Exercises 4

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Contents

1. Problem 1

Consider $f(x, y) = x^2 + y^2 - 4x - 6y$.

(a) Find the critical points of f.

(b) Classify them (minimum, maximum, or saddle point) using the Hessian.

1.1. Solution

(a) We first find the derivatives

$$f_{x(x,y)} = 2x - 4$$
$$f_{y(x,y)} = 2y - 6$$

We find the critical point(s) at

$$f_{x(x,y)} = 0$$
$$2x - 4 = 0 \Rightarrow x = \frac{4}{2} = 2$$
$$f_{y(x,y)} = 0$$
$$2y - 6 = 0 \Rightarrow x = \frac{6}{2} = 3$$

f only has 1 critical point on $(2, 3)$

(b) We know the hessian as

$$H(f)(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x y} \\ \frac{\partial^2 f}{\partial y x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

this results to

$$H(f)(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since we know that at the critical point

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

We can then find that: $(2 - \lambda) \cdot (2 - \lambda) = 0$ and $\lambda = 2$

This is then a local minumum by

Second-Derivative Test Hessian

At a critical point P_0, let $H = H(f)(P_0)$ be the Hessian matrix: Compute eigenvalues of H.

- All positive eigenvalues $\Rightarrow$ local minimum
- All negative eigenvalues $\Rightarrow$ local maximum
- Mixed signs $\Rightarrow$ saddle point
- Zero eigenvalue(s) $\Rightarrow$ test inconclusive

2. Problem 2

Let $f(u, v) = u^2 + 3uv$, $u(x, y) = x^2 - y$, and $v(x, y) = \sin(xy)$.

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ using the multivariale chain rule.

2.1. Solution

For this we use the multivariate chain rule for multivariable functions

Multivariate Chain Rule

For $f(g, h)$, with $g(x, y)$ and $h(u, v)$, then

- $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$
- $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= (2u + 3v) \cdot 2x + 3u \cdot (\cos(xy) \cdot y) \\
&= 4xu + 6xv + 3yu \cdot \cos(xy) \\
&= 4x(x^2 - y) + 6x(\sin(xy)) + 3y(x^2 - y) \cdot \cos(xy) \\
&= 4x^3 - 4xy + 6x\sin(xy) + 3yx^2\cos(xy) - 3y^2\cos(xy)
\end{aligned}$$

We also know that

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

So.. again

$$\begin{aligned}
\frac{\partial f}{\partial y} &= (2u + 3v) \cdot (-1) + 3u \cdot (\cos(xy) \cdot x) \\
&= -2u - 3v + 3xu \cdot \cos(xy) \\
&= -2(x^2 - y) - 3 \cdot (\sin(xy)) + 3x(x^2 - y) \cdot \cos(xy) \\
&= -2x^2 + 2y - 3 \cdot \sin(xy) + 3x^3 \cdot \cos(xy) - 3xy \cdot \cos(xy)
\end{aligned}$$

So finally

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 4x^3 - 4xy + 6x\sin(xy) + 3yx^2\cos(xy) - 3y^2\cos(xy) \\
\frac{\partial f}{\partial y} &= -2x^2 + 2y - 3 \cdot \sin(xy) + 3x^3 \cdot \cos(xy) - 3xy \cdot \cos(xy)
\end{aligned}$$

3. Problem 3

A rectangular box with square base and volume $V = 1\text{m}^3$ is to be built with the least surface area.

(a) Express the surface area $S(x, h)$ in terms of base side length x and height h.

(b) Eliminate one variable using the volume constraint.

(c) Use calculus to find the dimensions minimizing S.

3.1. Solution

(a) Since the box must have a base of x^2 (top and bottom) as well as 4 sides of $4xh$

$$S(x, h) = 2x^2 + 4xh$$

(b) Since volume is constrained, we know that $1\text{m}^2 = x^2 h$

becuase of this

$$h = \frac{1}{x^2}$$

then substitute h in the function for $S(x, h)$

this gives

$$S(x) = 2x^2 + \frac{4}{x}$$

(c) Take the derivative of $S(x)$ and solve for $\frac{\partial S}{\partial x} S(x) = 0$

