Exercice 1

$$\frac{\partial f}{\partial x} = z^2 - 4xy$$

$$\frac{\partial f}{\partial y} = -2x^2 + 2yz$$

$$\frac{\partial f}{\partial z} = 2xz + 2y^2$$

$\nabla f(1,1,z_0)$

$$= \langle \begin{pmatrix} z_0^2 - 4 & -2 + 2z_0 & 2z_0 + 2 \end{pmatrix},$$

$$(x-1,\ y-1,\ z-z_0) \rangle$$

$$= (z_0^2-4)(x-1) + (-2+2z_0)(z-1)$$

$+ (2z_0 + 2)(z - z_0)$

$= z_0^2 x - z_0^2 - 4x + 4$

$-\cancel{2z} + 2 + 2z_0 z - 2z_0^2 + \cancel{2z} - 2z_0$

$= z_0^2 x - z_0^2 - 4x + 6 + 2z_0 z - 2z_0^2$
$- 2z_0 = 0$

Exercice 2

(i) $f(x,y) = x^3 + y^3$

$g(x,y) = x^4 + y^4 - 32 = 0$

$\frac{\partial f}{\partial x} = 3x^2 \qquad \frac{\partial f}{\partial y} = 3y^2$

$\frac{\partial g}{\partial x} = 4x^3 \qquad \frac{\partial g}{\partial y} = 4y^3$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 3x^2 = \lambda 4x^3 \\ 3y^2 = \lambda 4y^3 \end{cases}$$

• $\lambda = 0 \Rightarrow x = 0,\ y = 0 \Rightarrow$ contrainte non satisfaite

Sinon $3 = \lambda 4 x \Rightarrow x = \frac{3}{4}\lambda$

$3 = \lambda 4 y \Rightarrow y = \frac{3}{4}\lambda$

$\Rightarrow x = y$

$\Rightarrow x = 2,\ y = 2$

ou $x = -2,\ y = -2$

car $x = y$

ii

$$\nabla f = (1 \quad 1 \quad 1)$$

$$\nabla g_1 = (2x \quad 2y \quad 2z)$$

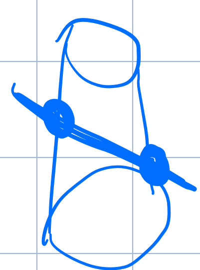
$$\nabla g_2 = (1 \quad -1 \quad 0)$$

$$\cancel{\begin{aligned} \nabla f &= \lambda_1 \nabla g_1 \\ \nabla f &= \lambda_2 \nabla g_2 \end{aligned}}$$

∇g_1 e ∇g_2 lin indep

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

porquoi?



$$x - y = 1$$
$$\Rightarrow x = y + 1$$
$$y = x - 1$$

$$\begin{cases} 1 = \lambda_1 2x + \lambda_2 \\ 1 = \lambda_1 2y - \lambda_2 \\ 1 = \lambda_1 2z \end{cases}$$

CHECK

$2y = 2x - 2$

$2y + 1 = 2x - 1$

$$\Rightarrow \lambda_1 = \frac{1}{2z}$$

$$\nabla f \, \nabla g = \frac{1+\lambda_2}{2y} = \frac{1-\lambda_2}{2x}$$

$\to 0$

$f = (0,0)$

$1 = \lambda_1(2y+2) + \lambda_2$

$1 = \lambda_1 2y - \lambda_2$

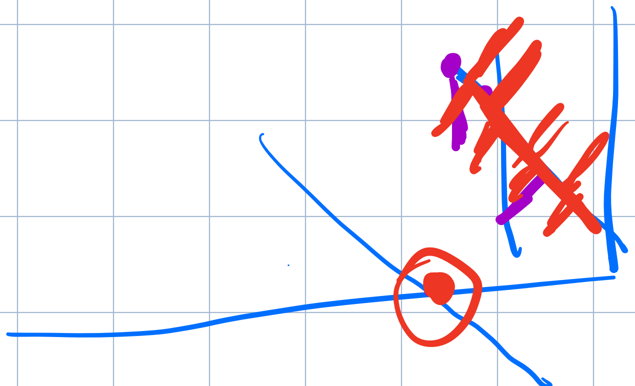
$\Rightarrow 0 = 2\lambda_1 + \lambda_2 + \lambda_2$

