Complement on geometric realisation

- Recall: $|\cdot| : sSet \longrightarrow Top$

  is the unique functor (up to natural iso)

  with
$$\begin{cases} * \ |\Delta^n| = \Delta^n_{top} \\ * \ |f| : \Delta^n_{top} \to \Delta^m_{top} \text{ is given by the unique linear map extending the one on vertices determined by } f. \\ * \ |\cdot| \text{ commutes with small colimits.} \end{cases}$$

- Let $X \in sSet$. Then we know

$$X = \operatorname*{colim}_{\int X} \Delta^n \quad \left(\int X \text{ category of elements} \right)$$

$$\Rightarrow |X| = \operatorname*{colim}_{\int X} |\Delta^n_{top}|$$

Like any colimit in a cocomplete category, it can be written as a coequalizer of maps between coproducts; coproducts in Top are disjoint unions and coequalizers are quotients by an eq. relation

$$\Rightarrow \quad |X| = \left(\coprod X_n \times \Delta^n_{top} \right) \Big/ \begin{array}{c} (\delta^*(x), t) \\ \wr \\ (x, \delta_*(t)) \\ \text{for } \delta \in \mathrm{Mor}(\Delta). \end{array}$$

Moreover as we saw, the skeletal filtration shows that $|X|$ is a CW-complex.

- The geometric realisation has an additional very useful exactness property:

(Non-) theorem: $|\cdot|$ ~~commutes with finite limits~~.

Alas this is not quite true... but almost!

Thm: a) $|\cdot|$ commutes with equalizers.

b) Let $X, Y \in sSet$. The canonical map

$$|X \times Y| \longrightarrow |X| \times |Y|$$

is a bijection of sets, and is an homeomorphism whenever $|X|$ or $|Y|$ is locally compact (for instance, when X or Y has finitely many non-degenerate simplices) $\square$

Cor: Let $f, g : X \to Y$ in sSet. If f is simplicially homotopic to g, i.e.

$\exists\ H : X \times \Delta^1 \to Y$ with $\begin{cases} H|_{X \times \Delta^{\{0\}}} = f \\ H|_{X \times \Delta^{\{1\}}} = g \end{cases}$

then $|f|$ is homotopic to $|g|$.

proof: $|X| \times [0,1] \overset{\text{Thm}}{\cong} |X \times \Delta^1| \xrightarrow{|H|} |Y|$

gives the required homotopy. $\square$

Here is why the thm is more complicated:

Top has a basic pathology which is inconvenient in many places in algebraic topology:

⚠ Top is not cartesian closed.

(i.e $A \times -$ does not have a right adjoint)

But it turns out this is not a problem in practice: there exists "convenient "full subcategories" $\mathrm{Top}' \subseteq \mathrm{Top}$ which:

- contain CW-complexes
- are complete and cocomplete.
- are cartesian closed.
- are closed under closed subspaces:
  ($A \in \mathrm{Top}'$, $B \subset A$ closed $\Rightarrow B \in \mathrm{Top}'$)
- are such that if $A, B \in \mathrm{Top}'$ and
  A is locally compact, then $A \underset{\mathrm{Top}}{\times} B \in \mathrm{Top}'$

Hence $A \underset{\mathrm{Top}}{\times} B = A \underset{\mathrm{Top}'}{\times} B$.

These conditions are satisfied eg for

- Top' = compactly generated spaces.
- Top' = compactly generated weakly Hausdorff spaces.

See the survey paper
"The category of CGWH spaces"
of Neil Strickland for details.

Let $\mathrm{Top}' \subseteq \mathrm{Top}$ be any full subcategory satisfying the above. We still write

$$|\cdot|: sSet \longrightarrow CW \subset \mathrm{Top}'$$

for the corestriction of the geom. real.

thm: $|\cdot|: sSet \longrightarrow \mathrm{Top}'$ preserves finite limits.

proof: It suffices to do equalizers and finite products. The case of finite products is the most interesting and useful, so I only give that part of the proof. (for equalizers, see [Gabriel-Zisman, III.3.3])

- First, we show that the canonical map $|\Delta^p \times \Delta^q| \longrightarrow |\Delta^p| \times |\Delta^q|$ is an homeomorphism for all p, q.

It is a continuous map between CW-complexes with $< \infty$ cells ($\Rightarrow$ compact Hausdorff) so it is enough to show it is a bijection.

