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Good Differential Forms for a Family of Schemes

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Abstract. We construct nicely behaved reflexive differential forms for the family $f : \mathrm{Spec}\ k[x, y, t, w]/(xy - tw) \to \mathrm{Spec}\ k[t]$ of schemes using logarithmic geometry. Our main result is that these differential forms remain reflexive under base change. Our methods are partially inspired by the Gross-Siebert program. We explain the concepts of log geometry from scratch, making the thesis accessible to the reader not familiar with log geometry.

Kurzfassung. Wir verwenden logarithmische Geometrie, um für die Familie $f : \mathrm{Spec}\ k[x, y, t, w]/(xy - tw) \to \mathrm{Spec}\ k[t]$ von Schemata reflexive Differentialformen mit gutem Verhalten zu konstruieren. Unser Hauptresultat ist, dass diese Differentialformen unter Basiswechsel reflexiv bleiben. Unsere Methoden sind teilweise durch das Gross-Siebert-Programm inspiriert. Die Arbeit ist außerdem auch eine Einführung in log-Geometrie.

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Eidesstattliche Erklärung

Hiermit versichere ich, Simon Felten, dass ich die Masterarbeit selbstständig, ohne fremde Hilfe verfasst und keine anderen als die angegebenen Hilfsmittel benutzt habe. Die Stellen der Arbeit, die dem Wortlaut oder dem Sinn nach anderen Werken entnommen wurden, sind unter Angabe der Quellen der Entlehnung kenntlich gemacht. Die Arbeit ist noch nicht veröffentlicht oder in gleicher oder anderer Form an irgendeiner Stelle als Prüfungsleistung vorgelegt worden.

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Deutschsprachige Zusammenfassung

Wenn $f : X \to S$ ein glatter und eigentlicher Morphismus von Schemata über einem Körper k der Charakteristik 0 ist, dann sind gemäß einem klassischen Resultat von Deligne (siehe [3]) die Garben $R^q f_* \Omega^p_{X/S}$ und $R^i f_* \Omega^\bullet_{X/S}$ lokal frei von endlichem Rang. Inspiriert durch die Idee des Gross-Siebert-Programms, Kohomologiedimensionen auf Entartungen glatter Varietäten zu berechnen, wollen wir dieses Resultat verallgemeinern, etwa auf die Entartung einer glatten Quartik in vier Ebenen, gegeben durch

$$\mathcal{X} = \{T_1(X^4 + Y^4 + Z^4 + W^4) - T_0 XYZW = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3 \xrightarrow{pr_1} \mathbb{P}^1 \ .$$

Da für allgemeine $f : X \to S$ der de-Rham-Komplex $\Omega^\bullet_{X/S}$ nicht mehr lokal frei ist, sondern z.B. Torsion haben kann, wollen wir ihn durch einen Komplex $W^\bullet_{X/S}$ mit besseren Eigenschaften ersetzen und hoffen, dass dann $R^q f_* W^p_{X/S}$ und $R^i f_* W^\bullet_{X/S}$ lokal frei sind. In dieser Masterarbeit beschränken wir uns darauf, $W^\bullet_{X/S}$ für

$$f : X := \mathrm{Spec}\ k[x,y,t,w]/(xy - tw) \to \mathrm{Spec}\ k[t] =: S$$

zu konstruieren und dessen Eigenschaften zu untersuchen. Der Morphismus ist ein étale lokales Modell für $\mathcal{X} \to \mathbb{P}^1$ an dessen interessantesten Punkten, und wir fassen diese Masterarbeit als Vorarbeit auf, um die Entartung für Morphismen zu zeigen, die lokal durch ähnliche Morphismen wie $f : X \to S$ beschrieben werden können.

Um (einen Kandidaten für) $W^\bullet_{X/S}$ zu konstruieren, statten wir X und S mit log-Strukturen aus und erhalten log-Schemata $X^\dagger$ und $S^\dagger$. Die log-Differentialformen $\Omega^1_{X^\dagger/S^\dagger}$ bilden *keine* kohärente Garbe, aber auf $U = X \setminus \{x = y = t = w = 0\}$ ist $\Omega^1_{U^\dagger/S^\dagger}$ lokal frei. Wir definieren $W^\bullet_{X/S} := j_* \Omega^\bullet_{U^\dagger/S^\dagger}$ für die Inklusion $j : U \subset X$. Die Garben $W^m_{X/S}$ sind dann reflexiv, und wir berechnen außerdem noch deren globale Schnitte. Unser Hauptresultat ist, dass für ein kartesisches Diagramm

$$\begin{array}{ccc} Y & \xrightarrow{c} & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{b} & S \end{array}$$

(mit affinem T) die Urbildgarbe $c^* W^m_{X/S}$ wieder reflexiv ist. Wir interpretieren dieses Resultat als Stabilität unserer reflexiven Differentialformen unter Basiswechseln, zumal da jetzt $c^* W^m_{X/S} \cong W^m_{Y/T}$, wenn wir letztere Garbe mit Hilfe der log-Struktur auf Y/T konstruieren. Unsere reflexiven Differentialformen sind jedoch i.A. nicht das Biduale der Kähler-Differentiale, d.h. $c^* W^m_{X/S} \ncong (\Omega^m_{Y/T})^{**}$.

Im Anhang beweisen wir, dass wir für ein kartesisches Diagramm von kohärenten log-Schemata einen kanonischen Homomorphismus $\Omega^\bullet_{X/S} \to c_* \Omega^\bullet_{Y/T}$ von Komplexen erhalten. Diese naheliegende Verallgemeinerung auf den log-Kontext eines klassischen Resultats konnten wir nirgends in der Literatur finden.

Contents

1 Introduction and Motivation

Let X/k be a smooth and proper scheme over a field k of characteristic 0. Inspired by a naive form of numerical homological mirror symmetry, we are interested in the dimensions of the cohomologies $H^q(X, \Omega^p_{X/k})$ and $\mathbb{H}^i(X, \Omega^\bullet_{X/k})$. By a classical result of Deligne [3], if we have a whole smooth and proper family

$$f : X \to S$$

defined over k, then $R^q f_* \Omega^p_{X/S}$ and $R^i f_* \Omega^\bullet_{X/S}$ are finite locally free and commute with base change, so we may attempt to compute the cohomology dimensions of X by putting it into a family and choose a particularly simple fiber for the computation. We aim to further simplify by computing on a singular, but even simpler fiber, so our setting is as follows: Let $f : X \to S$ be a flat and proper family of schemes (of relative dimension d). We assume that outside a single point $0 \in S$, the family is smooth, so we have a single singular fiber X_0/k. Now we like to compute the dimensions of $H^q(X_s, \Omega^p_{X_s/k})$ and $\mathbb{H}^i(X_s, \Omega^\bullet_{X_s/k})$ for a general fiber over $s \neq 0$ as the dimensions of a suitable cohomology on X_0. We cannot just choose $H^q(X_0, \Omega^p_{X_0/k})$ and $\mathbb{H}^i(X_0, \Omega^\bullet_{X_0/k})$ for the theorem of Deligne only applies to the smooth setting. By a suitable cohomology, we roughly mean that we have a class $\mathcal{L}$ of morphisms stable under base change and containing the families we are interested in, and for every $X/S \in \mathcal{L}$, we have a complex $W^\bullet_{X/S}$ of coherent sheaves on X such that the following conjecture holds:

Conjecture 1.1. *For every $(X \xrightarrow{f} S) \in \mathcal{L}$, the sheaves $R^q f_* W^p_{X/S}$ and $R^i f_* W^\bullet_{X/S}$ are finite locally free and commute with base change. Moreover, the spectral sequence*

$$R^q f_* W^p_{X/S} \Rightarrow R^{p+q} f_* W^\bullet_{X/S}$$

degenerates at E_1.

We do not attempt to prove this conjecture within this thesis, nor do we even attempt to define the class $\mathcal{L}$. Instead, we restrict ourselves to the example of the degeneration of a quartic surface to an arrangement of planes. We will explain the example in detail in the next section. At its most interesting points, this family is locally modeled on

$$\text{Spec } k[x, y, t, w]/(xy - tw) \to \text{Spec } k[t]$$

and this thesis is primarily about the correct definition of $W^\bullet_{X/S}$ on this affine family, and about its properties. The definition will be given at the end of the introduction. When searching for the correct definition, we will orientate ourselves on the local properties of the de Rham complex of smooth morphisms, and ignore the conjecture. We call differentials with nice local properties *good*, and the precise meaning of this will be elaborated in the course of the introduction. We hope to elaborate on the definition of $\mathcal{L}$ and a proof of Conjecture 1.1 elsewhere.

We like to mention that the Gross-Siebert program has been a great source of inspiration for us. It contains all the ideas of degenerating a smooth variety and defining new differential forms on these degenerations. We will end up using the same replacements as Gross and Siebert do. Also both the degeneration of the smooth quartic and its local model show up there, and our methods have been partially inspired by theirs.

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1.1 The Degeneration of a Smooth Quartic Surface

We give an example of the situation sketched above. Fix an algebraically closed field k of characteristic 0, and let us consider the well-known degeneration of a smooth quartic surface into an arrangement of planes $\{XYZW = 0\} \subset \mathbb{P}^3$. Namely, set

$$\mathcal{X} = \{T_0(X^4 + Y^4 + Z^4 + W^4) - T_1 XYZW = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3 ,$$

where X, Y, Z, W are homogeneous coordinates of $\mathbb{P}^3$, and T_0, T_1 are homogeneous coordinates of $\mathbb{P}^1$. Projecting to the first coordinate, we obtain a morphism $\varphi : \mathcal{X} \to \mathbb{P}^1$ which is flat, projective and surjective. Setting $0 = [0 : 1]$ and $\infty = [1 : 0]$, the two fibers are

$$\mathcal{X}_0 = \{XYZW = 0\} \subset \mathbb{P}^3 \quad \text{and} \quad \mathcal{X}_\infty = \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{P}^3 .$$

The fiber $\mathcal{X}_\infty$ is a smooth quartic surface. Moreover, there is an open $V \subset \mathbb{P}^1$ such that $\mathcal{X} \times_{\mathbb{P}^1} V \to V$ is smooth. Now setting

$$S := (V \cup \{0\}) \setminus \{\infty\} \subset \mathbb{A}^1 \subset \mathbb{P}^1$$

we obtain a family

$$f = \varphi \times_{\mathbb{P}^1} S : \quad X = \mathcal{X} \times_{\mathbb{P}^1} S \to S$$

having a single singular fiber $X_0 = \mathcal{X}_0$, and all other fibers are smooth. We see that $f : X \to S$ is a flat, projective, surjective Cohen-Macaulay morphism, the latter condition meaning that $f : X \to S$ is flat, and all its fibers are Cohen-Macaulay schemes. The central fiber consists of four copies of $\mathbb{P}^2$ intersecting in a total of six lines $\mathbb{P}^1$ like in Figure 1. We denote by $L \subset X_0$ the union of the six lines, and we set

$$Z := L \cap \{X^4 + Y^4 + Z^4 + W^4 = 0\} \subset \mathbb{P}^3 .$$

The set Z consists of 24 points, 4 on each line, and it turns out that Z is the singular locus of the total space X, i.e. $U = X \setminus Z$ is regular. The divisor $X_0 \cap U \subset U$ is a strict normal crossing divisor whereas $X_0 \subset X$ is not a normal crossing divisor since X is not regular in the points $z \in Z$. We mention that this means that we can consider $f : X \to S$ *logarithmically smooth* on U, but not in Z, but we won't elaborate on this.

As outlined above, we are searching for a nicely behaved complex $W^\bullet_{X/S}$ of sheaves replacing $\Omega^\bullet_{X/S}$, so we should search for a replacement of $\Omega^\bullet_{X/S}$ first locally. The most interesting points are the singularities $z \in Z$ of the total space X. Consider for

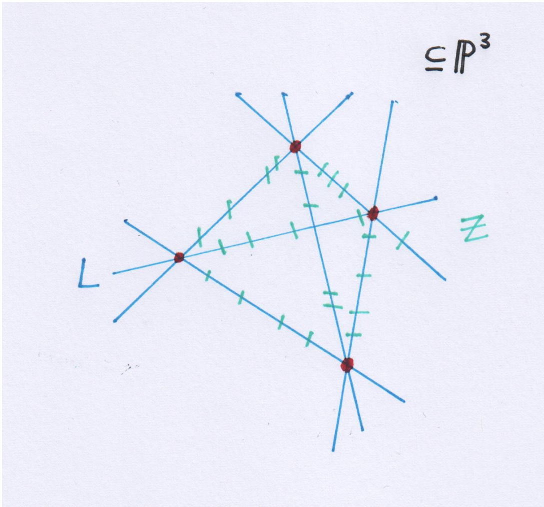

Figure 1: The central fiber X_0 consists of four copies of $\mathbb{P}^2$ intersecting in six lines $\mathbb{P}^1$.

example $p = [0 : 0 : z : 1] \in Z$ with $1 + z^4 = 0$. The point p is contained in the coordinate chart

$$U_{T_1 W} = \{T_1 \neq 0, W \neq 0\} \cong \mathbb{A}^1 \times \mathbb{A}^3 \subset \mathbb{P}^1 \times \mathbb{P}^3 ,$$

and on this chart, $\mathcal{X}$ is given by $t(x^4 + y^4 + z^4) - xyz = 0$. Consider the map

$$\begin{aligned} \psi : k[x', y', t', w'] &\to k[x, y, z, t] = \mathcal{O}(U_{T_1 W}) \\ x' \mapsto x, \quad y' \mapsto yz, \quad & t' \mapsto t, \quad w' \mapsto x^4 + y^4 + z^4 \end{aligned}$$

giving a map $\psi : U_{T_1 W} \to \mathbb{A}^4$ which turns out to be étale around $p \in U_{T_1 W}$. We see that $\psi^{-1}(\{x'y' - t'w' = 0\}) = \mathcal{X} \cap U_{T_1 W}$, so we obtain a commutative diagram

$$\begin{tikzcd}
S \arrow[d] & X \cap U_{T_1 W} \arrow[l] \arrow[d] & \\
\mathbb{A}^1 \arrow[d, equal] & \mathcal{X} \cap U_{T_1 W} \arrow[l] \arrow[r] \arrow[d, "\tilde{\psi}"] & U_{T_1 W} \arrow[d, "\psi"] \\
\mathbb{A}^1 & \{x'y' - t'w' = 0\} \arrow[l, "t \mapsto t'"'] \arrow[r] & \mathbb{A}^4
\end{tikzcd}$$

with $\tilde{\psi}$ étale around $p \in X \cap U_{T_1 W}$. We can summarize this as follows: The point

$p \in Z$ has an open neighbourhood $U_p \subset X$ that fits into a commutative diagram

$$\begin{tikzcd} U_p \arrow[r, "\tilde{\psi}"] \arrow[d] & \mathrm{Spec}\ B \arrow[dd] \\ X \arrow[d, "f"] & \\ S \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

with $B = k[x', y', t', w']/(x'y' - t'w')$ and $\tilde{\psi}$ étale. Moreover, we can find analogous diagrams for the remaining points of Z. Therefore, to understand $\Omega^\bullet_{X/S}$ for $f : X \to S$ and to find a good replacement, we should first understand differential forms for

$$\mathrm{Spec}\ k[x, y, t, w]/(xy - tw) \to \mathrm{Spec}\ k[t]$$

since this is a local model for a neighbourhood of $p \in Z \subset X$. The rest of the thesis is devoted to this family. After understanding Spec $B \to$ Spec $k[t]$, it will be straightforward how to define $W^\bullet_{X/S}$, but we won't elaborate on this within this thesis.

1.2 More on the Local Model

We fix notation and introduce first properties of the affine family that occurred as a local model in the previous section, and we will stick to the ring picture of that family for some time. In the case of colliding notation, the notation of this section is relevant for the rest of the thesis. Choose a base field k (not necessarily algebraically closed, arbitrary characteristic) and consider the k-algebras $A := k[t]$ and $B := k[x, y, t, w]/(xy - tw)$. For convenience we also introduce notation $C := k[x, y, t, w]$ and $F = xy - tw \in C$. The ring B becomes an A-algebra via $\phi : A \to B, t \mapsto t$. The ring B is integral and ϕ is injective, so B is torsion free as a module over A, whence flat. Geometrically, we get a flat family $f : X \to S$. We have for the localization A_t that $B \otimes_A A_t = B_t \cong k[x, y, t, t^{-1}]$, so the generic fiber of f is the plane $\mathbb{A}^2$. The central fiber is $B_0 := B \otimes_A A/(t) = B/(t) \cong k[x, y, w]/(xy)$, the union of two planes $\mathbb{A}^2$ intersecting in a line $L = \{x = 0, y = 0\}$. Thus we consider $f : X \to S$ a degeneration of the plane into the singular space $X_0 :=$ Spec B_0.

As a first step towards 'good' differential forms, we revisit the usual Kähler differentials $\Omega^m_{B/A}$. The closed embedding $(F) \hookrightarrow C \twoheadrightarrow B$ defined by the ideal $(F) \subset C$ yields an exact sequence

$$(F)/(F^2) \xrightarrow{\delta} \Omega^1_{C/A} \otimes_C B \to \Omega^1_{B/A} \to 0$$

of B-modules where $\delta \bar{F} = dF \otimes 1$. Since $\Omega^1_{C/A} = C \cdot dx \oplus C \cdot dy \oplus C \cdot dw$ and $dF = ydx + xdy - tdw$, we get

$$\Omega^1_{B/A} = (B \cdot dx \oplus B \cdot dy \oplus B \cdot dw) / (ydx + xdy - tdw) \ .$$

In Section 2.3 we will see that $\Omega^1_{B/A}$ is torsion free as a B-module. In particular, it is torsion free as an A-module, so flat over A. Though, these differentials do not behave like the differentials of a smooth family. E.g. consider the central fiber B_0 with differentials

$$\Omega^1_{B_0/k} = \Omega^1_{B/A} \otimes_A A/(t) = (B_0 \cdot dx \oplus B_0 \cdot dy \oplus B_0 \cdot dw)/(ydx + xdy) .$$

We see that $xdx \wedge dy = -ydx \wedge dx = 0$, but we will show that $dx \wedge dy \neq 0$, so $\Omega^2_{B_0/k}$ has torsion. We will also show that $dx \wedge dy \wedge dw \neq 0$, so we have $\Omega^3_{B_0/k} \neq 0$ and in particular $\Omega^3_{B/A} \neq 0$. So the differential forms for B/A behave much worse than differential forms for a smooth ring map, and we like to find a replacement of $\Omega^m_{B/A}$ that behaves as much like differentials of smooth maps as possible.

1.3 First Approaches to Good Differentials

A first approach are the *reflexive differentials*. Using the notation of e.g. [5], we set

$$\Omega^{[m]}_{B/A} := (\Omega^m_{B/A})^{**}$$

where $M^* := \mathrm{Hom}_B(M, B)$ are the module homomorphisms. Such differentials have been studied to some extent, and are usually used on normal spaces. Danilov uses these differential forms throughout the paper [2] for toric varieties and similar spaces and calls them *differentials in the sense of Zariski-Steenbrink*. Due to Danilov's contribution to these differentials, at least we also call them *Danilov differentials*. Steenbrink uses a closely related notion of differentials in [18]. In his context of V-manifolds, he gets that $\Omega^{[m]}_X = 0$ for $m > \dim X$, that the complex is a resolution of the constant sheaf $\underline{\mathbb{C}}$ and that some Hodge-to-deRham spectral sequence degenerates. This is a nice technical feature we like to have for good differential forms, but spectral sequences are beyond the scope of this thesis.

In our situation, it turns out that $\Omega^{[1]}_{B/A}$ - or more precisely the sheaf $\Omega^{[1]}_{X/S}$ associated to it - is locally free on $U := X \setminus \{P\}$ with $P \in X$ the point defined by the maximal ideal $\mathfrak{m}_P := (x, y, t, w) \subset B$. Thus the pull back to the central fiber $\Omega^{[1]}_{B/A} \otimes_A A/(t)$ is locally free on $X_0 \setminus \{P\}$, but we will see at the end of Section 6.5 that it is not locally free on X_0, and therefore also $\Omega^{[1]}_{X/S}$ is not locally free everywhere on X. Thus $\Omega^{[1]}_{B/A} \otimes_A A/(t)$ somehow marks the point P on X_0. This marking does not show up in the geometry of X_0 which is the union of two planes intersecting in a line. The scheme $X_0 \cong \mathbb{A}^1 \times \mathrm{Spec}\ k[x, y]/(xy)$ has no 'marked' point. On the other hand side, we expect the reflexive differentials $(\Omega^1_{B_0/k})^{**}$ of X_0 to reflect the symmetry of X_0, and indeed, we have $\Omega^{[1]}_{B/A} \otimes_A A/(t) \ncong (\Omega^1_{B_0/k})^{**}$. This is because $(\Omega^1_{B_0/k})^{**}$ is not locally free on $X_0 \setminus C$ where $C = \{w = 0\} \subset X_0$ is the union of two lines intersecting in $P \in X_0$. For a rigorous proof, first introduce notation $R := k[x, y]/(xy)$. Using the left half of the tensor product diagram

$$\begin{tikzcd}
R = k[x,y]/(xy) \arrow[r] & B_0 \arrow[r] & (B_0)_w \\
k \arrow[u] \arrow[r] & k[w] \arrow[u] \arrow[r] & k[w,w^{-1}] \arrow[u]
\end{tikzcd}$$

of flat ring homomorphisms, we obtain

$$\Omega^1_{B_0/k} \cong \Omega^1_{k[w]/k} \otimes_{k[w]} B_0 \oplus \Omega^1_{R/k} \otimes_R B_0 \ .$$

Because the bidual $(-)^{**}$ commutes with direct sums and the pullback along flat ring homomorphisms, we get

$$(\Omega^1_{B_0/k})^{**} \cong (\Omega^1_{k[w]/k})^{**} \otimes_{k[w]} B_0 \oplus (\Omega^1_{R/k})^{**} \otimes_R B_0 \ .$$

Now if we had $\Omega^{[1]}_{B/A} \otimes_A A/(t) \cong (\Omega^1_{B_0/k})^{**}$, then $(\Omega^1_{B_0/k})^{**} \otimes_{B_0} (B_0)_w$ was locally free (since $P \in \{w = 0\}$). Therefore $(\Omega^1_{R/k})^{**} \otimes_R (B_0)_w$ was flat over $(B_0)_w$ as a kernel of a surjection of flat modules, meaning it was locally free. The map $R \to (B_0)_w$ is faithfully flat, so flatness of $(\Omega^1_{B_0/k})^{**} \otimes_{B_0} (B_0)_w$ implies $(\Omega^1_{B_0/k})^{**}$ flat over R. This means $(\Omega^1_{R/k})^{**}$ locally free which is a contradiction, for a direct computation yields that this module is not locally free. We have included it in the appendix (see Section 8.1).

We see that it is not natural to take for all families the reflexive differentials as a replacement for Ω^1 since they do not satisfy a base change property. Nonetheless it turns out that $\Omega^{[1]}_{B/A} \otimes_A A/(t)$ is reflexive, so we may hope to consider it as a good reflexive replacement of $\Omega^1_{B_0/k}$ on the central fiber, and to consider $\Omega^{[1]}_{B/A}$ as the correct reflexive replacement of $\Omega^1_{B/A}$ on the whole family. Of course, we may just consider

$$\Omega^{[1]}_{B/A} \otimes_A T$$

as the correct replacement of $\Omega^1_{B\otimes_A T/T}$, but this is not intrinsic to the family $B \otimes_A T/T$. To find an intrinsic reflexive replacement, we have to add more data to the family to break the symmetry on the central fiber, yielding $\Omega^{[1]}_{B/A} \otimes_A A/(t)$ without (direct) reference to B/A. We will find these data in the realm of logarithmic geometry, and it will seem more by chance than by reason that $\Omega^{[1]}_{B/A}$ is the correct replacement of $\Omega^1_{B/A}$.

The most basic object of log geometry is a *log ring* $\alpha : M \to R$ where R is a ring (commutative with unit, as all our rings), M is a monoid (recall that a *monoid* is a set M together with an associative binary operation $+ : M \times M \to M$ that admits a neutral element $0 \in M$; our monoids are assumed commutative unless stated otherwise) and α is a monoid homomorphism, where we consider R as a monoid with its multiplication, forgetting the addition. The ring R is called the *underlying ring* of the log ring. If $\alpha : M \to R, \alpha' : M' \to R'$ are two log rings, then a morphism is given by a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & R \\ {\scriptstyle\theta_m}\downarrow & & {\scriptstyle\theta_r}\downarrow \\ M' & \xrightarrow{\alpha'} & R' \end{array}$$

where $\theta_m : M \to M'$ is a monoid homomorphism and $\theta_r : R \to R'$ is a ring homomorphism. Every ring R may be considered a log ring as $0 \to R$. Our first non-trivial example is

$$\mathbb{N} \to A = k[t], n \mapsto t^n \ .$$

It turns the base A of our family into a log ring. In the context of the log ring $\mathbb{N} \to A$, we will often write $Q = \mathbb{N}$. The most naive structure of a log ring that we may put on B such that we get a morphism of log rings is $\mathbb{N} \to B, n \mapsto t^n$ fitting in the diagram

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & B \\ \Big\| & & {\scriptstyle\phi}\uparrow \\ \mathbb{N} & \longrightarrow & A \end{array}$$

We call a morphism of log rings *strict*, if θ_m is an isomorphism (but this is not standard), so $(\mathbb{N} \to B)/(\mathbb{N} \to A)$ is strict. So now, what kind of differential may arise from a morphism of log rings? We have the notion of a *log derivation*: Let E be an R-module. Then a log derivation $(D, \Delta) : (R, M) \to E$ consists of a derivation $D : R \to E$ and a monoid homomorphism $\Delta : M \to E$ such that $\alpha(m) \cdot \Delta(m) = D(\alpha(m))$. Here we consider $(E, +)$ a monoid with its addition. This condition is modelled on the behaviour of the derivative of real valued functions: Set $R = \{f : (0, 1) \to \mathbb{R}_> \mid f \text{ is } \mathcal{C}^\infty\}$ and $M = R$ with multiplication as monoid operation. Set $D = \mathrm{d} : R \to R$ the usual derivative and $\Delta = \mathrm{dlog} : M \to R$ first taking the logarithm and then deriving. This yields a log derivation $(D, \Delta) : (R, M) \to R$ since $\mathrm{dlog}(f) = \frac{\mathrm{d}(f)}{f}$.

A log derivation $(D, \Delta) : (R', M') \to E'$ is called *relative* over $(\theta_r, \theta_m) : (R, M) \to (R', M')$, if we have $D \circ \theta_r = 0$ and $\Delta \circ \theta_m = 0$. Just as in the case of classical derivations, for a morphism $(R, M) \to (R', M')$ of log rings, there is a universal relative log derivation

$$(d, \delta) : (R', M') \to \Omega^1_{(R',M')/(R,M)} \ .$$

The module can be explicitly constructed as

$$\Omega^1_{(R',M')/(R,M)} = \left(\Omega^1_{R'/R} \oplus R' \otimes_{\mathbb{Z}} (M')^{gp}\right)/K \ ,$$

where $\Omega^1_{R'/R}$ are the classical relative differentials of $\theta_r : R \to R'$ and $(M')^{gp}$ is the *Grothendieck group* of M', that is the universal (abelian) group through which any monoid homomorphism $M' \to G$ for an abelian group G factors. The quotient is by

$$K = \langle (d\alpha'(m'), -\alpha'(m') \otimes m'), (0, 1 \otimes \theta_m(m)) | m' \in M', m \in M\rangle \ ,$$

the R'-submodule generated by such elements. The universal derivation is

$$\begin{aligned} d : R' \to \Omega^1_{(R',M')/(R,M)}, &\quad r' \mapsto [(dr', 0)] \\ \delta : M' \to \Omega^1_{(R',M')/(R,M)}, &\quad m' \mapsto [(0, 1 \otimes m')] \end{aligned}$$

We get an obvious map $\Omega^1_{R'/R} \to \Omega^1_{(R',M')/(R,M)}$ which is the homomorphism coming from the fact that the classical part d of the universal log derivation is a classical derivation.

Let us see what we get for $(A, \mathbb{N}) \to (B, \mathbb{N})$. We have $\mathbb{N}^{gp} = \mathbb{Z}$, and we get

$$\Omega^1_{(B,\mathbb{N})/(A,\mathbb{N})} = \left(\Omega^1_{B/A} \oplus B \otimes_{\mathbb{Z}} \mathbb{Z}\right) / \langle (d(t^{m'}), -t^{m'} \otimes m'), (0, 1 \otimes m) | m, m' \in \mathbb{N} \rangle$$

We have $d(t^{m'}) = 0$, so $K \subset 0 \oplus B \otimes_{\mathbb{Z}} \mathbb{Z}$, and $(0, 1 \otimes 1) \in K$, so $K = 0 \oplus B \otimes_{\mathbb{Z}} \mathbb{Z}$. This yields $\Omega^1_{(B,\mathbb{N})/(A,\mathbb{N})} = \Omega^1_{B/A}$ and we do not win much over the classical differentials $\Omega^1_{B/A}$. We also do not win anything on the fibers, as we explain now.

If $(R, M) \to (R', M')$ is a morphism of log rings, and $\psi : R \to S$ is a ring homomorphism, there is a canonical way to put a log structure on S and $S' := R' \otimes_R S$. Namely, we get a diagram

$$\begin{tikzcd} M' \arrow[r] & R' \arrow[r] & S' \\ M \arrow[r] \arrow[u] & R \arrow[r] \arrow[u] & S \arrow[u] \end{tikzcd}$$

and we take just the composition. This yields in our example a log structure on the fiber for any ring homomorphism $A \to T$, but as the reader may check we always get

$$\Omega^1_{(B \otimes_A T, \mathbb{N})/(T,\mathbb{N})} = \Omega^1_{B \otimes_A T/T} ,$$

so we really win nothing over the classical differentials since they are just the same. The problem here is that the morphism $(A, \mathbb{N}) \to (B, \mathbb{N})$ of log rings is strict.

The log structure on the base A seems correct since it somehow 'marks' $\{t = 0\}$, the locus over which the fiber is not smooth. Thus we try to change the log structure on B. We set

$$M_B := \{x^a y^b t^c w^d | a, b, c, d \geq 0\} \subset B$$

which is a monoid by multiplication. This is an analogue of $\mathbb{N} \to A = k[t]$, since it marks all the mononials. We can identify M_B also as a submonoid of $\mathbb{Z}^3$. Namely, consider

$$P := \langle (1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1) \rangle \subset \mathbb{Z}^3$$

the submonoid generated by these vectors, that is all their finite sums. We get a map

$$P \to M_B, \quad (1, 0, 0) \mapsto t,\ (1, 1, 0) \mapsto y,\ (1, 0, 1) \mapsto x,\ (1, 1, 1) \mapsto w$$

and the reader may check that this is a well-defined isomorphism of monoids. Later we will see an easier way to verify this, see Section 2.1. There the reader will find also nice pictures giving intuition on the monoid P. Setting $\theta_m : \mathbb{N} \to P, n \mapsto (n, 0, 0)$, we obtain a homomorphism of log rings

$$\begin{array}{ccc} P & \xrightarrow{\alpha_B} & B \\ {\scriptstyle\theta_m}\big\uparrow & & \big\uparrow{\scriptstyle\theta_r=\phi} \\ \mathbb{N} & \xrightarrow{\alpha_A} & A \end{array}$$

Let us compute the log differentials. We have $P^{gp} = \mathbb{Z}^3$, so we get

$$\Omega^1_{(B,P)/(A,\mathbb{N})} = \left(\Omega^1_{B/A} \oplus B \otimes_{\mathbb{Z}} \mathbb{Z}^3\right)$$
$$/\langle (d\alpha_B(p), -\alpha_B(p) \otimes p), (0, 1 \otimes (n, 0, 0)) | p \in P, n \in \mathbb{N}\rangle$$

We make an ansatz $W = B \cdot \frac{dx}{x} \oplus B \cdot \frac{dy}{y}$ as a free module on two generators. The motivation of the notation $\frac{dx}{x}$ will become clear soon. We set

$$\psi : W \to \Omega^1_{(B,P)/(A,\mathbb{N})}, \frac{dx}{x} \mapsto [(0, 1 \otimes (1, 0, 1))], \frac{dy}{y} \mapsto [(0, 1 \otimes (1, 1, 0))]$$

which is obviously a well-defined B-module homomorphism. On the other hand side, recalling that $\Omega^1_{B/A} = (Bdx \oplus Bdy \oplus Bdw)/(ydx + xdy - tdw)$, we can define a module homomorphism via

$$\begin{aligned} \psi' : \Omega^1_{(B,P)/(A,\mathbb{N})} &\to W \\ [(dx, 0)] &\mapsto x \cdot \frac{dx}{x} \\ [(dy, 0)] &\mapsto y \cdot \frac{dy}{y} \\ [(dw, 0)] &\mapsto w \cdot (\frac{dx}{x} + \frac{dy}{y}) \\ [(0, 1 \otimes (1, 0, 0))] &\mapsto 0 \\ [(0, 1 \otimes (0, 1, 0))] &\mapsto \frac{dy}{y} \\ [(0, 1 \otimes (0, 0, 1))] &\mapsto \frac{dx}{x} \end{aligned}$$

It is straightforward to check that ψ' is well-defined and that ψ and ψ' are inverse isomorphisms.

So this log structure yields indeed a free module of differential forms for B/A, the nicest we could hope for. The canonical map $\Omega^1_{B/A} \to \Omega^1_{(B,P)/(A,\mathbb{N})} \cong W$ is given by

$$dx \mapsto x \cdot \frac{dx}{x}, dy \mapsto y \cdot \frac{dy}{y}, dw \mapsto w \cdot (\frac{dx}{x} + \frac{dy}{y})$$

explaining the notation $\frac{dx}{x}$ etc. for the generators of our ansatz W.

Next let us see what $(A, \mathbb{N}) \to (B, P)$ yields on the fibers $(B \otimes_A T)/T$. We already have explained above how to construct a log structure on the fibers of a morphism of log rings. Using the explicit description of the log differential forms, we see that

$$\begin{aligned}\Omega^1_{(B\otimes_A T,P)/(T,\mathbb{N})} &= \left(\Omega^1_{B\otimes_A T/T} \oplus (B \otimes_A T) \otimes_{\mathbb{Z}} \mathbb{Z}^3\right) \\ &\quad /\langle (d\alpha_{B\otimes T}(p), -\alpha_{B\otimes T}(p) \otimes p), (0, 1 \otimes (n, 0, 0)) | p \in P, n \in \mathbb{N}\rangle \\ &= (B \otimes_A T) \otimes_B \left(\Omega^1_{B/A} \oplus B \otimes_{\mathbb{Z}} \mathbb{Z}^3\right) \\ &\quad /\langle (d\alpha_B(p), -\alpha_B(p) \otimes p), (0, 1 \otimes (n, 0, 0)) | p \in P, n \in \mathbb{N}\rangle \\ &= (B \otimes_A T) \otimes_B \Omega^1_{(B,P)/(A,\mathbb{N})}\end{aligned}$$

so we get a free module of differential forms for every base change to some T. Let us pay particular attention to the case $T = k[t, t^{-1}] = A_t$. This yields a family $k[t, t^{-1}] \to B_t \cong k[x, y, t, t^{-1}]$, so it is smooth with classical Kähler differentials

$$\Omega^1_{B_t/A_t} = B_t \cdot dx \oplus B_t \cdot dy \ .$$

The log structure $(B_t, P)/(A_t, \mathbb{N})$ gives rise to the differentials $W \otimes_B B_t = B_t \cdot \frac{dx}{x} \oplus B_t \cdot \frac{dy}{y}$ which give a free B_t-module, too. But the canonical map is given by

$$\Omega^1_{B_t/A_t} \to \Omega^1_{(B_t,P)/(A_t,\mathbb{N})}, dx \mapsto x \cdot \frac{dx}{x}, dy \mapsto y \cdot \frac{dy}{y}$$

so we have altered the differentials on the smooth fibers in a quite delicate way. The two modules $\Omega^1_{B_t/A_t}$ and $\Omega^1_{(B_t,P)/(A_t,\mathbb{N})}$ are isomorphic as B_t-modules, but they come both with a canonical (classical) derivation $d : B_t \to \Omega^1_{B_t/A_t}$ respective $d : B_t \to \Omega^1_{(B_t,P)/(A_t,\mathbb{N})}$ which do not agree under this isomorphism. Therefore we consider also this replacement as not good enough since a good replacement should give the original differentials on the smooth fibers.

1.4 Good Differentials

Regarding for which reasons we have discarded differentials so far, let us give now a wishlist what good differential forms should satisfy.

Definition 1.2. *A* system of good differential forms *for B/A is given by a choice of the following data:*

- *For every ring homomorphism $\sigma : A \to T$, setting $B_T := B \otimes_A T$, we have a complex $(W^\bullet_{B_T/T}, d^\bullet_T)$ of B_T-modules concentrated in degrees* $0, 1, 2$.
- *We have $W^0_{B_T/T} = B_T$.*
- *The differential $d^m : W^m_{B_T/T} \to W^{m+1}_{B_T/T}$ is T-linear, and $d^0_T : T \to W^1_{B_T/T}$ is a derivation.*

- *(Reflexivity) The modules $W^m_{B_T/T}$ are reflexive.*
- *(Functoriality) We have commutative diagrams*

$$\begin{array}{ccc} W^m_{B_T/T} & \xrightarrow{d^m_T} & W^{m+1}_{B_T/T} \\ {\scriptstyle\sigma^m}\uparrow & & {\scriptstyle\sigma^{m+1}}\uparrow \\ W^m_{B/A} & \xrightarrow{d^m_A} & W^{m+1}_{B/A} \end{array}$$

- *(Base Change) The map σ^m yields an isomorphism $W^m_{B/A} \otimes_B B_T \cong W^m_{B_T/T}$ via adjunction.*
- *(Kähler Property) Whenever B_T/T is smooth (as a morphism of schemes), the canonical homomorphism $\Omega^1_{B_T/T} \to W^1_{B_T/T}$ is an isomorphism.*
- *(Intrinsicality) It is possible to construct $(W^\bullet_{B_T/T}, d^\bullet_T)$ from additional 'intrinsic' data on B_T/T (leave this condition out to get a strict definition).*

Intrinsicality is a soft condition for we did not define what it means to be intrinsic. We may just consider the maps $A \to T, B \to B_T$ as intrinsic data, and define $W^m_{B_T/T} := \Omega^{[m]}_{B/A} \otimes_A T$ (if this satisfies all conditions). We won't do that. Instead we will satisfy the intrinsicality condition by constructing $W^m_{B_T/T}$ from some kind of log structure on B_T/T.

We have seen that the classical de Rham complex $(\Omega^\bullet_{B_T/T}, d^\bullet_T)$ is not concentrated in degrees $0, 1, 2$ and does not have reflexive pieces for $T = A$ and $T = A/(t)$. We have seen that the Danilov differentials $\Omega^{[1]}_{B_T/T}$ - when defined as the bidual on B_T - fail to satisfy the base change, since $\Omega^{[1]}_{B_0/k}$ and $\Omega^{[1]}_{B/A} \otimes_B B_0$ are not isomorphic by *any* isomorphism. The log structure $(B, \mathbb{N})/(A, \mathbb{N})$ yields just the classical differentials, and the log structure $(B, P)/(A, \mathbb{N})$ fails to satisfy the Kähler property (at least for the functoriality maps coming from the log structure). Finally, we consider $\Omega^{[m]}_{B/A} \otimes_B B_T$ as not intrinsic enough, and also reflexivity is not clear for these modules. Nonetheless the good differentials that we are going to construct will turn out equal to $\Omega^{[m]}_{B/A} \otimes_B B_T$ in case $m = 1$ as shown in Corollary 6.4 (and conjecturally there is an isomorphism for all m), but they are more closely related to the (log) geometry of B_T/T. In particular, the asymmetry of $\Omega^{[1]}_{B/A} \otimes_B B_0$ will also show up in the structure itself.

