AI511/MM505 Linear Algebra with Applications

Lecture 12 – Eigenvalues and Eigenvectors (cont.), Diagonalisation

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Outline

Recap

Similar matrices

Definition 3.1.3 of Johnston (2021)

We say that two matrices $A, D \in \mathbb{R}^{n \times n}$ are similar if there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that $A = PDP^{-1}$.

Eigenvalues and eigenvectors

Definition 3.3.1 of Johnston (2021)

Suppose A is a square matrix. A non-zero vector v is called an *eigenvector* of A if there is a scalar λ such that

$$Av = \lambda v.$$

Such a scalar λ is called the *eigenvalue* of A corresponding to v.

To find the eigenvalues and eigenvectors of a matrix,

(i) we first look for the eigenvalues by solving $\det(A - \lambda I) = 0$;

(ii) once we know the eigenvalues, we find the eigenvectors by solving the homogeneous system of linear equations

$$(A - \lambda I)v = 0$$

for each eigenvalue λ.

Example

- The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

  are $\lambda_1 = -1$ and $\lambda_2 = 6$.
- The eigenvectors corresponding to the eigenvalue $\lambda_1 = -1$ are the vectors of the form $v = c(-1, 1)$, where $v \in \mathbb{R}^2$, $c \in \mathbb{R}$, and $c \neq 0$.
- The eigenvectors corresponding to the eigenvalue $\lambda_1 = 6$ are the vectors of the form $v = c(2/5, 1)$, where $v \in \mathbb{R}^2$, $c \in \mathbb{R}$, and $c \neq 0$.

Characteristic polynomial

Definition 3.3.2 of Johnston (2021)

Suppose A is a square matrix. The function $p_A : \mathbb{R} \to \mathbb{R}$ defined by

$$p_A(\lambda) = \det(A - \lambda I)$$

is called the *characteristic polynomial* of A.

Definition 3.3.3 of Johnston (2021)

Suppose that $A \in \mathbb{R}^{n \times n}$ with eigenvalue λ. The *algebraic multiplicity* of λ is its multiplicity as a root of A's characteristic polynomial.

Eigenvalues and eigenvectors (cont.)

Eigenvalues and eigenvectors (cont.)

Complex eigenvalues

No real eigenvalues

- The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$$

  is given by

$$p_A(\lambda) = \lambda^2 + 2\lambda + 5.$$

- If we restrict our attention to $\lambda \in \mathbb{R}$, the quadratic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

  has no solutions (we have that $\Delta = b^2 - 4ac = -16 < 0$) which in turn would mean that matrix A has no eigenvalues.

Fundamental theorem of algebra

Theorem A.2.3 of Johnston (2021)

Every non-constant polynomial has at least one complex root.

- Hence, if we allow for complex eigenvalues, every matrix has at least one eigenvalue (which might be complex).
- In fact, every $n \times n$ matrix has *exactly* n complex eigenvalues, counting algebraic multiplicity.
- We need to specify that a matrix has exactly n complex eigenvalues counting algebraic multiplicity even if the entries of a matrix are real as we saw in the example.

Eigenvalues of symmetric matrices

A special case of Theorem 3.3.2 of Johnston (2021)

If $A \in \mathbb{R}^{n \times n}$ is symmetric, then all of its eigenvalues are real.

Eigenvalues and eigenvectors (cont.)

Eigenspaces and geometric multiplicity

Eigenspace

The set of all eigenvectors of a matrix $A \in \mathbb{R}^{n \times n}$ corresponding to a particular eigenvalue λ (together with the zero vector) is $\text{null}(A - \lambda I)$.

Definition 3.3.4 of Johnston (2021)

Suppose A is a square matrix with eigenvalue λ. The set of all eigenvectors of A corresponding to λ, together with the zero vector, is called the *eigenspace* of A corresponding to λ.

Instead of finding all eigenvectors corresponding to a particular eigenvalue, we typically just find a basis of that eigenspace.

Blackboard example: eigenspaces

- Compute bases of the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

- Recall that the eigenvalues of A are 2 (with algebraic multiplicity 1) and -1 (with algebraic multiplicity 2).
- Geometrically, this means that the eigenspace corresponding to $\lambda = 2$ is a line, while the eigenspace corresponding to $\lambda = -1$ is a plane.
- Every vector on that line is stretched by a factor of $\lambda = 2$, and every vector in that plane is reflected through the origin.

Geometric multiplicity

Definition 3.3.5 of Johnston (2021)

Suppose A is a square matrix with eigenvalue λ. The geometric multiplicity of λ is the dimension of its corresponding eigenspace.

In other words, the geometric multiplicity of the eigenvalue λ of A is nullity$(A - \lambda I)$.

Geometric multiplicity cannot exceed algebraic multiplicity

Theorem 3.3.3 of Johnston (2021)

For each eigenvalue of a square matrix, the geometric multiplicity is less than or equal to the algebraic multiplicity.

Blackboard example: smaller geometric multiplicity

Compute the eigenvalues and their algebraic as well as geometric multiplicities of the matrix

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Diagonalisation

Diagonalisable matrices

Definition 3.4.1 of Johnston (2021)

A matrix $A \in \mathbb{R}^{n \times n}$ is called diagonalisable if there is a diagonal matrix $D \in \mathbb{R}^{n \times n}$ and an invertible matrix $P \in \mathbb{R}^{n \times n}$ such that $A = PDP^{-1}$.

Theorem 3.4.1 of Johnston (2021)

Let $A \in \mathbb{R}^{n \times n}$ and suppose $P, D \in \mathbb{R}^{n \times n}$ are such that P is invertible and D is diagonal. Then $A = PDP^{-1}$ if and only if the columns of P are eigenvectors of A whose corresponding eigenvalues are the diagonal entries of D in the same order.

Blackboard example: diagnosable matrix

- Diagonalise the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}.$$

  Recall that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 6$ while the corresponding eigenspaces are spanned by $\{(-1, 1)\}$ and $\{(2, 5)\}$.
- How would we find the inverse of A once we have P, D, and P^{-1}?

Not all matrices are diagonalisable

- If $A \in \mathbb{R}^{n \times n}$, it may not be possible to construct D and P so that they have real entries, since A might have complex (non-real) eigenvalues and eigenvectors.
- Even if we allow for complex eigenvalues and eigenvectors, there may not be a way to choose eigenvectors so that the matrix P (whose columns are eigenvectors) is invertible.

Characterisation of diagonalisability

Theorem 3.4.3 of Johnston (2021)

Suppose $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

(a) A is diagonalisable over $\mathbb{R}$;

(b) there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (this basis is called an eigenbasis of A);

(c) the set of eigenvectors of A spans all of $\mathbb{R}^n$;

(d) the sum of the geometric multiplicities of the real eigenvalues of A is n.

Corollary

If $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues then it is diagonalisable.

Keep in mind that the above corollary only works in one direction.

Blackboard example: matrix that is not diagonalisable

Show that the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

cannot be diagonalised.

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.