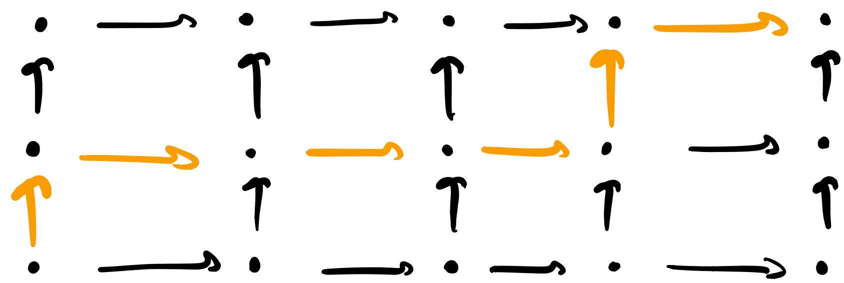
We sketch one combinatorial argument for this point (see [Gabriel-Zisman, II.5 and III.3] for details)

The poset $[p] \times [q]$ looks like:

$$[2] \quad \begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

$$[4]$$

The maximal chains (= totally ordered subsets) are all of length $p+q$ and look like:



There are $N = \binom{p+q}{q}$ such chains.

For a chain $C \simeq [p+q]$, the projections onto $[p]$ and $[q]$ define maps

$$[p+q] \rightrightarrows \begin{matrix}[p]\\ [q]\end{matrix},$$

Hence a $(p+q)$-element $x_C \in (\Delta^p \times \Delta^q)_{p+q}$

These are precisely the non-degenerate $(p+q)$-simplices of $\Delta^p \times \Delta^q$,

and this leads to a presentation

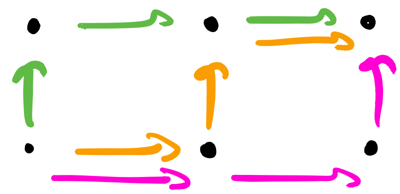
of $\Delta^p \times \Delta^q$ as an equalizer:

$$\coprod_{c \cap c'} \Delta^{|c \cap c'|} \rightrightarrows \coprod_{c} \Delta^{p+q} \xrightarrow{(x_c)} \Delta^p \times \Delta^q$$

We apply $|\cdot|$ and get

$$\coprod_{c \cap c'} \Delta_{top}^{|c \cap c'|} \rightrightarrows \coprod_{c} \Delta_{top}^{p+q} \longrightarrow |\Delta^p \times \Delta^q|$$

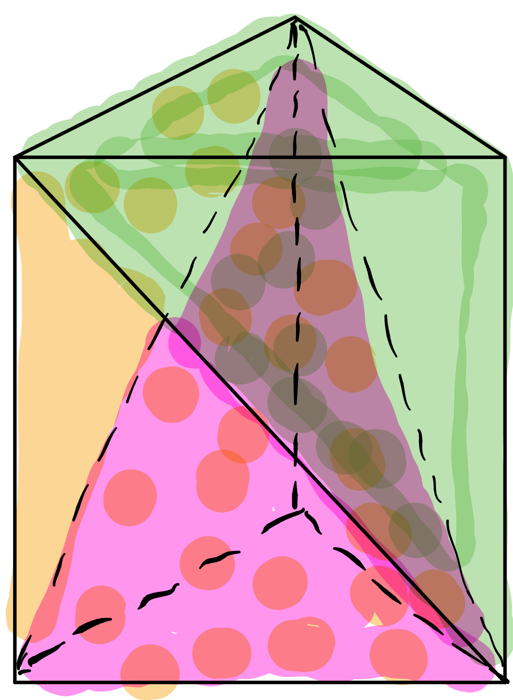
But the CW-complex $\Delta^p_{top} \times \Delta^q_{top}$ has a parallel presentation, induced by shuffle maps. At this point, I will just draw a picture and let you look at [Gabriel-Zisman] for details. Let $p=1$, $q=2$. Then $p+q=3$ and $N = \binom{3}{1} = 3$.

The maximal chains are 

The presentation of $\Delta^1 \times \Delta^2$ is

$$\Delta^1 \amalg \Delta^2 \amalg \Delta^2 \rightrightarrows \Delta^3 \amalg \Delta^3 \amalg \Delta^3 \rightrightarrows \Delta^1 \times \Delta^2$$

The geometric presentation of $\Delta^1_{top} \times \Delta^2_{top}$ is the decomposition of a prism into three pyramids:



Let's now do the general case.

