Exercice 1

notation?

(i) $$\frac{d^2y}{dx^2} = 1$$

$$\Leftrightarrow \quad d^2y = \int dx^2$$

$$y' = x + C$$

$$\Rightarrow y = \frac{1}{2}x^2 + C_1x + C_2$$

(ii) on résout l'équation caractéristique

$$\lambda^2 + 1 = 0$$

$$\Rightarrow a = i \,, \; b = -i$$

ici, $a \neq b$

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

(iii) $\lambda^2 - \lambda = 0$

$a = 1 \quad b = -1$

$y(x) = C_1 e^{x} + C_2 e^{-x}$

(iv) $\lambda^2 - 2\lambda + 1 = 0$

$a = 1 \quad b = 1$

$y(x) = C_1 e^{x} + C_2 x e^{x}$

Exercice 2

$$3\lambda^2 - 4\lambda + 1$$

$\Delta = 16 - 12 = 4$

$\lambda = \frac{4 \pm 2}{6}$

$\Rightarrow a = 1$

$b = 1/3$

$$y_1(x) = C_1 e^{x} + C_2 e^{x/3}$$

$$3\lambda^2 - 4\lambda + 2$$

$\Delta = 16 - 24$

$= -8$

$\sqrt{\Delta} = i\sqrt{8}$

$\Rightarrow a = \frac{4 + i\sqrt{8}}{6}$

$= \frac{2 + i\sqrt{2}}{3}$

$b = \frac{2 - i\sqrt{2}}{3}$

$$y_2(x) = C_1 e^{2/3 x} \cos\left(\frac{\sqrt{2}}{3}\right)$$

$$+ C_2 e^{2/3 x} \sin\left(\frac{\sqrt{2}}{3}\right)$$

$$3\lambda^2 - 4\lambda + 4/3$$

$$\Delta = 16 - 16 = 0$$

$$\lambda = \frac{4}{6} = \frac{2}{3}.$$

$$y(x) = e^{2/3x}(C_1 + C_2 x)$$

Exercice 3

On cherche d'abord par y_{hom} :

$$\frac{1}{3}\lambda^2 - \frac{3}{4} = 0$$

$$\Delta = -1 \qquad \lambda = \frac{\pm 3i}{2}$$
$$\sqrt{\Delta} = i$$

$$y_{hom}(x) = C_1 \cos\left(\frac{3}{2}x\right) + C_2 \sin\left(\frac{3}{2}x\right)$$

$$\frac{1}{3}y'' + \frac{4}{3}y = e^{2x}$$

$\lambda = 2$ n'est pas une sol.

$$y_{part} = e^{2x} T_n(x)$$

polynôme de degré 0

$$= e^{2x} C$$

$$\frac{1}{3}(y_{part})'' + \frac{4}{3} y_{part} = e^{2x}$$

$$\Rightarrow \frac{4C}{3} e^{2x} + \frac{4C}{3} e^{2x} = e^{2x}$$

$$4C + 4C = 3 \Rightarrow C = \frac{3}{8}$$

$y_{hom} + y_{part}$

$$= \frac{3}{8} e^{2x} + C_1 \cos(2x) + C_2 \sin(2x)$$

(ii) $y'' + 4y = 5\cos(2x)$

$$\lambda^2 + 4 = 0$$
$$\lambda = \pm 2i$$

$$y_{part} = Ax\cancel{\cos}(2x) + Bx\sin(2x)$$

$$(y_{part})' = A\cos(2x) - 2Ax\sin(2x) + B\sin(2x) + 2Bx\cos(2x)$$

$$(y_{part})'' = -2A\sin\cancel{(2x)} - 2\cancel{A\sin}(2x) - 4Ax\cancel{\cos}(2x) + 2B\cos(2x) + 2B\cos(2x) - 4Bx\sin(2x)$$

$$(y_{part})'' + 4y_{part} = 4B\cos(2x)$$

$$\Rightarrow B = \frac{5}{4}, \quad A = 0.$$

$$y = \frac{5}{4}x\sin(2x) + C_1\cos(2x) + C_2\sin(2x)$$

(iii) $\lambda^2 + 1 = 0$

$\lambda = \pm i$

2 solutions lin. indep

$$y_{hom}(x) = C_1 \cos(x) + C_2 \sin(x)$$

$\Rightarrow$ méthode de la variation de cste.

