ADVANCED BAYESIAN MODELING

Conjugate Prior Analysis

Review

Normal sample:

$$y = (y_1, \ldots, y_n)$$

$$y_1, \ldots, y_n \mid \mu, \sigma^2 \sim \text{iid N}(\mu, \sigma^2)$$

With σ^2 known, likelihood becomes

$$p(y \mid \mu) \propto \exp\left(-\frac{n}{2\sigma^2}(\mu - \bar{y})^2\right) \qquad -\infty < \mu < \infty$$

Look familiar?

Normal Prior

Try a normal prior distribution for μ:

$$\mu \sim \mathrm{N}(\mu_0, \tau_0^2)$$

$$p(\mu) \propto \exp\left(-\frac{1}{2\tau_0^2}(\mu - \mu_0)^2\right)$$

Then (Bayes' rule)

$$\begin{aligned} p(\mu \mid y) &\propto p(\mu)\, p(y \mid \mu) \\ &\propto \exp\left(-\frac{1}{2\tau_0^2}(\mu - \mu_0)^2 - \frac{n}{2\sigma^2}(\mu - \bar{y})^2\right) \end{aligned}$$

Completing the square,

$$p(\mu \mid y) \;\; \propto \;\; \exp\left(-\frac{1}{2\tau_n^2}(\mu - \mu_n)^2\right)$$

where

$$\mu_n \;\; = \;\; \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \qquad\qquad \frac{1}{\tau_n^2} \;\; = \;\; \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

So normal prior is conjugate for the normal mean-only model:

$$\mu \mid y \;\; \sim \;\; \mathrm{N}(\mu_n, \tau_n^2)$$

Remarks:

- A **precision** is the reciprocal of a variance, such as

$$\frac{1}{\tau_0^2} \qquad \frac{n}{\sigma^2} \qquad \frac{1}{\tau_n^2}$$

  (σ^2/n is sampling variance of $\bar{y}$)

- Posterior mean μ_n is weighted average of prior mean μ_0 and sample mean $\bar{y}$, with their precisions as weights

Example: Flint Data

$$y_i \quad = \quad \textit{logarithm} \text{ of first-draw lead level (ppb), for observation } i$$

```
> (n <- nrow(Flintdata))
[1] 271

> (ybar <- mean(log(Flintdata$FirstDraw)))
[1] 1.402925

> (s.2 <- var(log(Flintdata$FirstDraw)))
[1] 1.684078
```

So

$$n \;=\; 271 \qquad\qquad \bar{y} \;\approx\; 1.40 \qquad\qquad s^2 \;\approx\; 1.684$$

For demonstration, set

$$\sigma^2 \;=\; s^2 \;\approx\; 1.684$$

Choose prior mean

$$\mu_0 \;=\; \log(3) \;\approx\; 1.10$$

(approx. log-scale median from earlier official study)

Choose prior variance

$$\tau_0^2 \;=\; \sigma^2 \;\approx\; 1.684$$

(making the prior equivalent to one extra observation)

Compute posterior:

```
> sigma.2 <- s.2
> mu0 <- log(3)
> tau.2.0 <- sigma.2

> (mun <- (mu0/tau.2.0 + n*ybar/sigma.2) / (1/tau.2.0 + n/sigma.2))
[1] 1.401807

> (tau.2.n <- 1 / (1/tau.2.0 + n/sigma.2))
[1] 0.006191465
```

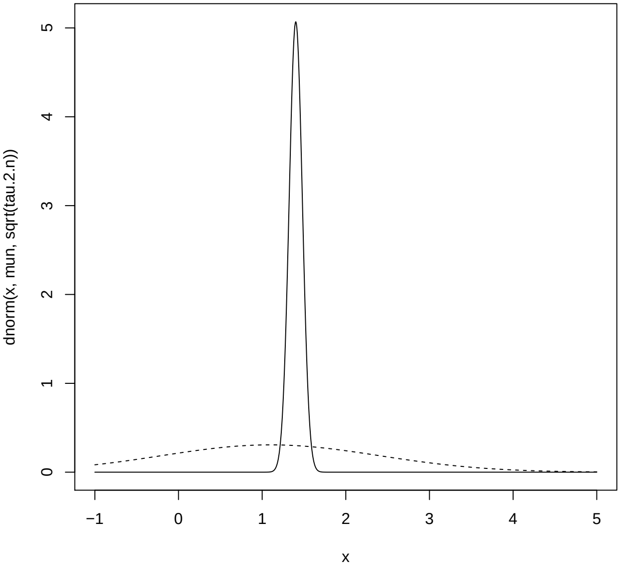
$$\mathrm{E}(\mu \mid y) \;=\; \mu_n \;\approx\; 1.40 \qquad\qquad \mathrm{var}(\mu \mid y) \;=\; \tau_n^2 \;\approx\; 0.0062$$

Posterior mean similar to $\bar{y}$.

Posterior variance much smaller than prior variance.

Plotting posterior and prior densities:

```
> curve(dnorm(x,mun,sqrt(tau.2.n)), -1, 5, n=1000)  # posterior
> curve(dnorm(x,mu0,sqrt(tau.2.0)), -1, 5, add=TRUE, lty=2)  # conjugate prior
```



dnorm(x, mun, sqrt(tau.2.n))
0
1
2
3
4
5
−1
0
1
2
3
4
5
x

95% posterior interval for μ:

$$\mu_n \;\pm\; 1.96 \cdot \sqrt{\tau_n^2}$$

```
> mun + c(-1,1) * 1.96 * sqrt(tau.2.n)
[1] 1.247582 1.556031
```

Can transform back to original (ppb) scale:

```
> exp(mun + c(-1,1) * 1.96 * sqrt(tau.2.n))
[1] 3.481915 4.739971
```

Note: <u>Not</u> mean on original scale, but possibly median