THE UNIVERSITY *of* EDINBURGH

Computer Security

INFR10067

Fall 2025

Cryptography

Asymmetric encryption

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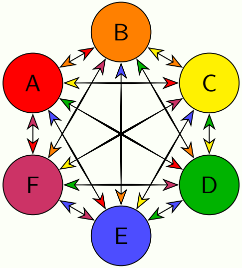
Introduction

So far: how two users can protect data using a shared secret key

- One shared secret key per pair of users that want to communicate



Our goal now: how to establish a shared secret key to begin with?

- Trusted Third Party (TTP)
- Diffie-Hellman (DH) protocol
- *RSA*
- ElGamal (EG)

Recap

Action items:

- ✓ Upload slides in advance (note that they might change before, and even after the lecture)
- ✓ Exercises: Those that struggle. please read this short chapter and do the exercises there: `https://joyofcryptography.com/pdf/chap0.pdf`
- ✓ **Office hours** are 11:15 Mon and Fri. Talk with me after lecture, especially MSc students. Offer to discuss Feistel networks, S-boxes, and key schedules stands.
- ✓ Request: What are challenges in modern cryptography?
  Trusted, distributed, fair and safe AI. **Digital and Decentralized Identity.** Post-quantum.
  Formalization of cryptography in theorem provers, e.g. Lean.

We finished MAC algorithms and authenticated encryption last time.

Online Trusted Third Party (TTP)

- Users U_1, U_2, U_3, ..., U_n, ...
- Each user U_i has a shared secret key K_i with the TTP
- U_i and U_j can establish a key $K_{i,j}$ with the help of the TTP
- $\{m\}_k$ denotes the symmetric encryption of m under the key k

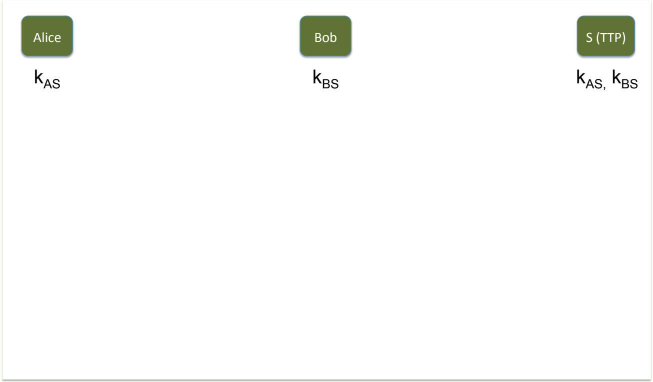


Figure: Paulson's variant of the Yahalom protocol

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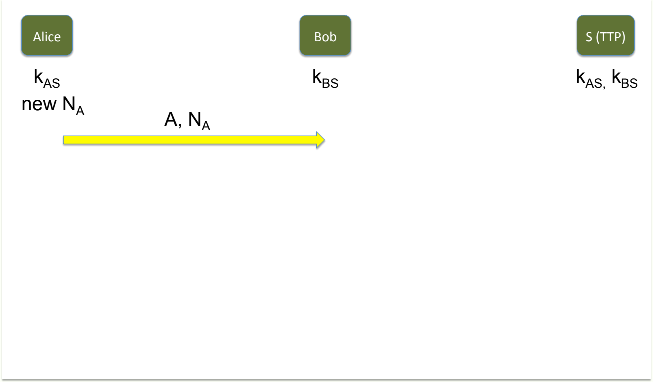

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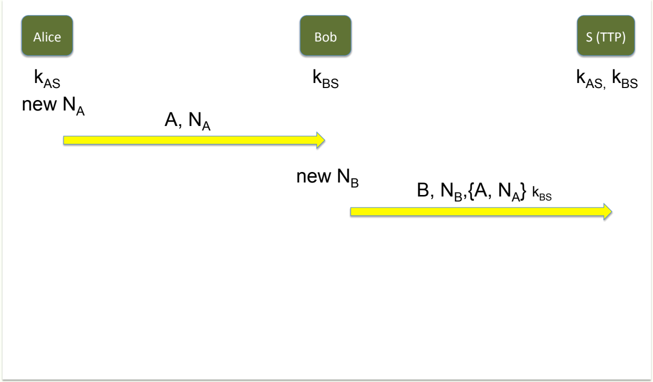

