Almost Log Smooth Families in the Gross-Siebert Program and Beyond

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Mirror Symmetry

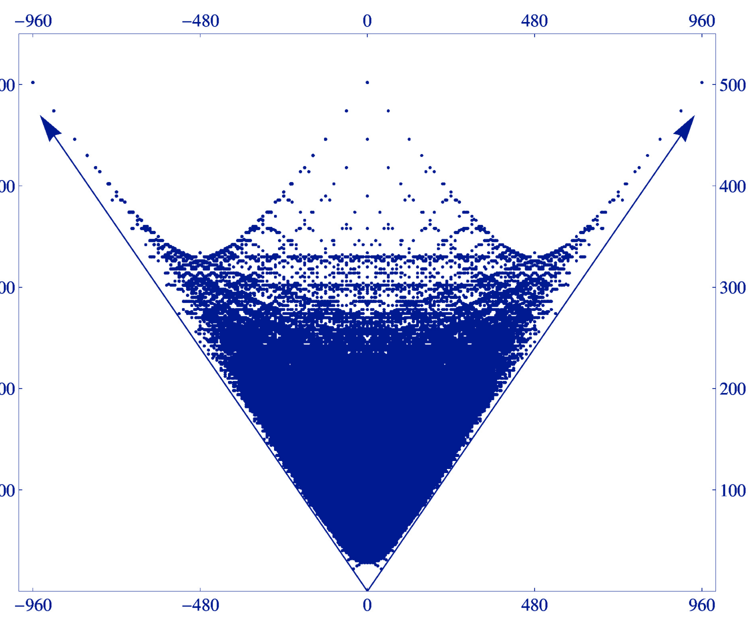
Consider a complex manifold X, that is a space glued from patches $U_\alpha \subset \mathbb{C}^n$, and assume that it is Calabi-Yau which is a technical condition on volume forms. We are interested in the cohomology groups $H^{p,q}$. They encode information on the geometry of X in a rather subtle way. The $H^{p,q}$ are finite-dimensional vector spaces, and we may write their dimensions $h^{p,q}$ in a diagram called Hodge diamond, e.g.

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{1,2} & & h^{2,1} & \\ h^{0,2} & & h^{1,1} & & h^{2,0} \\ & h^{0,1} & & h^{1,0} & \\ & & h^{0,0} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The right hand side is the Hodge diamond for a so called K3 surface, named after Kummer, Kähler, Kodaira and in analogy to the mountain K2. We may find pairs $(X, \check{X})$ of spaces with $h^{p,q}(X) = h^{n-p,q}(\check{X})$ for n the (complex) dimension of X and $\check{X}$, called mirror pairs since their Hodge diamond is mirrored. This may not be very interesting for K3 surfaces since their Hodge diamond is already symmetric, but there is for instance a pair of quintic threefolds with the following Hodge diamonds:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

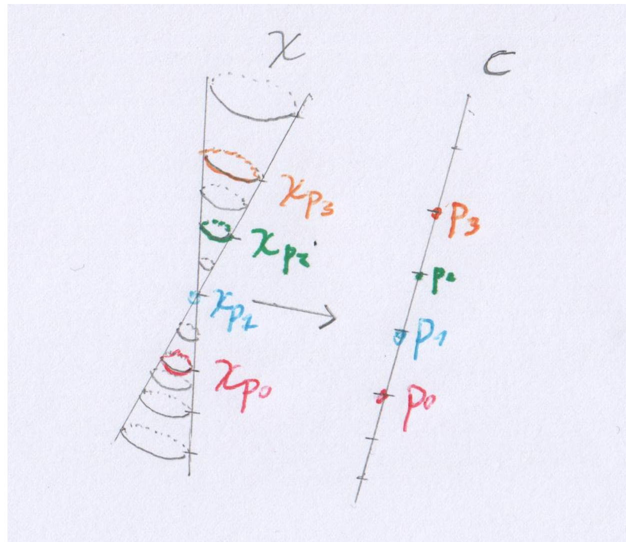
This may seem to be by chance, but we can find a lot of these examples. Indeed, a lot of threefolds are accessible by computation. We may plot a point $(2h^{1,1} - 2h^{1,2}, h^{1,1} + h^{1,2})$ for each threefold of them and obtain a diagram. We expect it to be symmetric along a vertical axis (since we always have $h^{p,q} = h^{q,p}$), and that's what we observe:



