Exercice 1

(i) $y'' - 5y' + 6y = (3x^2+1)\cos(x)$

$\lambda^2 - 5\lambda + 6 = 0$ $\qquad \lambda^2 - 5\lambda + p$

$\Delta = 25 - 24 = 1$

$\lambda = \dfrac{5 \pm 1}{2} = 3$ et 2

$y_{hom}(x) = C_1 e^{3x} + C_2 e^{2x}$

$a = 0$, $b = 1$, $n = 2$, $m = 0$

$y_{part}(x) = T_N(x)\cos(x) + S_N(x)\sin(x)$

$= (Ax^2 + Bx + C)\cos(x)$

$+ (Dx^2 + Ex + F)\sin(x)$

(ii) $y'' - 5y' + 6y = e^{3x}$

$y_{hom}(x) = C_1 e^{3x} + C_2 e^{2x}$

$c = 3$, $n = 0$, $m = 0$

↳ racine

$y_{part}(x) = Axe^{3x}$

(iii) $y'' - 3y' + 2y = xe^{3x} + 2x^2$

$\lambda^2 - 3y + 2 = 0$

$\lambda = 1 \quad \lambda = 2$

$y_{hom}(x) = C_1 e^{x} + C_2 e^{2x}$

$c = 3$, $n = 1$, $a = 0$, $b = 0$

$y_{part_1}(x) = (Ax + B)e^{3x}$

$$y_{part2}(x) = (Cx^2 + Dx + E)$$

$$y_{part}(x) = y_{part1}(x) + y_{part2}(x)$$

(iv) $c = 2$

$$y_{part1}(x) = Ax\, e^{2x}$$
$$y_{part2}(x) = e^{2x}(B\cos(x) + C\sin(x))$$

$$y_{part}(x) = y_{part1}(x) + y_{part2}(x)$$
$$= Axe^{2x} + e^{2x}\left[B\cos(x) + C\sin(x)\right]$$

$$y_{part}(x)' = Ae^{2x} + 2Axe^{2x}$$
$$+ 2e^{2x}\left[B\cos(x) + C\sin(x)\right]$$
$$+ e^{2x}\left[-B\sin(x) + C\cos(x)\right]$$

$$y_{part}''(x) = 2Ae^{2x} + 2Ae^{2x}$$

$$+ \; 4Ax\,e^{2x}$$
$$+ \; 4e^{2x}\left[B\cos(x) + C\sin(x)\right]$$
$$+ \; 2e^{2x}\left[-B\sin(x) + C\cos(x)\right]$$
$$+ \; 2e^{2x}\left[-B\sin(x) + C\cos(x)\right]$$
$$+ \; e^{2x}\left[-B\cos(x) - C\sin(x)\right]$$

$$(4A + 4Ax)$$
$$= e^{2x}\left(\cancel{2A + 2A + 4Ax}\right)$$

$$3e^{2x}$$
$$+ \left[B\cos(x) + C\sin(x)\right]\left(\cancel{4e^{2x} - 2e^{2x}}\right)$$
$$+ \left[-B\sin(x) + C\cos(x)\right]\left(4e^{2x}\right)$$

$$y_{part}''(x) - 3y'(x)$$

$$= Ae^{2x} - 2Axe^{2x}$$

$$- 3e^{2x}\left[B\cos(x) + C\sin(x)\right]$$
$$+ e^{2x}\left[-B\sin(x) + C\cos(x)\right]$$

$$y_{part}''(x) - 3y_{part}'(x) + 2y_{part}(x)$$

$$= Ae^{2x} + e^{2x}\left(-B\cos(x) - C\sin(x)\right.$$
$$\left. - B\sin(x) + C\cos(x)\right]$$

$$A = 1, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}$$

$$y_{part}(x) = Axe^{2x} + e^{2x}\left[B\cos(x) + C\sin(x)\right]$$

$$= xe^{2x} - \frac{1}{2}e^{2x}\cos(x) - \frac{1}{2}e^{2x}\sin(x)$$

$$y(x) = C_1 e^{x} + C_2 e^{2x}$$

$$+ xe^{2x} - \frac{1}{2}e^{2x}\cos(x) - \frac{1}{2}e^{2x}\sin(x)$$

$$y'(x) = C_1 e^{x} + 2C_2 e^{2x} + e^{2x} + 2xe^{2x} - e^{2x}\cos(x) + \frac{1}{2}e^{2x}\sin(x) - e^{2x}\sin(x) - \frac{1}{2}e^{2x}\cos(x)$$

$$y'(0) = \frac{1}{2} \Rightarrow C_1 + 2C_2 + 1 - 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow C_1 + 2C_2 = 1$$

$$y(0) = c_1 + c_2 - \frac{1}{2} = -\frac{1}{2}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & -1 \end{array}\right)$$

$$\Rightarrow y(x) = -e^{x} + e^{2x}$$

$$+ x e^{2x} - \frac{1}{2} e^{2x} \cos(x)$$

$$- \frac{1}{2} e^{2x} \sin(x)$$

(v) $y'' - 4y' + 4y = e^{x}\cos(x) - e^{x}\sin(x)$
$= e^{x}(\cos(x) - 1\sin(x))$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = 2$

$y_{hom}(x) = C_1 e^{2x} + C_2 e^{2x}$

$a = 1 \ , b = 1$

$\Rightarrow y_{part}(x) = e^{x}(A\cos(x) + B\sin(x))$

(vi) $f(x) = 3xe^{-2x} - e^{2x}$ $\lambda = -2$

$R_n(x) = 3x \ , n = 1$ | $R_n'(x) = -1$

$c = -2$ | $n = 0$

$c = 2$

$$y_{part}(x) = Ce^{2x} + (Ax+B)e^{-2x}x^2$$

Exercice 2

(i) $y' + y = \tan(x)$

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0 \quad \lambda = i \;, \lambda = -i$$

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

<u>variations de cste :</u>

$$y(x) = C_1(x)\cos(x) + C_2(x)\sin(x)$$

$$\begin{aligned} y'(x) = {} & C_1'(x)\cos(x) - C_1(x)\sin(x) \\ & + C_2'(x)\sin(x) + C_2(x)\cos(x) \end{aligned}$$

$$= 0$$

$$\begin{aligned} y''(x) = {} & -C_1'(x)\sin(x) \\ & - C_1(x)\cos(x) \\ & + C_2'(x)\cos(x) \\ & - C_2(x)\sin(x) \end{aligned}$$

$$\begin{cases} -C_1'(x)\sin(x) + C_2'(x)\cos(x) = \tan(x) \\ C_1'(x)\cos(x) + C_2'(x)\sin(x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -C_1'(x)\sin(x)\cos(x) + C_2'(x)\cos^2(x) = \sin(x) \\ C_1'(x)\sin(x)\cos(x) + C_2'(x)\sin^2(x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} C_2'(x)\sin^2(x) + C_2'(x)\cos^2(x) = \sin(x) \\ C_2'(x)\sin^2(x) = -C_1'(x)\sin(x)\cos(x \end{cases}$$

$$\int C_2'(x) = \int \sin(x)$$

$$\Rightarrow C_2(x) = -\cos(x)$$

$$\sin^2(x)$$

$$=$$

$$\Rightarrow \int C_1'(x) = \int \frac{-\sin^2(x)}{\cos(x)}$$

$$\Rightarrow C_1(x) = \int \frac{\cos^2(x) - 1}{\cos(x)}\,dx$$

$$= \int \cos(x) - \frac{1}{\cos(x)}\,dx$$

$$= \int \cos(x)\,dx - \int \frac{1}{\cos(x)}\,dx$$

$$\int \frac{1}{\cos(x)}\,dx = \int \frac{\cos(x)}{\cos^2(x)}\,dx$$

$$= \int \frac{\cos(x)}{1-\sin^2(x)}\,dx$$

$\sin(x) = u$
$dx\cos(x) = du$

$$= \int \frac{1}{1-u^2}\,du$$

$$= \int \frac{1}{(1-u)(1+u)}\,du$$

$$= \int \frac{1}{1-u} + \frac{1}{1+u}\,du$$

$$= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du$$

$$= \frac{1}{2}\left[-\ln|1-u| + \ln|1+u|\right]$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin(x)}{1-\sin(x)}\right|$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin(x))^2}{1-\sin^2(x)}\right|$$

$$= \frac{1}{2}\ln\left|\frac{[1+\sin(x)]^2}{\cos^2(x)}\right|$$

$$= \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$$

$$C_1(x) = \int \cos(x)\,dx - \int \frac{1}{\cos(x)}\,dx$$

$$= \sin(x) - \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$$

$$y(x) = C_1(x)\cos(x) + C_2(x)\sin(x)$$

$$= \left[\sin(x) - \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|\right]\cos(x)$$

$$- \cos(x)\sin(x)$$

$$= -\cos(x)\ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$$

(ii) en $\cos(x)=0$, $1+\sin(x)=0$, la fonction n'est plus définie

$$\Rightarrow x \in \frac{\pi}{2} + k\pi$$

$$\in \frac{3\pi}{2} + 2k\pi$$

$$\lim_{x \to \frac{\pi}{2}} -\cos(x) \ln\left|\frac{1+\sin(x)}{\cos(x)}\right|$$

$$= \lim_{x \to \frac{\pi}{2}} -\cos(x)\left[\ln|1+\sin(x)| - \ln|\cos(x)|\right]$$

$$= \lim_{x \to \frac{\pi}{2}} -\cos(x) \ln|1+\sin(x)| + \cos(x)\ln|\cos(x)|$$

$$\frac{\ln(\cos(x))}{\frac{1}{\cos(x)}} \overset{BH}{=} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\sin(x)}{\cos^2(x)}}$$

$$= \cos(x)$$

$$= 0$$

$$\lim_{x \to \frac{3\pi}{2}} -\cos(x)\left[\ln|1+\sin(x)| - \ln|\cos(x)|\right]$$

$$= -\cos(x) \ln(1+\sin(x)) + \cos(x)\ln(x)$$

$$\frac{-\ln(1+\sin(x))}{\frac{1}{\cos(x)}} \overset{BH}{=} \frac{\frac{-\cos(x)}{1+\sin(x)}}{\frac{\sin(x)}{\cos(x)^2}}$$

