

## II Weierstrass models of elliptic curves

- We start with elliptic curves. For this special case, there is a theory of models which is more elementary than both minimal regular models and Néron models

### 1) Weierstrass equations

def 1 Let  $R$  be a ring. A Weierstrass equation with coefficients in  $R$  is an homogeneous equation of the form

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

- Our goal is to derive the main properties of such equations with as little computations as possible, and then forget about the equations ! (not recommended if you actually want to compute things)

Lemma 2 Let  $k$  be a field,  $W \subseteq \mathbb{P}_k^2$  given by a Weierstrass equation  
Then  $W$  is geometrically integral, smooth at  $[0:1:0]$ .

Proof: - Writing the equation as  $F(x,y,z) = 0$ , we have

$$\frac{\partial F}{\partial z} = y^2 + a_1xy + 2a_3yz - a_2x^2 - 2a_4xz - 3a_6z^2$$

Hence  $\frac{\partial F}{\partial z}([0:1:0]) = 1 \neq 0 \Rightarrow W$  is smooth at  $[0:1:0]$ .

- $W \cap V(z) = \{[0:1:0]\}$ . Since  $V(z)$  intersects all irreducible components of  $W$  (2 curves in  $\mathbb{P}_k^2$  intersect!), and  $[0:1:0]$  is a smooth point, we get  $W$  irreducible.
- $W$  irreducible cubic curve. If  $W$  non-reduced, then necessarily  $F = L^3$ , and  $W$  everywhere non-reduced
- This argument applies over any field extension  
 $\Rightarrow W$  geometrically integral. □

prop 3 | Let  $(E, e)$  be an elliptic curve over a field  $k$ .  
 There exists  $x, y \in k(E)$  such that the  
 birational map  $\phi: E \longrightarrow \mathbb{P}_k^2, [x:y:z]$   
 is a closed immersion whose image is cut out by  
 a Weierstrass equation, and  $\phi(e) = [0:1:0]$  unique pt in the line at  $\infty$

pf of:  $\cdot RR \Rightarrow \forall n \geq 1, \dim H^0(E, \mathcal{O}(ne)) = n$ .

- Choose meromorphic functions  $x, y$  such that

$\{1, x\}$  basis of  $H^0(E, \mathcal{O}(2e))$ ,

$\{1, x, y\}$  basis of  $H^0(E, \mathcal{O}(3e))$ .

- Then  $\{1, x, y, x^2, xy, y^2, x^3\} \subseteq H^0(E, \mathcal{O}(6e))$

(Recall that if  $k = \mathbb{C}$ ,  $E^{\text{an}} = \mathbb{C}/\Lambda$  then can choose

$$x = P, y = P' \text{ with } P(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \text{ Weierstrass elliptic function.)}$$

$\Rightarrow \exists A_1, \dots, A_7 \in k$ , not all 0,

$$A_1 + A_2 x + A_3 y + \dots + A_7 x^3 = 0$$

- By considering pole order at  $e$  and using  $\dim H^0(E, \mathcal{O}(5e)) = 5$   
 see that  $A_6 \neq 0$  and  $A_7 \neq 0$ .
- Substitute  $x \mapsto -A_6 A_7 x$   $\Rightarrow F = 0$   
 $y \mapsto A_6 A_7^2 y$   $\Rightarrow$  (Weierstrass eqn<sup>o</sup>)  $\cdot A_6^3 A_7^4$
- Get  $\phi: E \dashrightarrow \mathbb{P}^2$ . Since  $E$  is smooth of dim  $\leq 1$  and  $\mathbb{P}^2$  is proper,  
 $\phi$  is automatically a morphism. The image of  $\phi$  is thus a closed  
 subset of  $\mathbb{P}^2$ , not reduced to a point, and contained in the integral  
 locus  $F = 0 \Rightarrow \text{Im}(\phi) = \{F = 0\}$ .

- By composing with proj to  $\mathbb{P}^1$  via  $[x:\gamma], [y:\gamma]$ , we see that  $\deg(\phi)$   
 divides 2 and 3  $\Rightarrow \deg(\phi) = 1$ .
- To conclude, it remains to show that  $\text{Im}(\phi)$  is smooth. By Lemma 2,  
 $\text{Im}(\phi)$  is an integral cubic curve in  $\mathbb{P}^2$ .

