leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xRightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since this is not an identity matrix there are an infinite amount of solutions

$$\begin{cases} x - z = 1 \\ y + z = 2 \end{cases} = \begin{cases} x = 1 + z \\ y = 2 - z \end{cases}$$

Therefore: $\{(1 + z, 2 - z, z) : z \in \mathbb{R}\}$

Problem 2

Suppose that

$$A = \begin{bmatrix} 3 & 8 & 1 & 7 \\ 2 & 9 & 10 & 2 \\ 1 & 1 & 8 & 8 \\ 2 & 2 & 5 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 2 & 4 & 8 \\ 3 & 4 & 2 & 4 \\ 6 & 10 & 3 & 6 \\ 10 & 9 & 2 & 4 \end{bmatrix}$$

Calculate the determinant of C, where $C = AB$

Answer

First calculate AB

$$AB = \begin{bmatrix} 121 & 111 & 45 & 90 \\ 121 & 158 & 60 & 120 \\ 138 & 158 & 46 & 92 \\ 150 & 152 & 47 & 94 \end{bmatrix}$$

Since $\text{col}_4 = 2 \cdot \text{col}_3$ the coloums are linearly dependent on eachother, because of this one dimension collapses and therefore $\det(AB) = 0$

Problem 3

Consider the subsets of $\mathbb{R}^3$

$$A = \{(8, 7, 6)\}, \quad B = \{(4, 9, 8), (-3, 5, 7)\}$$

and

$$C = \{(3, 7, 1), (8, 10, 9), (-4, 8, -14)\}$$

a) (10 pts) Determine which of the subsets A, B, and C are bases of some subspace of $\mathbb{R}^3$ and which are not bases of any subspace of $\mathbb{R}^3$.
b) (4 pts) Determine the dimensions of the subspaces that A, B, and C span.

Answer

a) To determine whether a set is a basis of some subspace, it needs to contain only linearly independent vectors.

   Set A = {(8,7,6)}:
   Has only 1 non-zero vector, so it has to be linearly independent.

   Therefore, **A is a basis of some subspace**.

   Set B = {(4,9,8), (-3,5,7)}:
   If B is a basis, then $(4, 9, 8) \neq k(-3, 5, 7)$, then $4 \neq -3k$, $9 \neq 5k$, $8 \neq 7k$. From the first equation, $k = -\frac{4}{3}$. But $9 = 5\left(-\frac{4}{3}\right) = -\frac{20}{3}$, so they're not scalar multiples.

   Therefore, **B is a basis of some subspace**

Set C = {(3,7,1), (8,10,9), (-4,8,-14)}:

```
import numpy as np
mat = np.array([
    [3, 7, 1],
    [8, 10, 9],
    [-4, 8, -14]
])

print('Matrix:')
print(mat)

# Calculate determinant with numpy
det_mat = np.linalg.det(mat)
print(f'Determinant: {det_mat}')

[OUTPUT]
Matrix:
[[  3   7   1]
 [  8  10   9]
 [ -4   8 -14]]
Determinant: 0.0
```

Since $\det(C) = 0$, the vectors are linearly dependent (at least one vector can be written as a linear combination at least one other). Therefore, **C is not a basis of any subspace**.

b) The dimension of span equals the number of linearly independent vectors:

dim(span(A)) = 1

$$\text{span}(A) = \left\{ c_1 \begin{pmatrix} 8 \\ 7 \\ 6 \end{pmatrix} : c_1 \in \mathbb{R} \right\}$$

(A has 1 linearly independent vector)

dim(span(B)) = 2

$$\text{span}(B) = \left\{ c_1 \begin{pmatrix} 4 \\ 9 \\ 8 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

(B has 2 linearly independent vectors)

dim(span(C)) = 2

Since C is not a basis (the vectors are linearly dependent), we need to find the rank of C to determine how many linearly independent vectors it actually contains. We row reduce the matrix formed by the vectors:

$$\begin{bmatrix} 3 & 7 & 1 \\ 8 & 10 & 9 \\ -4 & 8 & -14 \end{bmatrix} \overset{R_2 - \frac{8}{3}R_1}{\Longrightarrow} \begin{bmatrix} 3 & 7 & 1 \\ 0 & -\frac{26}{3} & \frac{19}{3} \\ -4 & 8 & -14 \end{bmatrix}$$

since $8 - 3x = 0 \iff x = \frac{8}{3}$

$$\overset{R_3 + \frac{4}{3}R_1}{\Longrightarrow} \begin{bmatrix} 3 & 7 & 1 \\ 0 & -\frac{26}{3} & \frac{19}{3} \\ 0 & \frac{52}{3} & -\frac{38}{3} \end{bmatrix}$$