$\Rightarrow 0 = 2\lambda_1 + 2\lambda_2$

$\Rightarrow 0 = \lambda_1 + \lambda_2$

$\Rightarrow -\lambda_1 = \lambda_2$

$$\begin{cases} 1 = \lambda_1(2x-1) \\ 1 = \lambda_1(2y+1) \\ 1 = \lambda_1 2z \end{cases}$$



$(0,0) = \lambda \nabla g = 0$

$$\Rightarrow \lambda_1 = \frac{1}{2x-1} = \frac{1}{2y+1} = \frac{1}{2z}$$

$$2x - 1 = 2y + 1 = 2z$$

$$\Rightarrow x - \frac{1}{2} = y + \frac{1}{2} = z$$

$$x^2 + (x-1)^2 + \left(x - \frac{1}{2}\right)^2 = 1$$

$$= x^2 + x^2 - 2x + 1 + x^2 - x + \frac{1}{4} - 1 = 0$$

$$= 3x^3 - 3x + \frac{1}{4}$$

$$\Delta = (-3)^2 - 4 \cdot \frac{1}{4} \cdot 3 = 9 - 3$$

$$= 6$$

$$x = \frac{1}{2} \pm \frac{1}{\sqrt{6}} \quad \Rightarrow y = \frac{\pm \cancel{\sqrt{6}}}{\sqrt{6}} - \frac{1}{2}$$

$$\Rightarrow z = \pm \frac{\cancel{\sqrt{6}}}{\sqrt{6}}$$

On a 2 candidats :

$$(x,y,z) = \left\{ \frac{1}{2} + \frac{1}{\sqrt{6}}, \frac{-1}{2} + \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\} \quad 1$$

$$\text{ou } (x,y,z) = \left\{ \frac{1}{2} - \frac{1}{\sqrt{6}}, \frac{-1}{2} - \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\} \quad 2$$

1 $\Rightarrow$ max, 2 $\Rightarrow$ min

Si $\lambda_2 = 0$ alors $\lambda_1 = \frac{1}{2k}$

et $x = y = z = k \Rightarrow$ impossible !

Si $\lambda_1 = 0 \Rightarrow \lambda_2 = 0 \Rightarrow$ impossible

nécessairement $\lambda_1 \neq 0$ et $\lambda_2 \neq 0$

Exercice 3

$$g(x,y,z) = 4x^2 + 3y^2 + 3z^2 + 2yz - 4x - 1 = 0$$

$$\nabla g(x,y,z)$$

$$= \begin{pmatrix} 8x - 4 \\ 6y + 2z \\ 6z + 2y \end{pmatrix}$$

$$\nabla f(x,y,z)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$
\begin{cases}
0 = \lambda(8x-4) \\
0 = \lambda(6y+2z) \\
1 = \lambda(6z+2y) \\
4x^2+3y^2+3z^2+2yz-4x-1=0
\end{cases}
$$

$\Rightarrow \lambda \neq 0$

$8x-4=0 \Rightarrow x = \dfrac{1}{2}$

$6y+2z=0$

$\Rightarrow y = -\dfrac{1}{3}z \Leftrightarrow z = -3y$

$$4\left(\frac{1}{2}\right)^2+3y^2+3(-3y)^2+2(-3y)y$$

$$-4\frac{1}{2}-1=0$$

$$\Leftrightarrow \cancel{1} + 3y^2 + 27y^2 - 6y^2 - 2 - \cancel{1} = 0$$
$$\Leftrightarrow 24y^2 - 2 = 0$$