- Assume that  $\text{Im}(\phi)$  is singular and let  $p$  be a singular point. Then  $p$  is of multiplicity exactly 2 (otherwise,  $\text{Im}(\phi)$  would be either  $\times$ ,  $\times^2$  or  $\times^3$ , hence not integral). This implies that the projection map  $\mathbb{P}^2 \setminus \{p\} \rightarrow \mathbb{P}^1$  restricts to a degree 1 rational map  $\text{Im}(\phi) \dashrightarrow \mathbb{P}^1$ , which composed with  $E \rightarrow \text{Im}(\phi)$  gives  $E \dashrightarrow \mathbb{P}^1$  of degree 1  $\{\}$   $\square$

rank: If  $\text{char}(k) \neq 2, 3$ , there are simplified forms of Weierstrass equations; since we try to develop a purely geometric theory this is not so important for us.

prop 4: Let  $W \subseteq \mathbb{P}_k^2$  be defined by a Weierstrass equation. Then  $W$  smooth  $\Leftrightarrow (W, [0:1:0])$  elliptic curve.

proof:

$\Leftarrow$  is clear.

$\Rightarrow$  follows from Hurwitz's genus formula and  $[0:1:0] \in W(k) \neq \emptyset$ .

Alternative argument for  $\Rightarrow$ : from Weierstrass equation, can write down

$$\omega = \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4} \in \Omega_{W/k}^1$$

and show (assuming  $W$  smooth) that  $\omega \in H^0(W, \Omega_{W/k}^1)$  and that  $\omega$  is everywhere non-vanishing.

This implies  $\Omega_{W/k}^1 \cong \omega \cdot \mathcal{O}$ , and that  $W$  is of genus 1  $\square$

Here are some further geometric properties of Weierstrass equations which are not strictly necessary for what follows and which we state without proofs.

- def 5
- The discriminant  $\Delta$  of a Weierstrass equation is  $\Delta = 2^{-4} \operatorname{disc}_X (\zeta(X^3 + a_2 X^2 + a_4 X + a_6) + (a_1 X + a_3)^2) \in k$ .
  - We also define  $c_4 = (a_1^2 + 4a_4)^2 - 2\zeta(2a_4 + a_1 a_3) \in k$

- prop 6
- Any two Weierstrass equation for the same curve are related by a change of variables of the form  $\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases}$  with  $u \in k^\times, r, s, t \in k$ .
  - Such a change has the effect that  $\begin{cases} c_4 \mapsto u^{-4} c_4 \\ \Delta \mapsto u^{-12} \Delta \end{cases}$ .

- prop 7
- If  $\operatorname{char}(k) \neq 2, 3$ , using such a change of variable, we can put the equation in the form:  $y^2 = x^3 + Ax + B$  and then  $\begin{cases} \Delta = -(4A^3 + 27B^2) \\ c_4 = AB \end{cases}$ .

- prop 8
- Let  $k$  be a field and  $W$  given by a Weierstrass equation.
  - (i)  $W$  non-singular  $\Leftrightarrow \Delta \neq 0$ .
  - (ii) If  $\Delta = 0, c_4 \neq 0$ , then  $W$  has a unique geometric singularity which is a node defined over  $k$ .  
(note that the branches at the node may not be defined over  $k$ .)
  - (iii) If  $\Delta = 0, c_4 = 0$ , then  $W$  has a unique geometric singularity which is a cusp. It is defined over  $k$  unless  $\operatorname{char}(k) \in \{2, 3\}$  and  $k$  is imperfect.

- All this can be found in [Silverman, chap III] except the discussion of non-national cusps on non-perfect fields which is nicely explained in [Conrad-models, p.15].

## 2) Weierstrass models

We now turn to models. For simplicity, we start with  $S = \text{Spec}(R)$ ,  $R$  discrete valuation ring,

In this case, it turns out that Weierstrass equations provide rather good models of elliptic curves.

Lemma 1: Let  $W/y$  be defined by a Weierstrass equation. Then  $W$  is isomorphic to a curve defined by a Weierstrass equation with coeffs in  $R$ .

Proof: Let  $v \in K$ . One checks that the substitution

$$\begin{cases} x = v^{-2}x' \\ y = v^{-3}y' \end{cases}$$
 acts on coefficients by  $a'_i = v^i a_i$

Hence by taking  $v_K(v) \gg 0$ , we can ensure that  $a_i \in R$ .

□

Def 2 Let  $(E, e)$  be an elliptic curve over  $K$ .  
A planar Weierstrass model (PWM) of  $(E, e)$  is a pair  $(W, i)$  with  $W \subseteq \mathbb{P}_R^2$  defined by a Weierstrass equation and  
 $i: W_K \xrightarrow{\sim} E$   
 $[0:1:0] \mapsto e$ .

Prop 3 Let  $W/S$  be a planar Weierstrass model.  
(i)  $W$  is proper flat over  $S$ .  
(ii)  $W_S$  has geometrically integral fibers, and is smooth at  $[0:1:0]$ . Equivalently,  $W_S$  is smooth at  $\varepsilon(e)$  for  $\varepsilon \in W(R)$  the unique section extending  $i^*(e)$ .  
(iii)  $W$  is normal.