* on peut factoriser par sin.

$$= {}^{*}\ \frac{-\cos^3(x)}{\underbrace{\sin(x)}_{-1} + \underbrace{\sin^2(x)}_{1}}$$

$\frac{3\pi}{2}$

$$= \frac{3\sin(x)\cos^2(x)}{\cos(x) - 2\sin(x)\cos(x)}$$

$$= \frac{3\cos(x)\cos^2(x) - 3\sin^2(x)\cos(x)}{\underbrace{-\sin(x)}_{-1} - 2\cos^2(x) + \underbrace{2\sin^2(x)}_{2}}$$

3

$$= 0$$

Exercice 3

(i)

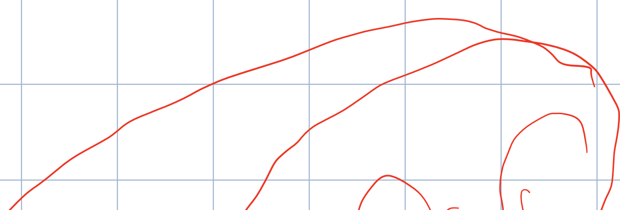
$$x^2 y'' + 3xy' + y = 2 + x^2$$

$$z = y(e^t) \qquad (g \circ f)'(x) = g'(f(x))\cdot f'(x)$$

$$z' = [y(e^t)]' = y'(e^t)\cdot e^t$$

$$z'' = [y'(e^t)\cdot e^t]' = e^t[y''(e^t)\cdot e^t] + y'(e^t)\cdot e^t$$
$$= e^{2t} y''(e^t) + y'(e^t)\cdot e^t$$

$$e^{2t}\cdot y''(e^t) \qquad z''$$
$$+ 3e^t \cdot y' \qquad +$$
$$+ y \qquad + z$$
$$= 2 + e^{2t}$$



$$x^2 y'' + 3xy' + y = 2 + x^2$$

$y(x)$

$$z = y(e^t)$$
$$z' = y'(e^t)\cdot e^t$$
$$z'' = y''(e^t)\cdot e^{2t} + y'(e^t)\cdot e^t$$

$y(t) \Rightarrow z$

$x = e^t$

$\frac{dx}{dt} \to z'$

$\frac{\frac{dx}{dt}}{\frac{dx}{dt}}$ e^t

$$x(t) = e^t$$

$$y' = \frac{dy}{dx} = \boxed{\frac{dy}{dt}}\cdot\frac{dt}{dx} = z'\cdot\frac{1}{e^t}$$

$$y'' = \frac{d^2y}{dx^2} = z''\cdot\frac{1}{e^{2t}} - z'\frac{1}{e^{2t}}$$
$$= [z'' - z']e^{-2t}$$

$$\cancel{e^{2t}}[z''-z']\cancel{e^{-2t}} + 3\cancel{e^t}\cdot z'\cdot\frac{1}{\cancel{e^t}}$$
$$+ z = 2 + e^{2t}$$

$$= z' + 2z' + z = 2 + e^{2t}$$

$\lambda^2 + 2\lambda + 1 = 0$ $\qquad \Delta = 2^2 - 4 \cdot 1 \cdot 1$

$= 0$

$\lambda = -1$

$z(t) = C_1 e^{-t} + C_2 t e^{-t}$

$2 + e^{2t}$

$c = 0$, $R_n(x) = 2$

$a = 2$, $b = 0$, $P_k(x) = 1$

$\Rightarrow z_{part}(t) = A + e^{2t} B$

$z_{part}'(t) = 2Be^{2t}$

$z_{part}''(t) = 4Be^{2t}$

$9B - 1 = 0$

$\Rightarrow B = \frac{1}{9}$

$6Be^{2t} + A + e^{2t}B = 2 + e^{2t}$

$\Leftrightarrow (9B - 1)e^{2t} + A = 2$

$$\Rightarrow y_{part} = \frac{1}{9}e^{2t} + 2$$

$$z(t) = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{9}e^{2t} + 2$$

$$z'(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} + \frac{2}{9}e^{2t}$$

$$z''(t) = C_1 e^{-t} - C_2 e^{-t} - C_2 e^{-t} + C_2 t e^{-t} + \frac{4}{9}e^{2t}$$

$$z''(t) + 2z'(t) + z$$

$$= \cancel{C_1 e^{-t}} \cancel{- 2C_2 e^{-t}} + \cancel{C_2 t e^{-t}} + \frac{4}{9}e^{2t} \cancel{- 2C_1 e^{-t}} + \cancel{2C_2 e^{-t}} \cancel{- 2C_2 t e^{-t}} + \frac{4}{9}e^{2t} + \cancel{C_1 e^{-t}} + \cancel{C_2 t e^{-t}} + \frac{1}{9}e^{2t} + 2$$

$$= 2 + e^{2t}$$

$= e^{2t} + 2$ $\qquad C_1, C_2 \in \mathbb{R}.$

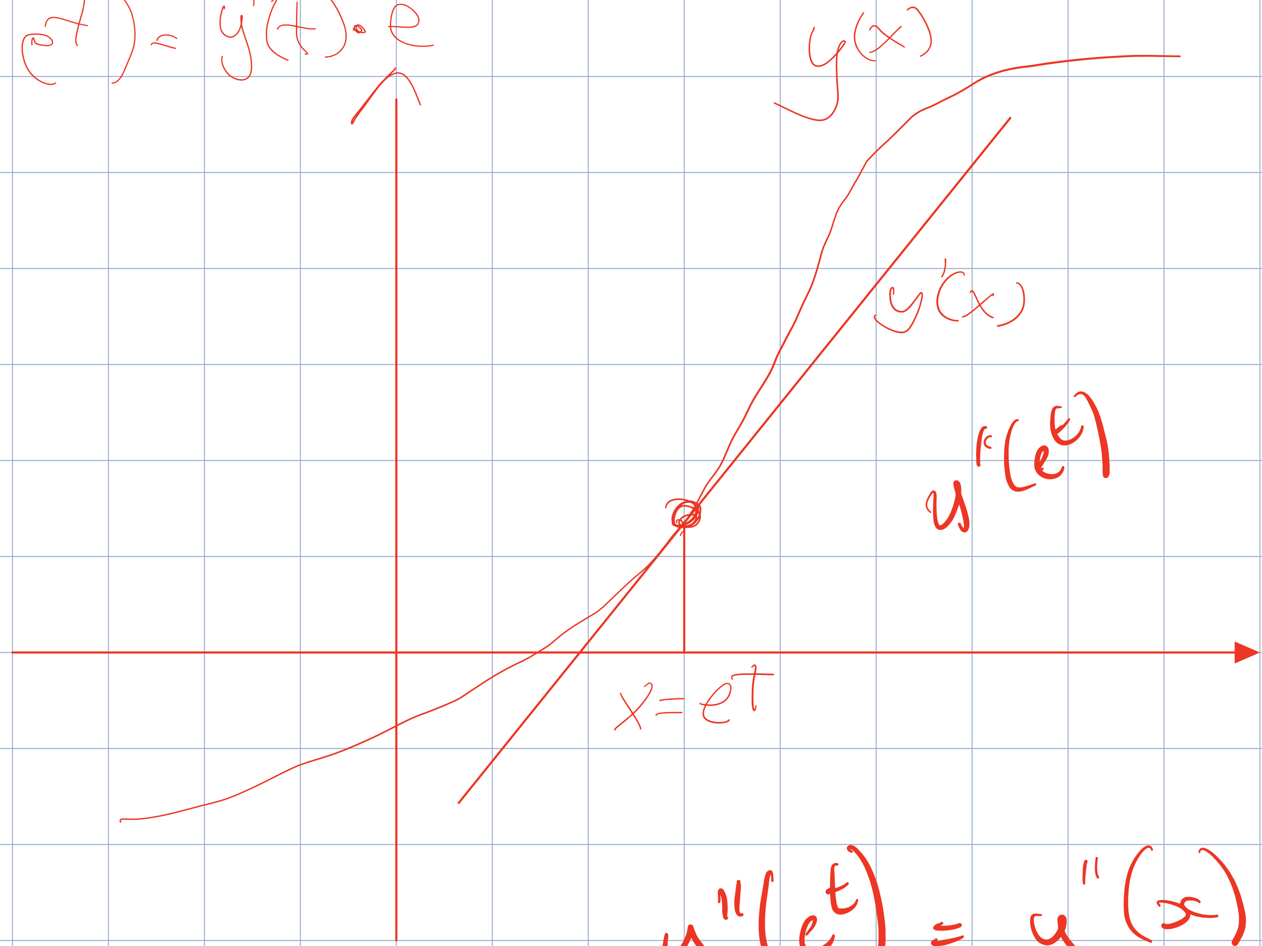
$\underline{y'(e^t)} = \underline{y'(x)}$