since $-4 + 3x = 0 \iff x = \frac{4}{3}$

$$\overset{-\frac{3}{26}R_2}{\Longrightarrow} \begin{bmatrix} 3 & 7 & 1 \\ 0 & 1 & -\frac{19}{26} \\ 0 & \frac{52}{3} & -\frac{38}{3} \end{bmatrix}$$

since $\left(-\frac{26}{3}\right) \cdot \left(-\frac{3}{26}\right) = 1$

$$\overset{R_3 - \frac{52}{3}R_2}{\Longrightarrow} \begin{bmatrix} 3 & 7 & 1 \\ 0 & 1 & -\frac{19}{26} \\ 0 & 0 & 0 \end{bmatrix}$$

since $\left(\frac{52}{3}\right) - \left(\frac{52}{3}\right) \cdot 1 = 0$ and $\left(-\frac{38}{3}\right) - \left(\frac{52}{3}\right) \cdot \left(-\frac{19}{26}\right) = 0$

Since we have 2 non-zero rows after row reduction, rank(C) = 2.

Therefore, dim(span(C)) = rank(C) = 2.

$$\operatorname{span}(C) = \left\{ c_1 \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 8 \\ 10 \\ 9 \end{pmatrix} + c_3 \begin{pmatrix} -4 \\ 8 \\ -14 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}$$

(C has exactly 2 linearly independent vectors among the 3 given)

Problem 4

Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

a) (2 pts) Show that

$$A = \frac{ww^T}{w^T w},$$

where $w = (1, 1)$.

b) (8 pts) Explain how $\mathbb{R}^2$ is transformed by the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_A(x) = Ax$ for $x \in \mathbb{R}^2$.

c) (8 pts) The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 0$ with their corresponding eigenvectors $2^{-1/2}(1, 1)$ and $2^{-1/2}(-1, 1)$ (they are given - you do not need to compute them). Let E_λ denote the eigenspace corresponding to the eigenvalue λ. Describe how the matrix A transforms non-zero vectors in E_{λ_1} and in E_{λ_2}. Make a connection with your response in part (b).

Answer

a) Given $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we need to show that $A = \frac{ww^T}{w^T w}$.

First, $w^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $w^T w = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + 1 = 2$.

Then,

$$ww^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 1 \\ 1 \cdot 1 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore:

$$A = \frac{ww^T}{w^T w} = \frac{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This matches the given matrix A.

b) The matrix $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a projection matrix that projects vectors onto the line $y = x$ in $\mathbb{R}^2$.

For any vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$:

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x + y \\ x + y \end{bmatrix} = \left(\frac{x + y}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This means the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is projected onto the line $y = x$, resulting in the point

$$\left(\frac{x + y}{2}, \frac{x + y}{2}\right)$$

.

c) The eigenvalues tell us how A transforms vectors in each eigenspace:

They must satisfy

$$Av = \lambda v$$

For vectors in E_{λ_1} (eigenvalue = 1):

- Any vector $v \in E_{\lambda_1}$ satisfies $Av = 1 \cdot v = v$
- These vectors are **unchanged** by the transformation
- $E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ (the line $y = x$)

For vectors in E_{λ_2} (eigenvalue = 0):

- Any vector $v \in E_{\lambda_2}$ satisfies $Av = 0 \cdot v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- These vectors are **collapsed to zero**
- $E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ (the line $y = -x$)

What does this say about the transformation

- Vectors already along $y = x$ stay unchanged (eigenvalue 1)
- Vectors along $y = -x$ get annihilated (eigenvalue 0)
- All other vectors get projected onto the line $y = x$

Problem 5

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and define the function $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by setting

$$f(x) = \frac{x^T A x}{x^T x}$$

for nonzero $x \in \mathbb{R}^n$.

a) (2 pts) What does the assumption that $A \in \mathbb{R}^{n \times n}$ is symmetric tell us about its eigenvalues?

b) (8 pts) Suppose that v is an eigenvector of A with its corresponding eigenvalue λ. Show that $f(v) = \lambda$, i.e., the value of f at an eigenvector is the corresponding eigenvalue.

c) (2 pts) Suppose that $X \in \mathbb{R}^{n \times p}$ with $n \geq p$. What is the size of the matrix $X^T X$? Is the matrix $X^T X$ symmetric?

d) (12 pts) Show that the eigenvalues of $X^T X$ are non-negative.

e) (12 pts) Suppose additionally that $X \in \mathbb{R}^{n \times p}$ is of full rank. Show that the eigenvalues of $X^T X$ are positive.

Answer

a)

> **A special case of Theorem 3.3.2 of Johnston (2021)**
>
> If $A \in \mathbb{R}^{n\times n}$ is symmetric, then all of its eigenvalues are real.

