AI511/MM505 Linear Algebra with Applications

Lecture 6 – Euclidean Vector Spaces (cont.), Linear Transformations

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Outline

Recap

Euclidean vector spaces (cont.)

- Bases
- Linear transformations
- Examples of linear transformations

Recap

Minors, cofactors, and determinant

Definition

If $A \in \mathbb{R}^{n\times n}$, then the (i,j)-*th minor*, M_{ij}, of A is the determinant of the $(n-1)\times(n-1)$-matrix obtained by deleting row i and column j. The (i,j)-*th cofactor* is the number $C_{ij} := (-1)^{i+j}M_{ij}$.

Definition

The determinant of $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.$$

Cofactor expansions

Theorem 3.2.8 of Johnston (2021)

Suppose $A \in \mathbb{R}^{n \times n}$. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \quad (1)$$
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} \quad (2)$$

for all $i, j = 1, \ldots, n$.

- (1) is called a cofactor expansion along the i-th row.
- (2) is called a cofactor expansion along the j-th column.

Invertibility and the determinant

Theorem

A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\det(A) \neq 0$.

Vectors

Definition

An ordered tuple $v = (v_1, \ldots, v_n)$ of real numbers $v_1, \ldots, v_n$ is called a *vector* (in dimension n) and the set of all such tuples is denoted $\mathbb{R}^n$.

If we give no indication otherwise, vectors are assumed to be column vectors

Dot product

Definition 1.2.1 of Johnston (2021)

Suppose $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$. Then their *dot product*, denoted by $v \cdot w$, is the quantity

$$v \cdot w \coloneqq v_1 w_1 + v_2 w_2 + \ldots v_n w_n.$$

The dot product can be interpreted geometrically as roughly measuring the amount of overlap between v and w.

The angle between two vectors on the unit circle in $\mathbb{R}^2$

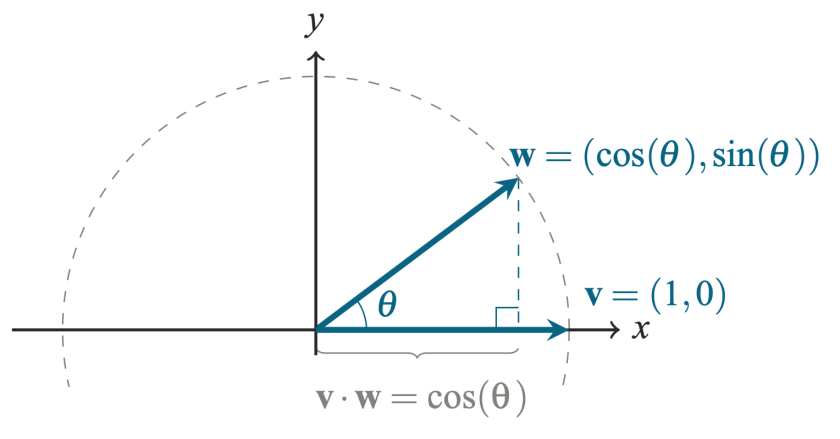
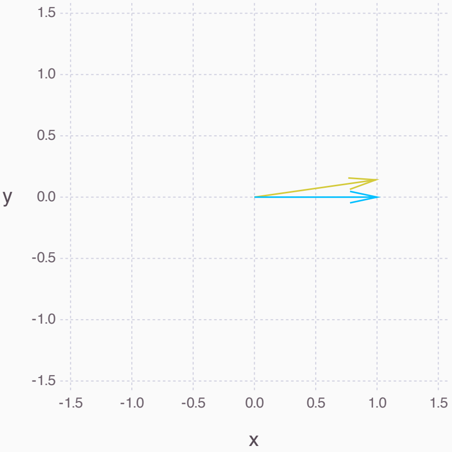


Figure 1.7 of Johnston (2021)

Vectors on the unit circle in $\mathbb{R}^2$: true or false?



(a) $v \cdot w > 0$; (b) $v \cdot w < 0$; (c) $v \cdot w = 0$; (d) $v \cdot w \approx 0$;

(e) $v \cdot w = 1$; (f) $v \cdot w \approx 1$; (g) $v \cdot w = -1$; (h) $v \cdot w \approx -1$.