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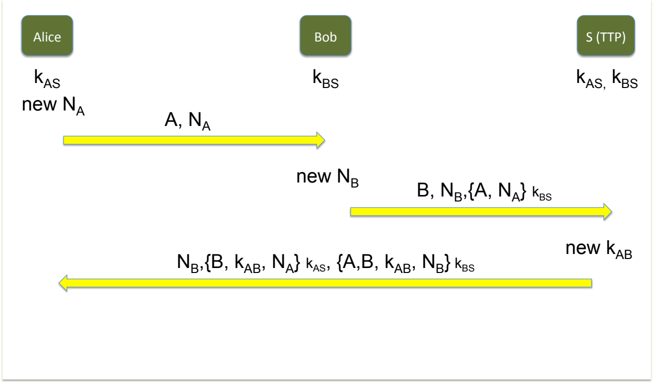

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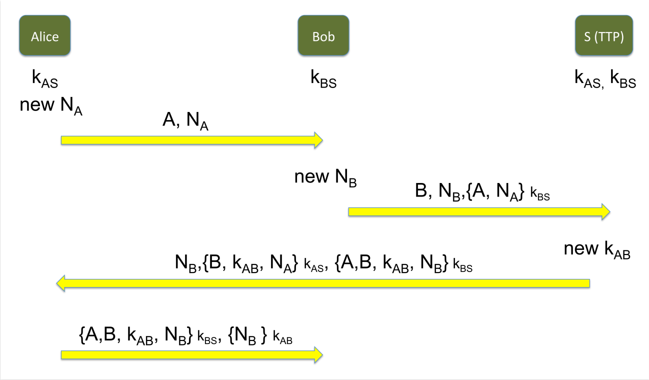


Alice

Bob

S (TTP)

k_{AS}

new N_A

k_{BS}

k_{AS}, k_{BS}

A, N_A

new N_B

$B, N_B, \{A, N_A\}_{k_{BS}}$

new k_{AB}

$N_B, \{B, k_{AB}, N_A\}_{k_{AS}}, \{A, B, k_{AB}, N_B\}_{k_{BS}}$

$\{A, B, k_{AB}, N_B\}_{k_{BS}}, \{N_B\}_{k_{AB}}$

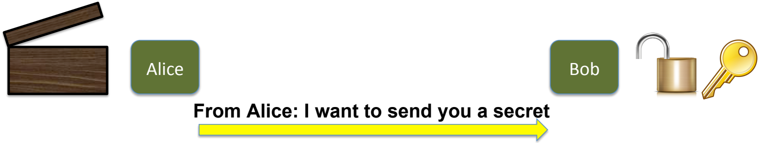
Figure: Paulson's variant of the Yahalom protocol

