Good Differential Forms for a Singular Family of Schemes

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The Family

We consider the family

$$f : X := \mathrm{Spec}\ k[x,y,t,w]/(xy - tw) \to \mathrm{Spec}\ k[t] =: S$$

of schemes defined by $t \mapsto t$. On $S^* := S \setminus \{0\}$, the family is constant with fiber $X_s \cong \mathbb{A}^2$ whence smooth. Contrarily, the central fiber $X_0 \cong \mathrm{Spec}\ k[x,y,w]/(xy)$ is the union of two planes meeting in a line, whence singular. We are interested in the relative de Rham complex

$$... \to 0 \to \mathcal{O}_X \xrightarrow{d^0} \Omega^1_{X/S} \xrightarrow{d^1} \Omega^2_{X/S} \xrightarrow{d^2} \Omega^3_{X/S} \xrightarrow{d^3} \Omega^4_{X/S} \to ...$$

constructed from the Kähler differential forms $\Omega^1_{X/S}$. Setting $B := \mathcal{O}_X(X) = k[x,y,t,w]/(xy - tw)$ the Kähler differential forms are explicitly given by

$$\Omega^1_{X/S}(X) \cong (B \cdot dx \oplus B \cdot dy \oplus B \cdot dw)/(ydx + xdy - tdw)$$

Not surprisingly, the properties of the de Rham complex of this family are not as good as of the de Rham complex of a smooth family. E.g. we have $\Omega^4_{X/S} = 0$, but $\Omega^3_{X/S} \neq 0$ is a torsion sheaf supported on X_0. Slogan:

$$\boxed{\Omega^\bullet_{X/S} \textbf{ is bad}}$$

We like to replace $\Omega^\bullet_{X/S}$ by another complex $W^\bullet_{X/S}$ that has better properties. We are not only interested in the family $f : X \to S$, but also in its fibers, so we like to have a complex $W^\bullet_{Y/T}$ for every cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{c} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{b} & S \end{array}$$

at once. We will call these complexes the complexes of *good differential forms*.

Good Differential Forms

What are the properties that good differential forms should satisfy? Inspired by the behaviour of smooth morphisms and by reflexive differential forms, we wish to have the following list of properties:

- $W^\bullet_{Y/T}$ is concentrated in degrees $0, 1, 2$.
- We have $W^0_{Y/T} = \mathcal{O}_Y$.
- The differential $d^m : W^m_{Y/T} \to W^{m+1}_{Y/T}$ is $g^{-1}\mathcal{O}_T$-linear, and $d^0_T : \mathcal{O}_Y \to W^1_{Y/T}$ is a derivation.
- (Reflexivity) The sheaves of modules $W^m_{Y/T}$ are reflexive.
- (Functoriality) We have commutative diagrams

$$\begin{array}{ccc} c_*W^m_{Y/T} & \xrightarrow{d^m_T} & c_*W^{m+1}_{Y/T} \\ {\scriptstyle \sigma^m}\uparrow & & \uparrow{\scriptstyle \sigma^{m+1}} \\ W^m_{X/S} & \xrightarrow{d^m_S} & W^{m+1}_{X/S} \end{array}$$

- (Base Change) The map σ^m yields an isomorphism $c^*W^m_{X/S} \cong W^m_{Y/T}$ via adjunction.
- (Kähler Property) Whenever Y/T is smooth, the canonical homomorphism $\Omega^1_{Y/T} \to W^1_{Y/T}$ constructed from the fact that $d^0_T : \mathcal{O}_Y \to W^1_{Y/T}$ is a $g^{-1}\mathcal{O}_T$-linear derivation is an isomorphism.