$$\Delta = -4(24)(-2) = 96 \cdot 2$$
$$= 192$$

$$y = \frac{\pm\sqrt{192}}{2 \cdot 24} = \pm\frac{\sqrt{3}}{6}$$

$$\Rightarrow z = \mp 3\left(\frac{\sqrt{3}}{6}\right)$$
$$= \mp\frac{\sqrt{3}}{2}$$

$$\Rightarrow x = \frac{1}{2}$$

$$\left\{\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right)\right\}$$

donc le max de f c'est $\frac{\sqrt{3}}{2}$ et le min c'est $-\frac{\sqrt{3}}{2}$

Exercice 4

(1)

$$x^2+y^2-32=0$$

$$\frac{\partial f}{\partial x} = 4x - y - 6 \qquad \frac{\partial^2 f}{\partial y \partial x} = -1$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial f}{\partial y} = -x + 4y - 6$$

$$\frac{\partial^2 f}{\partial y^2} = 4$$

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$$

Δ est $>0 \Rightarrow$ min

$r > 0$

$$\begin{cases} 4x - y - 6 = 0 \\ 4y - x - 6 = 0 \end{cases} \qquad 3x = 6 \Rightarrow x = 2$$

$$4x - 4y - y + x = 0$$

$$x(4+1) - y(4+1) = 0$$

$$5(x-y) = 0$$

$$\Rightarrow x = y$$

$x = 2,\ y = 2$

On check ensuite le bord

$$g = x^2 + y^2 - 32 = 0$$

$$\nabla g = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

$$\begin{cases} 4x - y - 6 = 2\lambda x \\ 4y - x - 6 = 2\lambda y \end{cases}$$

$$4x - 4y - y + x = 2\lambda(x - y)$$

$$5\cancel{(x-y)} = 2\lambda\cancel{(x-y)}$$

ou $x - y = 0$
$\Rightarrow x = y$

$$\ominus \quad 5 = 2\lambda \Rightarrow \lambda = \frac{5}{2}$$

8: $x=y$

$$x^2+y^2-32=0$$
$$2x^2-32=0$$

$$2(16)-32=0$$
$$\Rightarrow x=4 \quad y=4$$
$$x=-4 \quad y=-4$$

$$\begin{cases} 4x - y - 6 = 5x \\ \Leftrightarrow x = -y - 6 \end{cases}$$

$$\begin{cases} 4y - x - 6 = 5y \\ \Leftrightarrow y = -x - 6 \end{cases}$$

$$(-a-b)(-a-b)$$
$$= a^2 + ab + ab + b^2$$

$$x^2 + y^2 - 32 = 0$$

$$\Leftrightarrow x^2 + (-x-6)^2 - 32 = 0$$

$$\Leftrightarrow x^2 + x^2 + 12x + 36 - 32 = 0$$

$$\Leftrightarrow 2x^2 + 12x + 4 = 0$$

$$\Delta = (12)^2 - 4(2)(4)$$
$$= 144 - 32$$
$$= 112$$

$$\frac{-12 \pm 4\sqrt{7}}{4}$$

$$= \boxed{-3 \pm \sqrt{7}} = x$$

$$y = 3 \mp \sqrt{7} - 6$$

$$= -3 \mp \sqrt{7}$$

$$\left\{ \left(-3 \pm \sqrt{7}, -3 \mp \sqrt{7}\right), \left(\pm 4, \pm 4\right), \left(\pm 4, \mp 4\right) \right\}$$

(ii)

$$\frac{\partial f}{\partial x} = 2x - 2 \qquad \frac{\partial f}{\partial z} = 2z - 1$$

$$\frac{\partial f}{\partial y} = 2y + 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$D_1 > 0 \quad D_2 > 0 \quad D_3 > 0$ min

$$(x,y,z) = \left(1,-1,\frac{1}{2}\right)$$

$$\nabla g(x,y,z)$$

$$= \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\begin{cases} 2x-2 = \lambda 2x \\ 2y+2 = \lambda 2y \\ 2z-1 = \lambda 2z \\ \\ x^2+y^2+z^2-4=0 \end{cases}$$

