Exercice 1

(i) $\frac{\partial f}{\partial x} = -\sin(x)$

$\frac{\partial f}{\partial y} = 6y$

$\frac{\partial f}{\partial x} = 0 \Rightarrow x = \pi k \, , \, k \in \mathbb{Z}$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$

$\frac{\partial^2 f}{\partial x^2} = -\cos(x)$ $\qquad$ $\frac{\partial^2 f}{\partial y \partial x} = 0$

$\frac{\partial^2 f}{\partial x \partial y} = 0$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = 6$

$$\mathrm{Hess}\, f = \begin{pmatrix} -\cos(x) & 0 \\ 0 & 6 \end{pmatrix} = A$$

$$\det A = \begin{cases} -6 \,, & x = 2k\pi \\ 6 \,, & x = (2k+1)\pi \end{cases}$$

$x = 2k\pi \,, \; y = 0$

$\hookrightarrow \det A < 0 \rightarrow$ pco drehenm

$x = (2k+1)\pi, \; y = 0 \,, \; k \in \mathbb{Z}$

$\hookrightarrow \det A > 0, \; r > 0 \Rightarrow \min$

(ii)

$$\frac{\partial f}{\partial x} = 3x^2 + 2x + 2y$$

$$\frac{\partial f}{\partial y} = -3y^2 + 2x + 2y$$

$$3x^2 + \cancel{2x} + \cancel{2y} = -3y^2 + \cancel{2x} + \cancel{2y} = 0$$

$$\Leftrightarrow \quad 3(x^2 + y^2) = 0$$

$$\Rightarrow \quad x = 0, \ y = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 6x + 2 \qquad \frac{\partial^2 f}{\partial y \partial x} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = -6y + 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 2$$

$$\frac{\partial^3 f}{\partial x^3} = 6$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 0$$

$x x^2$

$y x^2$

x^2

$y x$

$x y^2$

$y y^2$

$$H = \begin{pmatrix} 6x+2 & 2 \\ 2 & -6y+2 \end{pmatrix}$$

$$H(0,0) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\det H = 0$$

⚠ $f(x,-x)$

$$= x^3 + x^3 + x^2 - 2x^2 + x^2$$
$$= 2x^3$$

on "trouve" une direction dans laquelle la fonction se comporte comme x^3, donc pas de min/max

(iii) $$\frac{\partial f}{\partial x} = -6x + y^2 = 0$$

$$\frac{\partial f}{\partial y} = 2xy - 4y^3 = 0$$

$$= 2y(x - 4y^2) = 0$$

$$\begin{cases} 2y = 0 \Rightarrow 0 \ ? \\ x = 4y^2 \quad ? \end{cases}$$

$$\cancel{-6(4y^2) + y^2 = 0}$$

$$\Rightarrow \quad y = 0, \quad x = 0$$

$$H = \begin{pmatrix} -6 & 2y \\ 2y & 2x - 12y^2 \end{pmatrix}$$

$$\det H = -12x + 68y^2$$

$$\Rightarrow (0,0) = 0$$

(iii) $$-(\underbrace{3x^2 - xy^2 + y^4}_{?})$$

$$3x^2 - xy^2 + y^4$$

$$= \frac{x^2+y^4}{2} \geq |xy^2| \qquad \frac{a+b}{2} > \sqrt{ab}$$

$$\Rightarrow x^2+y^4 \geq |xy^2|$$

$$\Rightarrow x^2+y^4+2x^2 \geq |xy^2|$$

$$\Rightarrow x^2+y^4+2x^2 - xy^2 \geq 0$$

$$\Rightarrow f(x,y) \leq 0 \Rightarrow \text{max (absolu)}$$

$$f(x,y) = -\frac{5}{2}x^2 - \frac{1}{2}x^2 + xy^2 - y^4$$

$$= \frac{-5}{2}x^2 - \left(\frac{1}{2}x^2 - xy^2 + y^4\right)$$

$$= \frac{-5}{2}x^2 - \left(\frac{1}{2}x + y^2\right)^2$$

$f(x,y) \leq 0$

Exercice 2

(i) On cherche les points stationnaires

$$\frac{\partial f}{\partial x} = -4x + 4y$$

$$\frac{\partial f}{\partial y} = -10y + 4x + 2z$$

$$\frac{\partial f}{\partial z} = -2z + 2y$$

(1) $x = y$

(2) $-10x + 4x + 2z = 0$

$\Leftrightarrow \quad -6x = -2z$

$\Leftrightarrow \quad z = 3x$

(3) $-6x + 2x = 0 \Rightarrow x = 0$

$\Rightarrow y = 0$

$\Rightarrow z = 0$

$(0,0,0)$

$$\frac{\partial^2 f}{\partial x^2} = -4 \qquad \frac{\partial^2 f}{\partial y \partial x} = 4 \qquad \frac{\partial^2 f}{\partial z \partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = -10 \qquad \frac{\partial^2 f}{\partial z \partial y} = 2$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = 2 \qquad \frac{\partial^2 f}{\partial z^2} = -2$$

