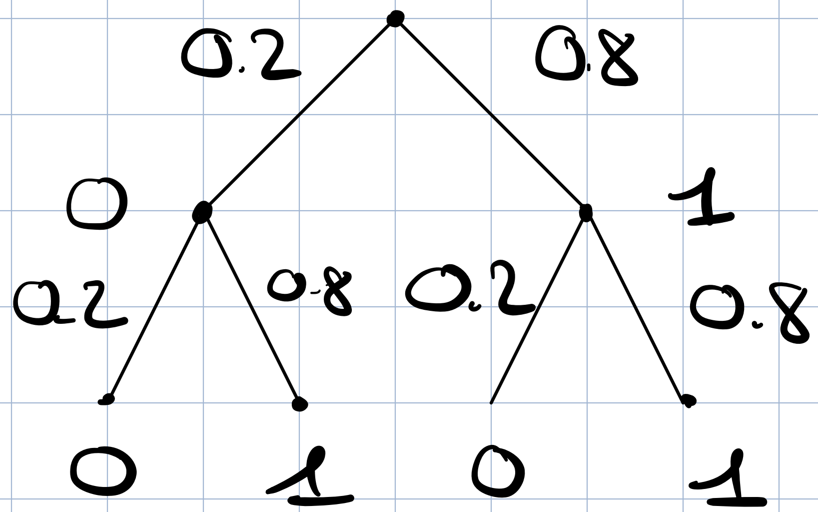
Exercise 4.1

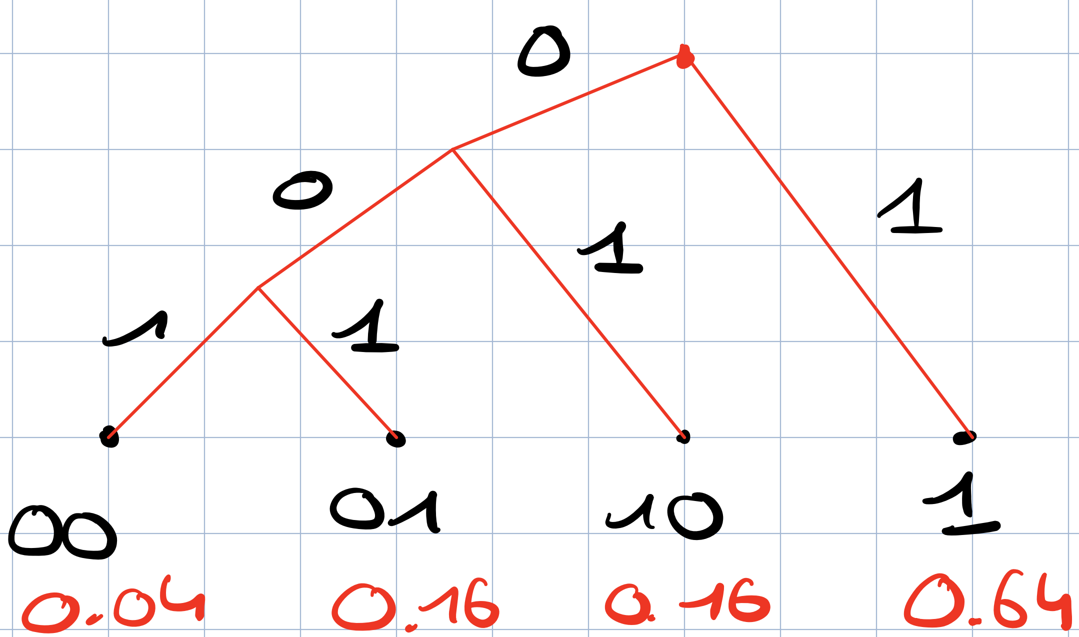
(1) $H(S) = \frac{1}{5} \cdot \log(5) + \frac{4}{5} \log\left(\frac{5}{4}\right)$

$= 0.72$

(2) Average length : 1

(3)

