Exercice 2

linéaire → d'ordre 1 $\quad ax+b$
quadratique → " 2 $\quad ax^2+bx+c$

Déf :

- $f(\bar{x}) \simeq F(\bar{a}) + F'(\bar{a}) + \frac{1}{2} F''(\bar{a})$

- $F'(\bar{a}) = \lim\limits_{t \to 0} \dfrac{f(\bar{a} + t(\bar{x} - \bar{a})) - f(\bar{a})}{t}$

$= Df(\bar{a}, (\bar{x} - \bar{a}))$ ← dérivée directionnelle

$= \langle \nabla f(\bar{a}), \bar{x} - \bar{a} \rangle$

$$f(\bar{x}) \simeq f(\bar{a}) + \nabla f(\bar{a}) \cdot (\bar{x} - \bar{a})$$
$$+ \frac{1}{2} (\bar{x} - \bar{a})^T \cdot Hf(\bar{a}) \cdot (\bar{x} - \bar{a})$$

$$f(x,y,z)$$

$$\simeq f(x_0,y_0,z_0)$$

$$+\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0)$$

$$+\frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0)$$

$$+\frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0)$$

$$+\frac{1}{2}\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial x\partial z} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y\partial y} & \frac{\partial^2 f}{\partial y\partial z} \\ \frac{\partial^2 f}{\partial z\partial x} & \frac{\partial^2 f}{\partial z\partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}(\bar{a})$$

$$\begin{pmatrix} c & e \\ d & f \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}$$
c

$$\cdot\begin{pmatrix} x-x_0 & y-y_0 & z-z_0 \end{pmatrix}$$

$$+\varepsilon\left(\|(x,y,z)-(x_0,y_0,z_0)\|^2\right)$$

Exercice 3

(i) $f(0,0) = 1$

$$\frac{\partial f}{\partial x} = 2xy + 2y - 5$$

$$\frac{\partial f}{\partial y} = x^2 + 2x + 6y$$

$$\frac{\partial f}{\partial x}(0,0) = -5 \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = \cancel{2y} \qquad \frac{\partial^2 f}{\partial y \partial x} = \cancel{2x} + 2$$

$$\frac{\partial^2 f}{\partial y^2} = 6 \qquad \frac{\partial^2 f}{\partial x \partial y} = \cancel{2x} + 2$$

$$f(x,y) = 1 - 5(x,y)$$

$$+ 1/2\,(x(0) + y(2) \quad x(2) + y(6))$$

$$\cdot \begin{pmatrix} x & y \end{pmatrix}$$

$$= 1 - 5(x,y) + \frac{1}{2}\left(\quad + 2xy \quad + 2xy + 6y^2 \right)$$

$$= 1 - 5x + 2xy + 3y^2$$

$$\varepsilon(x,y) = f(x,y) - p(x,y)$$

$$= x^2 y$$

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|\varepsilon(x,y)\|^2} = 0 \ ?$$

$$\lim_{r\to 0} \frac{\cancel{r^2}\cos^2(\varphi)\, r \sin\varphi}{\cancel{r^2}} = 0 \ \checkmark$$

ii) $f(x,y,z) = e^{x} + y\,sh(z)$

$$\frac{\partial f}{\partial x} = e^{x} \Rightarrow (0,0,0) = 1$$

$$\frac{\partial f}{\partial y} = sh(z) \Rightarrow (0,0,0) = 0$$

$$\frac{\partial f}{\partial z} = y\,ch(z) \Rightarrow (0,0,0) = 1$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x} \Rightarrow (0,0,0) = 1$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial z \partial x} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = ch(z) \Rightarrow (0,0,0) = 1$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y \partial z} = ch(z) \Rightarrow (0,0,0) = 1$$

$$\frac{\partial^2 f}{\partial z^2} = y\, sh(z) \Rightarrow (0,0,0) = 0$$

$$f(x,y,z) = 1$$

$$+ x + z$$

$$+ \frac{1}{2} \begin{pmatrix} x & z & y \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= 1 + x + z + \frac{1}{2} x^2 + yz$$

iii $f(x,y) = 3xy + x^2 - y + 5x - 3$

$$\frac{\partial f}{\partial x} = 3y + 2x + 5 \;\Big|\; (1,-2) \Rightarrow 1$$

$$\frac{\partial f}{\partial y} = 3x - 1 \;\Big|\; (1,-2) \Rightarrow 2$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \;\Big|\; (1,-2) = 2$$