By theorem, if a is symmentric, all of its eigenvalues must be real.

b) If $Av = \lambda v$ (since v is an eigenvector)

then

$$f(v) = \frac{v^T A v}{v^T v} = \frac{v^T \lambda v}{v^T v} = \lambda \frac{v^T v}{v^T v} = \lambda$$

The value of f at an eigenvenctor must be the corresponding eigenvalue

c) For a matrix $X \in \mathbb{R}^{n\times p}$, then $X^T X \in \mathbb{R}^{p\times n}\mathbb{R}^{p\times n}$

for this

$$(p \times n) \cdot (n \times p) = p \times p$$

So..

$$X^T X \in \mathbb{R}^{p\times p}$$

To check if $X^T X$ is symmetric we use the following theorem:

> **Theorem 1.3.4 of Johnston (2021)**
>
> Let A and B be matrices with sizes such that the operations below make sense and let c ∈ R be a scalar. Then
>
> (c) $(AB)^T = B^T A^T$

$$\left(X^T X\right)^T = X^T \left(X^T\right)^T = X^T X$$

Therefore $X^T X$ is symmetric.

d) Let λ be an eigenvalue of $X^T X$ with corresponding eigenvector $v \neq 0$.

Then $X^T X v = \lambda v$.

Taking the dot product of both sides with v will make isolating λ much easier:

$$v^T (X^T X v) = v^T (\lambda v)$$

$$v^T X^T X v = \lambda v^T v$$

Since $v^T v > 0$ (because $v \neq 0$), we can divide both sides by $v^T v$:

$$\lambda = \frac{v^T X^T X v}{v^T v}$$

Now, let $w = Xv$. Then:

$$v^T X^T X v = v^T X^T (Xv) = (Xv)^T (Xv) = w^T w = \|w\|^2 \geq 0$$

We know this from:

Definition 1.2.2: Length of a Vector

The length of a vector $v = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$, denoted by $\|v\|$, is the quantity

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

Now we know that $v^T v > 0$ and $v^T X^T X v \geq 0$, so..

$$\lambda = \frac{v^T X^T X v}{v^T v} \geq 0$$

Therefore, all eigenvalues of $X^T X$ has to be non-negative.

e) For the matrix $A \in \mathbb{R}^{n \times p}$ to be in full rank, A must be linearly indenpentent

$$\det(A) \neq 0$$

By contradriction

Assume that one of A's eigenvalues $\lambda = 0$

Then

$$\det(A) = \prod_{i=1}^{n} \lambda_i = 0 \quad (\perp)$$

By this, λ cannot be 0

And then

$$\lambda = \frac{v^T X^T X v}{v^T v} > 0$$

Therefore, the eigenvalues of a full rank matrix A must be positve

Problem 6

You are only required to respond true or false to the following statements.

a) (2 pts) A product of an invertible matrix and a non-invertible matrix is an invertible matrix.
b) (2 pts) Every subspace of $\mathbb{R}^n$ has infinitely many vectors.
c) (2 pts) If $A \in \mathbb{R}^{3\times 3}$ with $\det(A) = 3$, then $\text{rank}(A) = 3$.
d) (2 pts) 0 is an eigenvalue of every square matrix.
e) (2 pts) Every diagonalisable matrix is invertible.

Answer

a) **This is false.**

This can easily be proven by

$$\det(AB) = \det(A)\det(B)$$

By this, if A is non-invertable ($\det(A) = 0$), then AB will also be non-invertable ($\det(AB) = 0$)

b) **This is false.**

Since the zero subspace $\{0\}$ does not contain infinitely many vectors

c) **This is true.**

If $\det(A) = 3 \neq 0$, then A is invertible, which means A has full rank. For a 3×3 matrix, full rank means $\text{rank}(A) = 3$.

d) **This is false**

Proof by contradiction.

Let matrix A be invertible and this be the case

$$Av = \lambda v \iff Av = 0$$

Then

$$A^{-1}(Av) = A^{-1} \cdot 0 \implies Iv = 0 \implies v = 0 \quad (\perp)$$

Since v cannot be 0 (because it is an eigenvector) then $\lambda \neq 0$ for invertible matrices

e) **This is false.**

For any matrix to be diagonalisable it must satisfy:

$$A = PDP^{-1}$$

where D is diagonal and P is invertible.

Assume that A is invertible, then $\det(A) \neq 0$.

Now by contradiction

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

This matrix is diagonalizable (because $A = I^{-1}AI = A$).

And with

$$\det(A - AI) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 10 \\ 01 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda] \end{bmatrix}\right) = (1-\lambda)(-\lambda) = -\lambda(1-\lambda) = \lambda(\lambda-1)$$

setting this to 0

$$\lambda(\lambda - 1) = 0 \implies \lambda = 0, 1$$

But $\det(A) = 1 \cdot 0 - 0 \cdot 0 = 0$, so A is not invertible.