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Impact of Drag on Freefall
4 December 2020

In the investigation of the impact of drag on freefall, three objects, each with mass 0.500 kg, will be evaluated as each is vertically launched from the ground. Assume each object follows a perfectly vertical trajectory and does not undergo rotational motion.

Given:

The coefficient for drag is given by

$$D = \tfrac{1}{2} p c_d A \qquad (1)$$

where p is the density of the medium, A is the cross-sectional area of the object, and c_d is the coefficient of drag, dependent only on the shape of the object. In this case, the density of air, p, is 1.139 kg/m^3 at 25ºC at atmospheric pressure. For a sphere, $c_d = 0.47$.

The force of drag is given by

$$F = -Dv^2 \qquad (2a)$$

where v represents the velocity of the object and the object is moving in the +x direction. If it is moving in the -x direction, the force of drag is

$$F = Dv^2 \qquad (2b)$$

as the force due to drag is always acting in the direction opposite of motion.

Using Newton's Second Law of Motion and the Fundamental Theorem of Calculus yields an expression representing the velocity of an object dependent on its position:

$$v\,dv = \tfrac{F}{m}\,dx \qquad (3)$$

where F represents the net external force on the object, m represents its mass, and x represents its position.

The kinetic energy of an object in motion is given by

$$K = \tfrac{1}{2} m v^2 \qquad (4)$$

And the potential energy of an object is given by

$$U = mgx \qquad (5)$$

where g is the acceleration due to gravity.
The work done by a force on an object is given by

$$W = \int F\,dx \tag{6}$$

where F is the force doing the work.

Model:

In order to investigate the values of *speed*, *kinetic energy*, *potential energy*, and *work done by drag* as the object moves vertically, we will first derive these values as functions of the object's position. For the sake of simplicity, we will split this model into two parts: **Case 1: *going up***, and **Case 2: *going down***.

Let x_o be the initial height, v_o be the initial velocity, and x_m be the maximum height for each object's trajectory.

Case 1: *going up (with drag)*

When the object is traveling up, the forces due to gravity and drag (**Eq. 2a**) oppose its motion, thus, the net external force on the object is

$$F = -Dv^2 - mg \tag{7}$$

speed: Now that we have an expression for the net external force in this case, we can use **Eq. 3** to solve for the velocity. First, we must substitute the variable F found in **Eq. 3** for **Eq. 7**:

$$vdv = -(\tfrac{D}{m}v^2 + g)dx$$

Integrating this results in an expression for velocity. We will ignore the $\pm$ attached to the beginning of the square root because we are only speaking of motion in one direction:

$$v = \sqrt{(v_o{}^2 + \tfrac{mg}{D})e^{\frac{-2D}{m}(x-x_0)} - \tfrac{mg}{D}} \tag{8}$$

where v_o is the initial velocity the object is launched with. The introduction of this variable is justified by the definition of the velocity differential, dv, as its bounds are $[v_o, v]$. *Note* that we can now solve for the maximum height of the trajectory by setting $v = 0$. This results in an expression for the maximum height which we will use later:

$$x_m = -\frac{m\cdot ln(\frac{mg}{Dv_o{}^2+mg})}{2D} \tag{20}$$

kinetic energy: By including the newfound expression for v (**Eq. 8**) in **Eq. 4**, we can easily solve for the kinetic energy as a function of the object's position:

$$K = \tfrac{1}{2}m[(v_o{}^2 + \tfrac{mg}{D})e^{\frac{-2D}{m}(x-x_0)} - \tfrac{mg}{D}] \tag{9}$$

potential energy: The potential energy of the object is given by **Eq. 5**.

work done by drag: We know from **Eq. 6** that we can integrate the force of drag for this case (**Eq. 2a**) from the x_o to x:

$$W = -\int_{x_o}^{x} Dv^2 dx$$

which, after substituting the expression for v (**Eq. 8**), can be integrated to achieve an expression for the work done by drag:

$$W = mg(x - x_o) + \tfrac{m}{2D}\left[(Dv_o{}^2 + mg)(e^{\frac{-2D}{m}(x-x_o)} - 1)\right] \tag{10}$$

Case 2: ***going down (with drag)***

When the object is traveling down, the force due to drag (**Eq. 2b**) opposes its motion while the force due to gravity is what brings it back to the ground, thus, the net external force on the object is

$$F = Dv^2 - mg \tag{11}$$

speed: Now that we have an expression for the net external force in this case, we can use **Eq. 3** to solve for the velocity. Similarly to Case 1, we must first substitute the variable F found in **Eq. 3** for **Eq. 11**:

$$vdv = (\tfrac{D}{m}v^2 - g)dx$$

Integrating this results in an expression for velocity. Again, we will ignore the $\pm$ attached to the beginning of the square root because we are only speaking of motion in one direction:

$$v = \sqrt{\tfrac{mg}{D}(1 - e^{\frac{2D}{m}(x-x_m)})} \tag{12}$$

we can justify the introduction of the variable x_m by noting that the object's trajectory is bounded by $[x_m, x]$.

