Exercise 1

$n = 7$

$min = 2 \quad a_{min} = 3$

$j = 2 \quad a_2 = 3 \qquad\qquad j =$

1 comp

$j = 3 \qquad a_3 = 8$

1 comp

$j = 4 \qquad a_4 = 6$

1 comp

$j = 5 \qquad a_5 = 5$

1 comp

$j = 6 \quad a_6 = 1$

Exercise 1

même nb de comparaisons.

$$n + (n-1) + \dots + 1.$$

1 9 3 8 6 5 1 7
1 3 8 6 5 9 7
1 3 5 6 8 9 7
1 3 5 6 7 9 8
1 3 5 6 7 8 9

4

2 4 7 2 9 5 6 3

2 7 4 9 5 6 3

2 3 4 9 5 6 7   5

2 3 4 5 9 6 7

2 3 4 5 6 9 7

2 3 4 5 6 7 9

3 0

4 7 6 5 4 3 2 1

1 6 5 4 3 2 7   3

1 2 5 4 3 6 7

1 2 3 4 5 6 7

2 worst swap
eq. for comp.

Ex2

$$c^{n-3} \quad , \quad c > 1$$

$$3^n$$

Comme $c > 3^3 \geq 27$

on a ① > ②

$$n^{3\log_3(n)} \geq n^3 \log_3(n)$$

la puissance $\nearrow$ (+) que multipl:

$$n^{3x} \geq n^3 x,$$

or $x^n \nearrow$ + vite n^x

donc

$\boxed{1} > 2 > 3 > 4$

Ex 3

if $f_1(x)$ is $O(g_1(x))$

$$|f_1(x)| \leq C_1 |g_1(x)|$$

$$|f_2(x)| \leq C_2 |g_2(x)|$$

$$|f_1(x)| + |f_2(x)| \leq C_1 |g_1(x)| + C_2 |g_2(x)|$$

$$\Leftrightarrow |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)|$$

$$\leq C_1 |g_1(x) + g(x_2)|$$

$$\le C_1 |g_1(x)| + C_2 |g_2(x)|$$

$$\le (C_1 + C_2) \cdot \left(2 \left| \max(g_1(x), g_2(x)) \right| \right)$$

Ex 4

① f is $o(g(x))$

if $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$

$$\lim_{x \to \infty} \frac{5x}{x2} = \frac{\cancel{x}(5)}{\cancel{x}(x)} = 0.$$

② $\lim_{x \to \infty} \dfrac{2\cancel{x^2}}{\cancel{x^2}} = 2 \neq 0$

③ $\lim_{x \to \infty} \dfrac{1}{x} \cdot \dfrac{1}{x} = 0.$

Ex 5

$$f(0) = 0$$

$$f(n) = f(n-1) - 1$$

$$f(n+1) = f(n) - 1$$

$$0, -1, -2, -3$$

$$f(n) = -n.$$

Basis step

$$f(0) = 0 \quad \boxed{\text{Ok}}$$

Inductive step

$$f(k) = -k$$

$$f(k+1) = -k-1$$

$$= f(k) - 1 \quad \text{ok.}$$

$\Rightarrow$ therefore $f(n)$ is true $\forall n \in \mathbb{N}$

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) \underset{2^\circ}{=} 2f(n-2)$$

$$0,\ 1,\ 0,\ 2,\ 0,\ 4,\ 0$$

$$f(n) = \frac{2^{n-2} - (2^{n-2}) \cdot (-1)^{n+1}}{2}$$

$0, 1, 0, 2, 0, 4$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

$2^0 \qquad 2^1 \qquad 2^2$

$$\frac{2^{\frac{n-1}{2}} - \left(2^{\frac{n-1}{2}}\right) \cdot (-1)^n}{2}$$

$$= \frac{2^{\frac{n-1}{2}}\left(1 - (-1)^n\right)}{2}.$$

$$2^{\frac{n-3}{2}}\left(1 + (-1)^{n-1}\right)$$

Basis Step

$$f(0) = \frac{2^{1/2}(1-1)}{2} = 0$$

Inductive Step

We assume $P(k)$ is true.

- $f(n+1) = 2f(n-1)$.

