Other exercises

Exercise 11. A die is rolled twice resulting in an ordered pair (r_1, r_2) of independent random outcomes $r_1, r_2 \in \{1,2,3,4,5,6\}$, and the value $s = r_1 + 2r_2 - 4k \in \{1,2,3,4\}$ is computed, where $k \in \mathbf{Z}$.

- ◯ s is uniformly distributed over $\{1,2,3,4\}$.
- ◯ s is not uniformly distributed over $\{1,2,3,4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_1 + 3r_2$".
- ◯ s is not uniformly distributed over $\{1,2,3,4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_2 + 2r_1$".
- ◯ s is not uniformly distributed over $\{1,2,3,4\}$, but it is if all outcomes with $r_1 + r_2 = 7$ are discarded.

r_2 \ r_1	1	2	3	4	5	6
1	3	4	1	2	3	4
2	1	2	3	4	1	2
3	3	4	1	2	3	4
4	1	2	3	4	1	2
5	3	4	1	2	3	4
6	1	2	3	4	1	2

$9 \cdot 3$

$9 \cdot 1$

$9 \cdot 4$

$9 \cdot 2$

Habib

What is the probability that a 13-card bridge hand con-tains

a) all 13 hearts?

b) 13 cards of the same suit?c) seven spades and six clubs?

d) seven cards of one suit and six cards of a second suit?e) four diamonds, six hearts, two spades, and one club?

f) four cards of one suit, six cards of a second suit, twocards of a third suit, and one card of the fourth suit?

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a) 52 cards

$$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \ldots \frac{1}{49} \approx 1{,}57 \cdot 10^{-12}$$

b) $4 \cdot$ (answer a)

c) $$\frac{13!}{(13-7)!} = \frac{13!}{6!}$$ ways to pick 7 spades

$$\frac{13!}{(13-6)!} = \frac{13!}{7!}$$ ways to pick 6 clubs

(*English*) There are 6 elves and 6 goblins.

The elves are all different and unique, so their order matters.

The goblins are all the same, so their order is irrelevant.

The goblins are also afraid and therefore must have at least one goblin next to them.

The number of ways the elves and goblins can be put in a line equals

$2^4 \cdot 3^3 \cdot 5^2 \cdot 7.$

$2^5 \cdot 3^2 \cdot 5 \cdot 7^2.$

$2^8 \cdot 3^2 \cdot 5 \cdot 7.$

$2^6 \cdot 3^3 \cdot 5 \cdot 7$

- we can form blocks of 2 goblins

for example $E_1 B_2 E_2 B_2 E_3 E_4 E_5 E_6 B_2$

$\Rightarrow$ stars and bars $\binom{6+4-1}{4-1} = 84$

- or (disjoint), we can form blocks of 3 goblins

for example $E_1 B_3 E_2 E_3 E_4 B_3 E_5 E_6$

$\Rightarrow$ stars and bars $\binom{6+2-1}{2-1} = 28$

total number of ways to align $84 - 28 -$ intersec

intersec $=$ nb of ways to make blocks of 6.

for example $E_1 B_3 B_3 E_2 E_3 E_4 E_5 E_6$ is the same as $E_1 B_2 B_2 B_2 B_2 E_2 E_3 E_4 E_5 E_6$

$\Rightarrow$ stars and bars $\binom{6+1}{1} = 7$

$84 + 28 - 7$ ways to line them up.

$= 105.$

$105 \cdot 6!$ (number of ways to permute the elves)

$=$ answer a.

(English) Order the following functions so that each function is big-O of the next function. For example if f is $O(g)$ and g is $O(h)$, the order should be f, g, h. Justify your choices.

