AI511/MM505 Linear Algebra with Applications

Lecture 3 – Matrices and Matrix Operations (cont.), Inverses

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September 8, 2025

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Outline

Recap

Matrices and matrix operations (cont.)

- Some special matrices and operations
- Inverse of a matrix

Recap

Definition

An $m \times n$-matrix is an array of real numbers consisting of m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \tag{1}$$

where $a_{ij} \in \mathbb{R}$ with $i = 1, \dots, m$ and $j = 1, \dots, n$.

Some examples of matrices

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & -3 & \frac{1}{2} \\ \frac{1}{3} & 0 & -13 \end{bmatrix}}, \quad B = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \quad C = \underset{3\times 1}{\begin{bmatrix} 2 \\ 0 \\ \frac{1}{7} \end{bmatrix}}, \quad \text{and} \quad D = \underset{1\times 1}{\begin{bmatrix} \pi \end{bmatrix}}.$$

Matrix addition, scalar multiplication, and multiplication

- Matrix addition and scalar multiplication are performed element-wise.
- Algebraic properties of matrix addition and scalar multiplication coincide with our intuition.
- Matrices are multiplied in a weird way.

Matrix multiplication

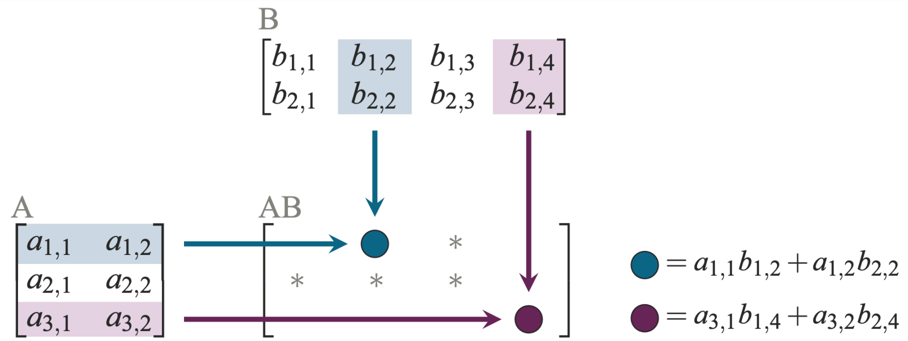

Figure 1.13 of Johnston (2021)

Matrices and matrix operations (cont.)

Properties of matrix multiplication

Theorem

Suppose that A, B, and C are matrices (with sizes such that the multiplications and additions below make sense) and let $c \in \mathbb{R}$ be a scalar. Then

(a) $(AB)C = A(BC)$ *(associativity);*

(b) $A(B + C) = AB + AC$ *(left distributivity);*

(c) $(A + B)C = AC + BC$ *(right distributivity);*

(d) $c(AB) = (cA)B.$

What is missing in this theorem?

Matrix multiplication is not commutative

- In general, AB is not necessarily equal to BA.
- Even if both AB and BA exist, they may or may not have the same size as each other, and even if they are of the same size, they still might not be equal each other.
- Consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

- Then $AB \neq BA$ even though $AB, BA \in \mathbb{R}^{2\times 2}$.
- An operation not being commutative should be our default assumption, since the order in which events occur matters.
- We would prefer to lose all of our money and then get a million dollars, rather than get a million dollars and then lose all of our money.

Matrices and matrix operations (cont.)

Some special matrices and operations

The zero matrix

- The zero matrix or the null matrix is a matrix all of whose entries are zero and is denoted by 0 or $0_{m,n}$ if we want to indicate the size of the zero matrix.
- For example,

$$0_{1,1} = \begin{bmatrix} 0 \end{bmatrix}, \quad 0_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad 0_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Alternatively, the zero matrix is denoted by O or $\mathbf{0}$ but we will use 0 to denote the zero matrix and the meaning should be clear from the context.
- Observe that

$$A \cdot 0 = 0 = 0 \cdot A \quad \text{and} \quad A + 0 = A = 0 + A$$

whenever the operations are defined.

Exercise: product of two non-zero matrices can be equal to 0

Consider two matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -1 & \frac{1}{2} \end{bmatrix}.$$

Show that $AB = 0$.