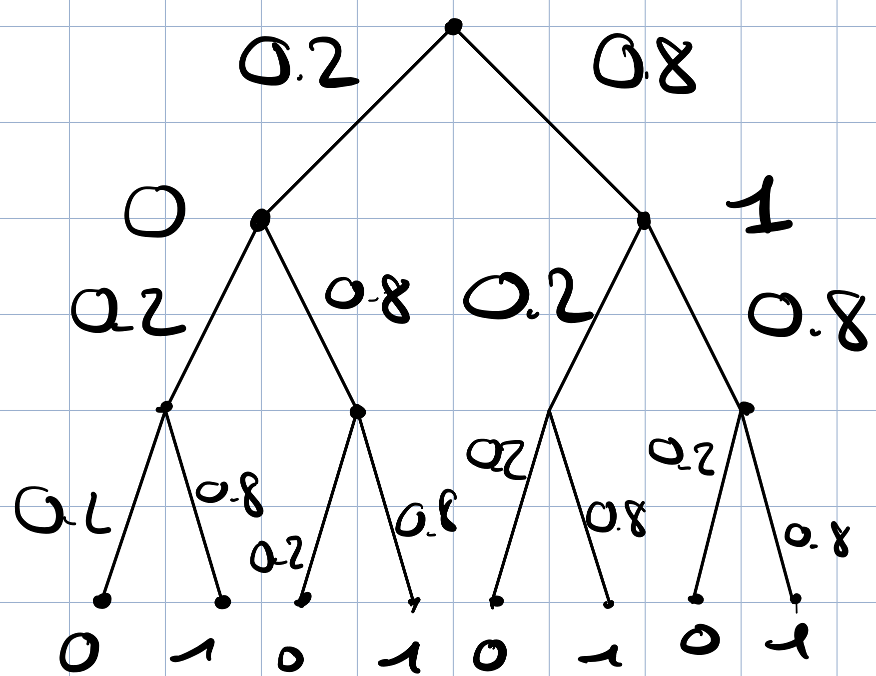


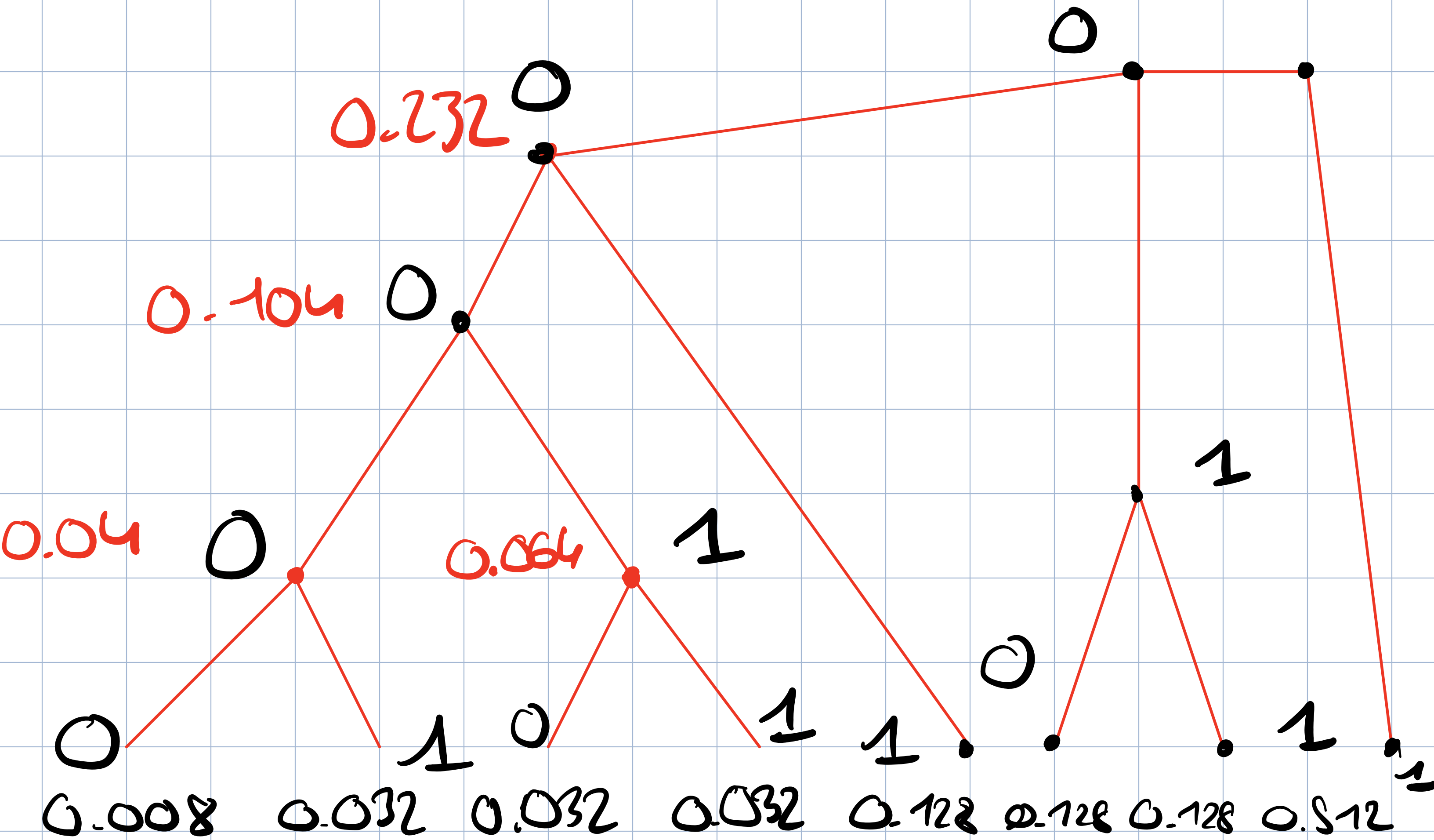


$$\left[\begin{array}{l} \quad 3 \cdot 0.04 \\ + \ 3 \cdot 0.16 \\ + \ 2 \cdot 0.16 \\ + \ 0.64 \end{array} \right] \cdot \frac{1}{2} = \underline{0.78}$$

