Exercises Rosen p. 491

Probability Theory

1

$$\begin{cases} 3p(T) = p(H) \\ 3p(T) + p(T) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} p(T) = \dfrac{1}{4} \\ p(H) = \dfrac{3}{4} \end{cases}$$

3

$$p(2) + p(4) = 3(4 \cdot p(0))$$

$$\Rightarrow 2p(b) = 12\, p(o)$$

$$\Leftrightarrow p(b) = 6\, p(o)$$

$$4 \cdot p(o) + 2 \cdot p(b) = 1$$

$$\Rightarrow (4 + 12)\, p(o) = 1$$

$$\Leftrightarrow p(o) = \frac{1}{16} \text{ and } p(b) = \frac{6}{16}$$

5

$x_1 + x_2 = 7$. stars and bars

$$\binom{7+2-1}{2-1} = 8 \text{ combinations}$$

But : • -2 because we can not have 7, then 0 or 0, then 7.

• the case 4 then 3 is more likely to happen :

$$p(4 \text{ then } 3) = \frac{2}{7} \cdot \frac{2}{7} = \frac{4}{49}$$

• for the other cases :

$$p(\text{the others}) = \frac{1}{7} \cdot \frac{1}{7} \cdot (8-2-1) = \frac{5}{49}$$

$$p(\text{total}) = p(\text{the others}) + p(4 \text{ then } 3)$$

$$= \frac{5}{49} + \frac{4}{49} = \frac{9}{49}.$$

(7)

(a) The probability is $1/2$ because each permutation has the same probability to happen, and there is the same nb of perm. with 1 before 4 and 4 before 1.

(b) / $\frac{1}{2}$

(c) Total nb of permutations: $4!$

If 4 precedes 1 and 2 then

4 123 $\rightarrow$ we can permute these $3!$

34 12 $\rightarrow$ we can permute these $2!$

$$\Rightarrow \quad \frac{3! + 2!}{4!} = \frac{1}{3}.$$

(d) 4 123 $\rightarrow$ 3! permutations

$$\Rightarrow \frac{3!}{4!} = \frac{1}{4}$$

(e) 42 13 $\rightarrow$ we can permute these 2!

$\hookrightarrow$ we can permute these 2!

or 4321 $\rightarrow$ we can permute these 2!

note that we can not permute 4 and 2, these are already counted (2341 ...)

$$\frac{2! + 2! + 2!}{4!} = \frac{1}{4}$$

9

(a) This only happens once $\frac{1}{26!} = p(E)$

(b) We fix z at the top. There are still 25! situations $\Rightarrow \frac{25!}{26!} = p(E)$

(c) symmetry: same nb of permutations w/ a before z and z before a $\Rightarrow p(E) = \frac{1}{2}$.

(d) we can view a and z as a block, and count the nb of situations. $\frac{25!}{26!} = \frac{1}{26} = p(E)$

(e) we can say that a, m, z are a unique block, we have only 23 "letters" $\frac{23!}{26!} = \frac{1}{15{,}600}$.

11 $p(E \cup F) \geq p(E) = 0{,}7.$

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$
$$= 0.7 + 0.5 - p(E \cap F)$$
$$= 1.2 - p(E \cap F) \leq 1$$

Therefore $p(E \cap F)$ is necessarily $\geq 0{,}2.$

13 $p(E \cup F) = p(E) + p(F) - p(E \cap F)$

$$\Leftrightarrow p(E \cap F) = -p(E \cup F) + p(E) + p(F)$$

$p(E \cup F) \leq 1 \quad \Leftrightarrow -p(E \cup F) \geq -1$

$$\Leftrightarrow -p(E \cup F) + p(E) + p(F) \geq -1 + p(E) + p(F)$$

Therefore, $p(E \cap F) \geq p(E) + p(F) - 1.$

15 We can prove this by induction:

Basis Step: $p(E_1 \cup E_2) = P(E_1) + P(E_2) \underbrace{- P(E_1 \cap E_2)}_{\geq 0}$] ok

Inductive Step:

$p(E_1 \cup \ldots \cup E_k) \leq p(E_1) + \ldots + p(E_k)$ holds. (*)

$p(E_1 \cup \ldots \cup E_k \cup E_{k+1})$

$= \underbrace{p(E_1 \cup \ldots \cup E_k)}_{\leq p(E_1) + \ldots + p(E_k)} + p(E_{k+1}) - \underbrace{P((E_1 \cup \ldots \cup E_k) \cap E_{k+1})}_{\geq 0}$

$\Rightarrow$ therefore $p(E_1 \cup \ldots \cup E_{k+1}) \leq p(E_1) + \ldots + p(E_{k+1})$

Conclusion: it works.