Proof: (i)-properness  $\Leftarrow$  projectivity.

- Flatness  $/S \Leftrightarrow \mathcal{O}_W$  torsion-free over  $R$ ; can be checked on affine patches; have to check  $\pi$  unif of  $R$  does not divide  $F$  in the factorial ring  $R[x, y]$ ; follows from monicity of Weierstrass eq.

(ii) follows from previous prop.

(iii) Want to apply Serre's criterion.

Recall  $(R, m)$  local ring. A regular sequence  $r_1, \dots, r_k \in m$  satisfies  $r_i$  non zero-divisor in  $R/(r_1, \dots, r_{i-1})$ .

The depth of  $R$  is the maximal length of a regular sequence.

Serre's criteria:

[St §310]  $X$  locally noetherian scheme. Then  $X$  normal iff

(R1)  $\text{RO}$   $\forall x \in X, \dim(\mathcal{O}_{X,x}) \leq 1 \Rightarrow \mathcal{O}_{X,x}$  regular

(S2)  $\text{S1}$   $\forall x \in X, \text{depth}(\mathcal{O}_{X,x}) \geq \min\{2, \dim(\mathcal{O}_{X,x})\}$ .

•  $W/S$  is flat,  $W_\gamma/\gamma$  is smooth,  $W_{\sigma}/\sigma$  is gen. smooth ( $\leq W_{\sigma}/\sigma$  is geom. reduced)

$\Rightarrow W \setminus \underbrace{W}_{S\text{-smooth locus}}^{\text{sm}}$  is a finite set of points in  $W_\sigma$

$\Rightarrow W$  is (R1).

•  $W$  is reduced, hence S1, so it remains to show that for all  $x$  codimension 2 pt, we have  $\text{depth}(\mathcal{O}_{W,x}) = 2$ . Such an  $x$  lies in  $W_\sigma$ .

• Lemma |  $(R, m)$  noeth. local ring,  $\pi \in m$ .  
 $[\text{St } \emptyset \neq R] \quad \pi \text{ non-zero divisor} \Rightarrow \text{depth}(R_{(\pi)}) = \text{depth}(R) - 1$ .

This reduces us to show  $\text{depth}(\mathcal{O}_{W_\sigma, x}) = 1$ .

$W_\sigma$  is reduced, hence (S1), and we are done.

[Alternative argument for (S2):  $W$  is an hypersurface in  $\mathbb{P}_S^2$ , hence local complete intersection, hence Cohen-Macaulay, hence (S2)  $\forall k$ .]

□

def | Let  $(E, e)/\gamma$  be an elliptic curve.  
An abstract Weierstrass model (AWM) of  $(E, e)$  is a pair  $(W, i)$  with  
•  $W/S$  proper flat with  $W$  normal and geometrically integral fibers.  
•  $i: W_K \xrightarrow{\sim} E$   
•  $W_\sigma$  smooth at  $\varepsilon(\sigma)$  for  $\varepsilon \in W(R)$   
unique section extending  $i^{-1}(e)$ .

• By the above, any planar Weierstrass model is an abstract Weierstrass model. Conversely:

thm 5 | Every abstract Weierstrass model is isomorphic to a planar one.

Proof Follows the same strategy as over a field with  $R/R$  replaced by Serre duality for  $W_\sigma$ .

• Serre duality: (all we need for this section is in Hartshorne!)

def 6 | Let  $k$  be a field,  $X/k$  projective equidimensional of dim.  $n \geq 0$ .  
A dualizing sheaf for  $X$  is a pair  $(\omega_X, t)$   
with •  $\omega_X$  coherent sheaf on  $X$   
•  $t : H^n(X, \omega_X) \xrightarrow{\sim} k$  (trace map)  
such that for any coherent sheaf  $\bar{J}^\dagger$  on  $X$  and  $i \in \mathbb{N}$ ,  
the natural pairing  

$$\mathrm{Ext}^i(\bar{J}^\dagger, \omega_X) \times H^{n-i}(X, \bar{J}^\dagger) \xrightarrow{\quad} H^n(X, \omega_X) \xrightarrow{t} k$$
  
is perfect.

