Exercises Rosen p. 518

Expected Value and Variance

1) $E(5X) = 5E(X) = 5 \cdot \frac{1}{2} = \frac{5}{2}$.

3) X : "6 appears" $\begin{cases} \text{yes} \Rightarrow 1 \\ \text{no} \Rightarrow 0 \end{cases}$

$$E(10X) = 10 \cdot \frac{1}{6} = \frac{5}{3}$$

5)

$$E(X+X) = 2E(X) = 2\left(\frac{2}{7} \cdot 3 + \frac{1}{7}(1+2+4+5+6)\right)$$

$$= 48/7$$

$$\begin{cases} p(3) = 2p(0) \\ 5 \cdot p(0) + 2p(0) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} p(3) = \frac{2}{7} \\ p(0) = \frac{1}{7} \end{cases}$$

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$$E(50X + 25Y)$$

$$= 50E(X) + 25E(Y)$$

$$= 50\left(\frac{9}{10} \cdot 2\right) + 25\left(\frac{8}{10} \cdot 4\right)$$

$$= \underline{170}$$

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N/A → depends on what linear search is.

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$$X \begin{cases} 1 & \text{if we get a 6 first} \\ 2 & \text{if ———— after the } 2^{nd} \text{ throws...} \\ \vdots & \\ 10 & \text{if we do not get a 6 after the nith throws} \end{cases}$$

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \left(\frac{5}{6} \cdot \frac{1}{6}\right)$$

$$+ 3 \cdot \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$+ 4 \cdot \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)$$

$$+ 10 \left(\frac{5}{6}\right)^9$$

$$\approx 5{,}03$$

13 $x_1 + x_2 = 7 \Rightarrow$ stars and bars $\binom{7+2-1}{2-1}$

$= 8 - 2 = 6.$ can not have 7,0 or 0,7

$$X \begin{cases} 1 & \text{if we get some } x_1 x_2 \in A \text{ at first} \\ 2 & \text{if } \text{———————— at } 2^{nd} \text{ throws} \\ \vdots & \end{cases}$$

Geometric distribution!

$$P(X=k) = \underbrace{(1-p)^{k-1}}_{\text{failures}} \underbrace{p}_{\text{success}} \qquad p = \frac{6}{36}$$

$$E(X) = \frac{1}{p} = \frac{36}{6} = 6.$$

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$$p(X \geq j)$$

$$= \sum_{k=j}^{\infty} p(X=k)$$

$$= \sum_{k=j}^{\infty} (1-p)^{k-1} p$$

like we did in exercise 13

$$= \left(\frac{p}{1-p}\right) \underbrace{\sum_{k=j}^{\infty} (1-p)^{k}}$$

$$\frac{a_0}{1-q} = \frac{(1-p)^j}{1-(1-p)}$$

$$= \frac{(1-p)^j}{p}$$

$$= \left(\frac{\cancel{p}}{1-p}\right)\left(\frac{(1-p)^j}{\cancel{p}}\right)$$

$$= (1-p)^{j-1}$$

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$$X \begin{cases} 1 & \text{if we get a prime at first} \\ 2 & \text{if ————— at } 2^{nd} \text{ roll} \\ \vdots & \end{cases}$$

Geometric distribution

$$E(X) = \frac{1}{p} \qquad p = \frac{1}{2302}$$

$$= \boxed{2302}$$

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$$E(X) = \frac{1}{6}(1+2+\ldots+6) = \frac{7}{2}$$

$$E(Y) = E(2X) = 7.$$

$$E(X)E(Y) = \frac{49}{2}$$

$x_1 \backslash x_2$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

La valeur moyenne de $X_1 \cdot Y = E(XY)$

$$\frac{1}{36}\left(1 \cdot \left(\sum_{k=2}^{7} k\right) + 2 \cdot \left(\sum_{k=3}^{8} k\right) + \dots \right) \simeq 27{,}42$$

$$E(X) \cdot E(Y) \neq E(XY)$$

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We know there are only these comb. left:

$x_1 \backslash x_2$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

→ same probability

$$E(X) = \frac{1}{10}\left(4 \cdot 9 + 3 \cdot 10 + 2 \cdot 11 + 12\right)$$

$$= \frac{1}{10}(100) = \underline{10}$$

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X : "average weight of a b. e. s."
F : "female"
M : "male"

$$E(X) = E(X/M) \cdot P(M) + E(X/F) \cdot P(F)$$

$$= 4200 \cdot 0{,}12 + 1100 \cdot 0{,}88$$

$$= 1472.$$

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R => number of runs

R => number of times we start a run.