$\frac{\partial S}{\partial x} S(x) = 4x - 4x^{-2} = 4x - \frac{4}{x^2}$

$$0 = 4x - \frac{4}{x^2}$$

$$\frac{4}{4} = x^3 = 1$$

S must have an critical point on $x = 1$

Now to find h

$$h = \frac{1}{(1)^2} = 1$$

To conclude. $x = 1$ and $h = 1$

Note that

$$S(1) = 2(1)^2 + \frac{4}{1^2} = 2 + 4 = 6\text{m}^2$$

4. Problem 4

Two products are manufactured in quantities q_1 and q_2 and sold at prices p_1 and p_2, respectively. The cost of producing them is given by

$$C = 2q_1^2 + 2q_2^2 + 10$$

(a) Find the maximum profit that can be made, assuming the prices are fixed.

(b) Find the rate of change of that maximum profit as p_1 increases.

4.1. Solution

(a) To maximize profit, we should first define the profit function

Profit must be

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$\pi(q_1, q_2) = R(q_1, q_2) - C(q_1, q_2)$$
$$\pi(q_1, q_2) = p_1 q_1 + p_2 q_2 - \left(2q_1^2 + 2q_2^2 + 10\right)$$
$$\pi(q_1, q_2) = p_1 q_1 + p_2 q_2 - 2q_1^2 - 2q_2^2 - 10)$$
$$\pi(q_1, q_2) = -2q_1^2 + p_1 q_1 - 2q_2^2 + p_2 q_2 - 10)$$

Now find $\frac{\partial \pi}{\partial q_1}$ and $\frac{\partial \pi}{\partial q_2}$

$$\frac{\partial \pi}{\partial q_1} = -4q_1 + p_1 \Longleftrightarrow q_1 = \frac{p_1}{4}$$
$$\frac{\partial \pi}{\partial q_2} = -4q_2 + p_2 \Longleftrightarrow q_2 = \frac{p_2}{4}$$

These are the profit-maximizing quantities

So…

$$\begin{aligned}\pi_{\max} = P\left(\tfrac{p_1}{4}, \tfrac{p_2}{4}\right) &= -2\left(\tfrac{p_1}{4}\right)^2 + p_1\left(\tfrac{p_1}{4}\right) - 2\left(\tfrac{p_2}{4}\right)^2 + p_2\left(\tfrac{p_2}{4}\right) - 10) \\ &= -\tfrac{p_1^2}{8} + \tfrac{p_1}{4} - \tfrac{p_2^2}{8} + \tfrac{p_2}{4} + 10 \\ &= \tfrac{p_1^2}{8} + \tfrac{p_2^2}{8} + 10\end{aligned}$$

5. Problem 5

Consider the function $f(x,y) = |xy|$.

(a) Use a computer to draw the graph of f. Does the graph look like a plane when we zoom in on the origin?

(b) Is f differentiable at $(x,y) \neq (0,0)$?

(c) Show that $f_x(0,0)$ and $f_y(0,0)$ exist.

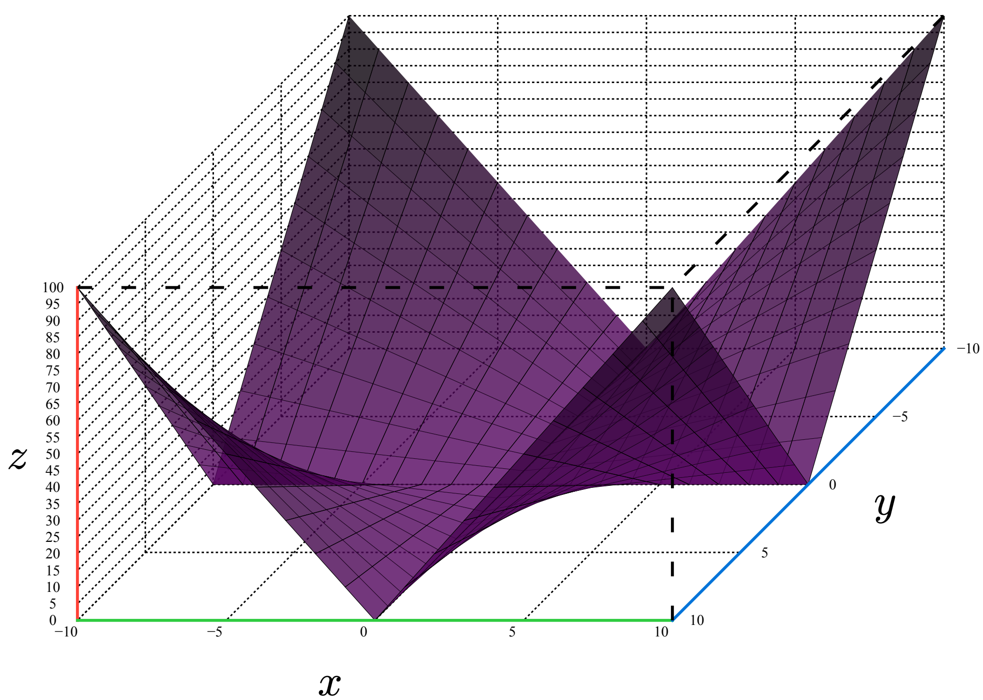
(d) Are f_x and f_y continuous at $(0,0)$?