$\uparrow$   
Ext cup-  
product

rank       $\omega_X$  is unique up to a unique iso if it exists.  
[Hartshorne, Proposition III.7.2]

thm 7 | 1) Let  $X/k$  be smooth projective of dimension  $n$ .  
Then  $\Omega_{X/k}^n := \bigwedge^n \Omega_{X/k}$  is a dualizing sheaf  
2) Let  $X \hookrightarrow Y$  be a closed embedding of noetherian  
equidimensional  $k$ -schemes, of dim.  $n$  and  $N$ .  
Assume •  $Y$  is smooth (hence by 1) has a dualizing sheaf)  
•  $X$  is Cohen-Macaulay.  
Then  $X$  has a dualizing sheaf given by  

$$\omega_X = i^* \mathrm{Ext}^{N-n}(i_* \mathcal{O}_X, \omega_Y).$$

ref: [Hartshorne, Cor. III.7.12 + Prop III.7.5 and its proof]  
+ Theorem III.7.6

- If we already knew that  $W$  was a planar Weierstrass model, we would see that  $W_6 \hookrightarrow \mathbb{P}_6^2$  has a dualizing sheaf  $\omega_6 := \omega_{W_6/\mathbb{P}_6}$  given by  $\omega_6 = i^* \text{Ext}^1(i_* \mathcal{O}_{W_6}, \mathcal{O}_{\mathbb{P}^2}(-3))$ .

Thm 8 |  $X/k$  projective equidimensional of dim  $n$ .  
 If  $X$  is Cohen-Macaulay, then  $X$  has a dualizing sheaf.

Proof: Apply Thm 7 to a projective embedding.

Cor 9 | Any | reduced proper curve/ $k$  | has a dualizing sheaf.  
Proof: - A proper curve over a field is projective (will review later)  
 - A | reduced curve over a field is Cohen-Macaulay | (because it is (S1)).  
 | l.c.i | ([Zim, 8.2.18])  $\square$

- So in particular  $W_6$  has a dualizing sheaf  $\omega_6$ .

Lemma 10 | i)  $\omega_6$  is torsion-free.  
 ii)  $\omega_6$  is generically invertible.

Proof: i) If  $\omega_6$  has torsion, it would contain a torsion subsheaf  $\ell \neq 0$ . By duality:  $\ell$  torsion on a curve

$$\text{Hom}(\ell, \omega_6) \cong H^1(W_6, \ell)^{\vee} \stackrel{\ell}{\leftarrow} \cong 0 \quad \{$$

ii) Pick a projective embedding  $W_6 \hookrightarrow \mathbb{P}_k^N$

By Thm 7.2), we have  $\omega_6 \cong i^* \text{Ext}^{N-1}(i_* \mathcal{O}_{W_6}, \mathcal{O}_{\mathbb{P}^N}(-N+1))$ .

If  $x \in W_6$  is a smooth point of  $W_6$ ,  $i$  is a regular immersion in a neighbourhood  $U$  of  $x$ , and a local computation then shows that  $\omega_6|_U$  is invertible (see proof of [Hartshorne, Theorem III.7.11])  $\square$

rk: By the same local comp, the dualizing sheaf of any projective l.c.i var/ $k$  is invertible; hence a posteriori  $\omega_6$  will actually be invertible.

• We can now start the proof in earnest.

By hypothesis,  $\varepsilon$  lies in  $W^{\text{reg}}(R)$ , hence  $\mathcal{J}_\varepsilon$  is invertible.

Write  $\mathcal{O}(\varepsilon) = \mathcal{J}_\varepsilon^{-1}$ ,  $\mathcal{O}(n\varepsilon) = (\mathcal{O}(\varepsilon))^{\otimes n}$ .

Write  $\bar{\varepsilon} \in W_6^{\text{reg}}(R)$  for the reduction, and  $\mathcal{O}(\bar{\varepsilon})$ ,  $\mathcal{O}(n\bar{\varepsilon}) \dots$

Lemma 11 |  $\forall n \geq 1, H^1(W_6, \mathcal{O}(n\bar{\varepsilon})) = 0$

Proof: • By duality:

$$\begin{aligned} H^1(W_6, \mathcal{O}(n\bar{\varepsilon})) &\cong \text{Hom}(\mathcal{O}(n\bar{\varepsilon}), \omega_6) \\ &\cong \text{Hom}(\mathcal{O}, \omega_6(-n\bar{\varepsilon})) \quad (\mathcal{O}(n\bar{\varepsilon}) \text{ invertible}) \end{aligned}$$

• Assume there is a non-zero map  $\mathcal{O} \xrightarrow{f} \omega_6(-n\bar{\varepsilon})$ . Since  $\omega_6(-n\bar{\varepsilon})$  is torsion-free and generically invertible,  $f$  is injective and its cokernel is torsion.