$$H(S^2) = H(S) + H(S) = 1.44$$

4

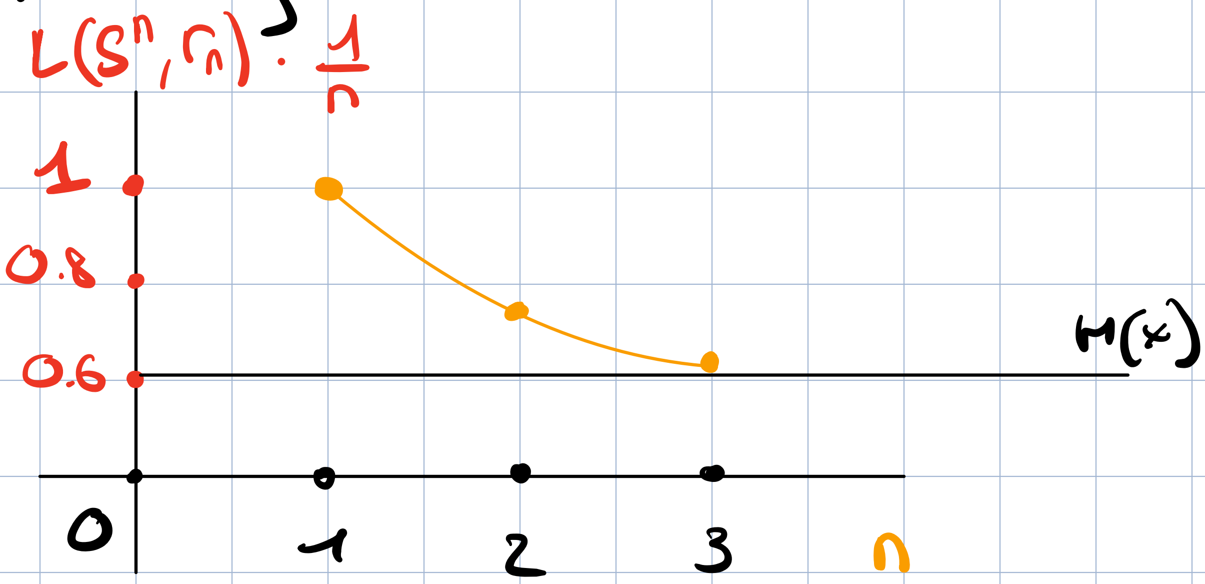




$$2.184 = L(S^3, \Gamma_3)$$

$$\simeq 0.728 \text{ avg}$$

S



Problem 4.2

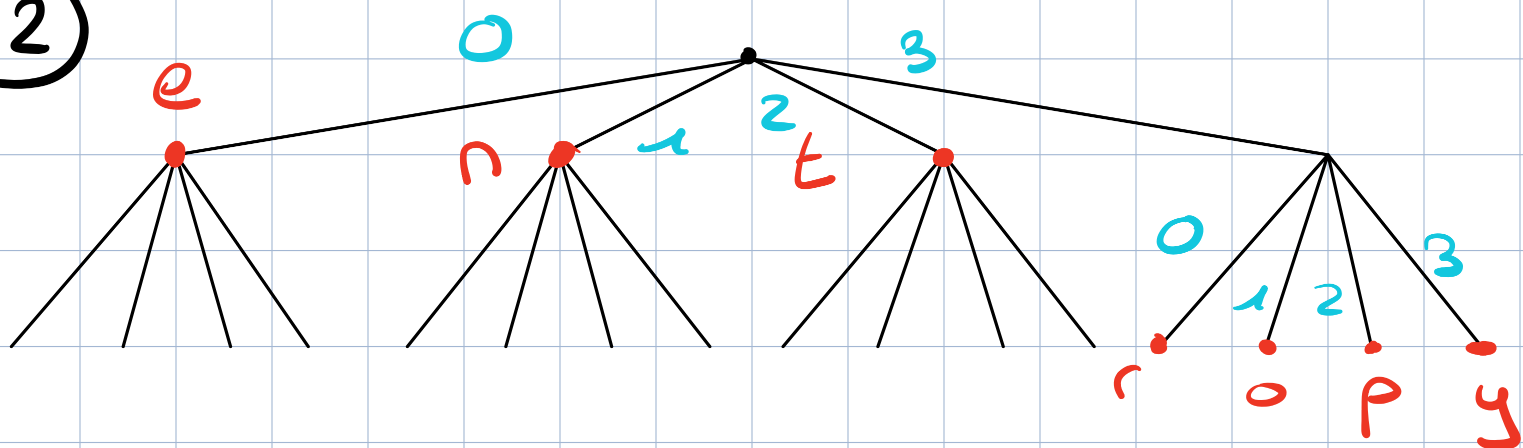
(1) $$\sum_{s \in S} D^{\ell(r(s))} \leqslant 1$$

$D = 2$ imp. $\frac{1}{2} + \frac{1}{2} + \dots > 1$

$D = 3$ imp. $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \dots > 1$

$D = 4$ $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ ✓

(2)



(3)

s	e	n	t	r	o	p	y
$\lvert\Gamma(s)\rvert$	1	1	1	2	2	2	2

.32 .3 .3 .02 .02 .02 .02

.04 .04

Problem 4.3

(1) $m = 3$ $\{0,1,2\}$

$H(Z) = 0$ (we know X, we know Y)

(2) $\{0,1,2,3\}$

$P(Z=1) = 1/4 \cdot 1/4 \cdot 4 = \frac{1}{4}$

$Z = 1$ $m = 4$

$\Rightarrow X + Y = 5$ $\begin{cases} x=3 & y=2 \\ y=2 & x=3 \end{cases}$

$X + Y = 1$ $\begin{cases} x=0 & y=1 \\ x=1 & y=0 \end{cases}$

(3) $m = 5$

$X + Y = 6$ $\begin{cases} x=4 & y=2 \\ x=3 & y=3 \\ x=2 & y=4 \end{cases}$

