Exercice 1

ⓐ $A = A^T$

$$A\vec{v} \cdot \vec{u} = (A\vec{v})^T \vec{u}$$
$$= \vec{v}^T A^T \vec{u}$$
$$= \vec{v}^T A \vec{u}$$
$$= \vec{v} \cdot A\vec{u}$$

ⓑ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = B$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$B\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1+v_2 \end{pmatrix}$$

$$B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1+u_2 \end{pmatrix}$$

$$(B\vec{v})\cdot\vec{u} = \begin{pmatrix} v_1 \\ v_1+v_2 \end{pmatrix}\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= v_1\cdot u_1 + v_1\cdot u_2 + v_2\cdot u_2$$

$$(B\vec{u})\cdot\vec{v} = \begin{pmatrix} u_1 \\ u_1+u_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= u_1\cdot v_1 + (u_1+u_2)\cdot v_2$$

$\rightarrow$ il manque $v_2\cdot u_1$

Exercice 2

ⓐ $\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}$

$\boxed{\begin{matrix} 2 \\ -2 \\ 5 \end{matrix}}$

$= (1-\lambda)(3-\lambda)(1-\lambda)$

$+ 3 + 3$

$- 3(3-\lambda)(3) - (1-\lambda)$

$- (1-\lambda)$

$= (3-\lambda-3\lambda+\lambda^2)(1-\lambda)$

$+ 6 - 9(3-\lambda) - (1-\lambda)$

$$= (3 - 4\lambda + \lambda^2)(1-\lambda)$$

$$+ 6 - 27 + 9\lambda - 1 - \lambda$$

$$= -18 - 4\lambda + \lambda^2 - 3\lambda$$

$$+ 4\lambda^2 - \lambda^3 + 9\lambda - 1 - \lambda$$

$$= -20 + 4\lambda + 5\lambda^2 - \lambda^3$$

5 est une valeur propre

$\Rightarrow$ 2 et -2.

$$\ker \begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix} \to \text{dimension } 1$$

$$= \text{vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$\ker \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \to \text{dimension } 1$$

$$= \text{vect}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\ker \begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix} \to \text{dim } 1$$

$$= \text{vect}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \end{pmatrix}$$

$$Q^T A Q = D$$

$$A = Q D Q^T$$

(b) $\det(A) = -1$.

une valeur propre est 1 .

$$\begin{array}{c} \begin{pmatrix} 0-\lambda & 1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \\ \begin{array}{ccc} + \quad & - \quad & + \end{array} \end{array}$$

$$(1-\lambda)\big((-\lambda)\cdot(-\lambda) - 1\big)$$

$$\Leftrightarrow (1-\lambda)(\lambda^2 - 1)$$

$$\Leftrightarrow (1-\lambda)(\lambda-1)(\lambda+1)$$

1 multiplicité 2, (-1) multiplicité 2

$$\ker\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mathrm{vect}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\ker\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \mathrm{vect}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

Exercice 3

(i) matrice diagonalisable car symétrique

$$A = Q D Q^T$$

$$= \begin{pmatrix} | & | & | \\ \vec{u}_1 & - & \vec{u}_n \\ | & | & \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \vec{u}_1^T & - & \vec{u}_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \vec{u}_1 & - & \vec{u}_n \end{pmatrix} \begin{pmatrix} \lambda_1 \vec{u}_1^T & - & \lambda_n \vec{u}_n^T \end{pmatrix}$$

$$= \lambda_1 \vec{u}_1^T \vec{u}_1 + \_\_\_\_ + \lambda_n \vec{u}_n^T \vec{u}_n$$

(ii) valeur propre est 2 , 0

$$\ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \text{vect}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \text{vect}\left\{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\right\}$$

$$\ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \text{vect}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \text{vect}\left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = A.$$

Exercice 4

ⓐ $$A = \begin{pmatrix} \vec{u}_1 & \dots & \vec{u}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \vec{u}_1^{\,T} \\ \vdots \\ \vec{u}_n^{\,T} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \vec{u}_1 & \dots & \lambda_n \vec{u}_n \end{pmatrix} \begin{pmatrix} \vec{u}_1^{\,T} \\ \vdots \\ \vec{u}_n^{\,T} \end{pmatrix}$$

$$= \lambda_1 \vec{u}_1 \vec{u}_1^{\,T} + \dots + \lambda_n \vec{u}_n \vec{u}_n^{\,T}$$

ⓑ 1 est une valeur propre de A.

$0 \cdot 0 \cdot 1 = 0 \Rightarrow$ somme des v.p.

1 multiplicité 2, (-1) mult. 1.

$$\ker(A - I) = \text{vect}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\ker(A + I) = \text{vect}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$Q = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$
+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
- \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
= P
$$

Exercice 5

- $A = A^T$
- $A = PDP^T$

$$A^{-1} = (PDP^T)^{-1}$$

$$= (P^T)^{-1}(D^{-1})(P)^{-1}$$

$$= PD^{-1}P^T$$

$= B$, B symétrique

Exercice 6

on connaît déjà la valeur propre 6

$$\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 48$$

produit des λ = 48

$$\cancel{6} \cdot \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \cancel{48}$$

$$\Leftrightarrow \quad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 8$$

et somme des λ = 12.

$$\lambda_1 + \lambda_2 + \lambda_3 = 6.$$

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 2.$$

$$\ker\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \operatorname{vect}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$\ker\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}$$

$$= \operatorname{vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/2 & 1/2 \\ -1/\sqrt{2} & 0 & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$