Problem 1

(1) Marginal distribution for X:

$$p_X(0) = p(X=0)$$

$$= 1/4 + 3/16 + 1/32$$

$$= 7/16 + 1/32$$

$$= 15/32$$

$$p_X(1) = 1/8 + 1/32 + 3/8$$

$$= 17/32$$

$$p_Y(A) = 1/4 + 1/8 = 3/8$$

$$p_Y(B) = 3/16 + 1/32 = 7/32$$

$$p_Y(C) = 1/32 + 3/8 = 13/32$$

(2) $p_{X/Y}(x/y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

$y = B$

We want

$$p_{X/Y}(0/B) = \frac{p_{X,Y}(0,B)}{p_Y(B)}$$

$$= \frac{3/16}{7/32}$$

$$= \frac{6}{7}.$$

$$p_{X/Y}(1/B) = \frac{1/32}{7/32} = \frac{1}{7}$$

$$p_{X|Y}(0|C) = \frac{p_{X,Y}(0,C)}{p_Y(C)}$$

$$= \frac{1/32}{13/32} = \frac{1}{13}$$

$$p_{X|Y}(1,C) = \frac{p_{X,Y}(1,C)}{p_Y(C)} = \frac{3/8}{13/32}$$

$$= \frac{12}{13}$$

③ $$p_{Y|X}(A,1) = \frac{p_{X,Y}(1,A)}{p_X(1)}$$

$$= \frac{1/8}{17/32} = \frac{4}{17}$$

$$p_{Y/X}(B,1) = \frac{p_{X,Y}(1,B)}{p_X(1)}$$

$$= \frac{1/32}{17/32} = \frac{1}{17}$$

$$p_{Y,X}(C,1) = \frac{p_{X,Y}(1,C)}{p_X(1)}$$

$$= \frac{3/8}{17/32} = \frac{12}{17}.$$

Problem 1.2

① AHH
AHT
ATH
ATT
BTT

② $p(\omega_1 = s_1 / \omega_0 = A) = ?$

$p(\omega_1 = H / \omega_0 = A) = 1/2$

$p(\omega_1 = T / \omega_0 = A) = 1/2$

$p(\omega_2 = s_2 / \omega_0 = A) = ?$

$p(\omega_2 = H / \omega_0 = A) = 1/2$

$p(\omega_2 = T / \omega_0 = A) = 1/2$

$p(\{\omega_1 = s_1\} \cap \{\omega_2 = s_2\}) = ?$

$\{s_1 = T, s_2 = T\} = 1/4$ $\{s_1 = H, s_2 = H\} = 1/4$

$\{s_1 = H, s_2 = T\} = 1/4$ $\{s_1 = T, s_2 = H\} = 1/4$

③ $p(\omega_1 = s_1) = p(\omega_1 = s_1 \mid \omega_0 = A) \cdot p(\omega_0 = A)$
$\quad + p(\omega_1 = s_1 \mid \omega_0 = B) \cdot p(\omega_0 = B)$

$p(\{\omega_1 = s_1\} \cap \{\omega_2 = s_2\})$

$= p(\{\omega_1 = s_1\} \cap \{\omega_2 = s_2\} \mid \omega_0 = A) \cdot p(\omega_0 = A)$

$+ p(\{\omega_1 = s_1\} \cap \{\omega_2 = s_2\} \mid \omega_0 = B) \cdot p(\omega_0 = B)$

		S_1	
		H	T
S_2	H	$p(\omega_1 = s_1) = \frac{1}{2}(\frac{1}{2}+0) = \frac{1}{4}$ $p(\omega_2 = s_2) = 1/2$ $p(\{\omega_1 = s_1\} \cap \{\omega_2 = s_2\})$ $= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$	$p(\omega_1 = s_1) = \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$ $p(\omega_2 = s_2) = \frac{1}{2}(\frac{1}{2}+0) = \frac{1}{4}$ $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{8}$
	T	$p(\omega_1 = s_1) = \frac{1}{2}(\frac{1}{2}+0) = \frac{1}{4}$ $p(\omega_2 = s_2) = \frac{1}{2}$ $p(\{\text{—}\}) = \frac{1}{8}$	$p(\omega_1 = s_1) = \frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$ $p(\omega_2 = s_2) = \frac{1}{2}(\frac{1}{2}+1) =$ $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{8}$

$p(E_1 | E_2) = p(E_1)$? $\to$ no.

if we know E_2 is H, then $p(E_1) \neq 0$ but $\frac{1}{2}$.

Problem 1.3

1. a) $p(\omega_1 = \omega_2)$

$$= \frac{1}{36} \cdot 6 = \frac{1}{6}.$$

b)

$\omega_2 \backslash \omega_1$	1	2	3	4	5	6
1						
2	✓					
3	✓	✓				
4	✓	✓	✓			
5	✓	✓	✓	✓		
6	✓	✓	✓	✓	✓	

$$p(\omega_1 < \omega_2) = \frac{15}{36}$$

$$p(\omega_1 \leq \omega_2) = \frac{21}{36}$$

2

$$p(E_1) = \frac{1}{6} = \left(6 \cdot \frac{1}{36}\right)$$

$Y_2 \backslash Y_1$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

~~$p(\{Y_1 = Y\}) = p(Y_2 = 0)$~~

- we want to see if:

$$p(E_1) \cdot p(E_2) = p(E_1 \cap E_2) \checkmark$$

- $p(E_1) = \frac{1}{6}$

$p(E_2)$ only happens when $Y_2 = 0$ (then $Y = Y_1$)

$$p(E_2) = \frac{1}{6}$$

$p(E_1 \cap E_2) = \frac{1}{36}$ because $E_1 = 0$ AND $E_2 = 0$

Problem 1.4

(1) A : "we throw a dice and we get 7"
B : " ——————————————— we get 1"
C : " ——————————————— we get 7"

