Independence

Recall independence of events A and B:

$$\Pr(A \cap B) \;=\; \Pr(A)\,\Pr(B)$$

Conceptually, this means A and B are probabilistically unrelated.

We now extend this notion to random variables.

Independent Random Variables

Formally, random variables U and V are **independent** if

$$\Pr(U \in D, V \in G) \;=\; \Pr(U \in D)\,\Pr(V \in G)$$

for *all* possible D and G.

Conceptually, U and V are probabilistically unrelated:

Any probability statement about U and V jointly should depend only on their marginal distributions.

If U and V are independent and have marginal densities that are either both discrete or both continuous, then they have a joint density

$$p(u, v) \quad = \quad p(u)\, p(v)$$

(Practically, we treat $p(u)\, p(v)$ as a joint density even when one is discrete and the other continuous.)

If U and V are independent, then the conditional distribution of U given $V = v$ is the same as its marginal distribution (and does not depend on v).

Hence, a marginal density for U is also a conditional density:

$$p(u \mid v) \;\; = \;\; p(u) \qquad \text{for all } v$$

Conceptually, knowing V should tell you nothing about U.

Independence extends to three or more random variables in the obvious way.

To denote independence, we might write

$$U, V, W \quad \sim \quad \text{indep. } \cdots$$

or perhaps

$$U, V, W \quad \sim \quad \text{iid } \cdots$$

when they are independent *and* identically distributed (iid).

Independence also extends to random vectors, e.g., (U, V) could be independent of W.

Combining conditioning and independence allows us to conveniently build joint distributions. For example:

$$\begin{aligned} U \mid V = v,\, W = w &\sim \mathrm{N}(v, w) \\ V &\sim \mathrm{N}(0, 1) \\ W &\sim \mathrm{Expon}(1) \end{aligned}$$

To complete the specification, we assume V and W are independent. (This will be the convention when only marginal distributions are specified.)

The joint density has the following structure:

$$p(u \mid v, w)\, p(v, w) \;=\; p(u \mid v, w)\, p(v)\, p(w)$$

Conditional Independence

U and V are **conditionally independent given** $W = w$ if their joint distribution conditional on $W = w$ specifies them as independent.

If this is true for *all* w, then U and V are **conditionally independent given** W.

In this case, when conditional densities exist, they may be taken to satisfy

$$p(u, v \mid w) \;=\; p(u \mid w)\, p(v \mid w)$$

$$p(u \mid v, w) \;=\; p(u \mid w) \qquad\qquad p(v \mid u, w) \;=\; p(v \mid w)$$

Conditional independence does not imply independence, nor vice versa.

Conditional independence extends to several random variables in the obvious way.

To signify that U and V are conditionally independent *and* (conditionally) identically distributed given $W = w$, we might write

$$U, V \mid W = w \quad \sim \quad \text{iid} \;\; \textit{some distribution involving } w$$