

# AI511/MM505 Linear Algebra with Applications

## Lecture 7 – Compositions, subspaces, linear combinations

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# Outline

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Recap

Euclidean vector spaces (cont.)

Examples of linear transformations (cont.)

Compositions of linear transformations

Simulated example

Subspaces

Linear combinations and independence

## Recap

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## The standard basis

- The set of vectors  $\{e_1, e_2, \dots, e_n\}$ , where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

is called the *standard basis* in  $\mathbb{R}^n$ .

- Any vector  $v \in \mathbb{R}^n$  can be written in terms of the vectors  $e_1, e_2, \dots, e_n$  in the following way

$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n. \tag{1}$$

## Matrix operators or matrix transformations

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Every matrix  $A \in \mathbb{R}^{m \times n}$  corresponds to a *matrix operator* or a *matrix transformation*  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by

$$T_A(x) := Ax$$

for  $x \in \mathbb{R}^n$ .

## Definition 1.4.1 of Johnston (2021)

A *linear transformation* is a function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  that satisfies the following two properties:

- (a)  $T(v + w) = T(v) + T(w)$  for all vectors  $v, w \in \mathbb{R}^n$ ;
- (b)  $T(cv) = cT(v)$  for all vectors  $v \in \mathbb{R}^n$  and all scalars  $c \in \mathbb{R}$ .

Visually speaking, a transformation is “linear” if it has two properties

- (i) all lines remain lines, without getting curved;
- (ii) the origin remains fixed in place.

# Matrices as linear transformations

## Theorem 1.4.1 of Johnston (2021)

A function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation if and only if there exists a matrix  $[T] \in \mathbb{R}^{m \times n}$  such that

$$T(v) = [T]v \quad \text{for all } v \in \mathbb{R}^n.$$

Furthermore, the unique matrix  $[T]$  with this property is called the *standard matrix* of  $T$ , and it is

$$[T] := \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}.$$

In simple terms, a transformation is linear if and only if it is a matrix transformation.

## Euclidean vector spaces (cont.)

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## Euclidean vector spaces (cont.)

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Examples of linear transformations (cont.)