The key point is that, in a cartesian closed category like Top' (but not Top!), finite products are left adjoints hence commute with colimits.

Let $X, Y \in sSet$. We write

$$X = \operatorname*{colim}_{\int X} \Delta^p, \quad Y = \operatorname*{colim}_{\int Y} \Delta^q$$

and we compute

$$|X \times Y| = \left| \operatorname*{colim}_{\int X} \Delta^p \times \operatorname*{colim}_{\int Y} \Delta^q \right|$$

$\times$ in sSet commutes with colim

$$\cong \left| \operatorname*{colim}_{\int X} \operatorname*{colim}_{\int Y} \Delta^p \times \Delta^q \right|$$

general fact about colimits

$$\cong \left| \operatorname*{colim}_{\int X \times \int Y} \Delta^p \times \Delta^q \right|$$

$|\cdot|$ commutes with colim

$$\cong \operatorname*{colim}_{\int X \times \int Y} |\Delta^p \times \Delta^q|$$

previous step

$$\cong \operatorname*{colim}_{\int X \times \int Y} |\Delta^p| \times |\Delta^q|$$

general fact about colim + "$\times$ commutes with colim in Top"

$$\cong \operatorname*{colim}_{\int X} |\Delta^p| \times \operatorname*{colim}_{\int Y} |\Delta^q|$$

$$\cong \left| \operatorname*{colim}_{\int X} \Delta^p \right| \times \left| \operatorname*{colim}_{\int Y} \Delta^q \right|$$

$|\cdot|$ commutes with colim.

$$\cong |X| \times |Y|$$

This (+ some verifications that this is indeed the canonical map) ends the proof. $\square$

2) ∞-categories

def 5 An ∞-category (or quasicategory) is a simplicial set $C \in sSet$ satisfying the inner horn extension property:

$\forall\, n \geq 2,\ \forall\, 0 < k < n,$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- A **functor** $F: C \to D$ between ∞-categories is simply a morphism of simplicial sets. This defines a (1-) category Cat_∞^1 of ∞-categories:
$$\mathrm{Cat}_\infty^1 \xhookrightarrow{\text{full}} \mathrm{sSet}.$$

- A **natural transformation** $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & D \end{tikzcd}$ is a morphism $\alpha: C \times \Delta^1 \to D$ with $\alpha|_{C \times \{0\}} = F$ and $\alpha|_{C \times \{1\}} = G$.

- A **natural isomorphism** $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & D \end{tikzcd}$ is a natural transformation α such that there exists $\beta: G \Rightarrow F$ and maps $t, t': C \times \Delta^2 \to D$ with

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\alpha"] \arrow[rr, "\mathrm{id}_F"'] \arrow[rr, phantom, bend left=20, "t"] & & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F \arrow[dr, "\alpha"] & \\ G \arrow[ur, "\beta"] \arrow[rr, "\mathrm{id}_G"'] \arrow[rr, phantom, bend left=20, "t'"] & & G \end{tikzcd}$$

- A functor $F : C \to D$ is a categorical equivalence (or an equivalence of ∞-categories) if there exists $G : D \to C$ and natural isomorphisms $\begin{cases} F \circ G \overset{\sim}{\Rightarrow} id_D \\ G \circ F \overset{\sim}{\Rightarrow} id_C \end{cases}$

Basic examples.

- By $\begin{cases} \text{the def. of Kan complexes} \\ \text{Prop 33 and Thm 34} \end{cases}$, we have fully faithful functors:

$$\begin{tikzcd} & \mathrm{Cat} \arrow[dr, hook, "N"] & & & \\ \mathrm{Grp} \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Cat}_\infty^1 \arrow[r, hook] & \mathrm{sSet}. \\ & \mathrm{Kan} \arrow[ur, hook] & & \end{tikzcd}$$

- We will see later that there are many examples which are not of these forms.

With this terminology we can formulate the main goal of this course:

* Develop "category theory" for ∞-categories, by introducing categorical concepts like (co)limits, representable functors, adjoints, etc. in such a way that:

  - they are compatible with usual category theory via the nerve.
  - they are invariant under categorical equivalences.

This leads to, as a secondary goal:

* Study the homotopy theory of categorical equivalences and the "∞-category of ∞-categories".