Goal of public-key encryption

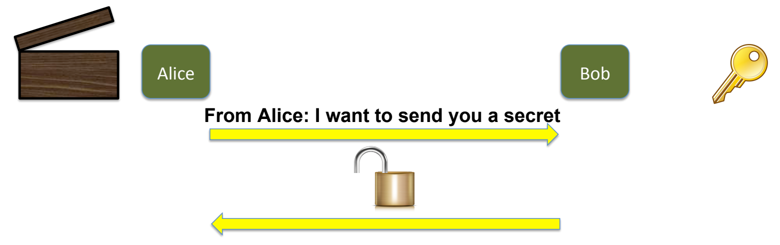
Alice

Bob

Goal of public-key encryption

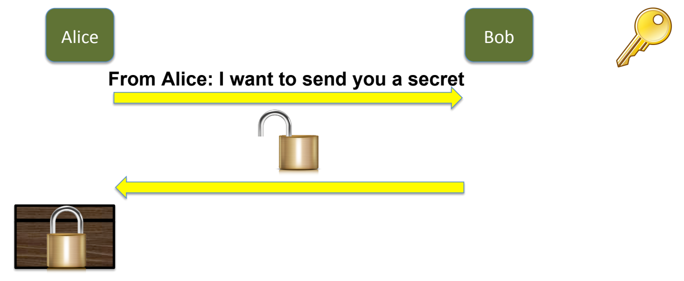


Goal of public-key encryption



From Alice: I want to send you a secret

Goal of public-key encryption



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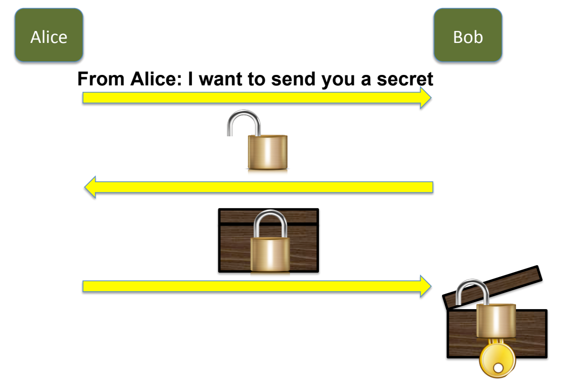
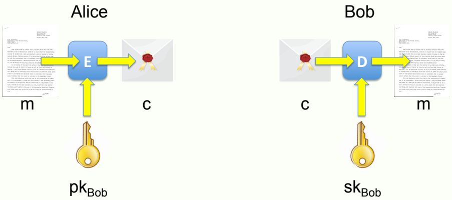
From Alice: I want to send you a secret

Goal of public-key encryption



Public-key encryption - Definition

- key generation algorithm: $G : \to \mathcal{K}_{pk} \times \mathcal{K}_{sk}$
  encryption algorithm $E : \mathcal{K}_{pk} \times \mathcal{M} \to \mathcal{C}$
  decryption algorithm $D : \mathcal{K}_{sk} \times \mathcal{C} \to \mathcal{M}$
  st. $\forall(sk, pk) \in G$, and $\forall m \in \mathcal{M}, D(sk, E(pk, m)) = m$



- the decryption key sk_{Bob} is secret (only known to Bob). The encryption key pk_{Bob} is known to everyone. And $sk_{Bob} \neq pk_{Bob}$

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Asymmetric encryption

We need a bit of number theory now

Primes

Definition

$p \in \mathbb{N}$ is a **prime** if its only divisors are 1 and p

Ex: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Theorem

Every $n \in \mathbb{N}$ has a **unique factorization** *as a product of prime numbers (which are called its factors)*

Ex: $23244 = 2 \times 2 \times 3 \times 13 \times 149$

Relative primes

Definition

a and b in $\mathbb{Z}$ are **relative primes** if they have no common factors

Definition

The Euler function $\phi(n)$ is the number of elements that are relative primes with n:

$$\phi(n) = |\{m \mid 0 < m < n \text{ and } \mathrm{gcd}(m, n) = 1\}|$$

- For p prime: $\phi(p) = p\text{-}1$
- For p and q primes: $\phi(p \cdot q) = (p\text{-}1)(q\text{-}1)$

Integers modulo n: $\mathbb{Z}_n$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n = \{0, \dots n\text{-}1\}$

$$\forall a \in \mathbb{Z}_n, \forall b \in \mathbb{Z},\ a \equiv b \pmod{n} \ \Leftrightarrow\ \exists k \in \mathbb{Z}.\ b = a + k \cdot n$$

- Modular inversion: the inverse of $x \in \mathbb{Z}_n$ is $y \in \mathbb{Z}_n$ *s.t.* $x \cdot y \equiv 1 \pmod{n}$. We denote x^{-1} the inverse of x mod n
  Ex: 7^{-1} in $\mathbb{Z}_{12}$: 7
  4^{-1} in $\mathbb{Z}_{12}$: 4 has no inverse in $\mathbb{Z}_{12}$

Theorem

Let $n \in \mathbb{N}$. Let $x \in \mathbb{Z}_n$. x has a inverse in $\mathbb{Z}_n$ iff $\gcd(x, n) = 1$

Inverse is computed using the Extended Euclidean Algorithm.