Write  $0 \rightarrow \mathcal{O}_{W_6} \xrightarrow{f} \omega_6(-n\bar{\varepsilon}) \rightarrow \mathcal{L} \rightarrow 0$

• We have  $X(\omega_6(-n\bar{\varepsilon})) = X(\mathcal{O}_{W_6}) + X(\mathcal{L}) = X(\mathcal{L}) \geq 0$

$\parallel \leftarrow$  flatness

$$X(\mathcal{O}_{W_6}) = 0$$

$\uparrow$   
ell. curve

• By duality,  $X(\omega_6(-n\bar{\varepsilon})) = -X(\mathcal{O}(n\bar{\varepsilon}))$

$$\underbrace{= -\deg(n\bar{\varepsilon}) - X(\mathcal{O}_{W_6})}_{(*)} \quad \parallel 0$$

$$\stackrel{(*)}{=} -n < 0 \quad \{$$

□

• Consider the  $R$ -module  $H^0(W, \mathcal{O}(n\varepsilon))$ .

It is of finite type over a DVR, hence direct sum of a free module and a torsion module. We want to show the torsion part vanishes.

{ We have seen that  $H^1(W_6, \mathcal{O}(n\bar{\varepsilon})) = 0$

{ We also have  $H^1(W_\gamma, \mathcal{O}(n\varepsilon_\gamma)) = 0$

By the standard result on cohomology and base change (e.g [Mumford, II.5. Cor 2]):

the formation of  $H^0(W, \mathcal{O}(n\varepsilon))$  commutes with arbitrary base change. In particular  
 $\left\{ \begin{array}{l} H^0(W, \mathcal{O}(n\varepsilon)) \otimes_R K \simeq H^0(W_\gamma, \mathcal{O}(n\varepsilon)) \text{ of dimension } n \text{ by Lemma 11 and its proof (*).} \\ H^0(W, \mathcal{O}(n\varepsilon)) \otimes_R K = H^0(W_\gamma, \mathcal{O}(n\varepsilon)) \text{ of dimension } n \text{ by the elliptic curve case.} \end{array} \right.$

So  $H^0(W, \mathcal{O}(n\varepsilon))$  is a free  $R$ -module of rank  $n$ .

- By the same argument as in the field case, we construct

$$W \longrightarrow \text{Proj}_S \left( \text{Sym} \left( H^0(W, \mathcal{O}(3\varepsilon)) \right) \right) \simeq \mathbb{P}_S^2$$

with image a planar Weierstrass model  $W'$  of  $E$ .

The following lemma then finishes the proof.

Lemma 12 | Let  $\phi: W \rightarrow W'$  be a morphism of abstract Weierstrass models. Then  $\phi$  is an isomorphism.

Proof:

• The map  $\phi: W \rightarrow W'$  is proper birational.  
 since  $\phi_\gamma$  is an iso. by hypothesis.  $\text{Im}(\phi) \left\{ \begin{array}{l} \text{is closed} \\ \text{contains } W'_\gamma \end{array} \right\} \Rightarrow \phi \text{ surjective.}$

The map  $\phi_\gamma$  is a non-constant map between integral curves, hence it is finite.

So  $\phi$  is finite birational. Since  $W'$  is normal, we deduce that  $\phi$  is an isomorphism. □

Lemma 13 | Let  $W, W'$  be planar Weierstrass model. Then (isomorphisms between  $W$  and  $W'$ ) as models are given by linear changes of coordinates  
 $\left\{ \begin{array}{l} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{array} \right. \quad \text{with } u \in R^\times, r, s, t \in R.$

There is in fact at most one such isomorphism.

Proof Follows from the intrinsic construction of the embedding  $W \hookrightarrow \mathbb{P}_R^2$  described in previous proof + explicit manipulation of equations as in field case. See [Conrad-models, Cor 2.9] □

## Over a more general Dedekind scheme:

- Let  $S$  be a Dedekind scheme and  $E/\gamma$  an elliptic curve. We can define AWMs for  $E$  in the same way, and, when  $S$  is affine, PWMs. (one should really allow coefficients in a line bundle on  $S$ , which would allow "PWM" beyond  $S$  affine, but I could not find a good reference).

Then any PWM is an AWM (some proof) and one can ask when the converse holds. This is not always true (ex. later!)

- Let  $S = \text{Spec}(R)$  affine Dedekind. The criterion for when an AWM is a PWM involves the relative dualizing sheaf  $\omega_{W/S}$ . We do not want to introduce too much machinery this early in the course, so let us just say that one can prove that  $W/S$  is a local complete intersection and that this implies that there exists  $\omega_{W/S}$  invertible sheaf on  $W$  with fibers the dualizing sheaves  $\omega_{W_s/S}$  for all  $s \in S$ .