History

- This definition is due to Boardman-Vogt

(1973) in the context of homotopy theory (homotopy coherent algebraic structures, infinite loop spaces). They proved some basic results which we will review soon.

- The idea of taking quasicategories as a model for $(\infty,1)$-categories is due to Joyal (late 90's) and he developped most of the results from the first half of this course. Then Lurie came and pushed the theory even further!

Terminology For $X_\bullet \in sSet$ (and in particular for ∞-categories), we call

- objects of $X_\bullet$ the elements of X_0
- (1-)morphisms of $X_\bullet$ ———— X_1

For $g \in X_1$, we say that the source (resp. the target) of f is $d_1(g)$ (resp. $d_0(g)$) and we write $g : d_1(g) \longrightarrow d_0(g)$.

- For $x \in X_0$, we write $id_x = s_0(x)$ and call it the identity morphism of x.

A basic tool we want to have in any "category theory" is duality.

def 6 The *order-reversing* functor

$$\rho : \Delta \longrightarrow \Delta \quad \text{is defined as}$$

the identity on objects and, for $f:[m]\to[n]$,

$$\rho(f)(i) := n - f(m-i).$$

Ex:

$$\begin{tikzcd}[row sep=small]
0 \arrow[r] & 0 \\
1 \arrow[r] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[uur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
0 \arrow[dr] & 0 \\
1 \arrow[r] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[ur] &
\end{tikzcd}$$

$$f \qquad\qquad\qquad \rho(f)$$

$\rho^* : sSet \longrightarrow sSet$ is the functor "precomposition by ρ". For $X_\bullet \in sSet$, the *opposite simplicial set* is $X_\bullet^{op} := \rho^*(X_\bullet)$. $\square$

Examples $*$ $(\Delta^n)^{op} \simeq \Delta^n$, $(I^n)^{op} \simeq I^n$, $(\partial\Delta^n)^{op} \simeq \partial\Delta^n$.

$*$ $(\Lambda^n_i)^{op} \simeq \Lambda^n_{n-i}$

$*$ $X_\bullet$ ∞-category (resp Kan) $\iff$ $X_\bullet^{op}$ ∞-category (resp Kan).

* $N(C^{op}) \simeq N(C)$ for $C \in Cat$.

- Let us discuss some (co)limits of ∞-categories.

Prop 7: 1) Arbitrary products and coproducts of ∞-categories (in sSet) are ∞-categories.

2) Filtered colimits of ∞-categories are ∞-categories.

proof: 1)

Let $\{C_\alpha\}_{\alpha \in J}$ be a family of ∞-categories.

- Let $0 \leq k \leq n$. We have

$$\begin{array}{ccc} sSet(\Delta^n, \prod_\alpha C_\alpha) & \longrightarrow & sSet(\Lambda^n_k, \prod_\alpha C_\alpha) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \prod_\alpha sSet(\Delta^n, C_\alpha) & \twoheadrightarrow & \prod_\alpha sSet(\Lambda^n_k, C_\alpha) \end{array}$$

product of surjections is a surjection

$\Rightarrow \prod_\alpha C_\alpha$ is an ∞-category.

- For the coproduct, we need to compute

$$sSet\left(\Lambda^n_k, \coprod C_\alpha\right) \quad \text{and} \quad sSet\left(\Delta^n, \coprod C_\alpha\right)$$

By Yoneda, $sSet\left(\Delta^n, \coprod C_\alpha\right) = \left(\coprod_\alpha C_\alpha\right)([n])$

colimits are objectwise

$$= \coprod_\alpha C_\alpha([n])$$

$$= \coprod_\alpha sSet(\Delta^n, C_\alpha)$$

So it suffices to show that the natural map

$$\coprod_\alpha sSet\left(\Lambda^n_k, C_\alpha\right) \longrightarrow sSet\left(\Lambda^n_k, \coprod_\alpha C_\alpha\right)$$

is a bijection. For this one can use

$$\Lambda^n_k = \coprod_{\substack{\Delta^{[n]\setminus\{i,j\}} \\ i \neq k}} \Delta^{[n]\setminus i}$$ and the Yoneda

trick above; the key point is that the various $\Delta^{[n]\setminus i}$ must be sent to <u>the same</u> C_α because they are connected via the $(n-2)$ - faces $\Delta^{[n] \setminus \{i,j\}}$. Or in other words, one can show that

$$\pi_0\left(\coprod C_\alpha \right) \simeq \coprod_\alpha C_\alpha \quad \text{while } \pi_0(\Lambda^n_k) \text{ has}$$

one element (Λ^n_k is connected).