(Français) Ordonnez les fonctions suivantes de façon à ce que chaque fonction soit un big-O de la fonction suivante. Par exemple, si f est $O(g)$ et g est $O(h)$, l'ordre doit être f, g, h. Justifiez vos choix.

$$f_1(n) = 2^{100000}$$
$$f_2(n) = \left(\log_2(n^2)\right)^2$$
$$f_3(n) = 0.001n!$$
$$f_4(n) = n^{0.01}$$

$$f_1(n) = 2^{1000.-}$$

$$f_2(n) = 2^{\log(\log(n^2)^2)}$$
$$= 2^{2\log(\log(n^2))}$$
$$= 2^{2\log(2\log(n))}$$
$$= 2^{2\log(2)\cdot 2\log(\log(n))}$$

so $\nearrow$ than f_1

$$f_4(n) = 2^{0.01\cdot\log(n)}$$

so $\nearrow$ than f_2

$$f_3(n) = 0.001n!$$

so $\nearrow$ than f_4

Question 8 : *This question is worth 2 points*

(English) The procedure **ExistsPair** takes as an input a list L of n integers and an integer d (we assume that the list is non empty: $n > 0$). **ExistsPair** returns True if there exists two elements in the list a_i and a_j such that their difference is equal to d. It returns False otherwise. Which of the following proposition is true?

(Français) La procédure **ExistsPair** prend en entrée une liste L de n entiers et un entier d (on suppose que la liste est non vide : $n > 0$). **ExistsPair** renvoie True s'il existe deux éléments dans la liste a_i et a_j tels que leur différence est égale à d. Elle renvoie False sinon. Laquelle des propositions suivantes est vraie ?

```
ExistsPair(L = a_1, a_2, ..., a_n: integers list, d: integer)
  L = MergeSort(L)
  for i = 1 to n do
     if Binary_Search(L, d + a_i) ≠ 0 then
        return True
  return False
```

- [ ] None of the other answers are correct.
  Aucune des autres réponses n'est correcte.
- [ ] The best case time complexity is $O(n \log_2(n))$ and the worst case time complexity is $O(n^2)$
  La complexité en temps est de $O(n \log_2(n))$ dans le meilleur cas et de $O(n^2)$ dans le pire cas.
- [ ] The best case time complexity is $O(n(\log_2(n))^2)$ and the worst case time complexity is $O(n^2)$
  La complexité en temps est de $O(n(\log_2(n))^2)$ dans le meilleur cas et de $O(n^2)$ dans le pire cas.
- [x] The best case time complexity is $O(n \log_2(n))$ and the worst case time complexity is $O(n \log_2(n))$
  La complexité en temps est de $O(n \log_2(n))$ dans le meilleur cas et de $O(n \log_2(n))$ dans le pire cas.

Merge sort : $n \log_2(n)$ (best and worst)

Binary search : $\log_2(n)$

$n \cdot$ binary searches : $n \log_2(n)$

$n \log_2(n) + (0$ (best) or $n \log_2(n)$ (worst))

1. (*français*) Étant donné un jeu standard de 52 cartes, quel est le nombre de mains distinctes de cinq cartes qui contiennent précisément quatre rangs ("kinds")?

(*English*) Given a standard deck of 52 cards, what is the number of different hands of five cards that contain precisely four kinds?

$4\binom{13}{2}13^3$.

$\binom{13}{4}4^4 \cdot 4 \cdot 3$.

$\binom{13}{4}4^4 \cdot 4$.

× $13\binom{4}{2}\binom{12}{3}4^3$.

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1}^3$$

7. (*français*) Pour une variable aléatoire Y, soit $E(Y)$ son espérance. Soit X une variable aléatoire arbitraire, alors

(*English*) Let $E(Y)$ for a random variable Y denote the expected value of Y. Let X be some arbitrary random variable, then

$\begin{cases} E(X^2) \neq E(X)^2. \\ E(X^2) \neq E(X)^2. \end{cases}$ ⊗

$\begin{cases} E(X^2) = E(X)^2 \leftrightarrow X \text{ est indépendente.} \\ E(X^2) = E(X)^2 \leftrightarrow X \text{ is independent.} \end{cases}$

$E(X^2) = E(X)^2 \leftrightarrow X$ est constante.
$E(X^2) = E(X)^2 \leftrightarrow X$ is constant.

then the variance is 0. (to get the intuition)