kinetic energy: Similarly to Case 1, by including the newfound expression for v (**Eq. 12**) in **Eq. 4**, we can easily solve for the kinetic energy as a function of the object's position:

$$K = \tfrac{1}{2}m[\tfrac{mg}{D}(1 - e^{\frac{2D}{m}(x-x_m)})] \tag{13}$$

potential energy: Again, the potential energy of the object is given by **Eq. 5**.

work done by drag: We know from **Eq. 6** that we can integrate the force of drag for this case (**Eq. 2b**) from x_m to x:

$$W = \int_{x_m}^{x} Dv^2 dx$$

which, after substituting the expression for v (**Eq. 12**), can be integrated to achieve an expression for the work done by drag:

$$W = mg(x - x_m) - \tfrac{m^2 g}{2D}(e^{\frac{2D}{m}(x-x_m)} - 1) \tag{14}$$

Case 3: ***going up (without drag)***

In this case, when the object is traveling up, the only force opposing it is its weight:

$$F = -mg \tag{15}$$

speed: Substituting F into **Eq. 3** with **Eq. 15**, we get

$$vdv = -gdx$$

which, by integration, provides us with our equation for speed:

$$v = \sqrt{v_o{}^2 - 2g(x - x_o)} \tag{16}$$

kinetic energy: Substituting **Eq. 16** into **Eq. 4**, we can see the kinetic energy is

$$K = \tfrac{1}{2}m[v_o{}^2 - 2g(x - x_o)] \tag{17}$$

potential energy: The potential energy of the object is given by **Eq. 5**.

Case 4: ***going down (without drag)***

In this case, when the object is traveling down, its weight acts in the same direction as its motion, the -x direction, thus, we will again use **Eq. 15** to represent the net external force.

speed: Substituting F into **Eq. 3** with **Eq. 15**, we get

$$vdv = -gdx$$

which, by integration, provides us with our equation for speed:

$$v = \sqrt{-2g(x - x_m)} \tag{18}$$

Despite this derivation being similar to that of **Eq. 16**, the end result differs because Case 3 is launched from the ground, while this case deals with the object as it falls from its maximum height back to the ground.

kinetic energy: Substituting **Eq. 18** into **Eq. 4**, we can see the kinetic energy is

$$K = \tfrac{1}{2}m[-2g(x - x_m)] \tag{19}$$

potential energy: The potential energy of the object is given by **Eq. 5**.

Application:

Let S_1, S_2, and S_3 be three spherical bodies of mass 0.500 kg with cross-sectional areas A_1, A_2, and A_3 where A_1 = 1.80 m^2, A_2 = 3.60 x 10^{-2} m^2, and A_3 = 7.20 x 10^{-4} m^2. Using **Eq. 1** yields a coefficient of drag (D) for each body to be used in computation.

By choosing a v_o for each spherical body, one less than, one equal to, and one greater than terminal velocity, we can first find the maximum height (x_m) using **Eq. 20**. After computing x_m, we know the domain of our function, $[x_o,\ x_m]$. We can then apply the various expressions for *speed*, *kinetic energy*, *potential energy*, and *work of drag* to each domain. The result of this very process can be found in **Figs. 1.A1-2.A3**. **Note that in the instances when there is no drag, the function is evaluated along the same domain for the sake of comparison.**

Interpretation:

Each figure shows that drag acts against the speed and kinetic energy of each object. **Fig. 1.A1.v1**, for example, shows both objects in Cases 1 and 3 being launched with the same velocity; however, by the time they both reach the end of their domain (x_m in the presence of drag), the object in Case 3 maintains a higher speed and higher kinetic energy.

For Cases 3 and 4, *both objects travel the same speed at the same values of displacement*. This is due to the conservation of gravitational energy. For example, **Fig. 1.A1.v1**'s Case 3 and **Fig. 2.A1.v1**'s Case 4 both share the same values for speed at a given displacement.

In **Figs. 1.A1-1.A3** *drag causes the speed to decrease with downward concavity*, thus, the object decelerates faster as its value decreases. On the other hand, its kinetic energy decreases with upward concavity, thus, as the object slows down, its kinetic energy decreases at a slower rate. This behavior of kinetic energy is because, as it slows, the force of drag decreases, drag does less work on the object, and the object loses less energy as it moves.

In **Figs. 2.A1-2.A3** *drag causes the speed and kinetic energy to increase with downward concavity*. This is because, as the object's speed approaches terminal velocity, it accelerates at a lower rate. In Case 4, while the speed increases with downward concavity, the kinetic energy increases linearly. This is because gravitational potential energy is converted to kinetic energy without any loss of energy.

Gravitational potential energy experiences no effect due to drag, as gravitational potential energy simply involves the distance of the object from some arbitrary position; thus, in every case, the plot of potential energy follows a linear path.