

AI511/MM505 Linear Algebra with Applications

Lecture 7 – Compositions, subspaces, linear combinations

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September 29, 2025

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Outline

Recap

Euclidean vector spaces (cont.)

- Examples of linear transformations (cont.)

- Compositions of linear transformations

- Simulated example

- Subspaces

- Linear combinations and independence

Recap

The standard basis

- The set of vectors $\{e_1, e_2, \dots, e_n\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

is called the *standard basis* in \mathbb{R}^n .

- Any vector $v \in \mathbb{R}^n$ can be written in terms of the vectors e_1, e_2, \dots, e_n in the following way

$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n. \tag{1}$$

Matrix operators or matrix transformations

Every matrix $A \in \mathbb{R}^{m \times n}$ corresponds to a *matrix operator* or a *matrix transformation* $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by

$$T_A(x) := Ax$$

for $x \in \mathbb{R}^n$.

Linear transformation

Definition 1.4.1 of Johnston (2021)

A *linear transformation* is a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that satisfies the following two properties:

- (a) $T(v + w) = T(v) + T(w)$ for all vectors $v, w \in \mathbb{R}^n$;
- (b) $T(cv) = cT(v)$ for all vectors $v \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.

Visually speaking, a transformation is “linear” if it has two properties

- (i) all lines remain lines, without getting curved;
- (ii) the origin remains fixed in place.

Matrices as linear transformations

Theorem 1.4.1 of Johnston (2021)

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if there exists a matrix $[T] \in \mathbb{R}^{m \times n}$ such that

$$T(v) = [T]v \quad \text{for all } v \in \mathbb{R}^n.$$

Furthermore, the unique matrix $[T]$ with this property is called the *standard matrix* of T , and it is

$$[T] := \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}.$$

In simple terms, a transformation is linear if and only if it is a matrix transformation.

Euclidean vector spaces (cont.)

Euclidean vector spaces (cont.)

Examples of linear transformations (cont.)

Orthogonal projection onto the direction of a unit vector u

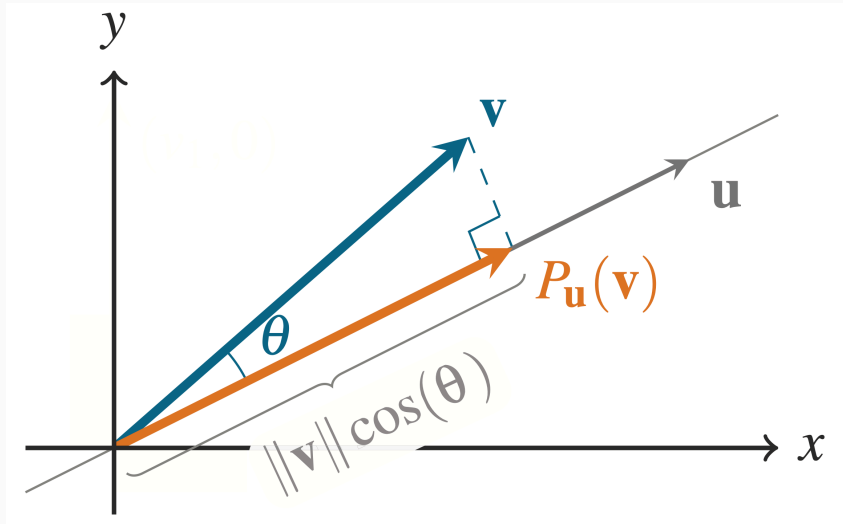


Figure 1.16 of Johnston (2021)

Orthogonal projection onto the direction of a unit vector u

- The projection of a vector v onto the line in the direction of u is denoted by $P_u(v)$, and it points a distance of $\|v\| \cos(\theta)$ (might be negative) in the direction of u , where θ is the angle between v and u .
- For $v, w \in \mathbb{R}^2$, we have that

$$\cos(\theta) = \frac{v \cdot w}{\|v\| \|w\|},$$

$$P_u(v) = \|v\| \cos(\theta) u = \|v\| \frac{(v \cdot u)}{\|v\| \|u\|} u = uu^T v$$

using the fact that $\|u\| = 1$ and $v \cdot u = u^T v$.