$$y(x) = C_1(x)\cos(x) + C_2(x)\sin(x)$$

$$\Rightarrow y'(x) = \overbrace{C_1'(x)\cos(x)} - C_1(x)\sin(x)$$
$$+ \underbrace{C_2'(x)\sin(x)} + C_2(x)\cos(x)$$

$0 =$

$$\Rightarrow y''(x) = \cancel{C_1''(x)\cos(x)} - \cancel{C_1'(x)\sin(x)}$$
$$- \cancel{C_1'(x)\sin(x)} - \cancel{C_1(x)\cos(x)}$$
$$+ \cancel{C_2''(x)\sin(x)} + \cancel{C_2'(x)\cos(x)}$$
$$+ \cancel{C_2'(x)\cos(x)} - \cancel{C_2(x)\sin(x)}$$

$$\cdot \sin(x)$$

$$\begin{matrix} u_1 \\ u_2 \end{matrix} \begin{cases} C_1'(x)\cos(x) + C_2'(x)\sin(x) = 0 \\ C_2'(x)\cos(x) - C_1'(x)\sin(x) = \dfrac{1}{\sin(x)} \end{cases}$$

$$\cdot \cos(x)$$

$$\begin{cases} C_1'(x)\cos(x)\sin(x) + C_2'(x)\sin^2(x) = 0 \\ C_2'(x)\cos^2(x) - C_1'(x)\sin(x)\cos(x) = \dfrac{\cos(x)}{\sin(x)} \end{cases}$$

$$\Rightarrow C_2'(x) = \frac{1}{\tan(x)}$$

$$\Rightarrow C_2(x) = \ln|\sin x|$$

$$\Rightarrow C_1'(x) = -1$$

$$\Rightarrow C_1(x) = -x$$

$$y(x) = y_{part}(x) + y_{hom}(x)$$

$$= -x\cos(x) + \ln|\sin x|\sin(x)$$

$$+ C_1\cos(x) + C_2\sin(x)$$

$C_1, C_2 \in \mathbb{R}$ $\qquad$ $x \in \left]0;\pi\right[$

(sinon log négatif)

Exercice 4

(i) $\lambda^2 + 2\lambda - 3$

$\Delta = 2^2 + 4 \cdot 3 = 16$

$\sqrt{\Delta} = 4$

$\lambda = \frac{-2 \pm 4}{2} = 1 = a$

$-3 = b$

$y_{hom}(x) = C_1 e^{x} + C_2 e^{-3x}$

(ii) méthode des coefficients indéterminés

0 n'est pas une racine

$\Rightarrow y_{part} = A\sin(3x) + B\cos(3x)$

$\Rightarrow (y_{part})' = 3A\cos(3x) - 3B\sin(3x)$

$\Rightarrow (y_{part})'' = -9A\sin(3x) - 9B\cos(3x)$

$= -9(y_{part})$

$$-9y_{part} + 6A\cos(3x) - 6B\sin(3x) - 3y_{part} = 5\sin(3x)$$

$$-12A\sin(3x) - 12B\cos(3x) + 6A\cos(3x) - 6B\sin(3x) = 5\sin(3x)$$

$$\sin(3x)(-12A - 6B) + \cos(3x)(-12B + 6A) = 5\sin(3x)$$

$$-12B + 6A = 0$$
$$-12A - 6B = 5$$

$$\left(\begin{array}{cc|c} 6 & -12 & 0 \\ -12 & -6 & 5 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 6 & -12 & 0 \\ -6 & -3 & 5/2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 6 & -12 & 0 \\ 0 & -15 & 5/2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 6 & -12 & 0 \\ 0 & 1 & -1/6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 6 & 0 & -2 \\ 0 & -1 & -1/6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & -1/6 \end{array}\right)$$