<u>thm 15</u>	<p>Let <math>\pi: W \rightarrow S</math> be an AWM. Then</p> <ul style="list-style-type: none"> <li>(i) <math>\pi_* \omega_{W/S} \cong (R^1 \pi_* \mathcal{O}_W)^\vee</math> (instance of <u>relative duality</u>)</li> <li>(ii) <math>R^1 \pi_* \mathcal{O}_W</math> and <math>\pi_* \omega_{W/S}</math> are locally free.</li> <li>(iii) <math>W</math> is a PWM <u>iff</u> <math>R^1 \pi_* \mathcal{O}_W</math> is free  <u>iff</u> <math>\pi_* \omega_{W/S}</math> is free.</li> </ul>
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cor 15:  $|S \text{ affine} + \text{Pic}(S) = 0 \Rightarrow \text{Every AWM is a PWM.}$

rmk: The criterion in terms of  $\pi_* \omega_{W/S}$  looks more complicated than the one for  $R^1 \pi_* \mathcal{O}_W$  but it is more directly related to computations with diff. forms.

### 3) Minimal Weierstrass models (Start again)

- Given an elliptic curve  $E$ , there are many non-isomorphic Weierstrass models, obtained by suitable transformations of  $x, y$ . It turns out there is a "best" one, the minimal Weierstrass model  $W^{\min}(E)$ .

There are several ways to pin it down:

- From a Weierstrass equation, one can extract the discriminant

$$\Delta := 2^{-4} \operatorname{disc} \left( 4(x^3 + a_2 x^2 + a_4 x + a_6) + (a_7 x + a_3)^2 \right) \in \mathbb{R}$$

Then  $v(\Delta) \in \mathbb{N}$  is an invariant of the model, and  $W^{\min}(E)$  is the unique model with minimal  $v(\Delta)$ .

$$(v(\Delta_{\min}) = 0 \Leftrightarrow E \text{ has good redct} \Leftrightarrow W^{\min}(E) \text{ smooth})$$

- Using the explicit form of the changes of variables,

We see that  $\begin{cases} \Delta' = v^{-12} \Delta \\ c_4' = v^{-4} c_4 \end{cases}$

Lemma 1  $v(\Delta) < 12$  or  $v(c_4) < 4 \Rightarrow W$  minimal.  
 $\Leftrightarrow (\operatorname{char}(k) \neq 2, 3)$  [Silverman, ex. VII.7.1]

def 2 Let  $E$  be an elliptic curve over  $K$  with minimal Weierstrass model  $W$ . We say that  $E$

- has good reduction if  $W_K$  is smooth ( $\Leftrightarrow v(\Delta) = 0$ )
- has multiplicative reduction if  $W_K$  has a node ( $\Leftrightarrow v(\Delta) > 0$   
 $v(c_4) = 0$ )
- has additive reduction if  $W_K$  has a cusp ( $\Leftrightarrow v(\Delta) > 0$   
 $v(c_4) > 0$ )

- For this approach see [Silverman, chap 7].

b) For any Weierstrass model  $W/S$ , one can prove that

$H^0(W^{\text{sm}}, \Omega_{W^{\text{sm}}/R}^1)$  is a free rk 1  $R$ -submodule

of  $H^0(E, \Omega^1)$ . [Conrad-models, thm 2.6] Then

$W^{\text{min}}(E)$  is the unique WM with maximal  $H^0(W^{\text{sm}}, \Omega^1)$ .

[Conrad-models, cor 2.10]

c)  $W^{\text{min}}(E)$  is the only Weierstrass model of  $E$

with rational singularities. [Conrad-models, Cor 8.4]

- Pt of view a) is classical and suitable for computations (incl. with a computer).
  - Pt of view b) is useful to relate Weierstrass models with the main objects of the course. (see later)
  - Pt of view c) is nice because of the distinguished role of rational singularities in resolution of singularities of surfaces (see later).
  - The relation between a) and b) is based on the direct computation:
- Lemma 3 | Let  $E/K$  be an elliptic curve with a fixed Weierstrass equation. We have  $\begin{cases} \Delta \in k^\times \\ \omega = \frac{dx}{2y + a_1x + a_3} \in H^0(E, \omega_{E/K}) \end{cases}$
- Then  $\Delta \omega^{\otimes 12} \in (H^0(E, \omega_{E/K}))^{\otimes 12}$  is independent of the choice of the equation.
- i.e. " $\Delta$  is a weight 12 modular form"!

## Semistable reduction for elliptic curves

- We look at the first instance where the properties of a model can be improved by passing to an extension of  $K$ .

thm 4: | Let  $E/K$  be an elliptic curve.  
| There exists a finite separable extension  $L/K$  such that  
|  $E_L$  has good or multiplicative reduction.  
| (elementary)

- Unfortunately I could not find an <sup>v</sup>equation-free proof of this in the literature, so I refer you to [Silverman, Prop 5.4.(c)].