1.5 Logarithmic Geometry

Our approaches so far to construct good differentials from log rings failed, and our best candidate is constructed without any reference to log rings. Though we claim that this candidate is natural from a log point of view. To explain this, we need more subtle log stuff than log rings. Namely, we need a geometric version of log rings, the *log schemes*. They are more flexible than log rings in the sense that not every log scheme is locally what is called the spectrum of a log ring, and they will allow us to define good differentials in a more natural way.

We start with prelog schemes. A *prelog scheme* consists of a scheme X, a sheaf of monoids $\mathcal{M}$ on X (for the Zariski topology, say) and a homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$ of sheaves of monoids where we consider $(\mathcal{O}_X, \cdot)$ a sheaf of monoids with its multiplication. The homomorphism α is called a *prelog structure*. If $V \subset X$ is open, then we get a log ring $\alpha(V) : \mathcal{M}(V) \to \mathcal{O}_X(V)$, so a prelog scheme is the geometric version of a log ring. To be a true log scheme, a prelog scheme needs to satisfy another technical condition. Namely, a prelog scheme $(X, \mathcal{M})$ is called a *log scheme*, if

$$\alpha|_{\alpha^{-1}(\mathcal{O}_X^*)} : \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism of sheaves of monoids, where $\mathcal{O}_X^*$ denotes the invertible functions on X. Every scheme X may be considered a log scheme by taking $\mathcal{M} = \mathcal{O}_X^*$ and for α the obvious inclusion. Such a log scheme is called *trivial*.

Mainly there are two important ways of constructing a log structure on a scheme. The first one is the *spectrum* of a log ring $\alpha : M \to R$. The underlying scheme of Spec $(M \to R)$ is Spec R, and we get a homomorphism $\beta : \underline{M} \to \mathcal{O}_X$ from the constant sheaf of monoids with stalk M to $\mathcal{O}_X$ induced by $M \to R$. This is a prelog structure, but it fails to satisfy the technical condition above since it does not take into account the invertibles of the different stalks of $\mathcal{O}_X$. This can be remedied by taking the so called *associated log structure* $\beta^a : (\underline{M})^a \to \mathcal{O}_X$ which is a log structure satisfying a universal property. This will be explained in more detail in Section 3 which gives a quick overview over basic log geometry.

The second important type of a log structure is constructed as follows: let $Z \subset X$ be a closed subscheme, and let $U = X \setminus Z$ be the complement. We define for an open $V \subset X$

$$\mathcal{M}_{U/X}(V) := \{g \in \mathcal{O}_X(V) \mid g|_{U \cap V} \text{is invertible in } \mathcal{O}_X(U \cap V)\}$$

and we take for $\alpha_{U/X} : \mathcal{M}_{U/X} \to \mathcal{O}_X$ the obvious inclusion. The reader may easily check that this is indeed a log structure. In the language of [17] (which we use) it is called the *compactifying* log structure of $U \subset X$, a name motivated by the idea to consider $(X, \mathcal{M}_{U/X})$ as a compactification of U in case X is proper. And indeed, $(X, \mathcal{M}_{U/X})$ is a trivial log scheme on U. Such log schemes have played some role e.g. in understanding the cohomology of non-compact schemes (and their associated complex analytic spaces). Theorem 2 and the Corollary after it in [9] are an early example of this involving only the log differentials, not the log structure which has not yet been defined that time (as far as we know). If $Z \subset X$ is a divisor, $\mathcal{M}_{U/X}$ is also called a *divisorial* log structure.

It turns out that these two constructions are sometimes related. On the base $S =$ Spec A of our example, we have that Spec $(\mathbb{N} \to A)$ is the compactifying log structure defined by $\{0\} \subset \mathbb{A}^1 = S$, and on the total space, Spec $(P \to B)$ is the compactifying log structure defined by $Z := \{xy = 0\} = \{tw = 0\}$. We illustrate that in more detail below, see also Section 3.1.

Of course, we have a notion of morphism of log schemes. It is what one might expect, namely a morphism from $(X, \mathcal{M}_X)$ to $(S, \mathcal{M}_S)$ consists of a morphism of schemes $f : X \to S$ and a morphism of monoid sheaves $f_m^\flat : \mathcal{M}_S \to f_*\mathcal{M}_X$ fitting in a commutative diagram

$$\begin{array}{ccc} f_*\mathcal{M}_X & \xrightarrow{f_*\alpha_X} & f_*\mathcal{O}_X \\ {\scriptstyle f_m^\flat}\uparrow & & {\scriptstyle f_r^\flat}\uparrow \\ \mathcal{M}_S & \xrightarrow{\alpha_S} & \mathcal{O}_S \end{array}$$

We denote the adjoint maps by $f_m^\sharp : f^{-1}\mathcal{M}_S \to \mathcal{M}_X$ and $f_r^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$, and usually we will leave out the indices m, r. The spectrum construction turns out to be functorial, so a morphism of log rings yields a morphism of log schemes.

We also have a notion of relative log derivation. If $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules, then a relative log derivation $(D, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}$ is a pair of sheaf maps $D : \mathcal{O}_X \to \mathcal{E}, \Delta : \mathcal{M}_X \to \mathcal{E}$ such that for any open $V \subset X$, the pair $(D(V), \Delta(V)) : (\mathcal{O}_X(V), \mathcal{M}_X(V)) \to \mathcal{E}(V)$ is a relative log derivation for the morphism

$$\begin{array}{ccc} \mathcal{M}_X(V) & \longrightarrow & \mathcal{O}_X(V) \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{M}_S(V) & \longrightarrow & f^{-1}\mathcal{O}_S(V) \end{array}$$

of log rings. Equivalently, we may consider the stalks instead. As in the case of log rings, there is a universal derivation $(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)}$. As for log rings, we have an explicit description: Let $\Omega^1_{X/S}$ be the classical sheaf of differentials, and $\mathcal{M}_X^{gp}$ the sheaf of (Grothendieck) groups associated to $\mathcal{M}_X$. Then we have

$$\Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)} = \left(\Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp})\right)/\mathcal{K}$$

where $\mathcal{K}$ is the sub-$\mathcal{O}_X$-module generated by

$$(d\alpha_X(m), -\alpha_X(m) \otimes m) \quad \text{and} \quad (0, 1 \otimes f^\sharp(n))$$

for $m \in \mathcal{M}_X, n \in f^{-1}\mathcal{M}_S$. The universal derivation is given by

$$\begin{aligned} d &: \mathcal{O}_X \to \Omega^1_{X/S} \to \Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)} \\ \delta &: \mathcal{M}_X \to \mathcal{M}_X^{gp} \to \mathcal{O}_X \otimes \mathcal{M}_X^{gp} \to \Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)} \end{aligned}$$

That this is indeed the universal relative log derivation is e.g. shown in [7][Lemma 1.9.]. If $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ arises from the spectrum construction from a morphism of log rings $(R, M) \to (R', M')$, then $\Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)}$ is the sheaf associated to the R'-module $\Omega^1_{(R',M')/(R,M)}$ which gives us an easy way to compute and understand log differential forms for such morphisms of log schemes. Much work in this paper is by the fact that it is in general not that easy to understand log differential forms.

We return to our example. On the base $S = \text{Spec } A$, we have a divisor $S_0 := \{t =$

$0\} = \{0\} \subset \mathbb{A}^1 = S$, and we set $S^+ := S \setminus S_0$. This yields the compactifying log structure $\mathcal{M}_S := \mathcal{M}_{S^+/S}$ defined by S_0, namely

$$\mathcal{M}_S(V) = \{g \in \mathcal{O}_S(V) \mid g|_{V \cap S^+} \in \mathcal{O}_S^*(V \cap S^+)\} \ .$$

As already noted above, one may show that there is an isomorphism Spec $(\mathbb{N} \to A) \cong (S, \mathcal{M}_S)$ fixing the underlying scheme S. From now on, if we write S, the log structure $\mathcal{M}_S$ is understood. Also Spec $(P \to B)$ can be written as a scheme with compactifying log structure. Setting

$$Z := \{xy = 0\} = \{tw = 0\} \subset X \text{ and } X^+ := X \setminus Z \ ,$$

we obtain a compactifying log structure $\mathcal{H} := \mathcal{M}_{X^+/X}$ on X. We will explain below why we choose the letter '$\mathcal{H}$' for its notation, and we reserve $\mathcal{M}_X$ for the log structure that we will finally use for our good differentials. Writing $X_h = (X, \mathcal{H})$ for the log scheme, we construct a morphism $h : X_h \to S$ of log schemes with underlying morphism $f : X \to S$ of schemes. To achieve this, we need to find a morphism $h_m^\flat : \mathcal{M}_S \to f_*\mathcal{H}$ of monoid sheaves on S fitting in a commutative diagram

$$\begin{tikzcd} f_*\mathcal{H} \arrow[r] & f_*\mathcal{O}_X \\ \mathcal{M}_S \arrow[r] \arrow[u, "h_m^\flat"] & \mathcal{O}_S \arrow[u, "f^\flat = h_r^\flat"'] \end{tikzcd}$$

It is an easy exercise with the definitions to check that there is a unique such $h_m^\flat$. We take it to define the morphism $h : X_h \to S$. One can show that there is an isomorphism Spec $(P \to B) \cong X_h$ fixing the underlying scheme X, and since $h : X_h \to S$ is the unique log morphism with underlying morphism $f : X \to S$ of schemes, it fits into a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\ (P \to B) \arrow[r, "\cong"] \arrow[d] & X_h \arrow[d, "h"] \\ \mathrm{Spec}\ (\mathbb{N} \to A) \arrow[r, "\cong"] & S \end{tikzcd}$$

This illustrates that there is some relationship between spectra of log rings and compactifying log structures, but it is by no means a one-to-one relationship.

At this point, we can see a geometric reason why $\Omega^1_{(B,P)/(\mathbb{N},A)}$ does not yield the classical differential forms on the smooth fibers. For, choose any $0 \neq s \in S$. Denoting the fiber of f over s by X_s, we see that $X_s \cap Z = \{xy = 0\} \subset X_s \cong \mathbb{A}^2$. We have not yet defined a log structure on the fiber in the geometric setting, but this non-empty intersection suggests that by putting the log structure $\mathcal{H}$ on X, we also changed the information on the fiber X_s, and that it is therefore natural to expect the log differentials of the compactifying log structure defined by $\{xy = 0\} \subset \mathbb{A}^2$ to show up in the fibers instead of the classical differentials. We would expect the classical differentials in the smooth fiber, if we had $X_s \cap Z = \emptyset$, for the compactifying log structure of $\emptyset$ is just the trivial log structure, and we consider a classical scheme as a log scheme with trivial log structure.

So, why not just taking a compactifying log structure for a closed subset $C \subset X$ with $X_s \cap C = \emptyset$ on the smooth fibers X_s? We take the most obvious choice, namely $C = X_0 := \{t = 0\} \subset X$ which is the only singular fiber. We obtain a compactifying log structure $\mathcal{M}_X$ on X, and as above, there is a unique morphism $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ of log schemes with underlying morphism $f : X \to S$ of schemes. This will be the right log structure to get good differentials. From now on, we write X for the log scheme $(X, \mathcal{M}_X)$, and we introduce notation $\underline{X} = (X, \mathcal{O}_X^*), \underline{S} = (S, \mathcal{O}_S^*)$ for the schemes with trivial log structures. We will use this underlined notation mainly for writing sheaves of classical differential forms, e.g. $\Omega^1_{\underline{X}/\underline{S}}$ is the sheaf associated to $\Omega^1_{B/A}$, whereas $\Omega^1_{X/S}$ denotes the sheaf of log differentials for the morphism $f : X \to S$ of log schemes.

So we get a sheaf $\Omega^1_{X/S}$ of differential forms, but this is still not a good replacement for $\Omega^1_{\underline{X}/\underline{S}}$. This is because $\Omega^1_{X/S}$ turns out to be *not* coherent, so it really fails to be 'good'. The noncoherence implies that $(X, \mathcal{M}_X)$ is not the spectrum of any log ring as we will see in Corollary 4.8 an example of a compactifying log structure that does not come from a log ring. The noncoherence has been observed by Gross and Siebert in [7][Ex. 1.11.], but we show it in more detail in Section 4.2. More precisely, we show that $\Omega^1_{X/S}$ is neither quasi-coherent nor of finite type. Thus now we are not only searching for a good replacement of the classical de Rham differentials, but also for a coherent replacement of the non-coherent log differentials $\Omega^1_{X/S}$. But remembering the Danilov differentials from the beginning, we try to resolve both issues at once by defining

$$W^1_{X/S} := \left(\Omega^1_{X/S}\right)^{**}$$

the bidual sheaf of $\Omega^1_{X/S}$. We call it the *log Danilov differentials* or *reflexive log differentials* of $f : X \to S$, and they turn out to form a coherent sheaf.

We like to have a whole log Danilov complex. To get a differential of the complex, we need another description of $W^1_{X/S}$. We will see that, setting $U := X \setminus \{P\}$ for the point P defined by $\mathfrak{m}_P = (x, y, t, w) \subset B$, the sheaf $\Omega^1_{X/S}$ is locally free on U, so we get $W^1_{X/S}|_U \cong \Omega^1_{X/S}|_U$ via the canonical homomorphism to the bidual, and writing $j : U \subset X$ for the inclusion, we will see that $W^1_{X/S} \cong j_*\Omega^1_{U/S}$ (since X is normal; more details are in the main text). Using this, we (re)define

$$W^m_{X/S} := j_*\Omega^m_{U/S}$$

and thus obtain a complex $(W^\bullet_{X/S}, d^\bullet)$ of reflexive sheaves from the log de Rham complex $(\Omega^\bullet_{U/S}, d^\bullet)$. We will compute the pieces $W^m_{X/S}$ in Section 6. As previously stated, it furthermore turns out that $\Gamma(X, W^1_{X/S}) \cong \Omega^{[1]}_{B/A}$.

Now we like to obtain a (sheafy version of) a system of good differentials in the sense of Definition 1.2. For every morphism $b : T \to S$ of log schemes, where the underlying scheme $\underline{T}$ is affine, we should have (in preliminary notation) a complex $(W^\bullet_T, d^\bullet_T)$ of sheaves on $\underline{Y} := \underline{T} \times_{\underline{S}} \underline{X}$ that is defined in a way 'intrinsic' to Y/T. At this point, the obvious approach works: The category of log schemes admits fiber products, so we can form a cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "b"] & S \end{tikzcd}$$

of log schemes. The underlying scheme of Y is indeed $\underline{Y}$ (which was defined as the fiber product of underlying schemes). Moreover, in such a cartesian diagram, we have an isomorphism $c^*\Omega^m_{X/S} \cong \Omega^m_{Y/T}$, so setting $V := c^{-1}(U)$, the sheaf $\Omega^m_{V/T}$ is locally free. We denote the inclusion $j : V \subset Y$ and define

$$(W^\bullet_{Y/T}, d^\bullet_T) := j_*(\Omega^\bullet_{V/T}, d^\bullet) \ .$$

To make this definition, we need that a differential $d^\bullet$ exists. We ensure the existence by requiring the technical condition of *coherence* for the log structure on T which will be explained at the end of Section 3.1. It turns out that the pieces $W^m_{Y/T}$ are reflexive sheaves, and that we have indeed a system of good differentials in the following sense:

Theorem 1.3. *The complexes $(W^\bullet_{Y/T}, d^\bullet_T)$ form a system of good differentials:*

- *For every morphism $b : T \to S$ of log schemes with $\underline{T}$ affine and T coherent, we have a complex $(W^\bullet_{Y/T}, d^\bullet_T)$ of coherent $\mathcal{O}_Y$-modules concentrated in degrees $0, 1, 2$.*
- *We have $W^0_{Y/T} = \mathcal{O}_Y$.*
- *The differential $d^m : W^m_{Y/T} \to W^{m+1}_{Y/T}$ is $g^{-1}\mathcal{O}_T$-linear, and $d^0_T : \mathcal{O}_Y \to W^1_{Y/T}$ is a derivation.*
- *(Reflexivity) The sheaves of modules $W^m_{Y/T}$ are reflexive.*
- *(Functoriality) We have commutative diagrams*

$$\begin{tikzcd} c_*W^m_{Y/T} \arrow[r, "d^m_T"] & c_*W^{m+1}_{Y/T} \\ W^m_{X/S} \arrow[r, "d^m_S"] \arrow[u, "\sigma^m"] & W^{m+1}_{X/S} \arrow[u, "\sigma^{m+1}"'] \end{tikzcd}$$

- *(Base Change) The map σ^m yields an isomorphism $c^*W^m_{X/S} \cong W^m_{Y/T}$ via adjunction.*
- *(Kähler Property) Whenever $\underline{Y}/\underline{T}$ is smooth (as a morphism of schemes), the canonical homomorphism $\Omega^1_{\underline{Y}/\underline{T}} \to W^1_{Y/T}$ constructed from the fact that $d^0_T : \mathcal{O}_Y \to W^1_{Y/T}$ is a $g^{-1}\mathcal{O}_T$-linear derivation is an isomorphism.*

The hardest part is the base change which will be achieved by a lengthy explicit calculation in Sections 6 and 7.

Let us return to why we chose the letter '$\mathcal{H}$' for the log structure. Namely, we have $X_0 \subset Z$, implying $\mathcal{M}_X \subset \mathcal{H}$ for the log structures. Therefore we consider $\mathcal{H}$ a *hull* around $\mathcal{M}_X$. Although we are not aware of any universal property it might have (like an injective hull of a module), we justify the name by the desire to wrap the log structure $\mathcal{M}_X$ that yields a non-coherent $\Omega^1_{X/S}$ into a log structure that yields coherent differential forms. Moreover, the inclusion $\mathcal{M}_X \subset \mathcal{H}$ yields a homomorphism $\Omega^1_{X/S} \to \Omega^1_{X_h/S}$ that induces inclusions $W^m_{X/S} \subset \Omega^m_{X_h/S}$ which turn out to be important for the base change result. Since $\Omega^1_{X_h/S}$ is coherent, we also call $\mathcal{H}$ a *coherent hull.* Moreover, $\mathcal{H}$ is coherent as a log structure.

1.6 Plan of the Thesis

The thesis is mainly organized towards a proof of Theorem 1.3. In Section 2 we investigate the underlying morphism of schemes of $f : X \to S$. We describe both X and S by a monoid, and explain how f is obtained from a monoid homomorphism between them. This gives a better understanding of the family. We furthermore prove basic results on the classical Kähler differentials of this family.

In Section 3 we give a brief introduction into basic log geometry with a special focus on log differential forms. The reader already acquainted with log geometry may skip the entire section besides the very last subsection starting on page 42.

In Section 4 we consider the log differentials of the morphism $f : X \to S$. We show that they are finite locally free on $U \subset X$, but not even quasi-coherent on the whole space X. This gives additional motivation for the replacement $W^1_{X/S}$, since we wish to have not only a better behaved replacement for $\Omega^1_{B/A}$, but also a coherent one for $\Omega^1_{X/S}$.

In Section 5 we introduce the system of differentials $(W^\bullet_{Y/T}, d^\bullet_T)$ and show its easier properties. In particular we show every sheaf $W^m_{Y/T}$ coherent and reflexive. Furthermore we introduce the so-called *coherent hull* $Y_h \xrightarrow{\gamma} Y$ which is obtained by base change of $X_h \to X$. It has the property that Y_h/T is log smooth whence the differentials $\Omega^1_{Y_h/T}$ are coherent.

Section 6 is devoted to the explicit computation of $W^m_{X/S}$ and related sheaves. The coherent hull $X_h \to X$ plays a crucial role to do this. We obtain a nice description of the sheaves in terms of P-graded $k[P]$-modules.

In Section 7 we use the explicit results of the previous section to prove the base change property announced in Theorem 1.3 which finishes the proof of that theorem.

In the appendix we compute the reflexive differentials of $k[x,y]/(xy)$ which we left out in the introduction, and give a short overview over basic monoid theory. We furthermore introduce the notion of relative log differential graded algebra to prove functoriality for the log de Rham complex which we could not find anywhere in the literature. Finally, we give an overview over notation that we use throughout the thesis.

2 The Family $f : X \to S$ as a Family of Schemes

This section is dedicated to classical non-log aspects of the family $f : X \to S$. We investigate the geometry of it more closely and obtain a description in terms of *monoid rings*. After this, we discuss the classical differential forms $\Omega^1_{B/A}$ and the reflexive differential forms $\Omega^{[1]}_{B/A}$.

2.1 Description by Monoids

We explain how to describe $f : X \to S$ in terms of monoids. If M is a monoid and R is a ring, then the so-called *monoid ring* $R[M]$ is constructed as follows: As an R-module, we set

$$R[M] := \bigoplus_{m \in M} R \cdot z^m \ ,$$

i.e. $R[M]$ is the free R-module on generators z^m. Now a multiplication is defined via $z^m \cdot z^{m'} := z^{m+m'}$ where addition is in the monoid. If $\theta : M \to M'$ is a monoid homomorphism, then we get functorially a ring homomorphism

$$R[\theta] : R[M] \to R[M'], z^m \mapsto z^{\theta(m)} \ .$$

Two easy examples of monoid rings are given by $R[0] \cong R$ and $R[\mathbb{N}] \cong R[X]$, the polynomial ring.

For our example, there are two important monoids: The first one is $Q := \mathbb{N}$, and its monoid ring over k is $k[Q] \cong k[t] = A$. The second important one is the monoid

$$P = \langle (1,0,0), (1,0,1), (1,1,0), (1,1,1) \rangle \subset \mathbb{Z}^3$$

that we already introduced in the introduction. It consists by definition of the finite sums of these four vectors. It consists moreover precisely of the integral points in the convex hull of the rays spanned by the four vectors in $\mathbb{R}^3$. We have included Figure 2 to give an intuition for the geometry of P.

Let us explain how P relates to B. Namely, we have a ring homomorphism

$$\chi : B \to k[P], x \mapsto z^{(1,0,1)}, y \mapsto z^{(1,1,0)}, t \mapsto z^{(1,0,0)}, w \mapsto z^{(1,1,1)}$$

which we like to show an isomorphism by constructing the inverse. To do so, observe $P \subset \mathbb{N}^3$ inducing an inclusion of rings $k[P] \subset k[\mathbb{N}^3] \cong k[u,v,w]$. This allows us to define a ring homomorphism to the localized ring B_t

$$\psi_t : k[P] \subset k[\mathbb{N}^3] \to B_t, z^{(1,0,0)} \mapsto t, z^{(0,1,0)} \mapsto \frac{y}{t}, z^{(0,0,1)} \mapsto \frac{x}{t} \ .$$

The localization map $B \to B_t$ is injective:

Lemma 2.1. *The localization map $B \to B_t$ is injective.*

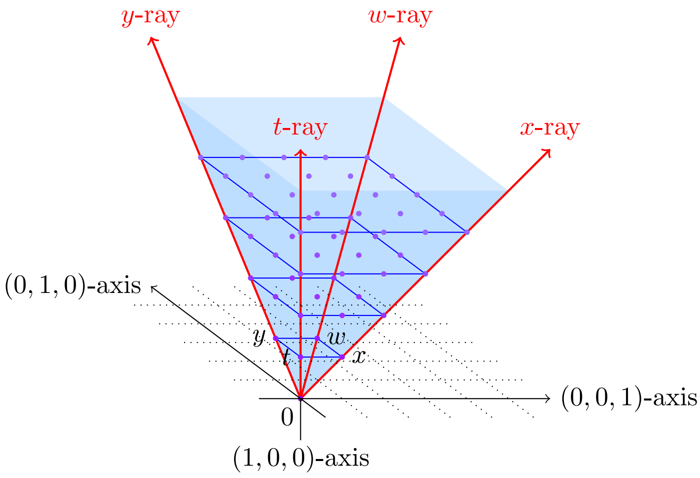

Figure 2: The four rays generated by the vectors span the cone in blue. The monoid P consists of the marked points, which are the integral points within the cone. We denote $t = (1,0,0), x = (1,0,1), y = (1,1,0), w = (1,1,1)$. The $(1,0,0)$-axis goes upwards.

Proof. Let $g \in C = k[x,y,t,w]$ with $[g] = 0 \in B_t$. Then $gt^n \in (xy - tw)$ for some $n \in \mathbb{N}$. Since $t \in C$ is prime and $t \nmid xy - tw$, this implies $g \in (xy - tw)$, so $[g] = 0 \in B$. $\square$

We have

$$\psi_t(z^{(1,0,0)}) = t \in B \subset B_t \qquad \psi_t(z^{(1,0,1)}) = x \in B \subset B_t$$
$$\psi_t(z^{(1,1,0)}) = y \in B \subset B_t \qquad \psi_t(z^{(1,1,1)}) = \frac{txy}{t^2} = w \in B \subset B_t$$

so there is a factorization $\psi : k[P] \to B$ of ψ_t. The maps $\chi : B \to k[P]$ and $\psi : k[P] \to B$ are obviously inverses of each other, so we obtain:

Lemma 2.2. *The homomorphism $\chi : B \to k[P]$ is an isomorphism.*

We see immediately that B is integral, for $k[P] \subset k[\mathbb{N}^3] \cong k[u,v,w]$ is integral. We can describe $f : X \to S$ by the monoids, namely we have a monoid homomorphism $\theta : Q \to P, n \mapsto (n,0,0)$ giving rise to a ring homomorphism

$$k[\theta] : k[Q] \to k[P]$$

which fits into a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\chi", "\cong"'] & k[P] \\ A \arrow[u, "\phi"] \arrow[r, "\cong"'] & k[Q] \arrow[u, "{k[\theta]}"'] \end{tikzcd}$$

Since B is integral, flatness of $\phi : A \to B$ follows as in the introduction (namely B is torsion free over a PID).

Remark 2.3. *Flatness can also be inferred from the fact that $\theta : Q \to P$ is a so-called* integral *monoid homomorphism, see e.g. [14][4.1].*

The description by monoids also shows that $f : X \to S$ is a toric morphism of toric varieties. As any toric variety, X is normal (see e.g. [4]), but its fibers are not normal, for $X_0 = \text{Spec } B_0$ is a reducible scheme.

Remark 2.4. *This situation is better for Cohen-Macaulayness. Using a theorem of Hochster (see [12]) that $k[M]$ is Cohen-Macaulay for appropriate monoids M, it is quite easy to prove that $f : X \to S$ has Cohen-Macaulay fibers which also implies that $B \otimes_A T$ is Cohen-Macaulay for any Cohen-Macaulay ring T. Noting that B_0 is Cohen-Macaulay, it is also easy to see directly that all fibers over k-valued closed points of S are Cohen-Macaulay. We won't need these facts explicitly, so we omit all proofs.*

Faces and Ideals

Interesting for the structure of $f : X \to S$ are also the *faces* of P. A face $F \subset P$ is a submonoid F such that for $x, y \in P$ with $x + y \in F$, we have $x, y \in F$. The trivial faces are 0 and P itself, and the faces are partially ordered by inclusion. The maximal non-trivial faces are called facets. The monoid P has four facets, the two facets

$$F_{tx} := P \cap \big(\mathbb{Z} \cdot (1,0,0) \oplus \mathbb{Z} \cdot (0,0,1)\big) \quad \text{and} \quad F_{ty} := P \cap \big(\mathbb{Z} \cdot (1,0,0) \oplus \mathbb{Z} \cdot (0,1,0)\big)$$

contain $\theta(Q)$ (which we identify from now on with Q) whereas the two facets

$$F_{wx} := P \cap \big(\mathbb{Z} \cdot (0,1,0) \oplus \mathbb{Z} \cdot (1,0,1)\big) \quad \text{and} \quad F_{wy} := P \cap \big(\mathbb{Z} \cdot (0,0,1) \oplus \mathbb{Z} \cdot (1,1,0)\big)$$

intersect Q in 0. See Figure 3 for a depiction of the facets. The facets can also be described by some group homomorphisms $h_i : P^{gp} = \mathbb{Z}^3 \to \mathbb{Z}$ for $i \in \{tx, ty, wx, wy\}$ such that $F_i = h_i^{-1}(0)$. Concretely, we set

$$\begin{aligned} h_{tx}(a,b,c) &= b \\ h_{ty}(a,b,c) &= c \\ h_{wx}(a,b,c) &= a - c \\ h_{wy}(a,b,c) &= a - b \end{aligned}$$

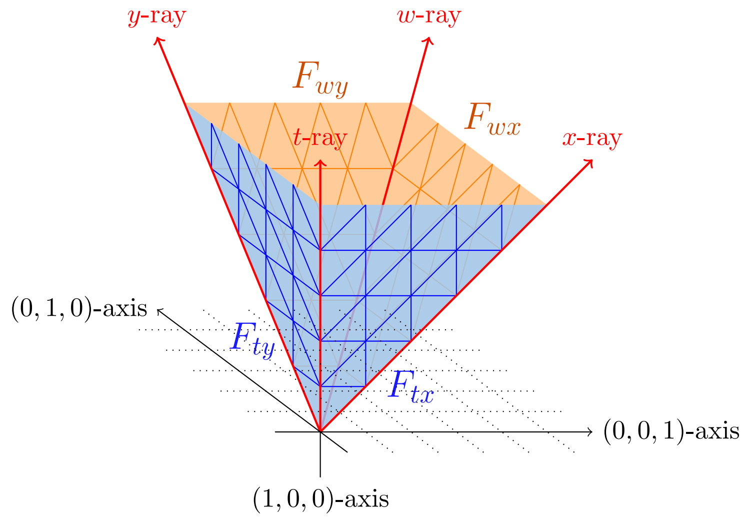

Figure 3: We see the four facets of P. We have the two faces F_{tx}, F_{ty} containing Q in blue, and the two maximal essential faces F_{wx}, F_{wy} in orange. The elements of the facets are the lattice points on the intersection of the lines of the grid. We also see $E = F_{wx} \cup F_{wy}$ in orange.

Moreover, we have $P = \bigcap_{i \in \{tx,ty,wx,wy\}} h_i^{-1}(\mathbb{N})$, that is, $P \subset P^{gp}$ is the locus where all the h_i are positive. We write h_i since we understand $h_i(p)$ as a height of $p \in P$ over the face F_i, and we call h_i a *height function.* They will be useful later to describe derivations and differentials of X, see Section 6. For convenience, we define the index subsets

$$L := \{tx, ty\} \quad \text{and} \quad M := \{wx, wy\} \ ,$$

i.e. if $\ell \in L$, then $Q \subset F_\ell$, and if $m \in M$, then $Q \not\subset F_m$.

A face F in P that does not contain $Q \subset P$ is called an *essential face*. The essential faces are partially ordered by inclusion, and since there are only finitely many faces, every essential face is contained in a maximal essential face. The maximal essential faces are precisely F_{wx} and F_{wy}, and we denote their union by

$$E := F_{wx} \cup F_{wy} \subset P \ .$$

Also for E, see Figure 3 . We obtain a bijection $\rho : E \times Q \to P, (e, q) \mapsto e + q$ which is compatible with the action of Q on both sets, i.e. $\rho(e, q + q') = \rho(e, q) + q'$. The reader is strongly encouraged to draw also his own pictures of all these sets and those that occur later. The isomorphism $E \times Q \cong P$ shows that $p \in (1,0,0) + P$ iff $p \notin E$. This can be used to prove the following lemma which seems a little bit random but is useful to prove that $\Omega^1_{X/S}$ is indeed not coherent in our example.

Lemma 2.5. *Let $b \in B$ and assume $bx, by \in (t)$. Then $b \in (t)$.*

Proof. We use the description of B as $k[P]$. It suffices to show the assertion for $b = z^p$ a monomial since a k-linear combination of such monomials is in (t) iff each summand is. As a k-vector space, we have

$$(t) = \bigoplus_{m \in \theta(1)+P} k \cdot z^m \ ,$$

so $z^p \in (t)$ iff $p \notin E$. Now if $bx, by \in (t)$, then $p + (1,1,0) \notin E, p + (1,0,1) \notin E$. Assume $p \in E$. Then either $p \in F_{wx}$ implying $p + (1,0,1) \in E$ or $p \in F_{wy}$ implying $p + (1,1,0) \in E$. This is a contradiction, so $p \notin E$ and $z^p \in (t)$ completing the proof. $\square$

Also relevant is the notion of an *ideal* $\mathfrak{b} \subset P$ in a monoid. The subset $\mathfrak{b} \subset P$ is an ideal, if $x \in P, y \in \mathfrak{b}$ implies $x + y \in \mathfrak{b}$. An ideal $\mathfrak{b} \subset P$ defines a ring ideal $I_{\mathfrak{b}} := k[\mathfrak{b}] := \bigoplus_{b \in \mathfrak{b}} k \cdot z^b \subset k[P]$ and hence a closed subscheme $V_{\mathfrak{b}} := \text{Spec } k[P]/k[\mathfrak{b}]$. For us most important are the ideals

$$\begin{aligned} \mathfrak{a}_0 &:= P \setminus E = (1,0,0) + P \\ \mathfrak{a} &:= P \setminus (F_{tx} \cup F_{ty} \cup F_{wx} \cup F_{wy}) = (2,1,1) + P \end{aligned}$$

Since $z^{(1,0,0)} = t$ and $z^{(2,1,1)} = xy = tw$, we see that $I_{\mathfrak{a}_0} = (t)$, so $V_{\mathfrak{a}_0} = X_0 = \{t = 0\}$, and $I_{\mathfrak{a}} = (xy) = (tw)$, so $V_{\mathfrak{a}} = Z = \{xy = 0\} = \{tw = 0\}$. In terms of the height functions h_i we have for $p \in P^{gp}$ that

$$\begin{aligned} p \in \mathfrak{a}_0 \quad &\Longleftrightarrow \quad h_{wx}(p) \geq 1, h_{wy}(p) \geq 1, h_{tx}(p) \geq 0 \text{ and } h_{ty}(p) \geq 0 \\ p \in \mathfrak{a} \quad &\Longleftrightarrow \quad \forall i \in \{tx, ty, wx, wy\} : h_i(p) \geq 1 \end{aligned}$$

We will use $\mathfrak{a}_0$ and $\mathfrak{a}$ to describe derivations on the log schemes X and X_h.

2.2 Extension of Functions on Y/T

In this section we prove a technical result that is a major step towards the reflexivity of $W^m_{Y/T}$. Set

$$U' := D(x) \cup D(y) \cup D(w) \subset X$$

where $D(x) = X \setminus \{x = 0\} = \text{Spec } B_x$ etc. We have $U' \subset U$, and for a morphism $b : T \to S$ from an affine scheme $T = \text{Spec } R$, we get a cartesian diagram

$$\begin{tikzcd} T \arrow[d, "b"'] & Y \arrow[l, "g"'] \arrow[d, "c"'] & V \arrow[l, "j"'] \arrow[d] & V' \arrow[l] \arrow[d] \\ S & X \arrow[l, "f"'] & U \arrow[l, "j"'] & U' \arrow[l] \end{tikzcd}$$

Since $W^m_{Y/T} = j_*\Omega^m_{V/T}$, we have $W^m_{Y/T}(Y) = W^m_{Y/T}(V)$ via the restriction map. To compare $W^m_{Y/T}$ with its bidual, we need this property for $\mathcal{O}_Y$ which we prove in this section. We call this property extension of functions since it means that a function $a \in \mathcal{O}_Y(V)$ can be extended to Y. Since this is easier to show for V' instead of V (because V' can be covered by only three affines), we show it for V' first and then conclude that it holds for V, too. The complement of $U' \subset X$ is

$$\begin{aligned} Z' := X \setminus U' &= \{x = 0, y = 0, w = 0\} \\ &\cong \text{Spec } k[x, y, t, w]/(xy - tw, x, y, w) = \text{Spec } k[t] \ , \end{aligned}$$

so $Z' \subset X$ has codimension 2. By [11][Prop. 1.6.], this implies that the restriction $\rho : \mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is bijective since X is integral and normal, and $\mathcal{O}_X$ is reflexive. We consider the Čech complex of the covering $U' = D(x) \cup D(y) \cup D(w)$ for the sheaf $\mathcal{O}_X$. Using the restriction $B \to \mathcal{O}_X(U')$, we obtain a sequence

$$0 \to B \xrightarrow{\alpha} B_x \oplus B_y \oplus B_w \xrightarrow{\beta} B_{xy} \oplus B_{xw} \oplus B_{yw}$$

where $\alpha(b) = (b, b, b)$ and $\beta(a_x, a_y, a_w) = (a_y - a_x, a_w - a_x, a_w - a_y)$. Since B is integral, α is injective, and since $\rho : \mathcal{O}_X(X) \to \mathcal{O}_X(U')$ is bijective, we get that the sequence is exact at $B_x \oplus B_y \oplus B_w$. Set

$$C := \text{coker}(\beta : B_x \oplus B_y \oplus B_w \to B_{xy} \oplus B_{xw} \oplus B_{yw})$$

the cokernel. The main task of this section is to show that C is a flat A-module. Since $A = k[t]$ is a principal ideal domain, it suffices to show C torsion free.

Let us first assume that we already know C flat over A. Then the exact sequence remains exact under tensoring with R over A, so setting $\Lambda := B \otimes_A R$, there is an exact sequence

$$0 \to \Lambda \to \Lambda_x \oplus \Lambda_y \oplus \Lambda_w \to \Lambda_{xy} \oplus \Lambda_{xw} \oplus \Lambda_{yw} \to C \otimes_A R \to 0$$

where we consider $x, y, w \in \Lambda$ via $B \to \Lambda$. The middle two terms belong to the Čech complex of $V' = c^{-1}(D(x)) \cup c^{-1}(D(y)) \cup c^{-1}(D(w))$, so

$$\Gamma(V', \mathcal{O}_Y) = \ker(\Lambda_x \oplus \Lambda_y \oplus \Lambda_w \to \Lambda_{xy} \oplus \Lambda_{xw} \oplus \Lambda_{yw}) \ .$$

We see that $\rho : \mathcal{O}_Y(Y) \to \mathcal{O}_Y(V')$ is indeed bijective.

It remains to show the $k[t]$-module C torsion free. In a first step we need an appropriate description of the localizations B_x, B_y etc. We use that $B \cong k[P]$, and we describe the localizations in terms of *monoid localizations*. If $f \in P$, then we define the localization of P in f as

$$P_f := \{p - nf | p \in P, n \in \mathbb{N}\} \subset P^{gp} = \mathbb{Z}^3 \ .$$

The idea of this localization is that any homomorphism to a monoid such that the image of f is invertible factors uniquely through P_f. It relates to the localization of rings via the isomorphism

$$k[P_f] \to k[P]_{z^f}, \quad z^{-f} \mapsto \frac{1}{z^f} \ .$$

Since $x = z^{(1,0,1)}$ etc. we can use this construction to describe B_x etc. To describe the monoids P_f, we introduce notation. For $p \in \mathbb{Z} \cup \{-\infty, \infty\}$, we define

$$T(p) := \bigoplus_{n \geq p} k \cdot t^n = \begin{cases} 0 & \text{if } p = \infty \\ k[t,t^{-1}] & \text{if } p = -\infty \\ k[t] \cdot t^p & \text{else} \end{cases} \qquad \subset k[t,t^{-1}] = k[\mathbb{Z}]$$

as a $k[t]$-submodule of $k[t,t^{-1}] = k[\mathbb{Z}]$. A function $p(a,b) : \mathbb{Z}^2 \to \mathbb{Z} \cup \{-\infty, \infty\}$ yields a $k[t]$-module

$$T_p := \bigoplus_{(a,b) \in \mathbb{Z}^2} T(p(a,b)) \cdot z^{(0,a,b)} = \bigoplus_{n \geq p(a,b)} k \cdot z^{(n,a,b)} \subset k[P^{gp}]$$

where we interpret $t = z^{(1,0,0)}$. The module T_p may be visualized as every lattice point (n,a,b) in $\mathbb{Z}^3$ marked if $n \geq p(a,b)$. Indeed, the localizations that are relevant for C are of this form. Namely, consider the following functions $p : \mathbb{Z}^2 \to \mathbb{Z} \cup \{-\infty, \infty\}$:

$$p(a,b;B_x) := \begin{cases} b & \text{if } a \geq 0 \\ \infty & \text{if } a < 0 \end{cases} \qquad p(a,b;B_y) := \begin{cases} a & \text{if } b \geq 0 \\ \infty & \text{if } b < 0 \end{cases}$$

$$p(a,b;B_w) := \begin{cases} b & \text{if } b \geq a \\ a & \text{if } a \geq b \end{cases} = \max(a,b)$$

$$p(a,b;B_{xy}) = -\infty, \qquad p(a,b;B_{xw}) = b, \qquad p(a,b;B_{yw}) = a$$

These functions describe the modules they are named after. We illustrate the method of showing it in the example B_x.

