AI511/MM505 Linear Algebra with Applications

Lecture 5 – Determinants, Euclidean Vector Spaces

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September 22, 2025

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Outline

Recap

Computing the inverse of a matrix

Theorem 2.2.5 of Johnston (2021)

Suppose $A \in \mathbb{R}^{n \times n}$. Then A is invertible if and only if there exists a matrix $E \in \mathbb{R}^{n \times n}$ such that the reduced REF of the block matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ is $\left[\begin{array}{c|c} I_n & E \end{array}\right]$. Furthermore, if A is invertible then it is necessarily the case that $A^{-1} = E$.

A simplifying theorem

Theorem 2.2.7 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{n \times n}$. If there exists a matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = I_n$ or $BA = I_n$, then A is invertible and $A^{-1} = B$.

Part (c) of Theorem 2.2.4 Johnston (2021)

If $A \in \mathbb{R}^{n \times n}$ is an invertible matrix, then for every $b \in \mathbb{R}^{n \times 1}$, the system of linear equations $Ax = b$ has exactly one solution $x = A^{-1}b$.

Theorem 2.1.1 of Johnston (2021)

Every system of linear equations has either

(a) no solutions;

(b) exactly one solution;

(c) infinitely many solutions.

Some special matrices

Diagonal matrices

Definition

$A \in \mathbb{R}^{n\times n}$ is said to be *diagonal* if $a_{ij} = 0$ for $i \neq j$.

Examples

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $A \in \mathbb{R}^{n\times n}$ is diagonal, we also write

$$A = \operatorname{diag}(\ a_1 \quad \dots \quad a_n\ ).$$

Diagonal matrices (cont.)

Matrix multiplication of diagonal matrices is performed elementwise

$$\operatorname{diag}\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \operatorname{diag}\begin{pmatrix} b_1 & \dots & b_n \end{pmatrix} = \\ = \operatorname{diag}\begin{pmatrix} a_1 b_1 & \dots & a_n b_n \end{pmatrix}.$$

Theorem

A diagonal matrix $D \in \mathbb{R}^{n\times n}$ is invertible if and only if $d_i \neq 0$ for all $i = 1, \dots, n$ and in which case $D^{-1} = \operatorname{diag}\begin{pmatrix} d_1^{-1} & \dots & d_n^{-1} \end{pmatrix}$.

It is straightforward to calculate powers of a diagonal matrix D since

$$D^k = \operatorname{diag}\begin{pmatrix} d_1^k & \dots & d_n^k \end{pmatrix}$$

for $k \geq 1$.

Triangular matrices

Definition

$A \in \mathbb{R}^{n\times n}$ is called *upper (lower) triangular* matrix if all entries below (above) the main diagonal are zero, i.e., $a_{ij} = 0$ for $i > j$ ($a_{ij} = 0$ for $i < j$).

Examples

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}}_{\text{Upper triangular}} \quad \text{and} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{\text{Lower triangular}}.$$

So diagonal means upper *and* lower triangular.

Properties of triangular matrices

Theorem

(a) *The transpose of a lower triangular matrix is upper triangular.*

(b) *The product of upper triangular matrices is upper triangular.*

(c) *An upper triangular matrix is invertible if and only if its diagonal entries are non-zero. If so, its inverse is also upper triangular.*

(d) *The same is true if "lower" and "upper" are swapped.*

Determinants

Determinants

Definition

Determinant

- Recall that for
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
we defined its determinant as $\det(A) = ad - bc$.
- We saw that $A \in \mathbb{R}^{2\times 2}$ is invertible if and only if $\det(A) \neq 0$.
- We now define the determinant of any $A \in \mathbb{R}^{n\times n}$.

Inductive definition

- For $A = [a] \in \mathbb{R}^{1\times 1}$, we define $\det([a]) := a$.
- For
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
we thus have
$$\begin{aligned} \det(A) &= a_{11} \det([a_{22}]) - a_{12} \det([a_{21}]) \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$
- General determinants are defined inductively.
- So assume that we know what the determinant of an $n \times n$-matrix is.
- To move up in size, we need *minors* and *cofactors*.

