Integration

AI503
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Lecture 11

Double Integral: Definition

Let $f(x, y)$ be a function on $R = [a, b] \times [c, d]$. The *definite integral* of f over R is

$$\int_R f \, dA = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \, \Delta x \Delta y,$$

if the limit exists. Then f is said to be **integrable** on R.

Iterated Integral

If f is integrable on $R = [a, b] \times [c, d]$, then

$$\int_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy.$$

To evaluate:

- Integrate with respect to x first (treat y constant).
- Then integrate the result with respect to y.

Integrals over Non-Rectangular Regions

We can define $\int_R f \, dA$ for other shapes (triangles, circles, etc.) by enclosing R in a rectangle and summing over subrectangles inside R.

Example

A city occupies a semicircular region of radius 3 km bordering the ocean. Find the average distance from points in the city to the ocean.

Example

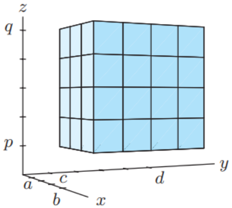
Sketch the region for

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx,$$

then reverse the order of integration.

Triple Integrals: Definition

We can define integrals over a solid region W in $\mathbb{R}^3$ by subdividing into boxes.



Using similar reasoning, express $\int_W f\, dV$ as a limit of Riemann sums.

Triple Integrals as Iterated Integrals

If f is integrable on $W = [a, b] \times [c, d] \times [p, q]$, then

$$\int_W f\, dV = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

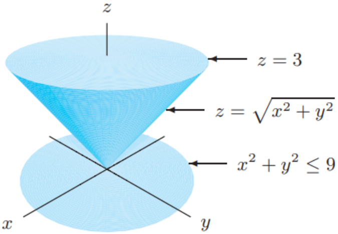
Other integration orders are also possible.

Example

Compute $\int_W (1 + xyz)\, dV$ over the cube $0 \leq x, y, z \leq 4$.

Example

Set up an iterated integral for the mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density $\delta(x, y, z) = z$.



Recall: single-variable u-substitution

Evaluate

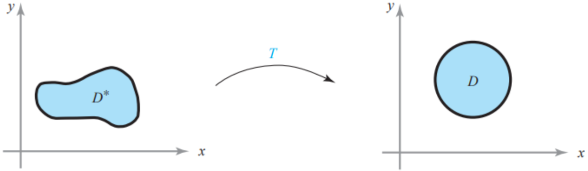
$$\int_0^1 xe^{x^2}\, dx.$$

(Recall: set $u = x^2$, $du = 2x\, dx$.)

We will learn how to generalize this to multivariate integration.

Mapping $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$

- Think of how T deforms subsets of D^* rather than the graph.
- **Image set:**
  $T(D^*) = \{(x, y) : (x, y) = T(u, v) \text{ for some } (u, v) \in D^*\}$.



Example: polar map

Let $D^* = [0,1] \times [0,2\pi]$ with coordinates (r,θ) and

$$T(r,\theta) = (r\cos\theta, r\sin\theta).$$

What is $T(D^*)$?

Example: polar map

Let $D^* = [0,1] \times [0,2\pi]$ with coordinates (r,θ) and

$$T(r,\theta) = (r\cos\theta, r\sin\theta).$$

What is $T(D^*)$? Answer: the unit disk $\{(x,y) : x^2 + y^2 \leq 1\}$.

Injective / surjective

One-to-one

T is one-to-one on D^* if $T(u, v) = T(u', v')$ implies $(u, v) = (u', v')$.

Onto

T is onto D if for every $(x, y) \in D$ there exists (u, v) with $T(u, v) = (x, y)$.

Linear maps and parallelograms

Theorem (linear map)

If A is 2×2 matrix with $\det A \neq 0$ and $T(u) = Au$, then T maps parallelograms to parallelograms and is bijective.

Example: $T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ on $D^* = [-1,1]^2$. Find $T(D^*)$.

Jacobian matrix

For $T(u, v) = (x(u, v), y(u, v))$ define

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}.$$

Provided the partial derivatives exist.

Jacobian of a linear map

Consider $T(u, v) = A \begin{pmatrix} u \\ v \end{pmatrix}$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$x = au + bv, \qquad y = cu + dv.$$

Compute the Jacobian matrix:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = A.$$

Hence for a linear map $T = Au$, the Jacobian matrix is A.

Change of variables formula (2D)

Theorem

Let $T : D^* \to D$ be one-to-one and onto (up to boundaries). For integrable f on D:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f\big(x(u,v),y(u,v)\big)\,|\det(J(u,v))|\,du\,dv.$$

Why the Jacobian?

Counterexample to naive formula without Jacobian:

Consider a linear map T that scales area by factor $c \neq 1$. Without the Jacobian area (and integrals) would be incorrect.

Polar coordinates

Apply the formula to $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Compute

$$\det(J(r,\theta)) = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

Hence

$$\iint_D f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Gaussian integral (application)

Use polar coordinates to compute

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

Standard trick: square the integral and use polar coordinates to obtain $\sqrt{\pi}$.

Jacobian in 3D

For $T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$,

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Change of variables formula (3D)

Let W and W^* be elementary region in $\mathbb{R}^3$ and let $T : W^* \to W$ be one-to-one and onto W (except possibly on a set that is the union of graphs of functions of two variables). Then for any integrable function $f : W \to \mathbb{R}$, we have

$$\begin{aligned}&\iiint_W f(x, y, z)\, dV \\ &= \iiint_{W^*} f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big)\ |J|\ du\, dv\, dw.\end{aligned}$$

Cylindrical coordinates

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. Let W^* consists of points in the form (r, θ, z) where $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1$. Find the image set and compute its Jacobian.

Spherical coordinates

Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \cos\theta$ and $z = \rho \cos\phi$. Compute its Jacobian and use this to derive the change of variable spherical cooridnate formula.

Final notes

- Check one-to-one condition or restrict domain when using change of variables.
- Always include absolute value of Jacobian.