Lemma 2.6. *We have $B_x \cong T_{p(a,b;B_x)}$.*

Proof. We need to show $k[P_{(1,0,1)}] = T_{p(a,b;B_x)}$. We have

$$k[P_{(1,0,1)}] = \bigoplus_{(n,a,b) \in \mathbb{Z}^3 :\ (n,a,b) \in P_{(1,0,1)}} k \cdot z^{(n,a,b)}$$

and

$$T_{p(a,b;B_x)} = \bigoplus_{(n,a,b) \in \mathbb{Z}^3 :\ n \geq p(a,b;B_x)} k \cdot z^{(n,a,b)}$$

so it suffices to show $(n,a,b) \in P_{(1,0,1)}$ iff $n \geq p(a,b;B_x)$. This is an easy computation.

First we assume $(n,a,b) \in P_{(1,0,1)}$ whence we have $\alpha, \beta, \gamma, \delta, \nu \geq 0$ with

$$\begin{aligned}(n,a,b) &= \alpha \cdot (1,0,0) + \beta \cdot (1,0,1) + \gamma \cdot (1,1,0) + \delta \cdot (1,1,1) - \nu \cdot (1,0,1)\\ &= (\alpha + \beta + \gamma + \delta - \nu, \gamma + \delta, \beta + \delta - \nu)\end{aligned}$$

We have $a = \gamma + \delta \geq 0$, so $p(a,b;B_x) = b$, and using $\alpha + \gamma \geq 0$ we obtain $n \geq b$.

Conversely assume $n \geq p(a,b;B_x)$. This is only possible if $p(a,b;B_x) \neq \infty$, so we have $a \geq 0$, and therefore $n - b \geq 0$. Writing

$$(n,a,b) = (n-b) \cdot (1,0,0) + (b-a) \cdot (1,0,1) + a \cdot (1,1,1)$$

we see that $(n,a,b) \in P_{(1,0,1)}$. $\square$

Not only the modules $B_x \oplus B_y \oplus B_w$ and $B_{xy} \oplus B_{xw} \oplus B_{yw}$ are $\mathbb{Z}^2$-graded via the above construction, but also the map β between them. For each $(a,b) \in \mathbb{Z}^2$, we have a map

$$\begin{aligned}\beta_{a,b} : &T(p(a,b;B_x)) \oplus T(p(a,b;B_y)) \oplus T(p(a,b;B_w))\\ &\to T(p(a,b;B_{xy})) \oplus T(p(a,b;B_{xw})) \oplus T(p(a,b;B_{yw}))\end{aligned}$$

given by the formula $(a_x, a_y, a_w) \mapsto (a_y - a_x, a_w - a_x, a_w - a_y)$. The reader may check directly that this formula makes sense, i.e. $T(p(a,b;B_x)) \subset T(p(a,b;B_{xy}))$ etc. More conceptually, these inclusions come from $B_x \subset B_{xy}$ etc. The direct sum of the maps $\beta_{a,b}$ is β which yields

$$C = \bigoplus_{(a,b)\in\mathbb{Z}^2} \operatorname{coker}(\beta_{a,b}) \ .$$

Thus to show C torsion free as a $k[t]$-module, it suffices to show every $\operatorname{coker}(\beta_{a,b})$ torsion free.

Fix $(a,b) \in \mathbb{Z}^2$ and write $\xi := p(a,b;B_x), \nu := p(a,b;B_y), \phi := p(a,b;B_w)$ for short. We are interested in the cokernel of

$$\beta' := \beta_{a,b} : T(\xi) \oplus T(\nu) \oplus T(\phi) \to T(-\infty) \oplus T(b) \oplus T(a) \ .$$

To show it torsion free, let $(b_{xy}, b_{xw}, b_{yw}) \in T(-\infty) \oplus T(b) \oplus T(a)$ and $0 \neq P \in k[t]$ with $P \cdot [(b_{xy}, b_{xw}, b_{yw})] = 0$ in the cokernel. This implies that we have $(a_x, a_y, a_w) \in T(\xi) \oplus T(\nu) \oplus T(\phi)$ with $\beta'(a_x, a_y, a_w) = P \cdot (b_{xy}, b_{xw}, b_{yw})$. If we can construct $(a'_x, a'_y, a'_w) \in T(\xi) \oplus T(\nu) \oplus T(\phi)$ with $\beta'(a'_x, a'_y, a'_w) = (b_{xy}, b_{xw}, b_{yw})$, then $[(b_{xy}, b_{xw}, b_{yw})] = 0$ in the cokernel, and torsion freeness follows. To achieve this, we need to distinguish five cases. For the computations recall that $T(p) \subset k[t, t^{-1}]$ for all p, so we can calculate with elements of different summands.

Case 1: $a < 0, b < 0, a \geq b$: In this case we have

$$\beta' : T(\infty) \oplus T(\infty) \oplus T(a) \to T(-\infty) \oplus T(b) \oplus T(a) \ ,$$

so $a_x = 0, a_y = 0$ and $P \cdot (b_{xy}, b_{xw}, b_{yw}) = (0, a_w, a_w)$. Since every $T(p)$ is torsion free, this implies $b_{xy} = 0$ and $b_{xw} = b_{yw} \in T(a)$. Setting $(a'_x, a'_y, a'_w) := (0, 0, b_{yw})$, we obtain $\beta'(a'_x, a'_y, a'_w) = (0, b_{yw}, b_{yw}) = (b_{xy}, b_{xw}, b_{yw})$.

Case 2: $a < 0, b < 0, a < b$: This case is very similar to Case 1. We have

$$\beta' : T(\infty) \oplus T(\infty) \oplus T(b) \to T(-\infty) \oplus T(b) \oplus T(a)$$

which means $a_x = 0, a_y = 0$. Thus $P \cdot (b_{xy}, b_{xw}, b_{yw}) = (0, a_w, a_w)$, so $b_{xy} = 0$ and $b_{xw} = b_{yw} \in T(b)$. Setting $(a'_x, a'_y, a'_w) := (0, 0, b_{xw})$, we get $\beta'(a'_x, a'_y, a'_w) = (0, b_{xw}, b_{xw}) = (b_{xy}, b_{xw}, b_{yw})$.

Case 3: $a < 0, b \geq 0$: In this case we have

$$\beta' : T(\infty) \oplus T(a) \oplus T(b) \to T(-\infty) \oplus T(b) \oplus T(a)$$

implying that $a_x = 0$, so $P \cdot (b_{xy}, b_{xw}, b_{yw}) = (a_y, a_w, a_w - a_y)$. We see that $P \cdot (b_{wx} - b_{xy}) = P \cdot b_{yw}$ whence $b_{xy} = b_{xw} - b_{yw}$. Since $b_{xw} \in T(b), b_{yw} \in T(a)$ and $T(b) \subset T(a)$, we have $b_{xy} \in T(a)$. Therefore $(0, b_{xy}, b_{xw}) \in T(\infty) \oplus T(a) \oplus T(b)$ and $\beta'(0, b_{xy}, b_{xw}) = (b_{xy}, b_{xw}, b_{xw} - b_{xy}) = (b_{xy}, b_{xw}, b_{yw})$.

Case 4: $a \geq 0, b < 0$: This case is very similar to Case 3. We have

$$\beta' : T(b) \oplus T(\infty) \oplus T(a) \to T(-\infty) \oplus T(b) \oplus T(a)$$

whence $a_y = 0$ and $P \cdot (b_{xy}, b_{xw}, b_{yw}) = (-a_x, a_w - a_x, a_w)$. This yields $P \cdot (b_{yw} + b_{xy}) = P \cdot b_{xw}$, so $b_{xy} = b_{xw} - b_{yw}$. Using that $b_{xw} \in T(b), b_{yw} \in T(a)$ and $T(a) \subset T(b)$, we get $b_{xy} \in T(b)$. Thus $(-b_{xy}, 0, b_{yw}) \in T(b) \oplus T(\infty) \oplus T(a)$ and we have $\beta'(-b_{xy}, 0, b_{yw}) = (b_{xy}, b_{xw}, b_{yw})$.

Case 5: $a, b \geq 0$: In this final case we have

$$\beta' : T(b) \oplus T(a) \oplus T(\phi) \to T(-\infty) \oplus T(b) \oplus T(a)$$

with $\phi = \max\{a, b\}$. Again $P \cdot (b_{xy} + b_{yw}) = P \cdot b_{xw}$ holds, so $b_{xy} + b_{yw} = b_{xw}$. Since $b_{xw} \in T(b)$ and $b_{yw} \in T(a)$, we obtain $(-b_{xw}, -b_{yw}, 0) \in T(b) \oplus T(a) \oplus T(\phi)$, and $\beta'(-b_{xw}, -b_{yw}, 0) = (b_{xw} - b_{yw}, b_{xw}, b_{yw}) = (b_{xy}, b_{xw}, b_{yw})$.

We conclude that $\operatorname{coker}(\beta_{a,b})$ and thus C is torsion free. Therefore we have proven the following proposition:

Proposition 2.7. *The restriction map $\rho : \mathcal{O}_Y(Y) \to \mathcal{O}_Y(V')$ is bijective.*

Recall that our primary interest was in the restriction to $V = c^{-1}(U)$, not to $V' \subset V$. To obtain the result for V, let $j : V' \subset Y$ be the inclusion. Both sheaves $\mathcal{O}_Y$ and $j_*\mathcal{O}_{V'}$ are quasi-coherent, so the proposition implies $\mathcal{O}_Y \cong j_*\mathcal{O}_{V'}$. In particular, for V we have (since $V' \subset V \subset Y$) that $(j_*\mathcal{O}_{V'})(X) = (j_*\mathcal{O}_{V'})(V) = (j_*\mathcal{O}_{V'})(V')$.

Corollary 2.8. *The restriction map $\rho : \mathcal{O}_Y(Y) \to \mathcal{O}_Y(V)$ is bijective.*

Remark 2.9. *In a more conceptual framework, the bijectivity of $\rho : \mathcal{O}_Y(Y) \to \mathcal{O}_Y(V')$ is by the fact that $f : X \to S$ is a Cohen-Macaulay morphism and the closed subset $X \setminus U'$ has codimension at least 2 in every fiber (by this we mean every base change to the spectrum of any field, and we say in this case it has* relative codimension *2).*

Since we could not find this result in the literature and our own proof requires some work, we chose to give a more direct proof for the concrete example.

2.3 The Classical Differentials $\Omega^m_{B/A}$

In this section we investigate the classical Kähler differentials $\Omega^m_{B/A}$ and prove the properties claimed in the introduction. We computed already in the introduction that

$$\Omega^1_{B/A} = (B \cdot dx \oplus B \cdot dy \oplus B \cdot dw)/(ydx + xdy - tdw) \ .$$

First, we like to show it torsion free as a B-module. We will need this fact later when proving that $\Omega^1_{X/S}$ is not coherent.

Lemma 2.10. $\Omega^1_{B/A}$ *is torsion free.*

Proof. Our strategy is to embed $\Omega^1_{B/A}$ into a torsion free module. The ring B is integral, so the localization map $B \to B_t$ is injective. Consider the B-module homomorphism

$$\chi : (B \cdot dx \oplus B \cdot dy \oplus B \cdot dw) /(ydx + xdy - tdw) \to B_t \cdot dx \oplus B_t \cdot dy$$
$$dx \mapsto dx,\ dy \mapsto dy,\ dw \mapsto \frac{xdy + ydx}{t}$$

We like to show χ injective. Assume $\chi(b_x dx + b_y dy + b_w dw) = 0$. This implies $b_x t + b_w y =$ and $b_y t + b_w x = 0$ in B_t, thus in B. Therefore $b_w x, b_w y \in (t)$ and $b_w \in (t)$ by Lemma 2.5. Since B is integral, there is a unique $a \in B$ with $at = b_w$. We get $b_x dx + b_y dy + b_w dw = -a(ydx + xdy - tdw)$, so χ is indeed injective.

Now if $\Omega^1_{B/A}$ was not torsion free, then there were $0 \neq b \in B, 0 \neq \omega \in \Omega^1_{B/A}$ with $\chi(b\omega) = 0$. Since $B_t \cdot dx \oplus B_t \cdot dy$ is a torsion free B-module, this implies $\chi(\omega) = 0$, so $\omega = 0$ contradicting the assumption. $\square$

As in the introduction next we look at the base change to the central fiber,

$$\Omega^1_{B_0/k} = (B_0 \cdot dx \oplus B_0 \cdot dy \oplus B_0 \cdot dw)/(ydx + xdy) \ .$$

We write $M := B_0 \cdot dx \oplus B_0 \cdot dy \oplus B_0 \cdot dw$ for short. Since B_0 is not integral, the question of torsion freeness of $\Omega^m_{B_0/k}$ is not really interesting. Of course $\Omega^0_{B_0/k} = B_0$ is not torsion free. In $\Omega^1_{B_0/k}$, we have $dx \neq 0$, but $y^2 dx = y(ydx + xdy) = 0$, so it is a torsion element. As stated in the introduction, also $\Omega^2_{B_0/k}$ has torsion. We saw that $xdx \wedge dy = 0$. Let us show now that $dx \wedge dy \neq 0$. Writing $I := (ydx + xdy) \subset M$, we get an exact sequence

$$0 \to I \to M \to \Omega^1_{B_0/k} \to 0$$

yielding an exact sequence

$$I \wedge M \to \bigwedge^2 M \to \Omega^2_{B_0/k} \to 0 \ .$$

Assume we had $dx \wedge dy = 0 \in \Omega^2_{B_0/k}$. Then there were $f_i \in B_0, m_i \in M$ with $\sum f_i(ydx + xdy) \wedge m_i = dx \wedge dy \in \bigwedge^2 M$. Denote by $(x, y) \subset B_0$ the ideal generated by these two elements. We have $ydx + xdy \in (x, y) \cdot M$ implying that $dx \wedge dy \in (x, y) \cdot \bigwedge^2 M$. Because $\bigwedge^2 M$ is a free B_0-module with basis $dx \wedge dy, dy \wedge dw, dw \wedge dx$, this means $1 \in (x, y)$ which is not the case. A similar argument shows that $dx \wedge dy \wedge dw \neq 0 \in \Omega^3_{B_0/k}$.

Lemma 2.11. *We have $\Omega^3_{B_0/k} \neq 0$. In particular $\Omega^3_{B/A} \neq 0$.*

Proof. We show that $dx \wedge dy \wedge dw \neq 0$ in $\Omega^3_{B_0/k}$. Keeping notation from above, we have an exact sequence

$$I \wedge \bigwedge^2 M \to \bigwedge^3 M \to \Omega^3_{B_0/k} \to 0 .$$

The module $\bigwedge^3 M$ is free with basis $dx \wedge dy \wedge dw$. Since $I \subset (x, y) \cdot M$, any element coming from $I \wedge \bigwedge^2 M$ is in $(x, y) \cdot \bigwedge^3 M$. Because $1 \notin (x, y)$, we have $dx \wedge dy \wedge dw \notin (x, y) \cdot \bigwedge^3 M$, so $dx \wedge dy \wedge dw \neq 0 \in \Omega^3_{B_0/k}$. Since $\Omega^3_{B_0/k} = \Omega^3_{B/A} \otimes_A A/(t)$, the second assertion is immediate. $\square$

We see that the Kähler differentials in our example do not behave as well as Kähler differentials do for smooth morphisms, for we have $\Omega^3_{B/A} \neq 0$ though the relative dimension of $f : X \to S$ is 2. At least we have $\Omega^m_{B/A} = 0$ for $m \geq 4$ since then $\bigwedge^m (B \cdot dx \oplus B \cdot dy \oplus B \cdot dw) = 0$.

3 Basic Facts in Log Geometry

This section is a brief introduction to log geometry and we will focus on log differential forms. The reader who is not acquainted to log geometry should read 1.5 in the introduction first. A good yet unpublished source for learning log geometry is [17] by Ogus. An older version is available on the internet, see [16]. A famous foundational work on log geometry is [14]. For lack of a published comprehensive work on the foundations, many papers include brief introductions to log geometry, e.g. [6],[13],[19]. Our basic objects of study are schemes X together with a so-called prelog structure. A *prelog structure* is a sheaf (for our purposes in the Zariski topology) of commutative monoids $\mathcal{M}$ on X and a monoid homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$ where $\mathcal{O}_X = (\mathcal{O}_X, \cdot)$ is considered with its monoid structure coming from multiplication in the rings. A prelog structure is called a *log structure* , if $\alpha : \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism, and in that case the pair $(X, \mathcal{M})$ is called a *log scheme.* If X is a log scheme (with $\mathcal{M}$ and α understood), then its underlying scheme is also denoted by $\underline{X}$.

If we have a prelog structure $\beta : \mathcal{M} \to \mathcal{O}_X$ on a scheme, there is always an *associated log structure* $\beta^a : \mathcal{M}^a \to \mathcal{O}_X$ constructed as the pushout

$$\begin{tikzcd}
\beta^{-1}(\mathcal{O}_X^*) \arrow[r, "\beta"] \arrow[d] & (\mathcal{O}_X^*, \cdot) \arrow[d, dotted] \arrow[dr] & \\
\mathcal{M} \arrow[r, dotted, "\epsilon"] \arrow[drr, "\beta"'] & \mathcal{M}^a \arrow[dr, dashed, "\beta^a"] & \\
& & (\mathcal{O}_X, \cdot)
\end{tikzcd}$$

in the category of sheaves of monoids. The construction yields a homomorphism $\epsilon : \mathcal{M} \to \mathcal{M}^a$ with the following universal property: If $\gamma : \mathcal{L} \to \mathcal{O}_X$ is a log structure, and $\phi : \mathcal{M} \to \mathcal{L}$ a monoid homomorphism with $\gamma \circ \phi = \beta$, then there is a unique monoid homomorphism $h : \mathcal{M}^a \to \mathcal{L}$ with $h \circ \epsilon = \phi$ and $\gamma \circ h = \beta^a$.

If $(X, \mathcal{M}), (Y, \mathcal{N})$ are log schemes, then a morphism $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ consists of a scheme morphism $f : X \to Y$ and a commutative diagram

$$\begin{tikzcd}
f_*\mathcal{M} \arrow[r, "f_*\alpha_X"] & f_*\mathcal{O}_X \\
\mathcal{N} \arrow[u, "f_m^\flat"] \arrow[r, "\alpha_Y"] & \mathcal{O}_Y \arrow[u, "f_r^\flat"']
\end{tikzcd}$$

of sheaves on Y. With composition defined in the obvious way, we obtain a category of log schemes.

This category admits fiber products which are constructed as follows: If $f : X \to Z$, $g : Y \to Z$ are morphisms of log schemes, set $W := X \times_T Y$ for the underlying scheme, and write $\mathrm{pr}_X : W \to X, \mathrm{pr}_Y : W \to Y, \mathrm{pr}_Z : W \to Z$ for the projections. We obtain a diagram

$$\begin{tikzcd}
\mathrm{pr}_Z^{-1}\mathcal{M}_Z \arrow[r] \arrow[d] & \mathrm{pr}_X^{-1}\mathcal{M}_X \arrow[d, dotted] \arrow[ddr] & \\
\mathrm{pr}_Y^{-1}\mathcal{M}_Y \arrow[r, dotted] \arrow[drr] & \mathcal{M} \arrow[dr, dashed, "\beta"] & \\
 & & \mathcal{O}_W
\end{tikzcd}$$

of sheaves of monoids on W, where $\mathcal{M}$ is defined to be the pushout, i.e. the diagram is cocartesian. From the universal property we get a prelog structure $\beta : \mathcal{M} \to \mathcal{O}_W$, and the log structure on W is the associated log structure $\beta^a : \mathcal{M}^a \to \mathcal{O}_W$.

If $(Y, \mathcal{N})$ is a log scheme and $f : X \to Y$ is a morphism of schemes, then from the composite $\mathcal{N} \to \mathcal{O}_Y \to f_*\mathcal{O}_X$ we get a monoid homomorphism $\beta : f^{-1}\mathcal{N} \to \mathcal{O}_X$ via adjunction. It is a prelog structure, and its associated log structure $\beta^a : f^*_{log}\mathcal{N} \to \mathcal{O}_X$ is called the *inverse image* or *pull back log structure*. We also have a morphism of log schemes $(X, f^*_{log}\mathcal{N}) \to (Y, \mathcal{N})$. This construction has the following universal property: Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of log schemes. Then there is a unique morphism of log schemes $g : (X, \mathcal{M}) \to (X, f^*_{log}\mathcal{N})$ which is the identity on underlying schemes and fits into a commutative diagram

$$\begin{tikzcd}
(X, \mathcal{M}) \arrow[r, dashed, "g"] \arrow[dr, "f"'] & (X, f^*_{log}\mathcal{N}) \arrow[d] \\
 & (Y, \mathcal{N})
\end{tikzcd}$$

A morphism $(X, f^*_{log}\mathcal{N}) \to (Y, \mathcal{N})$ of log schemes is called *strict*, if g constructed above is an isomorphism.

3.1 The Spectrum Construction and Coherent Log Structures

Already in the introduction we mentioned the spectrum construction for log rings. If $\alpha : M \to R$ is a log ring, then we set $X :=$ Spec R as underlying scheme. Denote the constant presheaf with value M on X by $\tilde{M}$. The map $\alpha : M \to R$ yields a homomorphism $\tilde{M} \to \mathcal{O}_X$ of presheaves of monoids, so we get a homomorphism $\beta : \underline{M} \to \mathcal{O}_X$ of sheaves, where $\underline{M}$ is the constant sheaf with fiber M. This is a prelog structure, and for Spec $(M \to R)$ we take its associated log structure $\alpha_X = \beta^a : \mathcal{M}_X := \mathcal{M}_{(R,M)} \to \mathcal{O}_X$.

If $\theta : (R, M) \to (R', M')$ is a homomorphism of log rings, then we obtain a morphism of log schemes $f : X' =$ Spec $(M' \to R') \to X =$ Spec $(M \to R)$. On underlying schemes, it is given by $\theta_r : R \to R'$. Now we obtain a diagram

$$\begin{tikzcd}
f_*\tilde{M}' \arrow[r] & f_*\underline{M}' \arrow[r] & f_*\mathcal{M}_{X'} \arrow[r] & f_*\mathcal{O}_{X'} \\
\tilde{M} \arrow[u, dotted, "\tilde{\theta}"] \arrow[r] & \underline{M} \arrow[r] & \mathcal{M}_X \arrow[r] & \mathcal{O}_X \arrow[u, "f_r^\flat"']
\end{tikzcd}$$

of sheaves on X. Note that $f_*\tilde{M}'$ is the constant presheaf on X with values M'. The morphism $\tilde{\theta}$ is constructed from $\theta_m : M \to M'$ in the obvious way, and the diagram is indeed commutative. In two further steps we can construct $f_m^\flat : \mathcal{M}_X \to f_*\mathcal{M}_{X'}$ using the construction of $\mathcal{M}_X$ as a pushout.

The most important examples of spectra of log rings are constructed as follows: Let R be a ring, and let M be a monoid. Then we obtain a log ring

$$M \to R[M], m \mapsto z^m$$

and we denote its spectrum by A_M. If $\theta : M \to M'$ is a monoid homomorphism, then we get a morphism of log rings $A_\theta : A_{M'} \to A_M$. The morphism $h : X_h \to S$ of our example is of this form for $M = Q, M' = P$ and $\theta : Q \to P$ as defined earlier.

We are interested in a connection between spectra of log rings and compactifying log structures as in the introduction. We use the following theorem which is mainly a consequence of [17][Thm. 1.9.4]. It is also closely related to [15][11.6], but to use this directly we would need a theory of log regularity at hand (which exists, but which we avoid to use in this thesis).

Theorem 3.1. *Let R be an integral and normal ring, and let M be a toric monoid. Let*

$$A_M = \text{Spec } (M \to R[M]) \ ,$$

and let $\mathcal{M} = \mathcal{M}_{(R[M],M)}$ be its log structure. Then $\mathcal{M}$ is (isomorphic to) the compactifying log structure defined by the open subset Spec $R[M^{gp}] \subset$ Spec $R[M]$.

In our example, we use this theorem to identify the log structures of X_h and S as some compactifying log structures as stated in the introduction. The field k is integral and normal, and the reader may check that $Q = \mathbb{N}$ and P are toric monoids (see Section 8.2 in the appendix for a definition of toric monoids). We need to compute Spec $k[Q^{gp}]$ and Spec $k[P^{gp}]$. To do so, we use monoid localizations as introduced in Section 2.2. We have $\mathbb{Z} = Q^{gp} = Q_1$ where Q_1 is the localization in $1 \in Q$, and $\mathbb{Z}^3 = P^{gp} = P_{(2,1,1)}$. This yields

$$k[Q^{gp}] = k[Q_1] = k[Q]_{z^1} \cong k[t]_t$$

so Spec $k[Q^{gp}] = S^+ = S \setminus S_0$, and

$$k[P^{gp}] = k[P_{(2,1,1)}] = k[P]_{z^{(2,1,1)}} \cong B_{xy} = B_{tw}$$

so Spec $k[P^{gp}] = X^+ = X \setminus Z$. Therefore we conclude:

Corollary 3.2. *The log scheme* Spec $(\mathbb{N} \to A)$ *is* S *with the compactifying log structure defined by* $S_0 \subset S$*, and* Spec $(P \to B)$ *is* X *with the compactifying log structure defined by* $Z \subset X$*.*

Now let us turn on to the notion of coherence. A log structure $\mathcal{M}$ on the scheme X is called *coherent*, if there is a covering $X = \bigcup_i U_i$ (in the Zariski topology for our purposes) and for every U_i a monoid M_i together with a monoid homomorphism $\beta_i : M_i \to \mathcal{M}(U_i)$ satisfying the following property: β_i induces a homomorphism $\underline{M_i} \to \mathcal{M}|_{U_i}$ from the locally constant sheaf, and the universal property of the associated log structure of $\underline{M_i} \to \mathcal{M}|_{U_i} \to \mathcal{O}_{U_i}$ yields a homomorphism $(\underline{M_i})^a \to \mathcal{M}|_{U_i}$. This homomorphism is required to be an isomorphism. In the case of $X = \mathrm{Spec}\ (M \to R)$, we take a single cover $U_1 = X$ and $M_1 = M$ to see:

Lemma 3.3. *The log scheme* Spec $(M \to R)$ *is coherent.*

Indeed a log scheme is coherent if and only if it admits locally a strict morphism to Spec $(M \to \mathbb{Z}[M])$ for some monoids M, but we won't prove that in detail here. This fact easily implies: Let $f : X \to Y$ be a strict morphism of log schemes, and let Y be coherent. Then X is coherent. Another important property is that the fiber product $X \times_Z Y$ of three coherent log schemes X, Y, Z is coherent.

3.2 Derivations and Differentials

This section is devoted to log derivations and differentials. We define them and cite their basic properties. After that, we prove some non-standard properties of these notions. We avoid the slightly complicated notion of log smoothness and instead introduce a notion of differential log smoothness which is sufficient for our purposes.

Recall the definitions from the introduction: If $(X, \mathcal{M})$ is a log scheme and $\mathcal{E}$ is a sheaf of $\mathcal{O}_X$-modules, then a *log derivation* with values in $\mathcal{E}$ is a pair (D, Δ) where $D : \mathcal{O}_X \to \mathcal{E}$ is a classical derivation and $\Delta : \mathcal{M} \to \mathcal{E}$ is a monoid homomorphism for the additive monoid structure on $\mathcal{E}$ such that $\alpha(m) \cdot \Delta(m) = D\alpha(m)$. If $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a log morphism, then a *relative log derivation* is a log derivation (D, Δ) such that for the canonical maps $f^\sharp : f^{-1}\mathcal{N} \to \mathcal{M}$ and $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, we have $D \circ f^\sharp = 0$ respective $\Delta \circ f^\sharp = 0$. The log derivations respective relative log derivations form a sheaf denoted by $\mathrm{Der}_X(\mathcal{E})$ respective $\mathrm{Der}_{X/Y}(\mathcal{E})$.

There is a sheaf $\Omega^1_{X/Y}$ representing the sheaf of relative derivations, that is there is a universal relative log derivation $(d, \delta) : (\mathcal{O}_X, \mathcal{M}) \to \Omega^1_{X/Y}$ such that any relative log derivation to $\mathcal{E}$ factors uniquely via a homomorphism $\phi : \Omega^1_{X/Y} \to \mathcal{E}$. As already mentioned in the introduction, this universal sheaf can be constructed explicitly as follows: Let $\Omega^1_{\underline{X}/\underline{Y}}$ be the classical sheaf of differentials, and $\mathcal{M}^{gp}$ the sheaf of (Grothendieck) groups associated to $\mathcal{M}$. Then we have

$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})\right) / \mathcal{K}$$

where $\mathcal{K}$ is the sub-$\mathcal{O}_X$-module generated by

$$(d\alpha_X(m), -\alpha_X(m) \otimes m) \text{ and } (0, 1 \otimes f^\sharp(n))$$

for $m \in \mathcal{M}, n \in f^{-1}\mathcal{N}$, see e.g. [7][Lemma 1.9.]. The universal derivation is given by

$$d : \mathcal{O}_X \to \Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X/Y},\ \delta : \mathcal{M} \to \mathcal{M}^{gp} \to \mathcal{O}_X \otimes \mathcal{M}^{gp} \to \Omega^1_{X/Y} .$$

The classical part $d : \mathcal{O}_X \to \Omega^1_{X/Y}$ of the universal log derivation (d, δ) is a derivation, so by the universal property of $\Omega^1_{\underline{X}/\underline{Y}}$ there is a unique homomorphism $e_{X/Y} : \Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X/Y}$ such that for the classical universal derivation $\underline{d} : \mathcal{O}_X \to \Omega^1_{\underline{X}/\underline{Y}}$ we have $e_{X/Y} \circ \underline{d} = d$. This map $e_{X/Y}$ coincides with the obvious map

$$\Omega^1_{\underline{X}/\underline{Y}} \to \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})\right)/\mathcal{K} .$$

The *relative log tangent sheaf* $\Theta^1_{X/Y}$ is defined as $\mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$. It is the sheaf of log derivations with values in $\mathcal{O}_X$. Dualizing $e_{X/Y}$, we obtain a map

$$e^*_{X/Y} : \Theta^1_{X/Y} \to \Theta^1_{\underline{X}/\underline{Y}}$$

which corresponds to forgetting the log part of a log derivation (D, Δ).

Next assume X log trivial, i.e. $\mathcal{M}_X = \mathcal{O}^*_X$. Now the forgetful map $\mathrm{Der}_{X/Y}(\mathcal{E}) \to \mathrm{Der}_{\underline{X}/\underline{Y}}(\mathcal{E})$ forgetting the log part is an isomorphism for any sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules since for a log derivation $(D, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}$, the log part Δ can be reconstructed from D as $\Delta(m) = m^{-1} \cdot D(m)$. Thus for log trivial X, the canonical map $e_{X/Y} : \Omega^1_{\underline{X}/\underline{Y}} \to \Omega^1_{X/Y}$ is an isomorphism.

If X, Y are coherent log schemes, then there is also a *log de Rham complex*. Namely, we form the exterior powers of $\Omega^1_{X/Y}$, which we denote by

$$\Omega^m_{X/Y} := \bigwedge^m_{\mathcal{O}_X} \Omega^1_{X/Y}$$

and call the *higher log differential forms*. The differentials $d^m : \Omega^m_{X/Y} \to \Omega^{m+1}_{X/Y}$ are characterized by the following proposition:

Proposition 3.4. *([17][V. Prop. 2.1.1])Let $f : X \to Y$ be a morphism of coherent log schemes, and for each m let $\Omega^m_{X/Y}$ be the m-th exterior power of $\Omega^1_{X/Y}$. Then there is a unique collection of homomorphisms of sheaves of abelian groups, called the exterior derivative $\{d^m : \Omega^m_{X/Y} \to \Omega^{m+1}_{X/Y} : m \in \mathbb{N}\}$ such that*

1. *$d^0 = d : \mathcal{O}_X \to \Omega^1_{X/Y}$, the classical part of the universal derivation.*
2. *$d^m d^{m-1}\omega = 0$, if ω is any section of $\Omega^{m-1}_{X/Y}$, and $d^1\delta(\mu) = 0$, if μ is any section of $\mathcal{M}_X$.*
3. *$d^{m+n}(\omega \wedge \omega') = (d^m\omega) \wedge \omega' + (-1)^m \omega \wedge (d^n\omega')$ for $\omega \in \Omega^m_{X/Y}$ and $\omega' \in \Omega^n_{X/Y}$.*

We will need to be careful with this proposition, for not all of our log structures are coherent. The cited source does not list Condition 1, but this condition is necessary, for otherwise $d^m = 0$ for all m would satisfy all conditions. Using $f^{-1}\mathcal{O}_Y$-linearity of $d^0 = d$, the third condition implies $f^{-1}\mathcal{O}_Y$-linearity of all d^m.

A nice property of the log de Rham complex is its functoriality: Let

$$\begin{array}{ccc} X' & \xrightarrow{c} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y' & \xrightarrow{b} & Y \end{array}$$

be a cartesian diagram of coherent log schemes. Then there is a homomorphism $\sigma^\bullet_{X'/X} : \Omega^\bullet_{X/Y} \to c_*\Omega^\bullet_{X'/Y'}$ of complexes of abelian sheaves, such that every $\sigma^m_{X'/X}$ is an $\mathcal{O}_X$-module homomorphism and $\sigma^1_{X'/X} : \Omega^1_{X/Y} \to c_*\Omega^1_{X'/Y'}$ is the unique homomorphism factoring the canonical log derivation $(\mathcal{O}_X, \mathcal{M}_X) \to c_*\Omega^1_{X'/Y'}$. Note that - contrary to $\Omega^\bullet_{X/Y} \to c_*\Omega^\bullet_{X'/Y'}$ - it does not make sense to ask for a homomorphism $c^*\Omega^\bullet_{X/Y} \to \Omega^\bullet_{X'/Y'}$ for c^* does not apply to the differentials of the complex which are not $\mathcal{O}_X$-linear. The functoriality property is explained in detail and proven in Section 8.3 in the appendix. We included the proof since we could not find any reference to the functoriality.

Basic Properties of Differentials and Differentially Log Smooth Morphisms

We recall some basic properties of log differential forms, and how to compute log differential forms. If $f : X \to Y, g : Y \to S$ are morphisms of log schemes, we get a relative log derivation $\mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*\Omega^1_{X/S}, \mathcal{M}_Y \to f_*\mathcal{M}_X \to f_*\Omega^1_{X/S}$ for Y/S, so a homomorphism $\Omega^1_{Y/S} \to f_*\Omega^1_{X/S}$. By adjunction, there is a homomorphism $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$. A relative log derivation for X/Y is also a relative log derivation for X/S, so we get a homomorphism $\Omega^1_{X/S} \to \Omega^1_{X/Y}$.

Proposition 3.5. *If $f : X \to Y, g : Y \to S$ are morphisms of log schemes, then we have an exact sequence*

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \ .$$

Proof. For *fine* log structures (i.e. coherent and integral), this is already in [14]. For the general case, see e.g. [16][IV. Prop. 2.3.1]. □

Later we will use this sequence to compute the log differential forms of $f : X \to S$ on U. A second crucial property is the base change property. It is the analogue of the base change for Kähler differentials and another motivation for proving base change of the log Danilov differentials later. Consider a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{c} & X \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{b} & S \end{array}$$

in the category of log schemes as constructed at the beginning of the section. The universal derivation $(d_Y, \delta_Y) : (\mathcal{O}_Y, \mathcal{M}_Y) \to \Omega^1_{Y/T}$ yields a derivation $(\mathcal{O}_X, \mathcal{M}_X) \to (c_*\mathcal{O}_Y, c_*\mathcal{M}_Y) \to c_*\Omega^1_{Y/T}$. This is a relative derivation for X/S, so we obtain a homomorphism $\Omega^1_{X/S} \to c_*\Omega^1_{Y/T}$. By adjunction, we get $c^*\Omega^1_{X/S} \to \Omega^1_{Y/T}$.

Proposition 3.6. *In the above situation, we have an isomorphism $c^*\Omega^1_{X/S} \cong \Omega^1_{Y/T}$.*

Proof. For fine log schemes, this is already in [14]. For the general case, see e.g. [16][IV. Prop. 1.3.1]. □

Our next concern is how to compute log differentials. We have the explicit construction, but this is a rather inefficient way of computing them. In the introduction we considered log differentials of log rings, and we have a functor associating a morphism of log schemes to a morphism of log rings. Indeed, the differentials of log rings compute the differentials of log schemes:

Proposition 3.7. *[17][IV. Cor. 1.2.6] or [16][IV. Cor. 1.1.11]. Let $\theta : (R, M) \to (R', M')$ be a morphism of log rings, and let $f : X' \to X$ be the associated morphism of spectra. Then $\Omega^1_{X'/X}$ is the quasi-coherent sheaf associated to $\Omega^1_{(R',M')/(R,M)}$.*

This result can be used to show the differentials of any morphism of coherent log schemes quasi-coherent:

Corollary 3.8. *[17][IV. Cor. 1.2.8] or [16][IV. Cor. 1.1.12]. Let $f : X \to Y$ be a morphism of coherent log schemes. Then $\Omega^1_{X/Y}$ is a quasi-coherent sheaf of modules.*

But unlike Kähler differentials, log differentials are not always quasi-coherent. Namely, as we show later, $\Omega^1_{X/S}$ of our example is not quasi-coherent (see Section 4.2).

Another method to compute log differential forms is to make an ansatz and show the universal property directly. We illustrate that method with the following example: Let k be a field, and set $X = \mathbb{A}^n_k$. Let $D_r = \{x_1 \cdot ... \cdot x_r = 0\}$ be a divisor for some $r \leq n$, and endow X with the compactifying log structure defined by D. Write $(\mathbb{A}^n, D_r)$ for this log scheme. First, we like to show the log scheme $(\mathbb{A}^n, D_r)$ coherent. To do so, consider the log ring

$$\mathbb{N}^r \to k[\mathbb{N}^n], (k_1, ..., k_r) \mapsto z^{(k_1,...,k_r,0,...,0)} .$$

We only sketch how to show Spec $(\mathbb{N}^r \to k[\mathbb{N}^n]) \cong (\mathbb{A}^n, D_r)$. Note that Proposition 3.1 does not apply directly. The log schemes $(\mathbb{A}^n, D_r)$ are relevant for us because X is locally of this form on U (as we will see in Section 4.1). We will need the coherence of X on U to construct the differentials of the complex $(W^\bullet_{Y/T}, d^\bullet_T)$ in Section 5.1.