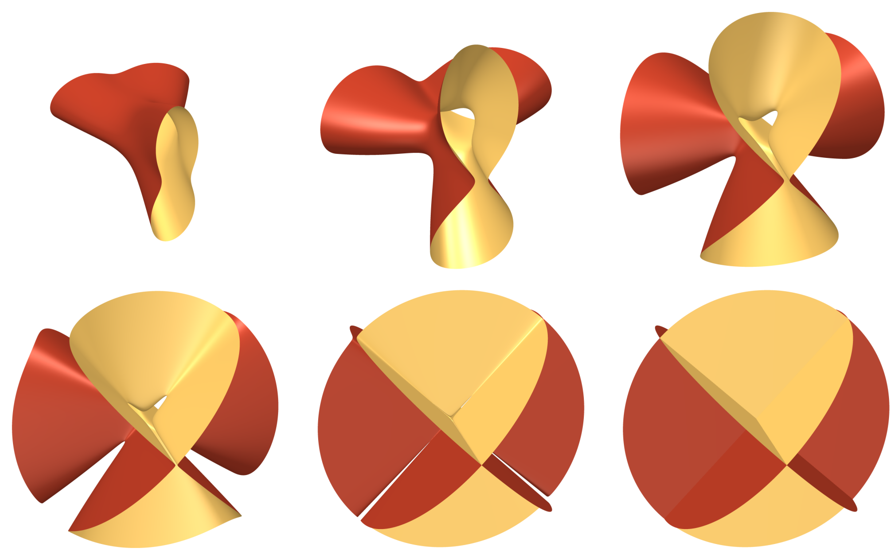
So we wonder why these mirror pairs exist, what is the underlying connection between them?

The Gross-Siebert Approach

The Gross-Siebert program is an approach to explain the existence of mirror pairs which uses degenerations. The idea of a degeneration is to deform an object to a simpler object, preserving the relevant properties. This is formalized by a curve C and a space $\mathcal{X}_p$ over each point $p \in C$ of the curve. A very simple (and not quite correct) example is the following degeneration:



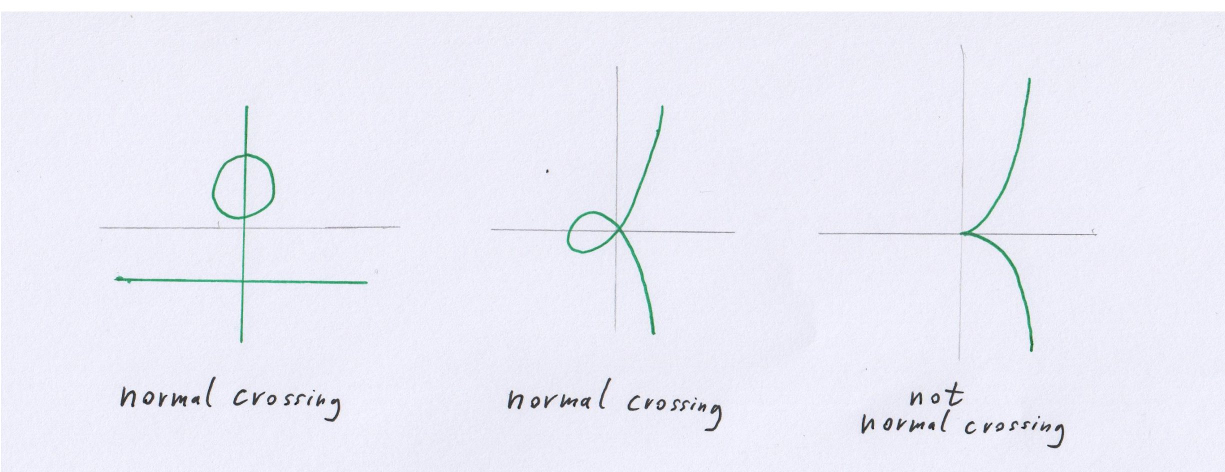
Here, we have a circle over each point, and the circles 'vary nicely' in the sense that all the circles together form a cone, which is called the total space of the degeneration. Now let us consider a toric degeneration. It consists very roughly of the open disk $\Delta \subset \mathbb{C}$ (it is a curve over $\mathbb{C}$ for it has 1 complex dimension) and a Calabi-Yau manifold $\mathcal{X}_p$ for every point $p \in \Delta$ such that they 'vary nicely' with $p \in \Delta$. For $p \neq 0$, $\mathcal{X}_p$ is a smooth manifold, but $\mathcal{X}_0$ is not. It is a degenerate CY obtained by gluing toric varieties. Toric varieties are a class of manifolds that can be described by combinatorial data.



