AI511/MM505 Linear Algebra with Applications

Lecture 8 – Subspaces, Linear Combinations, Bases and Coordinates, Dimension

Vaidotas Characiejus

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Department of Mathematics and Computer Science
University of Southern Denmark

Outline

Recap

Composition of linear transformations

- Suppose that we have two linear transformations

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad S : \mathbb{R}^m \to \mathbb{R}^p.$$

- Then the composition

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^p.$$

- $S \circ T$ is a linear transformation with the standard matrix given by

$$[S \circ T] = [S][T],$$

i.e., the standard matrix of $S \circ T$ is the matrix product of the standard matrices of S and T.

Example on the blackboard

- Consider two linear transformations

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}.$$

- The linear transformation B
  (i) rotates e_1 counterclockwise by 45° and extends its length to $\sqrt{2}$;
  (ii) rotates e_2 counterclockwise by 90° and extends its length to 2.
- The linear transformation A
  (i) flips e_1 and e_2 around 45° line;
  (ii) extends the length of e_2 to 2 (stretches the space horizontally by 2).
- We have that

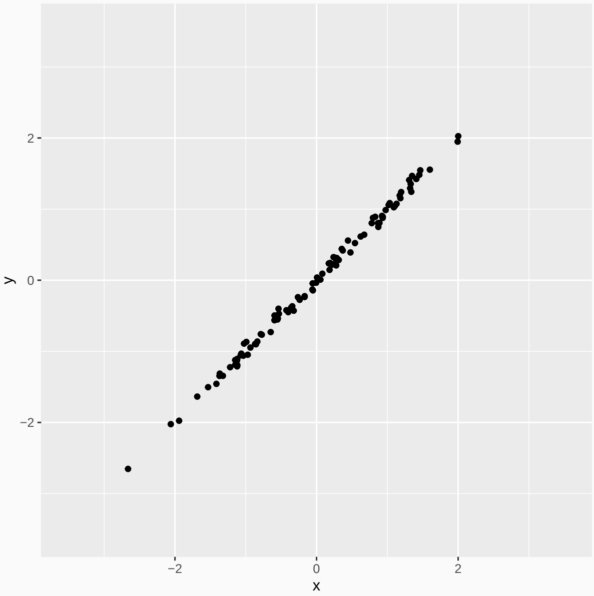
$$AB = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}.$$

Euclidean vector spaces (cont.)

Euclidean vector spaces (cont.)

Simulated example

Simulated data example in $\mathbb{R}^2$



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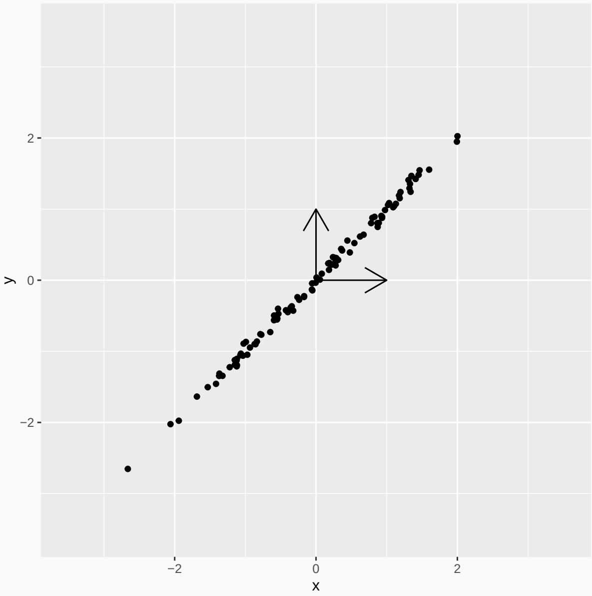