Lemma 3.9. *The log scheme* $(\mathbb{A}^n, D_r) \cong \mathrm{Spec}\ (\mathbb{N}^r \to k[\mathbb{N}^n])$ *is coherent.*

Sketch of Proof. We write $U_r := \mathbb{A}^n \setminus D_r$ and denote the log structure of $(\mathbb{A}^n, D_r)$ by $\mathcal{M}_{(\mathbb{A}^n, U_r)}$. Working towards Spec $(\mathbb{N}^r \to k[\mathbb{N}^n])$, we have a prelog structure

$$\underline{\mathbb{N}}^r \to \mathcal{O}_{\mathbb{A}^n}, (k_1, ..., k_r) \mapsto x_1^{k_1} \cdot ... \cdot x_r^{k_r} \ .$$

Since $(x_1^{k_1} \cdot ... \cdot x_r^{k_r})|_{U_r}$ is invertible, there is a factorization through $\mathcal{M}_{(\mathbb{A}^n, U_r)}$, whence the universal property of the associated log structure yields a further factorization

$$\underline{\mathbb{N}}^r \to (\underline{\mathbb{N}}^r)^a \xrightarrow{\phi} \mathcal{M}_{(\mathbb{A}^n, U_r)} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{A}^n} \ .$$

It suffices to show ϕ an isomorphism. Surjectivity is checked on standard opens $D(g)$ for $g \in k[x_1, ..., x_n]$ as will become clear in the proof of Lemma 3.10 below, and injectivity is checked on the stalks using that $\underline{\mathbb{N}}^r \to \mathcal{O}_{\mathbb{A}^n}$ is injective, both α and $\alpha \circ \phi$ are isomorphisms on the invertibles, and every element $v \in (\underline{\mathbb{N}}^r)^a_x$ can be written as a product $v = uk$ with $k \in \mathbb{N}^r = (\underline{\mathbb{N}}^r)_x$ and $u \in \mathcal{O}^*_{\mathbb{A}^n, x}$. Furthermore, we need to use the fact that $(\underline{\mathbb{N}}^r)^a$ is a sheaf of *integral* monoids. This property of monoids is explained in Section 8.2 in the appendix. □

Let further $Y = \mathrm{Spec}\ k$ with the trivial log structure, and there is an obvious morphism $f : X \to Y$. We are going to compute the log differentials $\Omega^1_{X/Y}$ with the method by ansatz. We could obtain the same morphism of log schemes as the morphism associated to $(0 \to k[0]) \to (\mathbb{N}^r \to k[\mathbb{N}^n])$, so another method to compute these differentials is given by the above proposition. As we see below, the method by ansatz is somewhat complicated but may yield very explicit results. The reader who is not interested in that method may skip the proof below (besides to complete the proof of the above lemma) and compute the differentials from log rings instead.

Lemma 3.10. $\Omega^1_{(\mathbb{A}^n, D_r)/k}$ *is a coherent finite locally free sheaf and we have*

$$\begin{aligned}\Gamma(\mathbb{A}^n, \Omega^1_{(\mathbb{A}^n, D_r)/k}) &= \bigoplus_{i=1}^r k[\underline{x}] \cdot \frac{dx_i}{x_i} \oplus \bigoplus_{j=r+1}^n k[\underline{x}] \cdot dx_j \\ &= k[\underline{x}] \cdot \frac{dx_1}{x_1} \oplus ... \oplus k[\underline{x}] \cdot \frac{dx_r}{x_r} \oplus k[\underline{x}] \cdot dx_{r+1} \oplus ... \oplus k[\underline{x}] \cdot dx_n\end{aligned}$$

For the universal derivations, we have $d(x_i) = dx_i$ *for* $1 \le i \le n$ *and* $\delta(x_i) = \frac{dx_i}{x_i}$ *for* $1 \le i \le r$.

Proof. Set $X = (\mathbb{A}^n, D_r)$ and write

$$\mathcal{D} := \bigoplus_{i=1}^r \mathcal{O}_X \cdot \frac{dx_i}{x_i} \oplus \bigoplus_{j=r+1}^n \mathcal{O}_X \cdot dx_j$$

the free sheaf on generators $\frac{dx_1}{x_1}, ..., dx_n$. We are going to define a relative log derivation $(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{D}$ and show that it is universal. Then we have a unique isomorphism $\Omega^1_{(\mathbb{A}^n,D_r)/k} \cong \mathcal{D}$ compatible with the derivations which yields the result.

In the first step we need to define the log derivation. Classical relative derivations $d : \mathcal{O}_X \to \mathcal{D}$ correspond to k-linear derivations $d : k[\underline{x}] \to \Gamma(X, \mathcal{D})$, and these can be defined by assigning a value to each x_i. We set $d(x_i) := x_i \cdot \frac{dx_i}{x_i}$ for $1 \leq i \leq r$ and $d(x_j) := 1 \cdot dx_j$ for $r+1 \leq j \leq n$. To define the log part δ, it suffices to define it on standard opens D_g for $g \in k[\underline{x}]$. The monoid $\mathcal{M}_X(D_g)$ consists of the elements of $k[\underline{x}]_g$ invertible in $k[\underline{x}]_{gx_1...x_r}$, so they are of the form $ux_1^{k_1}...x_r^{k_r}$ for $u \in \mathcal{O}_X^*(D_g)$. We define

$$\delta(ux_1^{k_1}...x_r^{k_r}) := u^{-1} \cdot d(u) + \sum_{i=1}^{r} k_i \cdot \frac{dx_i}{x_i} \ .$$

If we require $k_i = 0$ for invertible x_i's, then the representation is unique, so choosing this representation δ is well-defined. It is easy to see that we obtain a homomorphism $\delta : \mathcal{M}_X \to \mathcal{D}$ of monoid sheaves. It is straightforward to verify that $\alpha_X \cdot \delta = d \circ \alpha_X$, so it is indeed a log derivation. The derivation d is relative, and since Spec k is log trivial, there is no additional condition on δ for being a relative log derivation.

In the second step, we show that (d, δ) constructed above is universal. Let $(D, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}$ be a relative log derivation. We start with uniqueness of the factorization $\phi : \mathcal{D} \to \mathcal{E}$. Assume $\phi, \phi' : \mathcal{D} \to \mathcal{E}$ are two homomorphisms with $\phi \circ D = d, \phi \circ \delta = \Delta$ and $\phi' \circ d = D, \phi' \circ \delta = \Delta$. Then for $1 \leq i \leq r$, we have

$$\phi(\frac{dx_i}{x_i}) = \phi(\delta(x_i)) = \Delta(x_i) = \phi'(\delta(x_i)) = \phi'(\frac{dx_i}{x_i}) \ ,$$

and for $r+1 \leq j \leq n$, we have

$$\phi(dx_j) = \phi(d(x_j)) = D(x_i) = \phi'(d(x_j)) = \phi'(dx_j) \ ,$$

so $\phi = \phi'$. Note the difference in definition between dx_j as generator of $\mathcal{D}$ and $d(x_j)$ as image of x_j under d.

Now we turn to existence. We define $\phi(\frac{dx_i}{x_i}) := \Delta(x_i)$ and $\phi(dx_j) := D(x_j)$. Since $\mathcal{D}$ is a free $\mathcal{O}_X$-module, this gives a well-defined homomorphism $\phi : \mathcal{D} \to \mathcal{E}$. To show $\phi \circ d = D$ and $\phi \circ \delta = \Delta$, it suffices to do so on standard opens D_g. We start with the classical part D. We have two k-linear derivations

$$(\phi \circ d) \ , \ D : k[\underline{x}]_g \to \Gamma(D_g, \mathcal{E}) \ .$$

By construction, we have $\phi(d(x_i)) = \phi(x_i \cdot \frac{dx_i}{x_i}) = x_i\phi(\frac{dx_i}{x_i}) = x_i \cdot \Delta(x_i) = D(x_i)$ for $1 \leq i \leq r$ and $\phi(d(x_j)) = \phi(dx_j) = D(x_j)$ for $r+1 \leq j \leq n$. This implies $\phi \circ d(h) = D(h)$ for any $h \in k[\underline{x}]$. In particular, $\phi \circ d(g) = D(g)$ whence $\phi \circ d(g^{-1}) = D(g^{-1})$, whence we have $\phi \circ d = D$ on D_g. For the log part, we use the representation from above, and we have

$$\phi \circ \delta(ux_1^{k_1}...x_r^{k_r}) = \phi(u^{-1} \cdot d(u) + \sum_{i=1}^{r} k_i \cdot \frac{dx_i}{x_i}) = u^{-1}\phi(d(u)) + \sum_{i=1}^{r} k_i \cdot \phi(\frac{dx_i}{x_i})$$
$$= \Delta(u) + \sum_{i=1}^{r} k_i \cdot \Delta(x_i) = \Delta(ux_1^{k_1}...x_r^{k_r})$$

Therefore we have $\phi \circ \delta = \Delta$ which shows the existence of the factorization. This completes the proof. $\square$

We introduce a name for the property of $f : (\mathbb{A}^n, D_r) \to \operatorname{Spec} k$ shown by the lemma: We call a morphism $f : X \to Y$ of log schemes *differentially log smooth*, if $\Omega^1_{X/Y}$ is (coherent) finite locally free.

Remark 3.11. *The name is explained by the following fact: There is a notion of log smoothness defined by an infinitesimal lifting property. If $f : X \to Y$ is log smooth, then $\Omega^1_{X/Y}$ is finite locally free. Thus a morphism f is differentially log smooth, if its differentials behave as if f was log smooth. We will not use log smoothness itself in this thesis, but morphisms stated to be differentially log smooth are usually indeed log smooth. We decided to use differential log smoothness instead since it is sufficient for our purposes and easier accessible for the novice.*

Some Properties of Derivations

Here we investigate some properties of log derivations that will be used later. On a log scheme X, we have the forgetful map $\pi : \mathrm{Der}_X(\mathcal{O}_X) \to \mathrm{Der}_{\underline{X}}(\mathcal{O}_X)$ forgetting the log part of the absolute log derivation. For integral schemes with a compactifying log structure, it is injective:

Lemma 3.12. *Let X be a log scheme with $\underline{X}$ integral and having the compactifying log structure defined by $D \subsetneq X$. Then $\pi : \mathrm{Der}_X(\mathcal{O}_X) \to \mathrm{Der}_{\underline{X}}(\mathcal{O}_X)$ is injective.*

Proof. Set $X^* := X \setminus D$ and denote the log structure of X by $\mathcal{M}$. The map π is a homomorphism of sheaves of groups. To show it injective, assume (D, Δ) a log derivation defined on $\emptyset \neq V \subset X$ with $D = 0$. Because $\mathcal{M}$ is a trivial log structure on $X^* \cap V$, we have $\Delta|_{V \cap X^*} = 0$. Now assume $m \in \mathcal{M}(W)$ for $\emptyset \neq W \subset V$. We have $\Delta(m)|_{W \cap X^*} = \Delta(m|_{W \cap X^*}) = 0$, so $\Delta(m) = 0$ by injectivity of the restriction $\rho : \mathcal{O}_X(W) \to \mathcal{O}_X(W \cap X^*)$. Thus $\Delta = 0$ and π is injective. $\square$

Compare this also to the very similar proposition [7][Prop. 1.3.]. Under some additional (technical) hypotheses, we can determine the image of $\pi : \mathrm{Der}_X(\mathcal{O}_X) \to \mathrm{Der}_{\underline{X}}(\mathcal{O}_X)$. We found the criterion in [7][Ex. 1.4.].

Proposition 3.13. *Let X be a noetherian normal integral scheme, and let $\mathcal{I} \subset \mathcal{O}_X$ be a (coherent) sheaf of ideals defining a closed subscheme $Z \subset X$ which is reduced and the union of irreducible codimension 1 subschemes Z_i. Let $\mathcal{M}$ be the compactifying* *log structure defined by Z. Let $D : \mathcal{O}_X \to \mathcal{O}_X$ be a derivation. Then there is a log derivation (D, Δ) of $\mathcal{M}$ if and only if $D(\mathcal{I}) \subset \mathcal{I}$.*

Proof. Let η_i be the generic point of Z_i. First assume there is a log derivation (D, Δ). Let $U \subset X$ be open and $f \in \mathcal{I}(U)$. For η_i, the stalk $\mathcal{O}_{\eta_i}$ is a discrete valuation ring, and $\mathcal{I}_{\eta_i} \subset \mathcal{O}_{\eta_i}$ is a proper radical ideal, so $\mathcal{I}_{\eta_i} = (t_i)$ for a uniformizer $t_i \in \mathcal{O}_{\eta_i}$. Thus, if $\eta_i \in U$, $f = f' t_i^p$ for some $p > 0$. Now $t_i \in \mathcal{M}_{\eta_i}$, so $Df = t_i^p Df' + pf' t_i^{p-1} Dt = t_i^p (Df' + pf' \Delta(t_i))$. This shows $Df \in \mathcal{I}_{\eta_i}$. If $\eta \in U$ is any point with $\dim(\mathcal{O}_\eta) = 1$, then either $\eta = \eta_i$ for some i, or $\eta \in U \setminus Z$. In any case, $Df \in \mathcal{I}_\eta$. Thus there is an open $V \subset U$ with $Df|_V \in \mathcal{I}(V)$ and $\mathrm{codim}(V, U) \geq 2$. Consider the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$. The subset $V \cap Z \subset U \cap Z$ is dense and Z is reduced, so the restriction $\mathcal{O}_Z(U) \to \mathcal{O}_Z(V)$ is injective, and the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is bijective (by normality), so $\mathcal{I}(U) \to \mathcal{I}(V)$ is bijective. Thus $Df \in \mathcal{I}(U)$.

Conversely assume $D(\mathcal{I}) \subset \mathcal{I}$. Define $\Delta : \mathcal{M} \to \mathcal{K}_X, f \mapsto \frac{Df}{f}$. This is well-defined since X is integral and $f \neq 0$ (if f is not considered on the empty set). We like to show that it factors through $\mathcal{O}_X \subset \mathcal{K}_X$. It suffices to show $\frac{Df}{f} \in \mathcal{O}_\eta \subset \mathcal{K}_\eta$ for every η of codimension 1. If $\eta \notin Z$, then $f \in \mathcal{M}_\eta$ implies $f \in \mathcal{O}_\eta^*$, so $\frac{Df}{f} \in \mathcal{O}_\eta$. If $\eta \in Z$, then $\eta = \eta_i$ for some i. Then we have $f = f' t_i^p$ for some $f' \in \mathcal{O}_\eta^*$, and $Dt_i = t_i g$ for some $g \in \mathcal{O}_\eta$. Thus

$$\frac{Df}{f} = \frac{t_i^p Df' + p t_i^{p-1} f' Dt_i}{f' t_i^p} = \frac{Df' + pf'g}{f'} \in \mathcal{O}_\eta \ .$$

Therefore Δ exists and gives a log derivation (D, Δ). $\square$

Later we will use this to compute sheaves of derivations and their duals, the biduals of the log differential forms. If we have an absolute log derivation whose classical part is relative, often the log derivation itself is also relative:

Lemma 3.14. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of log schemes, let X be integral and assume $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ maps every $m \in \mathcal{M}_X$ to a non-zero divisor, e.g. $\mathcal{M}_X$ is compactifying for some divisor. Let $(D, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}$ be an absolute log derivation such that $D : \mathcal{O}_X \to \mathcal{E}$ is a relative derivation, and assume $\mathcal{E}$ torsion free. Then (D, Δ) is a relative log derivation.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
f^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"] & \mathcal{O}_X \arrow[r, "D"] & \mathcal{E} \\
f^{-1}\mathcal{M}_Y \arrow[r, "f^\sharp"] \arrow[u, "f^{-1}\alpha_Y"] & \mathcal{M}_X \arrow[r, "\Delta"] \arrow[u, "\alpha_X"] & \mathcal{E}
\end{tikzcd}$$

We need to show $\Delta \circ f^\sharp = 0$. For $m \in f^{-1}\mathcal{M}_Y$, we have

$$\alpha_X(f^\sharp(m)) \cdot \Delta(f^\sharp(m)) = D(\alpha_X \circ f^\sharp(m)) = D \circ f^\sharp(f^{-1}\alpha_Y(m)) = 0 \ ,$$

so since $\mathcal{E}$ is torsion free and $\alpha_X(f^\sharp(m))$ a non-zero divisor, we have $\Delta(f^\sharp(m)) = 0$. $\square$

4 The Log Differentials $\Omega^1_{X/S}$

In this section we investigate the log differentials $\Omega^1_{X/S}$ of the family $f : X \to S$. First we show that $\Omega^1_{X/S}$ is locally free on $U \subset X$, and that the log structure is coherent on U. Then we show that $\Omega^1_{X/S}$ is not coherent on the whole of X. This shows that $\Omega^1_{X/S}$ is not sufficient as good differentials, and the large locally free locus motivates the definition of $W^1_{X/S} := j_*\Omega^1_{U/S}$.

4.1 The Differentially Log Smooth Locus of $f : X \to S$

Recall that we call a log morphism $f : X \to Y$ *differentially log smooth* if $\Omega^1_{X/Y}$ is finite locally free. This section is devoted to showing that $f : X \to S$ in our example is differentially log smooth on

$$U := D(x) \cup D(y) \cup D(t) \cup D(w) = X \setminus \{P\} .$$

Here $D(x) = X \setminus V(x) = \text{Spec } B_x$ etc. The local freeness on U is not only important for the definition of $W^1_{X/S}$, but we also need it to show $\Omega^1_{X/S}$ non-coherent around $P \in X$ in the next section.

Let us begin with $D(x)$. We have $D(x) = \text{Spec } B_x$ with $B_x = k[x, x^{-1}, t, w]$. The scheme $D(x)$ has the compactifying log structure defined by the divisor $\{t = 0\}$, and recall that S has the compactifying log structure defined by $\{t = 0\} \subset S$. In particular, we see that X is coherent on $D(x)$, for it is isomorphic to a subscheme of $(\mathbb{A}^n, D_r)$ which is coherent by Lemma 3.9. We have morphisms $f : D(x) \to S$ and $g : S \to \text{Spec } k$. Thus we get an exact sequence

$$f^*\Omega^1_{S/k} \to \Omega^1_{D(x)/k} \to \Omega^1_{D(x)/S} \to 0 .$$

By Lemma 3.10, we have

$$\Gamma(S, \Omega^1_{S/k}) = k[t] \cdot \frac{dt}{t} .$$

We may also consider $D(x) \subset (\mathbb{A}^3, \{t = 0\})$, so Lemma 3.10 yields

$$\Gamma(D(x), \Omega^1_{D(x)/k}) = B_x \cdot \frac{dt}{t} \oplus B_x \cdot dw \oplus B_x \cdot dx .$$

Moreover, both sheaves are coherent. The construction of $f^*\Omega^1_{S/k} \to \Omega^1_{X/k}$ shows that this map is given by

$$B_x \otimes_{k[t]} k[t] \cdot \frac{dt}{t} \to B_x \cdot \frac{dt}{t} \oplus B_x \cdot dw \oplus B_x \cdot dx, \; \frac{dt}{t} \mapsto \frac{dt}{t} ,$$

so we conclude that $\Omega^1_{D(x)/S}$ is coherent and we have

$$\Gamma(D(x), \Omega^1_{D(x)/S}) = B_x \cdot dw \oplus B_x \cdot dx .$$

whence $\Omega^1_{X/S}$ is free on $D(x)$.

The case of $D(y)$ is very similar to $D(x)$. We have $D(y) = \mathrm{Spec}\ B_y$ with $B_y = k[y, y^{-1}, t, w]$. The scheme $D(y)$ has the compactifying log structure defined by the divisor $\{t = 0\}$, so X is coherent on $D(y)$. We get an exact sequence

$$f^*\Omega^1_{S/k} \to \Omega^1_{D(y)/k} \to \Omega^1_{D(y)/S} \to 0 \ ,$$

and using $D(y) \subset (\mathbb{A}^3, \{t = 0\})$, we compute

$$\Gamma(D(y), \Omega^1_{D(y)/k}) = B_y \cdot \frac{dt}{t} \oplus B_y \cdot dw \oplus B_y \cdot dy \ .$$

Again, $\Omega^1_{D(y)/S}$ is coherent, and we obtain its global sections as the cokernel of

$$B_y \otimes_{k[t]} k[t] \cdot \frac{dt}{t} \to B_y \cdot \frac{dt}{t} \oplus B_y \cdot dw \oplus B_y \cdot dy, \ \frac{dt}{t} \mapsto \frac{dt}{t} \ ,$$

that is we have

$$\Gamma(D(y), \Omega^1_{D(y)/S}) = B_y \cdot dw \oplus B_y \cdot dy \ .$$

We see that $\Omega^1_{X/S}$ is free on $D(y)$.

A bit more intricate is the case of $D(w)$. We have $D(w) = \mathrm{Spec}\ B_w$ with $B_w = k[x, y, w, w^{-1}]$, but now the log structure is given by the divisor $\{\frac{xy}{w} = 0\} = \{xy = 0\}$. That is, we have $D(w) \subset (\mathbb{A}^3, \{xy = 0\})$, so first X is coherent on $D(w)$, and secondly $\Omega^1_{D(w)/k}$ is coherent with

$$\Gamma(D(w), \Omega^1_{D(w)/k}) = B_w \cdot \frac{dx}{x} \oplus B_w \cdot \frac{dy}{y} \oplus B_w \cdot dw \ .$$

Now we see that $\Omega^1_{D(w)/S}$ is coherent. The morphism $f : D(w) \to S$ is given by $\phi : k[t] \to k[x, y, w, w^{-1}], t \mapsto \frac{xy}{w}$. On $D(w)$, we have a commutative diagram

$$\begin{tikzcd}
f^{-1}\Omega^1_{S/k} \arrow[r] & f^*\Omega^1_{S/k} \arrow[r, "\chi"] & \Omega^1_{D(w)/k} \\
f^{-1}\mathcal{M}_S \arrow[u, "\delta"] \arrow[rr, "\phi"] & & \mathcal{M}_{D(w)} \arrow[u, "\delta"']
\end{tikzcd}$$

This yields

$$\chi(\frac{dt}{t}) = \chi(\delta(t)) = \delta(\phi(t)) = \delta(\frac{xy}{w}) = \frac{dx}{x} + \frac{dy}{y} - w^{-1}dw \ .$$

We obtain an exact sequence

$$B_w \otimes_{k[t]} k[t]\frac{dt}{t} \ \to \ B_w \cdot \frac{dx}{x} \oplus B_w \cdot \frac{dy}{y} \oplus B_w \cdot dw \ \to \ B_w \cdot \frac{dx}{x} \oplus B_w \cdot \frac{dy}{y} \ \to \ 0$$

where the second map is given by $\frac{dx}{x} \mapsto \frac{dx}{x}, \frac{dy}{y} \mapsto \frac{dy}{y}, dw \mapsto \frac{wdx}{x} + \frac{wdy}{y}$. Therefore $\Omega^1_{D(w)/S}$ is free, and $\Omega^1_{X/S}$ is free on $D(w)$.

Finally we turn to $D(t)$. Since $D(t) \cap X_0 = \emptyset$, the log structure is trivial on $D(t)$ (whence coherent), and we have $\Omega^1_{D(t)/S} = \Omega^1_{\underline{D(t)}/\underline{S}}$. We have $D(t) = \text{Spec } B_t$ with $B_t = k[x, y, t, t^{-1}]$. The ring map $k[t] \to k[x, y, t, t^{-1}]$ is (classically) smooth, so we have that $\Omega^1_{D(t)/S} = \Omega^1_{\underline{D(t)}/\underline{S}}$ is finite locally free. Since being coherent is a local property of log schemes, and being finite locally free is a local property of sheaves, we conclude:

Proposition 4.1. *The log scheme X is coherent on $U = X \setminus \{P\}$, and the log differentials $\Omega^1_{X/S}$ are finite locally free on U.*

4.2 Noncoherence of $\Omega^1_{X/S}$

We show that $\Omega^1_{X/S}$ is not a coherent sheaf. Therefore our search for good differentials is also interesting from a log view point, since one may wish to have a coherent replacement of $\Omega^1_{X/S}$. Our proof is inspired by [7][Ex. 1.11.]. The main steps are as follows: Since X is log trivial on $X^* := X \setminus X_0$, the map $e_{X/S} : \Omega^1_{\underline{X}/\underline{S}} \to \Omega^1_{X/S}$ is an isomorphism on X^*. The classical differentials $\Omega^1_{\underline{X}/\underline{S}}$ are torsion free by Lemma 2.10 which then implies $e_{X/S}$ injective. In the next step, we will show $e_{X/S}$ surjective in the origin $P \in X$ by investigating the stalk $\mathcal{M}_{X,P}$ of the log structure. Now if $\Omega^1_{X/S}$ is coherent, then the cokernel $\mathcal{C} := \Omega^1_{X/S}/\Omega^1_{\underline{X}/\underline{S}}$ is coherent, so its support is closed and $P \notin \text{Supp } \mathcal{C}$. This means there is an open $V \ni P$ on which $e_{X/S}$ is an isomorphism. In particular we get an isomorphism $\Theta^1_{V/S} \to \Theta^1_{\underline{V}/\underline{S}}$ given by forgetting the log part of the derivation. We then give a classical derivation $D \in \Theta^1_{\underline{X}/\underline{S}}(X)$ that around P does not come from a log derivation contradicting the isomorphism.

We start with injectivity of $e_{X/S}$. The sheaf $\Omega^1_{\underline{X}/\underline{S}}$ is torsion free implying that $\Omega^1_{\underline{X}/\underline{S}}$ has injective restriction maps as long as we do not restrict to the empty set. We have $e_{X/S}|_{X^*} : \Omega^1_{\underline{X}/\underline{S}}|_{X^*} \cong \Omega^1_{X/S}|_{X^*}$ since X is log trivial on $X^* = X \setminus X_0$. If $V \subset X$ is open and non-empty, then we have a diagram

$$\begin{array}{ccc} \Omega^1_{\underline{X}/\underline{S}}(V) & \xrightarrow{e_{X/S}(V)} & \Omega^1_{X/S}(V) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\rho}\downarrow \\ \Omega^1_{\underline{X}/\underline{S}}(V \cap X^*) & \xrightarrow{e_{X/S}(V\cap X^*)} & \Omega^1_{X/S}(V \cap X^*) \end{array}$$

where in the lower line $e_{X/S}(V \cap X^*)$ is an isomorphism and on the left ρ is injective. Thus $e_{X/S}(V)$ is injective. We conclude:

Lemma 4.2. *The map $e_{X/S} : \Omega^1_{\underline{X}/\underline{S}} \to \Omega^1_{X/S}$ is injective.*

Our next goal is to show $e_{X/S}$ surjective in $P \in X$. Recall that $\mathcal{M}_X$ is the sheaf of functions invertible outside of X_0. Let $[f] \in \mathcal{M}_{X,P}$ be defined by a function $f \in$

$\mathcal{O}_X(V) =: A$ invertible on $V \setminus X_0$ with $V \cong \operatorname{Spec} A$ affine. For the vanishing locus we have $V(f) \subset X_0 \cap V = V(t)$, so $\operatorname{rad}(t) \subset \operatorname{rad}(f)$ in A. Therefore we can write $t^n = hf$ for some exponent n and $h \in A$.

In the next step we will prove that $[f] = [t]^k \cdot u \in \mathcal{O}_{X,P} = B_{\mathfrak{m}}$ for some $k \le n$ and a unit $u \in B^*_{\mathfrak{m}}$. Recall that $B \cong k[P]$ for the monoid P. More precisely we show:

Proposition 4.3. *Let* $t^n = \frac{g_1}{u_1} \cdot \frac{g_2}{u_2} \in k[P]_{\mathfrak{m}}$ *with* $g_1, g_2 \in k[P], u_1, u_2 \in k[P] \setminus \mathfrak{m}$. *Then there is a decomposition* $n = k + \ell$ *with* $k, \ell \ge 0$ *and* $c_1, c_2 \in k[P] \setminus \mathfrak{m}$ *with* $g_1 = t^k \cdot c_1, g_2 = t^\ell \cdot c_2$.

Proof. First note that $k[P] \setminus \mathfrak{m}$ are precisely the elements with non-zero $z^{(0,0,0)}$ coefficient. The ring $k[P]$ is integral, so the assumption $t^n = \frac{g_1}{u_1} \cdot \frac{g_2}{u_2}$ is equivalent to $t^n u_1 u_2 = g_1 g_2 \in k[P]$. Recall $t = z^{(1,0,0)}$, write $g_i = \sum \gamma^i_p \cdot z^p$ and set

$$k := \min\{m \in \mathbb{N} \mid \gamma^1_{(m,0,0)} \ne 0\}, \quad \ell := \min\{m \in \mathbb{N} \mid \gamma^2_{(m,0,0)} \ne 0\} \ .$$

This means g_1 has a t^k monomial, but no t^m monomial for $m < k$. Note that the indices k, ℓ are finite and that $k + \ell = n$. On $B = k[P]$, there are two gradings $\bigoplus_{m \ge 0} B^x_m$ and $\bigoplus_{m \ge 0} B^y_m$ given by

$$B^x_m := \left\{ \sum a_p z^p \mid a_{(p_1,p_2,p_3)} = 0 \text{ unless } p_1 - p_3 = m \right\}$$
$$B^y_m := \left\{ \sum a_p z^p \mid a_{(p_1,p_2,p_3)} = 0 \text{ unless } p_1 - p_2 = m \right\}$$

We have $x \in B^x_0$ and $y \in B^y_0$ explaining the notation. The gradings are compatible with the ring structure in the sense that $B^x_m \cdot B^x_{m'} \subset B^x_{m+m'}$ and $B^y_m \cdot B^y_{m'} \subset B^y_{m+m'}$.

With respect to the grading B^x, we can write

$$g_1 = g_1^{k',x} + g_1^{k'+1,x} + \ldots \quad \text{and} \quad g_2 = g_2^{\ell',x} + g_2^{\ell'+1,x} + \ldots$$

with $g_i^{m,x} \in B^x_m$ and $g_1^{k',x}, g_2^{\ell',x} \ne 0$. Thus

$$g_1 g_2 = g_1^{k',x} \cdot g_2^{\ell',x} + r \quad \text{with} \quad r \in \bigoplus_{m \ge k'+\ell'+1} B^x_m \ .$$

We have $0 \ne g_1^{k',x} \cdot g_2^{\ell',x} \in B^x_{k'+\ell'}$. Since $g_1 g_2 = t^n u_1 u_2$, the product $g_1 g_2$ has no component for B^x_m with $m < k + \ell$ showing that $k' + \ell' \ge n$. Since $k' \le k, \ell' \le \ell$, we have $k' = k, \ell' = \ell$. Thus

$$g_1 \in \bigoplus_{m \ge k} B^x_m = (t, y)^k \quad \text{and} \quad g_2 \in \bigoplus_{m \ge \ell} B^x_m = (t, y)^\ell \ .$$

Using the decomposition B^y instead, we obtain analogously

$$g_1 \in \bigoplus_{m \ge k} B^y_m = (t, x)^k \quad \text{and} \quad g_2 \in \bigoplus_{m \ge \ell} B^y_m = (t, x)^\ell$$

showing that $g_1 \in (t,x)^k \cap (t,y)^k = (t^k)$ and $g_2 \in (t^\ell)$. Therefore there are $c_1, c_2 \in k[P]$ with $g_1 = t^k \cdot c_1$ and $g_2 = t^\ell \cdot c_2$. Since g_1 has a t^k monomial, the element c_1 has a z^0 monomial whence $c_1 \in k[P] \setminus \mathfrak{m}$. Analogously, we have $c_2 \in k[P] \setminus \mathfrak{m}$. □

So there is a unit $u \in B_{\mathfrak{m}}^*$ with $[f] = [t]^k \cdot u$. For the stalk in P, we have

$$\Omega^1_{X/S,P} = \left(\Omega^1_{\underline{X}/\underline{S},P} \oplus (\mathcal{O}_{X,P} \otimes_{\mathbb{Z}} \mathcal{M}^{gp}_{X,P}) \right) / \mathcal{K}_P \ .$$

Now $(u^{-1}du, -1 \otimes u) = u^{-1}(du, -u \otimes u) \in \mathcal{K}_P$ and $(0, 1 \otimes [t]^k) \in \mathcal{K}_P$, so $(u^{-1}du, 0) = (0, 1 \otimes ([t]^k \cdot u)) = (0, 1 \otimes [f]) \in \Omega^1_{X/S,P}$. This shows $(0, 1 \otimes [f]) \in \mathrm{im}(e_{X/S,P})$, and since $e_{X/S,P}$ is an $\mathcal{O}_{X,P}$-module homomorphism, we get that $e_{X/S,P}$ is surjective. Thus we have shown:

Lemma 4.4. *The map $e_{X/S} : \Omega^1_{\underline{X}/\underline{S}} \to \Omega^1_{X/S}$ is surjective in $P \in X$.*

Now we define $\mathcal{C} := \mathrm{coker}(e_{X/S} : \Omega^1_{\underline{X}/\underline{S}} \to \Omega^1_{X/S})$. We have just proven that $\mathcal{C}_P = 0$. If $\Omega^1_{X/S}$ was coherent, then $\mathcal{C}$ was coherent. A coherent sheaf has a closed support, so assuming $\Omega^1_{X/S}$ coherent, there is an open $V \subset X$ with $P \in V$ and $\mathcal{C}|_V = 0$. Therefore $e_{X/S}$ is surjective on V, and we get $\Omega^1_{\underline{X}/\underline{S}}|_V \cong \Omega^1_{X/S}|_V$. Since $\Omega^1_{X/S}$ is locally free on U, this implies $\Omega^1_{\underline{X}/\underline{S}}$ locally free on $V' := V \setminus \{P\}$. Let $\bar{k}/k$ be an algebraic closure of k, and set $\bar{S} := S \times_k \bar{k}$ etc. and consider them as schemes (not log schemes). We get a morphism $\bar{f} : \bar{V}' \to \bar{S}$ such that $\Omega^1_{\bar{V}'/\bar{S}}$ is locally free. It is easy to see that U/k is nonsingular since each of $D(x), D(y), D(w), D(t)$ is nonsingular. Thus $\bar{V}'/\bar{k}$ is nonsingular, and also $\bar{S}/\bar{k}$ is nonsingular. Now [10][III. Prop. 10.4] implies that $\bar{f} : \bar{V}' \to \bar{S}$ is (classically) smooth of relative dimension 2. In particular, the central fiber $\bar{X}_0$ over $0 \in \bar{S} = \mathrm{Spec}\ \bar{k}[t]$ is nonsingular in $W \setminus \{P\}$ for a neighbourhood W of P in $\bar{X}_0$. But we have $\bar{X}_0 = \mathrm{Spec}\ \bar{k}[x,y,w]/(xy)$, an obvious contradiction. We conclude:

Proposition 4.5. *The sheaf of log differential forms $\Omega^1_{X/S}$ is not coherent.*

Now let us refine the above argument. The point about it is that the support of $\mathcal{C}$ is not closed, and that is not a matter of quasi-coherence, but of being *of finite type*. Recall the definition:

Definition 4.6. *(cf. [1][01B4, Def. 17.9.1]) A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called* of finite type*, if for every $x \in X$, there is an open $U \ni x$ with $\mathcal{F}|_U$ finitely generated, i.e. there is a surjection $\mathcal{O}_X|_U^{\oplus r} \to \mathcal{F}|_U$.*

So since the support of a finite type sheaf is closed, we have shown that $\mathcal{C}$ is not of finite type. It also shows that $\Omega^1_{X/S}$ is not of finite type, for otherwise $\mathcal{C}$ was of finite type. On a noetherian scheme coherent means of finite type and quasi-coherent, so how about quasi-coherence? We consider the extension by zero $j_!\mathcal{C}|_U$. It has an obvious map $j_!\mathcal{C}|_U \to \mathcal{C}$ which is an isomorphism on U. Since the stalks satisfy $(j_!\mathcal{C}|_U)_P = 0 = \mathcal{C}_P$, the map $j_!\mathcal{C}|_U \to \mathcal{C}$ is an isomorphism. Now if $\Omega^1_{X/S}$ was quasi-coherent, then $\mathcal{C}$ was quasi-coherent, so $\mathcal{C} = 0$ since $j_!\mathcal{C}|_U(X) = 0$. This would even imply $\Omega^1_{\underline{X}/\underline{S}} = \Omega^1_{X/S}$ on the whole space X, so again a contradiction.

Theorem 4.7. *The sheaf of log differential forms $\Omega^1_{X/S}$ is neither quasi-coherent nor of finite type.*

This has also implications for the coherence of the log structures involved. Namely, $\mathcal{M}_S$ on S is a coherent log structure since it is constructed from the spectrum construction. Now if $\mathcal{M}_X$ also was a coherent log structure, then the differential forms $\Omega^1_{X/S}$ would form a quasi-coherent sheaf.

Corollary 4.8. *The log structure $\mathcal{M}_X$ on X is* not *coherent.*

5 The System of Differentials $(W^\bullet_{Y/T}, d^\bullet_T)$

In our quest for good differentials for $f : X \to S$ in the introduction we arrived at considering
$$W^m_{X/S} := j_* \Omega^m_{U/S}$$
a candidate for good differential forms where $j : U \subset X$ is the inclusion. Moreover, for every morphism $b : T \to S$ of log schemes with T affine (and noetherian) and where the log structure on T is coherent we considered the cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "c"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "b"] & S \end{tikzcd}$$

of log schemes and – setting $V := c^{-1}(U)$ – attached to $b : T \to S$ the complex
$$(W^\bullet_{Y/T}, d^\bullet_T) := j_*(\Omega^\bullet_{V/T}, d^\bullet) \; .$$

Note that the restriction induces a canonical sheaf homomorphism
$$c^m_{Y/T} : \Omega^m_{Y/T} \to W^m_{Y/T}$$
which form a homomorphism of complexes $c_{Y/T} : (\Omega^\bullet_{Y/T}, d^\bullet) \to (W^\bullet_{Y/T}, d^\bullet_T)$. This section is devoted to investigating the basic properties of the complex $(W^\bullet_{Y/T}, d^\bullet_T)$.

5.1 Easy Properties

We start with easy properties of $(W^\bullet_{Y/T}, d^\bullet_T)$. The first point is existence of the differentials $d^\bullet_T$ of the complex. The log schemes U and S are coherent, and we can write $V = T \times_S U$ as a fiber product of log schemes. Because we assumed T coherent, this implies V coherent as a log scheme, and we can use Proposition 3.4 to construct differentials $d^m : \Omega^m_{V/T} \to \Omega^{m+1}_{V/T}$. Thus we really get a *complex* $(\Omega^\bullet_{V/T}, d^\bullet)$ of sheaves of groups on V to which we can apply the functor $j_*(-)$.

Remark 5.1. *Indeed, we assume T coherent to run this argument, for Proposition 3.4 requires coherence. In our situation, for non-coherent T there is a workaround using the factorization $(T, \mathcal{M}_T) \to (T, b^*_{log}\mathcal{M}_S) \to (S, \mathcal{M}_S)$. Namely, $b^*_{log}\mathcal{M}_S$ is coherent, and then we can construct the exterior derivative for $(T, \mathcal{M}_T)$ from that of $(T, b^*_{log}\mathcal{M}_S)$. However, we consider the case of coherent bases as general enough. We do not know whether there is an analogue of Proposition 3.4 for non-coherent log schemes.*

Next note that the pieces of the complex $(W^\bullet_{Y/T}, d^\bullet_T)$ are indeed $\mathcal{O}_Y$-modules. Using that the differentials of $(\Omega^\bullet_{V/T}, d^\bullet)$ are $(g^{-1}\mathcal{O}_T)|_V$-linear, it is straightforward to show that the differentials $d^\bullet_T$ are $g^{-1}\mathcal{O}_T$-linear. Since $V = c^{-1}(U)$, we have a cartesian diagram

$$\begin{array}{ccccc} V & \xrightarrow{j} & Y & \xrightarrow{g} & T \\ {\scriptstyle c}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow \\ U & \xrightarrow{j} & X & \xrightarrow{f} & S \end{array}$$

so by log base change (see Proposition 3.6) we have $\Omega^1_{V/T} \cong c^*\Omega^1_{U/S}$. Since the exterior power commutes with inverse images of sheaves, we have $\Omega^m_{V/T} \cong c^*\Omega^m_{U/S}$, too. We conclude that $\Omega^m_{V/T}$ is (finite) locally free, and $W^m_{Y/T}$ is locally free on $V \subset Y$:

Lemma 5.2. *The sheaf $W^m_{Y/T}$ is locally free on $V \subset Y$.*

More precisely, the concrete calculations of Section 4.1 show that $W^1_{Y/T}$ is locally free of rank 2 on V, so we have $(W^m_{Y/T})|_V = 0$ for $m \geq 3$. Now the definition of $W^m_{Y/T}$ yields $W^m_{Y/T} = 0$ for $m \geq 3$, and noting that of course $W^m_{Y/T} = 0$ for $m < 0$, we get that $(W^\bullet_{Y/T}, d^\bullet_T)$ is concentrated in degrees 0,1,2.

