Exercice 1

(i)

$$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{y}{x^2} = \frac{1}{y} - yx^{-2}$$

$$3^x$$

$$\frac{\partial f}{\partial y} = \frac{1}{x} - \frac{x}{y^2} \qquad e^{\log(3).x}$$

$$\mathcal{D}(f) = \mathbb{R} \setminus \{(x,y) \text{ t.q } xy = 0\}$$

$$\frac{\partial^2 f}{\partial^2 x} = 2yx^{-3}$$

$$\frac{\partial f}{\delta^2 y} = 2xy^{-3}$$

(ii)

$$\frac{\partial f}{\partial x} = y^{z} x^{(y^{z}-1)}$$

$$\frac{\partial f}{\partial y} = \left[e^{\log(x) \cdot y^{z}} \right]'$$

$$= \log(x) \cdot z y^{z-1} x^{y^{z}}$$

$[3^2]$

$2^{[3^2]} \neq [2^3]^2$

$\Rightarrow 2^9 \neq 2^6$

⚠ on ne peut pas considerer $k = x^y$

$$f(u) = x^u$$

$$f'(u) = x^u \log(x)$$

$$u(y,z) = y^z$$

$$u'(y,z) = y^z \log(y)$$

$\left(2^3\right)^2$

2^9

8^2

$\downarrow$

$[2^3]^2$

$$[f(u(y,z))]' = u'(y,z) \cdot f'(u(y,z))$$

$$[g(f(y,z))]' = f'(y,z)\, g'(f(y,z))$$

$$= y^z \log(y) \cdot x^{(y^z)} \log(x)$$

$$= \frac{\partial g}{\partial z}$$

Défine sur $\mathbb{R} \setminus \{(x,y,z) : x \leq 0, y \leq 0\}$

$y^z \in \mathbb{Q} \Rightarrow y^z = \frac{p}{q}$ et $q = 2k+1$, $k \in \mathbb{Z}$

(iii)

$$\frac{\partial f}{\partial x} = 2xy\cos(x^2y)\,ch(y-x)$$
$$-\sin(x^2y)\,sh(y-x)$$

$$ch = \frac{e^{x}+e^{-x}}{2}$$
$$(ch)' = \frac{1}{2}(e^{x}-e^{-x})$$

$$\frac{\partial f}{\partial} = x^2\cos(x^2y)\,ch(y-x)$$

∂y

$$+ \sin(x^2 y)\, sh(y-x)$$

$$\mathcal{D}(f) = \mathbb{R}^2$$

(iv) $$f(x,y,z) = sh\left(\frac{x+3yz}{z-2}\right)^2$$

$$\frac{\partial f}{\partial x} = \frac{2}{z-2} ch\left(\frac{x+3yz}{z-2}\right) sh\left(\frac{x+3yz}{z-2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{6z}{z-2} ch\left(\frac{x+3yz}{z-2}\right) sh\left(\frac{x+3yz}{z-2}\right)$$

$$\frac{\partial f}{\partial z} = 2\left[\frac{3y}{z-2} - \frac{x+3y}{(z-2)^2}\right]$$

$$ch\left(\frac{x+3yz}{z-2}\right) sh\left(\frac{x+3yz}{z-2}\right)$$

$$g(z) = \frac{x+3yz}{z-2}$$

$$g'(z) = \frac{(3y)(y-2) - (x+3yz)}{(z-2)^2}$$

$$= \frac{3y}{z-2} - \frac{x+3yz}{(z-2)^2}$$

Définies sur $\mathbb{R} \setminus \{(x,y,z),\ t.q\ z \neq 2\}$

Exercice 2

(i) ∇f est la matrice qui contient les les dérivées partielles x, y.

