Correlation Measures between Two Variables: Covariance and Correlation Coefficient

Variance for Single Variable

- The variance of a random variable *X* provides a measure of how much the value of *X* deviates from the mean or expected value of *X:*

$$\sigma^2 = \mathrm{var}(X) = E[(X-\mu)^2] = \begin{cases} \sum_x (x-\mu)^2 f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

  - where σ^2 is the variance of X, σ is called *standard deviation*

    μ is the mean, and μ = E[X] is the expected value of X
  - That is, variance is the expected value of the square deviation from the mean
  - It can also be written as: $\sigma^2 = \mathrm{var}(X) = E[(X-\mu)^2] = E[X^2] - \mu^2 = E[X^2] - [E(x)]^2$
- Sample variance is the average squared deviation of the data value x_i from the sample mean $\hat{\mu}$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

Covariance for Two Variables

- Covariance between two variables X_1 and X_2

$$\sigma_{12} = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E[X_1X_2] - \mu_1\mu_2 = E[X_1X_2] - E[X_1]E[X_2]$$

where $\mu_1 = E[X_1]$ is the respective mean or **expected value** of X_1; similarly for μ_2

- Sample covariance between X_1 and X_2: $\hat{\sigma}_{12} = \frac{1}{n}\sum_{i=1}^{n}(x_{i1} - \hat{\mu}_1)(x_{i2} - \hat{\mu}_2)$
- Sample covariance is a generalization of the sample variance:

$$\hat{\sigma}_{11} = \frac{1}{n}\sum_{i=1}^{n}(x_{i1} - \hat{\mu}_1)(x_{i1} - \hat{\mu}_1) = \frac{1}{n}\sum_{i=1}^{n}(x_{i1} - \hat{\mu}_1)^2 = \hat{\sigma}_1^{\,2}$$

- **Positive covariance:** If $\sigma_{12} > 0$
- **Negative covariance**: If $\sigma_{12} < 0$
- **Independence**: If X_1 and X_2 are independent, $\sigma_{12} = 0$ but the reverse is not true
  - Some pairs of random variables may have a covariance 0 but are not independent
  - Only under some additional assumptions (e.g., the data follow multivariate normal distributions) does a covariance of 0 imply independence

Example: Calculation of Covariance

- Suppose two stocks X_1 and X_2 have the following values in one week:
  - (2, 5), (3, 8), (5, 10), (4, 11), (6, 14)
- Question: If the stocks are affected by the same industry trends, will their prices rise or fall together?
- Covariance formula

$$\sigma_{12} = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E[X_1 X_2] - \mu_1 \mu_2 = E[X_1 X_2] - E[X_1]E[X_2]$$

- Its computation can be simplified as: $\sigma_{12} = E[X_1 X_2] - E[X_1]E[X_2]$
  - $E(X_1) = (2 + 3 + 5 + 4 + 6)/ 5 = 20/5 = 4$
  - $E(X_2) = (5 + 8 + 10 + 11 + 14) /5 = 48/5 = 9.6$
  - $\sigma_{12} = (2\times5 + 3\times8 + 5\times10 + 4\times11 + 6\times14)/5 - 4 \times 9.6 = 4$
- Thus, X_1 and X_2 rise together since $\sigma_{12} > 0$

Correlation between Two Numerical Variables

- **Correlation** between two variables X_1 and X_2 is the standard covariance, obtained by normalizing the covariance with the standard deviation of each variable