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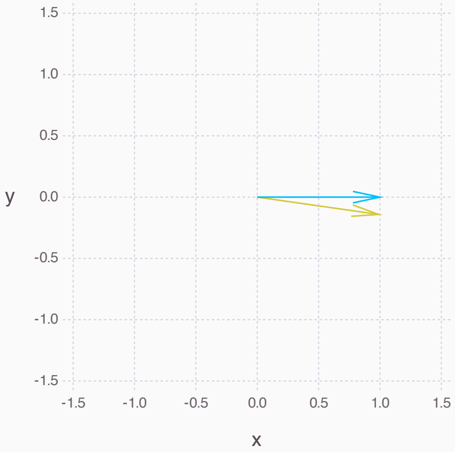

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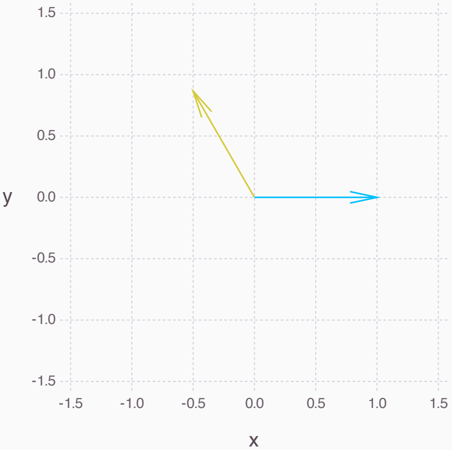

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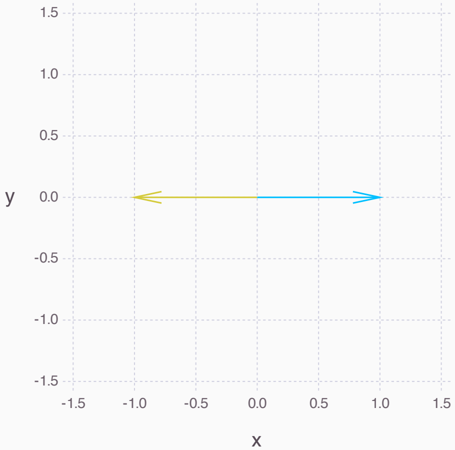

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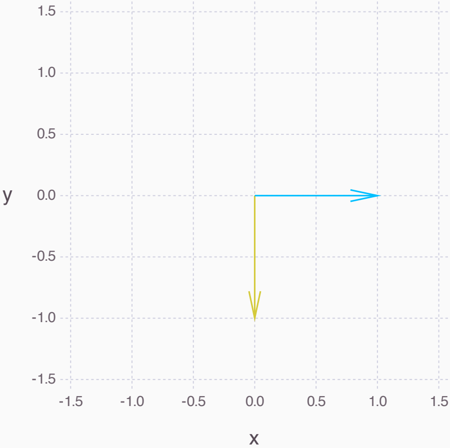
Vectors on the unit circle in $\mathbb{R}^2$: true or false?



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Euclidean vector spaces (cont.)

Dot product (cont.)

- Observe that
$$v \cdot w = v^T w = w^T v. \tag{3}$$
- When v and w are perpendicular, their dot product is equal to 0 and such vectors are called *orthogonal*.

Length of a vector

Definition 1.2.2 of Johnston (2021)

The *length* of a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, denoted by $\|v\|$, is the quantity

$$\|v\| := \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

$\|v\|$ is also called the Euclidean norm of the vector v.

Theorem 1.2.2 of Johnston (2021)

Suppose $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following properties hold:

(a) $\|cv\| = |c|\|v\|$;

(b) $\|v\| \geq 0$ with equality if and only if $v = 0$.

Unit vectors

- Vectors with length equal to 1 are called *unit vectors*.
- Any vector $v \neq 0$ can be *normalised* to have length equal to 1.
- We can write every non-zero vector $v \in \mathbb{R}^n$ in the form

$$v = \|v\|u,$$

where $u = v/\|v\|$ is the unique unit vector pointing in the same direction as v.

