AI511/MM505 Linear Algebra with Applications

Lecture 2 – Systems of Linear Equations (cont.), Matrices and Matrix Operations

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Outline

Recap

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- An equation is linear if it can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

  where $x_1, \ldots, x_n \in \mathbb{R}$ are variables and $a_1, \ldots, a_n \in \mathbb{R}$ as well as $b \in \mathbb{R}$ are constants.
- A system of two linear equations with two unknowns can have no solutions (lines are parallel), one solution (lines intersect at one point), or infinitely many solutions (lines overlap).
- The systems of linear equations in a triangular form are relatively easy to solve using back substitution.
- We can transform systems of linear equations without affecting the set of solutions.

Three operations

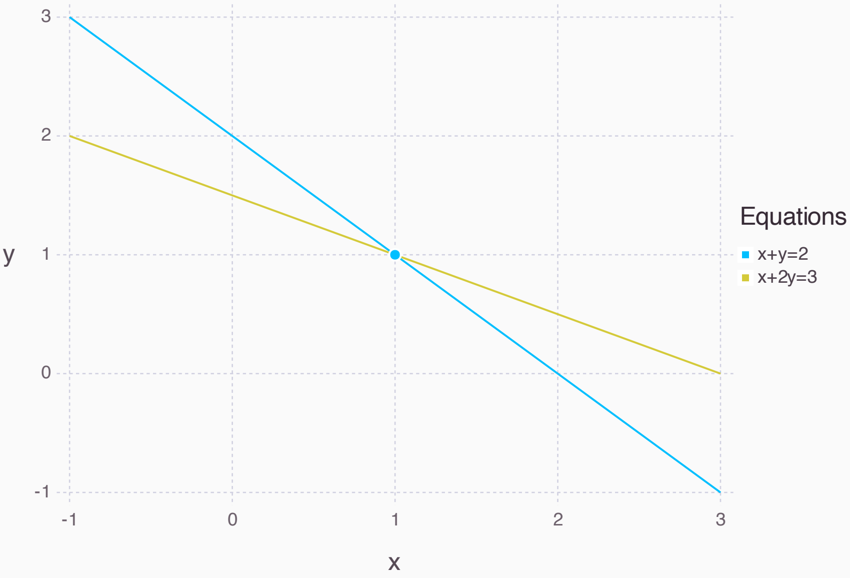
- The following three operations do not affect the set of solutions:
  (a) multiplying an equation by a non-zero scalar;
  (b) swapping the order of equations;
  (c) adding a scalar multiple of one equation to another.
- The system of linear equations

$$\begin{cases} x + 3y - 2z = 5; \\ \qquad 2y - 6z = 4; \\ \qquad\qquad 3z = 6 \end{cases} \tag{1}$$

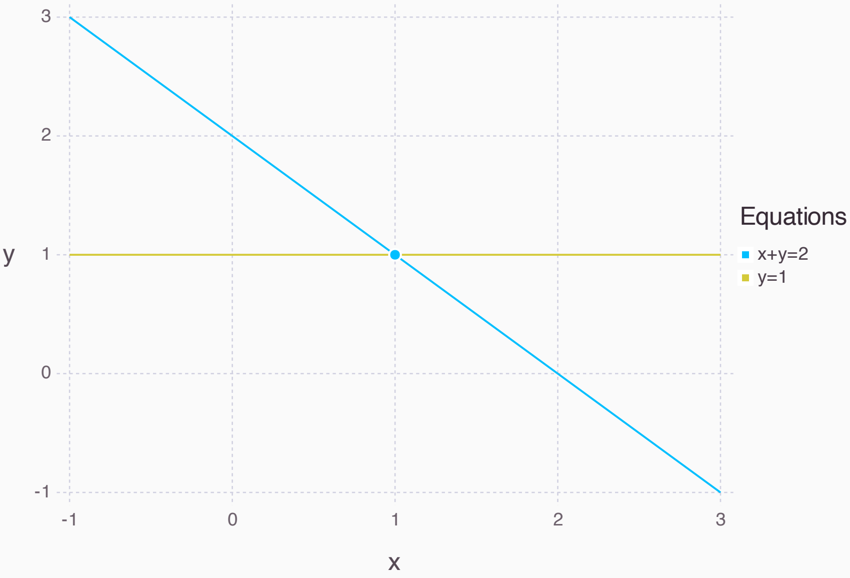
has the same set of solutions as the system of linear equations

$$\begin{cases} x + 3y - 2z = 5; \\ x + 5y - 8z = 9; \\ 2x + 4y + 5z = 12. \end{cases} \tag{2}$$

A system of two linear equations



A reduced system of two linear equations



Systems of linear equations (cont.)