$x + y = 1$ $\begin{cases} x = 1 & y = 0 \\ x = 0 & y = 1 \end{cases}$

m possibilities (from 0 to 4, Y will compensate).

$P(Z=1) = m \cdot \frac{1}{m} \cdot \frac{1}{m} = \frac{1}{m}$

$= P(z)$

④ $H(z) = \frac{1}{m} \cdot \log(m) \cdot m = \log(m)$

⑤ $H(Z/Y) = H(Z)$ même nombre de possibilités

Therefore Z indep. of Y?

Problem 4.4

(1) $H(S_n \mid S_0 \dots S_{n-1})$

$$H(S_n \mid S_{n-1} = 0) = \log(2)$$
$$H(S_n \mid S_{n-1} = 1) = 0$$

(2) Basis Step

$$p_{S_0}(0) = 0$$
$$= \frac{1}{3}(2-1) = 1.$$

Inductive Step

Let's assume $p_{S_n}(0) = \frac{1}{3}\left(2 + \frac{1}{(-2)^n}\right)$

$$ps_{n+1}(0) = ps_n(0) \cdot \frac{1}{2} + ps_n(1) \cdot 1$$

$p(0|0)$

$p(0|1)$

$$= \frac{1}{3}\left(2 + \frac{1}{(-2)^n}\right)\frac{1}{2} + \left(1 - \frac{1}{3}\left(2 + \frac{1}{(-2)^n}\right)\right)$$

$$= \frac{1}{3} + \frac{1}{(-2)^n 3} \cdot \frac{1}{2} + 1 - \frac{2}{3} - \frac{1}{3(-2)^n}$$

