Exercice 1

$$\bar{f}(u,v) = f(x(u,v), y(u,v))$$

nous on veut $f(x,y)$.

$\Rightarrow$ comme le changement de coordonnées est bijectif :

$$f(x,y) = \bar{f}(u(x,y), v(x,y))$$

$$g(x,y) = (u(x,y), v(x,y))$$

$$f(x,y) = (\bar{f} \circ g)(x,y)$$

$\Rightarrow$ on ne pourra pas trouver de forme explicite pour $f(x,y)$.

mais avec les matrices jacobiennes, on peut trouver le laplacien

Redo exercise 1 (correction)

$$J_g(x,y) = J_g\big(u(x,y), v(x,y)\big) \cdot J\big(u(x,y), v(x,y)\big)$$

$$\nabla f(x,y) = \nabla \bar{f}(u,v) \cdot \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \qquad (\cdot \quad \cdot)$$

$$\nabla f(x,y) = \left(\frac{\partial \bar{f}}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \bar{f}}{\partial v} \cdot \frac{\partial v}{\partial x} , \quad \frac{\partial \bar{f}}{\partial u} \frac{\partial u}{\partial y} \quad \frac{\partial \bar{f}}{\partial v} \cdot \frac{\partial v}{\partial y} \right)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{\partial \bar{f}}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \bar{f}}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\text{''} \right)$$

$u'v + uv'$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \bar{f}}{\partial u}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial \bar{f}}{\partial u} \cdot \frac{\partial^2 u}{\partial x^2}$$

$$+ \frac{\partial}{\partial x}\left(\frac{\partial \bar{f}}{\partial v}\right) \cdot \frac{\partial v}{\partial x} + \frac{\partial \bar{f}}{\partial v} \cdot \frac{\partial^2 v}{\partial x^2}$$

Comment dériver $\frac{\partial \bar{f}}{\partial u}$ par rapport à x ?

on dérive par rapport à u, puis v

→ on ne sait pas faire !

$$\frac{\partial}{\partial x}\left(\frac{\partial \bar{f}}{\partial v}\right) = \frac{\partial^2 f}{\partial u \partial x}$$

on fait

$$\frac{\partial^2 f}{\partial u \partial x} = \frac{\partial^2 f}{\partial u \partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial v}{\partial x}$$

Ce qui donne : $\frac{\partial^2 f}{\partial x^2} =$

$$\left(\frac{\partial^2 f}{\partial u \partial u} \cdot \overset{2}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial v}{\partial x}\right) \cdot \overset{2}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial f}{\partial u} \cdot \cancel{\frac{\partial^2 u}{\partial x^2}}$$

$$\left(\frac{\partial^2 f}{\partial u \partial u} \cdot \overset{2}{\cancel{\frac{\partial u}{\partial x}}} + \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{\partial v}{\partial x}\right) \cdot \overset{1}{\cancel{\frac{\partial v}{\partial x}}} + \frac{\partial f}{\partial v} \cdot \cancel{\frac{\partial^2 v}{\partial x^2}}$$

on connait

$$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial v}{\partial x} = 1$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} = 0$$

Exercice 3

$$\tilde{f} = (f \circ g)(r, \varphi)$$

$$g(r, \varphi) = \begin{pmatrix} x(r,\varphi) \\ y(r,\varphi) \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$\Delta f = \frac{\partial^2 \bar{f}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \bar{f}}{\partial \varphi^2} + \frac{1}{r} \frac{\partial \bar{f}}{\partial r}$$

Exercice 4

$$\begin{cases} -1 \leq \sin\theta \leq 1 \\ \Rightarrow 1 \geq \sin\theta\cos\varphi \geq -1 \\ \Rightarrow \rho \geq \rho\sin\theta\cos\varphi \geq -\rho \end{cases}$$

i))

ii) $$J_G = \begin{pmatrix} \frac{\partial G_1}{\partial \rho} & \frac{\partial G_1}{\partial \theta} & \frac{\partial G_1}{\partial \varphi} \\ \frac{\partial G_2}{\partial \rho} & \frac{\partial G_2}{\partial \theta} & \frac{\partial G_2}{\partial \varphi} \\ \frac{\partial G_3}{\partial \rho} & \frac{\partial G_3}{\partial \theta} & \frac{\partial G_3}{\partial \varphi} \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\varphi & \rho\cos\varphi\cos\theta & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\sin\varphi\cos\theta & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{pmatrix}$$

$$\det(J_G) = 0 \cdot (\text{—})$$

$$+ \rho \sin\theta \cos\varphi \cdot \begin{vmatrix} \sin\theta\cos\varphi & \rho\cos\varphi\cos\theta \\ \cos\theta & -\rho\sin\theta \end{vmatrix}$$