$\lambda = 1$ imp.

$$2x(1-\lambda) = 2$$

$$\Rightarrow x = \frac{1}{1-\lambda}$$

$$2y(1-\lambda) = -2 = \frac{-1}{1-\lambda}$$

$$2z(1-\lambda) = 1$$

$$\Rightarrow z = \frac{1}{2(1-\lambda)}$$

$$\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{4(1-\lambda)^2} = 4$$

$$= \frac{9}{4(1-\lambda)^2} = \cancel{4}$$

$$= \frac{9}{(1-\lambda)^2} = 16$$

$$\frac{9}{x} = 16 \Rightarrow x = \frac{9}{16}$$

$$(1-\lambda)^2 = \frac{9}{16} \Rightarrow 1-\lambda = \frac{-3}{4}$$

$$1-\lambda = \frac{3}{4}$$

$$-\lambda = \frac{-7}{4}$$
$$\lambda = \frac{7}{4}$$

$$\lambda = \frac{1}{4}$$

$$2x-2 = \lambda 2x$$
$$2x-2 = \frac{1}{2}x$$

$$\Rightarrow \quad x\left(2-\frac{1}{2}\right) = 2$$

$$\Rightarrow \quad x = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

$$2y+2 = \frac{1}{2}y$$

$$\Rightarrow \quad y\left(2-\frac{1}{2}\right) = -2 \quad \Rightarrow \quad -\frac{4}{3} = y$$

$$z = -\frac{2}{3}$$

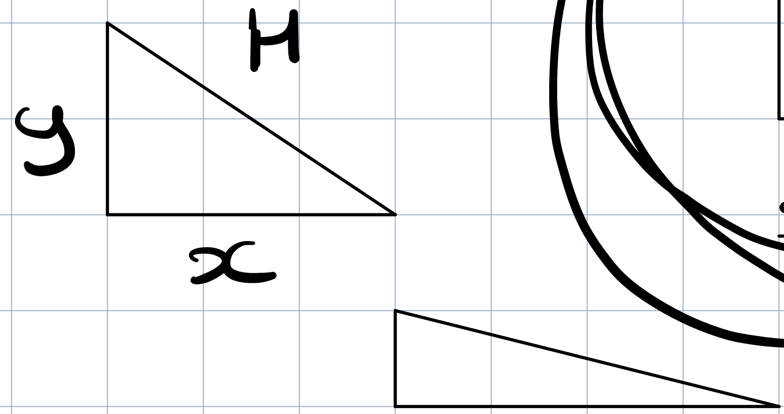
$$x' = \frac{4}{3}$$
$$y' = -\frac{4}{3}$$
$$z' = 2\text{u}3$$

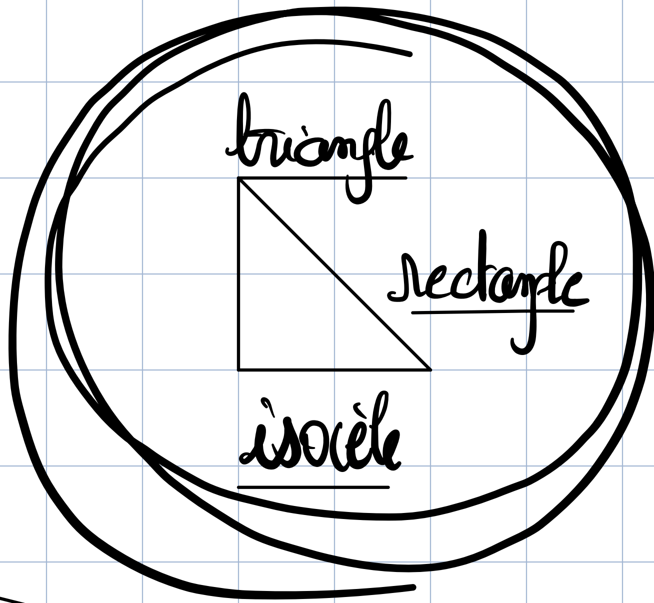
$\min\left(1, -1, -\frac{1}{2}\right)$ (dedans)

