AI511/MM505 Linear Algebra with Applications

Lecture 4 – Determinants

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Schedule

Week	Lectures	Exercise sessions	Part
36	AI511/MM505	No exercise sessions	Theory
37	AI511/MM505	AI511/MM505	Theory
38	**AI511**	**AI511**	**Applications**
39	AI511/MM505	AI511/MM505	Theory
40	AI511/MM505	AI511/MM505	Theory
41	AI511/MM505	AI511/MM505	Theory
42	Autumn break		
43	AI511/MM505	AI511/MM505	Theory
44	AI511	AI511/MM505	Applications/theory

- Week 38 is a short introduction to linear algebra in Python.
- Week 44 is split: lectures cover applications, exercises cover theory.

Outline

Recap

Matrices and matrix operations (cont.)

- Diagonal and triangular matrices
- Determinant

Recap

Systems of linear equations and the inverse of a matrix

- A system of linear equations can be compactly written as

$$Ax = b,$$

  where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^{n \times 1}$, and $b \in \mathbb{R}^{m \times 1}$.
- If $b = 0$, the system is called *homogeneous* and always has at least one solution.
- Can we somehow “divide” both sides of the equation by A so that we obtain the solution?

Inverse of a matrix

- We say $A \in \mathbb{R}^{n \times n}$ is *invertible* if there exists a matrix, which we denote A^{-1} and call the *inverse* of A, such that

$$AA^{-1} = A^{-1}A = I_n.$$

- The matrix

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

  is invertible while the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

  is not invertible.
- Why is B not invertible?
- If a matrix is invertible, its inverse is unique.

Characterising invertibility

Theorem (Theorem 2.2.4 of Johnston (2021))

Suppose that $A \in \mathbb{R}^{n \times n}$. The following are equivalent.

(a) *A is invertible.*

(b) *The linear system $Ax = 0$ has a unique solution.*

(c) *The reduced REF of A is I_n.*

- Why is this theorem useful? It gives us a way to check whether some matrix is invertible or not by transforming it into a reduced REF.
- This theorem does not tell how to find the inverse of A.

Elementary matrices

Definition

A square matrix $E \in \mathbb{R}^{n\times n}$ is called an *elementary matrix* if it can be obtained from the identity matrix via a single elementary row operation.

Examples

$$\underset{R_1\leftrightarrow R_2}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \quad \underset{R_3-3R_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}, \quad \text{and} \quad \underset{\frac{1}{2}R_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Elementary matrices (cont.)

Theorem (Theorem B.3.1 of Johnston (2021))

Suppose $A \in \mathbb{R}^{m \times n}$. If applying a single elementary row operation to A results in a matrix B, and applying that same elementary row operation to I_m results in a matrix E, then $EA = B$.

Theorem

Elementary matrices are invertible and their inverses are elementary.

Finding the inverse of a matrix

- According to Theorem 2.2.4 of Johnston (2021), A is invertible if and only if there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k \ldots E_1 A = I_n.$$

- If A is invertible, it follows that

$$E_k \ldots E_1 = E_k \ldots E_1 I_n = E_k \ldots E_1 A A^{-1} = I_n A^{-1} = A^{-1}.$$

- In simple terms, the elementary row operations that bring A to I_n simultaneously bring I_n to A^{-1}.

Computing the inverse of a matrix

Theorem (Theorem 2.2.5 of Johnston (2021))

Suppose $A \in \mathbb{R}^{n \times n}$. Then A is invertible if and only if there exists a matrix $E \in \mathbb{R}^{n \times n}$ such that the reduced REF of the block matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ is $\left[\begin{array}{c|c} I_n & E \end{array}\right]$. Furthermore, if A is invertible then it is necessarily the case that $A^{-1} = E$.

Example

- Suppose that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

- We showed that

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right].$$

Matrices and matrix operations (cont.)

Unique solution of a system of linear equations

Theorem (Theorem 1.6.2 of Anton, Rorres, and Kaul (2019))

If $A \in \mathbb{R}^{n \times n}$ is an invertible matrix, then for every $b \in \mathbb{R}^{n \times 1}$, the system of linear equations $Ax = b$ has exactly one solution $x = A^{-1}b$.

Proof.

Since $A(A^{-1}b) = b$, it follows that $x = A^{-1}b$ is a solution. Suppose that x_0 is another solution, i.e., $Ax_0 = b$. But then we have that $A^{-1}Ax_0 = A^{-1}b$ and $x_0 = A^{-1}b$. Hence, $A^{-1}b$ is a unique solution. The proof is complete. $\square$

Solutions of systems of linear equations

Theorem (Theorem 2.1.1 of Johnston (2021))

Every system of linear equations has either

(a) *no solutions;*

(b) *exactly one solution;*

(c) *infinitely many solutions.*

Proof.