- Hence, $P_u(v)$ is a linear transformation, and its standard matrix is given by

$$[P_u] = uu^T \in \mathbb{R}^{n \times n}.$$

Orthogonal projection onto the direction of a vector w

In general, we have that

$$P_w(v) = \frac{1}{\|w\|^2} ww^T v = \frac{ww^T}{w^T w} v$$

for two vectors $v, w \in \mathbb{R}^n$ provided that $w \neq 0$.

Euclidean vector spaces (cont.)

Compositions of linear transformations

Composition of linear transformations

Definition 1.4.2 of Johnston (2021)

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$ are linear transformations. Then *composition* of S and T is the function $S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ defined by

$$(S \circ T)(v) = S(T(v)) \quad \text{for all } v \in \mathbb{R}^n.$$

- T sends \mathbb{R}^n to \mathbb{R}^m and S sends \mathbb{R}^m to \mathbb{R}^p while the composition $S \circ T$ skips the intermediate step and sends \mathbb{R}^n directly to \mathbb{R}^p .
- Is $S \circ T$ linear? If so, what is its standard matrix?

Composition is linear

Theorem 1.4.2 of Johnston (2021)

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$ are linear transformations with standard matrices $[T] \in \mathbb{R}^{m \times n}$ and $[S] \in \mathbb{R}^{p \times m}$, respectively. Then $S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a linear transformation, and its standard matrix is

$$[S \circ T] = [S][T].$$

- In simple terms, composition of linear maps is a matrix product of standard matrices.
- This theorem is the main reason that matrix multiplication was defined in the seemingly bizarre way.
- Recommended to watch *Matrix multiplication as composition* of 3Blue1Brown.

Example on the blackboard

- Consider two linear transformations

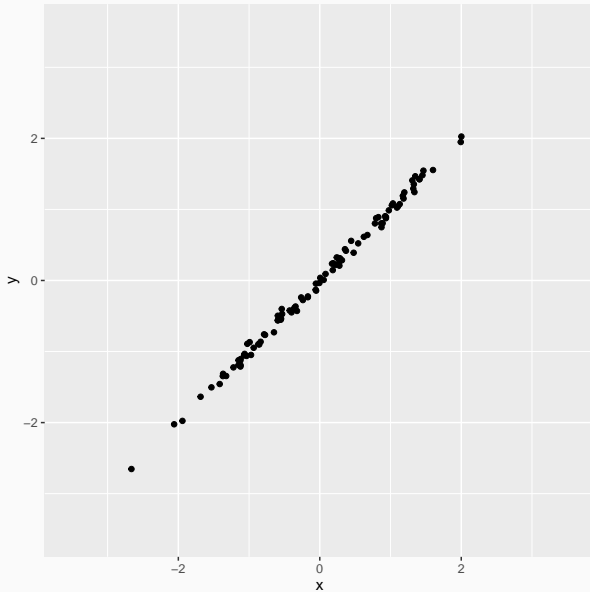
$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}.$$

- The linear transformation B
 - (i) rotates e_1 counterclockwise by 45° and extends its length to $\sqrt{2}$;
 - (ii) rotates e_2 counterclockwise by 90° and extends its length to 2.
- The linear transformation A
 - (i) flips e_1 and e_2 around 45° line;
 - (ii) extends the length of e_2 to 2 (stretches the space horizontally by 2).
- Find the standard matrix of the composition AB by describing where e_1 and e_2 are mapped and confirm your result by multiplying A and B .

