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Shannon's entropy for combining the efficiency results of different DEA models: Method and application

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ABSTRACT

Data envelopment analysis (DEA) was initially proposed by Charnes, Cooper, and Rhodes for evaluating the decision making units (DMUs), via calculating their relative efficiencies. To do this, there exist many different DEA models and different DEA viewpoints. Hence, selecting the best model (viewpoint) or the way of combining different models (viewpoints) for ranking DMUs is a main question in applied DEA. This paper provides a methodology, based upon Shannon's entropy formula, for combining the efficiency results of different DEA models (viewpoints). An application of the provided method in educational departments of a university in Iran is also illustrated.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique for measuring the relative efficiencies of a set of decision making units (DMUs) which consume multiple inputs to produce multiple outputs. This technique was initially proposed by Charnes et al. (CCR model) (Charnes, Cooper, & Rhodes, 1978) and was improved by other scholars, especially Banker et al. (BCC model) (Banker, Charnes, & Cooper, 1984). Nowadays, DEA has allocated to itself a wide variety of applied research and there is a great variety of DEA models for evaluation of DMUs, e.g., CCR, BCC, FDH (FDH-CRS, FDH-VRS, FDH-NDRS, and FDH-NIRS), SBM, RAM, Cost models, FG, ST, etc. (see books (Cooper, Sieford, & Tone, 2000; Ray, 2004; Thanassoulis, 2001) and the references therein). Also there exist different viewpoints for utilizing the DEA technique; for instance, input-oriented and output-oriented views (Cooper et al., 2000) or optimistic and pessimistic views (Entani, Maeda, & Tanaka, 2002).

It happens often that in an applied project we would like to rank the DMUs using DEA. Some questions arise: which model is more suitable? Which viewpoint is more desirable? It is clear that the rankings of units obtained by various models may not be the same. Each of the above-mentioned models and viewpoints has some valuable advantages which we would like not to ignore. Note that if we cannot identify the production frontier by preliminary surveys, it may be risky to rely on only one particular model (Cooper et al., 2000, p. 104). Hence, if the consequences of the performance analysis are important, it is wise to try different models and combine the results of the different models and viewpoints. To this end, this paper provides an approach which is based on Shannon's entropy formula and gives a complete ranking. This approach has some theoretical and applied advantages. For instance, in this method the most productive scale size (MPSS) units (Cooper et al., 2000) get the best rank and the interior points of the smallest production possibility sets (PPSs) which are inefficient in all models lie at the end of the ranking list. It is clear that combining the efficiency results of different DEA models provides a more realistic ranking compared with using any of the DEA models individually.

2. Provided approach

2.1. Method

It is assumed that we have a set of DMUs consisting of DMU_j; $j = 1,\dots,n$, with input–output vectors $(\mathbf{x}_j, \mathbf{y}_j)$; $j = 1,\dots,n$, in which $\mathbf{x}_j = (x_{1j},\dots,x_{mj})$ and $\mathbf{y}_j = (y_{1j},\dots,y_{sj})$. Also assume that these units have been evaluated by a set of different DEA models, say,[1]

$$M = \{M_1, M_2, \dots, M_k\} = \{\text{CCR}, \text{BCC} - \text{I}, \text{BCC} - \text{O}, \text{CCR} - \text{PV}, \dots\}$$

and the obtained efficiency results are as listed in the following matrix $E_{n\times k}$. Each row of E corresponds to a DMU and each of its columns corresponds to a model considered in M. Thus, E_{jl} exhibits the efficiency score of DMU_j obtained by model M_l for $j = 1,2,\dots,n$ and $l = 1,2,\dots,k$.

[1] The notations "*I*" and "*O*" stand for input and output orientations, respectively, and "*PV*" exhibits the pessimistic view. See (Cooper et al., 2000) for more details about the input- and output-oriented models and (Entani et al., 2002) for optimistic and pessimistic viewpoints in DEA.

