Homework 1

① a) Fixed $j$ and given $\omega_{ji} = 1, \forall i \in [1,9]$ and $b_j = 0$ then

$$z_j = step_j\left(0 + \sum_{i=1}^{9}(x_i\ 1)\right) = step_j\left(\sum_{i=1}^{9} x_i\right).$$

From the definition of step( ) we can notice that when $x_i, \forall i \in [1,9]$ is 1 then the sum is grater or equal to 1 and $y = 1$. This is the exact definition of OR operation: when at least one $x_i = 1$ then $y = 1$

b) Fixed $j$, $I \subseteq [1,9]$ and given $\omega_{ji} = 1, \forall i \in I$, $\omega_{ji} = -1, \forall i \in [1,9] \setminus I$ then

$$z_j = step_j\left(|I| + \sum_{i=1}^{9}(x_i \cdot \omega_{ji})\right) = step_j\left(|I| + \sum_{i \in I} -x_i + \sum_{i \in [1,9]\setminus I} x_i\right)$$

Here we can notice that when $x_i = 0$ ($\forall i \in I$) in the overall computation counts as 1, while when $x_i = 1$ ($\forall i \in I$) it cancels out counting as 0.
So it's still an OR operation, but inverted for $i \in I$

c) A possible weights choice is the following:

$\omega_{ji}$:

| $j \backslash i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 |
| 2 | 1 | -1 | -1 | 1 | -1 | -1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | -1 | -1 | 1 | -1 | -1 | 1 |
| 4 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | -1 | -1 |

| $b_j$ | $u_j$ |
|---|---|
| 4 | -1 |
| 4 | -1 |
| 4 | -1 |
| 4 | -1 |

$c = 4$

The 4 rows of the matrix represent the 4 possible cases to detect, so that $\omega_{ji} = -1$ only when the i-th pixel should be 1 in the j-th case, otherwise $\omega_{ji} = 1$
$c$ is set to 4 so that when one of the 4 cases is given to the network only one $z_j$ is 0 and the others are 1, so we can get $y = 1$ in our 4 cases.

② a) From the calculus rules we know that a composite function get derived as follows:

$$\frac{\partial \ell(f(x,w), g(x))}{\partial w} = \frac{\partial \ell(f(x,w), g(x))}{\partial f} \cdot \frac{\partial f(x,w)}{\partial w}$$

b) $\frac{\partial \ell(f(x,w), g(x))}{\partial f} = \frac{\partial (f-g)^2}{\partial f} = 2(f-g)$

$\frac{\partial f(x,w)}{\partial w} = \frac{\partial (wx)}{\partial w} = x$

$\rightarrow \frac{\partial (xw - x^3)^2}{\partial w} = 2(xw - x^3) \cdot x$

Now:

$$\int_{-1}^{1} 2x(xw - x^3)\,dx = 2\int_{-1}^{1} x^2 w - x^4\,dx = 2\left[\frac{x^3 w}{3} - \frac{x^5}{5}\right]_{-1}^{1} = 2\left[\frac{w}{3} - \frac{1}{5} + \frac{w}{3} - \frac{1}{5}\right] =$$

$$= \left(\frac{2w}{3} - \frac{2}{5}\right) \cdot 2$$

Let's impose it equal to 0: $2\left(\frac{2w}{3} - \frac{2}{5}\right) = 0$

$\leftrightarrow w = \frac{3}{5}$

c)