$$\begin{pmatrix} -4 & 4 & 0 \\ 4 & -10 & 2 \\ 0 & 2 & -2 \end{pmatrix} = H$$

$$\det \Delta_3 = \det H$$

$$\Delta_2 = \det \begin{pmatrix} -4 & 4 \\ 4 & -10 \end{pmatrix}$$

$$\Delta_1 = -4$$

$\Delta_1 < 0$

$\Delta_2 > 0$

$$\Delta_3 = -4 \begin{vmatrix} -10 & 2 \\ 2 & -2 \end{vmatrix} \quad (-4 \cdot 16)$$

$$+ 4 \begin{vmatrix} 4 & 0 \\ 2 & -2 \end{vmatrix} \quad (-4 \cdot 8)$$

$$= -96 < 0$$

$\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$

max local

ii

$$\frac{\partial f}{\partial x} = 4x - 3z^2$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3 \Rightarrow y^2 = 1$$
$$\Rightarrow y = \pm 1$$

$$\frac{\partial f}{\partial z} = -6xz + 6z = 6z(-x+1)$$

$$\Rightarrow z = 0$$
(alors $x = 0$)

$$\Rightarrow x = 1$$
$$\left(\text{alors } z = \pm \frac{2}{\sqrt{3}}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial x \partial z} = -6z$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 6y \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$\frac{\partial^2 f}{\partial z \partial x} = -6z \qquad \frac{\partial^2 f}{\partial z \partial y} = 0 \qquad \frac{\partial^2 f}{\partial z^2} = -6x+6$$

$$\begin{pmatrix} 4 & 0 & -6z \\ 0 & 6y & 0 \\ -6z & 0 & -6x+6 \end{pmatrix}$$

$$\Delta_1 = 4$$
$$\Delta_2 = 24y$$

$$\Delta_3 = 4(6y)(-6x+6)$$
$$-6z(-6y(-6z))$$

$$= 24y(-6x+6)$$

$$- 36\cdot 6 z^2 y$$

$$= 144y\left(-x+1+\frac{3}{2}z^2\right)$$

$\Delta_2 > 0$

$\Delta_3 > 0$

$\Delta_2 < 0$

$\Delta_3 < 0$

$H(0,1,0)$ $H(0,-1,0)$ $\Delta_2 > 0$ $\Delta_3 > 0$

$H\left(1,1,\frac{-2}{\sqrt{3}}\right)$ $H\left(1,1,\frac{2}{\sqrt{3}}\right)$

$H\left(1,-1,\frac{-2}{\sqrt{3}}\right)$ $H\left(1,-1,\frac{2}{\sqrt{3}}\right)$

$\Delta_2 > 0$

$\Delta_3 > 0$

$\Delta_2 < 0$

$\Delta_3 < 0$

$\Delta_2 < 0$

$\Delta_3 < 0$

nothing

max

Exercice 3

(i)

$$\frac{\partial f}{\partial x} = 2x - y - 1$$

$$\frac{\partial f}{\partial y} = -x + 2y - 1$$

points stationnaires :

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x - y - 1 = 0$$

$$\Leftrightarrow 2(-1+2y) - y - 1 = 0$$

$$\Leftrightarrow -2 + 4y - y - 1 = 0$$

$$\Leftrightarrow 3y - 3 = 0$$

$$\Leftrightarrow y = 1$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -x + 2y - 1 = 0$$

$$\Leftrightarrow x = -1 + 2y$$

$$\Leftrightarrow x = 1$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x} = -1 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$\det(H) > 0$, $r > 0 \Rightarrow$ min local