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$$E = \begin{array}{cc} \begin{array}{cccc} M_1 & M_2 & \dots & M_k \\ \downarrow & \downarrow & & \downarrow \end{array} & \\ \begin{pmatrix} E_{11} & E_{12} & \dots & E_{1k} \\ E_{21} & E_{22} & \dots & E_{2k} \\ \vdots & \vdots & & \vdots \\ E_{n1} & E_{n2} & \dots & E_{nk} \end{pmatrix} & \begin{array}{c} \leftarrow 1 \\ \leftarrow 2 \\ \vdots \\ \leftarrow \mathrm{n} \end{array} \end{array}$$

Now we would like to have the degree of the importance of each of the considered models, $M_l \in M$. To do this, one of the techniques which can be used is the AHP technique (Saaty, 1980; Saaty, 1986), which calculates the importance (value/weight) of considered models after some pairwise comparisons and examines the confidence by calculating a consistency rate. But we recommend obtaining the degree of importance of models considering the information of matrix E, using Shannon's entropy formula, and not only based on some comparisons. The concept of Shannon's entropy (Shannon, 1948; Taneja) has a central role in information theory, and sometimes refers to measure of uncertainty. This concept has been extended to different scientific fields, such as physics, social sciences, and so on. We use this formula to obtain the degree of importance of models in the following four steps:

Step 1. (Normalization) Set $\overline{E}_{jl} = \frac{E_{jl}}{\sum_{j=1}^{n} E_{jl}}$, $j = 1, \dots, n, l = 1, \dots, k$.

Step 2. Compute entropy e_l as

$$e_l = -e_0 \sum_{j=1}^{n} \overline{E}_{jl} \cdot \ln \overline{E}_{jl}, \quad l = 1, \dots, k,$$

where e_0 is the entropy constant and is considered equal to

$$e_0 = (\ln n)^{-1}.$$

Step 3. Set $d_l = 1 - e_l$ as the degree of diversification for $l = 1, \dots, k$.

Step 4. Set

$$w_l = \frac{d_l}{\sum_{t=1}^{k} d_t}, \quad l = 1, \dots, k$$

as the degree of importance of model M_l.

The following remark gives an interesting interpretation of using Shannon's entropy to calculate the value of models.

Remark 1. In fact, the above procedure for calculating the value of a model is based on the difference of efficiency scores obtained by the respective model. It is clear that if a model yields approximately equal scores for all units, then it does not have any considerable effect on the ranking, and hence it must be considered of a small degree of importance. Now we establish this fact mathematically. Considering $M_l \in M$, suppose that there exists a constant, c, such that

$$E_{jl} \to c,$$

for each $j = 1, 2, \dots, n$. Thus, $\overline{E}_{jl} \to \frac{1}{n}$, for each $j = 1, 2, \dots, n$. This implies that

$$e_l \to \frac{\sum_{j=1}^{n} \frac{1}{n} \ln \frac{1}{n}}{-\ln n} = 1.$$

Hence $d_l \to 0$ which implies that $w_l \to 0$, when $E_{jl} \to c$.

After calculating w_l for $l = 1, 2, \dots, k$ (using the above four steps), we calculate the following efficiency index which combines the efficiency scores (provides a weighted-sum of the efficiency scores) of all of the considered models, regarding the values of w_ls, and is suitable to provide a full ranking:

$$\beta_j = \sum_{l=1}^{k} w_l E_{jl}, \quad j = 1, \dots, n. \tag{1}$$

In the DEA literature, the envelopment form of the CCR model has the biggest feasible region and, hence, it has the minimum efficiency score compared with the other current radial models. Considering this fact, the following proposition provides this main property of the provided approach that the MPSS units (Cooper et al., 2000) get the first rank. Recall that DMU_o is MPSS if it is CCR-efficient.

Proposition 1

(i) $0 < \beta_j \leqslant 1$, $j = 1, \dots, n$.
(ii) If the CCR model belongs to M, which contains the radial DEA models with efficiency scores between zero and one, and DMU_o $o \in \{1, 2, \dots, n\}$ is MPSS, then $\beta_o = 1$.

Proof

(i) Straightforward.
(ii) Without loss of generality suppose that $M_1 = \mathrm{CCR}$. Since DMU_o is MPSS, it is CCR-efficient, and hence $E_{o1} = 1$. Since the CCR-efficiency score is equal or less than that of other models, $E_{ol} = 1$ for each $l \in \{1, 2, \dots, k\}$. Hence

$$\beta_o = \sum_{l=1}^{k} w_l = 1,$$

and the proof is complete. □

Corollary 1. *Regarding the above proposition, the MPSS units get the first rank position.*

2.2. Numerical illustration

In this subsection, we apply the data set of eight units ($A, B, \dots, H$), as listed in Table 1, to illustrate the applications of the provided technique. Each unit consumes one input to produce one output.