2) Let J be a filtered category and $C : J \longrightarrow sSet$ be a diagram so that each $X(\alpha)$ is an ∞-category. Once again we have

$$\left(\operatorname{colim}_J C \right)_n = \operatorname{colim}_J C([n])$$

because colimits are computed objectwise.

We want to show that the canonical map

$$\operatorname{Colim}_{\alpha \in J} sSet(\Lambda^n_k, C_\alpha) \longrightarrow sSet(\Lambda^n_k, \operatorname{Colim}_J C)$$

is a bijection. We will show this holds for Λ^n_k replaced by any $Y \in sSet$ with finitely many non-degenerate simplices.

Let $\mathcal{C} = \left\{ Y \in sSet \;\middle|\; \begin{array}{l} sSet(Y,-) \text{ commutes} \\ \text{with filtered colimits} \end{array} \right\}$

As remarked above, $\Delta^n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Let's show that $\mathcal{C}$ is closed under finite colimits. Let K be a finite category and $Y : K \to \mathcal{C}$ be a diagram.

We have

$$\begin{tikzcd}
\operatorname*{Colim}_{\alpha \in J} sSet\left(\operatorname*{Colim}_K Y, C_\alpha\right) \arrow[rr] \arrow[d, "\text{colim prop}", "\wr"'] & & sSet\left(\operatorname*{Colim}_K Y, \operatorname*{Colim}_J C\right) \arrow[d, "\text{colim prop.}", "\wr"'] \\
\operatorname*{Colim}_{\alpha \in J} \operatorname*{Lim}_{\beta \in K} sSet(Y_\beta, C_\alpha) \arrow[dr, "\wr", "\text{Filtered colimits commutes with finite limits in Set.}"'] & & \operatorname*{Lim}_{\beta \in K} sSet\left(Y_\beta, \operatorname*{Colim}_J C\right) \arrow[dl, "\wr"', "X(\alpha) \in \mathcal{C}"] \\
& \operatorname*{Lim}_{\beta \in K} \operatorname*{Colim}_{\alpha \in J} sSet(Y_\beta, C_\alpha) &
\end{tikzcd}$$

$\Longrightarrow \operatorname*{Colim}_K Y \in \mathcal{C}$.

- By the skeletal filtration,

$$\left\{ \begin{array}{l} \text{s. sets with } < \infty \\ \text{non-deg. simplices} \end{array} \right\} = \left\{ \begin{array}{l} \text{finite colimits of} \\ \text{standard simplices} \end{array} \right\} \subset \mathcal{C}$$

$\square$

Rmk: • The limits and colimits in the proposition have an important additional property (which we won't prove now): they are invariant under categorical equivalence! I.e, if we have categorical equivalences $C_\alpha \simeq D_\alpha$ for all α, then $\prod_\alpha C_\alpha \simeq \prod_\alpha D_\alpha$.

• Other types of limits and colimits in the 1-category Cat^1_∞, when they exist, may not satisfy this property! The situation will be corrected with (co)limits in the ∞-category of ∞-categories Cat_∞.

Given what we know about nerves, the following seems reasonable:

Slogan: An ∞-category is like the nerve of a category, except that composition of chains of composable morphisms is only well-defined up to homotopy, and these homotopies are compatible in a precise way.

In particular, it should be possible to get a 1-category by identifying all those homotopies. This amounts to giving a simple description of the fundamental category $\tau X_\bullet$ when $X_\bullet$ is an ∞-category. This was achieved by Boardman-Vogt.

Def 8: Let $X_\bullet \in sSet$, $x, y \in X_0$.
We say that $f, g : x \to y$ are left homotopic (resp. right homotopic), written $f \sim_\ell g$ (resp. $f \sim_r g$) if $\exists\, t \in X_2$ with:

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3ex, "t"] & & y \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3ex, "t"] & & y \end{tikzcd} \right)$$

$\square$

Left and right homotopy are not necc. equivalence relations; however we have

Lemma 9 Let C be an ∞-category. Then left and right homotopy coincide and is an equivalence relation.

Proof: Let $f, g, h : x \to y$ in $C(x, y)$.