System of linear equations

Recall that a general system of linear equations can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1; \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2; \\ \quad\vdots \qquad\qquad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases} \tag{3}$$

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Augmented matrix

- The coefficients defining system of linear equations (3) can be conveniently encoded in an array

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

- Such an array is called the augmented matrix of a system of linear equations.
- The numbers a_{ij} and b_i are called the entries of the augmented matrix.

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Example

The augmented matrix of system of linear equations (2) is given by

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right]. \tag{4}$$

Elementary row operations

- The transformation of (2) into (1) can be carried out using augmented matrix (4).
- (a), (b), and (c) correspond to operations where we modify the rows of the augmented matrix.
- We will use the following notation:
  - (i) multiplying row j by a non-zero scalar $c \in \mathbb{R}$ is denoted by cR_j;
  - (ii) swapping rows i and j is denoted by $R_i \leftrightarrow R_j$;
  - (iii) for any scalar $c \in \mathbb{R}$, replacing row i by $(\text{row } i) + c(\text{row } j)$ is denoted by $R_i + cR_j$.
- The three operations (i), (ii), and (iii) are called *elementary row operations*.
- Two (augmented) matrices are called *row equivalent* if one can be converted to the other via elementary row operations.

Example

By performing the following elementary row operations

$$
\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right]
\xrightarrow{R_2 - R_1}
\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 2 & 4 & 5 & 12 \end{array}\right]
$$
$$
\xrightarrow{R_3 - 2R_1}
\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & -2 & 9 & 2 \end{array}\right]
$$
$$
\xrightarrow{R_3 + R_2}
\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right], \tag{5}
$$

we obtain the augmented matrix corresponding to (1).

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Example

- By continuing to perform elementary row operations, we obtain

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2+6R_3]{R_1+2R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 2 & 0 & 16 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]. \quad (6)$$

- This form of the augmented matrix leads directly to the solution of the system of linear equations.

Method for solving systems of linear equations

This illustrates the most commonly-used method for solving systems of linear equations:

- use row operations to first make the matrix “triangular”;
- then either solve the system by back substitution or by performing additional row operations.

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Row echelon form

Definition

A (augmented) matrix is said to be in *row echelon form (REF)* if:

(a) all rows consisting entirely of zeros are below the non-zero rows;

(b) in each non-zero row, the first non-zero entry (called the *leading entry*) is to the left of any leading entries below it.

$$\begin{bmatrix} \star & * & * & * \\ 0 & 0 & \star & * \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & \star & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \star & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \star & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & * \end{bmatrix}$$

are in REF, where $\star$ is a non-zero leading entry and $*$ is an arbitrary (potentially zero) non-leading entry.

Reduced row echelon form

Definition

If a matrix that is in row echelon form also satisfies the following additional property, then it is in *reduced REF*:

(c) each leading entry equals 1 and is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

are in reduced REF, where $*$ is an arbitrary (potentially zero) non-leading entry.

Previous system of linear equations

- The two augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

  both are in REF but only the augmented matrix on the right is in *reduced* REF.
- Roughly speaking, a matrix is in REF if we can solve the associated system of linear equations via back substitution, whereas it is in reduced REF if we can just read the solution to the linear system directly from the entries of the matrix.

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Row-reduction and Gaussian elimination

- The process of transforming a matrix (or augmented matrix) into REF (or reduced REF) is called *row-reduction*.
- How do we perform row-reduction of a general matrix?
- *Gaussian elimination* is a method which can be used to transform any matrix into (not necessarily reduced) REF and thus solve the associated system of linear equations.

Gaussian elimination

(1) Position a leading entry. Locate the leftmost non-zero column of the matrix, and swap rows (if necessary) so that the topmost entry of this column is non-zero.