## Orthogonal projection onto the direction of a unit vector $u$

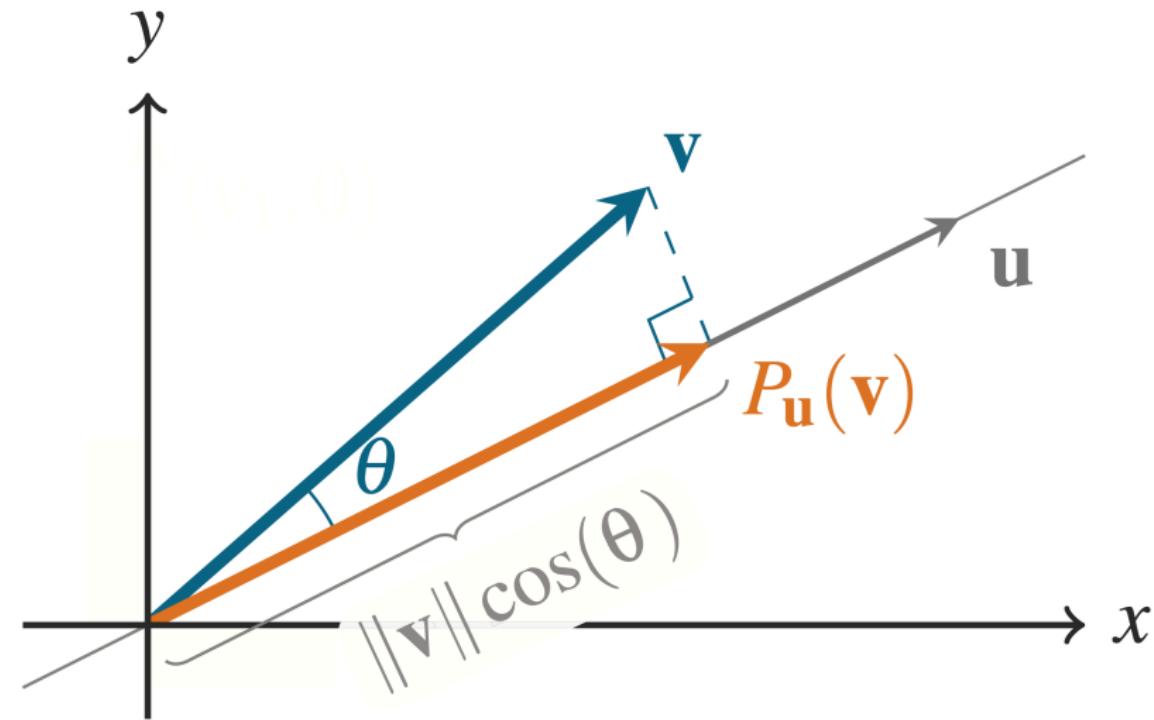


Figure 1.16 of Johnston (2021)

## Orthogonal projection onto the direction of a unit vector $u$

- The projection of a vector  $v$  onto the line in the direction of  $u$  is denoted by  $P_u(v)$ , and it points a distance of  $\|v\| \cos(\theta)$  (might be negative) in the direction of  $u$ , where  $\theta$  is the angle between  $v$  and  $u$ .
- For  $v, w \in \mathbb{R}^2$ , we have that

$$\cos(\theta) = \frac{v \cdot w}{\|v\| \|w\|},$$

$$P_u(v) = \|v\| \cos(\theta) u = \|v\| \frac{(v \cdot u)}{\|v\| \|u\|} u = uu^T v$$

using the fact that  $\|u\| = 1$  and  $v \cdot u = u^T v$ .

- Hence,  $P_u(v)$  is a linear transformation, and its standard matrix is given by

$$[P_u] = uu^T \in \mathbb{R}^{n \times n}.$$

## Orthogonal projection onto the direction of a vector $w$

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In general, we have that

$$P_w(v) = \frac{1}{\|w\|^2} w w^T v = \frac{w w^T}{w^T w} v$$

for two vectors  $v, w \in \mathbb{R}^n$  provided that  $w \neq 0$ .

## Euclidean vector spaces (cont.)

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Compositions of linear transformations

# Composition of linear transformations

## Definition 1.4.2 of Johnston (2021)

Suppose  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$  are linear transformations. Then *composition* of  $S$  and  $T$  is the function  $S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^p$  defined by

$$(S \circ T)(v) = S(T(v)) \quad \text{for all } v \in \mathbb{R}^n.$$

- $T$  sends  $\mathbb{R}^n$  to  $\mathbb{R}^m$  and  $S$  sends  $\mathbb{R}^m$  to  $\mathbb{R}^p$  while the composition  $S \circ T$  skips the intermediate step and sends  $\mathbb{R}^n$  directly to  $\mathbb{R}^p$ .
- Is  $S \circ T$  linear? If so, what is its standard matrix?

# Composition is linear

## Theorem 1.4.2 of Johnston (2021)

Suppose  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $S: \mathbb{R}^m \rightarrow \mathbb{R}^p$  are linear transformations with standard matrices  $[T] \in \mathbb{R}^{m \times n}$  and  $[S] \in \mathbb{R}^{p \times m}$ , respectively. Then  $S \circ T: \mathbb{R}^n \rightarrow \mathbb{R}^p$  is a linear transformation, and its standard matrix is

$$[S \circ T] = [S][T].$$

- In simple terms, composition of linear maps is a matrix product of standard matrices.
- This theorem is the main reason that matrix multiplication was defined in the seemingly bizarre way.
- Recommended to watch *Matrix multiplication as composition of 3Blue1Brown*.