$y(x) = y(e^t)$

$y'(e^t)$

$y'(t) =$

$y'(e^t) = y'(t) \cdot e^t$



$y''(e^t) = y''(x)$

$e^{2t} y'' + 3e^t y' + y = 2 + e^{2t}$

$y = y(e^t) = y(x)$

$z(t) = y(e^t)$

et ensuite on dérive z!

ii $v_1(x) = \frac{1}{x}$

$$v_1'(x) = \frac{-1}{x^2}$$

$$v_1''(x) = - \frac{-2x}{x^4} = \frac{2}{x^3}$$

$$x^2 \cdot \frac{2}{x^3} + 3x \cdot \frac{-1}{x^2} + \frac{1}{x}$$

$$= \frac{2}{x} - \frac{3}{x} + \frac{1}{x} = 0$$

trouver une deuxième solution linéairement indépendante :

$$v_2(x) = v_1(x) \int \frac{e^{-P(x)}}{v_1^2(x)} dx$$

$p(x) = \frac{3}{x}$ $\qquad P(x) = 3 \ln |x|$

$$= \frac{1}{x} \int e^{-3\ln|x|} x^2 dx$$

$$= \frac{1}{x} \int x^{-3} x^2 dx$$

$$= \frac{1}{x} \int \frac{1}{x} dx$$

$$= \frac{\ln(x)}{x}$$

$$y(x) = \frac{C_1}{x} + \frac{C_2 \ln(x)}{x}, \quad x > 0$$

On calcule le Wronskien :

$$W = \det \begin{pmatrix} 1/x & \ln(x)/x \\ -1/x^2 & \frac{1}{x^2} - \frac{\ln(x)}{x^2} \end{pmatrix}$$

$$= \frac{1}{x} \cdot \left(\frac{1}{x^2} - \frac{\ln(x)}{x^2} \right) - \frac{\ln(x)}{x} \left(\frac{-1}{x^2} \right)$$

$$= \frac{1}{x^3} - \frac{\ln(x)}{x^3} + \frac{\ln(x)}{x^3} = \frac{1}{x^3}$$

$$c_1(x) = -\int \frac{f(x)\, v_2(x)}{W}$$

$$= -\int \frac{\frac{[2+x^2]}{x^2}\, \frac{\ln(x)}{x}}{\frac{1}{x^3}}$$

$$= -\int [2+x^2]\ln(x)\,\cancel{x^2}$$

$$= -\int [2+x^2]\ln(x)\,dx$$

$$= -\int 2\ln(x)\,dx - \int x^2\ln(x)\,dx$$

$$= -2\left(x\ln(x) - x\right)$$

$$-\left(\frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3\,\frac{1}{x}\right)$$

$$= -2\left(x\ln(x) - x\right) - \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3$$

$$c_2(x) = \int \frac{f(x)\,u_1(x)}{W}$$

$$= \int \frac{\left(\frac{2+x^2}{x^2}\right)\frac{1}{x}}{\frac{1}{x^3}} = \int (2+x^2)\,\cancel{x^2}$$

$$= \int 2+x^2\,dx$$

$$= 2x + \frac{1}{3}x^3$$

Exercice 4

ouvert

(i) Soit $\bar{y} = (y_1, y_2, \dots y_n) \in E$

$B(\bar{y},$

Fermé

(ii) $CE = \{x \in \mathbb{R}^n, n \geq 1, \|x\| > \pi\}$

$\forall \bar{y} \in CE \Rightarrow$ la boule $\bar{B}\left(\bar{y}, \frac{1}{2}[\|y\| - \pi]\right)$

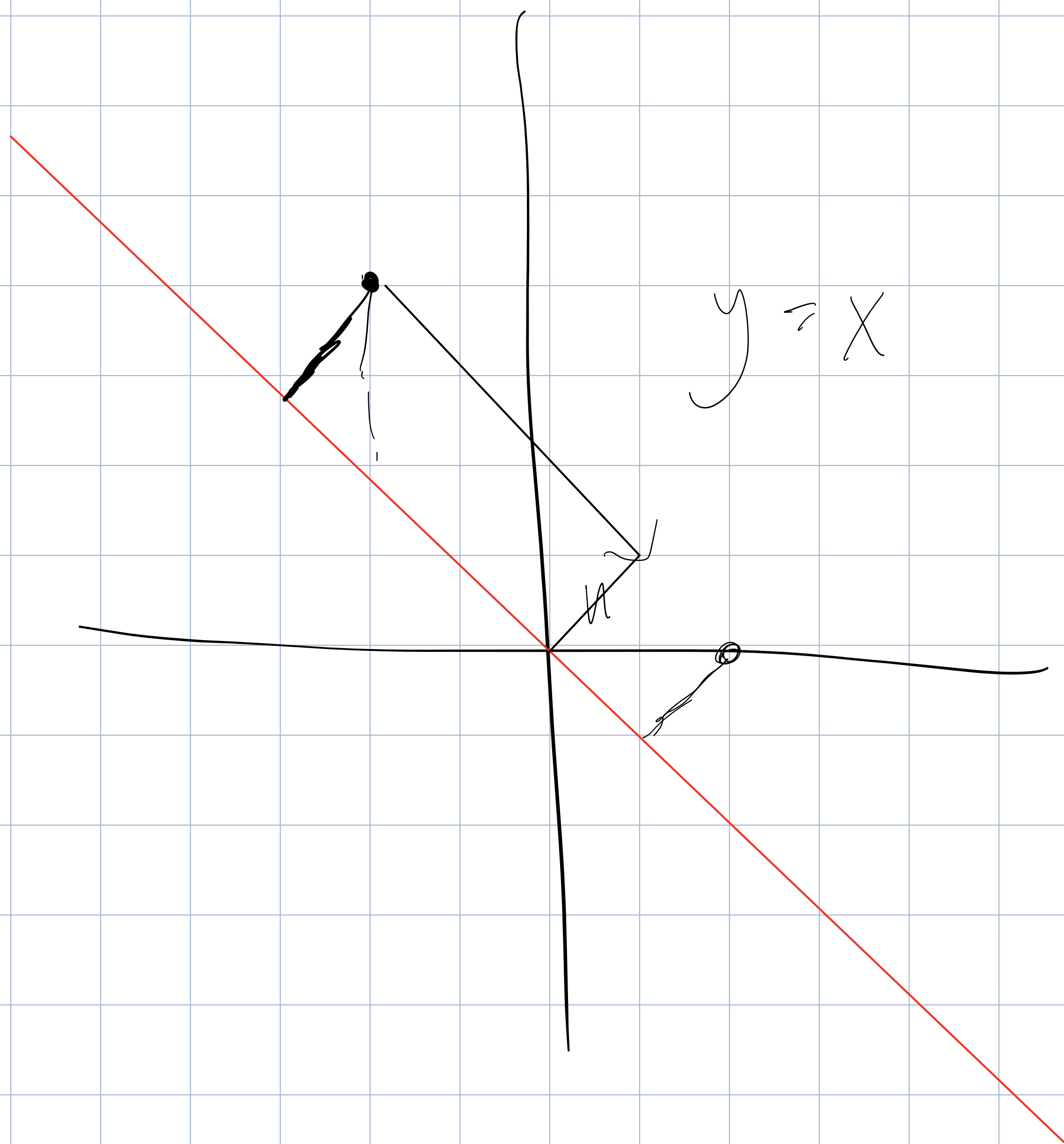
(iii) $CE = \{x \in \mathbb{R}^n : x_1 \leq 0 \cup x_2 = 0, \dots x_n\}$