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial g}{\partial x} + \frac{\partial h}{\partial x} \\ \frac{\partial (g+h)}{\partial y} \end{pmatrix}$$

(ii) $$\nabla f(x,y) = \begin{pmatrix} \frac{\partial g h + g \partial h}{\partial x} \\ \frac{\partial g h + g \partial h}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial g}{\partial x} h + \frac{\partial h}{\partial y} g \\ \frac{\partial g}{\partial y} h + \frac{\partial h}{\partial y} g \end{pmatrix}$$

$$= (\nabla g) \cdot h + (\nabla h) \cdot g$$

(iii) $$\nabla f(x,y) = \begin{pmatrix} \dfrac{\frac{\partial g}{\partial x} \cdot h - g \cdot \frac{\partial h}{\partial x}}{h^2} \\[2em] \dfrac{\frac{\partial g}{\partial y} \cdot h - g \cdot \frac{\partial h}{\partial y}}{h^2} \end{pmatrix}$$

$$= \frac{1}{h^2} \left(h \cdot \nabla g - g \cdot \nabla h \right)$$

Exercice 5

(i) surface $z = f(x,y)$

au point (x_0, y_0, z_0) où $z_0 = f(x_0, y_0)$

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\frac{\partial f}{\partial x} = 3yx^2 + 2x$$

$$\frac{\partial f}{\partial y} = x^3 + 2y$$

$$z = 3 + 5(x-1) + 3(y-1)$$

$$\Leftrightarrow z = 3 + 5x - 5 + 3y - 3$$

$$\Leftrightarrow\ 5 = 5x + 3y - z$$

(ii) $f(-1,1) = -\log(2)$

$P = (-1, 1, -\log(2))$

$$\frac{\partial f}{\partial x} = y\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right]$$

$$\frac{\partial f}{\partial y} = x\left[\log(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right]$$

$$z = -\log(2)$$
$$+ (\log(2) + 1)(x+1)$$
$$+ (-\log(2) - 1)(y - 1)$$
$$= \cancel{-\log(2)} + \log(2)x + \cancel{\log(2)} + x + 1$$

$$- y\log(2) + \log(2) - y + 1$$

$$= 2 + \log(2)(1 + x - y) + x - y$$

$$= 2 + (x-y)(\log(2)+1)$$

$$\left(g(u(x))\right)'$$

$$=$$

(iii)

$$z = x_0\, g\left(\frac{y_0}{x_0}\right)$$

$$\frac{\partial f}{\partial x} = g\left(\frac{y}{x}\right) + x\left[\frac{\partial g}{\partial x}\left(\frac{y}{x}\right)\cdot\frac{-y}{x^2}\right]$$

$$= g\left(\frac{y}{x}\right) - \frac{y}{x}\,\frac{\partial g}{\partial x}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x \frac{\partial g}{\partial y}\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$= \frac{\partial g}{\partial y}\left(\frac{y}{x}\right)$$

$$z = \cancel{x_0 g\left(\frac{y_0}{x_0}\right)}$$

$$+ \left[g\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0} g'\left(\frac{y_0}{x_0}\right) \right]\left[x - x_0 \right]$$

$$+ \left[g'\left(\frac{y_0}{x_0}\right) \right]\left[y - y_0 \right]$$

$$= -\frac{y_0}{x_0} g'\left(\frac{y_0}{x_0}\right) x + y_0 g'\left(\frac{y_0}{x_0}\right)$$

$$+ g'\left(\frac{y_0}{x_0}\right) y - y_0 g'\left(\frac{y_0}{x_0}\right)$$

$$= \frac{-y_0}{x_0} g'\left(\frac{y_0}{x_0}\right)x + g'\left(\frac{y_0}{x_0}\right)y$$

$$= 0$$

contient $\mathcal{Z}(0,0)$

Exercice 7

①

$$P(n) : \sum_{k=1}^{n} (k \cdot k!) = (n+1)! - 1.$$

Initialisation

P(1) est vérifiée :

$$1 = 2! - 1 = 1$$

Hérédité

$$P(\ell) \text{ vraie } \overset{?}{\Rightarrow} P(\ell+1) \text{ vraie}.$$

$$\sum_{k=1}^{\ell} k \cdot k! = (\ell+1)! - 1$$

$$\sum_{k=1}^{\ell+1} k \cdot k! = \sum_{k=1}^{\ell} k \cdot k! + [\ell+1][\ell+1]!$$

$$= (\ell+1)! - 1 + [\ell+1][\ell+1]!$$

$$= \underline{(\ell+2)! - 1}$$

$$\left[\begin{aligned} (\ell+2)! &= [\ell+2][\ell+1][\ell][\ell-1]\dots \\ &= (\ell+1)!(\ell+1) + (\ell+ \end{aligned} \right]$$

<u>Conclusion</u>

$P(n)$ est vérifiée pour $n=1$ et héréditaire donc vraie $\forall\, n \in \mathbb{N}^*$ par recurrence.