We assume $f(n) = \frac{2^{\frac{n-1}{2}}(1-(-1)^n)}{2}$

- $f(n+1) = \frac{2^{\frac{n}{2}}(1-(-1)^{n+1})}{2}$

$$= 2^{\frac{n-2}{2}}(1+(-1)^n)$$

$$f(n) = 2^{\frac{n-3}{2}}\left(1 + (-1)^{n-1}\right)$$

Ex 6

3

g_1 is $O(f)$ $\Leftrightarrow$

$$|g_1(x)| \leq C_1 |f(x)|$$

$$|g_2(x)| \leq C_2 |f(x)|$$

f is $\Omega(g_1)$ $\Leftrightarrow$

$$|g_1(x)| \geq C_3 |f(x)|$$

$$|g_2(x)| \geq C_4 |f(x)|$$

$$|g_1(x) + g_2(x)|$$

$$\leq |g_1(x)| + |g_2(x)|$$

$$\leq (C_1 + C_2) |f(x)| \Rightarrow (g_1 + g_2)(x)$$

is $O(f)$

INVALID

$$|g_1(x)| + |g_2(x)| \geq (C_3 + C_4)|f(x)|$$

$$|g_1(x)| + |g_2(x)| \geq |g_1(x) + g_2(x)| \geq (C_3 + C_4)|f(x)|$$

$g_1(x) = -5x$
$g_2(x) = 5x$

g_1 is $\Theta(x)$
g_2 is $\Theta(x)$

$(g_1 + g_2)$ is $\Theta(1)$
$\neq \Theta(x)$

because it is not $\Omega(x)$

2 + 4

g_1 is $O(f)$

$\Leftrightarrow$

$$|g_1(x)| \leq C_1 |f(x)|$$

$$|g_2(x)| \leq C_2 |f(x)|$$

f is $\Omega(g_1)$

$\Leftrightarrow$

$$|g_1(x)| \geq C_3 |f(x)|$$

$$|g_2(x)| \geq C_4 |f(x)|$$

$$|g_1 g_2| \leq C_1 C_2 |f^2|$$

ok parce qu'on multiplie par la valeur absolue

$$|g_1 g_2| \geq C_3 C_4 |f^2|$$

1

g_1 is $O(f)$ $\Leftrightarrow$

$$|g_1(x)| \leq c_1 |f(x)|$$

$$|g_2(x)| \leq c_2 |f(x)|$$

f is $\Omega(g_1)$ $\Leftrightarrow$

$$|g_1(x)| \geq c_3 |f(x)|$$

$$|g_2(x)| \geq c_4 |f(x)|$$

$$|g_1(x) + g_2(x)|$$

$$\leq |g_1(x)| + |g_2(x)|$$

$$\leq (c_1 + c_2) |f(x)| \Leftrightarrow (g_1 + g_2)(x) \text{ is } O(f)$$

$$|g_1(x)| + |g_2(x)| \geq (C_3 + C_4)\,|f(x)|$$

$$|g_1(x)| + |g_2(x)| \geq |g_1(x) + g_2(x)| \geq (C_3 + C_4)\,|f(x)|$$

supérieur à ça

donc aussi supérieur à ça

car quelq. > 0

(5)

$$\boxed{|g(x)| \leqslant C\,|f(x)| \qquad C > 0 \quad \forall x > k.}$$

$$\Leftrightarrow \quad 2^{|g(x)|} \leqslant 2^{C|f(x)|}$$

car g et f ≥ 0

$$\Leftrightarrow \quad 2^{g} \leqslant \left(2^{C}\right)^{f}$$

$$\leqslant C_2\, 2^{f}$$

$$? \qquad 1$$

$$2x \leqslant 4 \times 1x$$

$$2^{2x} < 2^{1x}$$

$$4^{x} < 2^{x}$$

Exercise 7

- $\log_m(k) = x \iff m^x = k$
- $\log_m(\ell) = \ell \iff m^y = \ell$

et $k > \ell \Rightarrow m^x > m^y$.