The pictures show different fibers of a degeneration. The degenerate fiber on the right in the lower line is a union of planes, the other fibers are smooth. We like to compare X and the mirror $\check{X}$ by putting both in a toric degeneration and comparing the degenerate fibers. In a toric degeneration, the numbers $h^{p,q}$ become functions on Δ since we have one for each fiber, and they are constant for $p \neq 0$, but it may attain another value on $p = 0$. Thus if we can find a new definition for $H^{p,q}$ such that the function $h^{p,q}$ is constant, then it suffices to compare the degenerate fibers to explain mirror symmetry.

Smooth and Log Smooth Spaces

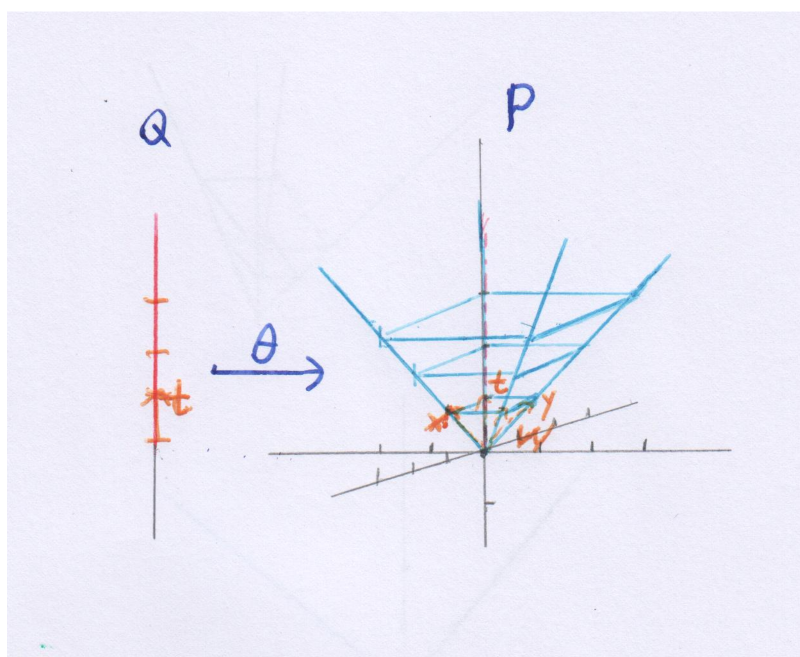
Consider again the degenerate fiber of the family above. It is not smooth in the intersection lines of the planes, so we may just take the complement of the lines to construct a new $H^{p,q}$. However, it turns out that this usually yields an infinite dimensional $H^{p,q}$, so the situation is even worse. We need a genuinely new definition of $H^{p,q}$. The key concept is that of a logarithmic structure: A logarithmic pair (X, D) consists of a space X and a closed subspace $D \subset X$.