We prove:

a) $f \underset{\ell}{\sim} f$.

b) $f \underset{\ell}{\sim} g$ and $g \underset{\ell}{\sim} h$ imply $f \underset{\ell}{\sim} h$.

c) $f \underset{\ell}{\sim} g$ implies $f \underset{r}{\sim} g$.

d) $f \underset{r}{\sim} g$ implies $g \underset{\ell}{\sim} f$.

<u>a)</u>: $t := f_{001}$ works:

$$\begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "f_{001}"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & {} & y \end{tikzcd}.$$

<u>b), c), d)</u>: They are proven in the same way:

- construct from the given 2-simplices a map $\Lambda^3_i \longrightarrow C$, with $i = \begin{cases} 1, & \text{for b) \& c)} \\ 2, & \text{for d)} \end{cases}$

- appeal to the inner horn extension property to get $\Delta^3 \to C$.

- restrict to the "new" 2-simplex.

It is easier in this case to display simplices so that the missing face is the back of the picture:

For b):

$$\begin{tikzcd}
x \arrow[dr, "id_x"] \arrow[dd, "id_x"'] \arrow[drr, bend left, "f"] \arrow[dd, phantom, shift left=6ex, "x_{000}"] & & \\
 & x \arrow[r, "g"'] \arrow[dl, "id_x"] \arrow[ur, phantom, "f \simeq_e g"] \arrow[dr, phantom, "g \simeq_e h"] & y \\
x \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

For c):

$$\begin{tikzcd}
x \arrow[dr, "id_x"] \arrow[dd, "f"'] \arrow[drr, bend left, "g"] \arrow[dd, phantom, shift left=6ex, "f \simeq_e g"] & & \\
 & x \arrow[r, "g"'] \arrow[dl, "g"] \arrow[ur, phantom, "g_{001}"] \arrow[dr, phantom, "g_{011}"] & y \\
x \arrow[urr, bend right, "id_x"'] & &
\end{tikzcd}$$

For d):

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[dd, "\mathrm{id}_x"'] \arrow[drr, bend left] \arrow[d, phantom, "f_{001}"] & & \\
 & y \arrow[r, "\mathrm{id}_y"] \arrow[ur, phantom, "f \simeq_R g"] & y \\
x \arrow[ur, "f"] \arrow[urr, bend right, "f"'] \arrow[rr, phantom, "f_{011}"] & &
\end{tikzcd}$$

Finally:

- c), d) $\Rightarrow$ $\simeq_e$ is symmetric.

+ a), b) $\Rightarrow$ $\simeq_e$ equivalence relation

+ c), d) $\Rightarrow$ $\simeq_e \, = \, \simeq_R$ $\square$

In this case we write $f \simeq g$ for $f \underset{R}{\simeq_e} g$.

and we write $[f]$ for the homotopy class of f in $C(x, y)$.

def 10: Let $C \in Cat_\infty^1$, $f : x \to y$, $g : y \to z$ and $h : x \to z$. We say that h is a composition of f and g if there is $t \in C_2$ with

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, phantom, "t"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z
\end{tikzcd}$$ $\square$

Prop 11: In an ∞-category, compositions exist; their homotopy class is well-defined and depends only on the homotopy classes of the morphisms being composed.

proof: • The existence is simply the extension property for $\Lambda^2_1 \hookrightarrow \Delta^2$.

• Let $\begin{cases} f \simeq f' : x \to y \\ g \simeq g' : x' \to y' \end{cases}$ and let $\begin{cases} h \text{ be a composition of } f \text{ and } g \\ h' \text{ ——————— } f' \text{ and } g'. \end{cases}$

We must prove that $h \simeq h'$. It is enough to treat separately the cases $f = f'$ and $g = g'$. By working in C^{op}, we reduce to $f = f'$. As before we construct a horn Λ^3_2, extend and restrict:

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[drr, bend left, "h"] \arrow[dd, "id_x"'] & & \\
& y \arrow[r, "g"] & z \\
x \arrow[ur, "f'"'] \arrow[urr, bend right, "h'"'] & &
\end{tikzcd}$$

$\square$

Prop 12: The resulting composition on homotopy classes of morphisms is associative and unital.

proof: Same method as Prop 11: left as exercise. $\square$

def 13 Let C be an ∞-category. Its homotopy category hC has:

$$\begin{cases} Ob(hC) = C_0 \\ hC(x,y) = C(x,y)/\simeq \end{cases}$$

By Prop 11-12, this is indeed a 1-category.

