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An improvement in DEA cross-efficiency aggregation based on the Shannon entropy

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Abstract

This paper improves a recently proposed Data Envelopment Analysis (DEA) cross-efficiency aggregation method based on the Shannon entropy. The weights for determining cross-efficiency are derived from minimizing the square distance of weighted cross-efficiency and weighted CCR efficiency. Our calculation example indicates that this method may produce inappropriate weights, which is significantly inconsistent with a widely accepted viewpoint. A variance coefficient method based on the Shannon entropy is presented to overcome the drawbacks of the DEA cross-efficiency aggregation method. In this study, comparisons of weights and cross-efficiency scores are provided.

Keywords: cross-efficiency; aggregation; weights; variance coefficient method

1. Introduction

Data Envelopment Analysis (DEA) cross-efficiency was originated by Sexton et al. (1986) and further investigated by Doyle and Green (1994), the main idea of which is to use DEA in a peer evaluation rather than a pure evaluation mode. The two prominent advantages of cross-efficiency evaluation are as follows:

1. cross-efficiency provides an efficiency ranking among all decision-making units (DMUs) to differentiate between good and poor performances;
2. cross-efficiency can eliminate the requirement of elicitation of weight restrictions from application area experts, thereby avoiding unrealistic DEA weighting schemes (Anderson et al., 2002; Liang et al., 2008).

In line with the classical denotations in DEA, we suppose that there are a set of DMUs and each DMU_j has m different inputs, x_{ij}, $i = 1, 2, \ldots, m$, and s different outputs, y_{rj}, $r = 1, 2, \ldots, s$. DEA

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Table 1
The cross-efficiency matrix

	Rated DMU_j			
Rating DMU_d	1	2	...	n
1	E_{11}	E_{12}	...	E_{1n}
2	E_{21}	E_{22}	...	E_{2n}
...	...	...	...	...
n	E_{n1}	E_{n2}	...	E_{nn}
Mean	$\bar{E}_1$	$\bar{E}_2$	...	$\bar{E}_n$

cross-efficiency is traditionally generated in a two-stage process. More specifically, in Stage 1, the self-evaluation efficiencies of each DMU are calculated according to the constant return-to-scale (CRS) DEA model developed by Charnes, Cooper and Rhodes (hereafter CCR) (1978); in Stage 2, weights derived from Stage 1 are applied to all peer DMUs to obtain the so-called cross-efficiency evaluation scores of each of the DMUs. Next, the mathematical models for the aforementioned two-stage process are presented.

Stage 1. The self-evaluation efficiency of DMU_d using the CRS DEA model is expressed as follows:

$$\max\ E_{dd} = \sum_{r=1}^{s} u_{rd} y_{rd}$$

$$\text{s.t.}\ \sum_{i=1}^{m} v_{id} x_{ij} - \sum_{r=1}^{s} u_{rd} y_{rj} \geq 0,\ j = 1, 2, \ldots, n.$$

$$\sum_{r=1}^{m} v_{id} x_{id} = 1. \tag{1}$$

$$u_{rd} \geq 0, r = 1, 2, \ldots, s$$

$$v_{id} \geq 0, i = 1, 2, \ldots, m,$$

where u_{rd} and v_{id} represent the weights for rth output and ith input for DMU_d, respectively. For each DMU_d, we obtain a set of optimal weights $v_{1d}^*, v_{2d}^*, \ldots, v_{md}^*, u_{1d}^*, u_{2d}^*, \ldots, u_{sd}^*$ by solving model (1).

Stage 2. The cross-efficiency of each DMU_j using the optimal weights of DMU_d, namely E_{dj} can be calculated as follows:

$$E_{dj} = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj}}{\sum_{i=1}^{m} v_{id}^* x_{ij}},\ d, j = 1, 2, \ldots, n. \tag{2}$$

Therefore, the cross-efficiency matrix (CEM) is presented in Table 1.

For each DMU_j, $j = 1, 2, \ldots, n$, an average of all E_{dj}, $d = 1, 2, \ldots, n$,

$$\bar{E}_j = \frac{1}{n}\sum_{d=1}^{n} E_{dj}, \tag{3}$$

is referred as the cross-efficiency score for DMU_j. This so-called cross-efficiency can also be deemed as the score aggregated with equal weights $1/n$.

