Exercice 1

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

On cherche les valeurs propres de $A^T A$

$$\det \begin{bmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{bmatrix}$$

$$1 \quad -1 \quad +\frac{1}{2}$$

$$\lambda' = \frac{\lambda}{10}$$

$$= 10^3 \det \underbrace{\begin{bmatrix} 8-\lambda' & 10 & 4 \\ 10 & 17-\lambda' & 14 \\ 4 & 14 & 20-\lambda' \end{bmatrix}}$$

$$(8-\lambda')(17-\lambda')(20-\lambda')$$

$$+ (10)(14)(4)$$

$$+ (4)(10)(14)$$

$$- (4)(17-\lambda')(4)$$

$$- (10)(10)(20-\lambda')$$

$$- (8-\lambda')(14)(14)$$

$$= \left((8)(17) - 8\lambda - 17\lambda + (\lambda)^2\right)(20-\lambda)$$

$$+ 2 \cdot 560 - 16(17-\lambda)$$

$$- 2000 + 100\lambda - 156(8-\lambda)$$

$$= (136 - 25\lambda' + (\lambda')^2)(20 - \lambda')$$

$$+ 1120 - 136 \cdot 8 + 16\lambda'$$
$$- 2000 + 100\lambda' - 1248 + 156\lambda'$$

$$= 2720 - 136\lambda' + \cancel{500}\lambda' + 20(\lambda')^2 - (\lambda')^3$$
$$- 2128 + 272\lambda'$$

$$= - 496 + 636\lambda' + 20(\lambda')^2 - (\lambda')^3$$

Je vais avoir autant d'intuition qu'à l'examen, $\lambda_1 = 0$, $\lambda_2 = 90$, $\lambda_3 = 360$.

$$\ker \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix} \qquad \lambda = 0$$

$$= \operatorname{vect} \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$\ker \begin{pmatrix} -10 & 100 & 40 \\ 100 & 80 & 140 \\ 40 & 140 & 110 \end{pmatrix} \qquad \lambda = 90$$

$$\to \begin{pmatrix} -10 & 100 & 40 \\ 0 & 1080 & 540 \\ 0 & 540 & 270 \end{pmatrix}$$

$$\to \begin{pmatrix} -10 & 100 & 40 \\ 0 & 0 & 0 \\ 0 & 540 & 270 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 10 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dots = \operatorname{vect}\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}\right\}$$

$$\ker\begin{pmatrix}-280 & 100 & 40\\100 & -190 & 140\\40 & 140 & -160\end{pmatrix} \qquad \lambda = 360$$

$$= \ker\begin{pmatrix}-28 & 10 & 4\\10 & -19 & 14\\4 & 14 & -16\end{pmatrix}$$

$$= \operatorname{vect}\left\{\begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$$

$$A\vec{v}_3 = \begin{bmatrix}4 & 11 & 14\\8 & 7 & -2\end{bmatrix}\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$$

$$= \begin{pmatrix}18\\6\end{pmatrix}$$

$$\|A\vec{v}_3\|^2 = 18^2 + 6^2$$

$$= 360 \quad \boxed{\text{Ok}}$$

$$A\vec{v}_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -9 \\ 27 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

$$\Sigma = \begin{pmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\|A\vec{v}_2\|} A\vec{v}_2 = \frac{1}{\sqrt{90}} \begin{pmatrix} 3 \\ -9 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\vec{u}_3 = \frac{1}{\|A\vec{v}_3\|} A\vec{v}_3 = \frac{1}{\sqrt{360}} \begin{pmatrix} 6 \\ 18 \end{pmatrix}$$

$$U = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{20} \\ -3/\sqrt{10} & -3/\sqrt{20} \end{pmatrix}$$

$U_{m\times m}$

Exercice 2

$V_{n\times n}$

$m\times n$

3×2

(i) $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$$

0 est une valeur propre

18 est une valeur propre

$$\ker\left(A^TA\right) = \text{vect}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \quad \lambda = 0$$

$$\ker\begin{pmatrix}-9 & -9\\-9 & -9\end{pmatrix} = \text{vect}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} \quad \lambda = 18$$

$$\vec{v}_1 = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

$$\vec{v_2} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$\Sigma = \begin{pmatrix}3\sqrt{2} & 0\\0 & 0\\0 & 0\end{pmatrix}$$