Renormalisation of a vector in $\mathbb{R}^2$ and the unit circle in $\mathbb{R}^2$

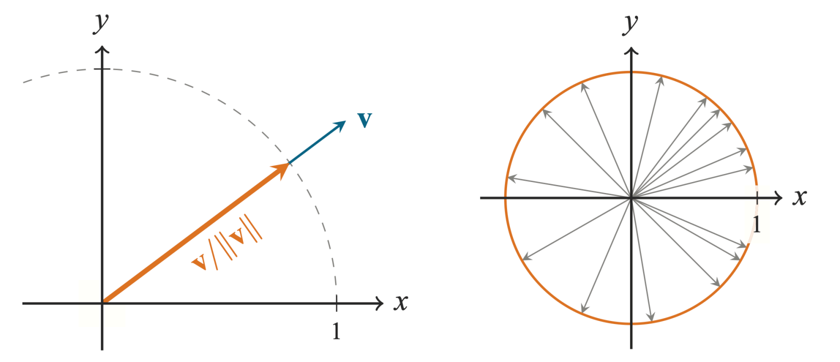


Figure 1.9 of Johnston (2021)

Euclidean vector spaces (cont.)

Bases

The standard basis

- The set of vectors $\{e_1, e_2, \ldots, e_n\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

  is called the *standard basis* in $\mathbb{R}^n$.
- Any vector $v \in \mathbb{R}^n$ can be written in terms of the vectors $e_1, e_2, \ldots, e_n$ in the following way

$$v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n. \tag{4}$$

- The standard basis vectors are sometimes denoted by $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ in $\mathbb{R}^3$.

Euclidean vector spaces (cont.)

Linear transformations

Functions and transformations

- Recall that a *function* is a rule that associates with each element of a set A one and only one element in a set B and we wite
$$f : A \to B.$$
- If f associates the element b with the element a, then we write
$$b = f(a)$$
and we say that b is the image of a under f or that $f(a)$ is the value of f at a.
- The set A is called the *domain* of f and the set B the *codomain* of f.
- The subset of the codomain that consists of all images of elements in the domain is called the *range* of f.

Section 1.8 of Anton, Rorres, and Kaul (2019)

Functions and transformations (cont.)

- In many applications the domain and codomain of a function are sets of real numbers.
- We will be concerned with functions for which the domain is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$ for some positive integers m and n.
- In this setting it is common to use italicised capital letters for functions, the letter T being typical.

Section 1.8 of Anton, Rorres, and Kaul (2019)

Definition 1 on page 78 of Anton, Rorres, and Kaul (2019)

If T is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, then we say that T is an *operator* or a *transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ or that T *maps* from $\mathbb{R}^n$ to $\mathbb{R}^m$, which we denote by writing

$$T : \mathbb{R}^n \to \mathbb{R}^m.$$

Matrices as transformations

- Consider now an $m \times n$-matrix A.
- For every vector $x \in \mathbb{R}^n$, the product

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ a_{21}x_1 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix}$$

  is a vector in $\mathbb{R}^m$.
- So we can think of A as a map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(x) := Ax$.
- T_A is called the *matrix operator* or *matrix transformation* associated with A.
- Recommended to watch *Linear transformations and matrices* of 3Blue1Brown.

Properties of matrix transformations

Theorem 1.8.1 of Anton, Rorres, and Kaul (2019)

For $A \in \mathbb{R}^{m \times n}$, $x, y \in \mathbb{R}^n$ and $k \in \mathbb{R}$, the following holds

(1) $T_A(0) = 0$;

(2) $T_A(k \cdot x) = k \cdot T_A(x)$ (homogeneity);

(3) $T_A(x + y) = T_A(x) + T_A(y)$ (additivity);

(4) $T_A(x - y) = T_A(x) - T_A(y)$.

- All these properties follow from our matrix multiplication rules. Note that (2) and (3) together imply (1) and (4).

Given some map $T : \mathbb{R}^n \to \mathbb{R}^m$, how can we know if it is a matrix transformation? To answer this question, we need to learn what a linear transformation is.

Definition 1.4.1 of Johnston (2021)

A *linear transformation* is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies the following two properties:

(a) $T(v + w) = T(v) + T(w)$ for all vectors $v, w \in \mathbb{R}^n$;

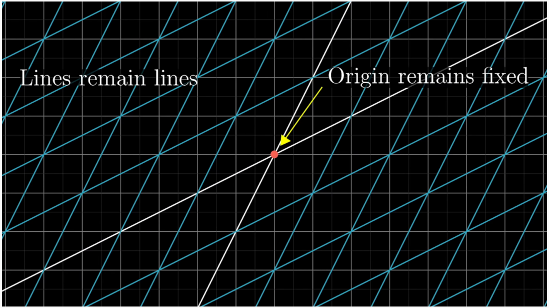
(b) $T(cv) = cT(v)$ for all vectors $v \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$.