Most of the extant studies on DEA cross-efficiency evaluations employ the equal weights to aggregate cross-efficiency, and thus generate the average cross-efficiency. However, the average cross-efficiency evaluations suffer several significant drawbacks.

1. Averaging cross-efficiency loses association with the weights (Despotis, 2002). The average cross-efficiency is not the only possibility to rate the units. One might also consider the range, variance, or the median as alternative ranking factors.
2. The average cross-efficiency is not a good choice, since it is not a Pareto solution (Wu et al., 2012). The average cross-efficiency has the equal weights to aggregate cross-efficiency roughly; this method has not considered the problem of Pareto solution. The equal weights have not achieved the optimal state, they still have an improvement possibility.
3. The self-evaluated efficiency of each DMU plays a distinct role in the final overall assessment and ranking, since it lies on the leading diagonal of the CEM (Wu et al., 2009). The weights assigned to them differ from one DMU to another.
4. The self-evaluated efficiencies fail to play a sufficient role in the final overall assessment and ranking (Wang and Chin, 2011). Self-evaluated efficiencies are much less weighted than peer-evaluated efficiencies, since each DMU has only one self-evaluated efficiency value, but multiple peer-evaluated efficiency values. When they are simply averaged together, the weight assigned to the self-evaluated efficiency is only $1/n$ if there are n DMUs to be evaluated, whereas the remaining weights, $(n-1)/n$, are given to those peer-evaluated efficiencies.
5. The average cross-efficiency scores sometimes fail to reflect the real performance of all DMUs, since this method simply aggregates them equally by ignoring their relative importance (Wang and Wang, 2013).

The existing studies have seldom emphasized on the question of how the decision makers can aggregate the CEM. To the best of our knowledge, there exist very few research articles on the determination of the ultimate cross-efficiency by summing up the weighted cross-efficiency values rather than simply averaging them. Soleimani-damaneh and Zarepisheh (2009) apply Shannon's entropy formula to aggregate the efficiency results of different DEA models. Wu et al. (2009) employ the Shapley value in cooperative game to determine the weights for DEA cross-efficiency aggregation. Wu et al. (2011) utilize the Shannon entropy to determine the weights for ultimate cross-efficiency scores. Wang and Chin (2011) propose the use of ordered weighted averaging operator weights for cross-efficiency aggregation, which allows the decision maker's optimism level toward best relative efficiency. Wang and Wang (2013) introduce three alternative approaches to determine the relative importance weights for aggregating DEA cross-efficiency.

In a recent paper, Wu et al. (2012) proposed the definition of the entropy value of cross-efficiency score, and then developed a distance entropy function between cross-efficiency score and CCR efficiency score to generate relative weights for cross-efficiency aggregation (hereafter called Wu's method). From Zeleny's viewpoint (1982, p. 187), "if all available alternatives score about equally with respect to a given attribute, then such an attribute will be judged unimportant by most decision makers. Such an attribute does not help in making a decision," it is reasonable to assign a small weight to a cross-efficiency score if the corresponding coefficients generate similar cross-efficiency scores across all DMUs. However, for typical DMU_d, $d = 1, 2, \ldots, n$, when $E_{d1}, E_{d2}, \ldots, E_{dn}$ are similar, or even identical, the weights obtained from Wu's method can break Zeleny's rule, which may lead to a situation in which the cross-efficiency scores are inappropriately calculated. The purpose of this paper is to improve Wu's method by developing a more adequate method to aggregate cross-efficiencies.

The remainder of this paper proceeds as follows. Section 2 reviews the methods developed by Wu et al. (2012). Section 3 proposes a variance coefficient method based on the Shannon entropy to provide an alternative for cross-efficiency aggregation. Section 4 presents a numerical example to illustrate the effectiveness of the proposed method. Section 5 concludes this study.