Multiplicative group of integers modulo n: $\mathbb{Z}_n^*$

- Let $n \in \mathbb{N}$. We define $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n \mid \gcd(x, n) = 1\}$
  Ex: $\mathbb{Z}_{12} = \{1, 5, 7, 11\}$
- Note that $|\mathbb{Z}_n^*| = \phi(n)$

Theorem (Euler)

$\forall n \in \mathbb{N}, \forall x \in \mathbb{Z}_n^*, x^{\phi(n)} \equiv 1 \pmod{n}$

Theorem (Euler)

$\forall p$ prime, $\mathbb{Z}_p^*$ is a cyclic group, i.e.

$$\exists g \in \mathbb{Z}_p^*, \{1, g, g^2, g^3, \dots, g^{p-2}\} = \mathbb{Z}_p^*$$

Intractable problems

- Factoring:
  input: $n \in \mathbb{N}$
  output: $p_1, \dots, p_m$ primes *st.* $n = p_1 \cdot \dots \cdot p_m$
- RSA Problem
  input: n *st.* $n = p \cdot q$ with $2 \leq p, q$ primes
  e *st.* $\gcd(e, \phi(n)) = 1$
  $m^e \mod n$
  output: m
- Discrete Logarithm Problem (DLP):
  input: prime p, generator g of $\mathbb{Z}_p^*$, $y \in \mathbb{Z}_p^*$
  output: x such that $y = g^x \pmod{p}$
- Diffie-Hellman Problem (DHP):
  input: prime p, generator g of $\mathbb{Z}_p^*$, $g^a \pmod{p}$, $g^b \pmod{p}$
  output: $g^{ab} \pmod{p}$

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Asymmetric encryption

Establish a key without a TTP

The Diffie-Hellman (DH) protocol

- Assumption: the DHP is hard in $\mathbb{Z}_p^*$
- Fix a very large prime p, and g generator of $\mathbb{Z}_p^*$

Alice	Bob
$a \overset{r}{\leftarrow} \{1, \ldots , p\text{-}1 \}$	$b \overset{r}{\leftarrow} \{1, \ldots , p\text{-}1 \}$

The Diffie-Hellman (DH) protocol

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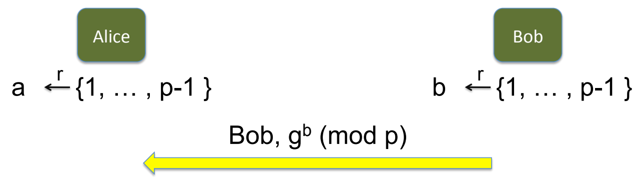

Alice

Bob

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Bob, g^b (mod p)

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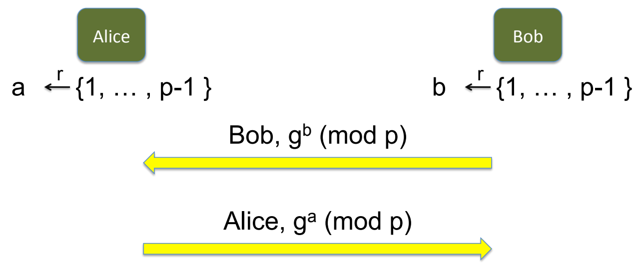

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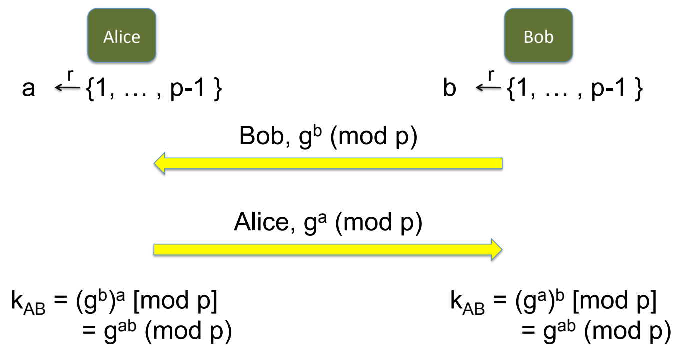