$F \cup F$

$\Rightarrow F$

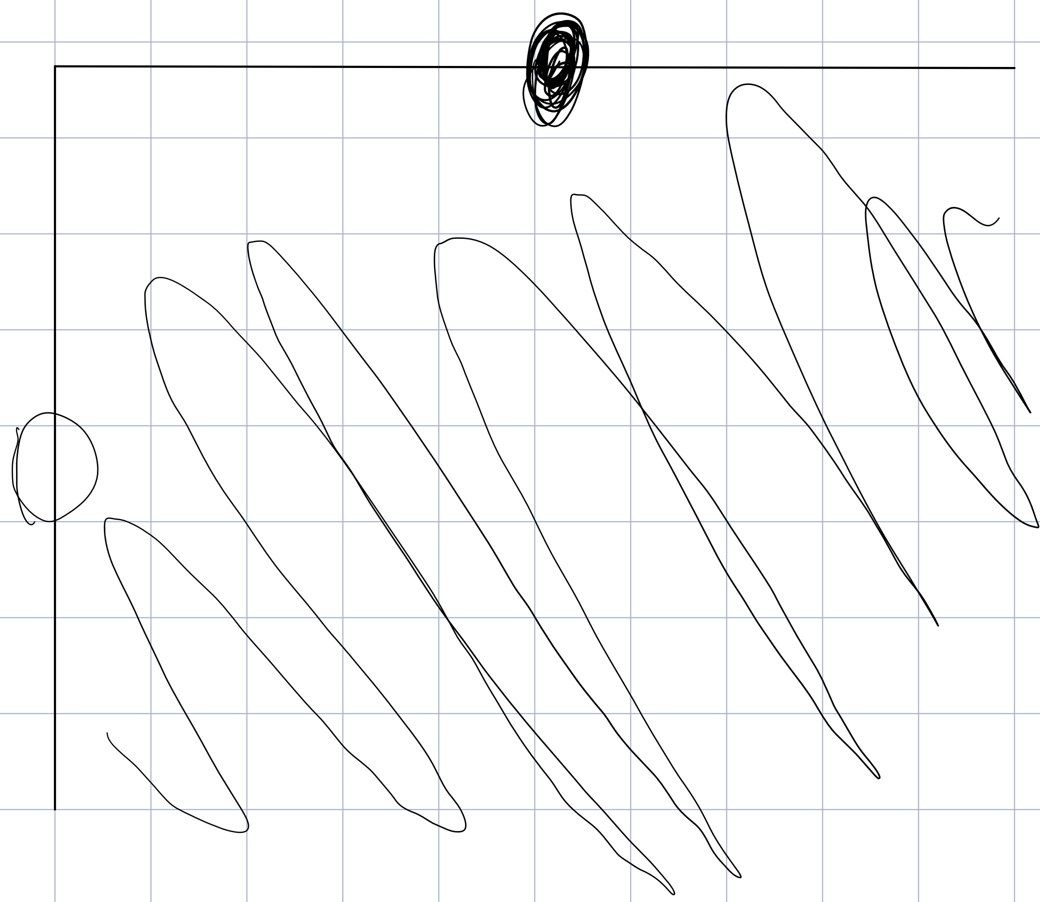
(iv) nil'an

(v)