## Situation over a Dedekind scheme:

- Let  $S$  be a general Dedekind scheme, and  $(E, e)_{/\gamma}$  be an elliptic curve. Then  $E$  admits a minimal AWM  $W^{\min}(E)_{/S}$  i.e., an AWM such that for every  $\sigma \in S^{(0)}$ ,  $W_S^{\min} \times_{\mathcal{O}_{S,\sigma}} \text{Spec}(\mathcal{O}_{S,\sigma}) / \text{Spec}(\mathcal{O}_{S,\sigma})$  is minimal.
  - Assume  $S$  affine. Then  $W^{\min}(E)$  is planar
    - iff  $\pi_* \omega_{W^{\min}(E)_{/S}}$  is free
    - iff the "Weierstrass ideal"  $\mathfrak{a}_{E/K} \subseteq R$  is principal.
- ( $\mathfrak{a}_{E/K}$  is defined as a product of local terms obtained from turning a fixed PBM into a minimal one at that point, see [Silverman, Chap VIII];  $\mathfrak{a}_{E/K} \sim \frac{1}{12} (\Delta \text{ of any PBM of } E)$ )

## Relationship with general theory:

- $W^{\min}(E)$  is neither the minimal regular model (it's not regular in general) nor the Néron model (it's not smooth in general).
- Let us sketch the relationship with the introduction. Write
  - $\mathcal{E}^{\text{reg}}$  for the minimal regular model of  $E$  over  $S$ .
  - $N(E)$  for the Néron model of  $E$  over  $S$ .

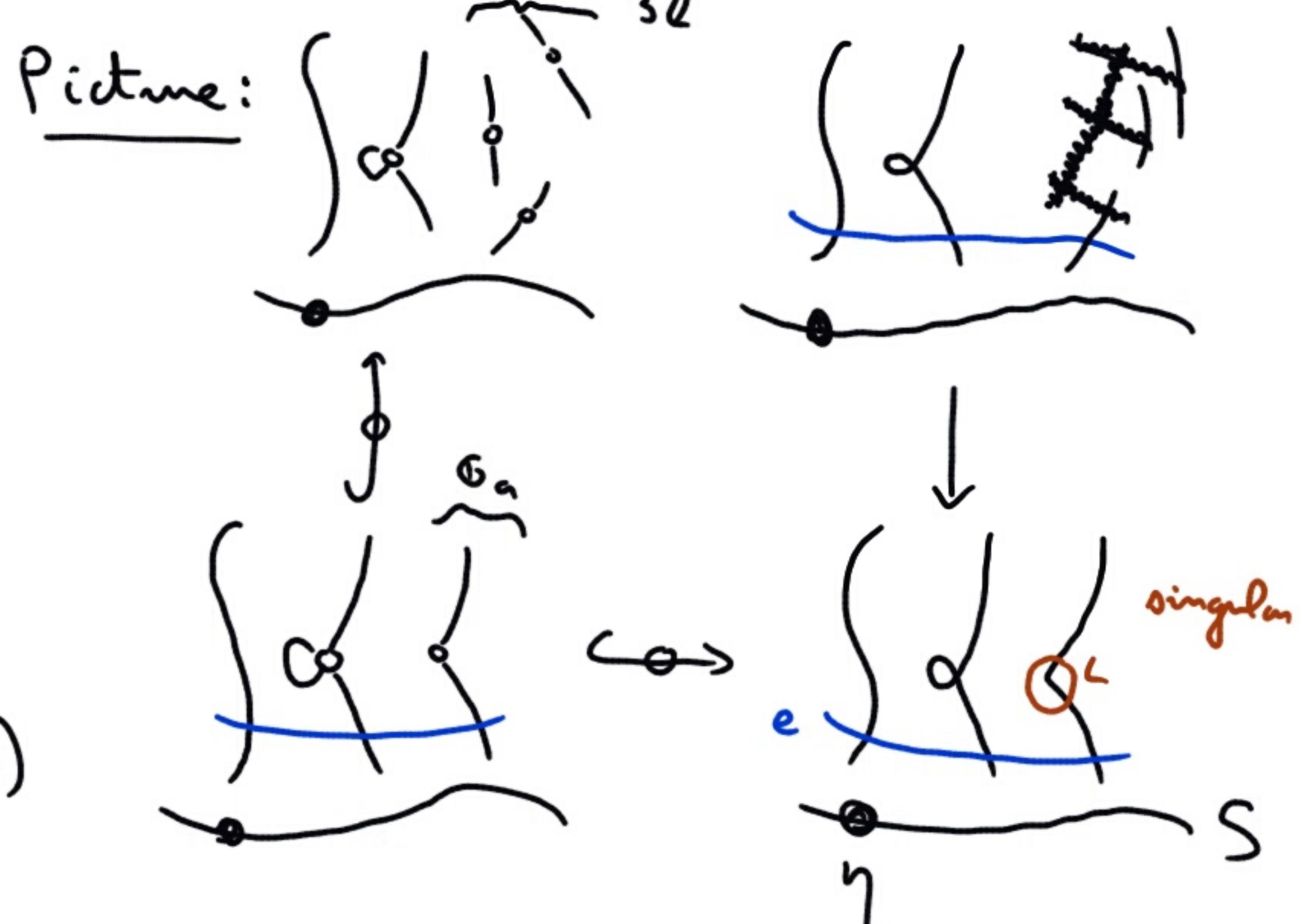
Then :