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$b \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

Bob, g^b (mod p)

Alice, g^a (mod p)

$k_{AB} = (g^b)^a$ [mod p]
$= g^{ab}$ (mod p)

$k_{AB} = (g^a)^b$ [mod p]
$= g^{ab}$ (mod p)

Man in the middle attack on DH

Alice

$a \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

Attacker

$a' \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

$b' \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

Bob

$b \xleftarrow{r} \{1, \ldots, p\text{-}1\}$

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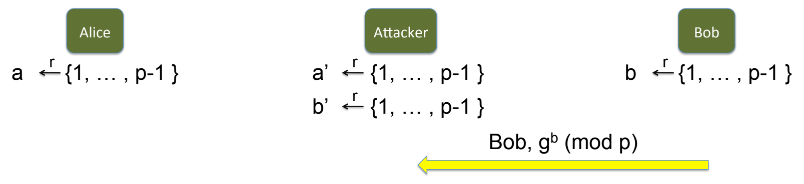

Alice

$a \xleftarrow{r} \{1, \ldots, p-1\}$

Attacker

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Bob

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Bob, $g^b \pmod{p}$

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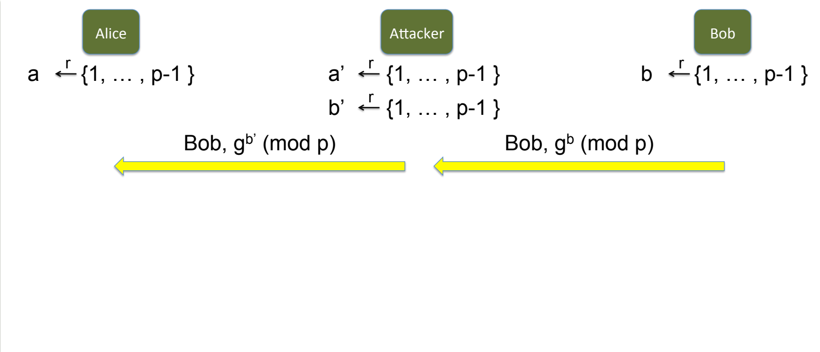

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Bob

$b \xleftarrow{r} \{1, \dots, p-1\}$

Bob, $g^{b'}$ (mod p)

Bob, g^{b} (mod p)

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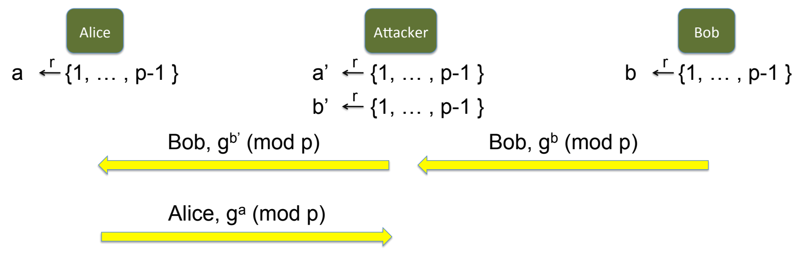

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Bob, g^{b} (mod p)

Alice, g^{a} (mod p)

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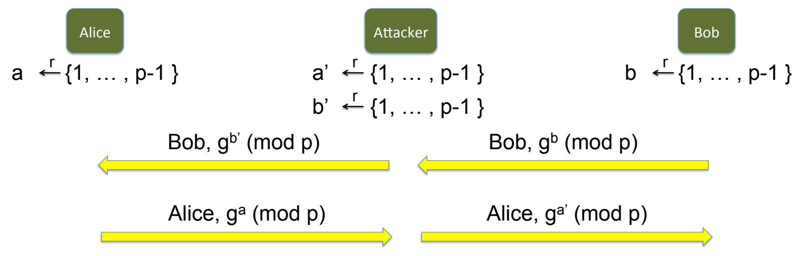