$$- \rho \sin\theta \sin\varphi \cdot \begin{vmatrix} \sin\theta\sin\varphi & \rho\sin\varphi\cos\theta \\ \cos\theta & -\rho\sin\theta \end{vmatrix}$$

$$= -\rho\sin\theta\cos\varphi \left(\sin\theta\cos\varphi\,\rho\sin\theta - \rho\cos\varphi\cos^2\theta\right)$$

$$+ \rho\sin\theta\sin\varphi\left(\sin\theta\sin\varphi\,\rho\sin\theta - \rho\sin\varphi\cos^2\theta\right)$$

(iii)

$$\bar{f} = (f \circ G)(\rho, \theta, \varphi)$$

$$J\bar{f} = \underbrace{J_f(G(\rho,\theta,\varphi)) \cdot J_g(\rho,\theta,\varphi)}$$

$$= \nabla \bar{f} = \left(\frac{\partial \bar{f}}{\partial \rho}, \frac{\partial \bar{f}}{\partial \theta}, \frac{\partial \bar{f}}{\partial \varphi}\right)$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

$$\begin{pmatrix} \sin\theta\cos\varphi & \rho\cos\varphi\cos\theta & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\sin\varphi\cos\theta & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{pmatrix}$$

$$\Rightarrow \quad (J_G)^{-1}\left(\frac{\partial \bar{f}}{\partial \rho}, \frac{\partial \bar{f}}{\partial \theta}, \frac{\partial \bar{f}}{\partial \varphi}\right) = \nabla f$$

$$\Rightarrow \quad (J_G)^{-1}\left(\frac{\partial \bar{f}}{\partial \rho}, 0, 0\right) = \nabla f$$

Exercice 2

(i) $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + 2y^2 \\ y/\sqrt{x} \end{pmatrix}$

$D = \{(x,y) \in \mathbb{R}^2 \text{ t.q } x > 0\}$

$\tilde{D} = \{(u,v) \in \mathbb{R}^2 \text{ t.q } u > 0\}$

$\begin{pmatrix} x^2 + 2y^2 \\ y/\sqrt{x} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$x = \sqrt{-2y^2 + u}$

$\frac{y}{\sqrt{x}} = v \Leftrightarrow y = v\sqrt{x}$

$$x^2 + 2(v - \sqrt{x})^2 = u$$
$$\Rightarrow \quad x^2 + 2v^2 x = u$$
$$\Rightarrow \quad x^2 + 2v^2 x - u = 0$$

$$\Delta = (2v^2)^2 - 4 \cdot 1 \cdot (-u)$$
$$= 4v^4 + 4u$$

$$\Rightarrow \quad x = \frac{-2v^2 \pm \sqrt{4v^4 + 4u}}{2}$$

$$x = -v^2 \pm \sqrt{v^4 + u}$$

$$y = v - \sqrt{-v^2 + \sqrt{v^4 + u}}$$

On veut $x > 0 \Rightarrow$ on garde $+$

$$G(u,v) = \begin{pmatrix} -v^2 + \sqrt{v^4 + u} \\ v - \sqrt{-v^2 + \sqrt{v^4 + u}} \end{pmatrix}$$

$$J \quad \left(\frac{\partial G_1}{} \quad \frac{\partial G_1}{} \right.$$

$$J_{G(u,v)} = \begin{pmatrix} \frac{\partial G_1}{\partial u} & \frac{\partial G_1}{\partial v} \\ \frac{\partial G_2}{\partial u} & \frac{\partial G_2}{\partial v} \end{pmatrix}$$