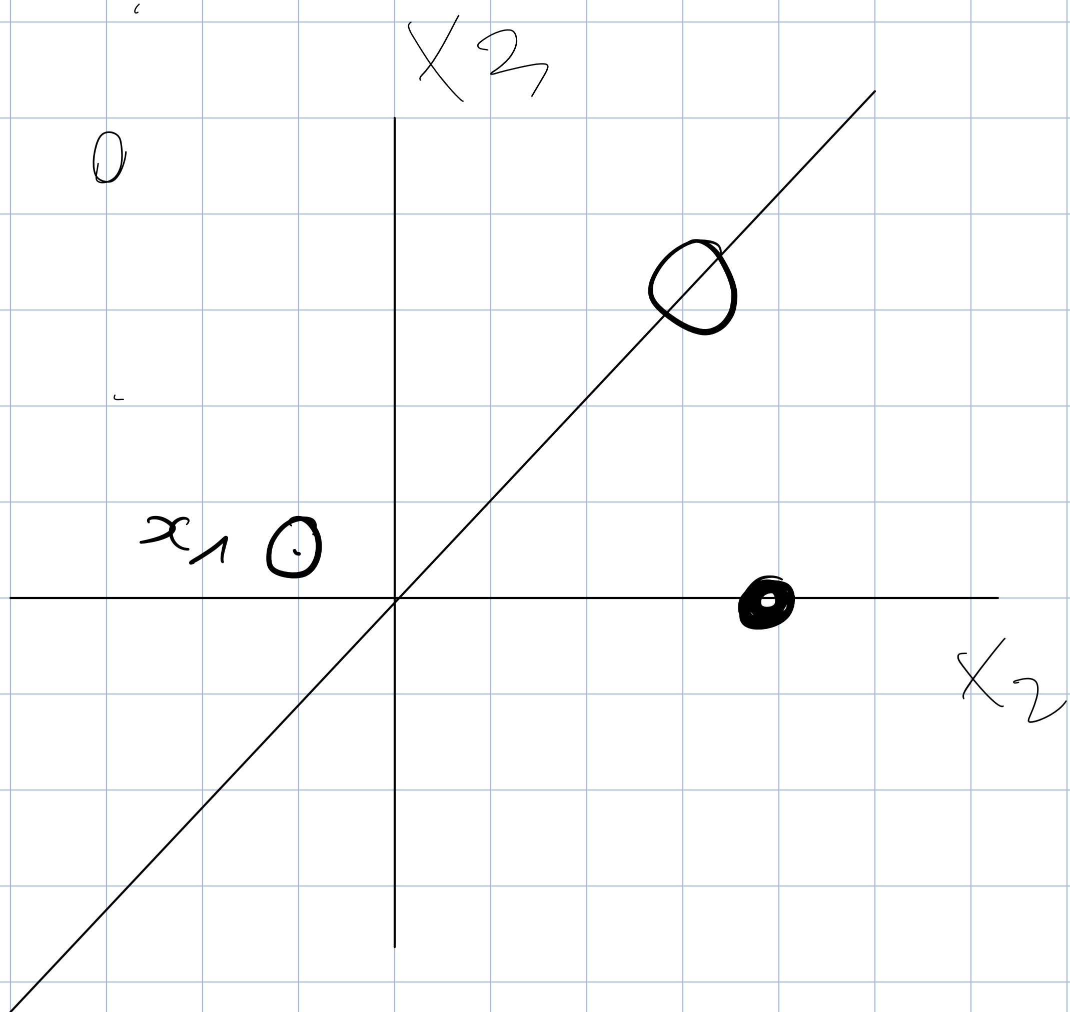


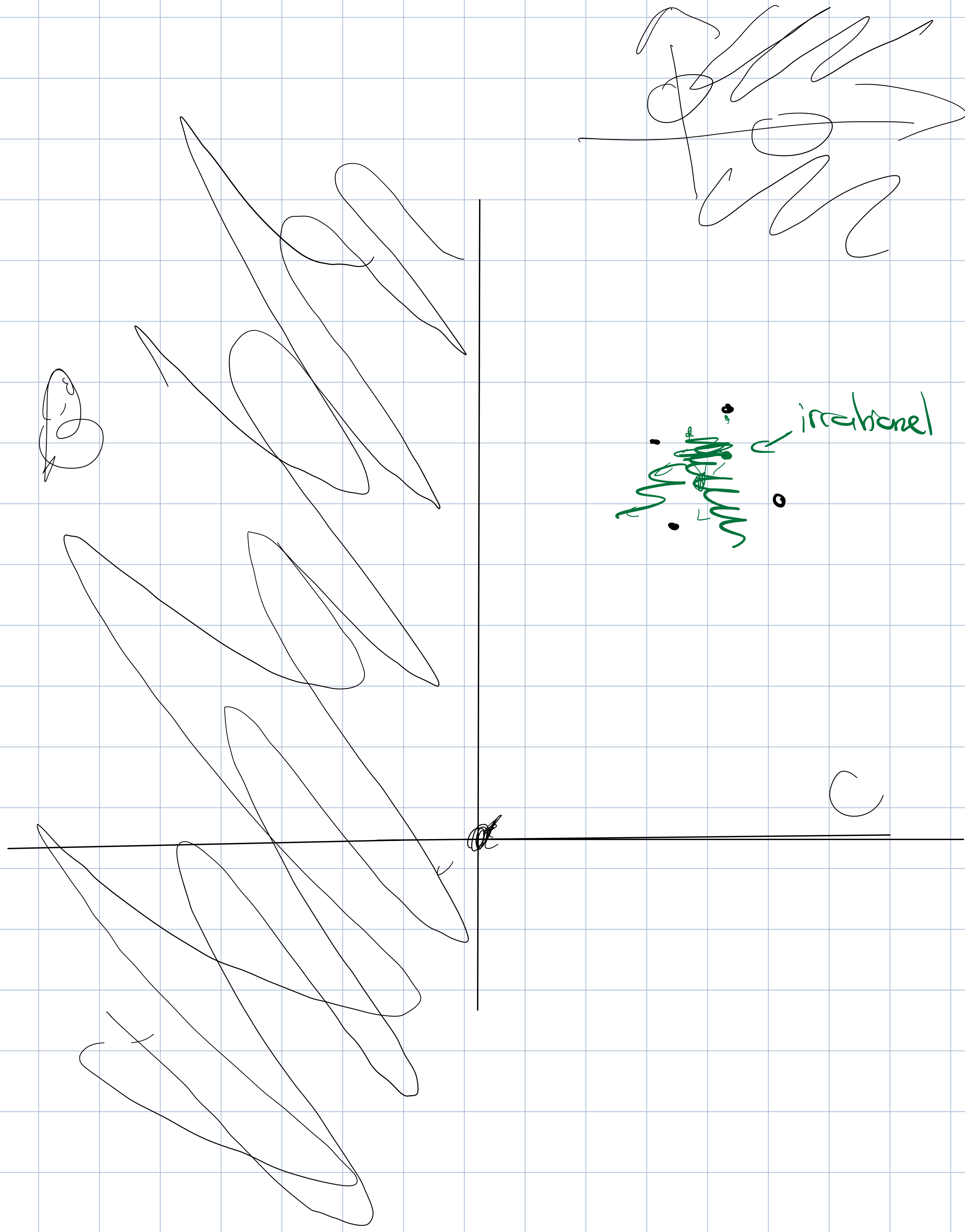
(viii) CE = $\left| \qquad \right|$

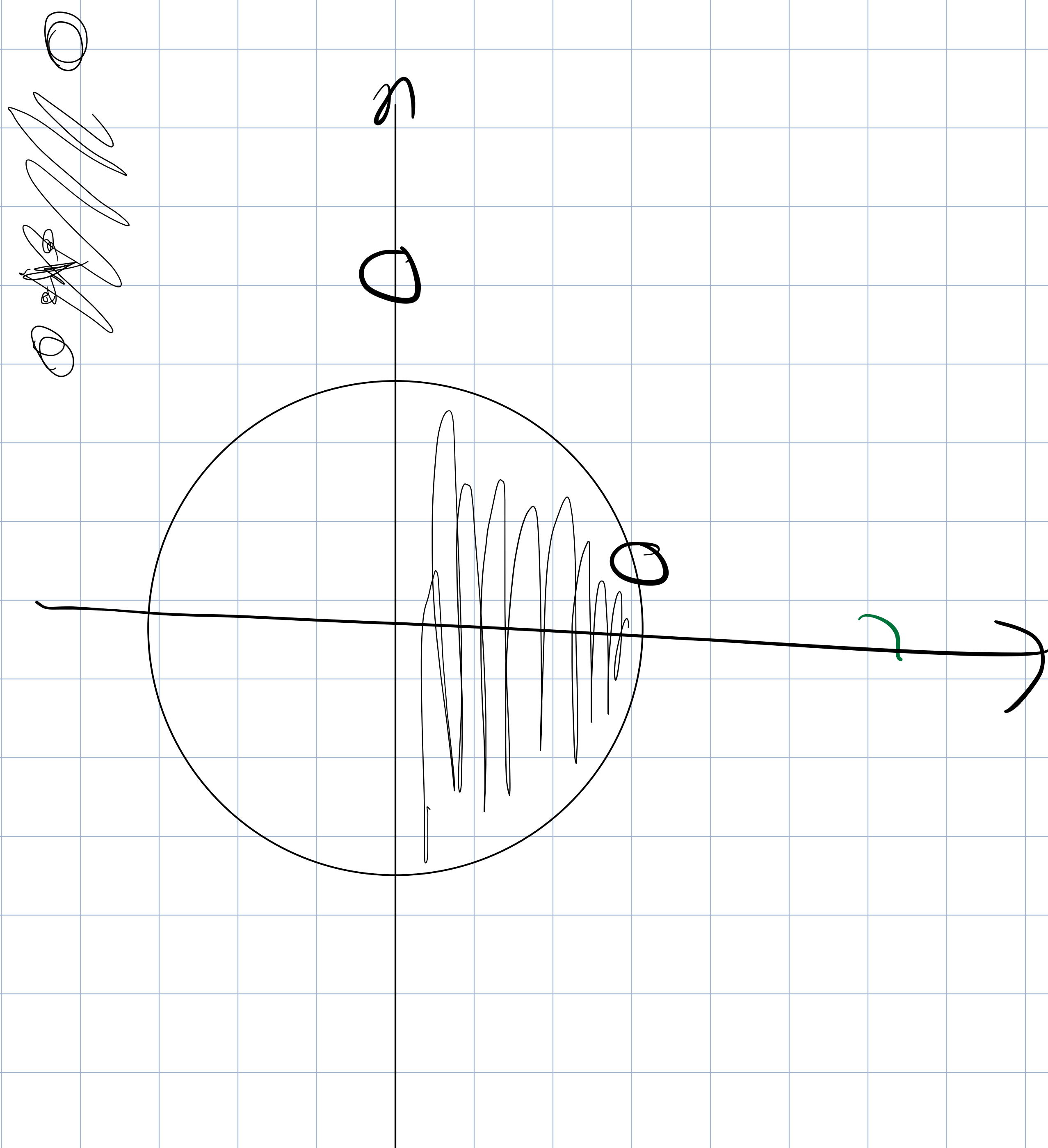
iv



v



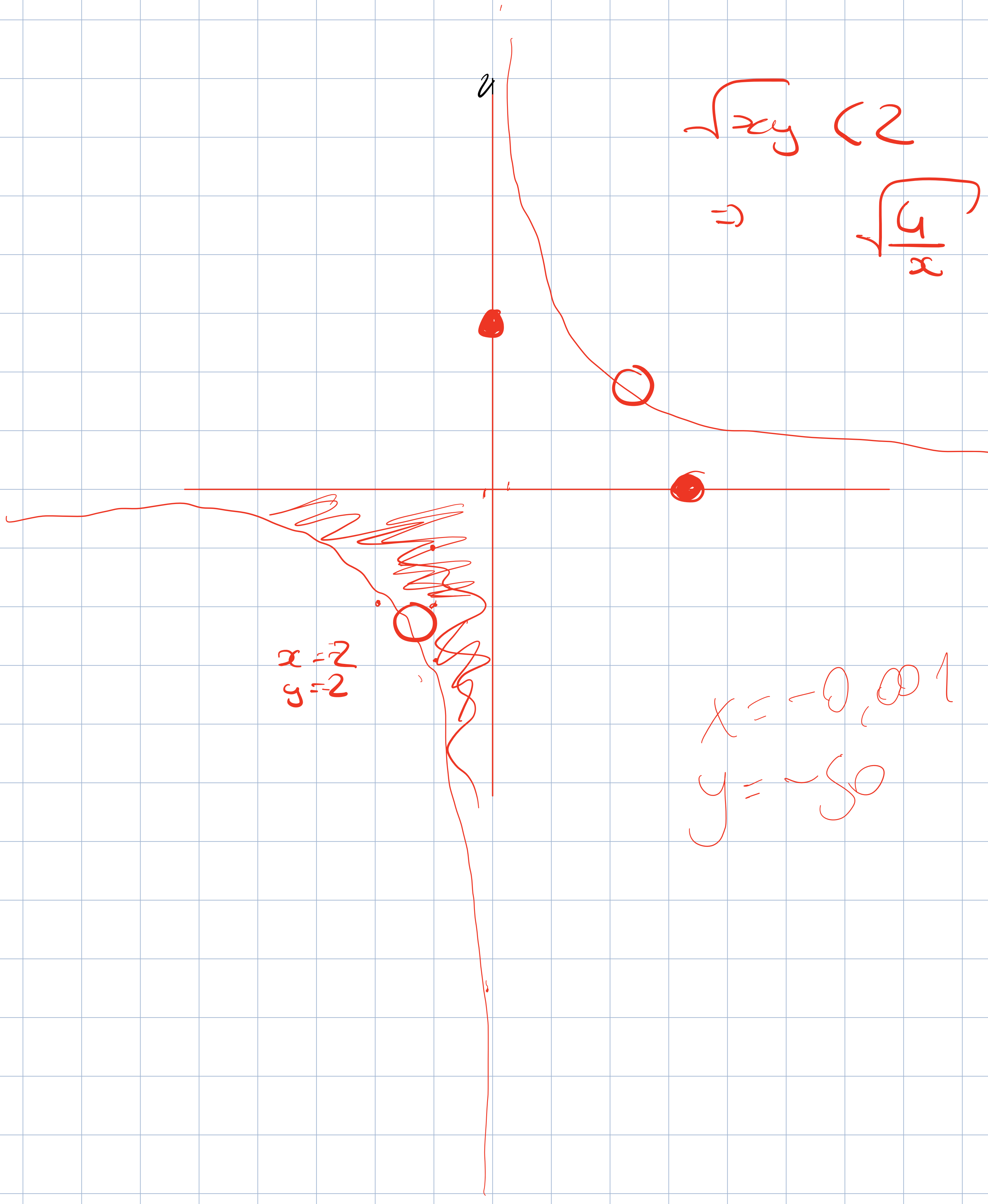

irrational



ouvert ?

$\sqrt{xy} < 2$

$\Rightarrow \sqrt{\frac{4}{x}}$



$x = -0{,}001$

$y = -50$

Exercice 5

Si deux nombres ont le même reste lorsqu'ils sont divisés par 9, alors leur différence est divisible par 9.