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Bob, g^{b} (mod p)

Alice, g^{a} (mod p)

Alice, $g^{a'}$ (mod p)

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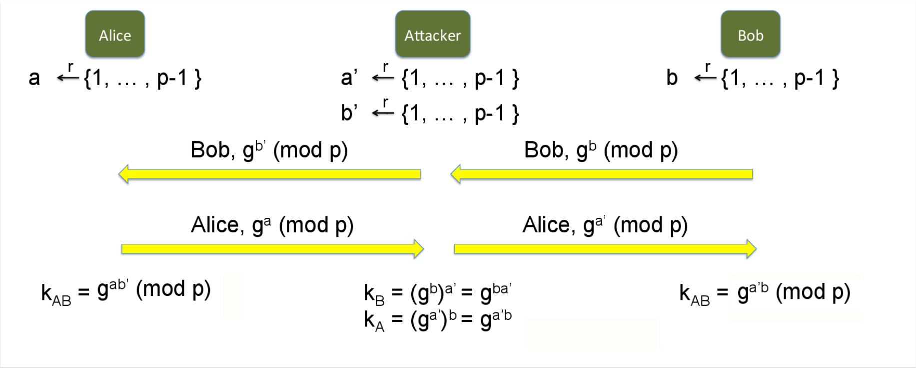

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Bob, $g^{b'} \pmod p$

Bob, $g^{b} \pmod p$

Alice, $g^{a} \pmod p$

Alice, $g^{a'} \pmod p$

$k_{AB} = g^{ab'} \pmod p$

$k_B = (g^b)^{a'} = g^{ba'}$

$k_A = (g^{a'})^b = g^{a'b}$

$k_{AB} = g^{a'b} \pmod p$

RSA trapdoor permutation

- $G_{RSA}() = (pk, sk)$ where $pk = (N, e)$ and $sk = (N, d)$
  and $N = p \cdot q$ with p, q random primes
  and $e, d \in \mathbb{Z}$ st. $e \cdot d \equiv 1 \pmod{\phi(N)}$
- $\mathcal{M} = \mathcal{C} = \mathbb{Z}_N$
- $RSA(pk, x) = x^e \pmod{N}$ where $pk = (N, e)$
- $RSA^{-1}(sk, x) = x^d \pmod{N}$ where $sk = (N, d)$
- Consistency: $\forall (pk, sk) = G_{RSA}(), \forall x, RSA^{-1}(sk, RSA(pk, x)) = x$

Proof: Let $pk = (N, e)$, $sk = (N, d)$. and $x \in \mathbb{Z}_N$. Easy case where x and N are relatively prime

$$\begin{aligned} RSA^{-1}(sk, RSA(pk, x)) &= (x^e)^d \pmod{N} \\ &= x^{e \cdot d} \pmod{N} \\ &= x^{1+k\phi(N)} \pmod{N} \\ &= x \cdot x^{k\phi(N)} \pmod{N} \\ &= x \cdot (x^{\phi(N)})^k \pmod{N} \\ &\overset{\text{Euler}}{=} x \pmod{N} \end{aligned}$$

How NOT to use RSA

(G_{RSA}, RSA, RSA^{-1}) is called raw RSA (or sometimes Textbook RSA).

Do not use raw RSA directly as an asymmetric cipher!