## Example on the blackboard

- Consider two linear transformations

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}.$$

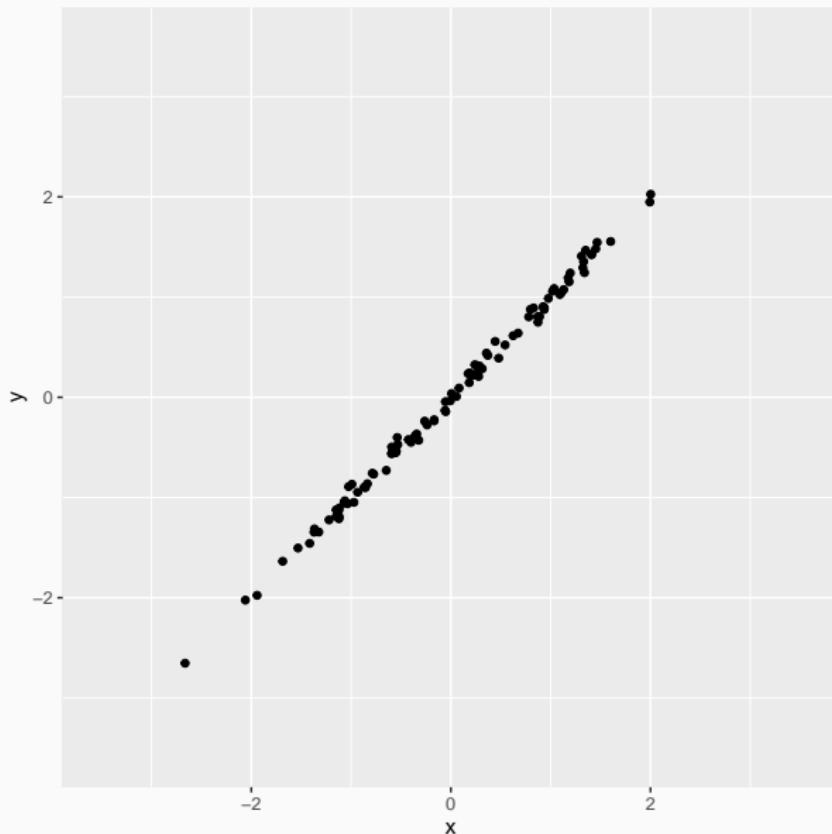
- The linear transformation  $B$ 
  - (i) rotates  $e_1$  counterclockwise by  $45^\circ$  and extends its length to  $\sqrt{2}$ ;
  - (ii) rotates  $e_2$  counterclockwise by  $90^\circ$  and extends its length to 2.
- The linear transformation  $A$ 
  - (i) flips  $e_1$  and  $e_2$  around  $45^\circ$  line;
  - (ii) extends the length of  $e_2$  to 2 (stretches the space horizontally by 2).
- Find the standard matrix of the composition  $AB$  by describing where  $e_1$  and  $e_2$  are mapped and confirm your result by multiplying  $A$  and  $B$ .

## Euclidean vector spaces (cont.)

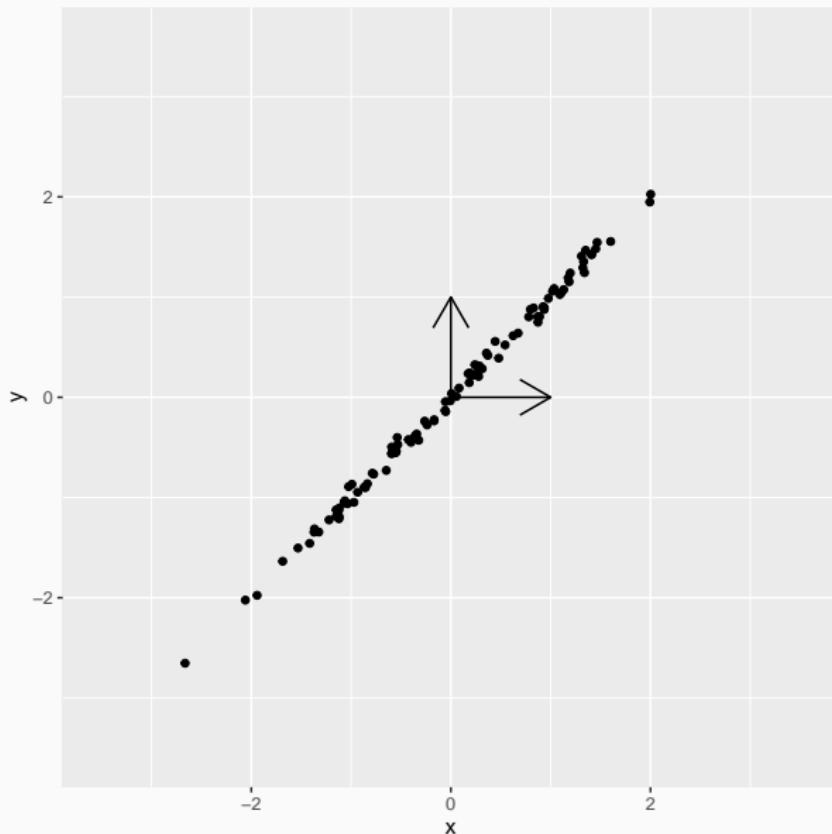
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Simulated example

