Multivariate functions

AI503
Shan Shan

Lecture 4

Multivariate functions

Definition

A **function of n variables** is a function whose domain is a subset A of $\mathbb{R}^n$ and whose range is a subset B of $\mathbb{R}$.

Examples

What are the domain and range for each of these functions?

1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = (x^2 + y^2)\log(xy)$
3. $f(x, y, z) = \log(z)x^2y^2$

Graph of a Function

Definition

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The graph of f is the subset of $\mathbb{R}^{n+1}$ given by

$$(x_1, \ldots, x_n, f(x_1, \ldots, x_n)).$$

- $n = 1$: graph is a curve in $\mathbb{R}^2$.
- $n = 2$: graph is a surface in $\mathbb{R}^3$.
- $n = 3$: graph is a hypersurface in $\mathbb{R}^4$

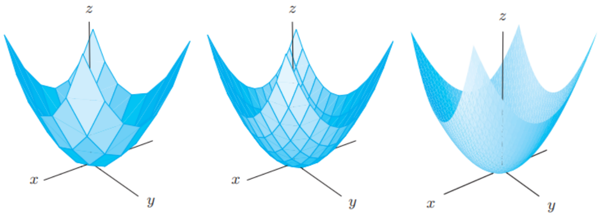
Example: Graph of $f(x, y) = x^2 + y^2$

Make a table of values of f.

	y						
x	−3	−2	−1	0	1	2	3
−3	18	13	10	9	10	13	18
−2	13	8	5	4	5	8	13
−1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

Example: Graph of $f(x, y) = x^2 + y^2$

We plot the points. For example, we plot (1, 2, 5) because f(1, 2) = 5. Then, we connect the points corresponding to the rows and columns in the table. This gives us a wire-frame picture of the graph. As we refine the grid, we get a surface called a *paraboloid*



Observe that if x is held fixed and y is allowed to vary, the graph dips down and then goes back up, just like the entries in the rows of the table.

Questions

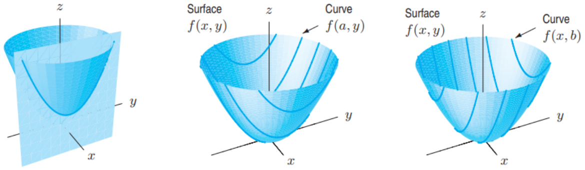
Can you describe in words the graphs of the following functions?

1. $g(x, y) = x^2 + y^2 + 5$
2. $h(x, y) = -x^2 - y^2$
3. $q(x, z) = x^2 + (z - 1)^2$

Cross Section

Definition

For $f(x, y)$, a **cross-section** is obtained by holding one variable fixed and letting the other vary.



Graphically, it is the intersection of the surface with a plane $x = c$ or $y = c$.

Example

Describe the cross-sections of the function $g(x, y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g.

Cross Sections in $\mathbb{R}^n$

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n)$. Fix some variables $x_i = a_i$ for $i \in I \subset \{1, \ldots, n\}$. The resulting function of the remaining variables is the **cross section**:

$$g(x_j : j \notin I) = f(x_1, \ldots, x_n) \quad \text{with } x_i = a_i \text{ fixed.}$$

Dimension of cross section: $n - |I|$

Example

Consider

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Fix $z = 1$: cross section is

$$g_1(x, y) = x^2 + y^2 + 1$$

Fix $y = z = 1$: cross section is

$$g_2(x) = x^2 + 2$$

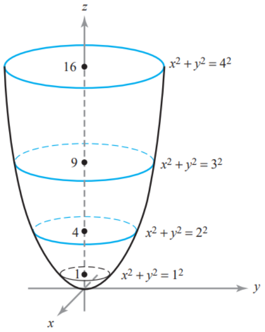
Level Set

Definition

A **level set** is a subset of the domain where f is constant:

$$f(x_1, \ldots, x_n) = c$$

A few level sets of the function $f(x, y) = x^2 + y^2$ are graphed below.



Examples

Describe the level sets of the functions

1. $f(x, y, z) = x^2 + y^2 + z^2$
2. $f(x, y, z) = x^2 + y^2 - z^2$

Linear Functions of Several Variables

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **linear** if it can be written as

$$f(x_1, x_2, \ldots, x_n) = c + m_1 x_1 + m_2 x_2 + \cdots + m_n x_n,$$

where $c, m_1, \ldots, m_n$ are constants.