9. (*français*) Le nombre de quintuples distincts $(x_1, x_2, x_3, x_4, x_5)$ d'entiers x_1, x_2, x_3, x_4, x_5 tels que $x_i \geq i$ pour $i \in \{1, 2, 3, 4, 5\}$ et $10 < x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$ est égal à

(*English*) The number of distinct tuples $(x_1, x_2, x_3, x_4, x_5)$ of integers x_1, x_2, x_3, x_4, x_5 such that $x_i \geq i$ for $i \in \{1, 2, 3, 4, 5\}$ and $10 < x_1 + x_2 + x_3 + x_4 + x_5 \leq 20$ equals

- ✓ 252.
- ○ 210.
- ○ 462.
- ○ 126.

$x_1 + x_2 + x_3 + x_4 + x_5 + x_{trash} = 20$

$1 + 2 + 3 + 4 + 5 > 10$

$20 - 5 - 4 - 3 - 2 - 1 = \underline{5}$

$\binom{10}{5} = 252$

10. (*français*) Soient $c > d \geq 2$ des constantes et soient les deux propositions

$$\text{la fonction } c^{\log_d(n)} \text{ est en } O(d^{\log_c(n)}) \qquad \text{et} \qquad \text{la fonction } \log_n(c) \text{ est en } \Theta(\log_n(d))$$

(pour $n \to \infty$),

(*English*) Let $c > d \geq 2$ be constants. Given the two statements

$$\text{the function } c^{\log_d(n)} \text{ is } O(d^{\log_c(n)}) \qquad \text{and} \qquad \text{the function } \log_n(c) \text{ is } \Theta(\log_n(d))$$

(for $n \to \infty$),

$$c^{\log_d(n)} = c^{\log_c(c) \cdot \log_d(n)} \qquad \times$$

$$d^{\log_c(n)} = c^{\log_c(d) \cdot \log_c(n)}$$

2. (*français*) Soient les propositions ci-dessous, où $d > 0$ est un entier constant,

$$n\log(n) \text{ est en } \Theta(\log(n!)) \qquad \sum_{i=0}^{d} n^i \text{ est en } O(n^d)$$

(*English*) Given the two statements below, where $d > 0$ is an integer constant,

$$n\log(n) \text{ is } \Theta(\log(n!)) \qquad \sum_{i=0}^{d} n^i \text{ is } O(n^d)$$

- Elles sont vraies toutes les deux.
They are both true.
- Seulement la première est vraie.
Only the first is true.
- Seulement la seconde est vraie.
Only the second is true.
- Elles sont fausses toutes les deux.
They are both false.

$$\sum_{i=0}^{n} \log(i) = \log(n!)$$

$$\sum_{i=0}^{n} \log(i) \leq \sum_{i=0}^{n} \log(n) = n \log(n)$$

therefore $\log(n!)$ is $\Omega(n \log(n))$

we want to show $\sum_{i=0}^{n} \log(i) \geq c \cdot n \cdot \log(n)$

$$\begin{aligned}
\sum_{i=0}^{n} \log(i) &= \sum_{i=0}^{\frac{n}{2}-1} \log(i) + \sum_{i=n/2}^{n} \log(i) \\
&\geq \sum_{i=n/2}^{n} \log(i) \\
&\geq \left(\frac{n}{2}\right) \log\left(\frac{n}{2}\right)
\end{aligned}$$

See this thread on maths exchange

https://math.stackexchange.com/questions/966928/proving-nlogn-is-ologn

14. (*français*) Étant donné un jeu standard de 52 cartes, quel est le nombre de mains distinctes de sept cartes si les couleurs sont ignorées.

(*English*) Given a standard deck of 52 cards, what is the number of different hands of seven cards if no distinction is made between different suits?

- ○ $\binom{19}{7} - \binom{13}{3}$.
- ✓ $\binom{19}{7} - 13\binom{14}{2}$.
- ○ $\binom{20}{7} - 13\binom{13}{2}$.
- ○ $\binom{52}{7}/4!$.

① Stars and bars (without taking into account that we can not have hands with more than 5 times the same kind).

$$\binom{7+13-1}{13-1} = \binom{19}{\;}$$