(e) Is f differentiable at $(0,0)$?

Hint: Consider the directional derivative $f_u(0,0)$ for $u = (i+j)\left(\sqrt{2}\right)$.

5.1. Solution

(a) The graph of $f(x,y) = |xy|$:



When we zoom in on the origin, the graph does NOT look like a plane - it has sharp ridges along the coordinate axes due to the absolute value function.

(b) Input $(0,0)$ into the $f_x(x,y)$ and $f_y(x,y)$

$$f_x(0,0) = \lim_{\varepsilon\to 0} \tfrac{f(\varepsilon,0)-f(0,0)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \tfrac{0-0}{\varepsilon} = \lim_{\varepsilon\to 0^+} \tfrac{0}{\varepsilon} = \lim_{\varepsilon\to 0^-} \tfrac{0}{\varepsilon} = 0$$
$$f_y(0,0) = \lim_{\varepsilon\to 0} \tfrac{f(0,\varepsilon)-f(0,0)}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \tfrac{0-0}{\varepsilon} = \lim_{\varepsilon\to 0^+} \tfrac{0}{\varepsilon} = \lim_{\varepsilon\to 0^-} \tfrac{0}{\varepsilon} = 0$$

They both **exists**

(c) First find f_x and f_y

$$\tfrac{\partial}{\partial x} f(x,y) = \tfrac{\partial}{\partial x}|xy| = |y|\tfrac{\partial}{\partial x}|x| = |y| \ \mathrm{sgn}(x)$$
$$\tfrac{\partial}{\partial y} f(x,y) = \tfrac{\partial}{\partial y}|xy| = |x|\tfrac{\partial}{\partial y}|y| = |x| \ \mathrm{sgn}(y)$$

where

$$\mathrm{sgn}(x) = \begin{cases} +1 & \text{if } x>0 \\ -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases}$$

So now check for $(x,y) \to (0,0)$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = |0| + \mathrm{sgn}(0) = 0$$
$$\lim_{(x,y)\to(0,0)} f_y(x,y) = |0| + \mathrm{sgn}(0) = 0$$

So both exists

(d) Since

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$$
$$\lim_{(x,y)\to(0,0)} f_y(x,y) = 0 = f_y(0,0)$$

They are both continuous at $(0,0)$

(e) For the directional derivative $f_u(0,0)$ for $\vec{u} = \frac{i+j}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$$f_u(x,y) = \nabla f(x,y) \cdot \vec{u} = \big(f_x(x,y), f_y(x,y)\big) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= f_x(x,y) \cdot \tfrac{1}{\sqrt{2}} + f_y(x,y) \cdot \tfrac{1}{\sqrt{2}}$$
$$= \tfrac{|y|\cdot \mathrm{sgn}(x)}{\sqrt{2}} + \tfrac{|x|\cdot \mathrm{sgn}(y)}{\sqrt{2}}$$
$$= \tfrac{|y|\cdot \mathrm{sgn}(x) + |x|\cdot \mathrm{sgn}(y)}{\sqrt{2}}$$
$$f_u(0,0) = \tfrac{|0|\cdot \mathrm{sgn}(0) + |0|\cdot \mathrm{sgn}(0)}{\sqrt{2}} = 0$$

So it is differentiable at $(0,0)$ with $f_u(0,0) = \mathbf{0}$