$$\frac{\partial G_1}{\partial u} = \frac{1}{2\sqrt{v^4+u}}$$

$$\frac{\partial G_2}{\partial u} =$$

$$v \frac{1}{2\sqrt{-v^2+\sqrt{v^4+u}}} \cdot \frac{1}{2\sqrt{v^4+u}}$$

$$f(g(u))$$

$$f(y) = v\sqrt{y}$$

$$g(u) = -v^2 + \sqrt{v^4+u}$$

$$f'(g(u)) \cdot g'(u)$$

$$f'(y) = v \frac{1}{2\sqrt{y}}$$

$$g'(u) = \frac{1}{2\sqrt{v^4+u}}$$

$$= \frac{v}{4\sqrt{-v^2+\sqrt{v^4+u}}\sqrt{v^4+u}}$$

$$\frac{\partial G_2}{\partial v} = \sqrt{-v^2+\sqrt{v^4+u}}$$

$$+ v \left[\frac{1}{2\sqrt{-v^2+\sqrt{v^4+u}}}\right]$$

$$\cdot \left[-2v + \frac{4v^3}{2\sqrt{v^4+u}}\right]$$

$$f(g(v))$$

$$= \sqrt{-v^2+\sqrt{v^4+u}}$$

$$f(v) = \sqrt{-v^2+g(v)}$$

$$f(y) = \sqrt{-v^2+y}$$

$$f'(y) = \frac{1}{2\sqrt{-v^2+y}}$$

$$g'(v) = \frac{4v^3}{2\sqrt{v^4+u}}$$

$$f'(g(v)) \cdot g'(v)$$

$$= \frac{D^2 B}{D} + \frac{-v^2 B}{D} + \frac{v^4}{D}$$

$$= \frac{(-v^2 + B)B - v^2 B + v^4}{D}$$

$$= \frac{-v^2 B + B^2 - v^2 B + v^4}{D}$$

$$= \frac{(-v^2 + B)}{D} = \frac{\left(-v^2 + \sqrt{v^4 + U}\right)^2}{\left(\sqrt{-v^2 + \sqrt{v^4 + U}}\right)\sqrt{v^4 + U}}$$

$$\frac{\partial G_1}{\partial v} = \frac{\partial\left(-v^2 + \sqrt{v^4 + U}\right)}{\partial v}$$

$$= -2v + \frac{4v^3}{2\sqrt{v^4 + U}}$$

$$= \frac{-\cancel{2}v\left(2\sqrt{v^4 + U}\right) + \cancel{4}^{2} v^3}{\cancel{2}\sqrt{v^4 + U}}$$

$$= \frac{-2v\left(\sqrt{v^4 + U} - v^2\right)}{\sqrt{v^4 + U}}$$

$$J_G(u,v) =$$

$$\begin{pmatrix} \dfrac{1}{2\sqrt{v^4+u}} & \dfrac{-2v\left(\sqrt{v^4+u}-v^2\right)}{\sqrt{v^4+u}} \\ \dfrac{v}{4\sqrt{-v^2+\sqrt{v^4+u}}\sqrt{v^4+u}} & \dfrac{\left(-v^2+\sqrt{v^4+u}\right)^2}{\left(\sqrt{-v^2+\sqrt{v^4+u}}\right)\sqrt{v^4+u}} \end{pmatrix}$$

$$\sqrt{v^4+u} = \sqrt{\frac{y^4}{x^2}+x^2+2y^2}$$

$$= \sqrt{\frac{y^4+x^4+2y^2x^2}{x^2}}$$

$$= \sqrt{\frac{(x^2+y^2)^2}{x^2}}$$

$$= \frac{x^2+y^2}{x^2}$$

$$\Rightarrow \frac{x^2}{x^2+y^2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a = \frac{1}{2}$$

$$b = -2 \frac{y}{\sqrt{x}} \left(- \frac{y^2}{x} + \frac{x^2+y^2}{x^2} \right)$$

$$= -2 \frac{y}{\sqrt{x}} \left(\frac{-y^2 x + x^2 + y^2}{x^2} \right)$$

$$=$$

(ii) $J_H = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix}$

$$= \begin{pmatrix} 2x & 4y \\ \frac{-\frac{y}{2\sqrt{x}}}{x} & \frac{1}{\sqrt{x}} \end{pmatrix}$$

$$= \begin{pmatrix} 2x & 4y \\ -\frac{y}{2x^{3/2}} & \frac{1}{\sqrt{x}} \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(J_H) = ad - bc$

$$= \frac{2x}{\sqrt{x}} + \frac{4y^2}{2x^{3/2}}$$

$$= \frac{1}{2(x^2+y^2)} \left(\; - \; - \; - \; \right)$$

C'est la même matrice. Pourquoi ?

$$J_H(\bar{a}) \cdot J_G(H(\bar{a})) = I$$

meilleure approximation linéaire de la fonction

$$\text{"}\simeq\text{"} \quad H(\bar{a}) \cdot G(H(\bar{a}))$$
$$\text{"}\simeq\text{"} \quad H(\bar{a}) \cdot H^{-1}(H(\bar{a})) = I$$

Exercice 6

Initialisation

$$f_0 = \frac{1}{2}\left(f_0 b + b f_0 \right)$$

$$= 0$$

Hérédité

$P(n,0)$ est vraie.