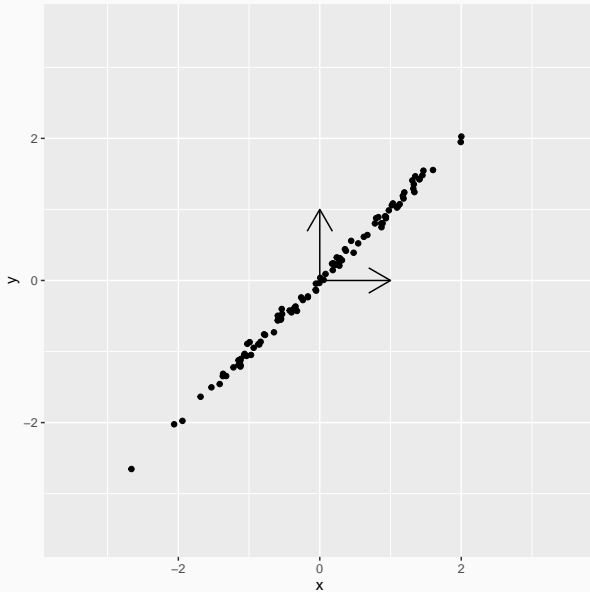
Euclidean vector spaces (cont.)

Simulated example

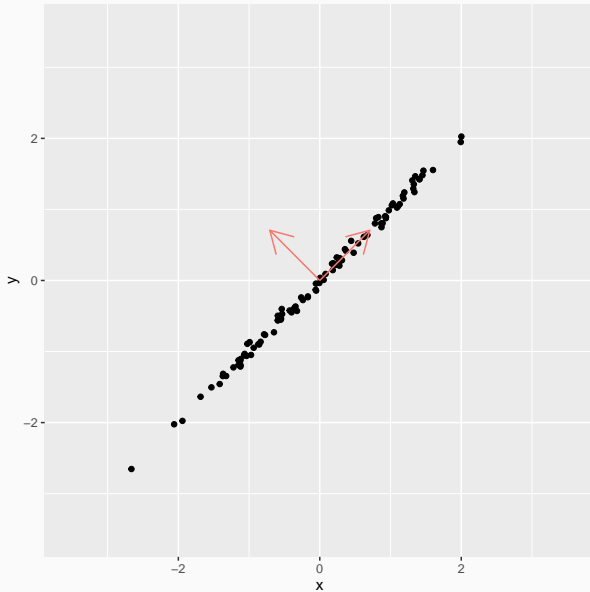
Simulated data example in \mathbb{R}^2



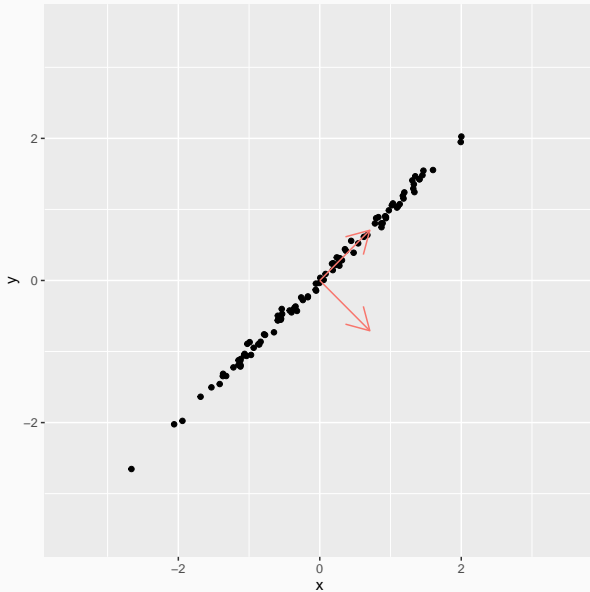
Simulated data example in \mathbb{R}^2



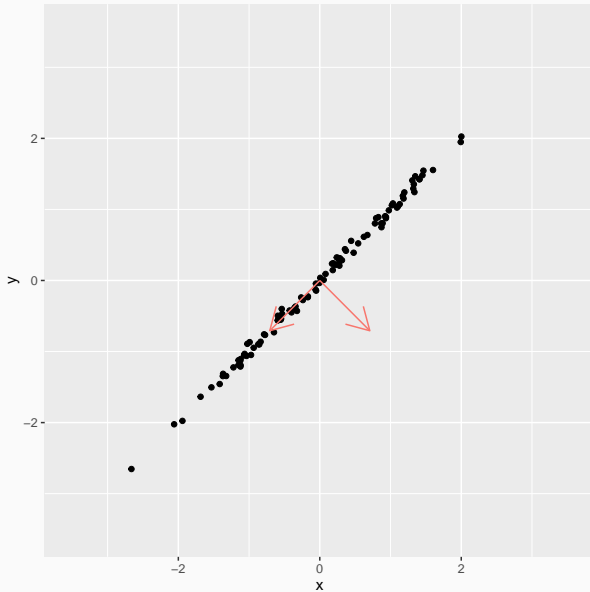
Simulated data example in \mathbb{R}^2



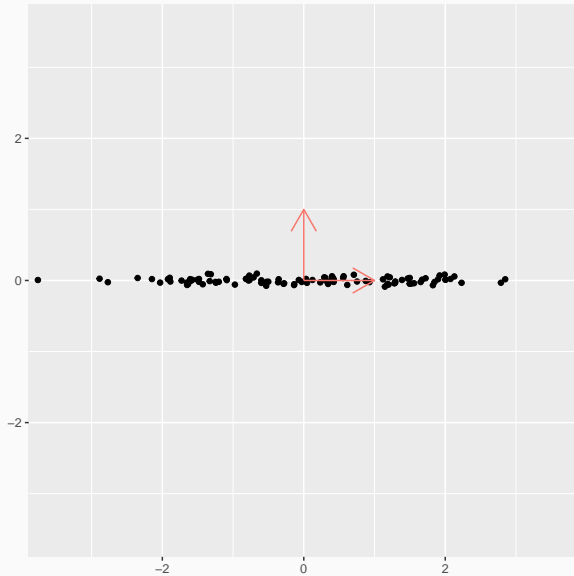
Simulated data example in \mathbb{R}^2



Simulated data example in \mathbb{R}^2



Simulated data example in \mathbb{R}^2



Questions

- What is this thing where the simulated vectors approximately live?
- What does it mean that we have a coordinate system for a Euclidean space and what role does it play in representing vectors?
- How do we transform the coordinates of a vector when changing from one coordinate system to another in a Euclidean space?

Euclidean vector spaces (cont.)

Subspaces