$$\Rightarrow y_{part} = \frac{-1}{3}\sin(3x) + \frac{-1}{6}\cos(3x)$$

$$y(x) = \frac{-1}{3}\sin(3x) + \frac{-1}{6}\cos(3x)$$

$$+ C_1 e^{x} + C_2 e^{-3x}$$

iii $y(0) = \frac{-1}{6} + C_1 + C_2 = 1$

$y'(0) = C_1 - 3C_2 - 1 = -\frac{1}{2}$

$$\left(\begin{array}{cc|c} 1 & 1 & 7/6 \\ 1 & -3 & 1/2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 7/6 \\ -1 & 3 & -1/2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0 & 4 & 2/3 \\ 1 & -3 & 1/2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0 & 1 & 1/6 \\ 1 & 0 & \underline{1} \end{array}\right)$$

$$y(x) = \frac{-1}{3}\sin(3x) + \frac{-1}{6}\cos(3x)$$

$$+ \quad e^{x} \quad + \frac{1}{6} e^{-3x}$$

Exercice 5

1) a) $y' - \tan(x)\, y = \cos(x)$

EDL1

b) autre

$\boxed{z = \frac{1}{y}} \iff y = \frac{1}{z}$

$z' = \frac{-y'}{y^2}$

$$y' + \frac{y}{x} = -xy^2$$

$$\Rightarrow \frac{y'}{y^2} + \frac{1}{xy} = -x$$

$$\Leftrightarrow -z' + zx = -x.$$

$$y = \frac{1}{z} \Rightarrow y^2 = \frac{1}{z^2}$$

$$\left(\frac{1}{z}\right)' + \frac{1}{zx} = \frac{-x}{z^2}$$

dérive

$$\Rightarrow \frac{-z'}{z^2} + \frac{1}{zx} = \frac{-x}{z^2}$$

$\cdot z^2$

$$\Rightarrow -z' + \frac{z}{x} = -x$$

EDL1

(c) $y' - \frac{1}{\tan(x)} y = 0$

EDVS

(d) $y'' + e^{x} y' - xy = 3x$

EDL1

(e) $y' = \sin(x)$ EDL1

(8) autre

$$u = x + y$$

$$u' = 1 + y'$$

$$\Rightarrow \quad y' + 1 - 1 = (x+y)^2$$

$$\Rightarrow \quad u' - 1 = u^2$$

$$\Rightarrow \quad \frac{u'}{u^2+1} = 0$$

(9) $xy' - y = y^3$

$\Rightarrow \quad xy' = y^3 + y$

$\Rightarrow \quad x = \dfrac{y^3+y}{y'}$

$\Rightarrow \quad \dfrac{1}{x} = \dfrac{y'}{y(y^2+1)}$ EDVS

(h) $y' + \frac{1}{x} y = e^x$ EDL1

(i) $y'' + \frac{2}{x} y' + y = 0$ EDVS

(j)

$$u = \frac{y}{x}$$

$$u' = \frac{-1}{x^2} \cdot y' +$$

$$z = \frac{1}{y}$$

$$y' = x^2 + 2xy + y^2$$

$$\Rightarrow \frac{y'}{y^2} = \frac{x^2}{y^2} + \frac{2x}{y} + 1$$

$$\Rightarrow -z' = z^2 x^2 + z \cdot 2x + 1$$

$$\frac{y}{x} = u \Rightarrow y = ux$$

$$u' = \frac{y'x - y}{x^2}$$

$y = x^4$

$$e^{2u} + \cancel{y}$$

$y' = u + xu'$

$$= \frac{x^2 u'}{y'x - y}$$

$$= \cancel{y} + xu'$$

due

$$\Leftrightarrow \quad xu' = e^{2u}$$

$$\Leftrightarrow \quad x = \frac{e^{2u}}{u'} \checkmark$$

Exercice 6

(i)

$$y_1' = a(x)(y_1)^2 + b(x)y_1 + c(x)$$

$$\left(y_1 + \frac{1}{v}\right)' = a(x)\left(y_1^2 + 2\frac{y_1}{v} + \frac{1}{v^2}\right)$$
$$+ b(x)\left(y_1 + \frac{1}{v}\right) + c(x)$$

$$\Rightarrow \cancel{a(x)(y_1)^2} + \cancel{b(x)(y_1)} + \cancel{c(x)} \quad \frac{-v'}{v^2}$$
$$= \cancel{a(x)(y_1)^2} + a(x)\left(2\frac{y_1}{v} + \frac{1}{v^2}\right)$$
$$+ \cancel{b(x)(y_1)} + b(x)\left(2\frac{1}{v}\right) + \cancel{c(x)}$$