Assume that x_1 and x_2 are two solutions (i.e., $Ax_1 = Ax_2 = b$) such that $x_1 \neq x_2$ and consider another vector $(1-c)x_1 + cx_2$ for $c \in \mathbb{R}$. Then

$$A((1-c)x_1 + cx_2) = (1-c)Ax_1 + cAx_2 = (1-c)b + cb = b$$

which means the system has an infinite number of solutions since $(1-c)x_1 + cx_2$ is a solution for any $c \in \mathbb{R}$. The proof is complete. $\square$

A simplifying theorem

Theorem (Theorem 2.2.7 of Johnston (2021))

Suppose that $A \in \mathbb{R}^{n\times n}$. If there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $AB = I_n$ or $BA = I_n$, then A is invertible and $A^{-1} = B$.

Proof.

If $BA = I_n$, then multiplying the linear system $Ax = 0$ on the left by B shows that $BAx = B0 = 0$, but also $BAx = I_n x = x$, so $x = 0$. The linear system $Ax = 0$ thus has $x = 0$ as its unique solution, so it follows from Theorem 2.2.4 of Johnston (2021) that A is invertible. Multiplying the equation $BA = I_n$ on the right by A^{-1} shows that $B = A^{-1}$, as desired. If $AB = I_n$, the argument is similar (hint: consider the transpose of AB). The proof is complete. $\square$

Matrices and matrix operations (cont.)

Diagonal and triangular matrices

Diagonal matrices

Definition

$A \in \mathbb{R}^{n \times n}$ is said to be *diagonal* if $a_{ij} = 0$ for $i \neq j$.

Examples

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $A \in \mathbb{R}^{n \times n}$ is diagonal, we also write

$$A = \mathrm{diag}(\ a_1 \quad \ldots \quad a_n\ ).$$

Diagonal matrices (cont.)

Matrix multiplication of diagonal matrices is performed elementwise

$$\operatorname{diag}\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \operatorname{diag}\begin{pmatrix} b_1 & \dots & b_n \end{pmatrix} =$$
$$= \operatorname{diag}\begin{pmatrix} a_1 b_1 & \dots & a_n b_n \end{pmatrix}.$$

Theorem

A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is invertible if and only if $d_i \neq 0$ for all $i = 1, \dots, n$ and in which case $D^{-1} = \operatorname{diag}\begin{pmatrix} d_1^{-1} & \dots & d_n^{-1} \end{pmatrix}$.

It is straightforward to calculate powers of a diagonal matrix D since

$$D^k = \operatorname{diag}\begin{pmatrix} d_1^k & \dots & d_n^k \end{pmatrix}$$

for $k \geq 1$.

Triangular matrices

Definition

$A \in \mathbb{R}^{n\times n}$ is called *upper (lower) triangular* matrix if all entries below (above) the main diagonal are zero, i.e., $a_{ij} = 0$ for $i > j$ ($a_{ij} = 0$ for $i < j$).

Examples

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{Upper triangular}} \quad \text{and} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{Lower triangular}}.$$

So diagonal means upper *and* lower triangular.

Properties of triangular matrices

Theorem

(a) *The transpose of a lower triangular matrix is upper triangular.*

(b) *The product of upper triangular matrices is upper triangular.*

(c) *An upper triangular matrix is invertible if and only if its diagonal entries are non-zero. If so, its inverse is also upper triangular.*

(d) *The same is true if "lower" and "upper" are swapped.*

Matrices and matrix operations (cont.)

Determinant

Determinant

- Recall that for
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
we defined its determinant as $\det(A) = ad - bc$.
- We saw that A is invertible if and only if $\det(A) \neq 0$.
- We now define the determinant of any $A \in \mathbb{R}^{n \times n}$.

Inductive definition

- For $A = [a] \in \mathbb{R}^{1\times 1}$, we define $\det([a]) := a$.
- For
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
we thus have
$$\begin{aligned} \det(A) &= a_{11}\det([a_{22}]) - a_{12}\det([a_{21}]) \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$
- General determinants are defined inductively.
- So assume that we know what the determinant of an $n \times n$-matrix is.
- To move up in size, we need *minors* and *cofactors*.

Minors, cofactors, and determinant

Definition

If $A \in \mathbb{R}^{n\times n}$, then the (i,j)*-th minor*, M_{ij}, of A is the determinant of the $(n-1)\times(n-1)$-matrix obtained by deleting row i and column j. The (i,j)*-th cofactor* is the number $C_{ij} := (-1)^{i+j}M_{ij}$.

Definition

The determinant of $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.$$

We also use the following notation

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} = \det\left(\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}\right).$$

Examples

- We have that
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a(-1)^{1+1}d + b(-1)^{1+2}c = ad - bc.$$
- Suppose that
$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}.$$
Show that $\det(A) = -1$.

Theorem (Theorem 3.2.8 of Johnston (2021))

Suppose $A \in \mathbb{R}^{n\times n}$. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \tag{1}$$
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} \tag{2}$$

for all $i, j = 1, \ldots, n$.

- (1) is called a cofactor expansion along the i-th row.
- (2) is called a cofactor expansion along the j-th column.

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). ***Elementary linear algebra: applications version.*** 12th edition. John Wiley & Sons, Inc.