$\max\left(-\frac{4}{3}, \frac{4}{3}, -\frac{2}{3}\right)$ (bords)

Exercices

1





$$\begin{cases} f(x,y) = x^2 + y^2 \\ \dfrac{x \cdot y}{2} = A \iff \dfrac{1}{2}xy - A = 0 \end{cases}$$

$\nabla f = \begin{pmatrix} 2x & 2y \end{pmatrix}$ $\nabla g \neq 0$

$y = x = 0$

$\nabla g = \begin{pmatrix} \frac{1}{2}y & \frac{1}{2}x \end{pmatrix}$ $\Rightarrow \in$

$$\begin{cases} 2x = \lambda \frac{1}{2} y \\ 2y = \lambda \frac{1}{2} x \\ \frac{1}{2} xy - A = 0 \end{cases}$$

$\Rightarrow \quad x = \lambda y$

$\Rightarrow \quad y = \lambda x$ $\Rightarrow x = y$ ✓

$\Rightarrow$

<u>triangle isocèle</u>

ii

$$\min f$$
$$\text{s.t. } z^2 = x^2+y^2 \text{ and } z = 1+x+y \iff (\ )$$

$$1+x+y=0$$

$$f(x,y)=\sqrt{x^2+y^2}$$

$$\underline{\sqrt{x^2+y^2} = 1+x+y}$$

$$\Rightarrow \cancel{x^2}+\cancel{y^2} = (1+x+y)^2$$
$$= (1+x)^2+2(1+x)(y)+y^2$$
$$= 1+2x+x^2+2y+2xy+y^2$$
$$= \cancel{x^2}+2x+2y+2xy+\cancel{y^2}+1$$

$$\Rightarrow \underline{2x+2y+2xy}+1 = 0$$
$$= g(x,y)$$

on veut minimiser f en respectant g

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\begin{cases} \dfrac{x}{\sqrt{x^2+y^2}} = \lambda(2+2y) \\[2ex] \dfrac{y}{\sqrt{x^2+y^2}} = \lambda(2+2x) \\[2ex] 2x+2y+2xy+1=0 \end{cases}$$

$$\frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} = 2y-2x$$

$$\Rightarrow \frac{x-y}{\sqrt{x^2+y^2}} = 2y-2x$$

$$\Rightarrow \frac{\cancel{x-y}^{\,(-1)}}{2\sqrt{x^2+y^2}} = \cancel{y-x}$$

if $y = x \longrightarrow$

if $y \neq x \longrightarrow \sqrt{x^2+y^2} = -\frac{1}{2} \longrightarrow \times$

$\Delta = y-x \neq 0 \longrightarrow \dfrac{\cancel{\Delta}}{2\sqrt{x^2+y^2}} = -\cancel{\Delta} \longrightarrow \sqrt{x^2+y^2} = -\frac{1}{2}$

$\longrightarrow \times$

$f'(x, y, z)$

$= z - f(x, y)$

then $x = y$

$2x + 2y + 2xy + 1 = 0$

$\Rightarrow 2x + 2x + 2x^2 + 1 = 0$

$\Rightarrow 2x^2 + 4x + 1 = 0$

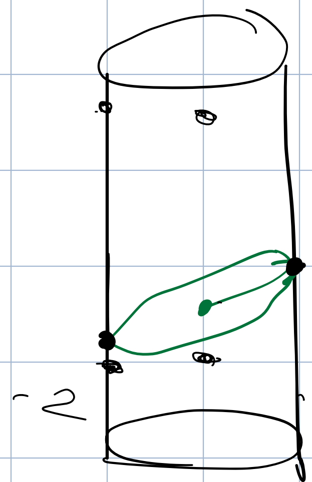
$\Delta = 16 - 4 \cdot 2 = 8$

$x = \dfrac{-4 \pm \sqrt{8}}{4} = -1 \pm \dfrac{2\sqrt{2}}{4}$

$= -1 \pm \dfrac{\sqrt{2}}{2}$

Exercice 6

notre fonct° c'est le plan (z en fonct° de x,y) et on veut maximiser en étant sur le cylindre