Corollary 2.8 yields an isomorphism $\mathcal{O}_Y \to j_*\mathcal{O}_V$, so we obtain

$$W^0_{Y/T} = j_*\Omega^0_{V/T} = j_*\mathcal{O}_V = \mathcal{O}_Y \ .$$

Since $d^0 : \mathcal{O}_V \to \Omega^1_{V/T}$ is a derivation, also $d^0_T : \mathcal{O}_Y \to W^1_{Y/T}$ is a derivation, for this property can be checked on the value of the sheaves on open subsets.

Now we turn to the functoriality property claimed in Theorem 1.3. The functoriality proven in Section 8.3 in the appendix applied to the coherent log schemes U, V, S, T yields a homomorphism $\sigma^\bullet_{V/U} : (\Omega^\bullet_{U/S}, d^\bullet) \to c_*(\Omega^\bullet_{V/T}, d^\bullet)$ of complexes of abelian sheaves. The maps $\sigma^m_{V/U}$ itself are homomorphisms of $\mathcal{O}_U$-modules. Applying $j_*(-)$ and noting that $j_*c_*\Omega^m_{V/T} = c_*j_*\Omega^m_{V/T} = c_*W^m_{Y/T}$ yields the desired commutative diagram

$$\begin{array}{ccc} c_*W^m_{Y/T} & \xrightarrow{d^m_T} & c_*W^{m+1}_{Y/T} \\ {\scriptstyle \sigma^m}\uparrow & & {\scriptstyle \sigma^{m+1}}\uparrow \\ W^m_{X/S} & \xrightarrow{d^m_S} & W^{m+1}_{X/S} \end{array}$$

5.2 Coherence and Reflexivity

By construction, $W^m_{Y/T}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. Our goal is to show it coherent and reflexive. We need the following extension property of quasi-coherent finite type sheaves on qcqs schemes:

Proposition 5.3. *Let X be a qcqs scheme, let $\mathcal{F}$ be a quasi-coherent sheaf on X, let $U \subset X$ be a quasi-compact open subset, and let $\mathcal{E} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf of finite type on U. Then there is a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ of finite type on X such that $\mathcal{E} = \mathcal{G}|_U$.*

Proof. This is [1][01PD, Lemma 27.22.2]. The original source seems to be [8][9.4.8]. □

Since noetherian schemes are qcqs and V is quasi-compact, we can apply this to $\Omega^m_{V/T} \subset W^m_{Y/T}|_V$. Thus there is a quasi-coherent finite type sheaf $\mathcal{G} \subset W^m_{Y/T}$ on Y with $\mathcal{G}|_V = \Omega^m_{V/T}$. This $\mathcal{G}$ is coherent since Y is noetherian. For a sheaf $\mathcal{F}$ we write $\mathcal{F}^* := \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$ for its dual and obtain a diagram

$$\begin{tikzcd}
 & j_*(W^m_{Y/T})|_V \arrow[rr] & & j_*(W^m_{Y/T})^{**}|_V \\
W^m_{Y/T} \arrow[ur] \arrow[rr] & & (W^m_{Y/T})^{**} \arrow[ur] & \\
 & j_*\mathcal{G}|_V \arrow[uu] \arrow[rr] & & j_*(\mathcal{G}^{**})|_V \arrow[uu] \\
\mathcal{G} \arrow[uu] \arrow[ur] \arrow[rr] & & \mathcal{G}^{**} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

with the bidual sheaves. Here the front sheet is constructed first, and then the back sheet is obtained by applying the functor $j_*(-)|_V$.

Since the dual of any coherent sheaf is coherent, $\mathcal{G}^{**}$ is coherent. If $\mathcal{E}$ is locally free, then $\mathcal{E} \to \mathcal{E}^{**}$ is an isomorphism, so $\mathcal{G} \to \mathcal{G}^{**}$ is an isomorphism on V, and also $W^m_{Y/T} \to (W^m_{Y/T})^{**}$ is an isomorphism on V. In particular, the back sheet consists only of isomorphisms. Furthermore, we have the following lemma:

Lemma 5.4. *Let $U \subset X$ be an open subscheme, and assume $\rho : \mathcal{O}_X \to j_*\mathcal{O}_U$ is an isomorphism. Let $\mathcal{F}$ be a sheaf of modules on X (which is not necessarily quasi-coherent). Then $\rho : \mathcal{F}^* \to j_*(\mathcal{F}^*)|_U$ is an isomorphism.*

Proof. The straightforward proof is left to the reader. □

In Section 2.2 we worked hard to obtain Corollary 2.8 which says that $\mathcal{O}_Y \to j_*\mathcal{O}_V$ is an isomorphism. Now applying the lemma to $\mathcal{F} = \mathcal{G}^*$ and $\mathcal{F} = (W^k_{X/S})^*$ we see that $(W^m_{Y/T})^{**} \to j_*(W^m_{Y/T})^{**}|_V$ and $\mathcal{G}^{**} \to j_*(\mathcal{G}^{**})|_V$ are isomorphisms. Moreover, $W^m_{Y/T} \to j_*(W^m_{Y/T})|_V$ is an isomorphism, so we get isomorphisms

$$W^m_{Y/T} \cong (W^m_{Y/T})^{**} \cong \mathcal{G}^{**}$$

which show $W^m_{Y/T}$ coherent. A sheaf $\mathcal{F}$ of modules is called *reflexive*, if it is coherent and isomorphic to its bidual, so we conclude:

Proposition 5.5. *The sheaves $W^m_{Y/T}$ are reflexive.*

Running the above argument with $\Omega^m_{Y/T}$ instead of $\mathcal{G}$ yields that $W^m_{Y/T} \cong (\Omega^m_{Y/T})^{**}$. This means that $W^m_{Y/T}$ are the log Danilov differentials, the bidual of the log differentials, as claimed in the introduction. The isomorphism can also be constructed from

$$(\Omega^m_{Y/T})^{**} \leftarrow \Omega^m_{Y/T} \to W^m_{Y/T} \ .$$

These two morphisms are isomorphisms on $V \subset Y$, and the two outer sheaves satisfy both $\mathcal{F} = j_*\mathcal{F}|_V$.

Also the log tangent sheaf $\Theta^1_{Y/T} := \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\Omega^1_{Y/T}, \mathcal{O}_Y)$ is coherent and reflexive. Namely, consider the dual of $c_{Y/T} : \Omega^1_{Y/T} \to W^1_{Y/T}$,

$$c^*_{Y/T} : (W^1_{Y/T})^* \to \Theta^1_{Y/T} \ .$$

It is an isomorphism on V, and by the above lemma, we have $\mathcal{F} = j_*\mathcal{F}|_V$ for $\mathcal{F} \in \{(W^1_{Y/T})^*, \Theta^1_{Y/T}\}$, whence $c^*_{Y/T}$ is an isomorphism. Therefore $\Theta^1_{Y/T}$ is coherent and even reflexive. The isomorphism can be interpreted as the fact that every log derivation to $\mathcal{O}_Y$ factors uniquely through $W^1_{Y/T}$.

5.3 The Kähler Property

In this section, we show the Kähler property for the differentials $W^1_{Y/T}$ claimed in Theorem 1.3, i.e. that $W^1_{Y/T}$ coincides with $\Omega^1_{\underline{Y}/\underline{T}}$ whenever $\underline{Y} \to \underline{T}$ is (classically) smooth. We consider the composite homomorphism

$$\Omega^1_{\underline{Y}/\underline{T}} \xrightarrow{e_{Y/T}} \Omega^1_{Y/T} \xrightarrow{c^1_{Y/T}} W^1_{Y/T}$$

and we like to show it an isomorphism whenever $\underline{g} : \underline{Y} \to \underline{T}$ is smooth as a morphism of schemes. We will achieve this by showing Y log trivial in a relative sense over T in this case (the appropriate notion is strictness of $g : Y \to T$), so $e_{Y/T}$ will be an isomorphism, and we will see that $c^1_{Y/T}$ is the identity.

We assumed T to be affine, so $T = \mathrm{Spec}\ R$ for an A-algebra R. We use the following lemma:

Lemma 5.6. *Let $\underline{g} : \underline{Y} \to \underline{T}$ be smooth. Then $t \in A = k[t]$ is invertible in R.*

Proof. Assume the contrary. Then $(t) \subset R$ is a proper ideal, so there is a maximal ideal $(t) \subset M \subset R$. This fits into a commutative diagram of rings

$$\begin{tikzcd}
 & & R \otimes_A B \arrow[dl] & & B \arrow[ll] \arrow[dl] \\
L \otimes_k B_0 & R/(t) \otimes_k B_0 \arrow[l] & & B_0 \arrow[ll] & \\
 & & R \arrow[uu] \arrow[dl] & & A = k[t] \arrow[ll] \arrow[uu] \arrow[dl] \\
L := R/M \arrow[uu] & R/(t) \arrow[l] \arrow[uu] & & k = A/(t) \arrow[ll] \arrow[uu] &
\end{tikzcd}$$

where every square is cocartesian. Smoothness is stable under base change and the ring $R \otimes_A B/R$ is smooth by assumption, so $L \otimes_k B_0/L$ is smooth. The quotient $L = R/M$ is a field since M is a maximal ideal, and we have $B_0 = k[x, y, w]/(xy)$, so $L \otimes_k B_0 = L[x, y, w]/(xy)$. This is obviously not smooth over L yielding a contradiction and showing that $t \in R$ is invertible. $\square$

Now since $t \in R$ is invertible, we get a factorization $k[t] \to k[t,t^{-1}] \to R$ corresponding to a factorization $T \to S^+ \subset S$ of schemes (since $S^+ = S \setminus \{0\} = \operatorname{Spec} k[t,t^{-1}]$) of classical schemes. Pulling back the log structure from S yields a factorization $(T, b_{log}^*\mathcal{M}_S) \to (S^+, \mathcal{M}_S|_{S^+}) \to (S, \mathcal{M}_S)$ of morphisms of log schemes. Using additionally the factorization $(T, \mathcal{M}_T) \to (T, b_{log}^*\mathcal{M}_S) \to (S, \mathcal{M}_S)$ of log schemes (coming from the universal property of pullback log structures as explained in Section 3), we obtain a cartesian diagram

$$\begin{tikzcd}
(Y,\mathcal{M}_Y) \arrow[r] \arrow[d, "g"'] & (Y, c_{log}^*\mathcal{M}_X) \arrow[r] \arrow[d, "\tilde{g}"'] & (X \setminus X_0, \mathcal{M}_X|_{X\setminus X_0}) \arrow[r] \arrow[d, "f"'] & (X, \mathcal{M}_X) \arrow[d, "f"'] \\
(T, \mathcal{M}_T) \arrow[r] & (T, b_{log}^*\mathcal{M}_S) \arrow[r] & (S^+, \mathcal{M}_S|_{S^+}) \arrow[r] & (S, \mathcal{M}_S)
\end{tikzcd}$$

of log schemes. The log structure $\mathcal{M}_S$ is compactifying for the divisor $S_0 = \{0\}$, so $\mathcal{M}_S|_{S^+} = \mathcal{O}_{S^+}^*$ the trivial log structure. Also $\mathcal{M}_X$ is compactifying with divisor X_0, so $\mathcal{M}_X|_{X\setminus X_0}$ is trivial. This implies $c_{log}^*\mathcal{M}_X$ and $b_{log}^*\mathcal{M}_S$ trivial, so we have $(Y, c_{log}^*\mathcal{M}_X) = \underline{Y}$, $(T, b_{log}^*\mathcal{M}_S) = \underline{S}$ and $\tilde{g} = \underline{g}$. Now the base change in the cartesian diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "g"'] & \underline{Y} \arrow[r, equal] \arrow[d, "\underline{g}"'] & (Y, c_{log}^*\mathcal{M}_X) \arrow[d, "\tilde{g}"'] \\
T \arrow[r] & \underline{T} \arrow[r, equal] & (T, b_{log}^*\mathcal{M}_S)
\end{tikzcd}$$

yields an isomorphism $\Omega^1_{\underline{Y}/\underline{T}} \cong \Omega^1_{Y/T}$ which coincides with $e_{Y/T} : \Omega^1_{\underline{Y}/\underline{T}} \cong \Omega^1_{Y/T}$.

Next observe that $X \setminus X_0 \subset U = X \setminus \{P\}$. This implies $V = Y$, so $W^1_{Y/T} = (\mathrm{id_X})_*\Omega^1_{Y/T}$, and $c^1_{Y/T}$ is the identity. We conclude:

Proposition 5.7. *If $\underline{g} : \underline{Y} \to \underline{T}$ is smooth, then the composition*

$$\Omega^1_{\underline{Y}/\underline{T}} \xrightarrow{e_{Y/T}} \Omega^1_{Y/T} \xrightarrow{c^1_{Y/T}} W^1_{Y/T}$$

is an isomorphism.

5.4 The Coherent Hull Y_h

In this section we explain the *coherent hull* of the log structure $\mathcal{M}_X$. It is mainly a tool to prove the base change property of the system of differentials $(W^\bullet_{Y/T}, d^\bullet_T)$.

Before we constructed the log structure $\mathcal{M}_X$ on $\underline{X}$ giving rise to the log scheme X, we constructed a log structure $\mathcal{H} = \mathcal{H}_X$ on $\underline{X}$. It has been constructed as the spectrum of a log ring, but is also the compactifying log structure defined by $Z \subset X$. Since $X_0 \subset Z$, a function $h \in \mathcal{O}_X(W)$ that is invertible on $W \setminus X_0$ is also invertible on $W \setminus Z \subset W \setminus X_0$. Therefore we get an inclusion $\mathcal{M}_X \subset \mathcal{H}_X$ and thus a sequence

$$X_h \xrightarrow{\gamma} X \xrightarrow{f} S$$

The underlying morphism of $f \circ \gamma$ is $f : X \to S$ as a morphism of schemes, and we observed in the introduction that there is a unique log morphism $h : X_h \to S$ with this property, so $f \circ \gamma = h$. Since $\mathcal{H}_X$ is a coherent log structure as explained at the end of Section 3.1, we call it a *coherent hull* around the non-coherent log structure $\mathcal{M}_X$. The differential forms $\Omega^1_{X_h/S}$ are the sheaf associated to $\Omega^1_{(B,P)/(A,Q)}$ because $h : X_h \to S$ comes from the morphism $(A, \mathbb{N}) \to (B, P)$ of log rings. In the introduction we computed

$$\Omega^1_{(B,P)/(A,Q)} \cong B \cdot \frac{dx}{x} \oplus B \cdot \frac{dy}{y}$$

which yields $\Omega^1_{X_h/S} \cong \mathcal{O}_X \cdot \frac{dx}{x} \oplus \mathcal{O}_X \cdot \frac{dy}{y}$. We see:

Lemma 5.8. *The morphism $h : X_h \to S$ is differentially log smooth.*

Next let us consider the following cartesian square of log schemes:

$$\begin{tikzcd}
Y_h \arrow[r, "\gamma'"] \arrow[d] & Y \arrow[r, "g"] \arrow[d, "c"'] & T \arrow[d, "b"'] \\
X_h \arrow[r, "\gamma"] & X \arrow[r, "f"] & S
\end{tikzcd}$$

We denote the log structure of Y_h by $\mathcal{H}_Y$. By base change we have

$$\Omega^1_{Y_h/T} \cong c^*\Omega^1_{X_h/S} \cong \mathcal{O}_Y \cdot \frac{dx}{x} \oplus \mathcal{O}_Y \cdot \frac{dy}{y} \ ,$$

so the composite $h' := g \circ \gamma'$ is also differentially log smooth, and Y_h is kind of a coherent hull around Y (but if T is not coherent, then also Y_h need not be coherent). From γ' and g, we obtain an exact sequence

$$\Omega^1_{Y/T} \xrightarrow{\lambda_{Y/T}} \Omega^1_{Y_h/T} \to \Omega^1_{Y_h/Y} \to 0 \ .$$

The homomorphism $\lambda_{Y/T} : \Omega^1_{Y/T} \to \Omega^1_{Y_h/T}$ is constructed using the universal property of $\Omega^1_{Y/T}$, namely we may restrict the universal derivation of Y_h/T to $\mathcal{M}_Y \to \mathcal{H}_Y$ and get a relative derivation for Y/T.

Applying $\bigwedge^m(-)$ yields a homomorphism $\Omega^m_{Y/T} \to \Omega^m_{Y_h/T}$, and setting $W^m_{Y_h/T} := j_*\Omega^m_{Y_h/T}|_V$, we get

$$\ell^m_{Y/T} : W^m_{Y/T} \to W^m_{Y_h/T} \ .$$

Note that $W^m_{Y_h/T} = \Omega^m_{Y_h/T}$ by Corollary 2.8. We denote the cokernel by

$$\mathcal{Q}^m_{Y/T} := \mathrm{coker}(\ell^m_{Y/T} : W^m_{Y/T} \to W^m_{Y_h/T}) \ .$$

This cokernel $\mathcal{Q}^m_{Y/T}$ is always a coherent sheaf (unlike $\Omega^1_{Y_h/Y}$). We may also consider the dual of $\lambda_{Y/T} : \Omega^1_{Y/T} \to \Omega^1_{Y_h/T}$ which we write

$$\ell^*_{Y/T} := \lambda^*_{Y/T} : \Theta^1_{Y_h/T} \to \Theta^1_{Y/T} \ .$$

It is given by restricting a relative log derivation to $\mathcal{M}_Y$ via $\mathcal{M}_Y \to \mathcal{H}_Y$. Dualizing this map we get $\ell^{**}_{Y/T}$ fitting in a commutative diagram

$$\begin{tikzcd}
(\Omega^1_{Y/T})^{**} \arrow[d, "\ell^{**}_{Y/T}"] & \Omega^1_{Y/T} \arrow[l] \arrow[r] \arrow[d] & W^1_{Y/T} \arrow[d, "\ell^1_{Y/T}"] \\
(\Omega^1_{Y_h/T})^{**} & \Omega^1_{Y_h/T} \arrow[l] \arrow[r] & W^1_{Y_h/T}
\end{tikzcd}$$

From this diagram we see that $\ell^{**}_{Y/T}|_V = \ell^1_{Y/T}|_V$ under the canonical isomorphisms of the source and target which implies $\ell^{**}_{Y/T} = \ell^1_{Y/T}$. This gives us a means to compute $\ell^1_{Y/T}$ because $\ell^*_{Y/T} : \Theta^1_{Y_h/T} \to \Theta^1_{Y/T}$ is relatively easy accessible.

6 The Differentials $(W^\bullet_{X/S}, d^\bullet_S)$

In this section we have a closer look at the differentials $W^m_{X/S}$ and $W^m_{X_h/S}$. After some general observations, this section is mainly devoted to compute them explicitly. Though we may access the differentials via the isomorphism $\Theta^1_{X/S} \to \Theta^1_{\underline{X}/\underline{S}}$ described in the first section below, and then use $\Omega^1_{B/A} = (B{\cdot}dx \oplus B{\cdot}dy \oplus B{\cdot}dw)/(ydx + xdy - tdw)$, we found it more enlightening to compute $\Theta^1_{X/S}$ from scratch using the description of $f : X \to S$ by the homomorphism of monoid rings $k[Q] \to k[P]$. This gives a strategy that can be easily generalized, but we won't do that within this paper. Our further strategy is as follows: We obtain $W^1_{X/S}$ as the dual sheaf of $\Theta^1_{X/S}$. Then we make an ansatz E^m for $W^m_{X/S}$ and show that it is indeed $W^m_{X/S}$ by showing it isomorphic to $\Omega^m_{\tilde{U}/S}$ on some $\tilde{U} \subset X$ and showing that the restriction $\rho : E^m(X) \to E^m(\tilde{U})$ is an isomorphism. We will also compute $W^m_{X_h/S}$ with the same method, and the canonical homomorphism $\ell^m_{X/S} : W^m_{X/S} \to W^m_{X_h/S}$. The ultimate goal of these calculations is to prove the base change result in Section 7 The main result of this section is the following

Theorem 6.1. *We have isomorphisms*

$$\Gamma(X, W^m_{X_h/S}) \cong \bigoplus_{p\in P} z^p \cdot \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k)$$

and

$$\Gamma(X, W^m_{X/S}) \cong \bigoplus_{p\in P} z^p \cdot \bigwedge_k^m \left(\bigcap_{p\in H_\ell} (H_\ell^{gp} \otimes k)/(Q^{gp} \otimes k) \right)$$

where the intersection is over $H_\ell \in \{F_{tx}, F_{ty}\}$ (we intersect the terms for those H_ℓ that satisfy $p \in H_\ell$). The canonical homomorphism $\ell^m_{X/S} : W^m_{X/S} \to W^m_{X_h/S}$ corresponds to the inclusions

$$\bigwedge_k^m \left(\bigcap_{p\in H_\ell} (H_\ell^{gp} \otimes k)/(Q^{gp} \otimes k) \right) \subset \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k) \ .$$

We conclude the section by calculating the modules given by the theorem more explicitly and we prove that $W^1_{X/S}$ is not locally free which shows that there is no locally free extension of $\Omega^1_{U/S}$ to the whole of X.

6.1 Generalities

First let us revisit the homomorphisms associated with the coherent hull a bit closer. On $X^+ = X \setminus Z$, both log structures $\mathcal{M}_X$ and $\mathcal{H}_X$ are trivial which means that the homomorphisms

$$\Omega^1_{\underline{X}/\underline{S}} \xrightarrow{e_{X/S}} \Omega^1_{X/S} \xrightarrow{\lambda_{X/S}} \Omega^1_{X_h/S}$$

become isomorphisms when restricted to X^+. Since $X^+ \subset U$, we see that also

$$\ell^m_{X/S} : W^m_{X/S} \to W^m_{X_h/S}$$

becomes an isomorphism when restricted to X^+. The sheaf $W^m_{X/S}$ is reflexive on the integral scheme X, so it is torsion free and has injective restrictions. Now the argument used in the proof of Lemma 4.2 yields $\ell^m_{X/S}$ injective.

Lemma 6.2. *The homomorphism $\ell^m_{X/S} : W^m_{X/S} \to W^m_{X_h/S}$ is injective.*

Thus we have a short exact sequence

$$0 \to W^m_{X/S} \to W^m_{X_h/S} \to \mathcal{Q}^m_{X_h/X} \to 0$$

which will play an important role in the proof of the base change property for the system of differentials $(W^\bullet_{Y/T}, d^\bullet_T)$. Namely, we will use it to show $c^*W^m_{X/S}$ reflexive.

Another interesting map is the composition

$$\Omega^1_{\underline{X}/\underline{S}} \xrightarrow{e_{X/S}} \Omega^1_{X/S} \xrightarrow{c_{X/S}} W^1_{X/S}$$

and its dual

$$(W^1_{X/S})^* \xrightarrow{c^*_{X/S}} \Theta^1_{X/S} \xrightarrow{e^*_{X/S}} \Theta^1_{\underline{X}/\underline{S}} \ .$$

As shown in Section 5.2, the homomorphism $c^*_{X/S}$ is an isomorphism. The map $e^*_{X/S}$ becomes an isomorphism when restricted to X^+, and $\Theta^1_{X/S}$ is reflexive, so $e^*_{X/S} : \Theta^1_{X/S} \to \Theta^1_{\underline{X}/\underline{S}}$ is injective. Since it is given by forgetting the log part of a derivation, it can also be seen injective using Lemma 3.12. Indeed, it is an isomorphism:

Proposition 6.3. *The map $e^*_{X/S} : \Theta^1_{X/S} \to \Theta^1_{\underline{X}/\underline{S}}$ is an isomorphism.*

Proof. We apply Proposition 3.13 with $\mathcal{I}$ the ideal sheaf associated to $(t) \subset B$. The ideal sheaf $\mathcal{I}$ defines $X_0 \subset X$ as a reduced union of irreducible codimension 1 subschemes, and it gives the log structure $\mathcal{M}_X$. Both sheaves are coherent, so it suffices to show $e^*_{X/S}$ surjective on global sections. Let $D \in \Theta^1_{\underline{X}/\underline{S}}(X)$ be a relative derivation. In particular $D(t) = 0$ which implies $D(\mathcal{I}) \subset \mathcal{I}$. Thus there is an absolute log derivation (D, Δ). It is straightforward to show that this (D, Δ) is also a relative log derivation whence $e^*_{X/S}$ is surjective. □

Dualizing the composite isomorphism, we see:

Corollary 6.4. *We have $(\Omega^1_{\underline{X}/\underline{S}})^{**} \cong W^1_{X/S}$. In particular, $\Omega^{[1]}_{B/A} \cong \Gamma(X, W^1_{X/S})$.*

This might be generalized to other m. Taking $\bigwedge^m(-)$ of $e_{X/S} : \Omega^1_{\underline{X}/\underline{S}} \to \Omega^1_{X/S}$, we get a map $e^m_{X/S} : \Omega^m_{\underline{X}/\underline{S}} \to \Omega^m_{X/S}$, so its bidual is a homomorphism $(e^m_{X/S})^{**} : (\Omega^m_{\underline{X}/\underline{S}})^{**} \to W^m_{X/S}$. It is an isomorphism on X^+, so by torsion freeness of $(\Omega^m_{\underline{X}/\underline{S}})^{**}$ it is injective.

Conjecture 6.5. *The homomorphism* $(e^m_{X/S})^{**} : (\Omega^m_{\underline{X}/\underline{S}})^{**} \to W^m_{X/S}$ *is an isomorphism for all* $m \geq 0$.

It should be possible (and not too hard) to settle this conjecture by some explicit calculation, but we won't do that here. We expect $m = 2$ to be the only somewhat more complicated case.

6.2 The Computation of $\Theta^1_{X/S}$

The goal of this section is to compute $\Theta^1_{X/S}$ from scratch, using methods very similar to that in [7][1.3. - 1.8.]. We will use $B \cong k[P]$ for the monoid P permanently. We have $P^{gp} \cong \mathbb{Z}^3$, and $k[P] \to k[P^{gp}]$ is a localization map corresponding to the inclusion $X^+ := X \setminus Z \subset X$. Namely, we have

$$k[P^{gp}] = k[P_{(2,1,1)}] = k[P]_{z^{(1,0,0)+(1,1,1)}} = B_{tw}$$

using the observation on monoid localizations in Section 2.2. On $\underline{X}$, there are three log structures we are interested in, namely $\mathcal{O}^*_X, \mathcal{M}_X$ and $\mathcal{H}_X$. We are going to describe them by *monoid ideals* of P. Recall that an *ideal* $\mathfrak{b} \subset P$ in a monoid is a (possibly empty) subset such that $x \in P, y \in \mathfrak{b}$ implies $x + y \in \mathfrak{b}$. For $\mathfrak{b} \subset P$ an ideal, $I_\mathfrak{b} := \bigoplus_{b \in \mathfrak{b}} k \cdot z^b \subset k[P]$ is a ring ideal defining a closed subscheme $V_\mathfrak{b} := \mathrm{Spec}\, k[P]/k[I_\mathfrak{b}] \subset X$ which in turn defines a compactifying log structure on $\underline{X}$. We write $X(\mathfrak{b}) = (\underline{X}, \mathcal{M}_\mathfrak{b})$ for the log scheme arising in this way. As shown in Section 2.1, we have $V_\mathfrak{a} = Z$ and $V_{\mathfrak{a}_0} = X_0$, so

$$X = X(\mathfrak{a}_0) \quad \text{and} \quad X_h = X(\mathfrak{a}) \ .$$

We have $V_P = \emptyset$, so $X(P) = \underline{X}$ considered as a log scheme with trivial log structure. By restricting the log part of a given derivation along the inclusions $\mathcal{O}^*_X \subset \mathcal{M}_X \subset \mathcal{H}_X$ we obtain homomorphisms $\Theta^1_{X_h/S} \to \Theta^1_{X/S} \to \Theta^1_{\underline{X}/\underline{S}}$ and similarly $\Theta^1_{X_h/k} \to \Theta^1_{X/k} \to \Theta^1_{\underline{X}/k}$ which fit into a commutative diagram

$$\begin{array}{ccccc}
\Theta^1_{X_h/S} & \longrightarrow & \Theta^1_{X/S} & \longrightarrow & \Theta^1_{\underline{X}/\underline{S}} \\
\downarrow & & \downarrow & & \downarrow \\
\Theta^1_{X_h/k} & \longrightarrow & \Theta^1_{X/k} & \longrightarrow & \Theta^1_{\underline{X}/k}
\end{array}$$

where on the vertical arrows we forget the fact that a relative derivation over S is relative. Note that $\Theta^1_{\underline{X}/\underline{S}}$ and $\Theta^1_{\underline{X}/k}$ are just the classical (relative) derivations as explained in Section 3.2. All the sheaves in the diagram are coherent and reflexive and all horizontal morphisms are isomorphisms on X^+, so the horizontal morphisms are injective by torsion freeness of the sheaves on the integral scheme X. The vertical morphisms are injective by construction. Since all the sheaves are torsion free, we get a diagram

$$\begin{tikzcd}
& \Gamma(X,\Theta^1_{X_h/S}) \arrow[rr] \arrow[dl] \arrow[dd] & & \Gamma(X,\Theta^1_{X/S}) \arrow[rr] \arrow[dl] \arrow[dd] & & \Gamma(X,\Theta^1_{\underline{X}/\underline{S}}) \arrow[dl] \arrow[dd] \\
\Gamma(X^+,\Theta^1_{X_h/S}) \arrow[rr] \arrow[dd] & & \Gamma(X^+,\Theta^1_{X/S}) \arrow[rr] \arrow[dd] & & \Gamma(X^+,\Theta^1_{\underline{X}/\underline{S}}) \arrow[dd] & \\
& \Gamma(X,\Theta^1_{X_h/k}) \arrow[rr] \arrow[dl] & & \Gamma(X,\Theta^1_{X/k}) \arrow[rr] \arrow[dl] & & \Gamma(X,\Theta^1_{\underline{X}/k}) \arrow[dl] \\
\Gamma(X^+,\Theta^1_{X_h/k}) \arrow[rr] & & \Gamma(X^+,\Theta^1_{X/k}) \arrow[rr] & & \Gamma(X^+,\Theta^1_{\underline{X}/k}) &
\end{tikzcd}$$

with all the homomorphisms injective. Our strategy is to compute $\Gamma(X^+,\Theta^1_{\underline{X}/k})$ and determine the images of some homomorphisms in it.

So let us start with the computation of the classical derivations on X^+, namely $\Gamma(X^+,\Theta^1_{\underline{X}/k})$. Considering the field k as its additive group, we denote by $\mathrm{Hom}(P^{gp},k)$ the group homomorphisms, considered as a k-vector space. We use this vector space to define a $k[P^{gp}]$-module

$$\bigoplus_{p\in P^{gp}} z^p\cdot \mathrm{Hom}(P^{gp},k)$$

via $z^{p'}\cdot(z^p\cdot\phi) := z^{p'+p}\cdot\phi$ and k-linear extension.

Proposition 6.6. *The map*

$$\chi : \bigoplus_{p\in P^{gp}} z^p\cdot \mathrm{Hom}(P^{gp},k)\to \Gamma(X^+,\Theta^1_{\underline{X}/k})\ ,$$

$$z^p\phi\mapsto (z^q\mapsto \phi(q)z^{p+q})$$

is a well-defined isomorphism of $k[P^{gp}]$-modules.

Proof. Write $\chi(z^p\phi)=\delta$. Then

$$\delta(z^{q+q'}) = z^{q'}\phi(q)z^{p+q} + z^q\phi(q')z^{p+q'} = z^{q'}\delta(z^q) + z^q\delta(q')\ ,$$

so δ is a derivation, and χ is well-defined. It is obviously a $k[P^{gp}]$-module homomorphism, so it remains to show χ bijective. Assume $\chi(\sum z^p\phi_p)=0$. Then $0=\chi(\sum z^p\phi_p)(z^q)=\sum\phi_p(q)z^{p+q}$, so $\phi_p(q)=0$ for all p,q whence $\sum z^p\phi_p=0$. Thus χ is injective. Now let $\delta : k[P^{gp}]\to k[P^{gp}]$ be a derivation. Then $\delta(z^q)=\sum_p a_{p,q}z^{p+q}$ for some coefficients $a_{p,q}\in k$. Set $\phi_p(q):=a_{p,q}$. Then

$$\sum_p a_{p,q+q'}z^{p+q+q'} = \delta(z^{q+q'}) = z^q\delta(z^{q'}) + z^{q'}\delta(z^q) = \sum_p (a_{p,q'}+a_{p,q})z^{p+q+q'}\ ,$$

so $\phi_p(q+q')=\phi_p(q)+\phi_p(q')$ and we have $\phi_p\in\mathrm{Hom}(P^{gp},\mathbb{Z})$. We have $P^{gp}\cong\mathbb{Z}^3$, so choose a basis $q_1,\ldots,q_3$ of P^{gp}. Then there are only finitely many p such that there is an i with $a_{p,q_i}\neq 0$. Let S be the set of these p. Then if $p\notin S$, we have $\phi_p=0$, and

$\chi(\sum_{p\in S} z^p\phi_p) = \delta$. Thus χ is surjective. $\square$

To describe the images of $\Gamma(X, \Theta^1_{X(\mathfrak{b})/k})$ we define some submodules of the P^{gp}-graded module $\bigoplus_{p\in P^{gp}} z^p \cdot \mathrm{Hom}(P^{gp}, k)$.

Definition 6.7. *Let $\mathfrak{b} \subset P$ be an ideal. Then for $p \in P^{gp}$, we define*

$$\begin{aligned} H(\mathfrak{b})_p := H(P,\mathfrak{b})_p := \{\phi \in \mathrm{Hom}(P^{gp}, k) | q \in P, p+q \notin P \Rightarrow \phi(q) = 0 \\ \text{and } q \in \mathfrak{b}, p+q \notin \mathfrak{b} \Rightarrow \phi(q) = 0\} \end{aligned}$$

as a subvector space of $\mathrm{Hom}(P^{gp}, k)$.

We use these vector spaces to define the submodules $H(\mathfrak{b}) \subset \bigoplus_{p\in P^{gp}} z^p\cdot\mathrm{Hom}(P^{gp}, k)$:

Lemma 6.8. *The k-vector space $H(\mathfrak{b}) := \bigoplus_{p\in P^{gp}} z^p \cdot H(\mathfrak{b})_p$ is a $k[P]$-submodule.*

Proof. We have to show that the scalar multiplication restricts to $H(\mathfrak{b})$. We have $z^q\cdot z^p\phi = z^{p+q}\phi$, whence it suffices to show $H(\mathfrak{b})_p \subset H(\mathfrak{b})_{p+q}$ for $q \in P$. Let $\phi \in H(\mathfrak{b})_p$ and let $q' \in P$ with $p + q + q' \notin P$, so $\phi(q + q') = 0$. If $p + q \in P$ then $p + q + q' \in P$ contradicting the assumption. So $p + q \notin P$ and $\phi(q) = 0$, whence $\phi(q') = 0$. The condition for $\mathfrak{b}$ is verified similarly. $\square$

And indeed, we have $H(P) = \Gamma(X, \Theta^1_{\underline{X}/k})$ under the restriction map $\rho : \Theta^1_{\underline{X}/k}(X) \to \Theta^1_{\underline{X}/k}(X^+)$:

Proposition 6.9. $\Gamma(X, \Theta^1_{\underline{X}/k})$ *corresponds to* $\bigoplus_{p\in P^{gp}} z^p \cdot H(P)_p$.

Proof. Since X is affine the global derivations are just the derivations of the coordinate ring $k[P]$, and the restriction map is given by extending to a commutative diagram:

$$\begin{array}{ccc} k[P] & \xrightarrow{D} & k[P] \\ \downarrow & & \downarrow \\ k[P^{gp}] & \xrightarrow{D} & k[P^{gp}] \end{array}$$

Thus $D : k[P^{gp}] \to k[P^{gp}]$ is the restriction of a global derivation, if $D : k[P] \to k[P^{gp}]$ maps to $k[P]$. If $D(z^q) = \sum_{p\in P^{gp}} \phi_p(q)z^{p+q}$, then $D(k[P]) \subset k[P]$ precisely if

$$\forall p \in P^{gp}\ \forall q \in P : p + q \notin P \Rightarrow \phi_p(q) = 0\ .$$

Since the two conditions in the definition of $H(\mathfrak{b})_p$ coincide for $\mathfrak{b} = P$, this means $\phi_p \in H(P)_p$ for every p. $\square$

To compute $\Gamma(X, \Theta^1_{X(\mathfrak{b})/k})$ we use Proposition 3.13:

Proposition 6.10. *Let $\mathfrak{b} \in \{\mathfrak{a}_0, \mathfrak{a}, P\}$. Then* $\Gamma(X, \Theta^1_{X(\mathfrak{b})/k})$ *corresponds to*

$$H(\mathfrak{b}) = \bigoplus_{p\in P^{gp}} z^p \cdot H(\mathfrak{b})_p\ .$$

Proof. The case $\mathfrak{b} = P$ has been settled previously. X is Noetherian, normal and integral. The log scheme $X(\mathfrak{b})$ has the compactifying log structure associated to the ideal sheaf defined by $k[\mathfrak{b}]$. The closed subschemes $X_0 = V_{\mathfrak{a}_0}$ and $Z = V_{\mathfrak{a}}$ are reduced and the union of codimension 1 irreducible subschemes. Thus a global derivation $D : \mathcal{O}_X \to \mathcal{O}_X$ comes from an absolute log derivation iff $D(\mathcal{I}) \subset \mathcal{I}$, where $\mathcal{I}$ is the ideal sheaf associated to $k[\mathfrak{b}]$. Note that this is equivalent to $D(k[\mathfrak{b}]) \subset k[\mathfrak{b}]$ on global sections. That is, we have to determine the derivations D with $D(k[\mathfrak{b}]) \subset k[\mathfrak{b}]$. Let $D = \chi(\sum \phi_p z^p)$, and let $q \in \mathfrak{b}$. Then $D(z^q) = \sum \phi_p(q) z^{p+q}$, so $D(z^q) \in k[\mathfrak{b}]$ iff $\phi_p(q) = 0$ for all p with $p + q \notin \mathfrak{b}$. This is the case iff $\phi_p \in H(\mathfrak{b})_p$. By Lemma 3.14, the absolute log derivations are relative, and the assertion follows. □

Now that we have computed the relative derivations over Spec k, we like to compute the relative derivations over $S = \text{Spec } k[Q]$ with $Q = \mathbb{N}$. Recall that $f : X \to S$ is given by $\theta : Q \to P, n \mapsto (n, 0, 0)$. Any relative derivation over S is also a relative derivation over Spec k, so we can determine them as a subset (as described above). We start determining the relative derivations of X^+ which are the classical relative derivations since X^+ is log trivial for all log structures we are interested in.