RSA is deterministic and malleable ⇒ not secure against chosen plaintext attacks

ISO standard: ISO/IEC 18033-2

Goal: build a CPA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})

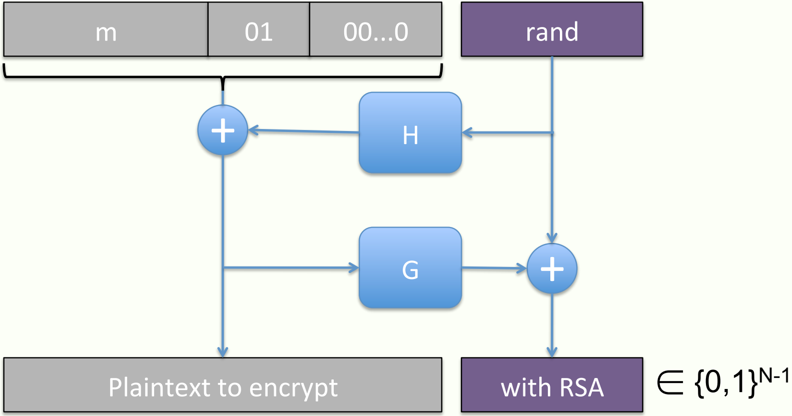
Let (E_s, D_s) be a symmetric encryption scheme over $(\mathcal{M}, \mathcal{C}, \mathcal{K})$
Let $H : \mathbb{Z}_N^* \to \mathcal{K}$

Build $(G_{RSA}, E_{RSA}, D_{RSA})$ as follows

- $G_{RSA}()$ as described above
- $E_{RSA}(pk, m)$:
    - pick random $x \in \mathbb{Z}_N^*$
    - $y \leftarrow RSA(pk, x)$ $(= x^e \mod N)$
    - $k \leftarrow H(x)$
    - return $y || E_s(k, m)$
- $D_{RSA}(sk, y||c) = D_s(H(RSA^{-1}(sk, y)), c)$

PKCS1 v2.0: RSA-OAEP (old version)

Goal: build a CCA secure asymmetric cipher using (G_{RSA}, RSA, RSA^{-1})



PKCS1 v2.0: RSA-OAEP (wiki version)

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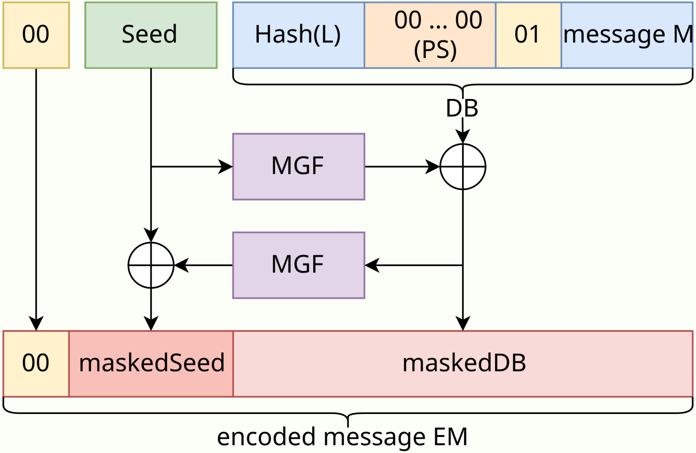

ElGamal (EG)

- Fix prime p, and generator $g \in \mathbb{Z}_p^*$
- $\mathcal{M} = \{0, \dots, p\text{-}1\}$ and $\mathcal{C} = \mathcal{M} \times \mathcal{M}$
- $G_{EG}() = (pk, sk)$ where $pk = g^d \pmod p$ and $sk = d$ and $d \xleftarrow{r} \{1, \dots, p\text{-}2\}$
- $E_{EG}(pk, x) = (g^r \pmod p),\ m \cdot (g^d)^r \pmod p)$ where $pk = g^d \pmod p$ and $r \xleftarrow{r} \mathbb{Z}$
- $D_{EG}(sk, x) = e^{\text{-}d} \cdot c \pmod p$ where $x = (e, c)$
- Consistency: $\forall (pk, sk) = G_{EG}(), \forall x, D_{EG}(sk, E_{EG}(pk, x)) = x$

Proof: Let $pk = g^d \pmod p$ and $sk = d$

$$\begin{aligned} D_{EG}(sk, E_{EG}(pk, x)) &= (g^r)^{\text{-}d} \cdot m \cdot (g^d)^r \pmod p \\ &= m \pmod p \end{aligned}$$