We have used four models, CCR, BCC-I, BCC-O, and FDH-VRS-I for assessing the considered DMUs, i.e.
$M = \{M_1, M_2, M_3, M_4\} = \{\mathrm{CCR}, \mathrm{BCC-I}, \mathrm{BCC-O}, \mathrm{FDH-VRS-I}\}$.

The efficiency scores obtained from these models have been summarized in matrix E:

$$E = \begin{array}{cc} \begin{array}{cccc} M_1 & M_2 & M_3 & M_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} & \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0.714 & 1 & 1 & 1 \\ 0.55 & 1 & 1 & 1 \\ 0.344 & 0.625 & 1 & 0.625 \\ 0.286 & 0.286 & 0.4 & 0.286 \\ 0.3 & 0.3 & 0.545 & 0.3 \end{pmatrix} & \begin{array}{c} \leftarrow \mathrm{A} \\ \leftarrow \mathrm{B} \\ \leftarrow \mathrm{C} \\ \leftarrow \mathrm{D} \\ \leftarrow \mathrm{E} \\ \leftarrow \mathrm{F} \\ \leftarrow \mathrm{G} \\ \leftarrow \mathrm{H} \end{array} \end{array}$$

Table 1
Data of the numerical example

DMUs	Input	Output
A	2	2
B	3	3
C	4	4
D	7	5
E	10	5.5
F	16	5.5
G	7	2
H	10	3

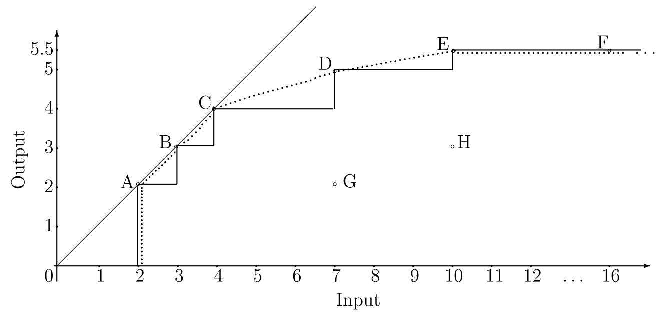

Fig. 1. The production possibility sets in the illustrative example.

From matrix E (and also Fig. 1) it can be seen that there are three MPSSs, A, B, C. Also it can be observed that the results of different models give different ranking results for units. The CCR frontier is the half-line segment passing through the origin and points A, B, C. The BCC frontier has been shown by dotted lines, and the FDH-VRS frontier is a stair case and has been shown by line segments.

Before discussing the provided procedure, we point out a main drawback of the existing super-efficiency-based ranking methods in the DEA literature (Adler, Friedman, & Sinuany-Stern, 2002; Andersen & Petersen, 1993). The so-called super-efficiency DEA (first provided by Andersen & Petersen (1993) and improved by others (Adler et al., 2002)) has the desirable feature of differentiating between some of the efficient DMUs that have identical efficiency scores equal to one in the basic DEA models. The ability to rank or differentiate between the efficient DMUs is of both theoretical and practical importance. But there is a main drawback in these models: It may happen that the MPSSs do not have the best rank among efficient units when they are assessed by super-efficiency-based models. For instance, in the above example, consider the BCC-I model. Using this model DMUs A, B, C, D and E are efficient. Now we calculate the super-efficiency score of these units by Andersen and Petersen's (1993) model. We have

$$\theta_E^{\text{super}} = 1.6 > \theta_A^{\text{super}} = 1.5 > \theta_C^{\text{super}} = 1.25 > \theta_D^{\text{super}} = 1.143 > \theta_B^{\text{super}} = 1.$$

It can be seen that B is MPSS, while it has the worst rank. Also E is not MPSS while it has the best rank.

Now we apply the provided approach. Applying Shannon's entropy procedure yields

$$w_1 = 0.376856665 \quad w_2 = 0.299709675$$
$$w_3 = 0.136350023 \quad w_4 = 0.187083635$$

and hence

$$\beta_A = \beta_B = \beta_C = 1, \quad \beta_D = 0.89221899181, \quad \beta_E = 0.83041449875,$$
$$\beta_F = 0.57023453451, \quad \beta_G = 0.30154390205,$$
$$\beta_H = 0.333405755635,$$

which leads to the following ranking:

$$A = B = C > D > E > F > H > G.$$

It can be seen that the MPSSs have the best rank.