$\Rightarrow x > y$.

donc $\log_m(k)$ is $O(\log_m(\ell))$

$$\log_m(k) = \frac{\log_2(k)}{\log_2(m)} = aX$$

$$\log_m(\ell) = \frac{\log_2(\ell)}{\log_2(m)} = bX$$

$\Rightarrow$ même ordre, VRAI

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

$$\log_a(b) = x$$

$$\Leftrightarrow \quad b = a^x$$

$$\Leftrightarrow \quad \log_c(b) = \log_c(a^x)$$

$$\Leftrightarrow \quad \log_c(b) = x \log_c(a)$$

$$\Leftrightarrow \quad x = \frac{\log_c(b)}{\log_c(a)}$$

$$\log_k(m) = \frac{\log_\ell(m)}{\log_\ell(k)}$$

$$k = \ell^{\log_\ell(k)}$$

$$\left(\ell^{\log_\ell(k)}\right)^{\frac{\log_k(m)}{\log_k(\ell)}} = \ell^{\log_\ell(k) \cdot \log_\ell(m)}$$

$$\left(\ell^{\log_k(m)}\right) = \ell^{\log_\ell(m) / \log_\ell(k)}$$

As $k > \ell \geq 2$,

$$k^{\log_\ell(m)} = m^c$$

with $c = \log_\ell(k) > 1$

$$\ell^{\log_k(m)} = m^{1/c} \qquad c < 1.$$

$\Rightarrow$ not $o\left(\ell^{\log_k(m)}\right)$

Exercise 8

f is $o(f)$

$$\iff \lim_{x\to\infty} \frac{f(x)}{f(x)} = 0$$

<u>faux</u>.

f is $o(g)$

$$\iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

$\exists x_0, \forall x \geq x_0, \; g(x) \geq f(x)$.

$\Rightarrow f$ is $O(g)$.

Exercise 9

$$\sum_{i=0}^{n} \quad 1+2+3+4+\dots+n \quad O(n^2)$$

$$\frac{n(n+1)}{2} \qquad \leq n+n+\dots+n$$

$$= n \cdot n$$

$$= n^2$$

ak

$$= a_0 n^0 + a_1 n^1 + \dots + a_n n^d$$

$$\leq D n^d + D n^d + \dots + D n^d$$

$$= D \cdot n^d \leq D \cdot \underline{n^{d+1}}$$

vrai ?

$$\geq n^0 + n^1 + \dots + n^d$$

$$= \frac{1 - n^{d+1}}{1 - n^d} =$$

$$a_0 n^0 + a_1 n^1 + \dots + a_n n^d$$

$$\leq D n^d + D n^d + \dots + D n^d$$

$$= D \cdot n^d \cdot n = D n^{d+1}$$

$$O(n^{d+1})$$

$$a_0 n^0 + a_1^1 + \dots + a_n n^d$$

$$\geq n^0 + n^1 + \dots + n^d$$

$$= \sum_{k=0}^{d} n^k = \frac{1 - n^{(d+1)}}{1 - n}$$

$$\geq n^{d+1}$$

$$\leq a_0 n^0 + a_1 n^1 + \dots + a_n n^d$$

$$a_1 n^1 + a_2 n^2 + a_3 n^3 + \dots + a_d n^d$$

$$\leq a_d n^d + a_d n^d + \dots + a_d n^d$$

$$= a_d n^d \cdot d$$

<u>Vrai</u>

Exercise 10

Basis Step

$$4! = 24 > 2^4 = 16$$

Inductive Step

We assume that $P(k): k! > 2^k$ is T.
Does it imply $P(k+1)$ T?

$$P(k+1): (k+1)! > 2^{k+1}$$

$$\iff k!\,(k+1) > 2^k \cdot 2$$

$$\iff k! > 2^k \cdot \boxed{\frac{2}{k+1}}$$

< 1

Exercise 11

$AB = BA$. $\quad P(k): AB^k = B^kA$

Basis Step

$AB^1 = B^1A \quad \boxed{\text{Ok}}$

Inductive step

$P(k+1): AB^{k+1} = B^{k+1}A$.

$\Leftrightarrow AB^kB = B^kBA$.

$\Leftrightarrow B^kAB = B^kBA$.

$\Leftrightarrow B^kBA = B^kBA$. Ok

$$\frac{f(x+h) - f(x)}{h}$$

$$-(-1) = \underline{\underline{1}}$$