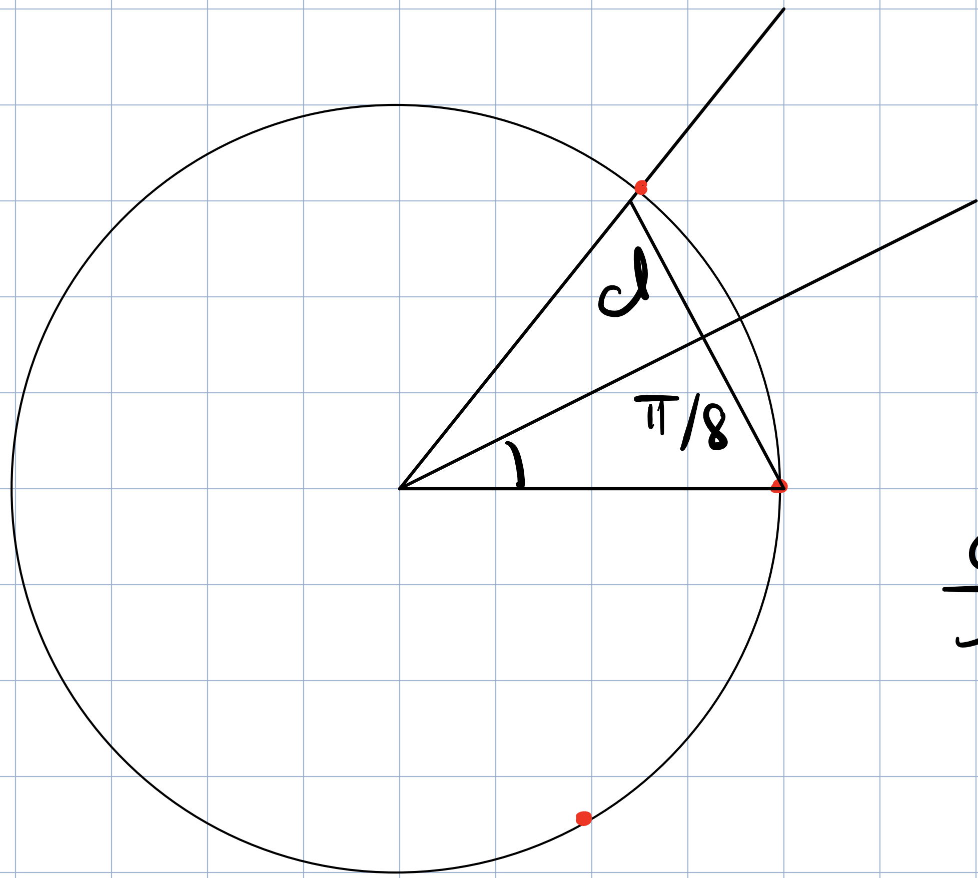
$a = 9k + r$
$b = 9l + r$
$a - b = 9(k-l)$
$b - a = 9(l-k)$.

$\Rightarrow$ on doit donc avoir au moins 12 nombres qui ont le même reste.

$\Rightarrow$ il y a 0, 1, 2, 3, 4, 5, 6, 7, 8 restes possibles

$\frac{100}{8} = 12.5 > 12$

ii



$$\frac{d}{1}$$

On doit placer 8 points sur un cercle de 2π. Ainsi il existe au moins 2 points tels que la distance sur l'arc de cercle entre eux est inférieure ou égale à $\frac{\pi}{4}$.

or la distance entre deux points situés à $\frac{\pi}{4}$ d'écart sur le cercle est < 1

$$2d = \sin\frac{\pi}{8} \cdot 2 = 0.76 \leq 1$$

(iii)

Soit s, t, u, v, x 5 entiers
Il y a 3 restes possibles après une division euclidienne par 3 ; 0, 1, 2.

Il y a donc :

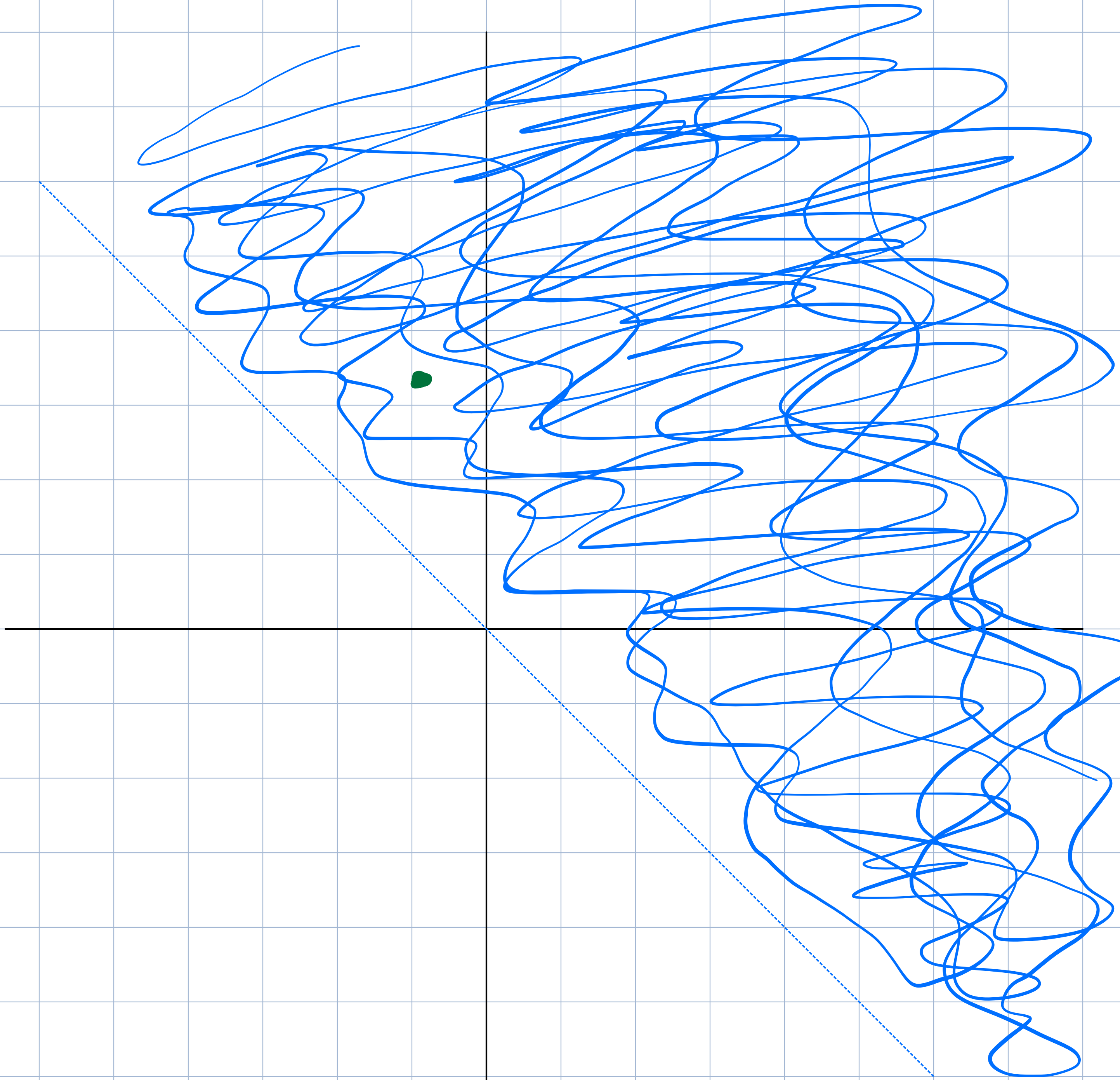
0, 0, 0, 1, 2

- soit ≥ 3 restes similaires et ≤ 2 restes diff
  $(3k + r + 3k' + r + 3k'' + r)$ divisible par 3

- soit ≥ 3 restes différents et ≤ 2 restes sim
  alors nécessairement on peut faire
  $(3k + 0 + 3k + 1 + 3k + 2)$
  divisible par 3.

Exercice 4

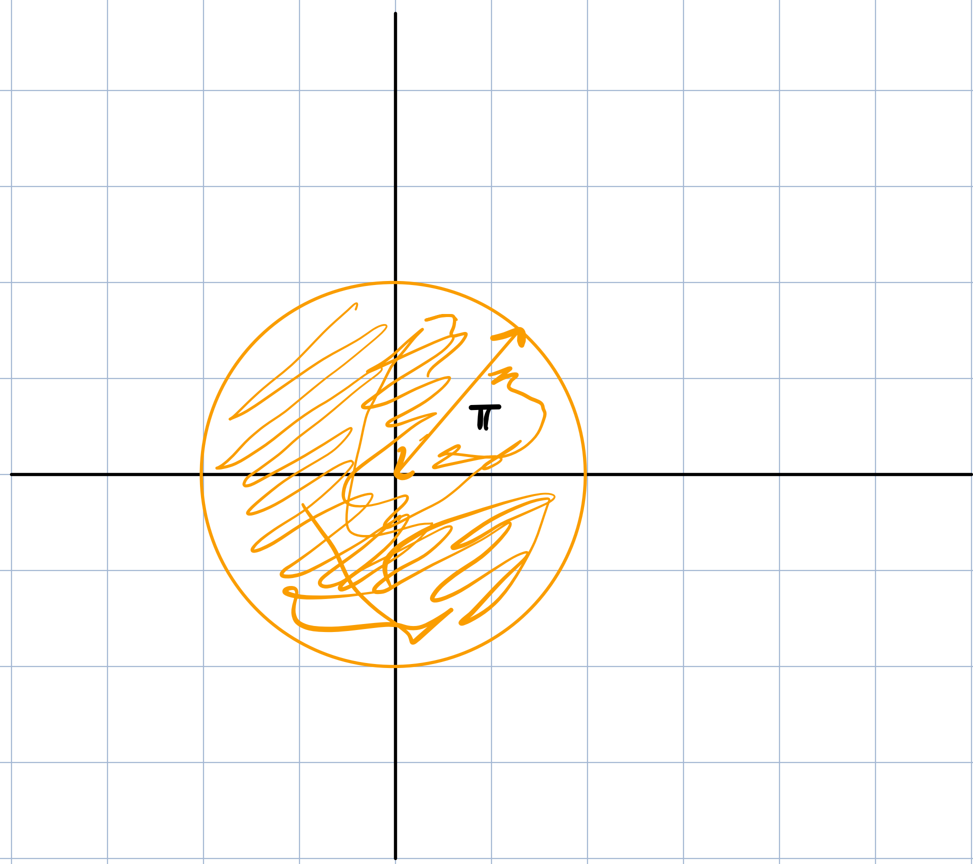
(i)