- There is a canonical morphism  $\mathcal{E}^{\text{reg}} \longrightarrow W^{\min}(E)$  of models that contracts all the irreducible components of the special fibers not meeting the unique section  $\varepsilon \in \mathcal{E}^{\text{reg}}(S)$  which extends  $e \in E(\gamma)$ .
- There is a canonical isomorphism  $(W^{\min}(E))^{\text{sm}} \simeq N(E)^{\circ}$  with  $(W^{\min}(E))^{\text{sm}}$  the  $S$ -smooth locus and  $N(E)^{\circ}$  is the union of the identity components of the fibers of the group scheme  $N(E)$ .
- There is a canonical isomorphism  $(\mathcal{E}^{\text{reg}})^{\text{sm}} \simeq N(E)$ .

rank:  $\begin{cases} \mathcal{E}^{\text{reg}} \text{ and } N(E) \text{ only depend on } E \text{ as a variety (not one)} \\ \text{while } W^{\min}(E) \text{ and } N(E)^{\circ} \text{ depend on } (E, e). \end{cases}$  (type IV\*)

Diagram:

$$\begin{array}{ccc} (\mathcal{E}^{\min})^{\text{sm}} \simeq N(E) & \hookrightarrow & \mathcal{E}^{\text{reg}} \\ \uparrow & & \downarrow \text{min resht of rings} \\ N(E)^{\circ} \simeq (W^{\min}(E))^{\text{sm}} & \hookrightarrow & W^{\min}(E) \end{array}$$



- These results and others will show up in the next instances of :

## Elliptic curves, the running example



- The other moral is that equations can fail us, already in the case of elliptic curves over global fields. Needless to say, for higher genus curves / higher dimensional abelian varieties, the situation will not improve and "abstract" algebraic geometry will be essential.

Examples:

- $S = \text{Spec } \mathbb{Z}$ : the situation is especially simple:  $\mathbb{Z} \text{ PID} \Rightarrow$  minimal PWM exists.
- + For any  $E/\mathbb{Q}$ , there exists a unique MPWM equation with  $a_1, a_3 \in \{0, 1\}$ ,  $a_2 \in \{-1, 0, 1\}$ . ("reduced min equat°")
- Here is a Weierstrass equation /  $\mathbb{Z}$ :  $W: y^2 = x^3 - 432$

We have  $\Delta_W = -2^{12} \cdot 3^9$ , and  $W_{\mathbb{F}_2}: y^2 = x^3$  is singular.

On the other hand, if we put  $x = 4x'$ ,  $y = 8y' - 4$ , we get

$$W': y'^2 - y' = x'^3 - 7 \quad \text{with } \Delta_{W'} = -3^9.$$

- So  $\begin{cases} W \text{ is not minimal at 2,} \\ W' \text{ is minimal at 2} (\Leftarrow \text{good red at 2}) \end{cases}$ .
- What about at 3? By looking at the possible changes of variables of Weierstrass equation, one sees that  $v_p(\Delta)$  at a prime  $p$  changes by multiples of 12. So  $v_3(\Delta_W) = v_3(\Delta_{W'}) = 9 < 12$  implies that both  $W$  and  $W'$  are minimal at 3;  $c_3(W) = 0 \Rightarrow$  additive reduction.
- In conclusion  $W'$  is the reduced MPWE of its generic fiber  $E$ .
- $E$  is a very cool elliptic curve:
  - it is isomorphic to the Fermat cubic  $x^3 + y^3 + z^3 = 0$
  - it has complex multiplication by  $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$
  - it is the modular curve  $X_0(27)$ .
- $E$  acquires semistable reduction after an extension of degree 12:  $\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{3})$
- $\text{Pic}(S) \neq 0$ : Let us describe an example of elliptic curve with no MPWM. We need  $\text{Pic}(S) \neq 0$ .

Given any number field  $K$  with class number  $> 1$ , a result

of Silverman states that there exists an elliptic curve with no MPWM. For instance, for  $K = \mathbb{Q}(\sqrt{-10})$ :

$$\text{Pic}(\mathcal{O}_K) = \langle (5, \sqrt{-10}) \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

$$E: y^2 = x^3 + 125$$

has Weierstrass ideal equal to  $(5, \sqrt{-10})$ , so

$E$  has no MPWM.