Lemma 6.11. *$\Gamma(X^+, \Theta^1_{\underline{X}/\underline{S}})$ corresponds to $\bigoplus_{p \in P^{gp}} z^p \cdot H_Q$, where*

$$H_Q = \{\phi | \forall q \in Q : \phi(q) = 0\} \subset \text{Hom}(P^{gp}, k) \ .$$

Proof. $D : k[P^{gp}] \to k[P^{gp}]$ is a relative derivation, iff $D(k[Q]) = 0$. Now $\sum \phi_p(q) z^{p+q} = 0$ for all $q \in Q$ is equivalent to $\phi_p(q) = 0$ for all $p \in P^{gp}, q \in Q$. □

Now we like to see the global sections of $\Theta^1_{X/S}$ and $\Theta^1_{X_h/S}$ as subsets of $\Gamma(X, \Theta^1_{X/k})$ and $\Gamma(X, \Theta^1_{X_h/k})$ which we have already computed. To do so, we introduce some notation.

Definition 6.12. *Let $\mathfrak{b} \subset P$ be an ideal. Then for $p \in P^{gp}$, we set*

$$\begin{aligned} H_Q(\mathfrak{b})_p := H(P/Q, \mathfrak{b})_p := \{\phi \in \text{Hom}(P^{gp}, k) | & q \in P, p + q \notin P \Rightarrow \phi(q) = 0 \\ & \text{and } q \in \mathfrak{b}, p + q \notin \mathfrak{b} \Rightarrow \phi(q) = 0 \\ & \text{and } q \in Q \Rightarrow \phi(q) = 0\} \ . \end{aligned}$$

Note that we just have $H_Q(\mathfrak{b})_p = H(\mathfrak{b})_p \cap H_Q$. These vector spaces give the desired module, also for $X(P)/\underline{S}$ (which we write $X(P)/S$ for short):

Proposition 6.13. *If $\mathfrak{b} \in \{\mathfrak{a}_0, \mathfrak{a}, P\}$, then $\Gamma(X, \Theta^1_{X(\mathfrak{b})/S})$ corresponds to*

$$\bigoplus_{p \in P^{gp}} z^p \cdot H_Q(\mathfrak{b})_p \ .$$

Proof. If (D, Δ) is a log derivation on $X(\mathfrak{b})$, it is a relative one iff D is a relative derivation by Lemma 3.14. The derivation D is relative, iff $D|_{X^+}$ is a relative derivation, for assume $D|_{X^+}$ a relative derivation and $g \in f^{-1}\mathcal{O}_S(V)$. Then $D(f^\sharp(g))|_{X^+\cap V} = 0$, so by integrality of X we have $D(f^\sharp(g)) = 0$. Thus $\Gamma(X, \Theta^1_{X(\mathfrak{b})/S})$ corresponds to

$$\bigoplus_{p\in P^{gp}} z^p \cdot H(\mathfrak{b})_p \cap \bigoplus_{p\in P^{gp}} z^p \cdot H_Q = \bigoplus_{p\in P^{gp}} z^p \cdot H_Q(\mathfrak{b})_p \ .$$

□

The proposition implies that we have a commutative diagram

$$\begin{tikzcd}
\Gamma(X, \Theta^1_{X(\mathfrak{a})/S}) \arrow[r] \arrow[d, "\cong"] & \Gamma(X, \Theta^1_{X(\mathfrak{a}_0)/S}) \arrow[r] \arrow[d, "\cong"] & \Gamma(X, \Theta^1_{X(P)/S}) \arrow[r] \arrow[d, "\cong"] & \Gamma(X^+, \Theta^1_{\underline{X}/k}) \arrow[d, "\cong"] \\
\bigoplus z^p \cdot H_Q(\mathfrak{a})_p \arrow[r] & \bigoplus z^p \cdot H_Q(\mathfrak{a}_0)_p \arrow[r] & \bigoplus z^p \cdot H_Q(P)_p \arrow[r] & \bigoplus z^p \cdot \mathrm{Hom}(P^{gp}, k)
\end{tikzcd}$$

showing inclusions $H_Q(\mathfrak{a})_p \subset H_Q(\mathfrak{a}_0)_p \subset H_Q(P)_p$ which are not obvious from the definitions. These inclusions describe the canonical maps on the level of sheaves. Since $e^*_{X/S} : \Theta^1_{X/S} \to \Theta^1_{\underline{X/S}}$ is an isomorphism, we obtain moreover $H_Q(\mathfrak{a}_0)_p = H_Q(P)_p$ which is also not obvious from the definitions.

A More Concrete Computation

Now we have computed the derivations in very technical terms, and we like to see these spaces more explicitly. We have to compute $H_Q(\mathfrak{b})_p$ for every $p \in P^{gp} = \mathbb{Z}^3$ which is a set of group homomorphisms $\phi : P^{gp} \to k$ satisfying some conditions depending on p. We are going to distinguish some cases depending on where in P^{gp} the element p is. To describe the cases, recall the description of the faces of P by the height functions $h_{tx}, h_{ty}, h_{wx}, h_{wy}$. In particular, we have

$$\begin{aligned}
P &= h_{tx}^{-1}(\{n \geq 0\}) \ \cap \ h_{ty}^{-1}(\{n \geq 0\}) \ \cap \ h_{wx}^{-1}(\{n \geq 0\}) \ \cap \ h_{wy}^{-1}(\{n \geq 0\}) \\
\mathfrak{a}_0 &= h_{tx}^{-1}(\{n \geq 0\}) \ \cap \ h_{ty}^{-1}(\{n \geq 0\}) \ \cap \ h_{wx}^{-1}(\{n \geq 1\}) \ \cap \ h_{wy}^{-1}(\{n \geq 1\}) \\
\mathfrak{a} &= h_{tx}^{-1}(\{n \geq 1\}) \ \cap \ h_{ty}^{-1}(\{n \geq 1\}) \ \cap \ h_{wx}^{-1}(\{n \geq 1\}) \ \cap \ h_{wy}^{-1}(\{n \geq 1\})
\end{aligned}$$

To save space, we introduce the notations

$$\begin{aligned}
a &= (0,0,1) \\
b &= (0,1,0) \\
t &= (1,0,0) \\
x &= (1,0,1) = a + t \\
y &= (1,1,0) = b + t \\
w &= (1,1,1) = a + b + t
\end{aligned}$$

for elements of P^{gp}. Now let $\phi \in H_Q(\mathfrak{b})_p \subset \mathrm{Hom}(P^{gp}, k)$. In any case for $p \in P^{gp}$, we have $t \in Q$, so $\phi(t) = 0$. We distinguish the following cases:

$$\boxed{\textbf{Case 1: } h_{wx}(p) < 0 \ \Rightarrow \ H_Q(\mathfrak{b})_p = 0}$$

We have $x, w \in P$, but $h_{wx}(x+p) < 0$ and $h_{wx}(w+p) < 0$, so $w+p, x+p \notin P$ and thus $\phi(x) = 0, \phi(w) = 0$. This yields $\phi = 0$, so we have $H_Q(\mathfrak{b})_p = 0$.

$$\boxed{\textbf{Case 2: } h_{wy}(p) < 0 \ \Rightarrow \ H_Q(\mathfrak{b})_p = 0}$$

This case is analogous using w and y. We get $H_Q(\mathfrak{b})_p = 0$.

$$\boxed{\textbf{Case 3: } h_{tx}(p) \leq -2 \ \Rightarrow \ H_Q(\mathfrak{b})_p = 0}$$

In this case, we have $x, y \in P$, but $h_{tx}(x+p) < 0, h_{tx}(y+p) < 0$, so $x+p, y+p \notin P$ and $\phi(x) = 0, \phi(y) = 0$. Thus $\phi = 0$, and we get $H_Q(\mathfrak{b})_p = 0$.

$$\boxed{\textbf{Case 4: } h_{ty}(p) \leq -2 \ \Rightarrow \ H_Q(\mathfrak{b})_p = 0}$$

This case is analogous and we get $H_Q(\mathfrak{b})_p = 0$.

In all the remaining cases we assume $h_{tx} \geq -1, h_{ty} \geq -1, h_{wx} \geq 0, h_{wy} \geq 0$ without further mentioning it.

$$\boxed{\textbf{Case 5: } h_{tx}(p) = -1, h_{ty}(p) \geq 0 \ \Rightarrow \ H_Q(\mathfrak{a})_p = 0, H_Q(\mathfrak{a}_0)_p = H_x}$$

In this case, we have $x \in P$, but $h_{tx}(x+p) = -1$, so $\phi(x) = 0$. If $\mathfrak{b} = \mathfrak{a}$, then $(2,1,1) \in \mathfrak{a}$, but $h_{tx}((2,1,1)+p) = 0$, so $(2,1,1)+p \notin \mathfrak{a}$ and $\phi(2,1,1) = 0$ implying $\phi = 0$. Then we have $H_Q(\mathfrak{a})_p = 0$. For $\mathfrak{b} = \mathfrak{a}_0$, we make an ansatz

$$H_x := \{\phi : P^{gp} \to k | \phi(t) = 0, \phi(a) = 0\} \ .$$

We have $H_Q(\mathfrak{a}_0)_p \subset H_x$. Now assume $\phi \in H_x$. If $q \in Q$, then $q = nt$ for some n, so $\phi(q) = 0$. If $q \in P, q+p \notin P$, then there is some h_i with $h_i(q+p) < 0$. This is only possible for h_{tx}, so we have $h_{tx}(q+p) = h_{tx}(q) - 1 < 0$. Thus $h_{tx}(q) = 0$ and we have $\phi(q) = 0$ since $q = \alpha t + \beta a$ for some coefficients α, β. If $q \in \mathfrak{a}_0$, but $q+p \notin \mathfrak{a}_0$, then similarly $h_{tx}(q+p) < 0$. Thus again $h_{tx}(q) = 0$, and we have $\phi(q) = 0$. This shows $\phi \in H_Q(\mathfrak{a}_0)_p$, so we have $H_Q(\mathfrak{a}_0)_p = H_x$.

$$\boxed{\textbf{Case 6: } h_{ty}(p) = -1, h_{tx}(p) \geq 0 \ \Rightarrow \ H_Q(\mathfrak{a})_p = 0, H_Q(\mathfrak{a}_0)_p = H_y}$$

This case is similar. We get $H_Q(\mathfrak{a})_p = 0$ and

$$H_Q(\mathfrak{a}_0)_p = H_y := \{\phi : P^{gp} \to k | \phi(t) = 0, \phi(b) = 0\} \ .$$

$$\boxed{\textbf{Case 7: } h_{tx}(p) = -1, h_{ty}(p) = -1 \ \Rightarrow \ H_Q(\mathfrak{b})_p = 0}$$

In this case, we have $x, y \in P$, but $h_{tx}(x+p) = -1, h_{ty}(y+p) = -1$, so $x+p, y+p \notin P$ and consequently $\phi(x) = 0, \phi(y) = 0$. Thus $\phi = 0$ and we have $H_Q(\mathfrak{b})_p = 0$.

$$\boxed{\textbf{Case 8: } p \in P \;\Rightarrow\; H_Q(\mathfrak{b})_p = H_{xy}}$$

This comprises all remaining cases. Our ansatz is

$$H_{xy} = \{\phi : P^{gp} \to k | \phi(t) = 0\} \ .$$

Such a ϕ satisfies all the conditions, for if $q \in P$, then also $q + p \in P$, and if $q \in \mathfrak{b}$, then also $q + p \in \mathfrak{b}$. Thus we have $H_Q(\mathfrak{b})_p = H_{xy}$.

Next note that we are in Case 5 precisely if

$$p \in (0,-1,0) + F_{tx} = \{(0,-1,0) + f | f \in F_{tx}\} \ ,$$

that we are in Case 6 precisely if $p \in (0,0,-1) + F_{ty}$ and that $H_{xy} = H_x \oplus H_y$. Thus we conclude:

Theorem 6.14. *We have*

$$\Gamma(X, \Theta^1_{X/S}) = \left(\bigoplus_{p \in P \cup (0,-1,0) + F_{tx}} H_x \cdot z^p \right) \oplus \left(\bigoplus_{p \in P \cup (0,0,-1) + F_{ty}} H_y \cdot z^p \right)$$

and

$$\Gamma(X, \Theta^1_{X_h/S}) = \bigoplus_{p \in P} H_{xy} \cdot z^p \ .$$

Note the isomorphism

$$\chi : \mathrm{Hom}(P^{gp}, k) \to k \oplus k \oplus k, \phi \mapsto (\phi(t), \phi(b), \phi(a)) \ .$$

We have $\chi(H_x) = 0 \oplus k \oplus 0$, $\chi(H_y) = 0 \oplus 0 \oplus k$ and $\chi(H_{xy}) = 0 \oplus k \oplus k$, and we recover that $\Theta^1_{X_h/S}$ is locally free of rank 2. Contrary to that, $\Theta^1_{X/S}$ is not locally free, for we will see in Proposition 6.36 that its dual $W^1_{X/S}$ is not locally free.

6.3 The Computation of $W^1_{X/S}$

Now we turn to

$$W^1_{X(\mathfrak{b})/S} = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Theta^1_{X(\mathfrak{b})/S}, \mathcal{O}_X) \ .$$

We have a similar strategy as for the derivations, namely we consider a commutative diagram

$$\begin{array}{ccccc}
\Gamma(X, W^1_{X(P)/S}) & \longrightarrow & \Gamma(X, W^1_{X(\mathfrak{a}_0)/S}) & \longrightarrow & \Gamma(X, W^1_{X(\mathfrak{a})/S}) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(X^+, W^1_{X(P)/S}) & \longrightarrow & \Gamma(X^+, W^1_{X(\mathfrak{a}_0)/S}) & \longrightarrow & \Gamma(X^+, W^1_{X(\mathfrak{a})/S})
\end{array}$$

obtained by dualizing the derivations. The lower line consists only of isomorphisms because all log structures coincide on X^+. Again, all homomorphisms are injective, so we are going to calculate $\Gamma(X^+, W^1_{X(P)/S})$ and then determine the images (after identifying every module in the lower line with it).

So let us compute $\Gamma(X^+, W^1_{X(P)/S})$ as the dual of $\Gamma(X^+, \Theta^1_{X(P)/S}$. The quotient $P^{gp}/Q^{gp} \cong \mathbb{Z}^2$ of the Grothendieck groups associated to the monoids is a group and as such a $\mathbb{Z}$-module.

Proposition 6.15. *The canonical map*

$$\begin{aligned} \psi : \quad & \bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes_{\mathbb{Z}} k \\ & \to \ \mathrm{Hom}_{k[P^{gp}]}(\Gamma(X^+, \Theta^1_{X(P)/S}), k[P^{gp}]) \ = \ \Gamma(X^+, W^1_{X(P)/S}) \\ & \quad z^p \cdot q \otimes s \mapsto (z^r \cdot \phi \mapsto s\phi(q) z^{p+r}) \end{aligned}$$

is an isomorphism of $k[P^{gp}]$-modules.

Proof. $(z^r \cdot \phi \mapsto s\phi(q)z^{p+r})$ gives a $k[P^{gp}]$-module homomorphism. Since $\phi \in H_Q$, it is independent of the representative of $q \in P^{gp}/Q^{gp}$. Thus ψ is a well-defined group homomorphism. It is also $k[P^{gp}]$-linear. Injectivity is straightforward using that $P^{gp} = \mathbb{Z}^3 \cong \mathbb{Z}^1 \oplus \mathbb{Z}^2 = Q^{gp} \oplus P^{gp}/Q^{gp}$. To show surjectivity, let $\chi : \Gamma(X^+, \Theta^1_{X^+/S}) \to k[P^{gp}]$ be a homomorphism. Then $\chi(z^r\phi) = \sum a_{r,p}(\phi) z^{p+r}$ for some coefficients $a_{r,p}(\phi)$. Note that $a_{r,p}(\phi) = a_{0,p}(\phi)$ by linearity of χ, and note also $a_{0,p}(\phi + \phi') = a_{0,p}(\phi) + a_{0,p}(\phi')$. Thus $a_{0,p} : H_Q \to R$ is a group homomorphism, and it is indeed k-linear. Next note the isomorphism

$$H_Q \to \mathrm{Hom}_k((P^{gp}/Q^{gp}) \otimes k, k), \quad \phi \mapsto (q \otimes s \mapsto s\phi(q)) \ .$$

$(P^{gp}/Q^{gp}) \otimes k$ is free whence reflexive, so there is a $t_p = \sum_i q^i_p \otimes s^i_p \in (P^{gp}/Q^{gp}) \otimes k$ with $a_{0,p}(\phi) = \phi(t_p) = \sum_i s^i_p \phi(q^i_p)$. Now we have

$$\psi(\sum_{p,i} z^p \cdot q^i_p \otimes s^i_p)(z^r\phi) = \sum_{p,i} s^i_p \phi(q^i_p) z^{p+r} = \sum_p a_{r,p}(\phi) z^{p+r} = \chi(z^r\phi)$$

showing surjectivity. □

Now we consider the restriction map

$$\Gamma(X, W^1_{X(\mathfrak{b})/S}) \to \Gamma(X^+, W^1_{X(\mathfrak{b})/S}) = \Gamma(X^+, W^1_{X(P)/S}) = \bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes_{\mathbb{Z}} k$$

which is injective. To determine its image, we introduce notation

$$Z^k_Q(\mathfrak{b})_p := \{t \in P^{gp} \otimes k \mid \forall \phi \in H_Q(\mathfrak{b})_p : \phi(t) = 0\} \subset P^{gp} \otimes k \ ,$$

where $\phi \in H_Q(\mathfrak{b})_p$ is extended to $P^{gp} \otimes k$ in the obvious way. The k in the notation $Z^k_Q(\mathfrak{b})_p$ refers to the chosen field k.

Definition 6.16. *For $\mathfrak{b} \in \{\mathfrak{a}, \mathfrak{a}_0, P\}$, we set $W(\mathfrak{b})_p = \left(\bigcap_{r : p+r \notin P} Z^k_Q(\mathfrak{b})_r\right) / (Q^{gp} \otimes k)$.*

Note that $P^{gp}/Q^{gp} \cong \mathbb{Z}^2$ is a flat $\mathbb{Z}$-module, so $Q^{gp} \otimes k \subset P^{gp} \otimes k$, and that for $\phi \in H_Q(\mathfrak{b})_p$ we have $\phi(Q) = 0$, so $Q^{gp} \otimes k \subset Z_Q^k(\mathfrak{b})_p$. Therefore the definition makes sense. Obviously $W(\mathfrak{b})_p \subset P^{gp} \otimes k / Q^{gp} \otimes k \cong (P^{gp}/Q^{gp}) \otimes k$, so we get at least a vector subspace

$$W(\mathfrak{b}) := \bigoplus_{p \in P^{gp}} z^p \cdot W(\mathfrak{b})_p \subset \bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes_{\mathbb{Z}} k \ .$$

This is indeed the image of the restriction map:

Proposition 6.17. *The sections $\Gamma(X, W^1_{X(\mathfrak{b})/S})$ correspond to*

$$W(\mathfrak{b}) = \bigoplus_{p \in P^{gp}} z^p \cdot W(\mathfrak{b})_p \ .$$

Proof. We have that $\tau \in \mathrm{Hom}_{k[P^{gp}]}(\Gamma(X^+, \Theta^1_{X(\mathfrak{b})/S}), k[P^{gp}])$ comes from a morphism in $\mathrm{Hom}_{k[P]}(\bigoplus_{p \in P^{gp}} z^p \cdot H_Q(\mathfrak{b})_p, k[P])$ precisely if it fits into a commutative diagram

$$\begin{array}{ccc} \bigoplus_{p \in P^{gp}} z^p \cdot H_Q(\mathfrak{b})_p & \xrightarrow{\tau} & k[P] \\ \big\downarrow & & \big\downarrow \\ \bigoplus_{p \in P^{gp}} z^p \cdot H_Q & \xrightarrow{\tau} & k[P^{gp}] \end{array}$$

This is equivalent to $\tau(z^p \cdot \phi) \in k[P]$ for $\phi \in H_Q(\mathfrak{b})_p$. Write $\tau = \psi(\sum_r z^r \cdot [t_r])$ where $t_r \in P^{gp} \otimes k$. Then $\tau(z^p \cdot \phi) = \sum_r \phi(t_r) z^{p+r}$, so $\tau(z^p \cdot \phi) \in k[P]$ iff $\phi(t_r) = 0$ for all r such that $r + p \notin P$. This means $t_r \in Z_Q^k(\mathfrak{b})_p$ for all p with $p + r \notin P$, i.e. $t_r \in \bigcap_{p : r+p \notin P} Z_Q^k(\mathfrak{b})_p$. This is equivalent to $[t_r] \in W(\mathfrak{b})_r$ and the result follows. $\square$

This proposition also *shows* that $\bigoplus_{p \in P^{gp}} z^p \cdot W(\mathfrak{b})_p \subset \bigoplus_{p \in P^{gp}} z^p \cdot (P^{gp}/Q^{gp}) \otimes_{\mathbb{Z}} k$ is a submodule. Since the homomorphisms

$$W^1_{X(P)/S} \to W^1_{X(\mathfrak{a}_0)/S} \to W^1_{X(\mathfrak{a})/S}$$

are injective, we have inclusions $W(P) \subset W(\mathfrak{a}_0) \subset W(\mathfrak{a})$ which in turn describe the injections.

Towards a Nicer Description

Our current description of $W^1_{X(\mathfrak{b})/S}$ is a bit wild, and we like to simplify it. Since we are working on an explicit example, we may try to compute it directly, but we like to do it more systematically, giving a glance on how to generalize. Since $W^1_{X(P)/S} \cong W^1_{X(\mathfrak{a}_0)/S}$, we concentrate on $\mathfrak{b} \in \{\mathfrak{a}, \mathfrak{a}_0\}$. Again the height functions h_i for the monoid P will play some role. For a subset $S \subset P^{gp}$, we introduce the following notation:

$$\mathbb{Z}\langle S \rangle := \{p \in P^{gp} \mid \exists n > 0 \exists s_i \in S : np = \sum \pm s_i\}$$

It is a subgroup of P^{gp}, and the cokernel of $\mathbb{Z}\langle S\rangle \to P^{gp}$ is torsion free. We need

$$Z_Q(\mathfrak{b})_p := \mathbb{Z}\langle q \in P^{gp} | (q \in P \wedge q + p \notin P) \vee (q \in \mathfrak{b} \wedge q + p \notin \mathfrak{b}) \vee q \in Q\rangle \subset P^{gp}$$

that is the set of the form $\mathbb{Z}\langle S\rangle$ generated by the elements $q \in P$ where a homomorphism $\phi : P^{gp} \to k$ needs to vanish to be in $H_Q(\mathfrak{b})_p \subset \mathrm{Hom}(P^{gp}, k)$. Since the cokernel of $\mathbb{Z}\langle S\rangle \to P^{gp}$ is torsion free, we have $Z_Q(\mathfrak{b})_p \otimes k \subset P^{gp} \otimes k$. For the $Z^k_Q(\mathfrak{b})_p$ defined above we have

$$Z^k_Q(\mathfrak{b})_p \subset Z_Q(\mathfrak{b})_p \otimes k \ .$$

For choose a basis $v_1, \dots, v_n$ of P^{gp} with $Z_Q(\mathfrak{b})_p = (v_1, \dots, v_m)$ for some m. Write $t = \sum s_i v_i$ and assume $t \notin Z_Q(\mathfrak{b})_p \otimes k$. Thus there is a number $k > m$ with $s_k \neq 0$. Define $\phi : P^{gp} \to k, v_i \mapsto \delta_{ki}$ for the Kronecker δ. We have obviously $\phi \in H_Q(\mathfrak{b})_p$ since $\phi(Z_Q(\mathfrak{b})_p) = 0$. Now $\phi(t) = s_k \neq 0$, so $t \notin Z^k_Q(\mathfrak{b})_p$.

Now observe that $Z_Q(\mathfrak{b})_0 = \mathbb{Z}\langle Q^{gp}\rangle = Q^{gp}$. Thus $Q^{gp} \otimes k \subset Z^k_Q(\mathfrak{b})_0 \subset Q^{gp} \otimes k$ and we have for $p \notin P$

$$\begin{aligned} W(\mathfrak{b})_p &= \left(\bigcap_{r : p + r \notin P} Z^k_Q(\mathfrak{b})_r \right) / (Q^{gp} \otimes k) \\ &\subset Z^k_Q(\mathfrak{b})_0 / (Q^{gp} \otimes k) = 0 \end{aligned}$$

We see that unlike for $\Theta^1_{X/S}$, the module $\bigoplus_{p \in P^{gp}} z^p \cdot W(\mathfrak{b})_p$ has only non-zero graded pieces for $p \in P$. To compute $W(\mathfrak{b})_p$ for $p \in P$, we need to understand $Z^k_Q(\mathfrak{b})_p$ better. Recall from Section 2 that facets are the maximal faces, and P has precisely four facets $F_{tx}, F_{ty}, F_{wx}, F_{wy}$.

Lemma 6.18. *Let $F \subset P$ be a facet with $\phi(F) = 0$ for all $\phi \in H_Q(\mathfrak{b})_p$. Then we have $F^{gp} \otimes k \subset Z^k_Q(\mathfrak{b})_p$. If $F^{gp} \otimes k \subsetneq Z^k_Q(\mathfrak{b})_p$, then $Z^k_Q(\mathfrak{b})_p = P^{gp} \otimes k$.*

Proof. $P^{gp}/F^{gp} \cong \mathbb{Z}$ is torsion free, so $F^{gp} \otimes k \subset P^{gp} \otimes k$. Now $\phi(\sum r_i p_i) = 0$ for $p_i \in F, r_i \in k$, so $F^{gp} \otimes k \subset Z^k_Q(\mathfrak{b})_p$. The submodule $I := Z^k_Q(\mathfrak{b})_p / F^{gp} \otimes k \subset P^{gp} \otimes k / F^{gp} \otimes k \cong k$ becomes an ideal under the isomorphism. Assume $I \neq 0$. Then $I = k$ since k is a field, and we have $Z^k_Q(\mathfrak{b})_p = P^{gp} \otimes k$. ☐

Also note that if $Z_Q(\mathfrak{b})_p = F^{gp}$, then $Z^k_Q(\mathfrak{b})_p = F^{gp} \otimes k$. Hence our strategy to compute some $Z^k_Q(\mathfrak{b})_p$ is to determine such facets, and then either show $Z_Q(\mathfrak{b})_p = F^{gp} \otimes k$, or find an element in $Z^k_Q(\mathfrak{b})_p \setminus F^{gp} \otimes k$. First we consider the case $\mathfrak{b} = \mathfrak{a}$. Recall that we have $\mathfrak{a} = P \setminus \bigcup_{F \subset P \text{ facet}} \subset P$. We use the height functions h_i for $i \in \{tx, ty, wx, wy\}$.

Lemma 6.19. *For $r \notin P$, we have $Z_Q(\mathfrak{a})_r = P^{gp} \otimes k$.*

Proof. Let $\{F_i = h_i^{-1}(0)\}$ be the set of all facets of P. We have $p \in P$ iff $h_i(p) \geq 0$ for all i. Thus there is a j with $h_j(r) < 0$. If $q \in F_j$, then $h_j(q + r) < 0$, so $q + r \notin P$ and $\phi(q) = 0$ for all $\phi \in H_Q(\mathfrak{a})_r$. This shows $F^{gp}_j \otimes k \subset Z^k_Q(\mathfrak{a})_r$. Next we distinguish two cases:

Case 1: $h_j(r) \leq -2$. Then choose a $p \in P$ with $h_j(p) = 1$. Thus $h_j(p+r) < 0$ and $\phi(p) = 0$ for all $\phi \in H_Q(\mathfrak{a})_r$ which means $p \otimes 1 \in Z_Q^k(\mathfrak{a})_r$. Consider the homomorphism $\ell = h_j \otimes k : P^{gp} \otimes k \to k$. Then $\ell(p \otimes 1) = 1$, so $p \otimes 1 \notin F_j^{gp} \otimes k$, and we have $Z_Q^k(\mathfrak{a})_p = P^{gp} \otimes k$.

Case 2: $h_j(r) = -1$. Here choose $p \in P$ with $h_j(p) = 1$ and $h_k(p) \geq 1$ for all k. Thus $p \in \mathfrak{a}$, and $h_j(p+r) = 0$, so $p + r \notin \mathfrak{a}$ implying $\phi(p) = 0$ for all $\phi \in H_Q(\mathfrak{a})_r$. Again, $Z_Q^k(\mathfrak{a})_r = P^{gp} \otimes k$. $\square$

Using the lemma, we get immediately:

Corollary 6.20. *We have $W(\mathfrak{a})_p = P^{gp} \otimes k / Q^{gp} \otimes k$ for $p \in P$, $W(\mathfrak{a})_r = 0$ for $r \notin P$. In particular, $W^1_{X_h/S} \cong \mathcal{O}_X^{\oplus d}$ is a free $\mathcal{O}_X$-module of rank $d := \mathrm{rk}(P^{gp}) - \mathrm{rk}(Q^{gp}) = 2$.*

Proof. Just use the definition $W(\mathfrak{a})_p = \left(\bigcap_{r : p + r \notin P} Z_Q^k(\mathfrak{a})_r \right) / Q^{gp} \otimes k$ and $W(\mathfrak{a})_p = 0$ for $p \notin P$. $\square$

This is of course the expected result since we already computed that $\Theta^1_{X_h/S} \cong \mathcal{O}_X^{\oplus 2}$. Next we like to carry out a similar computation for $\mathfrak{b} = \mathfrak{a}_0$. It is somewhat more complicated. Recall that $L = \{tx, ty\}$ is the set of indices such that $F_\ell \supset Q$ for $\ell \in L$, and $M = \{wx, wy\}$ is the set of indices such that $F_m \not\supset Q$ for $m \in M$.

Lemma 6.21. *Let $r \in P^{gp} \setminus P$. We have the following results on $Z_Q^k(\mathfrak{a}_0)_r$:*

- *If $h_i(r) \leq -2$ for $i \in L \cup M$, then $Z_Q^k(\mathfrak{a}_0)_r = P^{gp} \otimes k$.*
- *If $h_m(r) = -1$ for $m \in M$, then $Z_Q^k(\mathfrak{a}_0)_r = P^{gp} \otimes k$.*
- *If $h_\ell(r) = -1, h_{\ell'}(r) = -1$ for $\ell \neq \ell' \in L$, then $Z_Q^k(\mathfrak{a}_0)_r \supset F_\ell^{gp} \otimes k \cup F_{\ell'}^{gp} \otimes k$.*
- *If $h_\ell(r) = -1$ for $\ell \in L$, but $h_i(r) \geq 0$ for all $i \neq \ell$, then $Z_Q^k(\mathfrak{a}_0)_r = F_\ell^{gp} \otimes k$.*

Proof. First note that $F_i^{gp} \otimes k \subset Z_Q^k(\mathfrak{a}_0)_r$, if $h_i(r) \leq -1$. If $h_i(r) \leq -2$, choose a $p \in P$ with $h_i(p) = 1$. Then $h_i(p+r) \leq -1$, so $p + r \notin P$ and $\phi(p) = 0$ for all $\phi \in H_Q(\mathfrak{a}_0)_r$. Since $h_i(p) = 1$, we have $Z_Q^k(\mathfrak{a}_0)_r = P^{gp} \otimes k$. If $h_m(r) = -1$, choose a $p \in P$ with $h_m(p) = 1$, $h_i(p) \geq 1$ for all i. Then $p \in \mathfrak{a}_0$, but $h_m(p+r) = 0$, so $p+r$ is in a face not containing Q, whence $p + r \notin \mathfrak{a}_0$. Thus $p \otimes 1 \in Z_Q^k(\mathfrak{a}_0)_r \setminus F_m^{gp} \otimes k$. Nothing remains to show for the third statement.

In the last case, of course $F_\ell^{gp} \otimes k \subset Z_Q^k(\mathfrak{a}_0)_r$. Let us compute $Z_Q(\mathfrak{a}_0)_r$. We have $F_\ell \subset Z_Q(\mathfrak{a}_0)_r$, so $F_\ell^{gp} \subset Z_Q(\mathfrak{a}_0)_r$. We need to show that it contains no further elements. Let $q \in P, q + r \notin P$. For $i \neq \ell$, we have $h_i(q + r) \geq 0$, so $h_\ell(q + 1) \leq -1$. Thus $h_\ell(q + r) = -1$ and $h_\ell(q) = 0$ implying $q \in F_\ell$. Now let $q \in \mathfrak{a}_0, q + r \notin \mathfrak{a}_0$, but $q + r \in P$. Then $q + r \in \bigcup_{Q \not\subset G} G$ is in the union of the faces not containing Q. It is indeed in a facet not containing Q, so there is an $m \in M$ with $h_m(q + r) = 0$. Thus $h_m(q) = 0$ and $q \notin \mathfrak{a}_0$, a contradiction. Finally, if $q \in Q$, then $q \in F_\ell^{gp}$ since $Q \subset F_\ell$. Thus $Z_Q(\mathfrak{a}_0)_r = \mathbb{Z}\langle F_\ell^{gp} \rangle = F_\ell^{gp}$ since the cokernel of $F_\ell^{gp} \to P^{gp}$ is torsion free. This shows $Z_Q^k(\mathfrak{a}_0)_r = F_\ell^{gp} \otimes k$. $\square$

Now we can compute $W(\mathfrak{a}_0)_p$:

Corollary 6.22. *We have*

$$W(\mathfrak{a}_0)_p = \left(\bigcap_{\ell : p \in F_\ell} (F_\ell^{gp} \otimes k) \right) / Q^{gp} \otimes k$$

for $p \in P$, and $W(\mathfrak{a}_0)_p = 0$ otherwise.

Proof. By definition, we have $W(\mathfrak{a}_0)_p = \left(\bigcap_{r : p+r \notin P} Z_Q^k(\mathfrak{a}_0)_r \right) / Q^{gp} \otimes k$. Set

$$L_p := \{\ell \in L | \exists r : p + r \notin P \wedge h_\ell(r) = -1, h_i(r) \geq 0 \forall i \neq \ell\} \ .$$

Now $\bigcap_{r : r+p \notin P} Z_Q^k(\mathfrak{a}_0)_r \subset \bigcap_{\ell \in L_p} F_\ell^{gp} \otimes k$. Let $r \in P^{gp}$ such that $h_\ell(r) = -1, h_{\ell'}(r) = -1$ and $h_i(r) \geq 0$ for the other i, and such that $p + r \notin P$. Assume without loss of generality that $h_\ell(p) = 0$. Let $p' \in P$ with $h_\ell(p') = 0$ and $h_i(p') \geq 1$ for the other i. Then $h_\ell(p + p' + r) = -1$, so $p + p' + r \notin P$, and $h_\ell(p' + r) = -1, h_i(p' + r) \geq 0$. Thus $\ell \in L_p$ and $\bigcap_{\ell \in L_p} F_\ell^{gp} \otimes k \subset Z_Q^k(\mathfrak{a}_0)_r$ and we have

$$W(\mathfrak{a}_0)_p = \left(\bigcap_{r : p+r \notin P} Z_Q^k(\mathfrak{a}_0)_r \right) / Q^{gp} \otimes k = \left(\bigcap_{\ell \in L_p} (F_\ell^{gp} \otimes k) \right) / Q^{gp} \otimes k \ .$$

If $\ell \in L_p$, then $h_\ell(r) = -1$ and $h_i(r) \geq 0$ for $i \neq \ell$. Thus $h_i(p+r) \geq 0$ and $h_\ell(p+r) < 0$ since otherwise $p + r \in P$. This implies $h_\ell(p) = 0$, so $p \in F_\ell$. Conversely, if $p \in F_\ell$, then there is an $r \in P^{gp}$ with $h_\ell(r) = -1$ and $h_i(r) \geq 0$ for all $i \neq \ell$. Therefore $h_\ell(p + r) = -1$, so $p + r \notin P$. This shows $\ell \in L_p$ concluding the proof. $\square$

Now we see immediately in both cases $\mathfrak{b} = \mathfrak{a}$ and $\mathfrak{b} = \mathfrak{a}_0$:

Corollary 6.23. *Let $p \in P$ and $q \in Q$. Then $W(\mathfrak{b})_p = W(\mathfrak{b})_{p+q}$.*

Proof. For the case $\mathfrak{b} = \mathfrak{a}_0$ note that $p \in F_\ell$ iff $p + q \in F_\ell$. $\square$

Recall the isomorphism $E \times Q \to P$ of Q-sets, i.e. of sets with an action of $Q = \mathbb{N}$. Using the corollary we see:

Corollary 6.24. *The map*

$$\bigoplus_{e \in E} k[Q] \otimes_k W(\mathfrak{b})_e \to \bigoplus_{p \in P} z^p \cdot W(\mathfrak{b})_p, \quad z^q \otimes w_e \mapsto z^{e+q} \cdot w_e$$

is an isomorphism of $k[Q]$-modules. Thus $W^1_{X(\mathfrak{b})/S}$ is free as a $k[Q]$-module.

Proof. It is straightforward to check the isomorphism. The module on the left hand side is free since each $W(\mathfrak{b})_e$ is a vector space, so free, whence the direct sum is free. $\square$

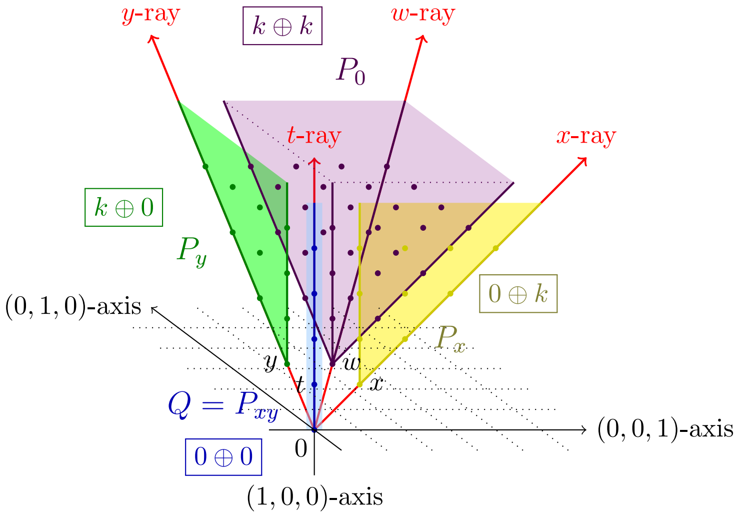

Figure 4: We see the four regions P_{xy}, P_x, P_y, P_0 of P. We imagine the vector spaces $W(\mathfrak{a}_0)_p$ sitting on the lattice points of the regions; their direct sum forms $\Gamma(X, W^1_{X/S})$.

We conclude this section with an explicit calculation of $\bigoplus_{p \in P^{gp}} z^p \cdot W(\mathfrak{a}_0)_p$. Since $L = \{tx, ty\}$, to compute

$$W(\mathfrak{a}_0)_p = \left(\bigcap_{\ell : p \in F_\ell} (F_\ell^{gp} \otimes k) \right) / Q^{gp} \otimes k \ ,$$

we have to consider only the two faces F_{tx} and F_{ty}. We subdivide P into four regions:

$$P_0 = P \setminus (F_{tx} \cup F_{ty}) \ , \quad P_x = F_{tx} \setminus Q \ , \quad P_y = F_{ty} \setminus Q \ , \quad P_{xy} = Q \ .$$

Then we have $P = P_0 \cup P_x \cup P_y \cup P_{xy}$ as a disjoint union. Using the isomorphism

$$P^{gp} \otimes k / Q^{gp} \otimes k \to k \oplus k, [(a,b,c)] \mapsto (b,c) \ ,$$

we get:

$$\begin{array}{ll} W(\mathfrak{a}_0)_p = (P^{gp} \otimes k)/(Q^{gp} \otimes k) = k \oplus k & \text{for } p \in P_0 \\ W(\mathfrak{a}_0)_p = (F_{tx}^{gp} \otimes k)/(Q^{gp} \otimes k) = 0 \oplus k & \text{for } p \in P_x \\ W(\mathfrak{a}_0)_p = (F_{ty}^{gp} \otimes k)/(Q^{gp} \otimes k) = k \oplus 0 & \text{for } p \in P_y \\ W(\mathfrak{a}_0)_p = (Q^{gp} \otimes k)/(Q^{gp} \otimes k) = 0 \oplus 0 & \text{for } p \in P_{xy} \end{array}$$

This can be visualized as shown in Figure 4. Moreover, we see that we have a decomposition $W^1_{X/S} = \mathcal{A}_x \oplus \mathcal{A}_y$ with $\Gamma(X, \mathcal{A}_x) = \bigoplus_{p \in P \setminus F_{ty}} z^p \cdot (0 \oplus k)$ and $\Gamma(X, \mathcal{A}_y) = \bigoplus_{p \in P \setminus F_{tx}} z^p \cdot (k \oplus 0)$ similar to the decomposition in Theorem 6.14.