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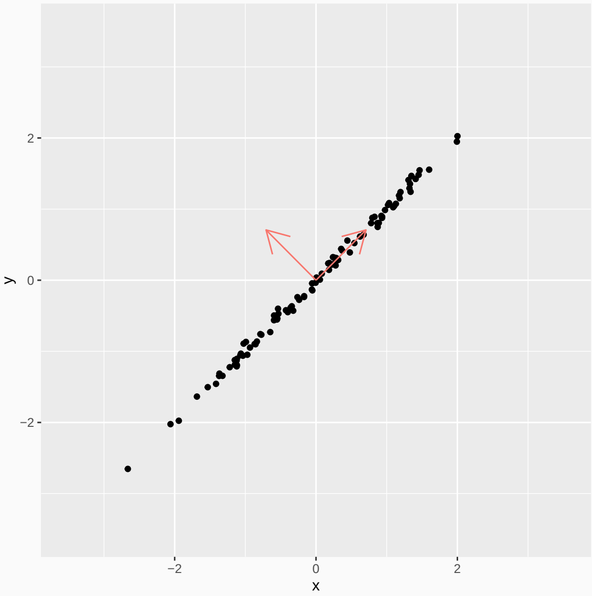

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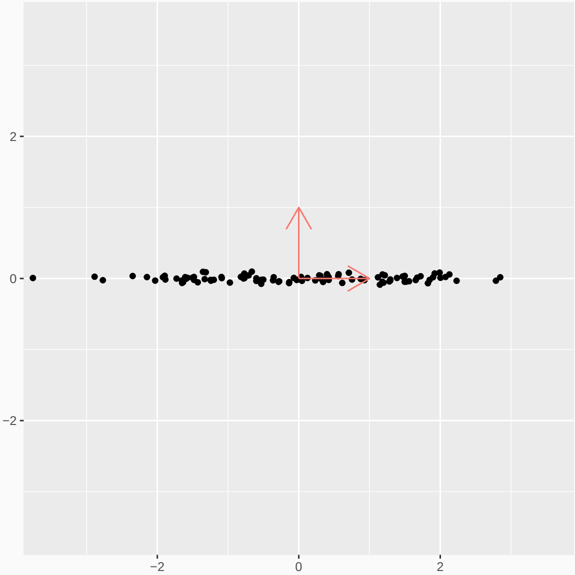

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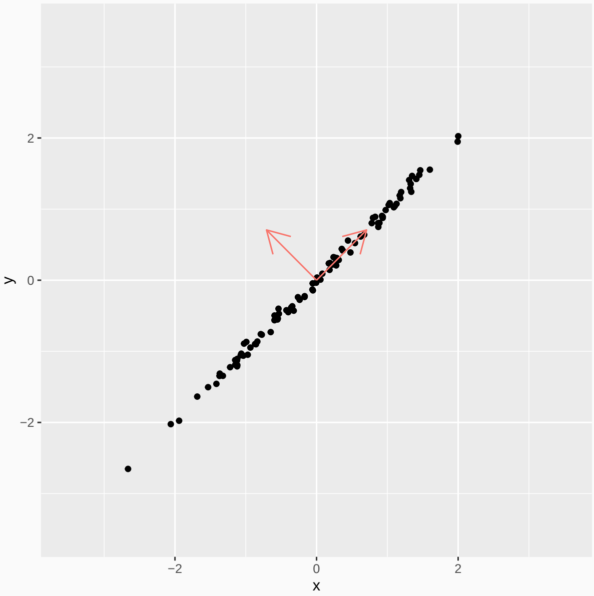

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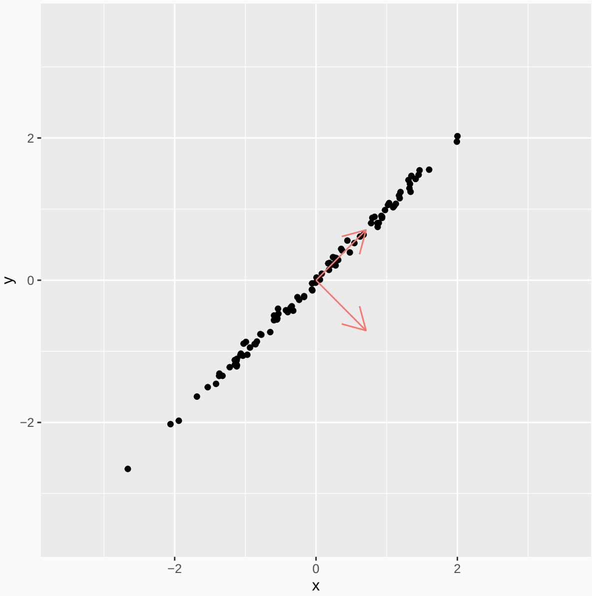