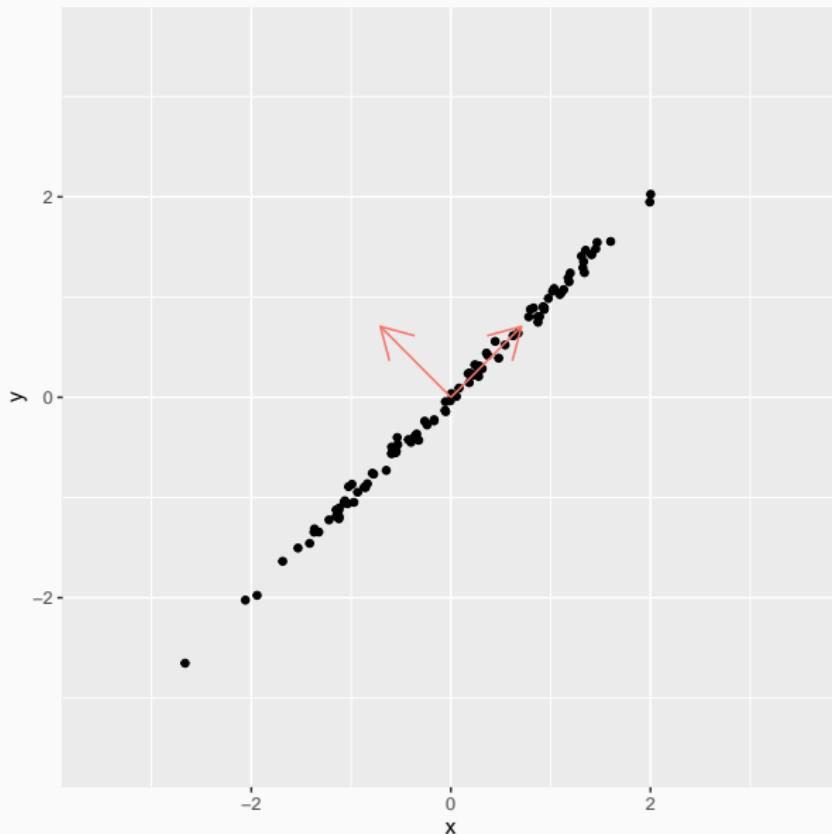
# Simulated data example in $\mathbb{R}^2$



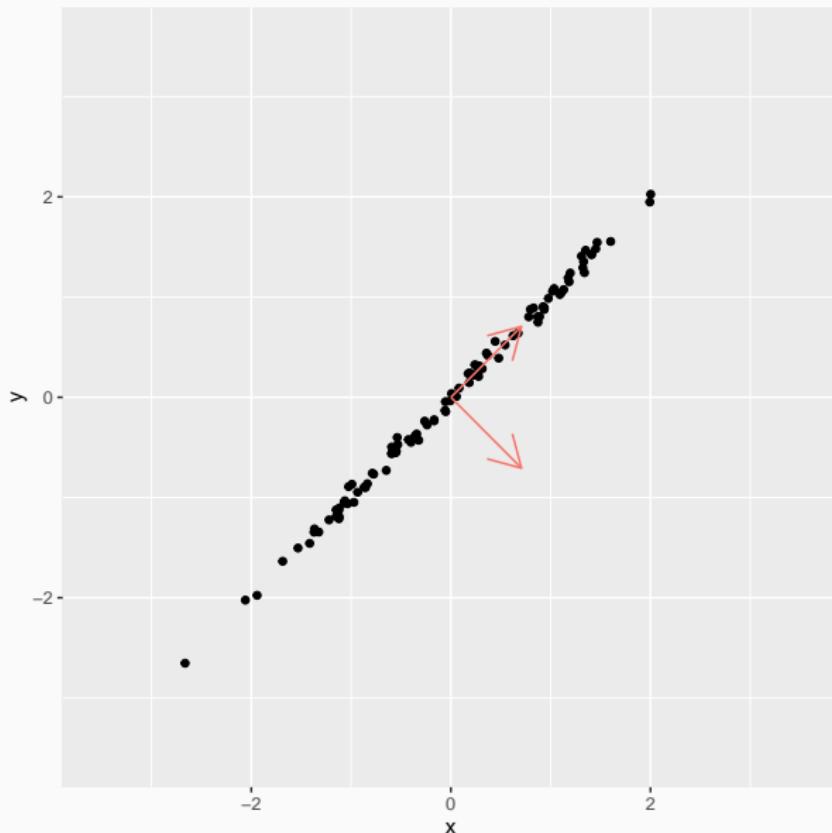
# Simulated data example in $\mathbb{R}^2$