$$\frac{\partial f}{\partial y \partial x} = 3 \;\Big|\; (1,-2) = 3$$

$$\frac{\partial f}{\partial x \partial y} = 3 \;\Big|\; (1,-2) = 0$$

$$\frac{\partial f}{\partial y^2} = 0 \;\Big|\; (1,-2) = 0$$

$$f(1,-2) = -6 + 1 + 2 + 5 - 3$$
$$= -1$$

$$f(x,y) = -1$$

$$+ (x-1) + 2(y+2)$$

$$= x + 2y + 4 - 2$$

$$= x + 2y + 2$$

$$= -1 + x - 1 + 2y + 2$$

$$+ (x-1)^2 + \frac{3}{2}(y+2)(x-1)$$

$$+ \frac{3}{2}(x-1)(y+2)$$

(iv) $f(x,y) = (\cos(x))^{1/2+\sin(y)}$

$$\frac{\partial f}{\partial x} = -\left[\frac{1}{2} + \sin(y)\right] \sin(x) \cos(x)^{-\frac{1}{2}+\sin y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{\ln(\cos(x)) \cdot \left(\frac{1}{2} + \sin(y)\right)} \right]$$

$$= \ln(\cos(x)) \cos(y) \cos(x)^{1/2 + \sin y}$$

$$\frac{\partial f}{\partial x}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^{0}$$

$$= -\frac{\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial y}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \ln\left(\frac{1}{2}\right) \frac{\sqrt{3}}{2} \frac{1}{2}$$

$$= -\ln(2) \frac{\sqrt{3}}{4}$$

$$f\left(\frac{\pi}{3},\frac{\pi}{6}\right)=\left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{1}{2}}=\frac{1}{2}$$

$$f(x,y)=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)$$
$$-\ln(2)\frac{\sqrt{3}}{4}\left(y-\frac{\pi}{6}\right)$$
$$=\frac{1}{2}-\frac{\sqrt{3}}{2}x+\frac{\sqrt{3}\pi}{6}$$
$$-\ln(2)\frac{\sqrt{3}}{4}y+\ln(2)\frac{\sqrt{3}\pi}{24}$$

(i)

Exercice 4

(i) $f(x,y,z) = e^{2xz+y}$

$h(x,y,z) = 2xy + z$

$$e^{s} = 1 + s + \frac{1}{2}s^2$$

$$f(x,y,z) = 1 + 2xy + z + \frac{1}{2}(2xy+z)^2$$

$$= 1 + 2xy + z + \frac{1}{2}z^2$$

(ii) $\sin(z) = \sin(3) + \cos(3)(z-3)$
$-\sin(3)(z-3)^2\frac{1}{2}$

$= \sin(3) + z\cos(3) - 3\cos(3)$

$- \frac{z^2}{2}\sin(3) + 3z\sin(3) - \frac{9}{2}\sin(3)$

$$h(x, y) = 2x + y^2$$

$$\sin(2x + y^2) = \sin(3) +$$

Exercice 1

(i) Leibniz

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\,dx\right)$$

$$= \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx + f(b(t),t)\,b'(t)$$

$$- f(a(t),t)\,a'(t)$$

$a(t) = 2 \Rightarrow a'(t) = 0$

$b(t) = 3 \Rightarrow b'(t) = 0$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\log(x)} \cdot x^t\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{\log(x)} e^{\ln(x) t}\right)$$

$$= x^t$$

$$\int_2^3 x^t \, dx = \left[\frac{x^{t+1}}{t+1}\right]_2^3$$

$$= \frac{3^{t+1}}{t+1} - \frac{2^{t+1}}{t+1}$$

$$= \frac{3^{t+1} - 2^{t+1}}{t+1} = F'(t)$$

(ii) $a(t) = t \Rightarrow a'(t) = 1$

$b(t) = t^2 \Rightarrow b'(t) = 2t$

$$\frac{\partial f}{\partial t} = \frac{2t}{x^2 + t^2}$$

$$\int_t^{t^2} \frac{2t}{x^2+t^2}\,dx$$

$$= 2t \int_t^{t^2} \frac{1}{x^2+t^2}\,dx$$

$$= 2t \left[\frac{1}{t} \arctan\left(\frac{x}{t}\right) \right]_t^{t^2}$$

$$= 2\operatorname{Arctan}(t) - \pi/2$$

$$\arctan(u)' = \left(\frac{1}{1+u^2}\right) \cdot u'$$

⚠ on dérive par rapport à x !

pas de $t' = \frac{-1}{t^2}$!