The identity matrix

- The matrix $I_n \in \mathbb{R}^{n \times n}$ given by

$$I_n := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

  is called the identity matrix.
- It satisfies

$$A \cdot I_n = A = I_n \cdot A$$

  for all $A \in \mathbb{R}^{n \times n}$.

Transpose

Definition

Let $A \in \mathbb{R}^{m\times n}$. Then the transpose of A, which we denote by A^T, is the $n \times m$-matrix whose (i,j)-entry is a_{ji}.

For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix},$$

i.e., the entries of A are reflected across its main diagonal and the rows of A are the columns of A^T.

Properties of the transpose

Theorem (Theorem 1.3.4 of Johnston (2021))

Let A and B be matrices with sizes such that the operations below make sense and let $c \in \mathbb{R}$ be a scalar. Then

(a) $(A^T)^T = A$;

(b) $(A+B)^T = A^T + B^T$;

(c) $(AB)^T = B^T A^T$;

(d) $(cA)^T = cA^T$.

Trace of a matrix

Definition (Definition 3.1.4 of Johnston (2021))

The *trace* of $A \in \mathbb{R}^{n \times n}$, denoted by $\operatorname{tr}(A)$, is the sum of its diagonal entries:

$$\operatorname{tr}(A) := a_{11} + a_{22} + \ldots + a_{nn}.$$

Theorem (Theorem 3.1.6 of Johnston (2021))

Suppose A and B are matrices whose sizes are such that the following operations make sense, and let c be a scalar.

(a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$;

(b) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$;

(c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Matrices and matrix operations (cont.)

Inverse of a matrix

Product of a matrix and a vector

Observe that if $A \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^{n\times 1}$, then $Ax \in \mathbb{R}^{m\times 1}$ and

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \ldots + a_{1n}x_n \\ a_{21}x_1 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n \end{bmatrix}.$$

Systems of linear equations using matrix notation

We can hence represent a system of linear equations compactly using the matrix notation in the following way

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Example: optimising a quadratic function (cont.)

- Recall that when we optimised the quadratic function, we had to solve the linear equation

$$2ax + b = 0$$

which can be equivalently written as

$$ax = -\frac{b}{2}.$$

- The trick here was to divide both sides of the equation by a to obtain the solution provided that $a \neq 0$.

Example: optimising a quadratic function (cont.)

- In the two-variable case, setting both partial derivatives to 0 leads to a system of two linear equations with two unknowns

$$\begin{cases} 2a_{11}x_1 + (a_{12}+a_{21})x_2 + b_1 = 0; \\ (a_{12}+a_{21})x_1 + 2a_{22}x_2 + b_2 = 0. \end{cases} \tag{2}$$

- Using matrix notation, system (2) can be compactly written as

$$(A + A^T)x + b = 0.$$

- If A is symmetric (i.e., $a_{12} = a_{21}$), the system becomes

$$2Ax + b = 0$$

or, equivalently,

$$Ax = -\frac{1}{2}b.$$

If we have a general system of linear equations

$$Ax = b,$$

can we somehow "divide" both sides of the equation by A so that we obtain the solution? How does $a \neq 0$ translate to the general setting when A is a matrix?

Definition (Definition 2.2.2 of Johnston (2021))

$A \in \mathbb{R}^{n \times n}$ is called *invertible* if there exists a matrix, which we denote by A^{-1} and call the *inverse* of A, such that

$$AA^{-1} = A^{-1}A = I_n.$$

Invertible matrices are also sometimes called *non-singular*, while non-invertible ones are called *singular*.

Examples

- Consider
$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}.$$
Then the inverse of A is given by
$$A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$
- Are there any other matrices that would also be an inverse of A?

Theorem

Suppose that $A \in \mathbb{R}^{n \times n}$ is invertible. The inverse of A is unique.

Proof.

Suppose that $A \in \mathbb{R}^{n \times n}$ has two inverses $B \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{n \times n}$. Then

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

so in fact these two inverses are equal. The proof is complete. $\square$

Properties of matrix inverses

Theorem (Theorem 2.2.2 of Johnston (2021))

Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix, let $c \neq 0$, and let k be a positive integer. Then

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$;

(b) cA *is invertible and* $(cA)^{-1} = \frac{1}{c}A^{-1}$;

(c) A^T *is invertible and* $(A^T)^{-1} = (A^{-1})^T$;

(d) A^k *is invertible and* $(A^k)^{-1} = (A^{-1})^k$.