- The graph of a linear function in n variables is an n-dimensional hyperplane in $\mathbb{R}^{n+1}$.
- Linear functions satisfy the superposition principle:

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}), \quad f(k\mathbf{x}) = kf(\mathbf{x}).$$

Example when $n = 2$

Find the equation of the plane passing through the points $(1, 0, 1)$, $(1, -1, 3)$, and $(3, 0, -1)$.

Example when $n = 2$

What is the minimum number of points required to uniquely determine a plane?

Example when $n = 2$

A convenient formula

If a plane has slope m in the x-direction, has slope n in the y-direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

Limit

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} = (a_1, \ldots, a_n)$. We say

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$

if $f(x_1, \ldots, x_n)$ is as close to L as we please whenever the distance from the point $(x_1, \ldots, x_n)$ to the point $(a_1, \ldots, a_n)$ is sufficiently small.

1. $\mathbf{x} \to \mathbf{a}$ means $\mathbf{x}$ approaches $\mathbf{a}$ but is not equal to $\mathbf{a}$.
2. The limit *does not exist* if no such L exists.
3. The limit laws from Lect 2 on addition, subtraction, multiplication, division, composition can be extended naturally to functions of several variables.

Limit (ε-δ Definition)

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} = (a_1, \ldots, a_n)$. We say

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies |f(\mathbf{x}) - L| < \varepsilon.$$

- As $\mathbf{x}$ gets arbitrarily close to $\mathbf{a}$, $f(\mathbf{x})$ gets arbitrarily close to L.
- Recall $\|\mathbf{x} - \mathbf{a}\| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$.
- The condition $0 < \|\mathbf{x} - \mathbf{a}\|$ ensures we consider points *other than* $\mathbf{a}$ itself.

Questions on Limits

In one-variable calculus, a limit exists at $x = a$ if the left-hand and right-hand limits match:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

Does this make sense for multivariate functions?

Questions on Limits

In one-variable calculus, a limit exists at $x = a$ if the left-hand and right-hand limits match:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

Does this make sense for multivariate functions?

Important

For $f(x, y)$, approaching a point (a, b) can occur along infinitely many paths:

$$(x, y) \to (a, b) \text{ along any curve in the domain.}$$

Questions on Limits

Does the following limit exists?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{xy}{x^2+y^2}$$

Questions on Limits

Does the following limit exists?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{xy}{x^2+y^2}$$

Conclusion: Matching "left and right" along axes is not enough. Limit may fail along other directions.

More Questions on Limits

Compute or determine whether the following limits exist:

1. $\lim_{(x,y)\to(0,0)} e^{-x-y}$
2. $\lim_{(x,y)\to(0,0)} x^2 + y^2$
3. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$

Proof of (3) using the ϵ and δ definition

Continuity

Definition

f is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

A function is continuous on a region U in $\mathbb{R}^n$ if it is continuous at each point in U. *There is no break in the graph of f!*

Using the limit laws, one can show that for functions of several variables, the following are continues

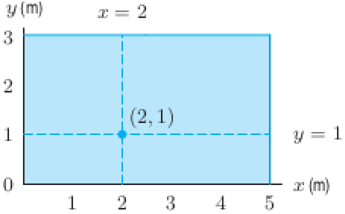
1. sums of continuous functions
2. products of continuous functions
3. compositions of continuous functions
4. quotient of two continuous functions wherever the denominator function is nonzero.

Examples

Is the function continuous in the given region?

1. $\dfrac{1}{x^2+y^2}$ on $-1 \le x, y \le 1$
2. $\dfrac{1}{x^2+y^2}$ on $1 \le x, y \le 2$
3. $\dfrac{y}{x^2+2}$ on $x^2+y^2 \le 1$

Imagine an unevenly heated thin rectangular metal plate lying in the xy-plane. Temperature at (x, y) is $T(x, y)$.



y (m)						
3	85	90	**110**	135	155	180
2	100	110	**120**	145	190	170
1	**125**	**128**	**135**	**160**	**175**	**160**
0	120	135	**155**	160	160	150
	0	1	2	3	4	5
	x (m)					

How does T vary near the point $(2, 1)$?

Partial Derivatives Intuition

Let's make the intuition more precise...

1. Along $y = 1$, consider $u(x) = T(x, 1)$.
2. By definition,

$$u'(2) = \lim_{h \to 0} \frac{T(2+h, 1) - T(2, 1)}{h}.$$

   Then $u'(2)$ measures rate of change of T in x direction at $(2, 1)$.
3. Denote $u'(2)$ as $T_x(2, 1)$. Approximate $T_x(2, 1)$ with $h = 1$.