16 if $p(E \cap F)$ are independent then

$$p(E \cap F) = p(E) \cdot p(F)$$

$$\Rightarrow p(\bar{E} \cap \bar{F}) = p(\overline{E \cup F})$$

only because E and F are independent

$$= 1 - p(E) - p(F) + P(E)P(F)$$

$$P(E) = 1 - p(E)$$

$$P(\bar{F}) = 1 - p(F)$$

$$\Rightarrow P(\bar{E})P(\bar{F}) = (1-p(E))(1-p(F))$$

$$= 1 - p(F) - p(E + P(E)P(F)$$

Therefore E and F indep $\Rightarrow$ $\bar{E}$ and $\bar{F}$ indep.

19 a

$$\frac{12}{144}$$ → nb de mois similaires

→ nb de combinaisons total de mois.

$$= \frac{1}{12}$$

b 1− (probability they are all distinct)

$$= 1 - \frac{12}{12} \cdot \frac{11}{12} \cdots \frac{12-n+1}{12}$$

c 5 (we try w/ this)

21

$1 -$ (all not Apr, 1 or 1 born on Apr, 1)

$$1 - \left(\binom{n}{0} \left(\frac{365}{366} \right)^n + \binom{n}{1} \left(\frac{365}{366} \right)^{n-1} \left(\frac{1}{366} \right) \right) > \frac{1}{2}$$

$$\Leftrightarrow \quad - \left(\binom{n}{0} \left(\frac{365}{366} \right)^n + \binom{n}{1} \left(\frac{365}{366} \right)^{n-1} \left(\frac{1}{366} \right) \right) > \frac{-1}{2}$$

$$\Leftrightarrow \quad \left(\frac{365}{366} \right)^n + n \left(\frac{365}{366} \right)^{n-1} \left(\frac{1}{366} \right) < \frac{1}{2}$$

$$\Rightarrow \quad n > 614$$

23 F: "first flip came up heads"
E: "3 of the next 4 flips are heads"

$$p(E) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \binom{4}{3}\left(\frac{1}{2}\right)^4$$

$$p(E/F) = \frac{p(E\cap F)}{p(F)} = \frac{\binom{4}{3}(1/2)^4 \cdot \frac{1}{2}}{1/2}$$

$$= \binom{4}{3}\cdot\left(\frac{1}{2}\right)^4$$

$$= 1/4$$

25 We can have 001, 100 or 000.
Total number of combinations $2^3 = 8$.

$$P(E) = \frac{3}{8}$$

27 E : " has children of both sexes"
F : " has at most 1 boy"

$A = \{G, B\}$ $A^n \rightarrow$ all the possible comb.

<u>$n = 2$</u> $A = \{GG, GB, BG, BB\}$

$p(E) = \frac{1}{2}$ $p(F) = \frac{3}{4}$

$p(E \cap F) = \frac{1}{2}$ $p(E) \cdot p(F) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$

$\Rightarrow$ not independent

$\underline{n = 4}$ $\quad A = \{GGGG, GGGB, GGBG, \dots\}$

$|A^4| = 2^4 = 16.$

$p(E) = \frac{14}{16}$ $\qquad p(F) = \frac{1}{16} + \boxed{\frac{4!}{3!} \cdot \frac{1}{16}}$

only girls

3 girls + 1 boy

$p(E \cap F) = \frac{4!}{3!} \cdot \frac{1}{16}$ $\qquad = \frac{5}{16}$

$= \frac{1}{4}$

$p(E) \cdot p(F) \neq p(E \cap F)$

29 $\quad \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1$

$= \frac{3}{16}$

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a) $\binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32} \simeq 3{,}1\%$

b) $\binom{5}{5}\left(\frac{49}{100}\right)^5 \simeq 2{,}8\%$

c) $1-\left(0{,}51-\frac{i}{100}\right)$

$= 1-0{,}51+\frac{i}{100}$

$= 0{,}49+\frac{i}{100}$

$p(E) = \left(\frac{50}{100}\right)\left(\frac{51}{100}\right)\left(\frac{52}{100}\right)\left(\frac{53}{100}\right)\left(\frac{54}{100}\right)$

$\simeq 3{,}8\%$

33 $p(E \cup F)$

$$= p(E) + p(F) - p(E \cap F)$$

$$= \frac{1}{2} + \frac{2^3}{2^5} - \frac{2^2}{2^5} = \frac{5}{8}$$

35

a) $\binom{n}{n}(p)^n = p^n$

b) $1 - p^n$

c) $p^n + \binom{n}{1}(p)^{n-1}(1-p)$

$$= p^n + n(p^{n-1})(1-p)$$

d) $1 -$ c