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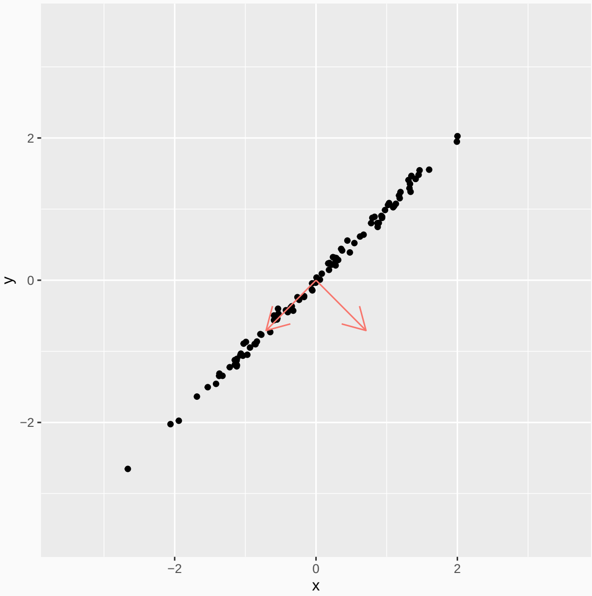

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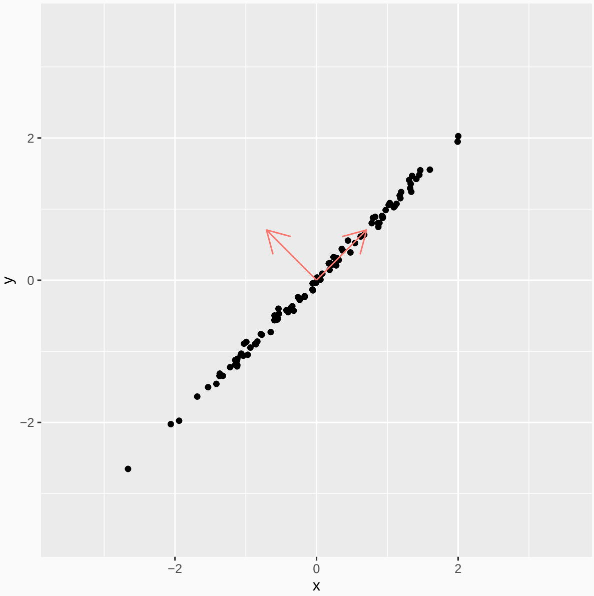

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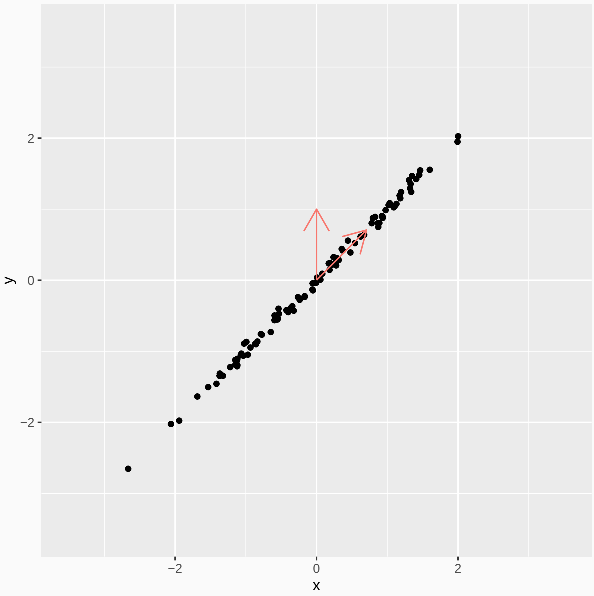