Now one may need a more complete ranking by sorting the positions of the MPSSs A, B, C. There are many ranking methods in the DEA literature for differentiating between efficient units (Adler et al., 2002). But the ranking results of all methods are not the same. Hence we recommend selecting some of the ranking methods using an AHP tree and then sketching a ranking procedure for A, B, C to sort them.

2.3. Some practical points

One of the factors that plays a crucial role in the computational requirements for using different DEA models is the number of considered models, k. It is evident that applying all of the current DEA models is expensive from a computational point of view. Thus, for selecting some models among current DEA models, we recommend constructing a hierarchical tree to the managers, then constructing the AHP matrices after performing the pairwise comparisons, and finally choosing the models with the highest priority. For depicting the hierarchical tree, to choose the convenient models, some criteria, say, computational complexity, interpretability, and so on, can be considered. Finally note that we can incorporate the weights obtained by AHP and the results of Shannon's entropy to obtain the degree of importance of models. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the weights obtained from AHP, Step 4 of Shannon's entropy can be modified as follows:

$$\bar{w}_l = \frac{\alpha_l d_l}{\sum_{t=1}^{k} \alpha_t d_t}, \quad l = 1, \ldots, k.$$

It is evident that the above formula incorporates the results of the AHP tree and Shannon's entropy.

3. Real application

In this section, an experimental study has been carried out to rank 20 educational departments of a university in Iran. This study has been done as a research project and the results are available in (Soleimani-damaneh). There are also some papers in the literature which deal with the applications of DEA in the educational environments, see, e.g., (Casu & Thanassoulis, 2006; Fandel, 2007; Fried, Lovell, & Turner, 1996) among others. Here, each educational department uses three inputs to produce three outputs as follows:

input-1: budget,
input-2: area of cultural, educational, and research space,
input-3: number of books in the library,
output-1: average of the scores of the students,
output-2: number of the students accepted for graduate studies (in the past two years),
output-3: satisfaction of students, staff, and professors.

The data for inputs and outputs has been adopted from *Information Office* of the university, except for output-3 whose values have been collected using questionnaires as qualitative data, categorized on a four-point scale (poor, medium, good, and excellent), and hence have been converted to quantitative data using the technique provided in (Jahanshahloo, Soleimani-damaneh, & Mostafaee, 2007). For evaluating and ranking the units, we have recommended 25 DEA models (including input-oriented, output-oriented, CRS, VRS, FDH, norm-based, optimistic-view-based, weight-restricted, ... models) to the managers of the university and the operational managers of the project. First we suggested that the managers use the AHP technique for selecting five models with respect to some criteria, such as: computational complexity, interpretability, validity of the efficiency scores as well as the projection points produced by models in the previous projects existing in the literature and the previous projects completed by our team, preliminary surveys about the RTS assumption of technology, etc. After collecting the managers' opinions and using the AHP technique, we decided to administer five selected models (denoted by $M_1, \ldots, M_5$) to the data. The obtained efficiency scores have been listed in the following matrix E. Note that in this matrix the units under consideration are denoted by $D1, D2, \ldots$.

$$\begin{array}{ccl} & \begin{array}{ccccc} M_1 & M_2 & M_3 & M_4 & M_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} & \\ E = & \begin{pmatrix} 1 & 0.82 & 1 & 0.9 & 1 \\ 0.95 & 1 & 1 & 0.8 & 0.6 \\ 0.86 & 0.5 & 1 & 0.65 & 0.45 \\ 1 & 0.8 & 0.8 & 1 & 1 \\ 0.9 & 0.82 & 0.85 & 1 & 0.88 \\ 0.8 & 0.67 & 0.65 & 0.75 & 0.8 \\ 1 & 0.82 & 0.82 & 0.62 & 1 \\ 1 & 1 & 1 & 0.44 & 0.9 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0.9 & 0.8 & 0.8 & 1 \\ 0.8 & 0.75 & 0.8 & 1 & 0.55 \\ 0.8 & 0.45 & 0.8 & 1 & 0.63 \\ 0.95 & 0.33 & 0.73 & 1 & 0.9 \\ 1 & 1 & 1 & 1 & 1 \\ 0.95 & 0.75 & 1 & 0.6 & 0.82 \\ 0.92 & 0.4 & 1 & 0.83 & 0.62 \\ 1 & 0.35 & 0.35 & 1 & 0.85 \\ 1 & 1 & 1 & 1 & 1 \\ 0.89 & 0.68 & 0.7 & 0.75 & 0.9 \\ 0.88 & 0.35 & 0.88 & 0.64 & 0.85 \end{pmatrix} & \begin{array}{l} \leftarrow \mathrm{D1} \\ \leftarrow \mathrm{D2} \\ \leftarrow \mathrm{D3} \\ \leftarrow \mathrm{D4} \\ \leftarrow \mathrm{D5} \\ \leftarrow \mathrm{D6} \\ \leftarrow \mathrm{D7} \\ \leftarrow \mathrm{D8} \\ \leftarrow \mathrm{D9} \\ \leftarrow \mathrm{D10} \\ \leftarrow \mathrm{D11} \\ \leftarrow \mathrm{D12} \\ \leftarrow \mathrm{D13} \\ \leftarrow \mathrm{D14} \\ \leftarrow \mathrm{D15} \\ \leftarrow \mathrm{D16} \\ \leftarrow \mathrm{D17} \\ \leftarrow \mathrm{D18} \\ \leftarrow \mathrm{D19} \\ \leftarrow \mathrm{D20} \end{array} \end{array}$$