Socks-and-shoes rule

Theorem (Theorem 2.2.3 of Johnston (2021))

Suppose that $A, B \in \mathbb{R}^{n \times n}$ are invertible. Then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Questions

- Are all matrices invertible?
- How to check if a general matrix is invertible?
- If a matrix is invertible, how do we find its inverse?

Invertibility of 2×2-matrices

Theorem

A 2×2-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$ and in this case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- This can be verified by a direct computation (exercise).
- The number $ad - bc$ is called the determinant of A; we shall return to this concept later.

Two equations and two unknowns revisited

- Consider a system of two linear equations and two unknowns

$$\begin{cases} ax + by = u; \\ cx + dy = v. \end{cases}$$

- This is equivalent to the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

- If $ad - bc \neq 0$, then the unique solution is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} du - bv \\ -cu + av \end{bmatrix}.$$

Definition

We say that a system of linear equations is *homogeneous* if it can be expressed in matrix form as

$$Ax = 0,$$

where $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^{n \times 1}$.

A homogenous system of linear equations always has at least one solution.

Characterising invertibility

Theorem (Theorem 2.2.4 of Johnston (2021))

Suppose that $A \in \mathbb{R}^{n\times n}$. The following are equivalent.

(a) *A is invertible.*

(b) *The linear system $Ax = 0$ has a unique solution.*

(c) *The reduced REF of A is I_n.*

- The reduced row echelon form tells us if A^{-1} exists but how do we find it?
- For that we need elementary matrices.

Elementary matrices

Definition

A square matrix $E \in \mathbb{R}^{n\times n}$ is called an *elementary matrix* if it can be obtained from the identity matrix via a single elementary row operation.

Examples

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_1 \leftrightarrow R_2}, \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{R_3 - 3R_1}, \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\frac{1}{2}R_2}.$$

Elementary row operations as matrix multiplication

Theorem (Theorem B.3.1 of Johnston (2021))

Suppose $A \in \mathbb{R}^{m\times n}$. If applying a single elementary row operation to A results in a matrix B, and applying that same elementary row operation to I_m results in a matrix E, then $EA = B$.

For example,

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=E} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{=A} = \underbrace{\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}}_{=EA}.$$

Inverses of elementary matrices

Theorem

Elementary matrices are invertible and their inverses are elementary.

Proof.

Suppose that E is an elementary matrix. Let E_0 be the elementary matrix obtained by performing the inverse row operation on I_n. Then, according to Theorem B.3.1 of Johnston (2021), E_0 transforms E back into I_n so that $E_0 E = I_n$. We also have that EE_0 and hence $E_0 = E^{-1}$. The proof is complete. $\square$

Finding the inverse of a matrix

- According to Theorem 2.2.4 of Johnston (2021), A is invertible if and only if a finite number of elementary row operations transforms A into I_n or, equivalently, there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k \ldots E_1 A = I_n.$$

- If A is invertible, it follows that

$$E_k \ldots E_1 = E_k \ldots E_1 I_n = E_k \ldots E_1 A A^{-1} = I_n A^{-1} = A^{-1}.$$

- In simple terms, the elementary row operations that bring A to I_n simultaneously bring I_n to A^{-1}.

Theorem (Theorem 2.2.5 of Johnston (2021))

Suppose $A \in \mathbb{R}^{n\times n}$. Then A is invertible if and only if there exists a matrix $E \in \mathbb{R}^{n\times n}$ such that the reduced REF of the block matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ is $\left[\begin{array}{c|c} I_n & E \end{array}\right]$. Furthermore, if A is invertible then it is necessarily the case that $A^{-1} = E$.

Exercise

- Suppose that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}.$$

- Find the inverse of A by bringing the augmented matrix

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

  to reduced REF.
- If the reduced REF is of the form $[I_3 \mid B]$, then A is invertible and $B = A^{-1}$. If not, then A is not invertible.

References

Johnston, Nathaniel (2021). ***Introduction to linear and matrix algebra.*** Springer Cham.