$t' = 0$!

$$v(x) = \tan^{-1}\left(\frac{x}{t}\right) \Rightarrow v'(x) = \frac{1}{1 + \frac{x^2}{t^2}} \cdot \frac{1}{t}$$

$$= \frac{-1}{t + \frac{x^2}{t}}$$

$$F'(t) = 2\operatorname{Arctan}(t) - \frac{\pi}{2}$$

$$+ f(b(t), t)\, b'(t)$$

$$- f(a(t), t)\, a'(t)$$

$$= 2\operatorname{Arctan}(t) - \pi/2 + \log(t^2(t^2+1))2t - \log(2t^2)$$

$f(t^2, t)$

$= \log(t^4 + t^2)$

$f(t, t)$

$= \log(t^2 + t^2)$

(iii) $a(t) = 1 \Rightarrow a'(t) = 0$

$b(t) = (t)^{1/3} \Rightarrow b'(t) = \frac{1}{3} t^{-2/3}$

$\frac{\delta f}{\delta t} = \frac{1}{x} x^3 e^{tx^3} = x^2 e^{tx^3}$

$\int_1^{\sqrt[3]{t}} x^2 e^{tx^3} \, dx$

$$= \int_1^{\sqrt[3]{t}} x^2 e^{tx^3} \, dx$$

$$= \left[\frac{1}{3t} e^{tx^3} \right]_1^{\sqrt[3]{t}}$$

$$= \frac{1}{3t} e^{t^2} - \frac{1}{3t} e^{t}$$

$$= \frac{1}{3t} \left(e^{t} \left(e^{t} - 1 \right) \right)$$

$$F'(t) = \frac{1}{3t} \left(e^{t^2} - e^{t} \right) + \frac{e^{t^2}}{3t}$$

$$= \frac{1}{3t} \left(2e^{t^2} - e^{t} \right)$$

$$f\left((t)^{1/3}, t \right) \left(\frac{1}{3} t^{-2/3} \right)$$

$$= e^{t^2}\left(\frac{1}{3}t^{-2/3}\right)t^{-1/3}$$

$$= \frac{e^{t^2}}{3t}$$

Exerice 5

(i) $$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\Rightarrow \nabla f(x,y) = (0,0) \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial^2 y} = 2$$

$$\det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2 \text{ et } 2 > 0$$

$\Rightarrow$ minimum local

(ii) $\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -2y$

$\frac{\partial^2 f}{\partial x^2} = 2 \quad \frac{\partial^2 f}{\partial y^2} = -2$

$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow$ pas d'extremum local

(iii) $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow$ pas d'el

(iv) $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow$ max loc

(v) $\frac{\partial f}{\partial x} = 4x^3 \quad \frac{\partial f}{\partial y} = 4y^3$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 \quad \frac{\partial^2 f}{\partial^2 y^2} = 12y^2$

$$\frac{\partial^2}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} (0) = 0$$

min local (on aurait pu redériver jusqu'à la dérivée 4ème, les dérivées pures complet)

(vi) $\begin{pmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix} \to$ pas d'extrem.

(ix) 0 n'est ni le min, ni le max, on peut trouver pire ou meilleur

x $\begin{pmatrix} -12x^2 & 0 \\ 0 & -12x^2 \end{pmatrix} \rightarrow$ max loc

Exercice 6

① Diagonaliser la matrice

$$(6-\lambda)(3-\lambda) - 4$$

$$= 18 - 6\lambda - 3\lambda + \lambda^2 - 4$$

$$= 14 - 9\lambda + \lambda^2$$

$$\Delta = 81 - 4 \cdot 14 = 81 - 56 = 25$$

$$\sqrt{\Delta} = 5$$

$$\lambda = \frac{9 \pm 5}{2} \begin{matrix} \nearrow 7 \\ \searrow 2 \end{matrix}$$

$$A\vec{v} = \lambda\vec{v} \Rightarrow A(\vec{v} - \lambda I) = 0$$

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 1 & 0 \end{array}\right)$$

$$-2x + y = 0$$

$$-2x = -y \quad \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$-x - 2y = 0 \Rightarrow -x = 2y$$

$$\begin{array}{cc} P & D \\ \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

$$\det(P) = 4 - 1 = 3$$

$$\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

ii

$$\frac{\partial f}{\partial x} = 6x - 2y$$

$$\frac{\partial f}{\partial y} = -2x + 3y$$

$$\frac{\partial^2 f}{\partial x^2} = 6 \qquad \frac{\partial^2 f}{\partial y^2} = 3$$

$$\frac{\partial^2 f}{\partial y \partial x} - -2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -2$$

$$\det(A) = 18 - 4 = \underline{14} > 0$$

$$6 > 0$$

$$\Rightarrow \underline{\text{min}}$$

iii

$$\begin{array}{cc} 2\times 2 & 2\times 1 \\ \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 2x - y \\ x - 2y \end{pmatrix}$$

$$\det(O^T) = 4+1 = 5$$

$$\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 2u+v \\ -u+2v \end{pmatrix}$$

$$x = \frac{2}{5}u + \frac{1}{5}v$$

$$y = \frac{-1}{5}u + \frac{2}{5}v$$

$$f(x,y) = 3\left(\frac{2}{5}u + \frac{1}{5}v\right)^2$$

$$-2\left(\frac{2}{5}u + \frac{1}{5}v\right)\left(\frac{-1}{5}u + \frac{2}{5}v\right)$$