Let $\bar{y} = (y_1, -y_1 + \delta, y_3, \ldots)$

$B(\bar{y}, \frac{\delta}{2}) \subset E \Rightarrow E$ ouvert

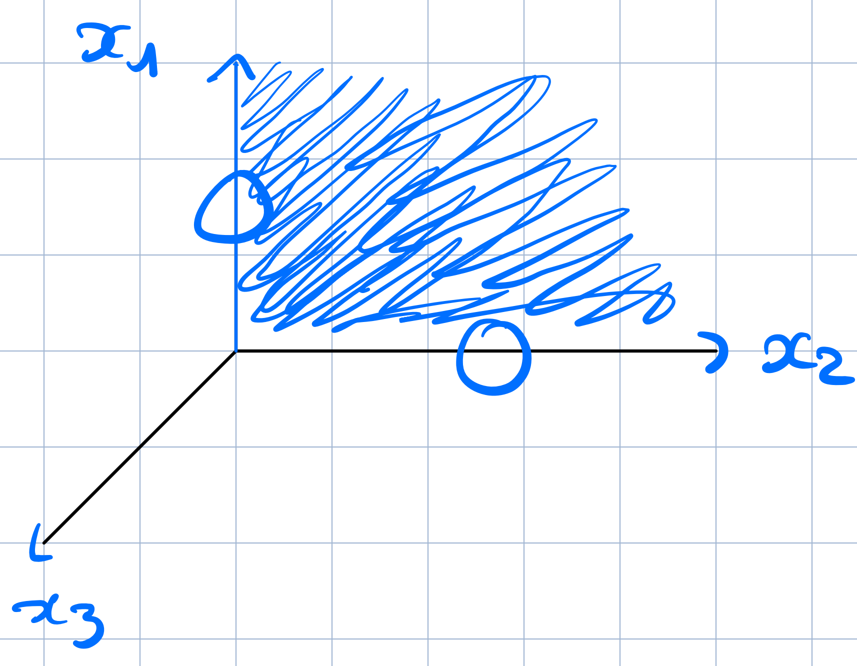
(ii)



Soit $\bar{y}$ tel que $\|\bar{y}\| = \pi + \delta$. $\delta > 0$

$\Rightarrow B\left(\bar{y}, \frac{\delta}{2}\right) \subset CE$ donc CE ouvert
donc E fermé

(iii)

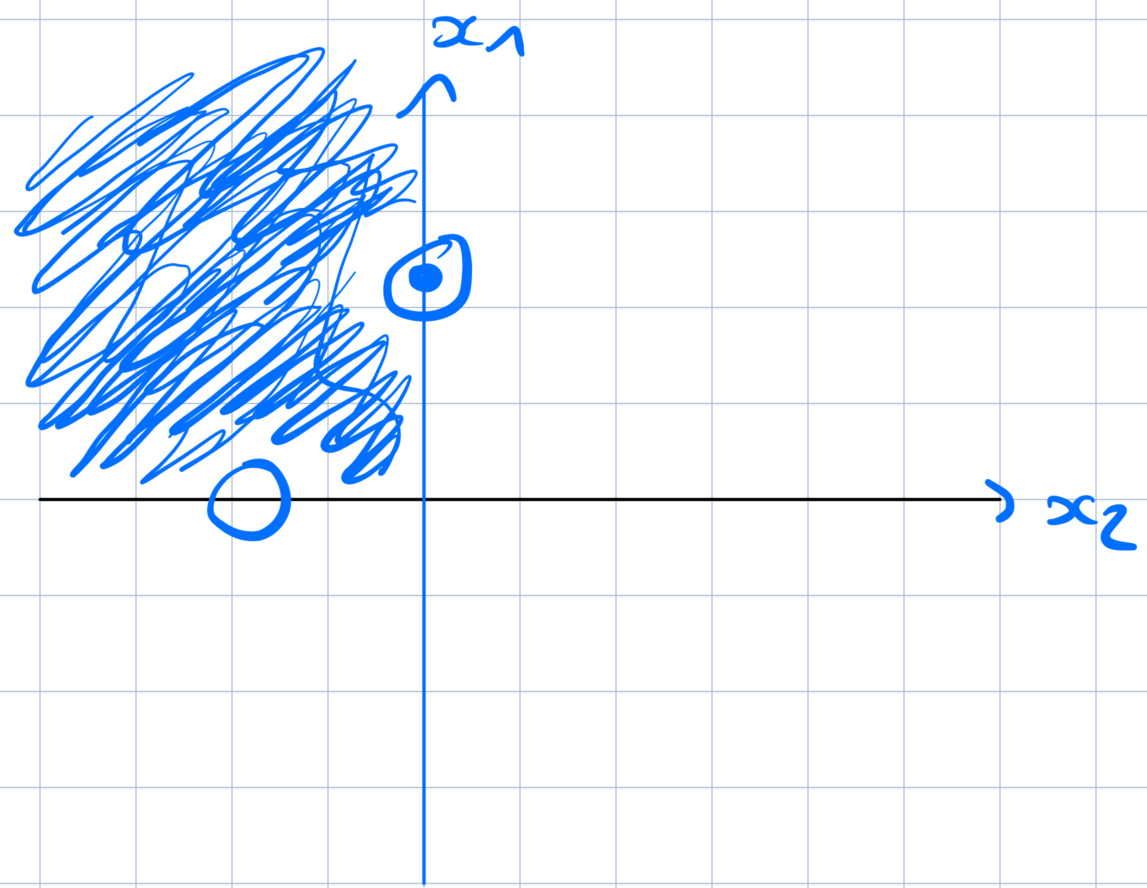


Soit $\bar{y} = \{ \underset{>0}{\delta} , \underset{\neq 0}{y_2} , y_3 \dots \}$

$\Rightarrow B(\bar{y}, \frac{1}{2}(\min(\delta, y_2))) \subset E$

$\Rightarrow$ E est ouvert

(iv)



on ne peut pas créer de boule

Soit $\bar{y} = (y_1, 0, y_2 \dots)$

Si on crée une boule, $\delta > 0$, $B(\bar{y}, \delta)$,

alors $\bar{y}' = (y_1, \frac{\delta}{2}, y_2 \dots) \notin E$.

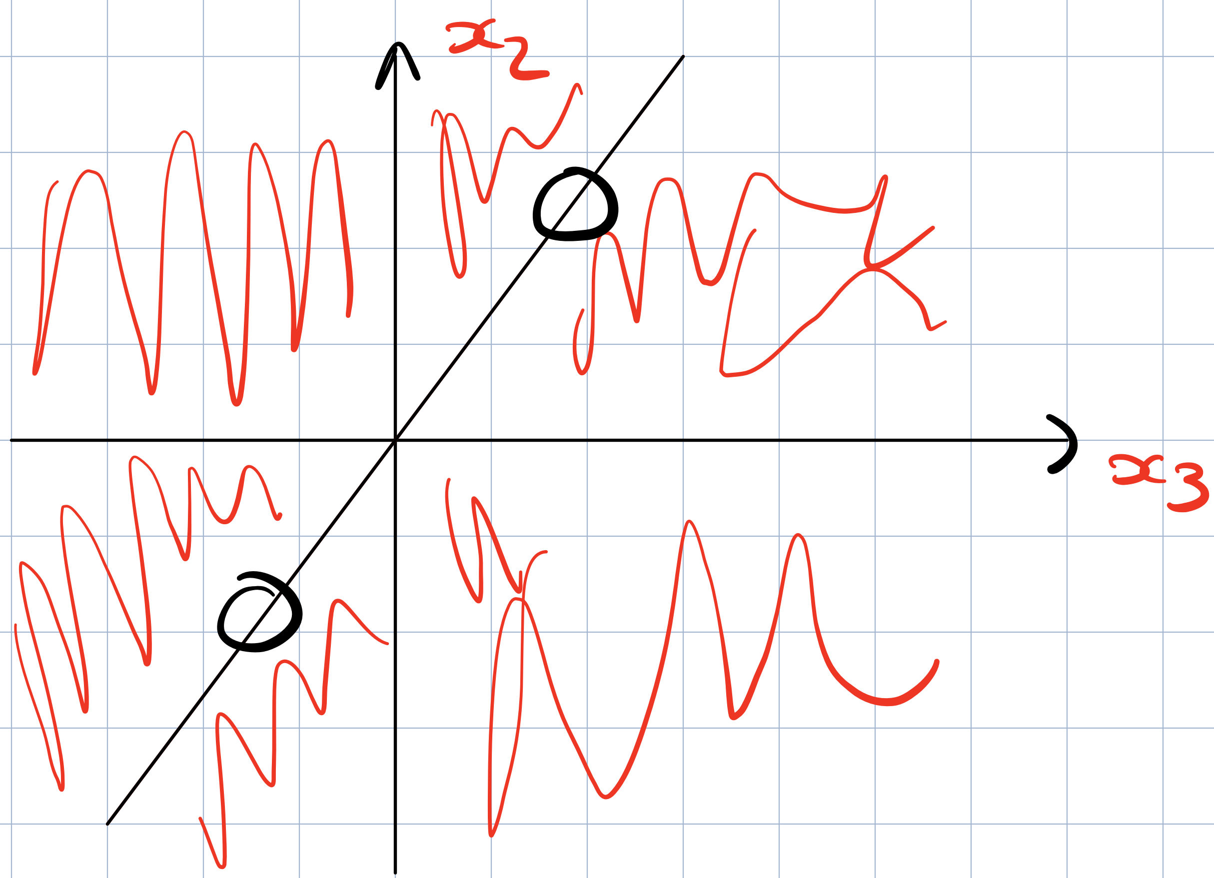
On ne peut pas créer de boule dans CE non plus..