6.4 The Computation of $W^m_{X/S}$

Now we turn to $W^m_{X/S} := j_* \Omega^m_{U/S}$. We will obtain a result very similar to [7][Prop. 1.12.] by Gross-Siebert, but we use the method of [2][§4] by Danilov. The plan of this section is as follows: We introduce a special kind of P^{gp}-graded modules, the Q-nice ones, and investigate their basic properties. Then for $\mathfrak{b} \in \{\mathfrak{a}, \mathfrak{a}_0\}$ we make an ansatz

$$E^m := E^m_{\mathfrak{b}} := \bigoplus_{p \in P} z^p \cdot \bigwedge_k^m W(\mathfrak{b})_p$$

for $W^m_{X(\mathfrak{b})/S}$ using the exterior power of vector spaces $\bigwedge_k^m W(\mathfrak{b})_p =: E^m(\mathfrak{b})_p$. The associated sheaf of E^m is denoted $\mathcal{E}^m := \mathcal{E}^m_{\mathfrak{b}}$. We get a diagram

$$\begin{array}{ccc} \bigwedge^m_{\mathcal{O}_X} W^1_{X(\mathfrak{b})/S} & \xrightarrow{\psi = \psi_{\mathfrak{b}}} & \mathcal{E}^m \\ {\scriptstyle \gamma = \gamma_{\mathfrak{b}}} \downarrow & & \\ W^m_{X(\mathfrak{b})/S} & & \end{array}$$

where γ is constructed from $\bigwedge^m W^1_{X(\mathfrak{b})/S} \leftarrow \bigwedge^m \Omega^1_{X(\mathfrak{b})/S} \to W^m_{X(\mathfrak{b})/S}$ by using that both morphisms are isomorphisms on U and that $W^m_{X(\mathfrak{b})/S} = j_* W^m_{X(\mathfrak{b})/S}|_U$. Thus γ is an isomorphism on U. The morphism ψ is given explicitly by

$$\psi : \bigwedge_{k[P]}^m \Gamma(X, W^1_{X(\mathfrak{b})/S}) \to E^m$$
$$z^{p_1} \cdot w_1 \wedge \cdots \wedge z^{p_m} \cdot w_m \mapsto z^{\sum p_i} \cdot w_1 \wedge \cdots \wedge w_m$$

where on the right hand side, the w_i are considered as elements of $W(\mathfrak{b})_{\sum p_i}$. It turns out that ψ is an isomorphism on

$$\tilde{U} := D(tx) \cup D(ty) \cup D(wx) \cup D(wy) \subset U \ .$$

Denoting $i : \tilde{U} \subset X$ the inclusion, we will see that $W^m_{X(\mathfrak{b})/S} = i_* W^m_{X(\mathfrak{b})/S}|_{\tilde{U}}$ and $\mathcal{E}^m = i_* \mathcal{E}^m|_{\tilde{U}}$. This then implies that $W^m_{X(\mathfrak{b})/S} \cong \mathcal{E}^m$.

We have inclusions $\bigwedge_k^m W^1(\mathfrak{a}_0)_p \to \bigwedge_k^m W^1(\mathfrak{a})_p$ yielding a map $\mathcal{E}^m_{\mathfrak{a}_0} \to \mathcal{E}^m_{\mathfrak{a}}$ which fits into a commutative diagram

$$\begin{tikzcd}
W^m_{X(\mathfrak{a}_0)/S} \arrow[d] & \bigwedge^m_{\mathcal{O}_X} W^1_{X(\mathfrak{a}_0)/S} \arrow[l, "\gamma_{\mathfrak{a}_0}"'] \arrow[r, "\psi_{\mathfrak{a}_0}"] \arrow[d] & \mathcal{E}^m_{\mathfrak{a}_0} \arrow[d] \\
W^m_{X(\mathfrak{a})/S} & \bigwedge^m_{\mathcal{O}_X} W^1_{X(\mathfrak{a})/S} \arrow[l, "\gamma_{\mathfrak{a}}"'] \arrow[r, "\psi_{\mathfrak{a}}"] & \mathcal{E}^m_{\mathfrak{a}}
\end{tikzcd}$$

This shows that $W^m_{X/S} \to W^m_{X_h/S}$ is given by the map constructed from the inclusions.

The Technical Notion of Q-Nice Modules

Let us introduce the Q-nice modules. Recall that a P^{gp}-graded $k[P]$-module M is given by a k-vector space M_s for every $s \in P^{gp}$ and a map $\phi_{r,s,q} : M_r \to M_s$ for $r + q = s$ which are 'functorial'. The module itself is as a vector space

$$M = \bigoplus_{s \in P^{gp}} z^s \cdot M_s$$

and the $k[P]$-module structure is given by $z^p \cdot (z^s \cdot m) := z^{p+s} \cdot \phi_{s,p+s,p}(m)$. The modules describing the differentials in the previous section are evidently P^{gp}-graded $k[P]$-modules of a particularly simple structure.

Definition 6.25. *A P^{gp}-graded $k[P]$-module M is called* Q-nice*, if $p \notin P$ implies $M_p = 0$, and if $p \in P, q \in Q$ implies $\phi_{p,p+q,q} : M_p \cong M_{p+q}$. A graded map of Q-nice modules $\rho : M \to N$ is given by homomorphisms $\rho_p : M_p \to N_p$ satisfying the obvious commutation relations.*

The sheaves $W^1_{X/S}$ and $W^1_{X_h/S}$ are described by Q-nice modules, and their imbedding is graded. Of course a graded map $\rho : M \to N$ gives rise to a homomorphism $\rho : \bigoplus_{s \in P^{gp}} z^s \cdot M_s \to \bigoplus_{s \in P^{gp}} z^s \cdot N_s$ of the actual $k[P]$-modules.

Lemma 6.26. *Let $\rho : M \to N$ be a graded homomorphism of Q-nice modules. Then* $\ker(\rho)$ *and* $\mathrm{coker}(\rho)$ *are canonically Q-nice, and the associated homomorphisms are graded.*

Proof. Just take kernels and cokernels at the level of the M_p. □

To compute $W^m_{X/S}$, we also need to know how to localize a Q-nice module M in $z^f \in k[P]$:

Lemma 6.27. *Let $f \in P$ and M a Q-nice module. Then $(M)_{z^f} = \bigoplus_{p \in P^{gp}} M_{f,p}$, where*

$$M_{f,p} = \varinjlim_{n \to \infty} M_{p+nf}$$

with the obvious structure of $k[P]$-module.

Proof. Consider the map $\zeta : (M)_{z^f} \to \bigoplus_{p\in P^{gp}} M_{f,p}, mz^p/z^{nf} \mapsto \bar{m}z^{p-nf}$, where $\bar{m} = i(m) \in M_{f,p}$ under the natural map $i : M_p \to M_{f,p}$. It is straightforward to check that this map is well-defined. If $\zeta(\sum m_i z^{p_i}/z^n f) = 0$, then $\bar{m}_i = 0$ for all i, so there are k_i with $\phi_{p_i,p_i+k_if,k_if}(m_i) = 0$. Setting $k = \max\{k_i\}$, this implies $z^{kf} \cdot (\sum m_i z^{p_i}/z^{nf}) = 0$, so ζ is injective. It is obviously surjective. $\square$

A property of Q-nice modules that will be very important for the base change in Section 7 is that it is free as a $k[Q]$-module. Recall that $E = F_{xw} \cup F_{yw} \subset P$ has the property that any $p \in P$ can be uniquely written as $p = e + q$ for $e \in E, q \in Q$. Since M_e is a k-vector space, $k[Q] \otimes_k M_e$ is a free $k[Q]$-module, and we may and do consider their direct sum

$$\bigoplus_{e\in E} z^e \cdot (k[Q] \otimes_k M_e)$$

where z^e is a variable for bookkeeping. This is a free $k[Q]$-module.

Lemma 6.28. *Let $M = \bigoplus_{p\in P^{gp}} z^p \cdot M_p$ be a Q-nice module defined by functions $\phi_{r,s,q} : M_r \to M_s$. Then*

$$\nu : \bigoplus_{e\in E} z^e \cdot (k[Q] \otimes_k M_e) \to \bigoplus_{p\in P^{gp}} z^p \cdot M_p,$$
$$(z^e \cdot (z^q \otimes m)) \mapsto z^{e+q} \cdot \phi_{e,e+q,q}(m)$$

is a well-defined isomorphism of $k[Q]$-modules. In particular, M is free and hence flat over $k[Q]$.

Proof. This is a generalization of Corollary 6.24 and straightforward using that $M_p = 0$ for $p \notin P$, that $M_p = M_{p+q}$ for $p \in P, q \in Q$ and that we have a unique decomposition $p = e + q$ for $p \in P$ with $e \in E, q \in Q$. $\square$

First Step: Understanding $\psi : \bigwedge^m_{\mathcal{O}_X} W^1_{X(\mathfrak{b})/S} \to \mathcal{E}^m$

We show that $\psi : \bigwedge^m_{\mathcal{O}_X} W^1_{X(\mathfrak{b})/S} \to \mathcal{E}^m$ is an isomorphism on $\tilde{U} \subset X$. We cover $\tilde{U}$ by the four standard opens

$$D(tx) = \text{Spec } B_{tx} = \text{Spec } k[P]_{z^{(2,0,1)}} = D(z^{(2,0,1)})$$
$$D(ty) = \text{Spec } B_{ty} = \text{Spec } k[P]_{z^{(2,1,0)}} = D(z^{(2,1,0)})$$
$$D(wx) = \text{Spec } B_{wx} = \text{Spec } k[P]_{z^{(2,1,2)}} = D(z^{(2,1,2)})$$
$$D(wy) = \text{Spec } B_{wy} = \text{Spec } k[P]_{z^{(2,2,1)}} = D(z^{(2,2,1)})$$

so it suffices to show that ψ becomes an isomorphism after localizing in z^f for

$$f \in \mathbb{S} := \{(2,0,1),(2,1,0),(2,1,2),(2,2,1)\}\ .$$

Note that every $f \in \mathbb{S}$ spans a facet H, $\langle(2,0,1)\rangle = F_{tx}$ etc. which is obtained as $H = h^{-1}(0)$ for a height function as explained in Section 2.1. First we compute the localizations of E^m. By $\langle H, Q\rangle \subset P$ we denote the face generated by H and Q. Since H is a facet, we have $\langle H, Q\rangle = H$ for $H \subset Q$ and $\langle H, Q\rangle = P$ for $Q \not\subset H$. This shows $W(\mathfrak{a}_0)_p = (\langle H, Q\rangle^{gp} \otimes k)/(Q^{gp} \otimes k)$ in case $p \in H$, but p not contained in any other facet.

Lemma 6.29. *Let $f \in \mathbb{S}$ with facet $H = \langle f\rangle$, and let $h : P^{gp} \to \mathbb{Z}$ be the associated height function with $h^{-1}(0) \cap P = H$. Then we have $\mathcal{E}^m(D(z^f)) = \bigoplus_{p\in P^{gp}} z^p \cdot E^m_{f,p}$ with*

$$E^m_{f,p} = \begin{cases} \bigwedge_k^m (P^{gp} \otimes k/Q^{gp} \otimes k), & \text{if } h(p) > 0 \\ E^m(\mathfrak{b}, H), & \text{if } h(p) = 0 \\ 0 \qquad\qquad , & \text{if } h(p) < 0 \end{cases}$$

where $E^m(\mathfrak{a}, H) = \bigwedge_k^m (P^{gp} \otimes k/Q^{gp} \otimes k)$ and $E^m(\mathfrak{a}_0, H) = \bigwedge_k^m (\langle H, Q\rangle^{gp} \otimes k/Q^{gp} \otimes k)$.

Proof. We use Lemma 6.27. Let $H_i \subset P$ be another facet and $h_i : P^{gp} \to \mathbb{Z}$ the associated height function. Since $f \notin H_i$, we have $h_i(f) > 0$, so if $h(p) > 0$, then $h_j(p + nf) > 0$ for all h_j (with $j \in \{tx, ty, wx, wy\}$) and sufficiently large n. Thus $E^m(\mathfrak{b})_{p+nf} = \bigwedge_k^m (P^{gp} \otimes k/Q^{gp} \otimes k)$ for $n >> 0$. If $h(p) = 0$, then $h(p + nf) = 0$ and $h_i(p + nf) > 0$ for $n >> 0$, so $p + nf \in H, p + nf \notin H_i$ and thus $E^m(\mathfrak{a})_{p+nf} = \bigwedge_k^m (P^{gp} \otimes k/Q^{gp} \otimes k), E^m(\mathfrak{a}_0)_{p+nf} = \bigwedge_k^m (\langle H, Q\rangle^{gp} \otimes k/Q^{gp} \otimes k)$. If $h(p) < 0$, then $h(p + nf) < 0$ for all n, so $E^m(\mathfrak{b})_{p+nf} = 0$. ☐

The result of the calculation holds in particular for $m = 1$ yielding the localization of $W^1_{X(\mathfrak{b})/S}$, and the localized map is given by

$$\psi_f : \bigwedge_{k[P_f]}^m \Big(\bigoplus_{p\in P^{gp}} z^p \cdot E^1_{f,p}\Big) \to \bigoplus_{p\in P^{gp}} z^p \cdot E^m_{f,p}$$
$$z^{p_1} \cdot e_1 \wedge \cdots \wedge z^{p_m} \cdot e_m \mapsto z^{\sum p_i} \cdot e_1 \wedge \cdots \wedge e_m$$

Here $e_1 \wedge \cdots \wedge e_m$ is formed by taking a representative $\tilde{e}_i \in W(\mathfrak{b})_{p_i+n_i f}$ and considering it an element of $W(\mathfrak{b})_{\sum p_i+\sum n_i f}$.

Lemma 6.30. *ψ_f is an isomorphism.*

Proof. It suffices to show ψ_f bijective. We start with surjectivity. We have an isomorphism $P^{gp}/Q^{gp} \cong \mathbb{Z}^r \oplus \langle H, Q\rangle^{gp}/Q^{gp}$ with $r \in \{0, 1\}$, and the latter summand is free. Thus we have a basis $v, v_1, \ldots, v_{d-1}$ of P^{gp}/Q^{gp} with $v_i \in \langle H, Q\rangle^{gp}/Q^{gp}$. This gives a basis of $P^{gp} \otimes k/Q^{gp} \otimes k$ with $v_i \in \langle H, Q\rangle^{gp} \otimes k/Q^{gp} \otimes k$. Let

$$z^p e_1 \wedge \cdots \wedge e_m \in \bigoplus_{p\in P^{gp}} z^p \cdot E^m_{f,p}$$

with $e_2, \ldots, e_m \in \langle H, Q\rangle^{gp}/Q^{gp} \subset \varinjlim W(\mathfrak{b})_{p+nf}$, and assume $h(p) \geq 0$. Thus $e_2, \ldots, e_m \in E^1_{f,0}$ and

$$z^p \cdot e_1 \wedge z^0 \cdot e_2 \wedge \cdots \wedge z^0 \cdot e_m \in \bigwedge_{k[P_f]}^m \Big(\bigoplus_{p\in P^{gp}} z^p \cdot E^1_{f,p}\Big)$$

is a well-defined element mapping to $z^p e_1 \wedge \cdots \wedge e_m$.

Turning to injectivity, let $g \in P$ be an element with $\langle g \rangle = P$, e.g. $g = (2,1,1)$. Thus ψ_g is the restriction of ψ to X^+. The map ψ_g is surjective, since ψ_f is (because it is a localization of ψ_f). Using Lemma 6.27, we check easily that

$$(E^m)_{z^g} = \bigoplus_{p \in P^{gp}} \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k) = \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k) \otimes_k k[P^{gp}] \ .$$

Thus ψ_g is an epimorphism of free $k[P^{gp}]$-modules of the same rank, and the kernel is a projective, hence locally free module of rank 0. Thus ψ_g is an isomorphism. The sheaf associated to $\bigwedge_{k[P_f]}^m (\bigoplus_{p \in P^{gp}} z^p \cdot E_{f,p}^1)$ is locally free on Spec $k[P_f]$, so it has injective restriction maps, and it follows that ψ_f is injective. $\square$

We conclude:

Corollary 6.31. *The homomorphism $\psi : \bigwedge_{\mathcal{O}_X}^m W^1_{X(\mathfrak{b})/S} \to \mathcal{E}^m$ is an isomorphism when restricted to $\tilde{U}$.*

Second Step: Direct Images from $\tilde{U}$

We show that $i_* W^m_{X(\mathfrak{b})/S}|_{\tilde{U}} = W^m_{X(\mathfrak{b})/S}$ and $i_* \mathcal{E}^m_{\mathfrak{b}}|_{\tilde{U}} = \mathcal{E}^m_{\mathfrak{b}}$. It is straightforward to show

$$X \setminus \tilde{U} = \{t = 0, w = 0\} \cup \{x = 0, y = 0\} \ ,$$

so the complement of $\tilde{U}$ has codimension 2 in X. This means $\mathcal{O}_X \to i_* \mathcal{O}_{\tilde{U}}$ is an isomorphism since X is a normal integral scheme.

Lemma 6.32. *We have $W^m_{X(\mathfrak{b})/S} \cong i_* W^m_{X(\mathfrak{b})/S}|_{\tilde{U}}$ induced by restriction.*

Proof. Since $\Omega^m_{U/S}$ is locally free, we have an isomorphism $\Omega^m_{U/S} \to i_* \Omega^m_{\tilde{U}/S}$ of sheaves on U via restriction. Applying $j_*(-)$ to the inclusion $j : U \subset X$ yields the desired result. $\square$

To show $i_* \mathcal{E}^m|_{\tilde{U}} \cong \mathcal{E}^m$ it suffices to show $\rho : \mathcal{E}^m(X) \to \mathcal{E}^m(\tilde{U})$ bijective. To describe the latter module, note that $E^m_{f,p} \subset \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k)$. For $\mathfrak{b} = \mathfrak{a}$, this is obvious. For $\mathfrak{b} = \mathfrak{a}_0$, this is since $(\langle H, Q \rangle^{gp} \otimes k)/(Q^{gp} \otimes k) \subset (P^{gp} \otimes k)/(Q^{gp} \otimes k)$ is a direct summand. Thus a family $(e_f)_{f \in \mathbb{S}}, e_i \in \mathcal{E}^m(\mathrm{Spec}\ k[P_f])$ gives an element of $\mathcal{E}^m(\tilde{U})$ iff all e_f coincide as elements of $\mathcal{E}^m(X^+)$, and we have

$$\mathcal{E}^m(\tilde{U}) = \bigcap_{f \in \mathbb{S}} \left(\bigoplus_{p \in P^{gp}} z^p \cdot E^m_{f,p} \right) = \bigoplus_{p \in P^{gp}} z^p \cdot \bigcap_{f \in \mathbb{S}} E^m_{f,p} \ .$$

Hence to show ρ bijective, it suffices to show $E^m(\mathfrak{b})_p \to \bigcap_{f \in S} E^m_{f,p}$ bijective. If $p \notin P$, then $E^m(\mathfrak{b})_p = 0$ and there is a facet H with $h(p) < 0$. This implies $E^m_{f_H,p} = 0$ for $f_H \in S$ with $\langle f_H \rangle = H$, and the map is bijective. If $p \in P$ and $\mathfrak{b} = \mathfrak{a}$, then $E^m_{f,p} = \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k)$ and the map is evidently bijective.

Corollary 6.33. *The homomorphism $\mathcal{E}_{\mathfrak{a}}^m \to i_*\mathcal{E}_{\mathfrak{a}}^m|_{\tilde{U}}$ induced by restriction is an isomorphism.*

We turn to $\mathfrak{b} = \mathfrak{a}_0$. If $p \in P$, then

$$\bigcap_{f \in \mathbb{S}} E_{f,p}^m = \bigcap_{p \in H} \bigwedge_k^m (\langle H, Q\rangle^{gp} \otimes k / Q^{gp} \otimes k) = \bigcap_{p \in H_\ell} \bigwedge_k^m (H_\ell^{gp} \otimes k / Q^{gp} \otimes k)$$

where the intersection on the right runs over $H_\ell \in \{F_{tx}, F_{ty}\}$. Since for k-vector spaces, the exterior power commutes with intersections, we obtain

$$\bigcap_{p \in H_\ell} \bigwedge_k^m (H_\ell^{gp} \otimes k / Q^{gp} \otimes k) = \bigwedge_k^m \left(\bigcap_{p \in H_\ell} (H_\ell^{gp} \otimes k / Q^{gp} \otimes k) \right) = \bigwedge_k^m W(\mathfrak{a}_0)_p \ .$$

We see immediately:

Corollary 6.34. *The homomorphism $\mathcal{E}_{\mathfrak{a}_0}^m \to i_*\mathcal{E}_{\mathfrak{a}_0}^m|_{\tilde{U}}$ induced by restriction is an isomorphism.*

Putting everything together, we get the central theorem of this section:

Theorem 6.35. *We have isomorphisms*

$$\Gamma(X, W_{X(\mathfrak{a})/S}^m) \cong \bigoplus_{p \in P} z^p \cdot \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k)$$

and

$$\Gamma(X, W_{X(\mathfrak{a}_0)/S}^m) \cong \bigoplus_{p \in P} z^p \cdot \bigwedge_k^m \left(\bigcap_{p \in H_\ell} (H_\ell^{gp} \otimes k)/(Q^{gp} \otimes k) \right)$$

where the intersection is over $H_\ell \in \{F_{tx}, F_{ty}\}$ (we intersect the terms for those H_ℓ that satisfy $p \in H_\ell$). The canonical homomorphism $\ell_{X/S}^m : W_{X(\mathfrak{a}_0)/S}^m \to W_{X(\mathfrak{a})/S}^m$ corresponds to the inclusions

$$\bigwedge_k^m \left(\bigcap_{p \in H_\ell} (H_\ell^{gp} \otimes k)/(Q^{gp} \otimes k) \right) \subset \bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k) \ .$$

6.5 The Sheaf $W_{X/S}^m$ Explicitly

In the previous section we computed $W_{X/S}^m$. Now we like to see it as explicitly as possible, generalizing what we did at the end of Section 6.3. We have

$$\bigwedge_k^0 \left(\bigcap_{p \in H_\ell} (H_\ell^{gp} \otimes k)/(Q^{gp} \otimes k) \right) = k \ ,$$

yielding $\Gamma(X, W^0_{X/S}) = k[P]$ as expected.

At the end of Section 6.3, we observed that $P^{gp} \otimes k / Q^{gp} \otimes k \cong k \oplus k$ and

$$\begin{array}{ll} E^1(\mathfrak{a}_0) = W(\mathfrak{a}_0)_p = k \oplus k & \text{for } p \in P_0 \\ E^1(\mathfrak{a}_0) = W(\mathfrak{a}_0)_p = 0 \oplus k & \text{for } p \in P_x \\ E^1(\mathfrak{a}_0) = W(\mathfrak{a}_0)_p = k \oplus 0 & \text{for } p \in P_y \\ E^1(\mathfrak{a}_0) = W(\mathfrak{a}_0)_p = 0 \oplus 0 & \text{for } p \in P_{xy} \end{array}$$

Thus we may write $\Gamma(X, W^1_{X/S}) = k[P \setminus F_{ty}] \oplus k[P \setminus F_{tx}]$ yielding a decomposition $W^1_{X/S} = \mathcal{A}_x \oplus \mathcal{A}_y$ with

$$\Gamma(X, \mathcal{A}_x) = \bigoplus_{p \in P \setminus F_{ty}} z^p \cdot (0 \oplus k)$$
$$\Gamma(X, \mathcal{A}_y) = \bigoplus_{p \in P \setminus F_{tx}} z^p \cdot (k \oplus 0)$$

Using this result, we see that in case $m = 2$ we have

$$\bigwedge_k^2 (P^{gp} \otimes k / Q^{gp} \otimes k) \cong \bigwedge_k^2 (k \oplus k) = k$$

and therefore

$$\begin{array}{ll} E^2(\mathfrak{a}_0) = k & \text{for } p \in P_0 \\ E^2(\mathfrak{a}_0) = 0 & \text{else} \end{array}$$

Since $P_0 = (1,1,1) + P = \{(1,1,1) + p | p \in P\}$ this implies

$$\mathcal{O}_X \cong W^2_{X/S}, \quad 1 \mapsto z^{(1,1,1)} \cdot 1$$

Because $m = 2$ is the relative dimension of the family $f : X \to S$, we might call it a *log Calabi-Yau* family due to the isomorphism above. Giving this name is also inspired by the so-called *toric log Calabi-Yau spaces* that are defined and studied in the Gross-Siebert program, see e.g. [6][Def. 4.3.]. For $m \geq 3$, we have $\bigwedge_k^3 (P^{gp} \otimes k / Q^{gp} \otimes k) = \bigwedge_k^3 (k \oplus k) = 0$, so we get $W^m_{X/S} = 0$.

We conclude this section by showing that $W^1_{X/S}$ is only reflexive, but not locally free. The method is essentially the same as the one employed in Section 8.1 in the appendix. Namely, assume $W^1_{X/S}$ was locally free. Then it had constant rank $\mathrm{rk}(W^1_{X/S}) = 2$. In particular, for the embedding $i : \{P\} \to X$, we had $\dim_k(i^* W^1_{X/S}(\{P\})) = 2$. If we write $A_x = \Gamma(X, \mathcal{A}_x)$ and $A_y = \Gamma(X, \mathcal{A}_y)$, then with $\mathfrak{m}_P = (x, y, t, w) \subset B \cong k[P]$ the ideal defining $\{P\} \subset X$, we have

$$i^* W^1_{X/S}(\{P\}) = (A_x \oplus A_y) / \mathfrak{m}_P \cdot (A_x \oplus A_y) = A_x / \mathfrak{m}_P A_x \oplus A_y / \mathfrak{m}_P A_y \ .$$

To compute its dimension, note that $\mathfrak{m}_P$ corresponds to the monoid ideal

$$((1,0,0),(1,0,1),(1,1,0),(1,1,1)) \subset P \ .$$

This yields

$$\mathfrak{m}_P \cdot A_x = z^{(1,0,0)} \cdot A_x + z^{(1,0,1)} \cdot A_x + z^{(1,1,0)} \cdot A_x + z^{(1,1,1)} \cdot A_x \subset A_x \ .$$

We have $A_x = k[P \setminus F_{ty}]$, and for any submonoid $P' \subset P$ and $p \in P$, we have $z^p \cdot k[P'] = k[p + P']$, implying that

$$\begin{aligned}\mathfrak{m}_P \cdot A_x = k\Big[\quad & (1,0,0) + (P \setminus F_{ty}) \ \cup \ (1,0,1) + (P \setminus F_{ty}) \\ & \cup (1,1,0) + (P \setminus F_{ty}) \ \cup \ (1,1,1) + (P \setminus F_{ty}) \quad \Big]\end{aligned}$$

Since $P \setminus F_{ty} = (1,0,1) + P \cup (1,1,1) + P$, it is easy (but maybe a bit tedious) to compute

$$\mathfrak{m}_P \cdot A_x = k[\ P \setminus (F_{ty} \cup \{(1,0,1),(1,1,1)\})\ ]$$

which shows that $\dim_k(A_x/\mathfrak{m}_P A_x) = 2$. Analogously, we have $\dim_k(A_y/\mathfrak{m}_P A_y) = 2$, so we get $\dim_k(i^* W^1_{X/S}(\{P\})) = 2 + 2 = 4$. We conclude:

Proposition 6.36. *The sheaf $W^1_{X/S}$ is not locally free.*

Finally, consider the pullback of $W^1_{X/S}$ to the central fiber $c : X_0 \subset X$. The sheaf $c^* W^1_{X/S}$ is not locally free, too. For consider the factorization $i : \{P\} \xrightarrow{m} X_0 \xrightarrow{c} X$. On the one hand side, $c^* W^1_{X/S}$ would have constant rank 2 implying that $\dim_k(m^* c^* W^1_{X/S}(\{P\})) = 2$. On the other hand side, we have computed

$$\dim_k(i^* W^1_{X/S}(\{P\})) = 4 \ .$$

Because $\Gamma(X, W^1_{X/S}) \cong \Omega^{[1]}_{B/A}$, we have furthermore that $\Omega^{[1]}_{B/A} \otimes_A A/(t)$ is not locally free on X_0 which was stated in the introduction.

7 The Base Change Property of $W^m_{X/S}$

In this section we prove the base change property for our differentials which we claimed in Theorem 1.3. We consider a diagram

$$\begin{array}{ccccc}
Y_h & \xrightarrow{\gamma'} & Y & \xrightarrow{g} & T \\
\downarrow & & c\downarrow & & b\downarrow \\
X_h & \xrightarrow{\gamma} & X & \xrightarrow{f} & S
\end{array}$$

as in Section 5.4. The functoriality of the complex of differentials endows us with a homomorphism

$$\sigma^m : W^m_{X/S} \to c_* W^m_{Y/T}$$

with adjoint

$$\tau^m : c^* W^m_{X/S} \to W^m_{Y/T} \ .$$

The goal of this section is to show that τ^m is an isomorphism. On $U \subset X$, the map σ^m is the homomorphism $\sigma^m_{V/U}$ constructed in Section 8.3 in the appendix, so on $V = c^{-1}(U) \subset Y$ the adjoint is $\tau^m_{V/U} : c^*\Omega^m_{U/S} \to \Omega^m_{V/T}$ which is an isomorphism by Proposition 8.8. To prove that τ^m is an isomorphism, let $j : V \subset Y$ be the inclusion and let us consider the diagram

$$\begin{array}{ccc}
c^* W^m_{X/S} & \xrightarrow{\tau^m} & W^m_{Y/T} \\
\rho\downarrow & & \rho'\downarrow\cong \\
j_* c^* W^m_{X/S} & \xrightarrow[\cong]{j_*\tau^m_{V/U}} & j_* W^m_{Y/T}
\end{array}$$

where $j_*\tau^m_{V/U}$ is an isomorphism as explained above, and ρ' is an isomorphism by the definition of $W^m_{Y/T}$. Thus to prove τ^m an isomorphism, it suffices to show $\rho : c^* W^m_{X/S} \to j_* c^* W^m_{X/S}$ an isomorphism.

Our means to prove the isomorphism is the exact sequence

$$0 \to W^m_{X/S} \to W^m_{X_h/S} \to \mathcal{Q}^m_{X_h/X} \to 0$$

observed in Section 6.1. It turns out to be universally exact, which means that the pullback under any base extension $b : T \to S$ is exact again. In particular, the pullback using c^* is exact, and applying the functor $j_*(-)$ yields a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c^* W^m_{X/S} & \xrightarrow{c^*\ell^m_{X/S}} & c^* W^m_{X_h/S} & \longrightarrow & c^*\mathcal{Q}^m_{X_h/X} & \longrightarrow & 0 \\
 & & \rho\downarrow & & \downarrow\cong & & \rho_Q\downarrow & & \\
0 & \longrightarrow & j_* c^* W^m_{X/S} & \xrightarrow{j_* c^*\ell^m_{X/S}} & j_* c^* W^m_{X_h/S} & \longrightarrow & j_* c^* \mathcal{Q}^m_{X_h/X} & &
\end{array}$$

where the middle vertical arrow is an isomorphism because $c^*W^m_{X_h/S} \cong \mathcal{O}_Y^{\oplus r}$ for some $r \geq 0$ is free and Corollary 2.8 applies. We see immediately that $\rho : c^*W^m_{X/S} \to j_*c^*W^m_{X/S}$ is injective, and to get surjectivity, it suffices to show $\rho_Q : c^*\mathcal{Q}^m_{X_h/X} \to j_*c^*\mathcal{Q}^m_{X_h/X}$ injective.

Universal Exactness

We explain why we get an exact sequence on Y. The scheme T is affine, so we write $R = \mathcal{O}_T(T)$ and have $T = \text{Spec } R$. We use the notion of Q-nice modules that we introduced in Section 6.4. Namely, we have a diagram

$$\begin{tikzcd} W^m_{X/S}(X) \arrow[r, hook, "\ell^m_{X/S}(X)"] \arrow[d, "\cong"] & W^m_{X_h/S}(X) \arrow[r, two heads] \arrow[d, "\cong"] & \mathcal{Q}^m_{X_h/X}(X) \arrow[d, "\cong"] \\ \bigoplus_{p\in P^{gp}} z^p \cdot E^m(\mathfrak{a}_0)_p \arrow[r, hook] & \bigoplus_{p\in P^{gp}} z^p \cdot E^m(\mathfrak{a})_p \arrow[r, two heads] & \bigoplus_{p\in P^{gp}} z^p \cdot (E^m(\mathfrak{a})_p/E^m(\mathfrak{a}_0)_p) \end{tikzcd}$$

on global sections coming from Theorem 6.35. The map $\ell^m_{X/S}(X)$ is given by a graded homomorphism of Q-nice modules, so its cokernel $\bigoplus_{p\in P^{gp}} z^p \cdot (E^m(\mathfrak{a})_p/E^m(\mathfrak{a}_0)_p)$ is a Q-nice module. As such, it is a flat $k[Q]$-module by Lemma 6.28. Now we have a commutative diagram

$$\begin{tikzcd} W^m_{X/S}(X) \otimes_B \mathcal{O}_Y(Y) \arrow[r, hook] \arrow[d, "\cong"] & W^m_{X_h/S}(X) \otimes_B \mathcal{O}_Y(Y) \arrow[r, two heads] \arrow[d, "\cong"] & \mathcal{Q}^m_{X_h/X}(X) \otimes_B \mathcal{O}_Y(Y) \arrow[d, "\cong"] \\ W^m_{X/S}(X) \otimes_A R \arrow[r, hook] & W^m_{X_h/S}(X) \otimes_A R \arrow[r, two heads] & \mathcal{Q}^m_{X_h/X}(X) \otimes_A R \end{tikzcd}$$

of R-modules. The lower line is exact since $\mathcal{Q}^m_{X_h/X}(X)$ is a flat $k[Q]$-module, so the upper line is exact. This implies

$$0 \to c^*W^m_{X/S} \to c^*W^m_{X_h/S} \to c^*\mathcal{Q}^m_{X_h/X} \to 0$$

exact as a sequence of $\mathcal{O}_Y$-modules since Y is affine.

7.1 The Map $\rho_Q : c^*\mathcal{Q}^m_{X_h/X} \to j_*c^*\mathcal{Q}^m_{X_h/X}$

To address injectivity of ρ_Q, we need to understand the coherent sheaf $c^*\mathcal{Q}^m_{X_h/X}$. As a first step towards this goal, we introduce the following description of $\mathcal{O}_Y(Y) = R \otimes_A B$: Restricting the ring homomorphism $k[Q] \to R$ to $Q \subset k[Q]$ yields a monoid homomorphism $\sigma : Q \to R$. Recall that $E = F_{xw} \cup F_{yw} \subset P$ has the property that any $p \in P$ admits a unique decomposition $p = e + q$ with $e \in E, q \in Q$.

Definition 7.1. *We define*

$$k[P]_\sigma := \bigoplus_{e\in E} R \cdot z^e$$

as a k-vector space and a ring structure is defined by

$$rz^e \cdot r'z^{e'} := \sigma(\tilde{q})rr'z^{\tilde{e}}$$

where $e + e' = \tilde{e} + \tilde{q}$ is the unique decomposition with $\tilde{e} \in E$ and $\tilde{q} \in Q$.

It is straightforward to show that $k[P]_\sigma$ is indeed a ring and moreover an R-algebra. It is isomorphic to $\mathcal{O}_Y(Y)$:

Lemma 7.2. *The map*

$$\zeta = \zeta_{k[P],\sigma} : k[P] \otimes_{k[Q]} R \to k[P]_\sigma, \quad z^p \otimes r \mapsto \sigma(q)rz^e$$

where $p = e + q$ is the unique decomposition, is an isomorphism of rings.

Proof. Since $\zeta(z^{p+q'} \otimes r) = \sigma(q + q')rz^e = \zeta(z^p \otimes \sigma(q')r)$, it is a well-defined group homomorphism. Since

$$\begin{aligned}\zeta((z^p \otimes r) \cdot (z^{p'} \otimes r')) &= \zeta(z^{p+p'} \otimes rr') = \sigma(\tilde{q} + q + q')rr'z^{\tilde{e}} \\ &= \sigma(q)rz^e \cdot \sigma(q')r'z^{e'} = \zeta(z^p \otimes r) \cdot \zeta(z^{p'} \otimes r')\end{aligned}$$

where $e + e' = \tilde{e} + \tilde{q}$, it is a ring homomorphism. Its inverse (say at the level of groups) is given by $\zeta^{-1}(rz^e) = z^e \otimes r$ (obviously well-defined) for a straightforward calculation yields $\zeta^{-1} \circ \zeta = \mathrm{Id}$ (the other one being obvious). Thus ζ is bijective and henceforth an isomorphism. $\square$

The module $\mathcal{Q}^m_{X_h/X}$ is Q-nice, so let us compute $M \otimes_{k[Q]} R$ for Q-nice modules M. Again, we make an ansatz.

Definition 7.3. *We define*

$$M_\sigma := \bigoplus_{e \in E} (M_e \otimes_k R) \cdot z^e$$

as a k-vector space. To define the action of $k[P]_\sigma$, recall that $M_p \cong M_{p+q}$. Then for $r \cdot z^e \in R \cdot z^e$ and $(m \otimes r')z^{e'} \in (M_{e'} \otimes_k R) \cdot z^{e'}$, we set

$$rz^e \cdot (m \otimes r')z^{e'} := (\phi_{e',e+e',e}(m) \otimes \sigma(\tilde{q})rr')z^{\tilde{e}} \in (M_{\tilde{e}} \otimes_k R) \cdot z^{\tilde{e}}$$

where $e + e' = \tilde{e} + \tilde{q}$ is the unique decomposition and $\phi_{e',e+e',e}(m)$ is considered as an element of $M_{\tilde{e}}$ on the right (i.e. as $\phi^{-1}_{\tilde{e},e+e',\tilde{q}}(\phi_{e',e+e',e}(m))$).

It is straightforward to check that M_σ is indeed a $k[P]_\sigma$-module. It is isomorphic to the tensor product $M \otimes_{k[Q]} R$:

Lemma 7.4. *The map*

$$\zeta = \zeta_{M,\sigma} : M \otimes_{k[Q]} R \to M_\sigma, \quad m_p z^p \otimes r \mapsto (\phi^{-1}_{e,p,q}(m_p) \otimes \sigma(q)r) \cdot z^e$$

is an isomorphism of $k[P]_\sigma$-modules.