Applying Shannon's entropy procedure yields

$w_1 = 0.02570824 \quad w_2 = 0.48725040$
$w_3 = 0.17329446 \quad w_4 = 0.12758164$
$w_5 = 0.18616525.$

Regarding the above-obtained weights, the values of β_js have been calculated by (1) and listed in Table 2. Regarding Table 2, we have the following ranking:

$$D9 = D14 = D18 > D8 > D1 > D2 > D10 > D12 > D4 > D5 > D7$$
$$> D15 > D11 > D19 > D6 > D16 > D13 > D3 > D20 > D17.$$

It can be observed that the MPSSs ($D9$, $D14$, $D18$) have the best rank. Also the interior points which are inefficient in all models ($D6$, $D19$, $D20$) are at the end of the ranking list. But the question may arise why units $D3$, $D6$, $D16$, $D17$, which have been recognized as efficient by at least one of the five considered models, follow some units which have been recognized as inefficient in all of the five considered models. To answer this question, regarding the obtained w_l values, the degree of importance of Model M_2 is much higher than that of other models and these four units, $D3$, $D6$, $D16$, $D17$, have very low scores when being assessed by M_2.

It is clear that combining the efficiency results of different DEA models provides more realistic ranking results for managers of the university compared with using each of the DEA models individually. For instance, if a DMU has a strict minimum input value for any input item, then it is recognized as an efficient unit when being assessed by some DEA models, say, BCC model without any weight restrictions, even if it is very poor with respect to other factors (see Theorem 4.3 in Cooper et al., 2000). In fact, in this case the model decides the efficiency score only based upon one factor. It is clear that this is not reasonable from an applied point of view, and this alarms us about using the results of one model individually. Note that almost all of the models have some such pitfalls. But using the combination of the results obtained from them compensates for these pitfalls.

Table 2
β_j values of the educational departments in the real application

DMU	β
D1	0.899536754
D2	0.898732150
D3	0.605731174
D4	0.867891018
D5	0.861390095
D6	0.704284189
D7	0.832620892
D8	0.909937746
D9	1.000000000
D10	0.891099730
D11	0.754612488
D12	0.874213788
D13	0.606850781
D14	1.000000000
D15	0.792359577
D16	0.613161417
D17	0.542721044
D18	1.000000000
D19	0.738751683
D20	0.585552728

4. Conclusions

It happens often that in an applied project we would like to rank the DMUs using DEA. Some questions arise: which model is more suitable? Which viewpoint is more desirable? It is clear that the rankings of units obtained by various models may not be the same. Each of the above-mentioned models and viewpoints has some valuable advantages which we would like not to ignore. But they have some shortcomings, too. For instance, if a DMU has a strict minimum input value for any input item, then it is recognized as an efficient unit when being assessed by some DEA models, say, BCC model without any weight restrictions, even if it is very poor with respect to other factors (see Theorem 4.3 in Cooper et al., 2000). In fact, in this case the model decides the efficiency score only based upon one factor. It is clear that this is not reasonable from an applied point of view, and this alarms us about using the results of one model individually. Note that almost all of the models have some such pitfalls. But using the combination of the results obtained from them compensates for these pitfalls. Regarding the above discussion, studying the approaches for combining the results of different DEA models is crucial, from an applied point of view. To this end, this paper provides an approach which is based on Shannon's entropy formula and gives a complete ranking. This approach has some theoretical and applied advantages.

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