$$z = 1 - \frac{y}{2} - \frac{x}{2}$$

$$\begin{cases} f(x,y) = z = 1 - y/2 - x/2 \\ x^2 + y^2 - 4 = 0 \quad = g_1 \\ 1 - \frac{y}{2} - \frac{x}{2} \quad = z = g_2 \end{cases}$$

$$\nabla g_1 = (2x \quad 2y)$$

$$\nabla f = \left(-\frac{1}{2} \quad -\frac{1}{2}\right)$$

$$\begin{cases} -\frac{1}{2} = \lambda 2x \Rightarrow x = \frac{-1}{4\lambda} \\ -\frac{1}{2} = \lambda 2y \Rightarrow y = \frac{-1}{4\lambda} \end{cases} \Rightarrow x = y$$

$$x^2 + y^2 - 4 = 0$$

$$= 2x^2 - 4 = 0$$

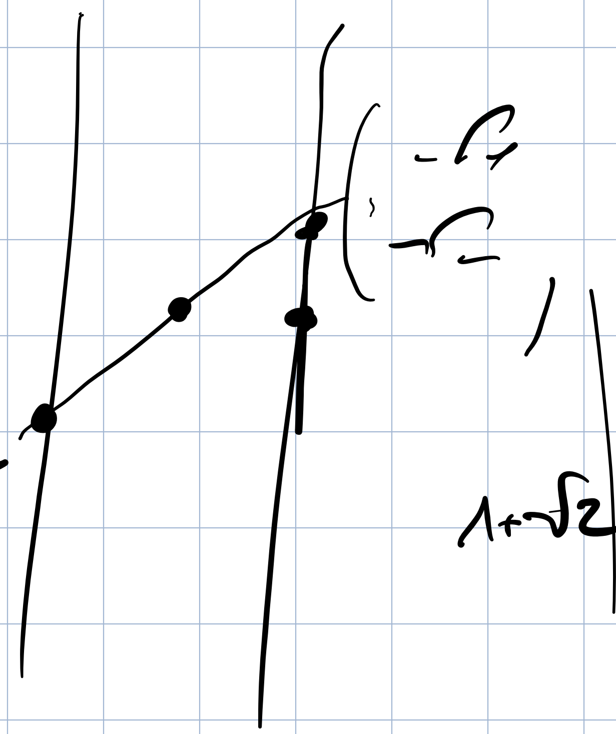
$$x = \sqrt{2}, \quad x = -\sqrt{2}$$

$$z = 1 - \frac{y}{2} - \frac{x}{2}$$

$$= 1 - \sqrt{2}$$

$$z = 1 + \sqrt{2}$$

$(\sqrt{2}, \sqrt{2}, 1-\sqrt{2}$



$$\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \underline{1-\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \\ \underline{1+\sqrt{2}} \end{pmatrix} \quad z = f(x, y) =$$

$$P \mid P = \begin{pmatrix} t \\ t \\ 1-t \end{pmatrix}$$

$$z_{moy} = \frac{1+\sqrt{2} + 1 - \sqrt{2}}{2} = 1$$

$$\begin{pmatrix} x \\ y \\ \cancel{1} \end{pmatrix} \cdot \begin{pmatrix} t \\ t \\ \cancel{1-t} \end{pmatrix} = 0$$