Minors, cofactors, and determinant

Definition

If $A \in \mathbb{R}^{n\times n}$, then the (i,j)-*th minor*, M_{ij}, of A is the determinant of the $(n-1)\times(n-1)$-matrix obtained by deleting row i and column j. The (i,j)-*th cofactor* is the number $C_{ij} := (-1)^{i+j}M_{ij}$.

Definition

The determinant of $A \in \mathbb{R}^{n\times n}$ is defined as

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.$$

We also use the following notation

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} = \det\left(\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}\right).$$

- We have that

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a(-1)^{1+1}d + b(-1)^{1+2}c = ad - bc.$$

- Suppose that

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}.$$

Show that $\det(A) = -1$.

Properties

Theorem 3.2.8 of Johnston (2021)

Suppose $A \in \mathbb{R}^{n \times n}$. Then

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \tag{1}$$
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj} \tag{2}$$

for all $i, j = 1, \ldots, n$.

- (1) is called a cofactor expansion along the i-th row.
- (2) is called a cofactor expansion along the j-th column.

Determinants

Computing determinants

Idea

- In general, it is tedious to calculate the determinant of a matrix.
- When we considered systems of linear equations, we saw that certain systems of linear equations are easier to solve.
- Our goal was to transform every system of linear equations into this particular type so that we can solve it in an easier way.
- Can we do something like this with determinants as well?

Determinants of triangular matrices

Theorem 3.2.4 of Johnston (2021)

If $A \in \mathbb{R}^{n \times n}$ is a triangular matrix, then

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} \cdot \ldots \cdot a_{nn}.$$

Example

$$\begin{vmatrix} 1 & 2 & -1 & 7 & 91 \\ 0 & 1 & 19 & 0 & \frac{1}{10} \\ 0 & 0 & 2 & 1 & -5 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 3 = 6.$$

Recall that a diagonal matrix is also triangular.

Matrix operations and determinants

- How do various matrix operations affect the determinant?
- $\det(A) = \det(A^T)$ since column expansion of A^T is the same as row expansion in A.

Theorem 2.2.3 of Anton, Rorres, and Kaul (2019)

Let $A \in \mathbb{R}^{n \times n}$. Then if B is the matrix obtained from A by

(a) multiplying a row with $k \in \mathbb{R}$, then $\det(B) = k \det(A)$;

(b) interchanging two rows, then $\det(B) = -\det(A)$;

(c) adding a multiple of one row to another, then $\det(B) = \det(A)$.

Clarification

Part (a) of Theorem 2.2.3 of Anton, Rorres, and Kaul (2019) means that

$$\det(B) = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \det(A)$$

or, equivalently,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{k} \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{k} \det(B).$$

Computing the determinant

- So, if we can bring A into triangular form in a few row operations then the determinant is easy to compute, as the product down the main diagonal (see Theorem 3.2.4 of Johnston (2021)).
- But we must keep track of how it changes along the way.

Example on the blackboard

Bring

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

into REF to show that the determinant of A is equal to 165.

Computing the determinant (cont.)

- The point is that cofactor expansion (i.e., the definition) is generally very slow and including some row operations may therefore be preferable for large matrices.
- According to Anton, Rorres, and Kaul (2019), cofactor expansion of a general 25×25-matrix will take millions of years on a computer.
- One can mix and match the methods as one sees fit.

Tips and tricks

- Instead of row operations, one can do column operations. This corresponds to row operations on A^T.
- If a matrix A has a row or column with a lot of zeros, then this often good for row/column cofactor expansion.
- If a matrix A has a zero row (or column) then $\det(A) = 0$.
- If two rows (columns) are proportional, then $\det(A) = 0$, since one can then create a zero-row (zero-column) by row operations.

Matrix algebra and determinants

- $\det(A+B)$ is not necessarily equal to $\det(A)+\det(B)$.
- Consider
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$
Then $\det(A+B) \neq \det(A)+\det(B)$.