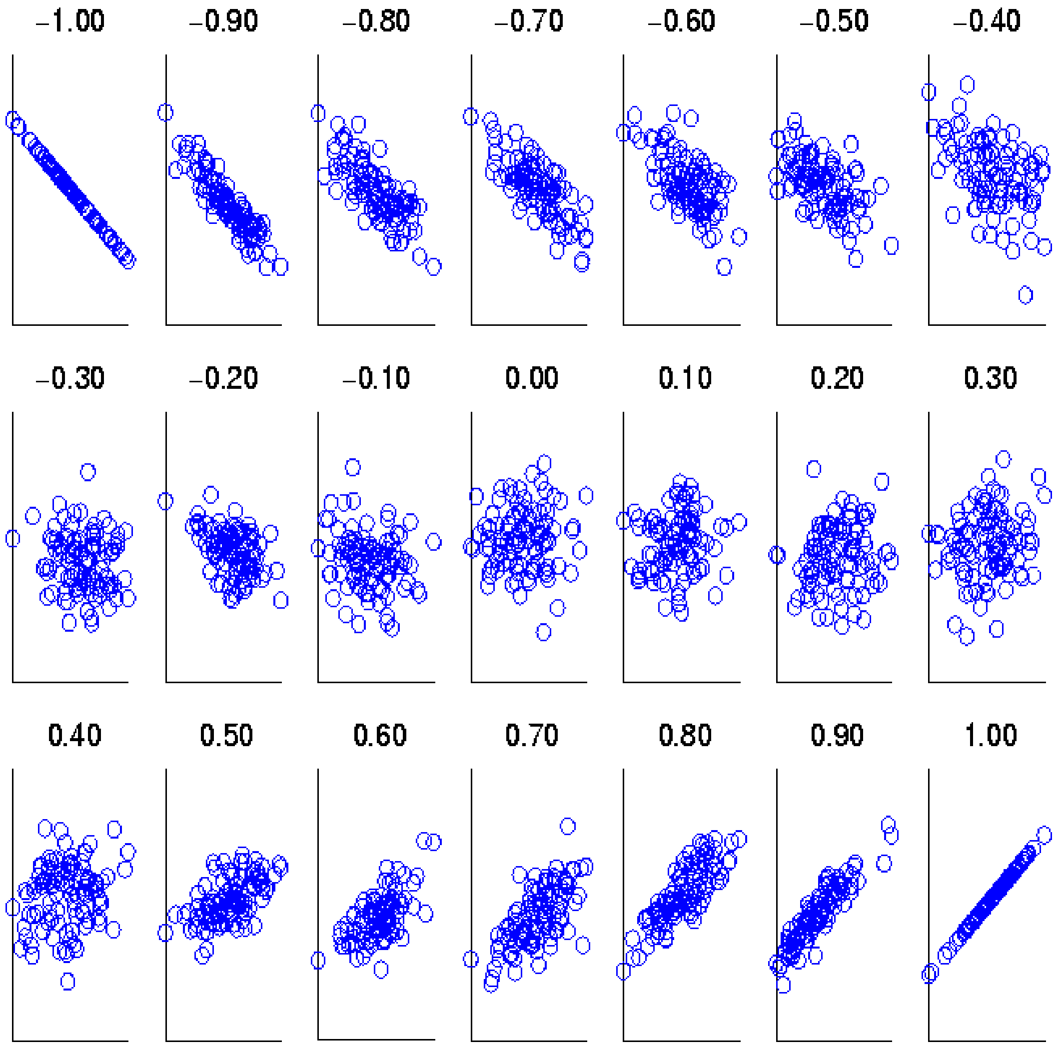
$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{\sigma_{12}}{\sqrt{\sigma_1^2 \sigma_2^2}}$$

- **Sample correlation** for two attributes X_1 and X_2: $$\hat{\rho}_{12} = \frac{\hat{\sigma}_{12}}{\hat{\sigma}_1 \hat{\sigma}_2} = \frac{\sum_{i=1}^{n} (x_{i1} - \hat{\mu}_1)(x_{i2} - \hat{\mu}_2)}{\sqrt{\sum_{i=1}^{n} (x_{i1} - \hat{\mu}_1)^2 \sum_{i=1}^{n} (x_{i2} - \hat{\mu}_2)^2}}$$

  where n is the number of tuples, μ_1 and μ_2 are the respective means of X_1 and X_2, σ_1 and σ_2 are the respective standard deviation of X_1 and X_2

- If $\rho_{12} > 0$: A and B are positively correlated (X_1's values increase as X_2's)
  - The higher, the stronger correlation
- If $\rho_{12} = 0$: independent (under the same assumption as discussed in co-variance)
- If $\rho_{12} < 0$: negatively correlated

Visualizing Changes of Correlation Coefficient



- Correlation coefficient value range: [–1, 1]
- A set of scatter plots shows sets of points and their correlation coefficients changing from –1 to 1

Covariance Matrix

- The variance and covariance information for the two variables X_1 and X_2 can be summarized as 2 X 2 covariance matrix as

$$\Sigma = E[(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T] = E[\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{pmatrix}(X_1-\mu_1 \quad X_2-\mu_2)]$$

$$= \begin{pmatrix} E[(X_1-\mu_1)(X_1-\mu_1)] & E[(X_1-\mu_1)(X_2-\mu_2)] \\ E[(X_2-\mu_2)(X_1-\mu_1)] & E[(X_2-\mu_2)(X_2-\mu_2)] \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

- Generalizing it to *d* dimensions, we have,

$$D = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dd} \end{pmatrix} \qquad \boldsymbol{\Sigma} = E[(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T] = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1d} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1} & \sigma_{d2} & \cdots & \sigma_d^2 \end{pmatrix}$$

Recommended Readings

- L. Kaufman and P. J. Rousseeuw, Finding Groups in Data: An Introduction to Cluster Analysis, John Wiley & Sons, 1990
- Mohammed J. Zaki and Wagner Meira, Jr.. Data Mining and Analysis: Fundamental Concepts and Algorithms. Cambridge University Press, 2014
- Jiawei Han, Micheline Kamber, and Jian Pei. Data Mining: Concepts and Techniques. Morgan Kaufmann, 3rd ed. , 2011
- Charu Aggarwal and Chandran K. Reddy (eds.). Data Clustering: Algorithms and Applications. CRC Press, 2014