Estimate rate of change in y direction.

Definition of Partial Derivatives

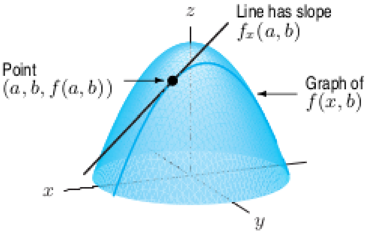
Partial derivatives of f at (a, b)

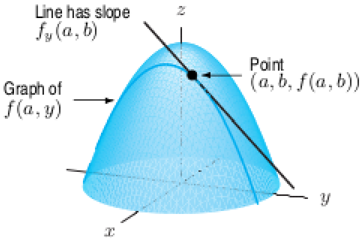
For all points at which the limit exist, we define the partial derivatives at the point (a, b) by

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

- Functions of (x, y): $f_x(x, y)$, $f_y(x, y)$.
- Alternative notation: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

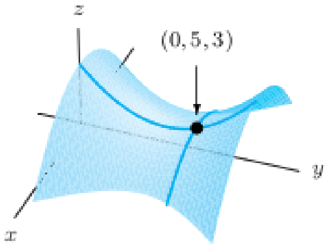
Geometric interpretation





Partial derivatives are the slope of tangent lines of the cross section.

Sign of Partial Derivatives



1. Sign of $f_x(0, 5)$?
2. Sign of $f_y(0, 5)$?

Computing Partial Derivatives

Since f_x and f_y are derivatives with one variable fixed, we can use ordinary differentiation formulas. Let

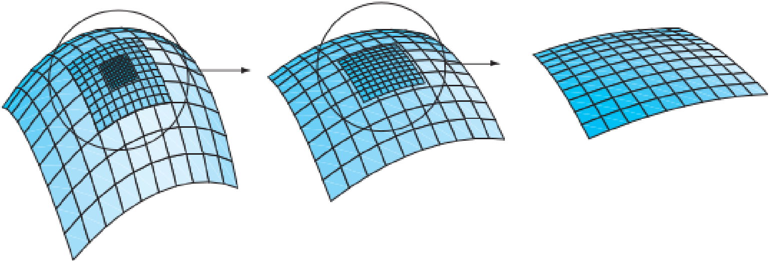
$$f(x, y) = \frac{x^2}{y+1}.$$

Find $f_x(3, 2)$.

Tangent Plane

Seeing a plane when we zoom in at a point tells us (provided the plane is not vertical) that $f(x, y)$ is closely approximated near that point by a linear function, $L(x, y)$.

The graph of the function $z = L(x, y)$ is the *tangent plane*.



Tangent Plane

Tangent Plane Equation

At (a, b), if f is differentiable:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example

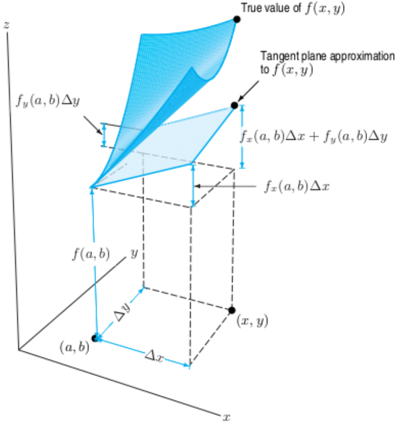
Find tangent plane to $f(x, y) = x^2 + y^2$ at $(3, 4)$.

Local Linearity

Tangent Plane Approximation

For (x, y) near (a, b):

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$



Example

Let $f(x, y) = x^2 + y^2$ and call the tangent plane you computed at (3,4) $L(x, y)$. We define the error function

$$E(x, y) = f(x, y) - L(x, y).$$

1. The error near (3,4) is given by $E(3 + h, 4 + k)$ where h, k are small.
2. What is the distance between $(3 + h, 4 + k)$ and $(3, 4)$?
3. Compare $|E(3 + h, 4 + k)|$ with $h^2 + k^2$.
4. Observe:
$$\frac{|E(x, y)|}{h^2 + k^2} \approx \text{constant}.$$
5. What can you say about the error near $(3, 4)$?

Local linearity

Tangent Plane Approximation

For (x, y) near (a, b):

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Error behavior: If f has continuous second derivatives, then

$$f(x, y) - L(x, y) = O\left(\|(x - a,\ y - b)\|^2\right).$$

That is, the error is proportional to the *square of the distance* from (x, y) to (a, b).