(ii) Soit $a_n = 25^n - 19^n$ $(P(n))$

$$25^n \equiv 1 \ [6] \qquad 19^n \equiv 1 \ [6]$$

Initialisation

$P(1)$:

Hérédité

$P(k)$ vraie $\Rightarrow$ $P(k+1)$ vraie ?

$$25^n - 19^n = 6k$$

$$25^{n+1} - 19^{n+1} =$$

(iii)

Initialisation

On peut composer 8 : (5cts) + (3cts)
9 : 3 x (3cts)
10 : (5cts) + (5cts)

Hérédité

Supposons que P(k), P(k+1), P(k+2) soient vraies:

on peut écrire P(k+3) comme P(k) + 3cts.

Conclusion P(n) est vérifiée pour i = 8, 9, 10 et héréditaire donc vraie $\forall n \geq 8 \in \mathbb{N}$ par recurrence.

Exercice 3

$x = r\cos$

(i) on vérifie que $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2} = 0.$

$$\lim_{(x,y)\to(0,0)} \frac{r^3\cos^2(\theta)\sin(\theta)}{r^2} = r\cos^2\theta\sin\theta$$

(ii)

$$\frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2) - x^2y(2x)}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y^2) - x^2y(2y)}{(x^2+y^2)^2}.$$

$$\lim_{t\to 0} \frac{f(x+t,0) - f(0,0)}{t} = \underline{0}$$

$$\lim_{t\to 0} \frac{f(0,y+t) - f(0,0)}{t} = \underline{0}$$

(iii) ~~dérivable $\Rightarrow \frac{\partial f}{\partial x}$ et $\frac{\partial f}{\partial y}$ continues~~ ✓

fonct° rationnelle
$\Rightarrow$ différentiable

$$\frac{\partial f}{\partial x}(\overline{0,0}) \quad \frac{\partial f}{\partial y}(\overline{0,0})$$

(iv)

$$f(0,0) \qquad (0,0)\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

$$f(x,y) = f(\bar{a}) + L_{\bar{a}}(\bar{x} - \bar{a}) + \varepsilon(x,y)\,\|\bar{x}-a\|$$

$$\lim_{(x,y)\to\bar{a}} \varepsilon(x,y) = 0$$

en 1D $\rightarrow$ différentiable $\Leftrightarrow$ dérivable

$\exists$ applicat° linéaire (DL d'ordre 1)

$\exists$ taux d'accroissement (limite)

$$f(0,0) + (0\ 0)\begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon(x,y)\|\bar{x}\|$$

$$= \varepsilon(x,y)\sqrt{x^2+y^2}$$

$$\Rightarrow f(x,y) = \varepsilon(x,y)\sqrt{x^2+y^2}$$

$$\Leftrightarrow \varepsilon(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$= \frac{x^2 y}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

Soit $x = \frac{1}{k}$, $y = \frac{1}{k}$, $k \to 0$

$$= \frac{\frac{1}{k^3}}{\frac{2}{k^2}\left(\frac{\sqrt{2}}{k}\right)} = \frac{1}{2\sqrt{2}} \neq 0$$

$$\varepsilon(x,y) = \frac{r^3(\cos^2\theta + \sin\theta)}{r^2(r)}$$

$$= \cos^2\theta + \sin\theta$$

v) non, car continue $\Rightarrow$ differentiable

Exercice 4

(1)

$$f(x,y) = \begin{cases} xy\log(x^2+y^2) & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x} = y\log(x^2+y^2) + xy\frac{2x}{x^2+y^2}$$

$$= y\left[\log(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right]$$

elle est continue sauf en $(0,0)$

$\frac{\partial f}{\partial x}(-1,1)$

$= \log(2) + \frac{2}{2}$

$\log(1+x) - x$

$x = r-1$

$$\frac{\partial f}{\partial y} = x\left[\log(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right]$$

