ADVANCED BAYESIAN MODELING

Varying-Coefficient Normal-Theory Model

For grouped regression data

$$y_{ij} \qquad X_{ij} \;=\; (x_{ij,1}, \ldots, x_{ij,K}) \qquad i = 1, \ldots, n_j \quad j = 1, \ldots, J$$

consider varying-coefficient regression

$$\mathrm{E}(y_{ij} \mid \theta, X_{ij}) \;\;=\;\; X_{ij}\beta^{(j)}$$

As in random effect models, often useful to treat groups as randomly sampled, so coefficient vectors

$$\beta^{(1)}, \;\ldots,\; \beta^{(J)}$$

are independent from the same distribution.

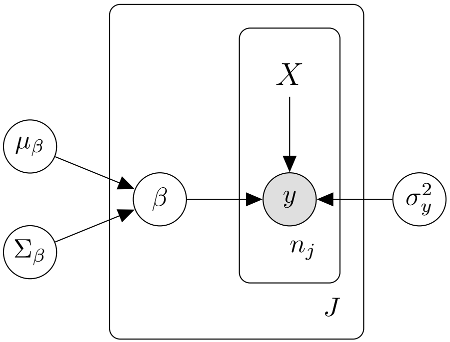
Normal-Theory Model

Consider sampling model

$$
\begin{aligned}
y_{ij} \mid \beta^{(j)}, \sigma_y^2, X_{ij} &\sim \text{indep. } \mathrm{N}\big(X_{ij}\beta^{(j)},\, \sigma_y^2\big) \\
\beta^{(j)} \mid \mu_\beta, \Sigma_\beta &\sim \text{iid } \mathrm{N}(\mu_\beta,\, \Sigma_\beta)
\end{aligned}
$$

Why make $\beta^{(j)}$ multivariate normal? Convenience, and partial conjugacy

DAG Model



Priors and hyperpriors? Later.

As a Mixed Model

The normal varying-coefficient model is a special case of a normal-theory mixed model:

$$y \mid \beta_f, \beta_r, \sigma_y^2, X_f, X_r \quad \sim \quad \mathrm{N}(X_f\beta_f + X_r\beta_r,\ \sigma_y^2 I)$$

- X_f has rows that are the X_{ij}s
- $\beta_f \;=\; \mu_\beta$
- X_r has just the pure interactions between all of the group indicators and all of the variables in X_f
- β_r contains all of the vectors $\beta^{(j)} - \mu_\beta$

Coefficient Covariance

$$\mathrm{var}\left(\beta^{(j)} \mid \mu_\beta, \Sigma_\beta\right) \;=\; \Sigma_\beta$$

What form should Σ_β have?

Should it be a multiple of the identity matrix?

No – that would assume all coefficients have same variance.

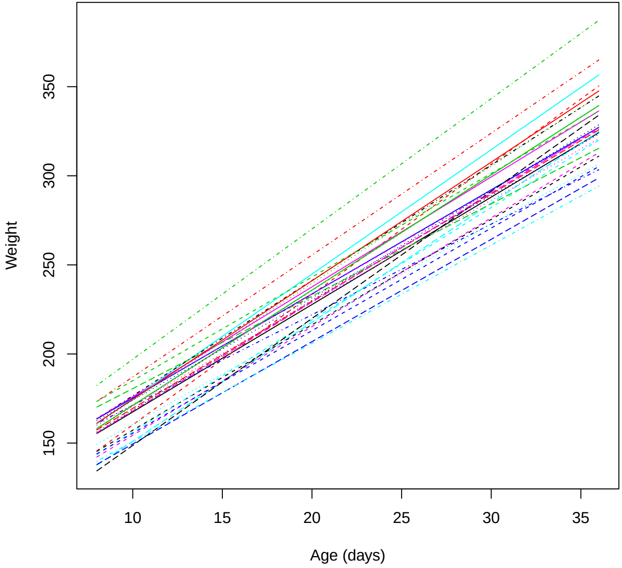
Should it be diagonal?

That would assume coefficients are independent, but ...

Consider again fitted simple linear regressions for rat growth curves ...

Rats with larger birth weights (intercepts) also seem to grow a bit faster (larger slopes).

So the intercept and slope of a rat are probably positively correlated.



In general, we must allow Σ_β to be any arbitrary (symmetric, positive definite) covariance matrix.

In the rat growth example, it could be

$$\Sigma_\beta \quad = \quad \begin{pmatrix} \sigma^2_{\beta_1} & \rho\,\sigma_{\beta_1}\sigma_{\beta_2} \\ \rho\,\sigma_{\beta_1}\sigma_{\beta_2} & \sigma^2_{\beta_2} \end{pmatrix}$$

where $-1 < \rho < 1$ is correlation between intercept and slope.

Prior

What should priors be? Prefer noninformative choices:

$$
\begin{aligned}
p(\sigma_y^2) &\propto (\sigma_y^2)^{-1} \qquad \sigma_y^2 > 0 \\
p(\mu_\beta) &\propto 1 \\
p(\Sigma_\beta) &\propto \text{?}
\end{aligned}
$$

Is there a natural choice for the Σ_β prior? ...