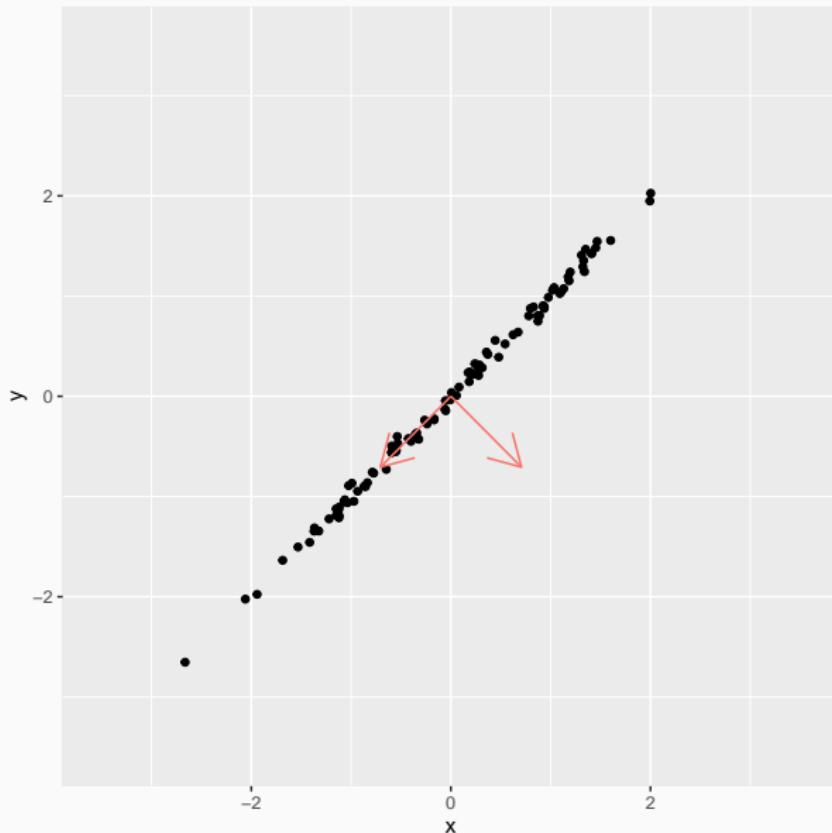
# Simulated data example in $\mathbb{R}^2$



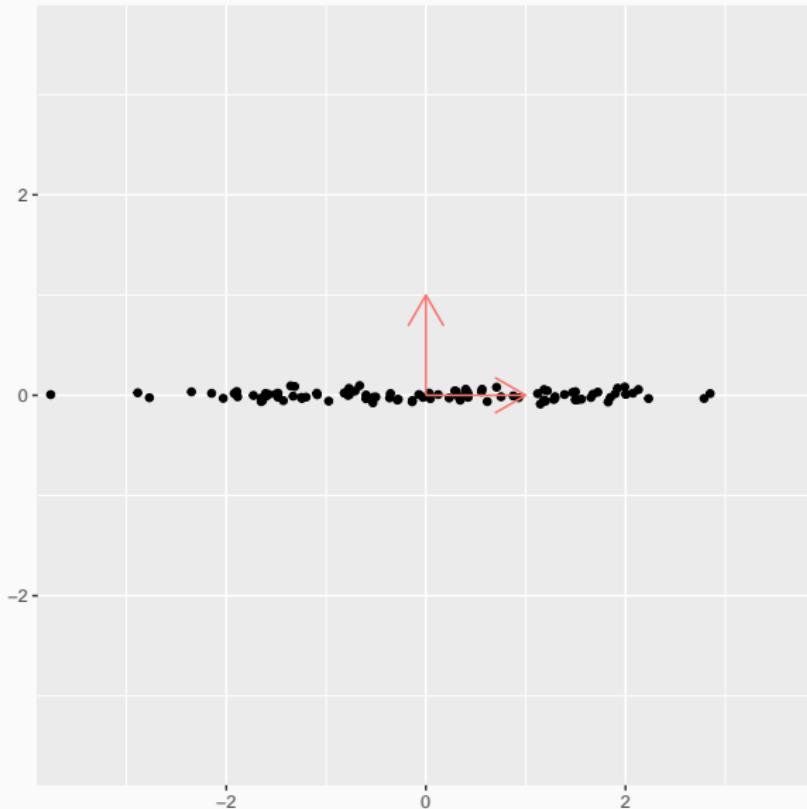
# Simulated data example in $\mathbb{R}^2$