$P(A \cap B \cap C) = 0$

$P(A \cap B) = 0 = P(A) \cdot P(B)$

$P(A \cap C) = 0 = P(A) \cdot P(C)$

$P(B \cap C) =$

A : " est pair "
B : " ≥ 3 "
C : " $\in \{2,5\}$ "

1 • $p(A \cap B) = \frac{2}{6} = \frac{1}{3}$

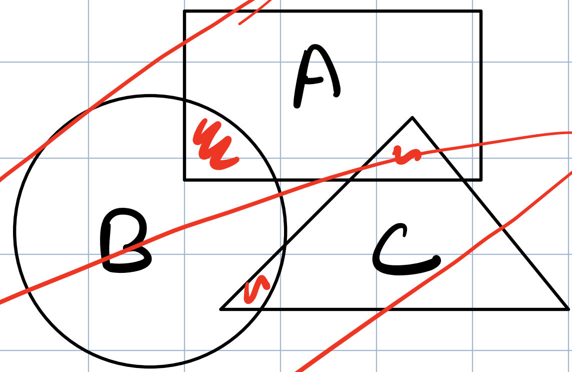
$p(A) \cdot p(B) = \frac{3}{6} \cdot \frac{4}{6} = \frac{12}{36} = \frac{1}{3}$

• $p(A \cap C) = \frac{1}{6}$

$p(A) \cdot p(C) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

• $p(B \cap C) = \frac{1}{6}$

$p(B) \cdot p(C) = \frac{4}{6} \cdot \frac{2}{6}$



A : « pair »

B : « $\{2,5\}$ »

C : « $\{1,2\}$ »

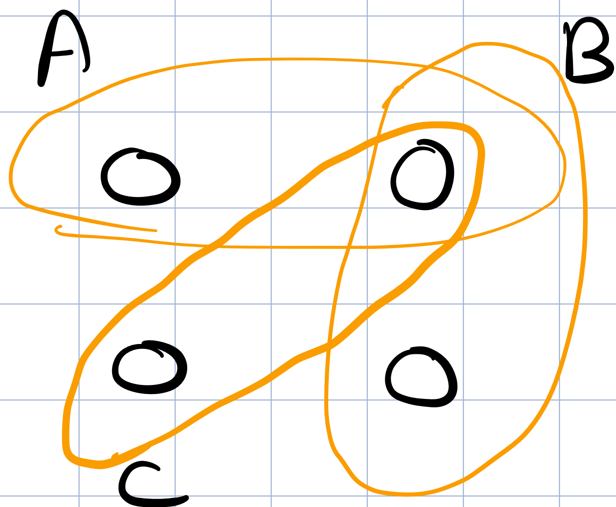
$$p(A \cap C) = \frac{1}{6}$$

$$p(A) \cdot p(C) = \frac{1}{2} \cdot \frac{2}{6} = \frac{1}{6}$$

$$p(B \cap C) = 1/6$$

$$p(B) \cdot p(C) =$$

Avec un dé à 6 faces → difficile de faire des events bien proportionnées



$p(C) = \frac{1}{2}$

$p(A) = \frac{1}{2}$ $\quad p(B) = \frac{1}{2}$

$p(A \cap C) = \frac{1}{4}$ $\quad p(B \cap C) = \frac{1}{4}$

$p(A) \cdot p(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$p(B) \cdot p(C) = \frac{1}{4}$ $\quad p(A) \cdot p(B) = \frac{1}{4}$

$p(A) \cdot p(B) \cdot p(C) = \frac{1}{8}$ $\quad P(A \cap B \cap C) = \frac{1}{4}$

Problem 1.5

- uniform distribution for the variable X $\left(\frac{1}{2^4}\right)$

$$\Rightarrow H_2(X) = -\sum_{s \in A} p_s(s) \log_2\left[p_s(s)\right]$$
$$= \sum_{s \in A} \frac{1}{2^4}(4)$$
$$= 2^4 \cdot \frac{1}{2^4}(4)$$
$$= 4 \text{ bits}$$

- $H_2(Y) = 2$ bits (same reasoning as 1)

- 15 max

X' ⇒ nombre binaires entre 0 et 15.

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & & & & \end{array}$$

$M_2(2)$

$$= 1 \cdot \left(\frac{4}{16} \cdot \log_2\left(\frac{16}{4}\right)\right)$$
$$+ 4 \cdot \left(\frac{3}{16} \cdot \log_2\left(\frac{16}{3}\right)\right)$$

$$= \frac{1}{4} \cdot \log_2(2^2)$$
$$+ 4 \cdot \frac{3}{16} \cdot \log_2\left(\frac{16}{3}\right)$$
$$= \frac{1}{2} \log_2(2) + \frac{3}{4} \cdot \left(\log_2(16) - \log_2(3)\right)$$
$$= \frac{1}{2} + \frac{3}{4}\left(4 - \log_2(3)\right)$$

$$= \frac{1}{2} + 3 - \frac{3}{4}\log_2(3)$$

$$= \frac{7}{2} - \frac{3}{4}\log_2(3)$$