$(-1,1)$ elle est continue sauf en $(0,0)$

$$\lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0 \quad \Big| \; \frac{\partial f}{\partial x}(0,0) = 0$$

$$\lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0 \quad \Big| \; \frac{\partial f}{\partial y}(0,0) = 0$$

$$\lim_{r\to 0} r\sin\varphi\left[\log(r^2) + \frac{2\cancel{r^2}\cos^2\varphi}{\cancel{r^2}}\right]$$

$$= \lim_{r\to 0} 2r\log r\sin\varphi + \underbrace{2r\sin\varphi\cos^2\varphi}_{\to 0}$$

$$= \lim_{r\to 0} \underbrace{2(r)(r-1)\sin\varphi}_{\to 0} = 0$$

(symétrie pour y)

$\Rightarrow$ dérivable de classe C^1

(ii) $$\frac{\partial^2 f}{\partial y \partial x} = \log(x^2+y^2) + \frac{2x^2}{x^2+y^2}$$

$$+ y\left(\frac{2y}{x^2+y^2} + \frac{4x(x^2+y^2) - 2x^2(2x)}{(x^2+y^2)^2}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \log(x^2+y^2) + \frac{2y^2}{x^2+y^2}$$

$$+ x\left(\frac{2x}{x^2+y^2} + \frac{4y(x^2+y^2) - 2y^2(2y)}{(x^2+y^2)^2}\right)$$

$$= \log(x^2+y^2) + \frac{2y^2}{x^2+y^2}$$

$$+ \frac{2x^2}{x^2+y^2} + \frac{4x^3y + \cancel{4y^3x} - \cancel{4y^3x}}{(x^2+y^2)^2}$$

$$= \overset{-\infty}{\log(x^2+y^2)} + \overset{0}{\frac{2y^2}{x^2+y^2}} + \overset{2}{\frac{2x^2}{x^2+y^2}}$$

$$+ \overset{0}{\frac{4x^3y}{(x^2+y^2)^2}} = -\infty$$

Les deux sont ègales. On veut prouver qu'elles ne sont pas continues.

(iii) oui car de classe C^1

(iv) dérivée directionnelle :

$$\nabla(\bar{a}) = \nabla(0,0) = (0,0)$$

$$\nabla(\bar{b}) = \nabla(\log(2)+1, -\log(2)-1)$$

$$\mathcal{D}f((0,0), \vec{v})$$

$$= \langle (0,0), (v_1, v_2) \rangle = 0$$

$$\mathcal{D}f((-1,1), \vec{v})$$

$$= \langle (\log(2)+1, -\log(2)-1), (v_1, v_2) \rangle$$

$$= v_1 \log(2) + v_1 - v_2 \log(2) - v_2$$

$$= \log(2)(v_1 - v_2) + v_1 - v_2$$

$$= (1 + \log(2))(v_1 - v_2)$$

Peu importe v_1 et v_2 donneront tjrs 0.

$$(1 + \log(2))(v_1 - v_2) = 0$$

$$\Rightarrow v_1 = v_2$$

$$\pm \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}} \right)$$

Exercice 6

$\boxed{\text{Q01}}$

f dérivable $\Rightarrow$ dérivées directionnelles existent

$\boxed{\text{VRAIE}}$

$\boxed{\text{Q02}}$

Faux

dérivées directionnelles existent $\not\Rightarrow$ f dérivable

$\boxed{Q03}$

Vrai

continuité des dérivées directionnelles
$\Rightarrow$ f dérivable

$\boxed{Q04}$

Vrai.

dérivées partielles existent et continues

$\Rightarrow$ f dérivable

$$\Rightarrow \mathcal{D}((x,y), \vec{v}) = \langle \nabla f(x,y), \vec{v} \rangle$$

continue car ∇ continue

$\boxed{\text{Q05}}$ Faux

$$D\left((x_0, y_0), \vec{v}\right)$$

$$= \langle \nabla f(x_0, y_0) , (v_1, v_2) \rangle$$

$$= v_1 \nabla f(x_0, y_0) + v_2 \nabla f(x_0, y_0)$$

if the gradient is null $\Rightarrow$ v_1, v_2 can be anything.