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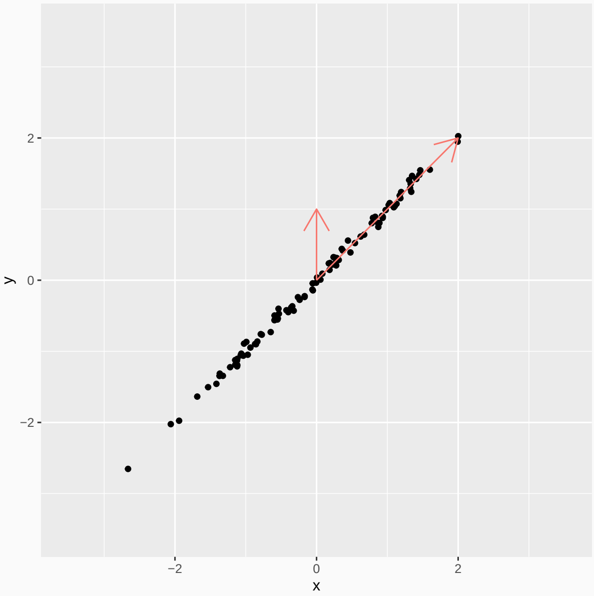


Questions

- What is this thing where the simulated vectors approximately live?
- What does it mean that we have a coordinate system for a Euclidean space and what role does it play in representing vectors?
- How do we transform the coordinates of a vector when changing from one coordinate system to another in a Euclidean space?

Euclidean vector spaces (cont.)

Subspaces

Definition 2.3.1 of Johnston (2021)

A *subspace* of $\mathbb{R}^n$ is a non-empty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ with the properties that

(a) if $v, w \in \mathcal{S}$, then $v + w \in \mathcal{S}$;

(b) if $v \in \mathcal{S}$ and $c \in \mathbb{R}$, then $cv \in \mathcal{S}$.

- (a) ensures that subspaces are "flat".
- (b) makes it so that they are "infinitely long".
- Observe that every subspace contains the zero vector (think why this is the case).

Definition 2.3.2 of Johnston (2021)

Suppose $A \in \mathbb{R}^{m \times n}$.

(a) The *range* of A is the subspace of $\mathbb{R}^m$, denoted by $\operatorname{range}(A)$, that consists of all vectors of the form Ax.

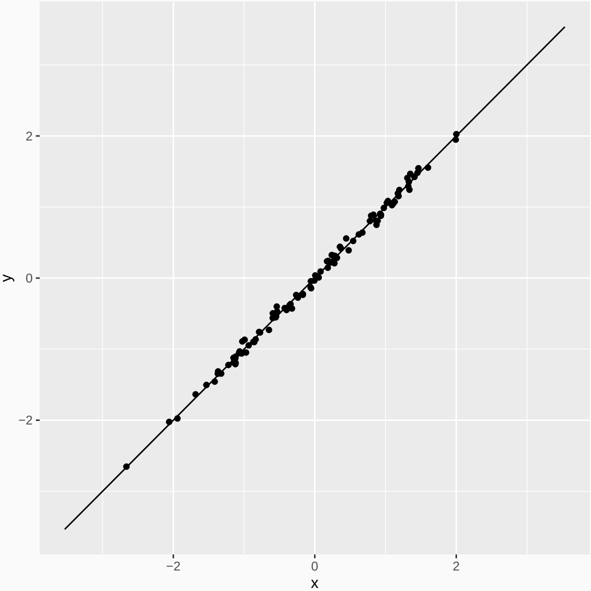
(b) The *null space* of A is the subspace of $\mathbb{R}^n$, denoted by $\operatorname{null}(A)$, that consists of all solutions x of the system of linear equations $Ax = 0$.

Is the set of solutions of $Ax = b$ with $b \neq 0$ a subspace in $\mathbb{R}^n$?

Are $\operatorname{range}(A)$ and $\operatorname{null}(A)$ actually subspaces?

- $\mathbf{range}(A)$ is non-empty since $0 \in \mathbf{range}(A)$. If $Ax, Ay \in \mathbf{range}(A)$, then $Ax + Ay = A(x + y) \in \mathbf{range}(A)$. If $Ax \in \mathbf{range}(A)$, then $cAx = A(cx) \in \mathbf{range}(A)$.
- $\mathbf{null}(A)$ is non-empty since $0 \in \mathbf{null}(A)$. If $x, y \in \mathbf{null}(A)$, then $x + y \in \mathbf{null}(A)$ since $A(x + y) = Ax + Ay = 0 + 0 = 0$. If $x \in \mathbf{null}(A)$, then $cx \in \mathbf{null}(A)$, since $A(cx) = cAx = c0 = 0$.