$$\Rightarrow \quad u' = -a(x)(2y_1u+1) - b(x)(u)$$

$$\Leftrightarrow \quad u' + u(2a(x)y_1 + b) = -a$$

(ii) $y_1 = 1$ et $y_1 = -1$

$$y' = \frac{4-4y^2}{3x}$$

$$y' = \frac{1}{x}\cdot\frac{4}{3} - \frac{4}{3x}y^2$$

$$u' + \left(\frac{-8}{3x}\right)u = \frac{4}{3x}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{8}{3x} u + \frac{4}{3x}$$

$$= \frac{1}{x}\left(\frac{4}{3} + \frac{8}{3} u\right)$$

$$\Rightarrow \int \frac{du}{\frac{8}{3}\left(\frac{1}{2} + u\right)} = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{3}{8} \int \frac{du}{1+2u} = \int \frac{1}{x} dx$$

$$\Rightarrow \quad \ln\left|\frac{1}{2} + u\right| = \frac{8}{3} \ln(x) + C$$

$$\Rightarrow \quad \frac{1}{2} + u = x^{8/3} \cdot C$$

$$\Rightarrow \quad u = x^{8/3} \cdot \underbrace{C}_{>0} - \frac{1}{2}$$

$$= C x^{8/3} - \frac{1}{2}$$

Exercice 7

(i) $\sqrt{8} = 4\sqrt{2}$. Comme 4 est rationnel, alors $\sqrt{8}$ est rationnel ssi $\sqrt{2}$ est rationnel.

Supposons que $\sqrt{2} = \frac{p}{q}$ avec p et q premiers entre eux, $p, q \in \mathbb{N}$, $q \neq 0$.

$$\Rightarrow 2 = \frac{p^2}{q^2}$$

$$\Leftrightarrow 2q^2 = p^2$$

$$\Rightarrow p = 2k' \quad (*)$$

(*) car si $p = 2k' + 1$

$$\Rightarrow p^2 = 4k'^2 + 2k' + 1$$

(impossible)

donc p est pair, donc $q^2 = \frac{4(k')^2}{2}$

$$= 2(k')^2$$

$\Rightarrow$ alors q est pair (*)

donc $\sqrt{2} \neq \frac{p}{q}$, p, q premiers entre eux

donc $\sqrt{8}$ irrationnel.

(ii) $\sqrt{37} + \sqrt{18} = \frac{p}{q}$

$$\Rightarrow \sqrt{37} = \frac{p}{q} - \sqrt{18}$$

$$19 = \frac{p}{q}\left(\sqrt{37} - \sqrt{18}\right)$$

$$= \frac{p}{q}\left(\frac{p}{q} - 2\sqrt{18}\right)$$

$$= \frac{p^2}{q^2} - \frac{2p}{q}\sqrt{18}$$

$$\Rightarrow -\frac{2p}{q}\sqrt{18} = 19 - \frac{p^2}{q^2}$$

$$\Rightarrow \sqrt{18} = -\frac{q^2 19}{2pq} + \frac{p^2}{2pq}$$

$$= \frac{p^2 - q^2 19}{2pq}$$

$$\Rightarrow 3\sqrt{2} = \frac{p^2 - q^2 19}{2pq}$$

rationel
donc on doit rationel
que $\sqrt{2}$ est rationel

$\Rightarrow$ FAUX.

donc $\sqrt{18} - \sqrt{37}$
irrationel

iii) $\lim_{n \to \infty} x_n = x$

$\Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ t.q $\forall n > n_0$:

$$|x_n - x| < \varepsilon$$

Supposons par l'absurde que

$x > b$.

On pose $\varepsilon = \frac{x-b}{2}$. Alors

$\forall \varepsilon > \frac{x-b}{2}, \exists n_1 \in \mathbb{N}$ t.q $\forall n > n_1$:

$$|x_n - x| < \varepsilon$$

(c'est-à-dire que la suite est comprise entre $\frac{x-b}{2}$ et $-\left(\frac{x-b}{2}\right)$)