STAT 578: Advanced Bayesian Modeling

Week 7 – Lesson 2

Gibbs Sampling

Fall 2019

A Gibbs Sampler Derivation

Normal Sample

Consider this simple Bayesian model for a normal sample:

$$\begin{aligned} y_1, \ldots, y_n \mid \mu, \sigma^2 &\sim \text{iid N}(\mu, \sigma^2) \\ \mu &\sim \text{N}(\mu_0, \tau_0^2) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

Note: μ and σ^2 are independent under the prior.

Prior is not conjugate but is partially conjugate for both μ and σ^2.

Conditional Posteriors

To implement the Gibbs sampler, need to sample from

$$p(\mu \mid \sigma^2, y) \qquad \text{and} \qquad p(\sigma^2 \mid \mu, y).$$

By partial conjugacy, these densities will be normal and scaled inverse chi-square, which are easy to directly sample.

For Mean

The conditional posterior for μ is just the posterior for μ assuming σ^2 is fixed.

Recall, for the mean-only normal model,

$$\mu \mid \sigma^2, y \;\sim\; \mathrm{N}\big(\mu_n(\sigma^2),\, \tau_n^2(\sigma^2)\big)$$

where

$$\mu_n(\sigma^2) \;=\; \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \qquad\qquad \frac{1}{\tau_n^2(\sigma^2)} \;=\; \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

For Variance

We could use formulas for the variance-only model (BDA3, Sec. 2.6), but let's derive directly, for illustration ...

Recall likelihood:

$$p(y \mid \mu, \sigma^2) \quad \propto \quad \frac{1}{(\sigma^2)^{n/2}} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \exp\left(-\frac{n}{2\sigma^2}(\mu - \bar{y})^2\right)$$

Prior density (BDA3, Table A.1):

$$p(\sigma^2) \quad \propto \quad (\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \qquad\qquad \sigma^2 > 0$$

To identify the conditional posterior, note that it is proportional to the full posterior:

$$p(\sigma^2 \mid \mu, y) \quad \propto \quad p(\mu \mid y)\, p(\sigma^2 \mid \mu, y) \quad = \quad p(\mu, \sigma^2 \mid y)$$

where the proportionality is in σ^2 only.

Thus, in σ^2,

$$\begin{aligned} p(\sigma^2 \mid \mu, y) &\propto p(\mu, \sigma^2 \mid y) \\ &\propto p(\mu)\, p(\sigma^2)\, p(y \mid \mu, \sigma^2) \\ &\propto p(\sigma^2)\, p(y \mid \mu, \sigma^2) \end{aligned}$$

Substituting,

$$
\begin{aligned}
p(\sigma^2 \mid \mu, y) \;&\propto\; (\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \\
&\qquad \cdot \frac{1}{(\sigma^2)^{n/2}} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \exp\left(-\frac{n}{2\sigma^2}(\mu-\bar{y})^2\right) \\
&\propto\; (\sigma^2)^{-(\nu_n/2+1)} \exp\left(-\frac{\nu_n\sigma_n^2(\mu)}{2\sigma^2}\right) \qquad\qquad \sigma^2 > 0
\end{aligned}
$$

where

$$
\nu_n \;=\; \nu_0 + n \qquad\qquad \nu_n\sigma_n^2(\mu) \;=\; \nu_0\sigma_0^2 + (n-1)s^2 + n(\mu-\bar{y})^2
$$

This is the $\mathrm{Inv}\text{-}\chi^2\big(\nu_n,\, \sigma_n^2(\mu)\big)$ density.

Algorithm

To use this Gibbs sampler:

Choose starting values μ^0, $(\sigma^2)^0$

For $t = 1, 2, \ldots$

1. Sample
$$\mu^t \mid (\sigma^2)^{t-1} \quad \sim \quad \mathrm{N}\Big(\mu_n\big((\sigma^2)^{t-1}\big),\, \tau_n^2\big((\sigma^2)^{t-1}\big)\Big)$$
2. Sample
$$(\sigma^2)^t \mid \mu^t \quad \sim \quad \text{Inv-}\chi^2\big(\nu_n,\, \sigma_n^2(\mu^t)\big)$$

For implementation, note

$$\text{Inv-}\chi^2\big(\nu_n,\, \sigma_n^2(\mu)\big) \quad = \quad \text{Inv-gamma}\big(\nu_n/2,\, \nu_n\sigma_n^2(\mu)/2\big).$$