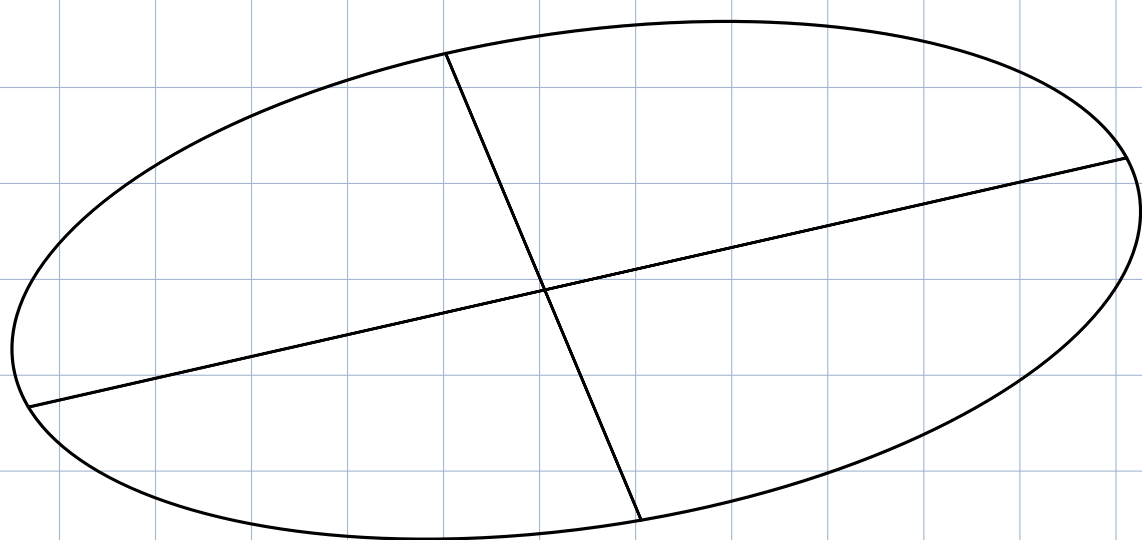
$$xt + yt + \cancel{1-t} = 0$$

demi grand

$$\begin{pmatrix} -a \\ a \\ 1 \end{pmatrix} \begin{pmatrix} t \\ t \\ 1-t \end{pmatrix}$$

$$x = -y$$

$$y = -x$$



Exercice 7

$$f(r,h) = \pi r^2 h$$

$$2\pi r^2 + 2\pi h r - S = 0$$

$$\nabla f = \begin{pmatrix} 2\pi r h & \pi r^2 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 4\pi r + 2\pi h & 2\pi r \end{pmatrix}$$

$$\begin{cases} \cancel{2}\cancel{\pi} r h = \lambda \left(\overset{2}{\cancel{4}} \cancel{\pi} r + \cancel{2} \cancel{\pi} h \right) \\ \pi r^2 = \lambda (2\pi r) \end{cases}$$

$$\Leftrightarrow \begin{cases} r h = \lambda (2r + h) \\ \cancel{\pi} r \cancel{^2} = 2\lambda \cancel{\pi} \cancel{r} \Leftrightarrow r = 2\lambda \Rightarrow \lambda = \frac{r}{2} \end{cases}$$

$$rh = \lambda(2r + h)$$

$$\Leftrightarrow \cancel{r}h = \frac{\cancel{r}}{\cancel{2}}\left(\cancel{2}r + \frac{h}{2}\right)$$

