Integration

AI503
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Lecture 10

Integration by Parts

Recall the **Product Rule:**

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x) + C.$$

Hence,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Integration by Parts Formula

Let $u = f(x)$ and $dv = g'(x)\,dx$. Then

$$\int u\,dv = uv - \int v\,du.$$

For definite integrals:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Examples

Evaluate:

1. $\displaystyle\int_0^1 xe^x\, dx.$
2. $\displaystyle\int_1^2 \ln x\, dx.$

Extending to Two Variables

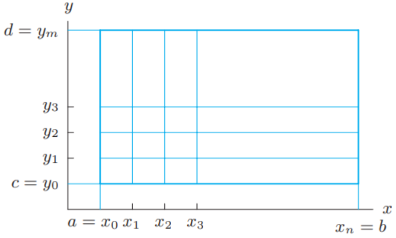
We extend the definition of the integral to a function $f(x, y)$ defined on

$$a \leq x \leq b, \quad c \leq y \leq d.$$

Divide both intervals into n and m equal parts:

$$\Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{m}.$$

- Number of subrectangles:
- Area of each subrectangle:



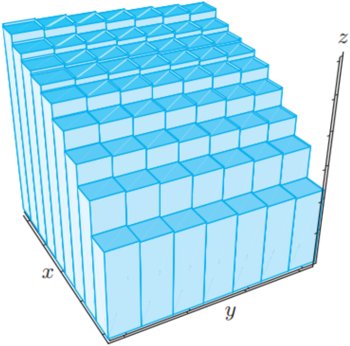
Riemann Sums for Two Variables

Let M_{ij} and L_{ij} denote the max and min of f on the ij-th rectangle. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle:

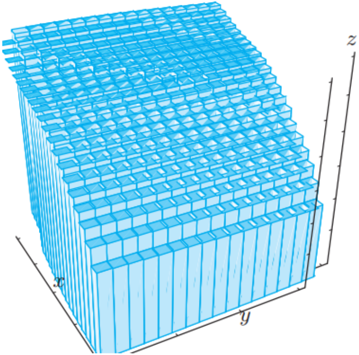
$$\sum_{i=1}^{n}\sum_{j=1}^{m} L_{ij}\Delta x\Delta y \leq \sum_{i=1}^{n}\sum_{j=1}^{m} f(u_{ij}, v_{ij})\Delta x\Delta y \leq \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\Delta x\Delta y$$

Question:

- How does $\sum_{ij} f(u_{ij}, v_{ij})\Delta x\Delta y$ look geometrically?
- What happens as $n, m \to \infty$?



z
x
y



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Double Integral: Definition

Let $f(x, y)$ be a function on $R = [a, b] \times [c, d]$. The *definite integral* of f over R is

$$\int_R f \, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(u_{ij}, v_{ij}) \, \Delta x \Delta y,$$

if the limit exists. Then f is said to be **integrable** on R.

Interpretation and Notation

- $dA = dx\,dy$ represents the area of an infinitesimal rectangle.

$$\int_R f\,dA = \int_R f(x,y)\,dx\,dy$$

- $\int_R f\,dA$ gives the volume under the surface $z = f(x,y)$ over R.
- If f is continuous (or has finitely many jump discontinuities), it is integrable on R.

Questions

1. If $g(x, y) = 1$, what is $\int_R g \, dA$?
2. How do we compute the average value of f over R?

Iterated Integral

If f is integrable on $R = [a, b] \times [c, d]$, then

$$\int_R f\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

To evaluate:

- Integrate with respect to x first (treat y constant).
- Then integrate the result with respect to y.

Example: Iterated Integral

Let $f(x, y) = x^2 y$, $R = [0, 1] \times [0, 1]$. Compute $\int_R f \, dA$.

$$\int_0^1 \int_0^1 x^2 y \, dx \, dy$$

Example: Iterated Integral

What if we switch the order of integration?

$$\int_0^1 \int_0^1 x^2 y \, dy \, dx$$

Why does the result stay the same?

Example: Volume Problem

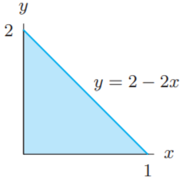
A building is 8 m wide and 16 m long. It has a flat roof 12 m high at one corner, and 10 m high at each adjacent corner. Find the volume of the building.

Integrals over Non-Rectangular Regions

We can define $\int_R f \, dA$ for other shapes (triangles, circles, etc.) by enclosing R in a rectangle and summing over subrectangles inside R.

Example

The density at (x, y) of a triangular metal plate is $\delta(x, y)$. Express its mass as an iterated integral.



Limits on Iterated Integrals

Rules:

1. The limits of the outer integral must be constants.
2. The limits of the inner integral can depend on the outer variable.

Example

A city occupies a semicircular region of radius 3 km bordering the ocean. Find the average distance from points in the city to the ocean.

Example

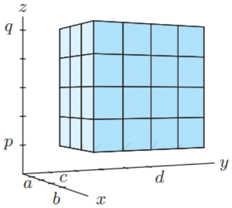
Sketch the region for

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx,$$

then reverse the order of integration.

Triple Integrals: Definition

We can define integrals over a solid region W in $\mathbb{R}^3$ by subdividing into boxes.



Using similar reasoning, express $\int_W f\,dV$ as a limit of Riemann sums.

Triple Integrals as Iterated Integrals

If f is integrable on $W = [a, b] \times [c, d] \times [p, q]$, then

$$\int_W f\, dV = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

Other integration orders are also possible.

Example

Compute $\int_W (1 + xyz)\, dV$ over the cube $0 \leq x, y, z \leq 4$.

Example

Set up an iterated integral for the mass of a solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, with density $\delta(x, y, z) = z$.