Therefore $R = \sum_{j=1}^{n} I_j$

$$I_j = \begin{cases} 0 & \text{if the current run is already started or } a_j \text{ is a failure} \\ 1 & \text{if a new run starts at position } j \end{cases}$$

$$E(I_1) = p \cdot 1 + 0 \cdot (1-p)$$

$$= p.$$

$$E(I_j) = \begin{cases} 1 & \text{if the } (j-1)\text{th trial is a failure and the } j\text{th trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$

probability of $(1-p)p$.

$$E(I_j) = (1-p)p.$$

$$E(R) = E(I_1) + \dots + E(I_n)$$

$$= p + (n-1)(1-p)p$$

$\downarrow$

from 2 to n

27 Variance for Bernoulli

$$10V(X) = 10\left(E(X^2) - E(X)^2\right)$$

$X^2 = X$ because $X = \begin{cases} 1 \text{ if success} \\ 0 \text{ if fail} \end{cases}$

$$= 10(p - p^2)$$

$$= 10(0{,}5 - 0{,}25)$$

$$= 10(0{,}25)$$

$$= 2{,}5$$

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$$X_n = X_T - X_H$$

ⓐ $E(X_n) = E(X_T) - E(X_H)$

$$= 1/2 - 1/2$$

$$= 0$$

ⓑ Bernoulli

Sample space :

$$\{-n, (-n+2), \dots, -2, 0, 2, \dots, (n-2), n\}$$

<u>E.g</u>

1/4	HH	-2
1/4	HT	0
1/4	TH	0
1/4	TT	2
1/8	HHH	-3
1/8	HHT	* -1
1/8	HTH	-1
1/8	HTT	1
1/8	THH	-1
1/8	THT	1
1/8	TTH	1
1/8	TTT	3

<u>Therefore</u> :

Let i be the number of tails.
Then $n-i$ is the nb of heads.

$$X_n = X_T - X_H$$
$$= i - n + i = -n + 2i$$

We want the probability to reach a specific value of X_n, we can see * that it is not uniform!

$$P(X_n = -n+2i) = \binom{n}{i}\left(\frac{1}{2}\right)^n = \binom{n}{i}\frac{1}{2^n}$$

for example -1 is reached 3 times (*)
with $\underline{1}$ success among 3 throws $\binom{3}{\underline{1}}$

$$E(X^2) = \sum_{i=0}^{n} p(X_n = -n+2i)(-n+2i)^2$$

$$= \sum_{i=0}^{n} p(X_n = -n+2i)(n^2 - 4in + 4i^2)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^n} (n^2 - 4in + 4i^2)$$

$$= \frac{1}{2^n} \left(n^2 \sum_{i=0}^{n} \binom{n}{i} - 4 \sum_{i=0}^{n} i \binom{n}{i} + 4 \sum_{i=0}^{n} i^2 \binom{n}{i} \right)$$

$$= \frac{1}{2^n} \left(n^2 2^n - 4n^2 2^{n-1} + 4n(n+1) 2^{n-2} \right)$$

$$= -n^2 + n(n+1) = n$$

* $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^{n-i} \cdot 1^i = \sum_{i=0}^{n} \binom{n}{i}$

* $\sum_{i=0}^{n} i \binom{n}{i} = \underbrace{\sum_{i=0}^{n/2} n \binom{n}{i}}_{\text{because it's symmetric}} = n \sum_{i=0}^{n/2} \binom{n}{i} = n \frac{1}{2} 2^n$

$$= 0\binom{n}{0} + 1\binom{n}{1} + \dots + (n-1)\binom{n}{n-1} + n\binom{n}{n}$$

$$= \binom{n}{0}(0+n) + (1+(n-1))\binom{n}{1} + \dots = n\left(\binom{n}{0} + \dots + \binom{n}{n}\right)$$

* we should ask Joachim