Homework 12

Tanja Käser

Exercise 1. Let A_n^k for $n \in \mathbf{Z}_{\geq 0}$ denote the set of binary strings of length n that do not contain k consecutive zeros.

1. Derive a recurrence relation for $|A_n^3|$.
2. Derive a recurrence relation for $|A_n^k|$ with $k > 3$.

(1) $A_{n-3} + 100$

$A_{n-2} + 10$

$A_{n-1} + 1$

distinct (see cheat sheet)

$$|A_n^3| = |A_{n-1}^3| + |A_{n-2}^3| + |A_{n-3}^3|$$

(2) $$|A_n^k| = |A_{n-1}^k| + \dots + |A_{n-k}^k|$$

Exercise 3.

1. Solve the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$, with $a_0 = 1$, $a_1 = 3$, $a_2 = 7$
2. Solve the recurrence relation $a_n = -a_{n-1} + a_{n-2} + a_{n-3}$, with $a_0 = 3$, $a_1 = -4$, $a_2 = 9$

Hint: When trying to solve a polynomial of degree 3, we advise that you check for binomial coefficients or trivial roots such as 0, 1, or -1. After finding the trivial roots, you can factorize the polynomial to find the other roots.

(1) $r^n - 3r^{n-1} + 3r^{n-2} - r^{n-3} = 0$

$\Rightarrow$ divide by r^{n-3}

$\Rightarrow \quad r^3 - 3r^2 + 3r - 1 = 0$

$\Leftrightarrow \quad (r-1)(r-1)(r-1) = 0$

$a_n = (\alpha_{0,1} + \alpha_{1,1} n + \alpha_{1,2} n^2)\,(1)^n$

$a_0 = \alpha_{0,1} = 1$

$a_1 = (1 + \alpha_{1,1} + \alpha_{1,2}) = 3$

$a_2 = (1 + 2\alpha_{1,1} + 4\alpha_{1,2}) = 7$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 4 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

$\Rightarrow \quad a_n = (1 + n + n^2)$

2 $\quad a_n = -a_{n-1} + a_{n-2} + a_{n-3}$

$\Rightarrow \quad r^n + r^{n-1} - r^{n-2} - r^{n-3} = 0$

$\Rightarrow \quad r^3 + r^2 - r - 1 = 0$

$\Leftrightarrow \quad (r-1)(r^2 + 2r + 1) = 0$

$\Leftrightarrow \quad (r-1)(r+1)(r+1) = 0$

$$a_n = (\alpha_{1,0}) + (\alpha_{2,0} + \alpha_{2,1} n)(-1)^n$$
$$a_0 = \alpha_{1,0} + \alpha_{2,0} = 3$$
$$a_1 = \alpha_{1,0} - \alpha_{2,0} - \alpha_{2,1} = -4$$
$$a_2 = \alpha_{1,0} + \alpha_{2,0} + 2\alpha_{2,1} = 9$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & -1 & -1 & -4 \\ 1 & 1 & 2 & 9 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & -1 & -1 & -4 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Rightarrow a_n = 1 + (2 + 3n)(-1)^n$$

Exercise 2. How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

We first want to count the bit strings with both conditions:

1. 000 S 1111 → 2 situations
2. 1111 S 000 → 2 situations
3. 1111 000 S → 2 situations −1 from 2
4. S 000 1111 → 2 situations −1 from 1
5. 000 1111 S → 2 situations −1 from 1
6. S 1111 000 → 2 situations −1 from 2

⇒ total of 8 situations

To count the number of bit strings containing 3 consecutive zeros:

- we assume we know how many B.S. of length $n-1$ contains three consecutive zeros. ($B.S30$)

So, we can:

- take all the $B.S30_{n-1}$ and add a 1 to them.

it would leave us with the same number of BS30,n-1.

- take all the $BS30_{n-2}$ and add a 10, then it would leave us with the same number of BS30,n-2.
- take all the $BS30_{n-3}$ and add a 000, then it would create 2^{n-3} new BS30,n valid! (not counted yet because they are the only one counted ending with three zeros).

$$|A_n| = |A_{n-1}| + |A_{n-2}| + 2^{n-3}.$$

We do the same thing for 1s:

$$|B_n| = |B_{n-1}| + |B_{n-2}| + |B_{n-3}| + 2^{n-4}$$

$\Rightarrow |A_8| = 107$, $|B_8| = 48$ (not really a formula to compute them, we have to use a script).

$$|A_8| + |B_8| - 8 = 147$$