valid argument form

true for any p and q

Modus Ponens

$$\text{premises}\begin{cases} p \to q \\ p \end{cases}$$

$$\therefore q \quad \} \text{conclusion}$$

$$\boxed{p \land (p \to q) \to q}$$

$\Downarrow$

tautology
(never false)

Argument

- sequence of propositio
  - propositions are called premises
  - last statement is the conclusion
  - valid if premises imply conclusion

- argument form → valid no matter what we put in it
- inference rules are simple argument forms

Step 1

Step 2

⋮

Step n

∴ Conclusion

Inference rule

eg

$$\frac{\begin{array}{c} p \\ p \to q \end{array}}{q}$$

Argument form

$$\frac{\begin{array}{c} r \wedge s \\ (r \wedge s) \to q \end{array}}{q}$$

Premises

$$\text{Assumption} \begin{cases} p \\ p \to (r \wedge s) \\ (r \wedge s) \to q \end{cases}$$

Modus Ponens

① $(r \wedge s)$

Modus Ponens

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② q

$$\begin{cases} p_1 \\ p_2 \\ \vdots \\ p_n \\ (p_1 \wedge p_2 \wedge \dots \wedge p_n) \to q \end{cases}$$

① $(p_1 \wedge p_2)$ conj. rule

② $(p_1 \wedge p_2 \wedge p_3)$ conj. rule

③ $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$ conj. rule

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④ MP.

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$

Modus tollens

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Addition

$$\boxed{\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$$

$p \to \neg q$

$\neg q \to r$

$p \to r$

① $p \wedge (p \to q)$ conclude q

p } Simplif. from ①

$(p \to q)$

q } MP ②

Universal instantiation

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

Universal generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x\, P(x)}$$

Existential generalization

$$\frac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$$

Existential instantiation

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some } c}$$

$$\exists x\, (C(x) \wedge \neg B(x))$$
$$\forall x\, (C(x) \rightarrow P(x))$$

$C(a) \wedge \neg B(a)$	E.I
$C(a)$	Simplif.
$C(a) \rightarrow P(a)$	U.I.
$P(a)$	M.P.

$\neg B(a)$ Simplif.

$P(a) \wedge \neg B(a)$ Conjunction.

$\exists x (\neg B(x) \wedge P(x))$ E.G.

$\neg p$

$s \to p$

$t \to p$

$\neg p \to \neg s$

$\neg p \to \neg t$

Mathematical proof proven:

: truth table

Theorem: proven:

- definitio
- axioms
- other theorems
- rules of inference

Axiom: given as true

Informal proofs

→ more errors

Lemma: "helping theorem"

Corollary:

Conjecture

Prove theorem

to prove $\forall x(P(x) \to Q(x))$

we prove $P(c) \to Q(c)$ for an arbitrary c.
with no assumption

then U. Generalisation $\to \forall x$

Proving $p \to q$

- $\boxed{\text{trivial proof}}$ if we know q true
- $\boxed{\text{vacuous proof}}$ if p is false

Proof by contraposition

$p \to q$
$\neg q \to \neg p$

Proof by contradiction

$(\equiv \neg(p \to q))$

Assume $p \wedge \neg q$ to show $p \to q$

then show that

$$(p \wedge \neg q) \to \underbrace{(r \wedge \neg r)}_{\text{impossible.}}$$

e.g n^2 is an odd integer then n is odd

① n^2 is an odd integer and n is even.

② $n^2 = 2k+1$ $\quad$ $n = 2k'$

③ $n^2 = 4k'^2$

(4) n^2 even

contradiction $\Rightarrow$ ↯ $\Leftarrow$

Contraposition vs Contradiction

Proof by cases

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \to q$$

use the fact that

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \to q$$

$$\equiv \left[(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)\right]$$

$$|xy| = |x| \cdot |y|$$

$$x, y \geq 0$$

$$\left.\begin{array}{l} x \geq 0, \ y < 0 \\ x < 0, \ y \geq 0 \end{array}\right\} \text{can be the same}$$

$$x < 0, \ y < 0$$

Counterexamples

Proofs of equivalence

Existence Proofs $\exists x P(x)$

- **Constructive** find one^a^ that works

- **Nonconstructive** show $\neg \exists x P(x)$ leads to a contradiction

Uniqueness proof

① Existence $P(x)$

② Uniqueness $P(x) \wedge Q(\ )$

$\rightarrow$