$$\Leftrightarrow h = r + \frac{h}{2}$$

$$\Leftrightarrow \frac{1}{2}h = r \quad \Rightarrow \quad h = 2r$$

$$2\pi r^2 + 2\pi h r - S = 0$$

$$\Leftrightarrow 2\pi r^2 + 2\pi(2r)r - S = 0$$

$$\Leftrightarrow 2\pi r^2 + 4\pi r^2 - S = 0$$

$$\Leftrightarrow r^2(4\pi - 2\pi) - S = 0$$

$$\Leftrightarrow r^2(2\pi) - S = 0$$

$$\Leftrightarrow r^2(2\pi) = S$$

$$\Leftrightarrow r^2 = \frac{S}{2\pi}$$

$$\Leftrightarrow \quad r = \sqrt{\frac{S}{2\pi}}$$

Exercice 8

DISJONCTION DE CAS

(i) $\Leftrightarrow \big|\, |x+5| - |x-1| \,\big| \leq 6$

$x < -5 \Rightarrow$

$$|-(x+5) - (-(x-1))| \leq 6$$
$$\Rightarrow |-x-5+x-1| \leq 6$$
$$\Rightarrow |-6| \leq 6 \quad \checkmark$$

$x < 1,\ x \geq -5$

$$|x+5 - (-(x-1))| \leq 6$$

$$\Rightarrow |x+5+x-1| \leq 6$$

$$\Rightarrow |2x+4| \leq 6$$

On prend le max de x, $x=1$

$$|2\cdot 1+4| = |6| \checkmark$$

On prend le min de x, $x=-5$

$$|2\cdot(-5)+4| = |-6| \checkmark$$

$x > 1$

$$|x+5 \; -(x-1)|$$

$$= |x+5-x+1| = |6| \checkmark$$

(ii) La quantité minimum qu'il faudra mettre dans les poches est :

$$0+1+2+\dots+8+9$$

$$= \frac{n(n+1)}{2}$$

$$= \frac{9(10)}{2} = 45 > 44$$

impossible

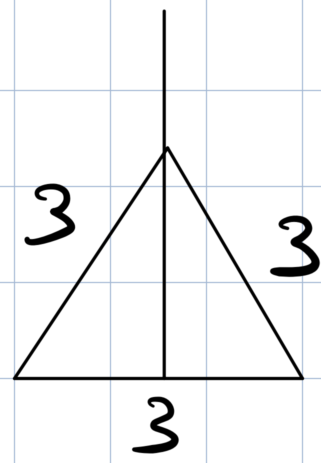
(iii) Si n est un carré d'un nombre naturel

n pair $\begin{cases} n = (2k+1)^2 \\ \quad = 4k^2 + 4k + 1 \\ \quad = 4(k^2+k)+1 \end{cases}$

n impair $\begin{cases} n = (2k)^2 \\ \quad = 4k^2 \end{cases}$

Le reste de la division par 4 est toujours 1 et 0 pour un carré d'un nombre naturel par contraposée.

(iv)



Supposons que ce ne soit pas le cas (chaque point a une boule de rayon 1 autour de elle)

aire $\geq$ 10 alors que non

l'aire de ce triangle est $\frac{3}{2} \cdot \frac{\sqrt{27}}{2}$

$$= \frac{3\sqrt{27}}{4}$$

$$\left(\frac{3}{2}\right)^2 + h^2 = 3^2$$

$$\Rightarrow h^2 = 3^2 - \left(\frac{3}{2}\right)^2$$

1 4

$$= 9 - \frac{9}{4}$$

$$= \frac{36-9}{4} = \frac{27}{4}$$

$$\Rightarrow h = \frac{\sqrt{27}}{2}$$

(v) Pour chaque élément, on a le choix entre :

- garder l'élément
- ne pas le garder

Ainsi on a d'abord 2 possibilités pour le premier, puis 2 de nouveau, etc. 2^n fois.

(vi) Il y a le même nombre de sous-ensembles pairs et impairs de A → on divise par 2 pour enlever la moitié impaire.

$2^{2-1} = 2$

$\{\}$
$\{0,1\}$
$\{0\}$
$\{1\}$

$A = \{0,1\}$

$\{\}$
$\{0\}$
$\{0,1\}$
$\{0,2\}$
$\{1,2\}$
$\{0,1,2\}$
$\{1\}$
$\{2\}$

$A = \{0,1,2\}$