(2) Zero out the leading entry's column. Use the "addition" row operation to create zeros in all entries below the leading entry from Step 1.

(3) Repeat until we cannot. Partition the matrix into a block matrix whose top block consists of the row with a leading entry from Step 1, and whose bottom block consists of all lower rows. Repeat Steps 1 and 2 on the bottom block.

Gaussian elimination (cont.)

- The Gaussian elimination transforms the matrix into REF.
- We actually used the Gaussian elimination when we transformed (4) into (5).
- To further transform the matrix into *reduced* REF, we can do so via an extension of Gaussian elimination called *Gauss-Jordan* elimination, which consists of just one additional step.

(4) Reduce even more, from right-to-left. Starting with the rightmost leading entry, and moving from bottom-right to top-left, use the “multiplication” row operation to change each leading entry to 1 and use the “addition” row operation to create zeros in all entries above the leading entries.

Gauss-Jordan elimination (cont.)

We used the Gauss-Jordan elimination when we transformed (5) into (6).

Theorem

Every matrix can be row-reduced to one, and only one, matrix in reduced REF.

Non-reduced REF are not unique, i.e., applying a different sequence of row operations might get us to a different REF.

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Some examples

- Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

- Is this matrix in a reduced REF? What can we say about the solutions of the system of linear equations behind this augmented matrix?
- A system of linear equations has no solutions if and only if its REF's have a row that looks like

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b \end{array}\right]$$

for some $b \neq 0$.

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Some examples (cont.)

- Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{7}$$

- Is this matrix in a reduced REF? What can we say about the solutions of the system of linear equations behind this augmented matrix?
- A system of linear equations has infinitely many solutions if and only if it is consistent and the number of leading entries in its REF's is less than the number of variables.

Leading and free variables

- The variables corresponding to columns containing a leading entry are called leading variables (x and z in (7)) and the other variables are called free variables (y in (7)).
- The solution set of a system with infinitely many solutions can always be described by solving for the leading variables in terms of the free variables, and each free variable corresponds to one “dimension” or “degree of freedom” in the solution set.
- For example, if there is one free variable then the solution set is a line, if there are two then it is a plane, and so on.

Flowchart

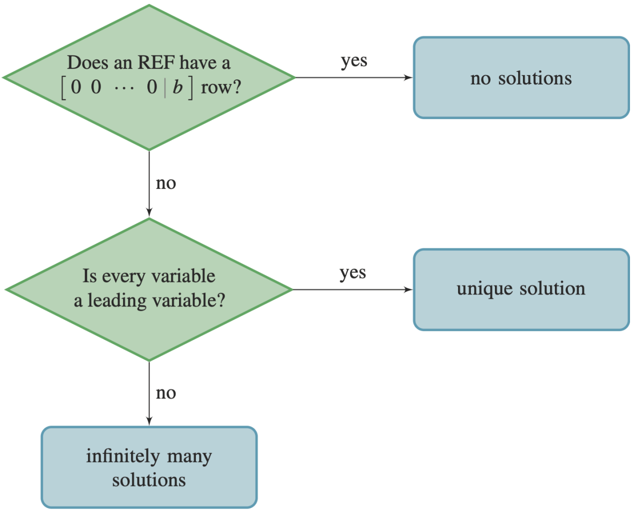

Figure 2.6 of Johnston (2021)

Matrices

Systems of linear equations and matrices

- Any system of linear equations can be encoded into an augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

- It therefore makes sense to understand the general nature of matrices better.

Matrices

Definition

An $m \times n$-matrix is an array of real numbers consisting of m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{8}$$

where $a_{ij} \in \mathbb{R}$ with $i = 1, \dots, m$ and $j = 1, \dots, n$.

