

AI511/MM505 Linear Algebra with Applications

Lecture 1 – General Information, Systems of Linear Equations

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General information

Teachers and teaching assistant

Teacher of the theory part

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Teacher of the applications part

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Two course codes

- AI511 is a 7.5 ECTS course that has two parts:
 - a 5 ECTS theory part (taught by Vaidotas Characiejus);
 - a 2.5 ECTS applications part (taught by Henry Kirveslahti).
- The 5 ECTS course MM505 coincides with the theory part of AI511.

Lectures and exercise sessions

Week	Lectures	Exercise sessions
36	AI511/MM505	No exercise sessions
37	AI511/MM505	AI511/MM505
38	AI511	AI511
39	AI511/MM505	AI511/MM505
40	AI511/MM505	AI511/MM505
41	AI511/MM505	AI511/MM505
42	Autumn break	
43	AI511/MM505	AI511/MM505
44	TBA later	AI511/MM505

- Week 38 is compulsory only for the students of AI511.
- The complete schedule of the applications part will be announced by the applications part teacher.

Exercise sessions

- There will be weekly (except week 36) exercise sessions.
- The exercises will be made available one week before the exercise session.
- You will have to prepare for each of the exercise sessions by going over a set of exercises and problems in advance.
- It is not a problem if you are not able to solve all of the exercises completely before the exercise session.
- However, you are expected to get familiar with the exercises, consider possible solutions and get as far as you can.
- The teaching assistant will lead the exercise sessions and help you solve the exercises.

Take-home assignment

- The exam of the theory part of AI511 and MM505 will consist of a set of pen-and-paper problems to solve at home.
- The problems are expected to be posted in week 45 and you will have 72 hours to complete the assignment.
- This exam will have to be taken by the students of AI511 and by the students of MM505.
- For the students of AI511, there will be additional assignments in the applications part.

5 ECTS theory part

- The theory part of the course has 5 ECTS credits.
- What does it mean that it is 5 ECTS credits?
- 1 ECTS corresponds to 25 – 30 hours of work (see here).
- The time allocation looks as follows

Lectures	Exercise sessions	Outside classroom
24 hours	12 hours	89–114 hours

- The time outside clasroom includes going over the lecture material and exercises for the exercise sessions as well as preparing for and taking the exam.

- Course announcements will be posted on itslearning.
- Lecture and exercise session material will be posted on
<https://imada.sdu.dk/u/characiejus/linearalgebra2025/>
- The take-home assignment of the theory part will have to be downloaded and submitted on Digital Eksamens.

Textbook

Nathaniel Johnston (2021). *Introduction to linear and matrix algebra*.
Springer Cham

Animated series

Linear Algebra: An introduction to visualizing what matrices are really doing by 3Blue1Brown (freely available [here](#)).

Topics

- Systems of linear equations.
- Matrix operations, inverses, determinants.
- Vector spaces, basis, coordinates, linear independence.
- Linear transformations, eigenvalue problems, diagonalisation.
- Inner product and orthogonality.

Systems of linear equations

Linear algebra

- Linear algebra is used everywhere.
- For example, in machine learning¹, statistics, optimisation², signal processing, computer vision, etc.
- The central problem of linear algebra is solving systems of linear equations.
- Why would we be interested in systems of linear equations?

¹AI512: Introduction to Machine Learning

²AI505: Optimisation

Systems of linear equations

Motivation

Example: optimising a quadratic function

- Consider the following function

$$f(x) = ax^2 + bx + c \quad (1)$$

for $x \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$ are known constants.

- If $a \neq 0$, this function has a unique minimum or a unique maximum.
- How can we find x , where the value of the function is the smallest or the largest?

Example: optimising a quadratic function (cont.)

- The derivative³ is given by

$$f'(x) = 2ax + b$$

for $x \in \mathbb{R}$.

- By setting the derivative equal to 0, we obtain

$$2ax + b = 0. \tag{2}$$

- (2) is a *linear equation* that we need to solve to find a point where the function is at its unique minimum or at its unique maximum.
- Provided that $a \neq 0$, the solution of equation (2) is given by

$$x = -\frac{b}{2a}.$$

³For the students of Artificial Intelligence: you will soon get familiar with differentiation in AI503: Calculus if you are not already

Example: optimising a quadratic function (cont.)

- Consider another function

$$f(x_1, x_2) = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c,$$

for $x_1, x_2 \in \mathbb{R}$, where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c \in \mathbb{R}$ are known constants.

- Quadratic functions are behind probably the most commonly used loss functions (quadratic loss functions) in statistics and machine learning.
- We ask the same question: how can we find x_1 and x_2 , where the value of the function is the smallest or the largest?

Example: optimising a quadratic function (cont.)

- The partial derivatives⁴ are given by

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 2a_{11}x_1 + (a_{12} + a_{21})x_2 + b_1$$

and

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = (a_{12} + a_{21})x_1 + 2a_{22}x_2 + b_2.$$

- Setting both partial derivatives to 0, leads to a system of two linear equations with two unknowns

$$\begin{cases} 2a_{11}x_1 + (a_{12} + a_{21})x_2 + b_1 = 0; \\ (a_{12} + a_{21})x_1 + 2a_{22}x_2 + b_2 = 0. \end{cases} \quad (3)$$

⁴For the students of Artificial Intelligence: partial derivatives will be covered in AI503: Calculus

Example: optimising a quadratic function (cont.)