Simulated data example in $\mathbb{R}^2$



Euclidean vector spaces (cont.)

Linear combinations and independence

Linear combination and span

Definition 1.1.3 of Johnston (2021)

A *linear combination* of the vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is any vector of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k,$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Definition 2.3.3 of Johnston (2021)

If $B = \{v_1, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of those vectors is called their *span*, and it is denoted by $\mathrm{span}(B)$ and $\mathrm{span}(v_1, \ldots, v_k)$.

Recommended to watch *Linear combinations, span, and basis vectors* of 3Blue1Brown.

Span (cont.)

Theorem 2.3.1 of Johnston (2021)

Let $v_1, \ldots, v_k \in \mathbb{R}^n$ with $k \geq 1$. Then $\mathsf{span}(v_1, \ldots, v_k)$ is a subspace of $\mathbb{R}^n$.

Theorem 2.3.2 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{m \times n}$ has columns $a_1, \ldots, a_n$. Then

$$\mathsf{range}(A) = \mathsf{span}(a_1, \ldots, a_n).$$

Spanning set and linear independence

- If $\mathcal{S}$ is a subspace and $\mathcal{S} = \text{span}\{v_1, \ldots, v_n\}$ for some $v_1, \ldots, v_n$, then $\{v_1, \ldots, v_n\}$ is called a *spanning set* of $\mathcal{S}$.
- Spanning sets are not unique. That is, we may have

$$\mathcal{S} = \text{span}\{v_1, \ldots, v_n\} = \text{span}\{v_1', \ldots, v_n'\}.$$

  This happens if and only if each v_i can be written as a linear combination of the vectors $v_1', \ldots, v_m'$ and vice versa.
- Some spanning sets include redundant vectors. For example,

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Linear dependence and independence

Definition 2.3.4 of Johnston (2021)

A set of vectors $\mathcal{S} = \{v_1, v_2, \ldots, v_k\}$ is *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, at least one of which is not zero, such that

$$c_1 v_1 + \ldots + c_k v_k = 0.$$

If $\mathcal{S}$ is not linearly dependent then it is called *linearly independent*.

- A set consisting of a single nonzero vector is linearly independent.
- A set containing the vector 0 is linearly dependent.
- A set with two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Euclidean vector spaces (cont.)

Bases and coordinates

Bases revisited

Definition 2.4.1 of Johnston (2021)

A basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ is a set of vectors in $\mathcal{S}$ that

(a) spans $\mathcal{S}$;

(b) is linearly independent.

- A basis is "big enough" to span the subspace, but it is not "so big" that it contains redundancies.
- We have already seen the standard basis of $\mathbb{R}^n$, i.e., $\{e_1, \ldots, e_n\}$.
- Every subspace other than $\{0\}$ (see Remark 2.4.1 of Johnston (2021)) has infinitely many different bases. For example,

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^2$.

Uniqueness of linear combinations

Theorem 3.1.1 of Johnston (2021)

Let $\mathcal{S} \subset \mathbb{R}^n$ with basis B. For every vector $v \in \mathcal{S}$, there is exactly one way to write v as a linear combination of the vectors from B.

Proof.

Suppose that some vector $v \in V$ can be written as

$$v = c_1v_1 + c_2v_2 + \ldots + c_nv_n \quad \text{and} \quad v = k_1v_1 + k_2v_2 + \ldots + k_nv_n.$$

Subtracting the first equation from the second equation, we obtain

$$0 = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \ldots + (c_n - k_n)v_n.$$

Linear independence of B implies that $c_1 = k_1, \ldots, c_n = k_n$. $\square$

Coordinates

Definition 3.1.1 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, $B = \{v_1, v_2, \ldots, v_k\}$ is a basis for $\mathcal{S}$, and $v \in \mathcal{S}$. Then the unique scalars $c_1, c_2, \ldots, c_k$ for which

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

are called the *coordinates* of v with respect to B, and the vector

$$[v]_B := (c_1, c_2, \ldots, c_k)$$

is called the *coordinate vector* of v with respect to B.