Soit $\bar{z} = (0, y_2 \dots)$

$B(\bar{z}, \delta) \ni (\frac{\delta}{2}, y_2 \dots) \notin CE$

$\Rightarrow$ ni ouvert, ni fermé

V



Soit $\bar{y} = (0, x_2, x_2 + \Delta, \dots)$

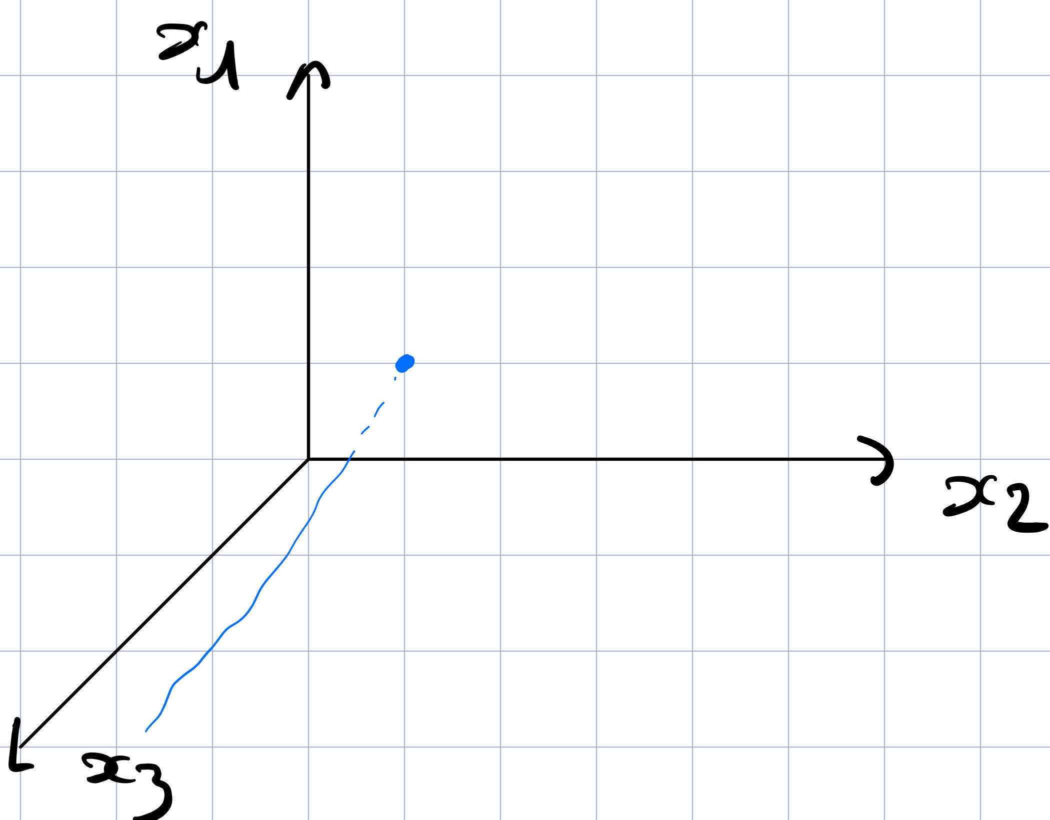
$\forall \delta > 0, \ B(\bar{y}, \delta) \ni \left(\frac{\delta}{2}, x_2, x_2 + \Delta, \dots\right)$
donc on ne peut pas créer de boule.

$CE : x_1 \neq 0$ ou $x_2 = x_3$

Soit $\bar{z} = (0, x_2, x_2 \dots) \in CE.$

$B(\bar{z}, \delta)$ pose pb : $\bar{z}' = (0, x_2 + \frac{\delta}{2}, x_2) \notin CE.$

vi



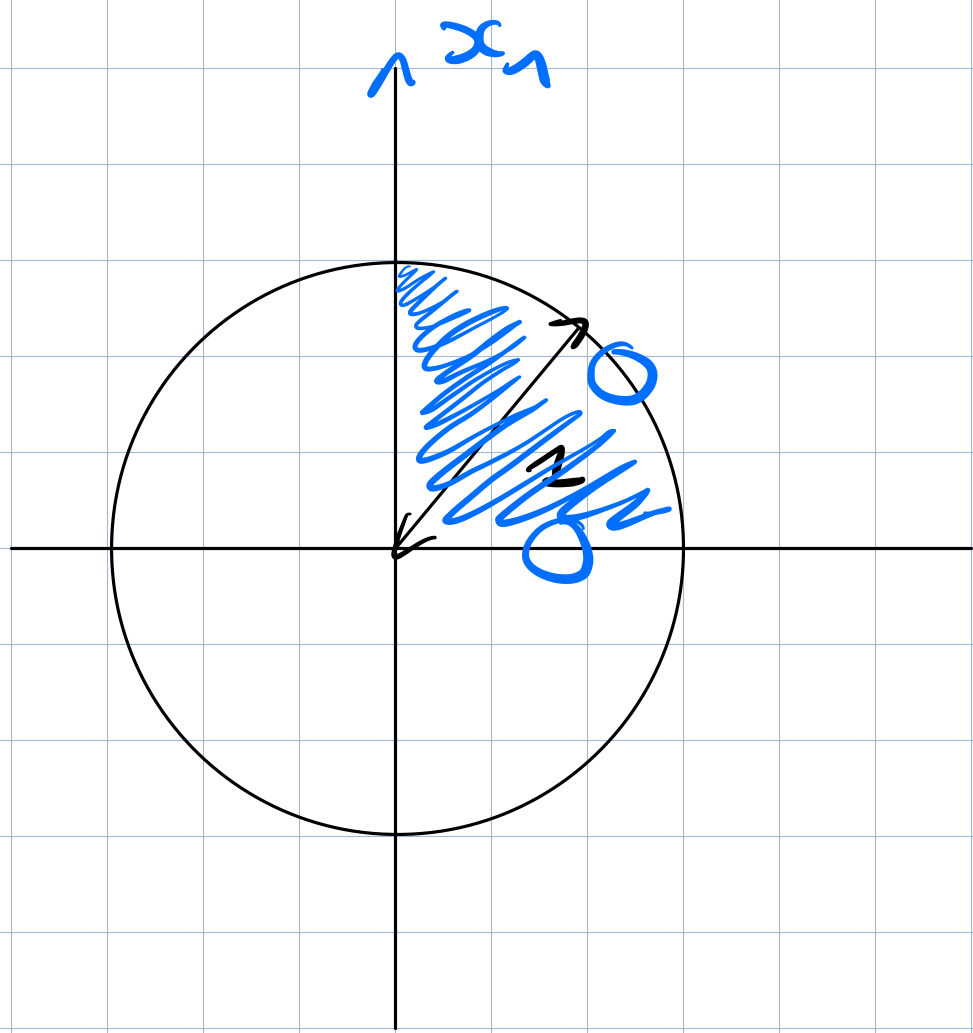
Soit $\bar{y} = (2 + \varepsilon_1, -2 + \varepsilon_2, \dots)$

$B(\bar{y}, \frac{1}{2}\min(\varepsilon_1, \varepsilon_2)) \subset CE$

$\Rightarrow E$ fermé

vii ni ouvert, ni fermé

viii



Soit $\bar{y} = (\varepsilon_1, x_2, x_3 \dots)$
et $\|\bar{y}\| = 1 - \Delta$

$B(\bar{y}, \frac{1}{2}\min(\varepsilon_1, \Delta))$