Some examples of matrices

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & -3 & \frac{1}{2} \\ \frac{1}{3} & 0 & -13 \end{bmatrix}}, \quad B = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad C = \underset{3\times 1}{\begin{bmatrix} 2 \\ 0 \\ \frac{1}{7} \end{bmatrix}}, \quad \text{and} \quad D = \underset{1\times 1}{\begin{bmatrix} \pi \end{bmatrix}}.$$

Some notation and terminology

- The set of all $m \times n$ matrices whose entries are real numbers is denoted by $\mathbb{R}^{m\times n}$.
- If $m = n$, the matrices are called *square*.
- Matrix (8) can also be denoted by $(a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ or simply (a_{ij}), where a_{ij} is the entry in the i-th row and j-th column of A.
- The number of rows and columns that a matrix has are collectively referred to as its *size*.
- Two matrices A and B are equal if and only if they have the same size and the entries are the same: $a_{ij} = b_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Row and column vectors

- $1 \times n$- and $m \times 1$-matrices play a special role and are called *row-* and *column vectors*, respectively.
- For example,

$$\underbrace{\begin{bmatrix} 1 & 0 & \frac{1}{2} & \sqrt{2} \end{bmatrix}}_{\text{row vector}} \quad \text{and} \quad \underbrace{\begin{bmatrix} 2 \\ -3 \\ \frac{4}{7} \end{bmatrix}}_{\text{column vector}} .$$

Matrix addition and scalar multiplication

Definition

Suppose that $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$. The *sum* $A + B$ and *scalar multiplication* cA are the $m \times n$ matrices whose (i, j)- entries, for each $1 \le i \le m$ and $1 \le j \le n$, are

$$(A + B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (cA)_{ij} = ca_{ij},$$

respectively.

Matrix subtraction is defined as

$$A - B = A + (-1)B$$

and the negative of a matrix is defined as

$$-A = (-1)A.$$

Section 1.3 of Johnston (2021)

Exercise: matrix addition and scalar multiplication

Suppose that

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Compute

(a) $A + B$; (b) $2A - 3B$; (c) $A + 2C$.

Properties of matrix addition and scalar multiplication

Theorem (Theorem 1.3.1 of Johnston (2021))

Suppose that $A, B, C \in \mathbb{R}^{m\times n}$ and $c, d \in \mathbb{R}$. Then

(a) $A + B = B + A$ *(commutativity);*

(b) $(A + B) + C = A + (B + C)$ *(associativity);*

(c) $c(A + B) = cA + cB$ *(distributivity);*

(d) $(c + d)A = cA + dA$ *(distributivity);*

(e) $c(dA) = (cd)A$.

None of these algebraic properties are meant to be surprising.

Matrix multiplication

Definition

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, then their *product* $AB \in \mathbb{R}^{m\times p}$ whose (i,j)-entry, for each $1 \leq i \leq m$ and $1 \leq j \leq p$ is

$$(ab)_{ij} := a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

- This looks like a somewhat strange definition, but there are good reasons as we shall see later when we study linear maps.
- The matrix product AB only makes sense if A has the same number of columns as B has rows (such matrices are called conformable for multiplication).
- Observe that $\underset{m\times n}{A}$, $\underset{n\times p}{B}$, and $\underset{m\times p}{AB}$.

Matrix multiplication (cont.)

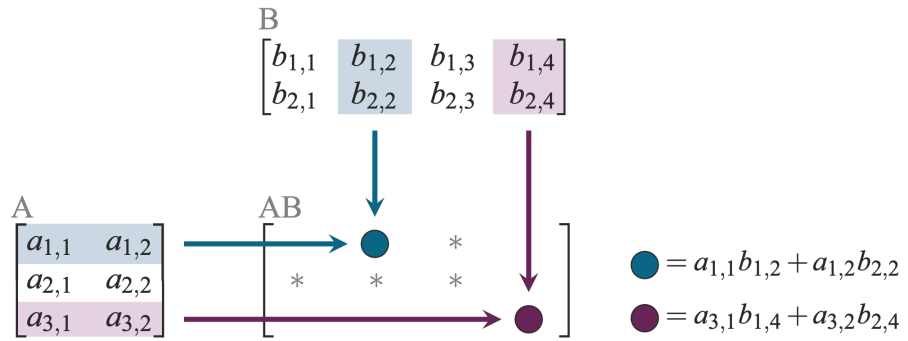

Figure 1.13 of Johnston (2021)

Exercises: matrix multiplication

Suppose that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

Compute (if possible)

(a) AB; (b) AC; (c) BA; (d) BC.

Section 1.3 of Johnston (2021)