2. Wu's method

This section reviews Wu's method for DEA cross-efficiency aggregation. After normalizing the CEM in Table 1, the entropy values of cross-efficiency scores are defined as follows:

Definition 1. *For DMU_j, the entropy value of cross-efficiency score E_{dj} is defined as*

$$h_{dj} = -e_{dj} \ln e_{dj}, \tag{4}$$

where

$$e_{dj} = \frac{E_{dj}}{\sum_{j=1}^{n} E_{dj}}.$$

Definition 2. *For DMU_j, the distance entropy function between cross-efficiency score and CCR efficiency score is defined as*

$$b_{dj} = h_{dj} - h_{jj},\ j = 1, 2, \ldots, n, \tag{5}$$

where h_{dj} and h_{jj} are the entropy value of cross-efficiency score and CCR efficiency score, respectively.

Definition 3. *For DMU_j, Wu's method considers the square of the distance of weighted cross-efficiency score and weighted CCR efficiency score as the evaluation criteria:*

$$\min z_j = \sum_{d=1}^{n} \left(b_{dj}\lambda_d\right)^2$$

$$\text{s.t.} \sum_{d=1}^{n} \lambda_d = 1,\ \lambda_d \geq 0. \tag{6}$$

For all DMUs, Wu et al. (2012) propose the following mathematical programing to derive weights:

$$\min Z = \sum_{j=1}^{n} \sum_{d=1}^{n} \left(b_{dj}\lambda_d\right)^2$$

$$\text{s.t.} \sum_{d=1}^{n} \lambda_d = 1, \lambda_d \geq 0. \tag{7}$$

The global optimal solutions to (7) are

$$\lambda_d = \frac{1}{\left[\sum_{d=1}^{n} \left(\sum_{j=1}^{n} \left(b_{dj}\right)^2\right)^{-1}\right] \left[\sum_{j=1}^{n} \left(b_{dj}\right)^2\right]}. \tag{8}$$

Consequently, the ultimate cross-efficiency can be aggregated as

$$E_j^{cross} = \sum_{d=1}^{n} E_{dj}\lambda_d, \, j = 1, 2, \ldots, n. \tag{9}$$

In what follows, we present an example to demonstrate the inconsistence between Wu's method and Zeleny's standpoint. Consider the following CEM with four DMUs:

$$E = \begin{pmatrix} 1.0000 & 0.9992 & 0.6917 & 0.8330 \\ 0.9771 & 1.0000 & 0.6180 & 0.7841 \\ 1.0000 & 0.9592 & 1.0000 & 0.9268 \\ 0.9999 & 0.9998 & 0.9997 & 1.0000 \end{pmatrix}. \tag{10}$$

It is observed that using the weights of DMU 4 derived from CCR model, all DMUs may generate very similar cross-efficiency scores. Therefore, on the strength of Zeleny's statement, the weights associated with DMU 4 should be considerably small during the calculation of cross-efficiency. However, weights obtained from Wu's method are

$$W = (0.000001, 0.000000, 0.000018, 0.999980)^T. \tag{11}$$

Contrary to Zeleny's viewpoint, Wu's method unreasonably assigns the largest weight to the least important DMU. Therefore, we can say that Wu's method is inappropriate or applicable to aggregate DEA cross-efficiency. In what follows, we seek to develop a new method based on the Shannon entropy for cross-efficiency aggregation.

3. Proposed method

In this section, a variation coefficient method based on the Shannon entropy is proposed to overcome the drawbacks from Wu's method, the rationale of which is assigning weights according to the dispersion degree of the data. In line with Yu and Lai (2011), for the entropy values of cross-efficiency scores matrix $H_{n\times n}$ derived in Section 2, the working procedure for aggregation is indicated as follows:

1. *Mean computation*. For rating DMU_d, the mean value can be computed by

$$\overline{h}_d = \frac{1}{n}\sum_{j=1}^{n} h_{dj}, d = 1, 2, \ldots, n. \tag{12}$$

2. *Standard deviation computation*. For rating DMU_d, the standard deviation value can be calculated by

$$\sigma_d = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\left(h_{dj} - \overline{h}_d\right)^2}, d = 1, 2, \ldots, n. \tag{13}$$

3. *Variation coefficient computation*. Relied on the obtained mean value and standard deviation value, the variation coefficient for rating DMU_d can be expressed by

$$\delta_d = \frac{\sigma_d}{\overline{h}_d}, d = 1, 2, \ldots, n. \tag{14}$$

4. *Weights determination*. For rating DMU_d, the weights can be determined by

$$\lambda_d = \frac{\delta_d}{\sum_{d=1}^{n}\delta_d}, d = 1, 2, \ldots, n. \tag{15}$$

5. *Cross-efficiency aggregation*. For rated DMU_j, the cross-efficiency can be aggregated by (9).