Theorem

If $A, B \in \mathbb{R}^{n\times n}$, *then* $\det(AB) = \det(A)\det(B)$.

Determinants

What can determinants be used for?

Invertibility and the determinant

Theorem

A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det(A) \neq 0$.

Proof.

We prove that if A is invertible, then $\det(A) \neq 0$. If A is invertible then,

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

and hence $\det(A) \neq 0$. $\square$

Cramer's rule

Theorem 3.A.2 of Johnston (2021)

Suppose $A \in \mathbb{R}^{n \times n}$ is invertible and $b \in \mathbb{R}^{n \times 1}$. Define A_j to be the matrix that equals A, except its j-th column is replaced by b. Then the linear system $Ax = b$ has a unique solution x, whose entries are

$$x_j = \frac{\det(A_j)}{\det(A)} \quad \text{for all} \quad 1 \leq j \leq n.$$

Exercise

Use Cramer's rule to solve

$$\begin{cases} x_1 + + 2x_3 = 6; \\ -3x_1 + 4x_2 + 6x_3 = 30; \\ -x_1 - 2x_2 + 3x_3 = 8. \end{cases}$$

Euclidean vector spaces

Euclidean vector spaces

Vectors

Vectors

Definition

An ordered tuple $v = (v_1, \ldots, v_n)$ of real numbers $v_1, \ldots, v_n$ is called a *vector* (in dimension n) and the set of all such tuples is denoted $\mathbb{R}^n$.

Recommended to watch *Vectors, what even are they?* of 3Blue1Brown.

Row vectors and column vectors

- When we say that v is a vector, we typically do not specify whether it is a column vector (i.e., $\mathbb{R}^{n\times 1}$) or a row vector (i.e., $\mathbb{R}^{1\times n}$).
- When we think of vectors geometrically, this difference is often unimportant.
- However, when we perform matrix multiplication, the difference between row vectors and column vectors is important.
- When the shape of a vector matters and we give no indication otherwise, vectors are assumed to be column vectors.

Dot product

Definition 1.2.1 of Johnston (2021)

Suppose $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$. Then their *dot product*, denoted by $v \cdot w$, is the quantity

$$v \cdot w \coloneqq v_1 w_1 + v_2 w_2 + \ldots v_n w_n.$$

The dot product can be interpreted geometrically as roughly measuring the amount of overlap between v and w.

The angle between two vectors on the unit circle in $\mathbb{R}^2$

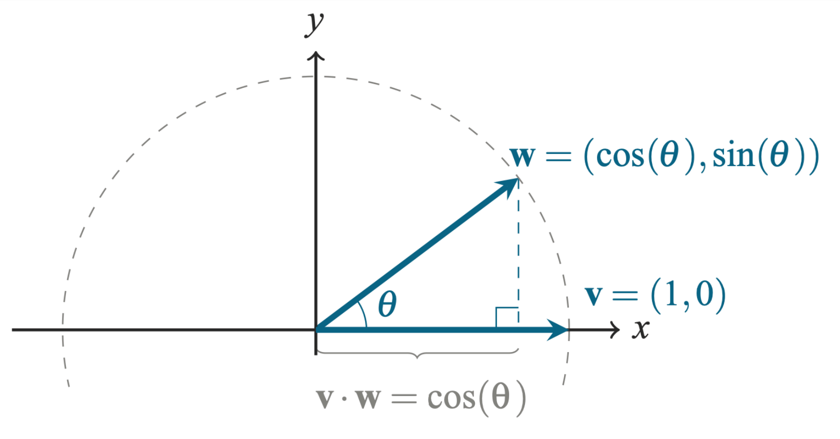

Figure 1.7 of Johnston (2021)

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.

Anton, Howard, Chris Rorres, and Anton Kaul (Sept. 2019). ***Elementary linear algebra: applications version.*** 12th edition. John Wiley & Sons, Inc.