- When does a unique minimum or a unique maximum exist (corresponds to $a \neq 0$ in (1))?
- If either a minimum or a maximum exists, does the function have a minimum (corresponds to $a > 0$ in (1)) or a maximum (corresponds to $a < 0$ in (1))?
- If either a unique minimum or a unique maximum exists, how can we find it (i.e., how can we solve (3))?
- We need to know linear algebra to answer these questions.

Better notation

- Can we also use some better notation to make the problem clearer?
- We will learn that we can actually write

$$f(x) = x^T A x + b^T x + c,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

and A^T denotes the transpose of the matrix A .

- The system of equations that we need to solve can then be compactly written as

$$(A + A^T)x + b = 0,$$

where $0 = (0 \ 0)^T$.

Better notation (cont.)

If $a_{12} = a_{21}$ (i.e., we will learn that such matrices are called symmetric), system of linear equations (3) can be expressed as

$$2Ax + b = 0 \tag{4}$$

which looks essentially identical to (2).

Wrap-up

- Systems of linear equations appear in problems that are not linear per se (optimisation of a quadratic function).
- Optimising a quadratic function boils down to solving a system of linear equations.
- The matrix A determines how function f behaves (i.e., whether it has a unique minimum, a unique maximum, or neither).
- We also need to find the value of $x = (x_1 \ x_2)^T$ (if it exists) that satisfies (3) which means that we need to solve system of equations (3).
- It is a good idea to have a simpler problem in mind when we consider more complicated problems (compare (2) with (4)).

Systems of linear equations

Definitions

Linear equation

Definition

A *linear equation* in n variables x_1, x_2, \dots, x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where a_1, a_2, \dots, a_n and b are constants called the *coefficients* of the linear equation.

- The following equations are all linear:

$$x + 3y = 4, \quad \sqrt{3}x - y = \sqrt{5}, \quad \text{and} \quad \cos(1)x + \sin(1)y = 2.$$

- The following equations are all *not* linear:

$$\sqrt{x} + 3y = 4, \quad \cos x + \sin y = 0, \quad \text{and} \quad 4y + 2xz = 1.$$

Flat objects

- Geometrically, we can think of linear equations as representing lines and planes (and higher-dimensional flat shapes that we cannot quite picture).
- For example, the general equation of a line in \mathbb{R}^2 is

$$ax + by = c,$$

and the general equation of a plane in \mathbb{R}^3 is

$$ax + by + cz = d,$$

both of which are linear equations.

Section 2.1 of Johnston (2021)

System of linear equations

Definition

A system of linear equations (or a linear system) is a finite set of linear equations, each with the same variables x_1, x_2, \dots, x_n .

A general system of linear equations can be written as

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2; \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{array} \right. \quad (5)$$

Section 2.1 of Johnston (2021)

Solutions and solution sets

Definition

A *solution* of a system of linear equations consists of n real numbers

$$s_1, s_2, \dots, s_n$$

such that the substitution $x_1 = s_1, \dots, x_n = s_n$ makes all m linear equations satisfied simultaneously.

Definition

The *solution set* of a system of linear equations is the set of all solutions of the system.

Definition

A system of linear equations that has at least one solution is called *consistent* and *inconsistent* otherwise.

Systems of linear equations

Two equations and two unknowns

Two equations and two unknowns

- Consider the following system of linear equations

$$\begin{cases} 5x + y = 3; \\ 2x - y = 4. \end{cases} \quad (6)$$

- One may solve this by isolating one variable in one of the equations and substituting it into the other equation to obtain $x = 1$ and $y = -2$.
- (6) is equivalent to

$$\begin{cases} y = -5x + 3; \\ y = 2x - 4. \end{cases} \quad (7)$$

- (7) describes two lines in the plane that intersect at the point $(1, -2)$ which is the solution of system of linear equations (6).

Existence and number of solutions

- System of equations (6) has exactly one solution.
- There might be no solution:

$$\begin{cases} 2x + y = 3; \\ 2x + y = 5. \end{cases} \quad (8)$$

- There might be infinitely many solutions:

$$\begin{cases} 2x + y = 3; \\ 4x + 2y = 6. \end{cases} \quad (9)$$

- What happens geometrically when there is no solution and when there are infinitely many solutions?
- There are only three possibilities: a system of linear equations can have none, one, or infinitely many solutions.

Systems of linear equations

Row echelon form and Gauss-Jordan elimination

Triangular shape and back substitution

- If the linear system has a “triangular” form, then solving it is fairly intuitive.
- For example,

$$\left\{ \begin{array}{l} x + 3y - 2z = 5; \\ 2y - 6z = 4; \\ 3z = 6. \end{array} \right. \quad (10)$$

- The solution is $z = 2$, $y = 8$, and $x = -15$.
- The procedure that we used to solve the example is called *back substitution*, and it worked because of the triangular shape of the linear system.
- Our goal is to put *every* system of equations into this triangular form.

Transformations of systems of linear equations

- Consider another system of linear equations

$$\begin{cases} x + 3y - 2z = 5; \\ x + 5y - 8z = 9; \\ 2x + 4y + 5z = 12. \end{cases} \quad (11)$$

- Can we somehow transform (11) into the triangular shape?
- The following three operations do not affect the set of solutions (think why this is the case):
 - multiplying an equation by a non-zero scalar;
 - swapping the order of equations;
 - adding a scalar multiple of one equation to another.
- We can actually transform (11) into (10) without affecting the set of solutions (example on the blackboard).