When we work with coordinates or coordinate vectors, we simply use the basis vectors in the order written. If $B = \{v_1, v_2, \ldots, v_k\}$, then the "first" basis vector is v_1, the "second" basis vector is v_2, and so on.

Examples of coordinates

- For $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have

$$[v]_B = (v_1, \ldots, v_n)$$

  with respect to the standard basis $B = \{e_1, \ldots, e_n\}$.
- The coordinates of $v = (5, 0)$ with respect to the basis

$$B' := \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

  of $\mathbb{R}^2$ are $[v]_{B'} = (10, -5)$.

Euclidean vector spaces (cont.)

Dimension

Number of vectors in a basis

Theorem 2.4.1 of Johnston (2021)

Let $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ and suppose that $B, C \subset \mathcal{S}$ are finite sets with the properties that B is linearly independent and $\mathrm{span}(C) = \mathcal{S}$. Then

$$|B| \leq |C|.$$

Corollary 2.4.2 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$. Every basis of $\mathcal{S}$ has the same number of vectors.

Proof.

Suppose that B and C are bases of $\mathcal{S}$. Then $|B| \leq |C|$ since B is linearly independent and $\mathrm{span}(C) = \mathcal{S}$. On the other hand, $|C| \leq |B|$ since C is linearly independent and $\mathrm{span}(B) = \mathcal{S}$. Hence, $|B| = |C|$. $\square$

Definition 2.4.2 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$. The number of vectors in a basis of $\mathcal{S}$ is called the *dimension* of $\mathcal{S}$ and is denoted by $\dim(\mathcal{S})$.

We formally define the dimension of $\{0\}$ to be 0.

Examples of dimensions

- $\dim(\mathbb{R}^n) = n$.
- Lines are 1-dimensional since a single vector acts as a basis of a line.
- Planes in $\mathbb{R}^3$ are 2-dimensional since two non-parallel vectors form a basis of a plane.

Creating a basis

Theorem 2.4.3 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ and $B \subset \mathcal{S}$ is a finite set of vectors.

(a) If B is linearly independent, then there is a basis C of $\mathcal{S}$ with $B \subset C$.

(b) If B spans $\mathcal{S}$, then there is a basis C of $\mathcal{S}$ with $C \subset B$.

We can always toss away vectors from a spanning set until it is small enough to be a basis, and we can always add new vectors to a linearly independent set until it is big enough to be a basis.

Basis

If one already knows $\dim(\mathcal{S})$, then checking if something is a basis becomes easier.

Theorem 2.4.4 of Johnston (2021)

Suppose $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ and $B \subset \mathcal{S}$ is a set containing k vectors.

(a) If $k \neq \dim(\mathcal{S})$ then B is not a basis of $\mathcal{S}$.

(b) If $k = \dim(\mathcal{S})$, then the following are equivalent

  (i) B spans $\mathcal{S}$;

  (ii) B is linearly independent;

  (iii) B is a basis of $\mathcal{S}$.

For instance, any two non-proportional vectors in $\mathbb{R}^2$ are a basis.

Proof

Proof.

B spans $\mathcal{S} \implies B$ is linearly independent. Suppose otherwise, i.e., $\exists v \in B$ such that $v \in \mathsf{span}(B \setminus \{v\})$. Then $\mathsf{span}(B \setminus \{v\}) = \mathsf{span}(B) = \mathcal{S}$. But this is a contradiction since no set with $k-1$ elements can span a k-dimensional subspace.

B is linearly independent $\implies \mathsf{span}(B) = \mathcal{S}$. Suppose otherwise, i.e., $\exists v \in \mathcal{S}$ such that $v \notin \mathsf{span}(B)$. Then the set $B \cup \{v\}$ has $k+1$ vectors and still is linearly independent. But this is a contradiction since a set with $k+1$ vectors cannot be linearly independent in an k dimensional subspace. $\square$

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.