The Construction of Good Differential Forms

Our approach is to construct good differential forms for the family $f : X \to S$ using log geometry. In a first step, we like to see $f : X \to S$ as a morphism $f : X^\dagger \to S^\dagger$ of log schemes, and then we like to construct $W^\bullet_{X/S}$ from the relative log differential forms $\Omega^1_{X^\dagger/S^\dagger}$.
The restricted morphism $f : X^* = X \setminus X_0 \to S^*$ is smooth, so by the Kähler property, good differential forms satisfy

$$\Omega^1_{X^*/S^*} \cong (W^1_{X/S})|_{X^*} \ .$$

Therefore we like to choose the log structures such that $S^\dagger|_{S^*}$ and $X^\dagger|_{X^*}$ are (log)trivial since this implies $\Omega^1_{X^*/S^*} \cong (\Omega^1_{X^\dagger/S^\dagger})|_{X^*}$, and the most straightforward choices are the divisorial log structures defined by $S_0 \subset S$ and $X_0 \subset X$ which are the complements of the (log)trivial loci, i.e. to define

$$\mathcal{M}_X(W) := \{g \in \mathcal{O}_X(W) | g|_{W \cap X^*} \in \mathcal{O}^*_X(W \cap X^*)\}$$

for an open $W \subset X$, and analogously for S.
The log differentials $\Omega^1_{X^\dagger/S^\dagger}$ are not quite the correct choice for $W^1_{X/S}$. The sheaf $\Omega^1_{X^\dagger/S^\dagger}$ is locally free on $U := X \setminus \{P\}$ where $P \in X$ is the point defined by $x = y = t = w = 0$, but it turns out that

$$\Omega^1_{X^\dagger/S^\dagger} \text{ is neither quasi-coherent nor of finite type}$$

around P. We remedy this situation by defining $W^1_{X/S} := j_*\Omega^1_{U^\dagger/S^\dagger}$ where $j : U \subset X$ is the inclusion. Now we can also define the whole complex

$$\boxed{W^\bullet_{X/S} := j_*\Omega^\bullet_{U^\dagger/S^\dagger}}$$

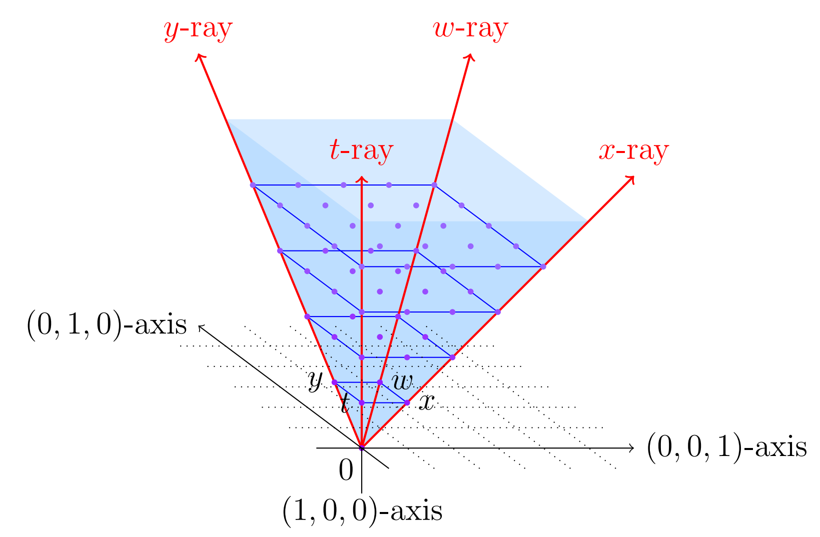
since we have a log de Rham complex $\Omega^\bullet_{U^\dagger/S^\dagger}$ on U (whereas $\Omega^\bullet_{X^\dagger/S^\dagger}$ on the whole space X may not have the structure of a complex since the differential is usually constructed only for coherent log schemes). The complexes $W^\bullet_{Y/T}$ for a base extension are defined analogously, using the open $V := T \times_S U \subset Y$.

A Nice Description by Monoids

To prove the properties of $W^\bullet_{X/S}$, in particular the base change property, we need an explicit description of the family $f : X \to S$ in 'combinatorial' terms. First of all, we need two monoids

- $Q := \mathbb{N}$
- $P := \langle (1,0,0), (1,0,1), (1,1,0), (1,1,1) \rangle \subset \mathbb{Z}^3$ the submonoid generated by the four vectors.