The above pictures show examples of log pairs. We have $X = \mathbb{C}^2$ the ambient plane and D is the subspace drawn in green. A log pair is smooth, if the underlying space X is smooth, and if D is normal crossing. The first two examples are normal crossing, the third one is not. Furthermore, any space Y with a map $f : Y \to X$ is considered a log space via this map. Now setting $X = \mathcal{X}$ the total space of the degeneration and $D = \mathcal{X}_0$ the central fiber $(\mathcal{X}, \mathcal{X}_0)$ is a log pair, and every fiber $\mathcal{X}_p$ becomes a log space via the embedding $\mathcal{X}_p \to \mathcal{X}$. This log structure can be used to define a new $H^{p,q}_{log}$.

- For $p \neq 0$, we have $\mathcal{X}_p \cap D = \emptyset$, so the log structure on $\mathcal{X}_p$ is considered trivial. We have $H^{p,q}_{log} = H^{p,q}$.
- D is normal crossing, so $(\mathcal{X}, D)$ is log smooth, wherever the total space $\mathcal{X}$ is smooth.
- $\mathcal{X}$ is smooth only outside some small set $Z \subset \mathcal{X}_0$ of singularities. Thus $\mathcal{X}_0$ is log smooth only on $U = \mathcal{X}_0 \setminus Z$.
- Z is small enough to use $H^{p,q}_{log}(U)$ as a good replacement for $H^{p,q}(\mathcal{X}_0)$.

In the example, Z is a finite set of isolated points, so it is indeed smaller than the singular set of $\mathcal{X}_0$ which consists of lines. The same phenomenon also occurs in the following example:



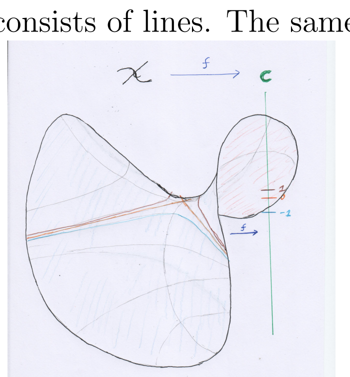
We have a smooth saddle surface $\mathcal{X}$ mapping to a line C. All fibers $\mathcal{X}_p$ with $p \neq 0$ are smooth, indeed they are hyperbolas. The central fiber $\mathcal{X}_0$ is the union of two lines, so it has a singularity. Thus although $f : \mathcal{X} \to C$ has a singular fiber, the total space $\mathcal{X}$ is smooth.

Almost Log Smooth Families

From the observation above, we introduce the notion of an almost log smooth family which (at least morally) generalizes the degenerations of Gross-Siebert. An almost log smooth family consists roughly of:

- A base space S (e.g. a curve).
- A space $\mathcal{X}_p$ with a log structure for each point $p \in S$, forming a total space $\mathcal{X}$.
- A large enough open subset $U \subset \mathcal{X}$, such that $\mathcal{X}_p \cap U$ is log smooth.

That it is not log smooth everywhere explains the name 'almost'. We want to classify all almost log smooth families, and this is already possible locally. Consider a cone P in $\mathbb{R}^n$ and the embedding of a subcone Q, and assume that they are spanned by integral vectors:



The cone P is spanned by the integral vectors x, y, t, w, faces are spanned by for instance two of them, like t, x. Q is also a cone, spanned by one vector t. The embedding θ maps t in Q to t in P.

There is a natural way to associate to these data an almost log smooth family. The first idea to do so is to interpret integral vectors in the cone as functions on a space. The faces will then describe (functions on) hypersurfaces, and these hypersurfaces give the log structure. And indeed, *every* almost log smooth family looks locally like the almost log smooth families constructed from cones. This gives a local description of als families which is important to settle deeper questions. One of these is the following - it is still open:

Conjecture: *The numbers $h^{p,q}_{log}(p) = \dim H^{p,q}_{log}(\mathcal{X}_p \cap U)$ are for every als family a constant function on the base S.*

This is the case for some class of families in the Gross-Siebert program, but we conjecture it more general.