$f(1,1) = 1 - 1 + 1 - 1 - 1 = -1$

$f(0,0) = 0$

$\cancel{f(3,0) = 9 - 0 - 3 =}$ trop dur

$$\frac{3}{2}\cdot\frac{3}{2}=\frac{9}{4}$$

$$\overbrace{(x+y)^2}^{3^2} - \overbrace{3xy}^{0 \le \cdot - \le \frac{9}{4}\cdot 3} - (\underbrace{x+y}_{3})$$

$$x^2 + 2xy - y^2 - 3xy - x - y \checkmark$$

$$3^2 - 3 = 6$$

$$6 \le (x+y)^2 - (x+y) \le 6$$

$$6 - \frac{21}{4} \le \underline{\qquad\qquad} \le 6$$

max $(3,0)$, $(0,3)$
min $(1,1)$

(ii)

$$\frac{\partial f}{\partial x} = 4x - y - 6$$

$$\frac{\partial f}{\partial y} = -x + 4y - 6$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 4x - y - 6 = 0$$

$$\Rightarrow y = -6 + 4x$$

$$\Rightarrow y = -6 + 8 = 2$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 4y = 6 + x$$

$$\Leftrightarrow 4(-6 + 4x) = 6 + x$$

$$\Leftrightarrow -24 + 16x = 6 + x$$

$$\Leftrightarrow 15x = 30$$

$$\Leftrightarrow x = 2$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x} = -1 \qquad \frac{\partial^2 f}{\partial y^2} = 4$$

$$H = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

$\det(H) > 0 \qquad 4 > 0 \Rightarrow$ min local

$$f(x, \sqrt{32 - x^2})$$

$$\cancel{2\underbrace{(x^2+y^2)}_{32} - xy - 6(x+y)}$$

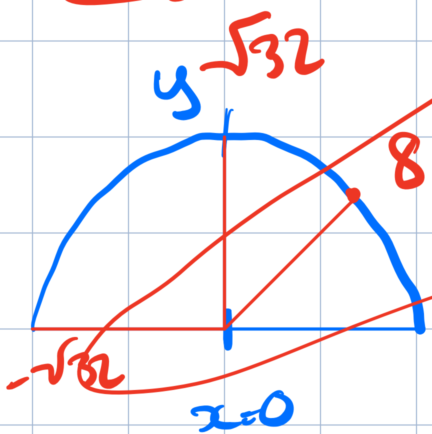
$$\cancel{g(x,y) = 64 - xy - 6x - 6y}$$

$$\frac{\partial g}{\partial x} = -y-6 \Rightarrow y = -6$$

$$\frac{\partial g}{\partial y} = -x-6 \Rightarrow x = -6$$

$$2\underbrace{(x^2+y^2)}_{32} - xy - 6(x+y)$$

$x+y$



$$xy \begin{cases} \text{max} & \sqrt{16}=4,\ \sqrt{16}=4 \Rightarrow 16 \\ \text{min} & -\sqrt{16}=4,\ -\sqrt{16}=4 \Rightarrow -16 \end{cases}$$

$$x+y \begin{cases} \text{max} & 8 \\ \text{min} & x=-\sqrt{32} \quad y=0 \end{cases}$$

$$f(-\sqrt{32},0) = 64 + 6\sqrt{32} \approx 97$$

$$f(-4,4) = 64+16-6(0) = 80$$

$$f(4,4) = 64-16-6\cdot 8$$
$$= 64-48-16 = 0$$

$$64 \leq 2(x^2+y^2) \leq 64$$

$$\Leftrightarrow \quad 64-16 \leq 2(x^2+y^2)-xy \leq 64+16$$

$$\Leftrightarrow \quad 64-16-6(8) \leq \_\_\_\_\_\_ \leq 64+16+6\cdot 8$$

$=$

$$f(2,2) = 2\cdot 4-4+2\cdot 4-24-24$$
$$= 8-4+8-48 = 12-48$$
$$= 36$$

$(4,-4)$

$f(-4,4)$ max

$f(4,4)$ min

$f(2,2)$ nothing

Exercice 4

$$\frac{\partial f}{\partial x} = 0 \Rightarrow z = -1$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \text{chicken attack}$$