Intuition

Visually speaking, a transformation is “linear” if it has two properties

(i) all lines must remain lines, without getting curved;

(ii) the origin must remain fixed in place.



Linear transformations and matrices by 3Blue1Brown

Standard matrix of a linear transformation

Theorem 1.4.1 of Johnston (2021)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists a matrix $[T] \in \mathbb{R}^{m \times n}$ such that

$$T(v) = [T]v \quad \text{for all} \quad v \in \mathbb{R}^n.$$

Furthermore, the unique matrix $[T]$ with this property is called the *standard matrix* of T, and it is

$$[T] := \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}.$$

Observe that $[T]$ denotes a matrix.

Partial proof

Proof.

We prove that any linear transformation can be represented as $T(v) = [T]v$ for all $v \in \mathbb{R}^n$. Using (4) and the assumption that T is linear,

$$\begin{aligned} T(v) &= T(v_1 e_1 + v_2 e_2 + \ldots + v_n e_n) \\ &= v_1 T(e_1) + v_2 T(e_2) + \ldots + v_n T(e_n) \\ &= \begin{bmatrix} T(e_1) & T(e_2) & \ldots & T(e_n) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= [T]v. \end{aligned}$$

$\square$

Example on the blackboard

- To find the standard matrix $[T]$ of a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, compute the column vectors $T(e_1), \ldots, T(e_n) \in \mathbb{R}^m$ to form the columns of $[T]$, i.e.,

$$[T] = \begin{bmatrix} T(e_1) & \ldots & T(e_n) \end{bmatrix}.$$

- Find the standard matrix of the following linear transformations
  (a) $T(v_1, v_2) = (v_1 + 2v_2, 3v_1 + 4v_2)$;
  (b) $T(v_1, v_2, v_3) = (3v_1 - v_2 + v_3, 2v_1 + 4v_2 - 2v_3)$.

Euclidean vector spaces (cont.)

Examples of linear transformations

Simplest linear transformations

- The *zero transformation* $0 : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$0(v) = 0$$

  for all $v \in \mathbb{R}^n$.
- The *identity transformation* $I : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$I(v) = v$$

  for all $v \in \mathbb{R}^n$.
- What are the standard matrices of these transformations?

Section 1.4.2 of Johnston (2021)

Orthogonal projections onto the x- and y-axes

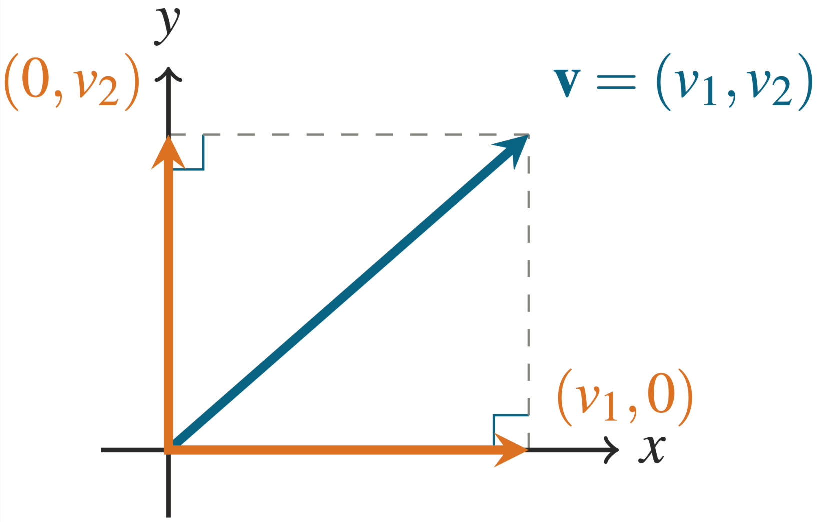


Figure 1.16 of Johnston (2021)

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). ***Elementary linear algebra: applications version.*** 12th edition. John Wiley & Sons, Inc.