To illustrate the applicability of this variation coefficient method, we reexamine the simple example presented in Section 2, and obtain the optimal weights as follows:

$$W^* = (0.403725, 0.508601, 0.087365, 0.000309)^T. \tag{16}$$

From Zeleny's viewpoint, the variation coefficient method takes for granted the assignment of a very small weight ($w_4 = 0.000309$) with respect to least important DMU. Thus, we are confident to say that our method, in a sense, outperforms Wu's method.

4. Numerical examples

For illustrating the effectiveness of the proposed method, we recalculate the numerical example about a flexible manufacturing system (FMS) present by Wu et al. (2012).

Shang and Sueyoshi (1995) address the problem of selecting the most appropriate FMS for a manufacturing organization, whose input and output data are given in Table 2. Two inputs and four outputs are listed as follows:

Inputs:
x_1: the annual operating and depreciation cost (in units of \$100,000);
x_2: the floor space requirements of each specific system (in thousands of square feet);

Table 2
Data drawn from Shang and Sueyoshi (1995)

DMU	Inputs		Outputs			
	x_1	x_2	y_1	y_2	y_3	y_4
1	17.02	5.0	42	45.3	14.2	30.1
2	16.46	4.5	39	40.1	13.0	29.8
3	11.76	6.0	26	39.6	13.8	24.5
4	10.52	4.0	22	36.0	11.3	25.0
5	9.50	3.8	21	34.2	12.0	20.4
6	4.79	5.4	10	20.1	5.0	16.5
7	6.21	6.2	14	26.5	7.0	19.7
8	11.12	6.0	25	35.9	9.0	24.7
9	3.67	8.0	4	17.4	0.1	18.1
10	8.93	7.0	16	34.3	6.5	20.6
11	17.74	7.1	43	45.6	14.0	31.1
12	14.85	6.2	27	38.7	13.8	25.4

Table 3
Cross-efficiency matrix

Rating DMU	Rated DMU 1	2	3	4	5	6	7	8	9	10	11	12
1	1.0000	0.9992	0.6917	0.8330	0.8539	0.3712	0.4338	0.6349	0.1415	0.4262	0.7866	0.6495
2	0.9771	1.0000	0.6180	0.7841	0.7824	0.3222	0.3731	0.5719	0.1301	0.3607	0.7361	0.5998
3	1.0000	0.9592	0.9824	0.9268	1.0000	0.9316	1.0000	0.9271	0.4222	0.7605	0.9767	0.8011
4	0.9550	0.9813	0.7836	1.0000	0.9899	0.5012	0.5502	0.6760	0.2573	0.4898	0.7457	0.7358
5	0.8787	0.8786	0.7746	0.9248	1.0000	0.3977	0.4633	0.5720	0.0945	0.4013	0.6705	0.7191
6	0.7066	0.6992	0.8370	0.8818	0.8889	1.0000	0.9912	0.7668	0.8911	0.7307	0.6789	0.6851
7	0.6835	0.6611	0.8713	0.8839	0.9325	0.9900	1.0000	0.7473	0.7792	0.7454	0.6541	0.6997
8	1.0000	0.9766	0.9488	1.0000	1.0000	0.9624	1.0000	0.9614	0.7528	0.8334	0.9507	0.7943
9	0.4697	0.4721	0.5383	0.5924	0.5575	0.7931	0.7507	0.5540	1.0000	0.5576	0.4618	0.4407
10	0.7645	0.7043	0.9045	0.9564	1.0000	0.9513	1.0000	0.8596	0.8489	0.9536	0.7140	0.7200
11	1.0000	0.9664	0.9236	0.8962	0.9271	0.9542	1.0000	0.9449	0.6721	0.7829	0.9831	0.7594
12	1.0000	0.9844	0.9529	1.0000	1.0000	0.9674	1.0000	0.9503	0.7237	0.7954	0.9528	0.8012

Outputs:
y_1: the improvements in qualitative benefits (%);
y_2: the work in process reduced (10);
y_3: the average number of tardy jobs reduced (%);
y_4: the average yield increased (100).