$$f_n = \frac{1}{2}\left(f_n b + \ell_n f_0 \right)$$

$$= \frac{1}{2} f_n b$$

$$f_{n+1} = \frac{1}{2} f_{n+1} b \quad \checkmark \quad \text{car } b = 2$$

$$P(n,0) \Rightarrow P(n+1, 0)$$

$\forall m,n \quad P(m,n) \Rightarrow P(m,n+1)$

$$f_{m+n} = \frac{1}{2}\left(f_m l_n + l_m f_n\right)$$

$$f_{m+n-1} = \frac{1}{2}\left(f_m l_{n-1} + l_m f_{n-1}\right)$$

$$f_{m+n+1} = \frac{1}{2}\left(f_m l_{n+1} + l_m f_{n+1}\right) \, ?$$

$$= \frac{1}{2}\left(f_m\left(l_n + l_{n-1}\right) + l_m\left(f_n + f_{n-1}\right)\right)$$

$$= \frac{1}{2}\left(f_m l_n + f_m l_{n-1} + l_m f_n + l_m f_{n-1}\right)$$

$$= f_{m+n} + f_{m+n-1} \quad \checkmark$$

<u>Conclusion</u> $P(n)$ vérifiée $\forall n,m \in \mathbb{N}^*$ par récurrence.

Exercice 7

(i)

$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n$

Initialisation

$P(0,0)$

$$f_1 = f_1 \cdot f_1 + f_0 f_0 = 1 \checkmark$$

Hérédité

$P(m,0)$ vraie.

$\Rightarrow P(m+1,0)$

$$f_{m+2} = f_{m+2} f_1 + f_{m+1} f_0$$
$$= f_{m+2} \checkmark$$

$f_{m+n+1} = f_{m+1} f_{n+1} + f_m f_n$

$$\forall m,n \quad P(m,n) \Rightarrow P(m,n+1)$$

$$P(m,n) \Rightarrow P(m,n+1)$$

$$f_{m+(n+1)+1}$$
$$= f_{m+1} f_{n+2} + f_m f_{n+1}$$
$$= f_{m+1}(f_{n+1} + f_n) + f_m f_{n+1}$$
$$= f_{m+1}(f_{n+1} + f_n) + f_m f_n$$
$$+ f_m f_{n-1}$$
$$= f_{m+1} f_{n+1} + f_{m+1} f_n$$

$+ f_m f_n + f_m f_{n-1}$

$= f_{m+n+1} + f_{m+n}$ définition de f

$= f_{m+n+2}$

(ii)

$$f_{(t+1)k} = f_{tk+1} f_k + f_{tk} f_{k-1}$$

Initialisation

$P(0)$ $\qquad f_k = f_1 f_k + f_0 f_{k-1} = f_k$

Hérédité

$P(t)$ vraie.

On veut démontrer $P(t+1)$, c'est-à-dire

$$f_{(t+2)k} = f_{tk+k+1} f_k + f_{tk+k} f_{k-1}$$

$$= f_{tk+1} f_{k+1} + f_{tk} f_k$$

$$f_{(t+2)k} = f_{(t+1)k+1} f_k + f_{(t+1)k} f_{k-1}$$

$$\iff f_{(t+2)k} = f_{tk+k+1} f_k + f_{tk+k} f_{k-1}$$

$$= \left(f_{tk+1} f_{k+1} + f_{tk} f_k\right) f_k + f_{tk+k} f_{k-1}$$

$=$

Exercise 5

(i)

(i) $g(1,1) = (1,1)$

$$\frac{\partial g_1}{\partial x} = 2x$$

$$\frac{\partial g_1}{\partial y} = -2y$$

$$\int 2x = x^2 + c_1$$

$$\int -2y = -y^2 + c_2$$

$$\int 2y = 2yx + c_3$$

$$\int 2x = 2yx + c_4$$

$g_1(x,y) = (x^2 - y^2 + 1,\ 2xy - 1)$

$\Rightarrow c_1 = 1$
$c_2 = -1$

localement bijective : ex x^2 partout sauf en $(0,0)$

$g_1(-1,2) = (-2, -5)$

on veut $(x,y) \Leftrightarrow (w,z)$

il faut trouver sur quel intervalle g_1 ou g_2 est bijective

$\Rightarrow$ le déterminant de la Jacobienne doit être différent de zéro

$\det(J_g(x,y)) = 4x^2 + 4y^2$

ii

$$\int y^2 \, dx = xy^2 + c_1$$

$$\int 4x \, dx = 2x^2 + c_2$$

$$\int 2xy \, dy = xy^2 + c_3$$

$$\int -1 \, dy = -y + c_4$$

$$g_2(x,y) = (xy^2, 2x^2 - y)$$

$$g_2(-1,2) = (-4, 2-2)$$

$$= (-4, 0)$$