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \quad \lambda = 18$$

$$A\vec{v}_1 = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2/6 \\ -4/6 \\ 4/6 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\|A\vec{v}_1\|} A\vec{v}_1$$

$$= \frac{1}{1} \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$\vec{v}_2 = \vec{0}$

$\Rightarrow$ problème : on veut une base orthogonale pour U.

on choisit

$$U_{temp} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$U_{E_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \left\| \frac{1}{3} \hat{U}_{E_2} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9}}$$

$$= \frac{\sqrt{5}}{3}$$

$$U_{E_3} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \left\| \frac{1}{3} U_{E_3} \right\| = \frac{\sqrt{2}}{3}$$

$$\begin{pmatrix} 1/3 & -6/\sqrt{5} & 0 \\ -2/3 & 0 & 3/\sqrt{2} \\ 2/3 & 3/\sqrt{5} & 8/\sqrt{2} \end{pmatrix} = U$$

$$\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \Sigma$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = V^T$$

(ii)

$m \times n$

2×3

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

U 2×2 Σ 2×3 V^T 3×3

$\Rightarrow A^T A \Rightarrow 3 \times 3$ ✗

$$AA^T = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$$

Σ valeur propre = 34
valeur propre 25
$34 - 25 = 9$.

on cherche les vecteurs propres:

$$\ker(AA^T - 25I) \qquad \boxed{\lambda = 25}$$

$$= \ker\begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}$$

$$= \text{vect}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$\ker(AA^T - 9I)$$

$$= \ker\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}$$

$$= \text{vect}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

on a les colonnes de U

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

on cherche les vecteurs V

$$A^T \vec{u}_1 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{1}{5\sqrt{2}} \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} \\ \end{pmatrix}$$

$$\begin{pmatrix}-1/-\sqrt{2}\end{pmatrix}$$

$$A^T \vec{u_2} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \Rightarrow \vec{v_2} = \boxed{\frac{1}{3\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}}$$

on cherche $\vec{v_3}$ pour compléter la base. $\vec{x_3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2}(\vec{x}_3)$$

$$\text{proj}_{W_2}(\vec{x}_3) = (\vec{x}_3 \cdot \vec{v}_1)\, \vec{v}_1 + (\vec{x}_3 \cdot \vec{v}_2)\, \vec{v}_2$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{2\cancel{\sqrt{2}}}{3} \cdot \frac{1}{3\cancel{\sqrt{2}}} \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{2}{9}\begin{pmatrix}-1\\1\\4\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}-2/9\\2/9\\8/9\end{pmatrix}}$$

$$\begin{pmatrix}1\\1\end{pmatrix} - \left(\frac{\quad}{\quad} \right) = \begin{pmatrix}2/9\\-2/9\\1/9\end{pmatrix} = \begin{pmatrix}2/27\\-2/27\\1/27\end{pmatrix} \Rightarrow \vec{v_3}$$

$$\vec{x_3} \cdot \vec{v_1} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\vec{x_3} \cdot \vec{v_2} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\vec{v_3} = \begin{pmatrix}2/27\\-2/27\\1/27\end{pmatrix} \quad \boxed{\text{ok}}$$

Exercice 3

① Si A ne possède pas n valeurs singulières non nulles

$$\Leftrightarrow \quad A = \sigma_1 \vec{u_1} \vec{v_1}^T + \dots + \sigma_r \vec{u_r} \vec{v_r}^T$$

$\Rightarrow$ A efface une dimension

$\Rightarrow$ A n'est pas inversible.

2

$$(A^{-1}) = (V^T)^{-1} (\Sigma)^{-1} (U)^{-1}$$

Exercice 4

a AB

=

Exercice 8

a) $\begin{pmatrix} a & b & c \end{pmatrix}$ 1×3

$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ 1×3

b) 1×1 1×1

c) $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$

$=$

$\vec{v} \cdot \vec{u} = \vec{v}^T \vec{u}$

$\boxed{\text{oui}}$

(d) 3×3

3×3

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{(1\ 2\ 3)}$$

$$= a + 2b + 3c$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}^{(a\ b\ c)} = a + 2b + 3c.$$

oui, mais pourquoi ?