Definition 2.3.1 of Johnston (2021)

A *subspace* of \mathbb{R}^n is a non-empty set \mathcal{S} of vectors in \mathbb{R}^n with the properties that

- (a) if $v, w \in \mathcal{S}$, then $v + w \in \mathcal{S}$;
- (b) if $v \in \mathcal{S}$ and $c \in \mathbb{R}$, then $cv \in \mathcal{S}$.

- (a) ensures that subspaces are “flat”.
- (b) makes it so that they are “infinitely long”.
- Observe that every subspace contains the zero vector (think why this is the case).

Definition 2.3.2 of Johnston (2021)

Suppose $A \in \mathbb{R}^{m \times n}$.

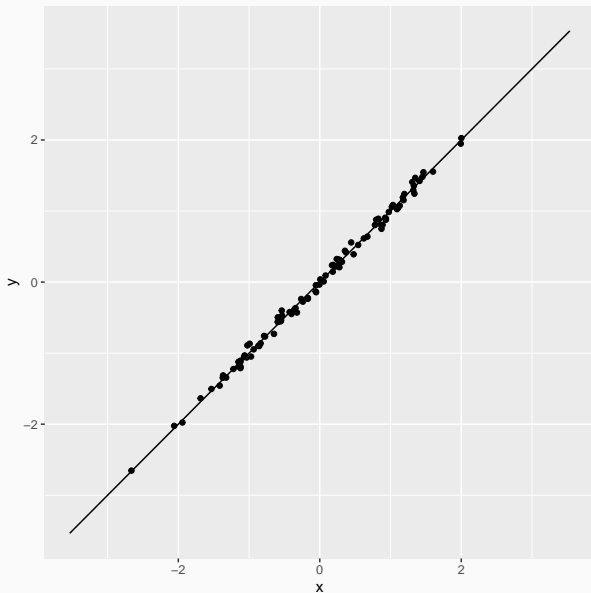
- (a) The *range* of A is the subspace of \mathbb{R}^m , denoted by $\text{range}(A)$, that consists of all vectors of the form Ax .
- (b) The *null space* of A is the subspace of \mathbb{R}^n , denoted by $\text{null}(A)$, that consists of all solutions x of the system of linear equations $Ax = 0$.