$$= \frac{-1}{3} + 1 + \frac{1}{(-2)^{n+1}}$$

$$= \frac{1}{3}\left(2 + \frac{1}{(-2)^{n+1}}\right)$$

Conclusion : it works

3) $H(S_n | S_{n-1})$

$$= H(S_n | S_{n-1} = 0)\, p_{S_{n-1}}(0) + H(S_n | S_{n-1} = 1)\, p_{S_{n-1}}(1)$$

$$= \log(2) \cdot \frac{1}{3}\left(2 + \frac{1}{(-2)^n}\right)$$

$$+ 0 \cdot (\_)$$

$$H^*(S) = \lim_{n\to\infty} H(S_n | S_1 \dots S_{n-1})$$

$$= \lim_{n\to\infty} H(S_n | S_{n-1})$$

$$= \log(2) \cdot \frac{2}{3}$$

✓ exists

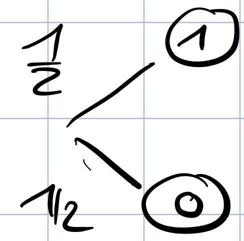
$$\lim_{n\to\infty} H(S_n) = -p_{S_n}(0) \log(p_{S_n}(0)) - (1 - p_{S_n}(0)) \log(1 - p_{S_n}(0))$$

$$= \frac{2}{3} \log\left(\frac{3}{2}\right) + \frac{1}{3} \log(3)$$

$$= \log(3) - \frac{2}{3}\log(2)$$

④ $C_k = S_{I(k)+1}$

After a zero,



$$p(C_k=0) = \overbrace{p(\overset{C_k}{S_{I(k)+1}}=0 \mid S_{I(k)}=0)}^{\boxed{\frac{1}{2}}}\ \overbrace{p(S_{I(k)}=0)}^{\boxed{1}}$$

$$+ \underbrace{p(\overset{C_k}{S_{I(k)+1}}=0 \mid S_{I}(k)=1)}_{\frac{1}{2}}\ \underbrace{p(S_{I(k)}=1)}_{\boxed{0}}$$

$$= \frac{1}{2}$$

⑤ $p_{C_k} = p_{S_{I(k-1)+1}}$

$$p_{C_k \mid C_1 \dots C_{k-1}} = p_{S_{I(k)+1} \mid S_0, \dots \mid S_{I(k-1)+1}, \dots}$$

$= \begin{cases} 1^{er} \text{ cas : si } S_{I(k-1)+1} = \boxed{S_{I(k)}} \text{ et} \\ \text{alors on a déjà l'info, c'est 0 !} \\ \\ 2^{ème} \text{ cas : si } S_{I(k-1)+1} \neq S_{I(k)}, \\ \text{alors on s'en fiche, } S_{I(k-1)+1} \text{ indep de } S_{I(k)} \end{cases}$

⑥ $$H^*(C) = \lim_{n\to\infty} H(C_k \mid C_1 \dots C_{k-1})$$

$$= \lim_{n\to\infty} H(C_k)$$

$$= 1$$

⑦

$\cancel{S_1 : 1101110 1}$

$\cancel{S_2 : 110110 11}$

$C : 11$

actually impossible strings

$S_3 : 000010100$

$S_4 : 0100$

$C : 00110$

yes

000010100

01001001

0010000100010

01001001

Problem 4.3

① $H(Z)=0$ (soit le log, soit la proba sera nulle)

② $A = \{0, 1, 2, 3\}$

$$P(Z=1) = P(X=3, Y=2) + P(X=2, Y=3) + P(X=1, Y=0) + P(X=0, Y=1)$$
$$= p(X=3)\cdot p(Y=2) + \dots$$

Soit $\bar{x}$ le nombre qu'il manque à ajouter à x pour avoir le bon z.

$$= \sum_x p_X(Y=\bar{x}, X=x)$$
$$= \sum_x p_Y(\bar{x})\cdot p_X(x)$$
$$= \frac{1}{4}\underbrace{\sum_x p_Y(\bar{x})}_{1} = \frac{1}{4}$$

③ for every z, there will be a matching y for every x to arrive at the result.
Therefore: $P_Z(z) = \frac{1}{m}$

④ $H(Z) = \frac{1}{m} \cdot \log(m) \cdot m = \log(m)$

⑤ $H(Z/Y) = H(Z)$. As we said, knowing Y will not modify any probability, as X is uniform, we can still get all the results with a probability $\frac{1}{m}$.

Therefore Z and Y are independent.