Proof. The well-definedness as a group homomorphism is similar to the previous lemma. k-linearity is obvious. $k[P]_\sigma$-linearity is a straightforward calculation (the homomorphisms ϕ are omitted for simplicity):

$$\begin{aligned}&\zeta((z^p \otimes r)\cdot(mz^{p'}\otimes r')) = \zeta(mz^{p+p'}\otimes rr') = \zeta(mz^{\bar{e}+\bar{q}+q+q'}\otimes rr')\\ &= (m\otimes\sigma(\bar{q}+q+q')rr')z^{\bar{e}} = \sigma(q)rz^e\cdot(m\otimes\sigma(q')r')z^{e'} = \zeta(z^p\otimes r)\cdot\zeta(mz^{p'}\otimes r')\end{aligned}$$

where $p = e+q, \quad p' = e'+q', \quad e+e' = \bar{e}+\bar{q}$. We have that $\zeta^{-1}((m\otimes r)z^e) = mz^e\otimes r$ is the inverse group homomorphism, so ζ is bijective and thus an isomorphism. □

This description allows us to compute $c^*\mathcal{Q}^m_{X_h/X}$. To address injectivity of ρ_Q, recall that in Section 2.2 we considered

$$U' := D_X(x)\cup D_X(y)\cup D_X(w)\subset U$$

where $D_X(b) := \text{Spec } B_b \subset X$ for $b\in B$ are the standard opens. Its base change to T is

$$V' := D_Y(x)\cup D_Y(y)\cup D_Y(w)\subset V$$

where we consider $x,y,w\in k[P]_\sigma = \mathcal{O}_Y(Y)$. To show injectivity of ρ_Q, it suffices to prove the restriction

$$\hat{\rho} : c^*\mathcal{Q}^m_{X_h/X}(X)\to c^*\mathcal{Q}^m_{X_h/X}(V')$$

injective whence we like to understand the individual restrictions

$$\hat{\rho}_f : c^*\mathcal{Q}^m_{X_h/X}(X)\to c^*\mathcal{Q}^m_{X_h/X}(D_Y(z^f))$$

for $z^f\in\{x,y,w\}$ better. Since every $D_Y(z^f)$ is affine, these restrictions are given by the localization maps. Observe that for $z^f\in\{x,y,w\}$, we have $f\in E$. Therefore our next task is to understand the localizations $M_\sigma\to(M_\sigma)_{z^f}$ better, where $f\in E$ and M is a Q-nice module.

So let M be a Q-nice module, and fix $f\in E$. For any $e\in E$, we construct a directed system $\tilde{M}_{e,n,f}$ for $n\in\mathbb{N}$. Namely, we decompose $e+nf = \tilde{e}_n+\tilde{q}_n$ with $\tilde{e}_n\in E, \tilde{q}_n\in Q$, and we set

$$\tilde{M}_{e,n,f} := M_{\tilde{e}_n}\otimes_k R$$

as a k-vector space. Note that also the summands of M_σ are of this form. We like to define homomorphisms $\ell_{n,m} : \tilde{M}_{e,n,f}\to\tilde{M}_{e,n+m,f}$. To do so, observe that $\tilde{e}_n + mf = \tilde{e}_{n+m} + (\tilde{q}_{n+m}-\tilde{q}_n)$ with $\tilde{q}_{n+m}-\tilde{q}_n\in Q\subset Q^{gp}$. Thus we have a map $\hat{\phi}_{n,m} : M_{\tilde{e}_n}\to M_{\tilde{e}_n+mf}\cong M_{\tilde{e}_{n+m}}$ coming from the structure of Q-nice module, and we define

$$\ell_{n,m} : \tilde{M}_{e,n,f}\to\tilde{M}_{e,n+m,f}, \quad \mu\otimes r\mapsto\hat{\phi}_{n,m}(\mu)\otimes\sigma(\tilde{q}_{n+m}-\tilde{q}_n)r\ .$$

It is a straightforward computation to show that we have indeed a directed system, and we denote its colimit by $\tilde{M}_{f,e}$.

Proposition 7.5. *Let $f \in E$ and let M be Q-nice. Then we have a k-linear factorization*

$$M_\sigma \xrightarrow{\theta} M_{\sigma,f} := \bigoplus_{e \in E} \tilde{M}_{f,e} \cdot z^e \xrightarrow{\eta} (M_\sigma)_{z^f}$$

of the localization map, where the latter map η is injective.

Proof. For $(\mu \otimes r) \cdot z^e \in (M_e \otimes_k R) \cdot z^e \subset M_\sigma$, we set $\theta((\mu \otimes r)z^e) := [\mu \otimes r] \cdot z^e$ where $[\mu \otimes r] \in \tilde{M}_{f,e}$ is the image of $\mu \otimes r$ under the canonical map to the limit. Next let $\tilde{t}z^e \in M_{\sigma,f}$. Then $\tilde{t} \in \tilde{M}_{f,e}$, so there is a $k \in \mathbb{N}$ and a $t \in \tilde{M}_{e,k,f}$ with $[t] = \tilde{t}$. Set

$$\eta(\tilde{t}z^e) := tz^{e'}/z^{kf} \ ,$$

where $e' + q' = e + kf$. This does not depend on the choice of t, and it is indeed a well-defined group homomorphism. The composition $\eta \circ \theta$ is obviously the localization map, and it remains to show η injective. So assume $\eta(\sum \tilde{t}_i z^{e_i}) = 0$ (with all e_i distinct). Let $t_i \in \tilde{M}_{e_i,k_i,f}$ such that $[t_i] = \tilde{t}_i$ and write $e_i + k_i f = e'_i + q_i$. Then

$$0 = \eta(\sum \tilde{t}_i z^{e_i}) = \sum t_i z^{e'_i}/z^{k_i f} \ ,$$

whence $\sum z^{kf - k_i f + nf} \cdot (t_i z^{e'_i}) = 0$ for $k = \sum k_i$ and some $n \in \mathbb{N}$. Write $e_i + kf + nf = \hat{e}_i + \hat{q}_i$. The summand for i is contained in $(M_{\hat{e}_i} \otimes R) z^{\hat{e}_i}$. Since the e_i are all distinct, the $\hat{e}_i$ are all distinct, so $\tilde{t}_i = [t_i] = 0$. Therefore, η is injective. $\square$

It is particularly easy to compute $\tilde{M}_{f,e}$, if $e, f \in F \subset E$ are in a common face of P. Namely, then $e + nf \in E$, so $\ell_{n,m}(\mu \otimes r) = \phi_{e+nf,e+nf+mf,mf}(\mu) \otimes r$ which implies

$$\tilde{M}_{f,e} = \varinjlim_{n \to \infty} \tilde{M}_{e,n,f} = \varinjlim_{n \to \infty} (M_{e+nf} \otimes_k R) = (\varinjlim_{n \to \infty} M_{e+nf}) \otimes_k R \ .$$

With this preparation we turn to $c^* \mathcal{Q}^m_{X_h/X}$. For abbreviation, we define $\mathcal{Q} := \mathcal{Q}^m_{X_h/X}(X)$ and $\mathcal{A} = \{(1,0,1), (1,1,0), (1,1,1)\} \subset E$ where of course z^f is x, y, w respectively for $f \in \mathcal{A}$. The $k[P]$-module $\mathcal{Q}$ is Q-nice, and its pieces are

$$\mathcal{Q}_p = E^m(\mathfrak{a})_p / E^m(\mathfrak{a}_0)_p = \left[\bigwedge_k^m (P^{gp} \otimes k / Q^{gp} \otimes k) \right] / \left[\bigwedge_k^m (\bigcap_{p \in H_\ell} H_\ell^{gp} \otimes k / Q^{gp} \otimes k) \right] \ .$$

We see that the structure maps $\phi_{p,p+p',p'} : \mathcal{Q}_p \to \mathcal{Q}_{p+p'}$ from the Q-nice structure are surjective. To prove $\hat{\rho} : c^* \mathcal{Q}^m_{X_h/X}(Y) \to c^* \mathcal{Q}^m_{X_h/X}(V')$ injective, observe $c^* \mathcal{Q}^m_{X_h/X}(Y) = \mathcal{Q}_\sigma$. Now let

$$\mu = \sum_i t_i \cdot z^{e_i} \in \mathcal{Q}_\sigma$$

with $t_i \in \mathcal{Q}_{e_i} \otimes_k R$ and all e_i distinct, and assume $\hat{\rho}(\mu) = 0$. Because $V' = \bigcup_{f \in \mathcal{A}} D_Y(z^f)$, this is equivalent to $\hat{\rho}_f(\mu) = 0$ for every $f \in \mathcal{A}$ where $\hat{\rho}_f : \mathcal{Q}_\sigma \to (\mathcal{Q}_\sigma)_{z^f}$

is the localization map. By Proposition 7.5, it factors as

$$\mathcal{Q}_\sigma \xrightarrow{\theta_f} \mathcal{Q}_{\sigma,f} = \bigoplus_{e\in E} \tilde{\mathcal{Q}}_{f,e} \cdot z^e \xrightarrow{\eta_f} (\mathcal{Q}_\sigma)_{z^f}$$

with η_f injective whence we have $\theta_f(\mu) = 0$ for every $f \in \mathcal{A}$. This shows $[t_i] = 0 \in \tilde{\mathcal{Q}}_{f,e_i}$ for the canonical map

$$[-] : \tilde{\mathcal{Q}}_{e_i,0,f} = \mathcal{Q}_{e_i} \otimes_k R \to \tilde{Q}_{f,e_i} = \varinjlim \tilde{\mathcal{Q}}_{e_i,n,f} \ .$$

Thus to prove $\hat{\rho}$ injective, it suffices to show:

Lemma 7.6. *Let $t \in \mathcal{Q}_e \otimes_k R$ and assume $[t] = 0 \in \tilde{\mathcal{Q}}_{f,e}$ for every $f \in \mathcal{A}$. Then $t = 0$.*

Proof. Set $\mathcal{K}_{f,e} := \ker(\mathcal{Q}_e \otimes_k R \to \tilde{\mathcal{Q}}_{f,e})$. We show

$$\bigcap_{f\in\mathcal{A}} \mathcal{K}_{f,e} = 0 \ .$$

In case $e, f \in F \subset E$ are in a common face, $\tilde{\mathcal{Q}}_{f,e}$ is particularly simple, for it is the colimit of the directed system

$$\mathcal{Q}_e \otimes_k R \xrightarrow{\phi_{e,e+f,f}\otimes R} \mathcal{Q}_{e+f} \otimes_k R \xrightarrow{\phi_{e+f,e+2f,f}\otimes R} \mathcal{Q}_{e+2f} \otimes_k R \to ...$$

which is obtained by tensoring the directed system

$$\mathcal{Q}_e \xrightarrow{\phi_{e,e+f,f}} \mathcal{Q}_{e+f} \xrightarrow{\phi_{e+f,e+2f,f}} \mathcal{Q}_{e+2f} \to ...$$

The latter consists of finite dimensional vector spaces and surjective maps, so it becomes stationary. Indeed, for $n \geq 1$ we have $e + nf \in H_\ell$ iff $e, f \in H_\ell$ showing that $\mathcal{Q}_{e+f} = \mathcal{Q}_{e+2f} = ...$ Since the former directed system becomes stationary at the same object, we have

$$\mathcal{K}_{f,e} = \ker(\mathcal{Q}_e \otimes_k R \to \mathcal{Q}_{e+f} \otimes_k R) = \ker(\mathcal{Q}_e \to \mathcal{Q}_{e+f}) \otimes_k R \ .$$

We define

$$\mathcal{A}_e := \{f \in \mathcal{A} \mid e, f \in F \subset E \text{ are in a common face}\} \subset \mathcal{A}$$

and compute that

$$\begin{aligned}\bigcap_{f\in\mathcal{A}} \mathcal{K}_{f,e} \subset \bigcap_{f\in\mathcal{A}_e} \mathcal{K}_{f,e} &= \bigcap_{f\in\mathcal{A}_e} [\ker(\mathcal{Q}_e \to \mathcal{Q}_{e+f}) \otimes_k R] \\ &= [\bigcap_{f\in\mathcal{A}_e} \ker(\mathcal{Q}_e \to \mathcal{Q}_{e+f})] \otimes_k R\end{aligned}$$

We have

$$\ker(\mathcal{Q}_e \to \mathcal{Q}_{e+f}) = \left[\bigwedge_k^m (\bigcap_{e+f\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k)\right] / \left[\bigwedge_k^m (\bigcap_{e\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k)\right]$$

The intersection $\bigcap_{f\in\mathcal{A}_e}$ commutes with the quotient and also with the exterior power, so we get

$$\begin{aligned}&\bigcap_{f\in\mathcal{A}_e} \ker(\mathcal{Q}_e \to \mathcal{Q}_{e+f})\\ &= \left[\bigwedge_k^m (\bigcap_{f\in\mathcal{A}_e}\bigcap_{e+f\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k)\right] / \left[\bigwedge_k^m (\bigcap_{e\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k)\right]\end{aligned}$$

whence to show $\bigcap_{f\in\mathcal{A}} \mathcal{K}_{f,e} = 0$ it suffices to show that

$$\mathcal{B}_1(e) := \{H_\ell | \exists f\in\mathcal{A}_e : e+f\in H_\ell\} = \{H_\ell | e\in H_\ell\} =: \mathcal{B}_2(e)$$

for this implies

$$\bigcap_{f\in\mathcal{A}_e}\bigcap_{e+f\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k = \bigcap_{e\in H_\ell} H_\ell^{gp}\otimes k/Q^{gp}\otimes k\ .$$

The face H_ℓ runs through F_{tx}, F_{ty}, so $\mathcal{B}_1(e), \mathcal{B}_2(e) \subset \{F_{tx}, F_{ty}\}$. We distinguish four cases:

Case 1: $e\in F_{tx}, e\in F_{ty}$. Thus $e\in F_{tx}\cap F_{ty}\cap E = 0$, so $e=0$. We have $e+(1,0,1)\in F_{tx}$ and $e+(1,1,0)\in F_{ty}$, so $\mathcal{B}_1(e) = \mathcal{B}_2(e)$.

Case 2: $e\in F_{tx}, e\notin F_{ty}$. This means $\mathcal{B}_2(e) = \{F_{tx}\}$. We have $e+(1,0,1)\in F_{tx}$, but $e+f\notin F_{ty}$ since $e\notin F_{ty}$ showing that $\mathcal{B}_1(e) = \{F_{tx}\}$.

Case 3: $e\notin F_{tx}, e\in F_{ty}$. This means $\mathcal{B}_2(e) = \{F_{ty}\}$. We have $e+(1,1,0)\in F_{ty}$, but $e+f\notin F_{tx}$ since $e\notin F_{tx}$ showing that $\mathcal{B}_1(e) = \{F_{ty}\}$.

Case 4: $e\notin F_{tx}, e\notin F_{ty}$. We see $\mathcal{B}_2(e) = \emptyset$, and also $e+f\notin F_{tx}, F_{ty}$ since $e\notin F_{tx}, F_{ty}$.

In any case, we have $\mathcal{B}_1(e) = \mathcal{B}_2(e)$ showing the assertion. □

Putting the arguments together, we conclude:

Corollary 7.7. *The restriction $\hat\rho : c^*\mathcal{Q}^m_{X_h/X}(Y) \to c^*\mathcal{Q}^m_{X_h/X}(V')$ is injective.*

Moreover, we can conclude the base change theorem:

Theorem 7.8. *The homomorphism $\tau^m : c^*W^m_{X/S} \to W^m_{Y/T}$ constructed via adjunction from $\sigma^m : W^m_{X/S} \to c_*W^m_{Y/T}$ is an isomorphism.*

8 Appendix

8.1 Reflexive Differentials of $k[x,y]/(xy)$

In the introduction we claimed that the reflexive differentials of $k[x,y]/(xy)$ over k are not locally free. The proof is an easy computation which we give here.

Lemma 8.1. *Setting $R := k[x,y]/(xy)$, the R-module $(\Omega^1_{R/k})^{**}$ is not locally free.*

Proof. We start by computing $(\Omega^1_{R/k})^{**}$. We have

$$E := \Omega^1_{R/k} = (R \cdot dx \oplus R \cdot dy)/(ydx + xdy) \; .$$

Consider the R-modules

$$P_x := \left\{ \sum_{i \geq 1} \alpha_i x^i \mid \alpha_i \in k \right\} = \mathrm{Ann}(y) \subset R$$

$$P_y := \left\{ \sum_{i \geq 1} \alpha_i y^i \mid \alpha_i \in k \right\} = \mathrm{Ann}(x) \subset R$$

and define a homomorphism

$$\eta : E^* \to P_x \oplus P_y \; , \phi \mapsto (\phi(dx), \phi(dy))$$

It is well-defined since for $\phi : E \to R$, the element

$$y\phi(dx) = \phi(-xdy) = -x\phi(dy) \in R$$

is in $(x) \cap (y) = 0$. It is straightforward to check η an isomorphism. For the next step, note the isomorphisms

$$\theta_x : \mathrm{Hom}(P_x, R) \to P_x, \phi \mapsto \phi(x)$$

$$\theta_y : \mathrm{Hom}(P_y, R) \to P_y, \phi \mapsto \phi(y)$$

which yield an isomorphism $E^{**} \cong P_x \oplus P_y$.

Let $\mathcal{E} := \tilde{E}$ be the sheaf on $X = \mathrm{Spec}\; R$ associated to E. Assume $\mathcal{E}^{**}$ was locally free. Then it was locally free of constant rank since X is connected, and considering the smooth locus, we see that $\mathrm{rk}(\mathcal{E}^{**}) = 1$, for X is a (reducible) curve. Therefore also for the maximal ideal $\mathfrak{m}_0 := (x,y)$, we obtain

$$\dim(E^{**} \otimes R/\mathfrak{m}_0) = 1 \; .$$

But we see that

$$\dim(E^{**} \otimes R/\mathfrak{m}_0) = \dim(P_x \otimes R/\mathfrak{m}_0) + \dim(P_y \otimes R/\mathfrak{m}_0) = 2$$

showing that E^{**} is not locally free. □

8.2 Basic Theory of Monoids

We introduce some basic theory of monoids as a starter for the reader unfamiliar with monoids. A *monoid* is a set M together with a binary operation $+ : M \times M \to M$ which is associative and commutative. There is an element $0 \in M$ such that $m+0 = m$ for any $m \in M$. Such an element is necessarily unique. Examples of monoids include:

- $M_1 = \{0\}$ with $0 + 0 = 0$
- $M_2 = \mathbb{N}$ with the usual addition
- $M_3 = \{0, 1\}$ with $0 + 0 := 0,\ 0 + 1 := 1,\ 1 + 1 := 1$
- $M_4 = G$ any abelian group
- $M_5 = \mathbb{N}^r$ with the usual addition for $r \geq 0$

If $n \in \mathbb{N}$ and $m \in M$, then we denote by $nm := n \cdot m := m + ... + m$ the n-fold sum.

If M, M' are two monoids, then a *monoid homomorphism* $\theta : M \to M'$ is a map $\theta : M \to M'$ of sets satisfying $\theta(m + n) = \theta(m) + \theta(n)$ for all $m, n \in M$ and $\theta(0) = 0$. Unlike in the case of groups, the second condition of a homomorphism does not follow from the first, as the example of the map $M_1 \to M_3, 0 \mapsto 1$ shows which is not a monoid homomorphism.

A monoid M is called *finitely generated*, if there is a surjective monoid homomorphism $\mathbb{N}^r \to M$ for some $r \geq 0$. It is called *integral*, if for $m, n, n' \in M$, we have that $m + n = m + n'$ implies $n = n'$. A monoid that is both finitely generated and integral is called *fine*.

If M is a monoid, then we may consider equivalence classes of pairs (m, m') with $m, m' \in M$ where $(m, m') \sim (n, n')$ iff there is $k \in M$ with $m + n' + k = m' + n + k$. With the obvious addition, these pairs form an abelian group M^{gp} which is called the *Grothendieck group* of M. There is a canonical monoid homomorphism

$$M \to M^{gp}, \quad m \mapsto (m, 0) \ ,$$

and any monoid homomorphism $M \to G$ for an abelian group G factors uniquely through it. A monoid M is integral precisely if $M \to M^{gp}$ is injective. If $\theta : M \to M'$ is a monoid homomorphism, then the universal property of the Grothendieck group yields a group homomorphism $\theta^{gp} : M^{gp} \to (M')^{gp}$.

An integral monoid M is called *saturated*, if for any $m \in M, p \in M^{gp}$ and $n \in \mathbb{N}_{>0}$ with $(m, 0) = n \cdot p$ we have that $p \in M$, i.e. $p = (p', 0)$ for some (unique) $p' \in M$. A monoid M is called *toric*, if it is fine, saturated and $M^{gp} \cong \mathbb{Z}^r$ for some $r \geq 0$. The name is motivated by the fact that such monoids arise in toric geometry. Namely, if $C \subset \mathbb{R}^n$ is a rational convex polyhedral cone, then $C \cap \mathbb{Z}^n$ is a toric monoid.

8.3 RLDGAs and Functoriality for $\Omega^\bullet_{X/Y}$

The goal of this section is to establish functoriality of the log de Rham complex for morphisms $f : X \to Y$ of coherent log schemes. We found it nowhere in the literature, so we do it here. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{c} & X \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y' & \xrightarrow{b} & Y \end{array}$$

of coherent log schemes. Functoriality means that we have a homomorphism

$$\sigma^\bullet_{X'/X} : \Omega^\bullet_{X/Y} \to c_*\Omega^\bullet_{X'/Y'}$$

of complexes of abelian sheaves. This homomorphism should be compatible with composition in the sense that for a cartesian diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{c'} & X' & \xrightarrow{c} & X \\ {\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y'' & \xrightarrow{b'} & Y' & \xrightarrow{b} & Y \end{array}$$

we should have $c_*\sigma^\bullet_{X''/X'} \circ \sigma^\bullet_{X'/X} = \sigma^\bullet_{X''/X} : \Omega^\bullet_{X/Y} \to (c \circ c')_*\Omega^\bullet_{X''/Y''}$. Our approach to it is via a universal property of the whole log de Rham complex, not only of the differentials. Modelled on the behaviour of the log de Rham complex, we define the following notion:

Definition 8.2. *Let $f : X \to Y$ be a morphism of log schemes. Then a* relative log differential graded algebra *for X/Y, rldga for short, is a quintuple $(\mathcal{E}^\bullet, D^\bullet, \wedge, \sigma, \Delta)$ s.t.*

- *$\mathcal{E}^m$ is a sheaf of abelian groups for $m \geq 0$.*
- *$D^m : \mathcal{E}^m \to \mathcal{E}^{m+1}$ is a homomorphism of sheaves of abelian groups.*
- $D^{m+1} \circ D^m = 0$
- *$\wedge : \mathcal{E}^m \times \mathcal{E}^n \to \mathcal{E}^{m+n}$ defines an associative and graded commutative product.*
- *$D^{m+n}(e \wedge e') = D^m(e) \wedge e' + (-1)^m e \wedge D^n(e')$ for $e \in \mathcal{E}^m, e' \in \mathcal{E}^n$.*

In particular $(\mathcal{E}^0, \wedge)$ is a commutative ring, and any $\mathcal{E}^m$ is a module over it.

- *$\sigma : \mathcal{O}_X \to \mathcal{E}^0$ is a ring homomorphism.*
- *$\Delta : \mathcal{M}_X \to \mathcal{E}^1$ is a monoid homomorphism (for the group structure on $\mathcal{E}^1$).*
- *$(D^0 \circ \sigma, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}^1$ is a relative log derivation for X/Y.*

- $D^1 \circ \Delta = 0$

If $\mathcal{E} = (\mathcal{E}^\bullet, D_E^\bullet, \wedge, \sigma_E, \Delta_E)$ and $\mathcal{F} = (\mathcal{F}^\bullet, D_F^\bullet, \wedge, \sigma_F, \Delta_F)$ are two rldgas, then a morphism is a family of group homomorphisms $\phi^m : \mathcal{E}^m \to \mathcal{F}^m$ compatible with differentials and the product s.t. $\sigma_F = \phi^0 \circ \sigma_E$ and $\Delta_F = \phi^1 \circ \Delta_E$. Rldgas for X/Y form a category $RLDGA(X/Y)$.

The homomorphism $\sigma : \mathcal{O}_X \to \mathcal{E}^0$ turns every $\mathcal{E}^m$ into an $\mathcal{O}_X$-module. The primary example of an rldga is the log de Rham complex of a morphism $f : X \to Y$ of coherent log schemes:

Proposition 8.3. *Let $f : X \to Y$ be a morphism of coherent log schemes. Then the log de Rham complex $(\Omega^\bullet_{X/Y}, d^\bullet, \wedge, \mathrm{id}, \delta)$ is an rldga for X/Y.*

Proof. This is straightforward using Proposition 3.4. □

We also have a direct image construction for rldgas. Namely, consider a cartesian square of coherent log schemes

$$\begin{array}{ccc} X' & \xrightarrow{c} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y' & \xrightarrow{b} & Y \end{array}$$

and let $\mathcal{E} = (\mathcal{E}^\bullet, D^\bullet, \wedge, \sigma, \Delta)$ be an rldga for X'/Y'. Then $c_*(\mathcal{E}^\bullet, D^\bullet)$ is a complex of abelian sheaves having moreover an associative and graded commutative product. We set $\sigma_{c_*\mathcal{E}} := c_*\sigma \circ c^\flat : \mathcal{O}_X \to c_*\mathcal{O}_{X'} \to c_*\mathcal{E}^0$ and obtain $\Delta_{c_*\mathcal{E}}$ as composition $c_*\Delta \circ c^\flat : \mathcal{M}_X \to c_*\mathcal{M}_{X'} \to c_*\mathcal{E}^1$. All the conditions are more or less easy to check, so this is indeed an rldga for X/Y. The direct image construction extends to morphisms of rldgas in the obvious way yielding a functor $c_* : RLDGA(X'/Y') \to RLDGA(X/Y)$. The reader may easily check:

Lemma 8.4. *For a cartesian square*

$$\begin{array}{ccccc} X'' & \xrightarrow{c'} & X' & \xrightarrow{c} & X \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y'' & \xrightarrow{b'} & Y' & \xrightarrow{b} & Y \end{array}$$

of coherent log schemes, we have $(c \circ c')_* = c_* \circ c'_* : RLDGA(X''/Y'') \to RLDGA(X/Y)$ as equality of functors (not just up to natural transformation).

The direct image construction applies in particular to the log de Rham complex $\Omega^\bullet_{X'/Y'}$ yielding more examples of rldgas for X/Y.

We only introduced the notion of rldga to get a universal property of the log de Rham complex. Explicitly, we state it as follows:

Theorem 8.5. *Let $f : X \to Y$ be a morphism of coherent log schemes, and let $(\mathcal{E}^\bullet, D^\bullet)$ be an rldga. Then there is a unique morphism of rldgas $\phi^\bullet : (\Omega^\bullet_{X/Y}, d^\bullet) \to (\mathcal{E}^\bullet, D^\bullet)$.*

As a corollary, this theorem yields the functoriality of the log de Rham complex. Namely, we obtain a homomorphism

$$\sigma^\bullet_{X'/X} : (\Omega^\bullet_{X/Y}, d^\bullet_Y) \to c_*(\Omega^\bullet_{X'/Y'}, d^\bullet_{Y'})$$

of complexes of abelian sheaves s.t. $\sigma^0_{X'/X}$ coincides with $\mathcal{O}_X \to c_*\mathcal{O}_{X'}$ and $\sigma^1_{X'/X}$: $\Omega^1_{X/Y} \to c_*\Omega^1_{X'/Y'}$ is the homomorphism factoring the log derivation

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r] & c_*\mathcal{O}_{X'} \arrow[r, "c_* d_{Y'}"] & c_*\Omega^1_{X'/Y'} \\
\mathcal{M}_X \arrow[u] \arrow[r] & c_*\mathcal{M}_{X'} \arrow[u] \arrow[ur, "c_*\delta_{Y'}"'] &
\end{tikzcd}$$

These homomorphisms are compatible with the composition of cartesian squares, for both $c_*\sigma^\bullet_{X''/X'} \circ \sigma^\bullet_{X'/X}$ and $\sigma^\bullet_{X''/X}$ are morphisms of rldgas $\Omega^\bullet_{X/Y} \to (c \circ c')_*\Omega^\bullet_{X''/Y''}$.

The Proof of the Theorem

We start the proof of Theorem 8.5 with existence. First we set $\phi^0 := \sigma : \mathcal{O}_X \to \mathcal{E}^0$. Since $(D^0 \circ \sigma, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}^1$ is a relative log derivation, we obtain a module homomorphism $\phi^1 : \Omega^1_{X/Y} \to \mathcal{E}^1$ satisfying $\phi^1 \circ d^0 = D^0 \circ \sigma$ and $\phi^1 \circ \delta = \Delta$. The former equality means that we have a commutative diagram

$$\begin{tikzcd}
\Omega^1_{X/Y} \arrow[r, "\phi^1"] & \mathcal{E}^1 \\
\mathcal{O}_X \arrow[u, "d^0"] \arrow[r, "\phi^0=\sigma"] & \mathcal{E}^0 \arrow[u, "D^0"']
\end{tikzcd}$$

To construct ϕ^m for $m \geq 2$, consider

$$\psi : T(\Omega^1_{X/Y}) = \bigoplus_{m\geq 0} (\Omega^1_{X/Y})^{\otimes m} \to \bigoplus_{m \geq 0} \mathcal{E}^m,$$
$$\omega_1 \otimes ... \otimes \omega_m \mapsto \phi^1(\omega_1) \wedge ... \wedge \phi^1(\omega_m)$$

from the tensor algebra $T(\Omega^1_{X/Y})$. It is a homomorphism of associative algebras and $\psi(\omega \otimes \omega) = 0$ for any $\omega \in \Omega^1_{X/Y}$ whence it defines a homomorphism

$$\phi : \Lambda^\bullet \Omega^1_{X/Y} \to \bigoplus_{m \geq 0} \mathcal{E}^m$$

which decomposes into homomorphisms $\phi^m : \Omega^m_{X/Y} \to \mathcal{E}^m$. Obviously, for $m = 0, 1$, we recover ϕ^0, ϕ^1 as previously defined. We summarize:

Lemma 8.6. *We have $\mathcal{O}_X$-module homomorphisms $\phi^m : \Omega^m_{X/Y} \to \mathcal{E}^m$ satisfying*

$\phi^{m+n}(\omega \wedge \omega') = \phi^m(\omega) \wedge \phi^n(\omega')$ *for* $\omega \in \Omega^m_{X/Y}, \omega' \in \Omega^n_{X/Y}$.

Since we have already shown that $\sigma = \phi^0 \circ \mathrm{id}$ and $\Delta = \phi^1 \circ \delta$, it remains to show $\phi^\bullet$ compatible with the differentials, i.e. $\phi^{m+1} \circ d^m = D^m \circ \phi^m$. We saw this above for $m = 0$. For any m, it suffices to consider the stalks. For the case $m = 1$, we need the following preparation:

Lemma 8.7. *Let $x \in X$ be a point and $\omega \in (\Omega^1_{X/Y})_x$. Then there are $g_i, f_i, h_j \in \mathcal{O}_{X,x}$ and $m_j \in \mathcal{M}_{X,x}$ with*

$$\omega = \sum_i g_i d^0 f_i + \sum_j h_j \delta(m_j) \ .$$

Proof. We have the explicit description

$$\Omega^1_{X/Y} = \Big(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})\Big)/\mathcal{K} \ .$$

On the stalk in x, the projection map $W := \Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}) \to \Omega^1_{X/Y}$ is surjective, and the maps d, δ factor through W via $d(f) = (df, 0)$ and $\delta(m) = (0, 1 \otimes m)$. Thus it suffices to give such a decomposition for any $w \in W_x$. To do so, decompose $w = \tilde{\omega} + \tilde{m}$ with $\tilde{\omega} \in \Omega^1_{\underline{X}/\underline{Y}}$ and $\tilde{m} \in \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp}$. The elements $d(f)$ generate the classical differentials $\Omega^1_{\underline{X}/\underline{Y}}$, and any element in $\mathcal{M}^{gp}_{X,x}$ is a difference of elements in $\mathcal{M}_{X,x}$ showing the claim. □

This enables us to show the commutativity of the diagram

$$\begin{array}{ccc}
\Omega^2_{X/Y} & \xrightarrow{\phi^2} & \mathcal{E}^2 \\
{\scriptstyle d^1}\uparrow & & {\scriptstyle D^1}\uparrow \\
\Omega^1_{X/Y} & \xrightarrow{\phi^1} & \mathcal{E}^1
\end{array}$$

Namely both compositions are linear, and it suffices to show equality on the stalks for elements of the form $gd^0 f$ and $h\delta(m)$. Using the axioms of an rldga, we compute

$$\begin{aligned}
\phi^2 d^1(gd^0 f) &= \phi^2(d^0 g \wedge d^0 f + g d^1 d^0 f) = \phi^1 d^0 g \wedge \phi^1 d^0 f \\
&= D^0 \phi^0 g \wedge \phi^1 d^0 f + \phi^0 g \wedge D^1 \phi^1 d^0 f = D^1(\phi^0 g \wedge \phi^1 d^0 f) \\
&= D^1 \phi^1 (g d^0 f)
\end{aligned}$$

and

$$\begin{aligned}
\phi^2 d^1(h\delta m) &= \phi^2(d^0 h \wedge \delta m + h \wedge d^1 \delta m) = \phi^1 d^0 h \wedge \phi^1 \delta m \\
&= D^0 \phi^0 h \wedge \Delta m = D^0 \phi^0 h \wedge \Delta m + \phi^0 h \wedge D^1 \Delta m \\
&= D^1(\phi^0 h \wedge \phi^1 \delta m) = D^1 \phi^1 (h \delta m)
\end{aligned}$$

We conclude that $\phi^2 \circ d^1 = D^1 \circ \phi^1 : \Omega^1_{X/Y} \to \mathcal{E}^2$ as homomorphisms of abelian sheaves. Finally we turn to the case $m \geq 2$. On the stalks, every element $w \in (\Omega^m_{X/Y})_x$ can be written as a finite sum $w = \sum_i \omega_1^i \wedge \cdots \wedge \omega_m^i$ with $\omega_k^i \in (\Omega^1_{X/Y})_x$. Thus it suffices to show

$$\phi^{m+1} d^m(\omega_1 \wedge \dots \wedge \omega_m) = D^m \phi^m(\omega_1 \wedge \dots \wedge \omega_m) \ .$$

We calculate

$$\begin{aligned}
\phi^{m+1} d^m(\omega_1 \wedge \dots \wedge \omega_m) &= \phi^{m+1}(\sum_{i=1}^m (-1)^{i+1} \omega_1 \wedge \dots \wedge d^1\omega_i \wedge \dots \wedge \omega_m) \\
&= \sum_{i=1}^m (-1)^{i+1} \phi^1\omega_1 \wedge \dots \wedge \phi^2 d^1\omega_i \wedge \dots \wedge \phi^1\omega_m \\
&= \sum_{i=1}^m (-1)^{i+1} \phi^1\omega_1 \wedge \dots \wedge D^1\phi^1\omega_i \wedge \dots \wedge \phi^1\omega_m \\
&= D^m(\phi^1\omega_1 \wedge \dots \wedge \phi^1\omega_m) = D^m\phi^m(\omega_1 \wedge \dots \wedge \omega_m)
\end{aligned}$$

and see that indeed $\phi^{m+1} \circ d^m = D^m \circ \phi^m : \Omega^m_{X/Y} \to \mathcal{E}^{m+1}$ as homomorphisms of abelian sheaves. This completes the proof of existence.

To prove uniqueness, assume $\phi^\bullet : (\Omega^\bullet_{X/Y}, d^\bullet) \to (\mathcal{E}^\bullet, D^\bullet)$ is another morphism of rldgas. We have $\psi^0 \circ \mathrm{id}_{\mathcal{O}_X} = \sigma : \mathcal{O}_X \to \mathcal{E}^0$, so $\psi^0 = \phi^0$. Since $\psi^\bullet$ is compatible with the $\wedge$-product, every ψ^m is not only a group homomorphism, but an $\mathcal{O}_X$-module homomorphism. Moreover, we have

$$\psi^1 \circ d^0 = D^0 \circ \psi^0 = D^0 \circ \phi^0 = \phi^1 \circ d^0$$

and $\psi^1 \circ \delta = \Delta = \phi^1 \circ \delta$, so $\psi^1 = \phi^1$ is the unique module homomorphism factoring the log derivation $(D^0 \circ \phi^0, \Delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \mathcal{E}^1$. Now we have

$$\psi^m(\omega_1 \wedge \dots \wedge \omega_m) = \psi^1\omega_1 \wedge \dots \wedge \psi^1\omega_m = \phi^1\omega_1 \wedge \dots \wedge \phi^1\omega_m = \phi^m(\omega_1 \wedge \dots \wedge \omega_m)$$

showing that $\psi^m = \phi^m$ since every element of $\Omega^m_{X/Y}$ is locally a sum of elements of the form $\omega_1 \wedge \dots \wedge \omega_m$. This concludes the proof of the theorem.

The Adjoint of $\sigma^\bullet_{X'/X}$

Each map $\sigma^m_{X'/X} : \Omega^m_{X/Y} \to c_*\Omega^m_{X'/Y'}$ is indeed an $\mathcal{O}_X$-module homomorphism, so it admits an adjoint

$$\tau^m_{X'/X} : c^*\Omega^m_{X/Y} \to c^*c_*\Omega^m_{X'/Y'} \to \Omega^m_{X'/Y'} \ .$$

These adjoints are compatible with the graded multiplication, so we obtain a homomorphism

$$\tau^\bullet_{X'/X} : c^*\Lambda^\bullet\Omega^1_{X/Y} \to \Lambda^\bullet\Omega^1_{X'/Y'}$$

of graded (commutative) algebras. There is an isomorphism $\Lambda^\bullet c^*\Omega^1_{X/Y} \cong c^*\Lambda^\bullet\Omega^1_{X/Y}$ which is the identity in degree 1, so we have a homomorphism

$$\Lambda^\bullet c^*\Omega^1_{X/Y} \to \Lambda^\bullet\Omega^1_{X'/Y'} .$$

Proposition 3.6 shows that it is an isomorphism in degree 1, so by generalities on $\Lambda^\bullet(-)$, we see that $\tau^\bullet_{X'/X}$ is an isomorphism. This proves:

Proposition 8.8. *For a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "c"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd}$$

of coherent log schemes, we have that $\tau^m_{X'/X} : c^*\Omega^m_{X/Y} \to \Omega^m_{X'/Y'}$ *is an isomorphism.*

8.4 Overview of Notation

In the introduction (and later on) we fixed some notation for the whole paper. Here we give an overview over these definitions.

k	a field
A	$:= k[t]$
B	$:= k[x, y, t, w]/(xy - tw)$
C	$:= k[x, y, t, w]$
F	$:= xy - tw \in C$
$\phi : A \to B$	$t \mapsto t$
B_0	$:= B \otimes_A A/(t)$
X	$:= \text{Spec } B$
S	$:= \text{Spec } A$
$f : X \to S$	defined by ϕ
S_0	$:= \{t = 0\} = \{0\} \subset S$
S^+	$:= S \setminus S_0$
X_0	$:= \text{Spec } B_0$ is the central fiber of f
$\mathfrak{m}_P$	$:= (x, y, t, w) \subset B$ an ideal
$P \in X$	point defined by $\mathfrak{m}_P$; may be considered $P \in X_0 \subset X$
Q	$:= \mathbb{N}$ as a monoid
P	$:= \langle (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1) \rangle \subset \mathbb{Z}^3$ submonoid induced by these four vectors; is not the same as the point P
$\theta : Q \to P$	$n \mapsto (n, 0, 0)$
U	$:= X \setminus \{P\}$ an open subset
U'	$:= D(x) \cup D(y) \cup D(w) \subset X$ a smaller open subset
Z	$:= \{xy = 0\} = \{tw = 0\} \subset X$ a closed subset
X^+	$:= X \setminus Z$ an open subset
$\mathcal{M}_S$	compactifying log structure on S defined by $\{0\} \subset \mathbb{A}^1 = S$
$\mathcal{M}_X$	compactifying log structure on X defined by $X_0 \subset X$
$\mathcal{H}$	compactifying log structure on X defined by $Z \subset X$
X_h	$:= (X, \mathcal{H})$ log scheme with $\mathcal{H}$
$\gamma : X_h \to X$	morphism defined by $\mathcal{M}_X \subset \mathcal{H}$
$h : X_h \to S$	composition $f \circ \gamma$ with $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$

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