z détermine la croissance de x

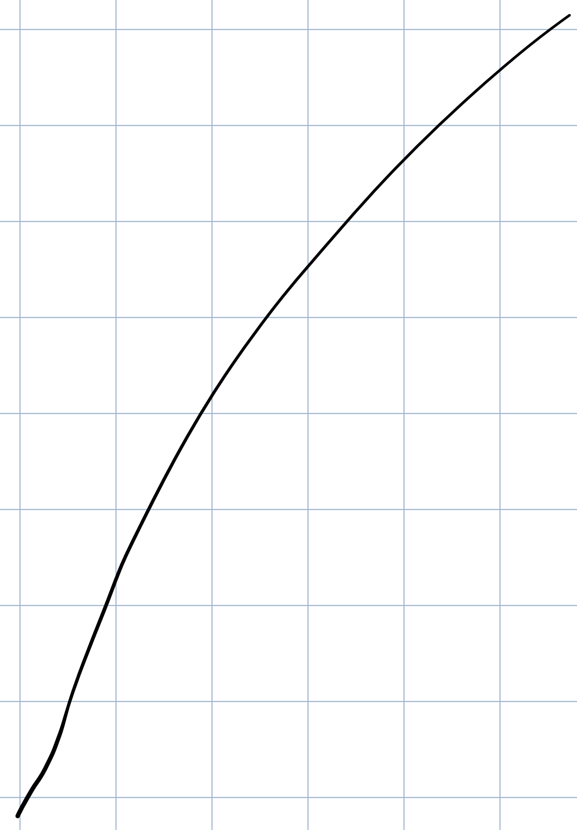
$$f(x,y,z) = xz + x - y + 2z + C$$

$$f(0,0,0) = C \Rightarrow C = 3$$

$$f(x,y,z) = z(x+2) + x - y + 3$$

max est qd $y = 0$, $x = a, y = b$

min est qd $y = b$, $x = 0$, $z = 0$



Exercice 3

$d_1 d_2 d_3$

$d_1 = y_0$
$d_2 = x_0$

$y_0 x_0$

$$x + y - a = 0$$

$$d(P, r) = \frac{(x_0 + y_0 - a)}{\sqrt{2}}$$

$$x_0 y_0 \left(\frac{|x_0+y_0-a|}{\sqrt{2}}\right) \Rightarrow -x^2y_0 - x_0y_0^2 + ax_0y_0$$

car $a > x_0, y_0$

$$= \frac{|x_0^2y_0 + x_0y_0^2 + ax_0y_0|}{\sqrt{2}}$$

$$f(x,y) = -x^2y - xy^2 + axy$$

$$\frac{\partial f}{\partial x} = -2xy - y^2 + ay$$

$$\frac{\partial f}{\partial y} = -x^2 - 2xy + ax$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow -y^2 - 2xy + ay = 0$$

$$\Leftrightarrow y(-y-2x+a) = 0$$

$$\Rightarrow y = 0 \text{ ou } -y-2x+a = 0$$

$$\Rightarrow -y = 2x - a$$

$$\Rightarrow y = -2x + a$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -x^2 - 2xy + ax = 0$$

$$\Rightarrow x = 0 \text{ ou } x = -2y + a$$

$$= -2(-2x+a)$$

$+a$

$= 4x - 2a + a$

$\Rightarrow 3x = a$

$\Rightarrow x = \frac{1}{3}a$

$y = \frac{-2}{3}a + a = \frac{1}{3}a$

Exercice 6

①

$$2x^3 - x^2y^4 + 2y^3 + 3x - 2 = 0$$

qd $x = 0$, alors $y = 1$

$$\left.\frac{\partial f}{\partial y}\right|_{x=0,\, y=1} = -4x^2y^3 + 6y^2 = 6y^2$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0,\, y=1} = 6x^2 - 2xy^4 + 3 = 3$$

$$f'(x) = -\frac{3}{6y^2} = -\frac{1}{2}$$

(ii) $y + 2 = 0 \Rightarrow y = -2$

$$\frac{\partial f}{\partial x} = e^y + ye^x \Big|_{\substack{x=0 \\ y=-2}} e^{-2} - 2$$

$$\frac{\partial f}{\partial y} = xe^y + e^x \Big|_{\substack{x=0 \\ y=-2}} 1$$

$$f'(0) = -\left(\frac{e^{-2} - 2}{1}\right)$$

$$= -e^{-2} + 2$$

Exercice 7

(i) $\mathrm{Tr}(AB)$ c'est la somme des valeurs propres de AB

AB a les mêmes valeurs que BA

(ii)

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$ABC = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

$$CA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BAC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercice 8

trace $= r+t$ (par def)

det prod valeurs pr $\Rightarrow > 0$

$rt - s^2 > 0$

donc r et t doivent avoir le même signe (négatif)

donc $r < 0$