# Simulated data example in $\mathbb{R}^2$



# Simulated data example in $\mathbb{R}^2$



## Questions

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- What is this thing where the simulated vectors approximately live?
- What does it mean that we have a coordinate system for a Euclidean space and what role does it play in representing vectors?
- How do we transform the coordinates of a vector when changing from one coordinate system to another in a Euclidean space?

## Euclidean vector spaces (cont.)

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### Subspaces

# Subspaces

## Definition 2.3.1 of Johnston (2021)

A *subspace* of  $\mathbb{R}^n$  is a non-empty set  $\mathcal{S}$  of vectors in  $\mathbb{R}^n$  with the properties that

- (a) if  $v, w \in \mathcal{S}$ , then  $v + w \in \mathcal{S}$ ;
- (b) if  $v \in \mathcal{S}$  and  $c \in \mathbb{R}$ , then  $cv \in \mathcal{S}$ .

- (a) ensures that subspaces are “flat”.
- (b) makes it so that they are “infinitely long”.
- Observe that every subspace contains the zero vector (think why this is the case).

# Matrix subspaces

## Definition 2.3.2 of Johnston (2021)

Suppose  $A \in \mathbb{R}^{m \times n}$ .

- (a) The *range* of  $A$  is the subspace of  $\mathbb{R}^m$ , denoted by  $\text{range}(A)$ , that consists of all vectors of the form  $Ax$ .
- (b) The *null space* of  $A$  is the subspace of  $\mathbb{R}^n$ , denoted by  $\text{null}(A)$ , that consists of all solutions  $x$  of the system of linear equations  $Ax = 0$ .

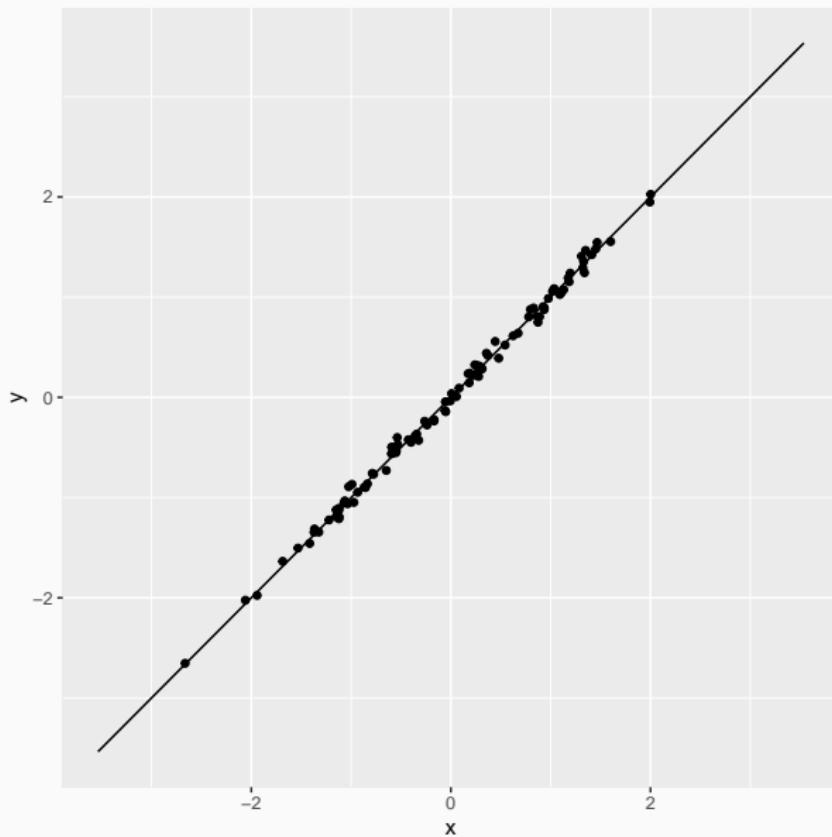
Is the set of solutions of  $Ax = b$  with  $b \neq 0$  a subspace in  $\mathbb{R}^n$ ?

