Homework 11

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Exercise 1. How many different words can we make by rearranging the letters in the word PASSWORD?

$\Rightarrow$ perm. w/ indist. objects

$$\frac{8!}{2!} = 20\,160$$

Exercise 2. An exam has 12 questions, with 4 possible answers for each question. How many students should complete the exam to ensure that at least 3 students will submit the exact same answers?

pigeonhole principle

$$\left\lceil \frac{N}{k} \right\rceil = r \quad , \quad k = 4^{12}, \quad r = 3$$

$$\Rightarrow N = (r-1)k + 1 = 2 \cdot 4^{12} + 1 = 33\,554\,433$$

Exercise 3. You are leading the organization of a complex school event, which includes multiple activities like setting up booths, managing teams, and sending invitations.

1. The event invitations include a unique identifier consisting of 3 distinct letters from a to f included, and 2 distinct digits from 0 to 3 included. Additionally, each invitation has one of five different colors and can be printed in one of three different fonts. How many unique invitations can be created?
2. You have an unlimited supply of 6 different types of booths to choose from. You need to arrange five booths in a row, but at least two of them must have a different type. In how many different ways can you arrange the 5 booths in a row?
3. You need to form a team of 4 volunteers from a group of 6 bachelor students and 5 master students. The team must have at least one master student. The team will be given 4 hats: 2 red, one blue, and one green. How many different team compositions and hats assignments are possible?
4. You have an unlimited supply of 12 different gift types: $\{G_1, G_2, \cdots G_{12}\}$, and you need to create gift packages containing exactly 5 gifts. Each package must include a gift of type G_5. How many different gift packages can be prepared?
5. Finally, you're tasked with designing unique name badges. Due to a unique design theme, each badge name must include all vowels ('A', 'E', 'I', 'O') exactly once, and all consonants ('C', 'L', 'B') exactly twice. How many different badge names can be designed under these conditions?

1

$$\frac{6!}{(6-3)!} \cdot \frac{4!}{(4-2)!} \cdot \binom{5}{1} \cdot \binom{3}{1}$$

$$= 21\,600$$

2

permutations with rep. 6^5 situations, minus the 6 situations with the same booth.

$$6^5 - 6 = 7770$$

3 $\left(\binom{5}{4} + \binom{6}{1} \cdot \binom{5}{3} + \binom{6}{2} \cdot \binom{5}{2} + \binom{6}{3} \cdot \binom{5}{1}\right)$

$$\cdot \left(\frac{4!}{2!}\right) = 3\,780$$

4 comb. with repetitions (see check sheet stars and bars)

$$\binom{4+12-1}{12-1} = \underline{1365}$$

we want to create 11 separators for the 12 types of gifts

5 $\dfrac{(4+6)!}{2!\,2!\,2!}$

total nb of letters

nb of common letters

Exercise 5. The number of distinct n-tuples $(x_0, x_1, x_2, \ldots, x_{n-1})$ of integers x_0, x_1, x_2, …, x_{n-1} such that $x_i \geq i$ for $0 \leq i < n$ and $\sum_{i=0}^{n-1} x_i \leq \frac{n(n+1)}{2}$ equals

- ○ $\binom{2n}{n}$.
- ○ $\binom{2n+1}{n}$.
- ○ $\binom{2n-1}{n}$.
- ○ $\binom{2n-1}{n-1}$.

we first place all the required nb in each x_i.

That means we have already placed

$\frac{(n-1)(n)}{2}$ elements (because we start at 0)

$$x_0 + x_1 + \ldots + x_{n-1} + x_{trash} = \frac{n(n+1)}{2}$$

$\frac{n(n+1)}{2} - \frac{(n-1)(n)}{2} = n$ left to place among $n+1$ boxes.

box count − 1

$$\binom{n + (n+1) - 1}{(n+1) - 1} = \binom{2n}{n}$$

box count − 1 : nb of separators