$$+ 3/2\left(2/5v - 1/5u\right)^2$$

$$= 3\left(\frac{4}{25}u^2 + \frac{4}{25}uv + \frac{1}{25}v^2\right)$$
$$-2\left(\frac{-2}{25}u^2 + \frac{4}{25}uv - \frac{1}{25}uv + \frac{2}{25}v^2\right)$$
$$+\frac{3}{2}\left(\frac{4}{25}v^2 - \frac{4}{25}uv + \frac{1}{25}u^2\right)$$

$$= \cancel{\frac{12}{25}}u^2 + \cancel{\frac{12}{25}}uv + \cancel{\frac{3}{25}}v^2$$
$$+ \cancel{\frac{4}{25}}u^2 - \cancel{\frac{8}{25}}uv + \cancel{\frac{2}{25}}uv - \cancel{\frac{4}{25}}v^2$$
$$+ \frac{12}{50}v^2 - \frac{12}{50}uv + \frac{3}{50}u^2$$

$$= \frac{32}{50}\cancel{\frac{16}{25}}u^2 + \frac{12}{50}\cancel{\frac{6}{25}}uv - \frac{2}{50}\cancel{\frac{1}{25}}v^2$$
$$+ \frac{12}{50}v^2 - \frac{12}{50}uv + \frac{3}{50}u^2$$
$$= \frac{35}{50}u^2 + \frac{1}{5}v^2$$

$$\frac{\partial f}{\partial u} = \frac{35}{25} u$$
$$\frac{\partial f}{\partial v} = \frac{2}{25} v$$
$$\frac{\partial^2 f}{\partial u^2} = \frac{35}{25} \qquad \frac{\partial^2}{\partial v^2} = \frac{2}{25}$$
$$\frac{\partial^2 f}{\partial v \partial u} = 0 \qquad \frac{\partial^2 f}{\partial u \partial v} = 0$$

$$\begin{pmatrix} \frac{35}{25} & 0 \\ 0 & \frac{2}{25} \end{pmatrix}$$

det > 0
$r > 0$
$\Rightarrow$ min local

(iv)
$$\frac{\partial f}{\partial x} = 4y \qquad \frac{\partial f}{\partial y} = 4x$$
$$\frac{\partial f}{\partial y \partial x} = 4 \qquad \frac{\partial f}{\partial x \partial y} = 4$$

$$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$\det < 0$
pas d'extremum

$$(0-\lambda)(0-\lambda) - 16$$

$$= \lambda^2 - 16 = 0$$

$$\lambda = \pm 4$$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$O^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x + y \\ x - y \end{pmatrix}$$

$u = x + y$

$v = x - y$

Exercice 7

(i)

sum $P(n,k)$
$P(n,k-1)$

Initialisation

(a) $f_{n+1} = f_n f_0 + f_{n+1} f_1$

$= f_{n+1}$ ✓

$P(n,0)$ vraie $\forall n \in \mathbb{N}$

(b)

$P(n+1, m)$

$(n+1), (m)$

$f_{m+(n+1)+1} = f_{n+1} f_m + f_{n+2} f_{m+1}$

$= f_{n+1} f_m + (f_n + f_{n+1}) f_{m+1}$

$= f_{n+1} (f_{m+2}) + f_n f_{m+1}$

$(n), (m+1)$

$f_{(m+1)+n+1} = f_n f_{m+1} + f_{n+1} f_{m+2}$

(c) $P(n, 0)$ ✓

$\downarrow$

$P(n, 0+1)$

$0+2$

etc.

$n+1 \quad 1+1$ etc.

$1+2$

(ii)

$P(0,0)$

$$f_0 + (-1)^0 f_0 = \ell_0 f_0 \quad \checkmark$$

$P(n,0)$

$$f_n + (-1)^0 f_n = \ell_0 f_n \quad \checkmark$$

$P(n+1, k)$

$$f_{(n+1)+k} + (-1)^k f_{(n+1)-k} = \ell_k f_{n+1}$$

$$= \ell_k (f_{n+2} - f_n)$$

$= -\ell_k f_n + \ell_k f_{n+2}$

$P(n,k)$

$\Rightarrow P(n,k+1)$

$$f_{n+(k+1)} + (-1)^k \underset{-}{(-1)} f_{n-k-1} \stackrel{?}{=} \ell_{k+1} f_n$$

$$= \ell_k f_n + \ell_{k-1} f_n$$

$P(n,k) + P(n,k-1) \to P(n,k+1)$

$f_{n+k} + \underset{+}{(-1)^k} f_{n-k}$ → same as $k+1$

$+ f_{n+k-1} + \underset{+}{(-1)^{k-1}} f_{n-k+1}$

$$= f_{n+k+1} + (-1)^k(-1) f_{n-k-1} = \ell_{k+1} f_n$$