We also have a monoid homomorphism $\theta : Q \to P, n \mapsto (n,0,0)$. This situation is visualized as follows:



Now mapping $k[t] \to k[Q], t \mapsto z^1$ and

$$\begin{array}{c} k[x,y,t,w]/(xy - tw) \to k[P] \\ x \mapsto z^{(1,0,1)} \\ y \mapsto z^{(1,1,0)} \\ t \mapsto z^{(1,0,0)} \\ w \mapsto z^{(1,1,1)} \end{array}$$

we obtain a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}\ k[P] & \xrightarrow{\cong} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\ k[Q] & \xrightarrow{\cong} & S \end{array}$$

The Coherent Hull

Another crucial ingredient to prove the base change property is the coherent hull. The log differentials $\Omega^1_{X^\dagger/S^\dagger}$ are not a coherent sheaf, so the log structure $\mathcal{M}_X$ is not coherent. We like to relate it to a coherent log structure. Namely, set

$$Z := \{xy = 0\} = \{tw = 0\} \subset X \ .$$

We have $X_0 \subset Z$, and Z also defines a divisorial log structure $\mathcal{H}_X$ on X, giving rise to a log scheme $M^\dagger = (X, \mathcal{H}_X)$. Since $X_0 \subset Z$, we obtain an inclusion $\mathcal{M}_X \subset \mathcal{H}_X$ of log structures, a morphism $h : M^\dagger \to X^\dagger$ of log schemes and hence a canonical map

$$\boxed{\Omega^1_{X^\dagger/S^\dagger} \to \Omega^1_{M^\dagger/S^\dagger}}$$

Moreover, $\Omega^1_{M^\dagger/S^\dagger}$ turns out to be locally free. Therefore we call $M^\dagger$ a *coherent hull* around $X^\dagger$. After applying $\Lambda^\bullet$ and $j_*(-)$, the above map yields a universally exact sequence

$$0 \to W^\bullet_{X/S} \xrightarrow{\ell^\bullet_{X/S}} \Omega^\bullet_{M^\dagger/S^\dagger} \to \mathcal{Q}^\bullet_{M/X} \to 0$$

which means that it remains exact after any base change $b : T \to S$. Here $\mathcal{Q}^\bullet_{M/X}$ denotes the cokernel of the module homomorphism. Writing $c : Y \to X$ for the projection of the base change, we see that the map

$$\rho : c^*W^m_{X/S} \to j_*(c^*W^m_{X/S}|_V)$$

induced by restriction to $V \xrightarrow{j} Y$ is injective since $c^*\Omega^m_{M^\dagger/S^\dagger} \to j_*(c^*\Omega^m_{M^\dagger/S^\dagger})|_V$ is bijective. To show ρ an isomorphism, it suffices to show the corresponding restriction map of $c^*\mathcal{Q}^m_{M/X}$ injective. We achieve this by explicit calculation using the monoid description.

Explicit Calculations

Using the description in terms of monoids, we can compute the good differentials:

Theorem 1. *We have isomorphisms*

$$\Gamma(X, \Omega^m_{M^\dagger/S^\dagger}) \cong \bigoplus_{p \in P} z^p \cdot \Lambda^m_k(P^{gp} \otimes k / Q^{gp} \otimes k)$$

and

$$\Gamma(X, W^m_{X/S}) \cong \bigoplus_{p \in P} z^p \cdot \Lambda^m_k\Big(\bigcap_{p \in H_\ell} H^{gp}_\ell \otimes k / Q^{gp} \otimes k\Big)$$

where the intersection is over $H_\ell \in \{F_{tx}, F_{ty}\}$ (we intersect the terms for those H_ℓ that satisfy $p \in H_\ell$). The canonical homomorphism $\ell^m_{X/S} : W^m_{X/S} \to \Omega^m_{M^\dagger/S^\dagger}$ corresponds to the inclusions

$$\Lambda^m_k\Big(\bigcap_{p \in H_\ell} H^{gp}_\ell \otimes k / Q^{gp} \otimes k\Big) \subset \Lambda^m_k(P^{gp} \otimes k / Q^{gp} \otimes k) \ .$$

In this theorem, $F_{tx} \subset P$ is the face generated by $t, x \in P$ and $F_{ty} \subset P$ is the face generated by $t, y \in P$. We can think of these modules as k-vector spaces sitting on the points of the monoid P. We use this explicit description to complete the proof of the base change property as explained on the left. We get:

Main Theorem: *The complexes $W^\bullet_{Y/T}$ are good differential forms.*