Is the set of solutions of $Ax = b$ with $b \neq 0$ a subspace in \mathbb{R}^n ?

Are $\text{range}(A)$ and $\text{null}(A)$ subspaces?

- $\text{range}(A)$ is non-empty since $0 \in \text{range}(A)$. If $Ax, Ay \in \text{range}(A)$, then $Ax + Ay = A(x + y) \in \text{range}(A)$. If $Ax \in \text{range}(A)$, then $cAx = A(cx) \in \text{range}(A)$.
- $\text{null}(A)$ is non-empty since $0 \in \text{null}(A)$. If $x, y \in \text{null}(A)$, then $x + y \in \text{null}(A)$ since $A(x + y) = Ax + Ay = 0 + 0 = 0$. If $x \in \text{null}(A)$, then $cx \in \text{null}(A)$, since $A(cx) = cAx = c0 = 0$.

Simulated data example in \mathbb{R}^2



Euclidean vector spaces (cont.)

Linear combinations and independence

Linear combination and span

Definition 1.1.3 of Johnston (2021)

A *linear combination* of the vectors $v_1, \dots, v_k \in \mathbb{R}^n$ is any vector of the form

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k,$$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$.

Definition 2.3.3 of Johnston (2021)

If $B = \{v_1, \dots, v_k\}$ is a set of vectors in \mathbb{R}^n , then the set of all linear combinations of those vectors is called their *span*, and it is denoted by $\text{span}(B)$ and $\text{span}(v_1, \dots, v_k)$.

Recommended to watch *Linear combinations, span, and basis vectors* of 3Blue1Brown.

Theorem 2.3.1 of Johnston (2021)

Let $v_1, \dots, v_k \in \mathbb{R}^n$ with $k \geq 1$. Then $\text{span}(v_1, \dots, v_k)$ is a subspace of \mathbb{R}^n .

Theorem 2.3.2 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{m \times n}$ has columns a_1, \dots, a_n . Then

$$\text{range}(A) = \text{span}(a_1, \dots, a_n).$$

Spanning set and linear independence

- If \mathcal{S} is a subspace and $\mathcal{S} = \text{span}\{v_1, \dots, v_n\}$ for some v_1, \dots, v_n , then $\{v_1, \dots, v_n\}$ is called a *spanning set* of \mathcal{S} .
- Spanning sets are not unique. That is, we may have

$$\mathcal{S} = \text{span}\{v_1, \dots, v_n\} = \text{span}\{v'_1, \dots, v'_m\}.$$

This happens if and only if each v_i can be written as a linear combination of the vectors v'_1, \dots, v'_m and vice versa.

- Some spanning sets include redundant vectors. For example,

$$\mathbb{R}^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Linear dependence and independence

Definition 2.3.4 of Johnston (2021)

A set of vectors $\mathcal{S} = \{v_1, v_2, \dots, v_k\}$ is *linearly dependent* if there exist scalars $c_1, c_2, \dots, c_k \in \mathbb{R}$, at least one of which is not zero, such that

$$c_1 v_1 + \dots + c_k v_k = 0.$$

If \mathcal{S} is not linearly dependent then it is called *linearly independent*.

- A set consisting of a single nonzero vector is linearly independent.
- A set containing the vector 0 is linearly dependent.
- A set with two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.



Johnston, Nathaniel (2021). *Introduction to linear and matrix algebra*. Springer Cham.