## Are $\text{range}(A)$ and $\text{null}(A)$ subspaces?

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- $\text{range}(A)$  is non-empty since  $0 \in \text{range}(A)$ . If  $Ax, Ay \in \text{range}(A)$ , then  $Ax + Ay = A(x + y) \in \text{range}(A)$ . If  $Ax \in \text{range}(A)$ , then  $cAx = A(cx) \in \text{range}(A)$ .
- $\text{null}(A)$  is non-empty since  $0 \in \text{null}(A)$ . If  $x, y \in \text{null}(A)$ , then  $x + y \in \text{null}(A)$  since  $A(x + y) = Ax + Ay = 0 + 0 = 0$ . If  $x \in \text{null}(A)$ , then  $cx \in \text{null}(A)$ , since  $A(cx) = cAx = c0 = 0$ .

# Simulated data example in $\mathbb{R}^2$



## Euclidean vector spaces (cont.)

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Linear combinations and independence

# Linear combination and span

## Definition 1.1.3 of Johnston (2021)

A *linear combination* of the vectors  $v_1, \dots, v_k \in \mathbb{R}^n$  is any vector of the form

$$c_1v_1 + c_2v_2 + \dots + c_kv_k,$$

where  $c_1, c_2, \dots, c_k \in \mathbb{R}$ .

## Definition 2.3.3 of Johnston (2021)

If  $B = \{v_1, \dots, v_k\}$  is a set of vectors in  $\mathbb{R}^n$ , then the set of all linear combinations of those vectors is called their *span*, and it is denoted by  $\text{span}(B)$  and  $\text{span}(v_1, \dots, v_k)$ .

Recommended to watch *Linear combinations, span, and basis vectors* of 3Blue1Brown.

## Span (cont.)

### Theorem 2.3.1 of Johnston (2021)

Let  $v_1, \dots, v_k \in \mathbb{R}^n$  with  $k \geq 1$ . Then  $\text{span}(v_1, \dots, v_k)$  is a subspace of  $\mathbb{R}^n$ .

### Theorem 2.3.2 of Johnston (2021)

Suppose that  $A \in \mathbb{R}^{m \times n}$  has columns  $a_1, \dots, a_n$ . Then

$$\text{range}(A) = \text{span}(a_1, \dots, a_n).$$

## Spanning set and linear independence

- If  $\mathcal{S}$  is a subspace and  $\mathcal{S} = \text{span}\{v_1, \dots, v_n\}$  for some  $v_1, \dots, v_n$ , then  $\{v_1, \dots, v_n\}$  is called a *spanning set* of  $\mathcal{S}$ .
- Spanning sets are not unique. That is, we may have

$$\mathcal{S} = \text{span}\{v_1, \dots, v_n\} = \text{span}\{v'_1, \dots, v'_n\}.$$

This happens if and only if each  $v_i$  can be written as a linear combination of the vectors  $v'_1, \dots, v'_m$  and vice versa.

- Some spanning sets include redundant vectors. For example,

$$\mathbb{R}^2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

# Linear dependence and independence

## Definition 2.3.4 of Johnston (2021)

A set of vectors  $\mathcal{S} = \{v_1, v_2, \dots, v_k\}$  is *linearly dependent* if there exist scalars  $c_1, c_2, \dots, c_k \in \mathbb{R}$ , at least one of which is not zero, such that

$$c_1v_1 + \dots + c_kv_k = 0.$$

If  $\mathcal{S}$  is not linearly dependent then it is called *linearly independent*.

- A set consisting of a single nonzero vector is linearly independent.
- A set containing the vector  $0$  is linearly dependent.
- A set with two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

# References

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-  Johnston, Nathaniel (2021). *Introduction to linear and matrix algebra*. Springer Cham.