According to models (1)–(3), we derive the CEM in Table 3, and each diagonal element in the CCR efficiency scores corresponding to each DMU.

Table 4 and Fig. 1 report and compare the results derived from Wu et al. (2012) and the variation coefficient method based on the Shannon entropy.

Table 4
Results comparisons

	Wu's method			Our method		
DMU	Weights	Scores	Ranking	Weights	Scores	Ranking
1	0.007410	0.9935	5	0.152984	0.8870	2
2	0.006169	0.9084	7	0.163844	0.8844	3
3	0.044615	0.9202	6	0.070713	0.7635	5
4	0.012161	0.9559	3	0.118618	0.8702	4
5	0.006663	0.9713	2	0.163491	0.8924	1
6	0.061969	0.9440	4	0.046419	0.6062	10
7	0.051863	0.9795	1	0.050565	0.6484	9
8	0.314890	0.9057	8	0.031508	0.6884	7
9	0.011322	0.7233	12	0.083381	0.3958	12
10	0.103538	0.7993	10	0.043337	0.5467	11
11	0.147678	0.8880	9	0.039887	0.7449	6
12	0.231722	0.7642	11	0.035252	0.6778	8

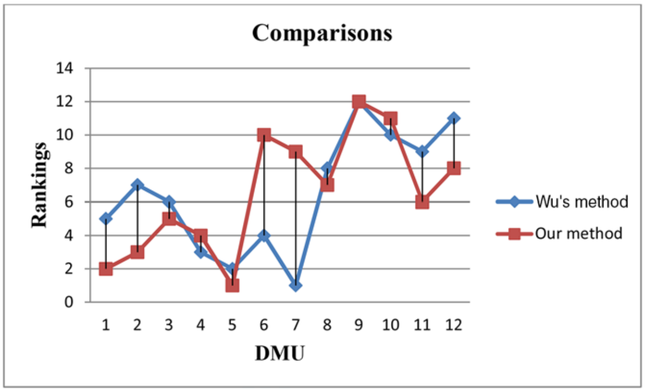

Fig. 1. Ranking comparison.

It is noted that both methods provide almost completely different rankings of DMUs, one exception is DMU 9, which is simultaneously identified as the least important DMU. More specifically, Wu's method ranks DMU 7 at the first position, which is ranked at the ninth position according to our method; while our method ranks DMU 5 at the first, which is ranked at the second position based on Wu's method. In addition, Wu's method assigns the largest weight to DMU 8, while our method deems DMU 2 as the most important, the CEM with respect to who varies in the highest degree (Var = 0.0033). These differences are originated from different modeling mechanisms.

In addition, for testifying the robustness of the ranking generated from our approach, we alternatively employ the Shannon entropy approach presented by Yu and Lai (2011) to determine the weights associated with each rating DMU to aggregate cross-efficiency. The final ranking is as follows: DMU 2, DMU 1, DMU 5, DMU 4, DMU 11, DMU 3, DMU 12, DMU 8, DMU 7, DMU 6, DMU 10, and DMU 9, which ranks six DMUs at the same positions as in our method, but only one DMU is compared with Wu's method. Therefore, we reasonably conclude that our approach is more robust than Wu's method.

5. Concluding remarks

Recently, Wu et al. (2012) presented a DEA cross-efficiency aggregation method based on the Shannon entropy to eliminate the drawbacks of traditional average cross-efficiency computation. The optimal weights used to determine ultimate cross-efficiency were derived from minimizing the square distance of weighted cross-efficiency and weighted CCR efficiency. However, from Zeleny's perspective, Wu's method may generate unreasonable results. To deal with this issue, we propose a variance coefficient method based on the Shannon entropy to determine more promising weights for DEA cross-efficiency aggregation, and compare the weights and cross-efficiency scores derived from both methods. It is natural that different methods will lead to different results, and the adoption of a specific method heavily depends on the decision maker's preference and decision environment.

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