Prop 14: The construction of the homotopy category defines a functor $h : Cat^1_\infty \longrightarrow Cat$ which is a left adjoint to $N : Cat \longrightarrow Cat^1_\infty$.

($\Leftrightarrow$ $\tau C \simeq hC$ naturally in C)

proof: . The functoriality follows from the fact that for $F : C \to D$, $g \simeq g'$ in $C \Rightarrow F(g) \simeq F(g')$ which is clear.

. Let C be an ∞-category. We construct a natural equivalence (in fact isomorphism) of categories $\tau C \xrightarrow[\varphi]{\sim} hC$.

Because of Prop 3, to define φ it is enough to define $\begin{cases} C_0 \longrightarrow \mathrm{Ob}\, hC \\ C_1 \longrightarrow \mathrm{Mor}\, hC \end{cases}$

satisfying some relations given by identities and C_2.

We put $C_0 = \mathrm{Ob}\, hC$

$$C_1 \longrightarrow \mathrm{Mor}\, hC \,, \quad \gamma \longmapsto [\gamma]$$

and the relations are satisfied by def 13.

- By construction, φ is $\begin{cases} \text{bijective on objects} \\ \text{surjective on morphisms.} \end{cases}$

It remains to show φ is faithful.

Because of the lifting property for $\Lambda^2_1 \hookrightarrow \Delta^2$, any morphism in τC can be written as $\overline{\gamma}$ for $\gamma \in C_1$. Now suppose that $\gamma, \gamma' : x \to y$ satisfy $[\gamma] = [\gamma']$. By definition, there is an homotopy

$$\begin{tikzcd} x \arrow[d, equal] \arrow[dr, "\gamma'"] & \\ x \arrow[r, "\gamma"'] \arrow[ur, phantom, "\sigma"] & y \end{tikzcd}$$

, but this also implies $\mathrm{id}_x \circ \overline{\gamma'} = \overline{\gamma}$ in τC and we are done. $\square$

Prop 15: Let C, C' be ∞-categories. Then the canonical map $h(C \times C') \to hC \times hC'$ is an isomorphism of categories.

proof: It is a bijection on objects and surjective on morphisms, so it remains to see that it is injective on morphisms. If $([f],[g]) = ([f'],[g'])$ in $hC \times hC'$, we get triangles $t \in C_2$, $t' \in C_2'$ giving homotopies, but then $(t,t') \in C_2 \times C_2'$ implies that $[(f,g)] = [(f',g')]$ in $h(C \times C')$. $\square$

Cor 16: Let $F: C \to C'$ be a an equivalence of ∞-categories. Then $hF: hC \to hC'$ is an equivalence of categories.

proof: It is enough to show that a natural transformation $\alpha: F \Rightarrow G$ of functor $F, G: C \to D$ induces a natural transformation $h\alpha: hF \Rightarrow hG$, in such a way that

$$\begin{cases} h(\alpha \circ \beta) = h(\alpha) \circ h(\beta) \\ h(id) = id. \end{cases}$$

We leave this as an exercise, using

$$h(C \times \Delta^1) \underset{\text{Prop 15}}{\simeq} hC \times [1] \qquad \square$$

def 17 Let C be an ∞-category.

A morphism $f \in C_1$ is an isomorphism if $[f] \in Mor(hC)$ is an isomorphism, i.e if $\exists\, g \in C_1$ with $[f] \circ [g]$ and $[g] \circ [f]$ identities in hC.

Ex: • $\forall x \in C_0$, $id_x = s_0(x)$ is an iso.

• If C is a 1-category, then isomorphisms in NC are exactly the isos. in C.

def 18: An ∞-groupoid is an ∞-category in which every morphism is an iso.

Lemma 19: Kan complexes are ∞-groupoids.

proof: Same argument as part ⅱ) of Prop 4.

$\square$

The converse is true but much more difficult: we will see (at least parts of) the proof later.

thm 20: ∞-groupoids are Kan complexes.

Rmk: This will eventually imply that the (not yet defined) ∞-category of ∞-groupoids is equivalent to the ∞-category of Kan complexes, which by simplicial homotopy theory is equivalent to the ∞-cat. of topological spaces. $\Rightarrow$ Grothendieck's homotopy hypothesis holds in the quasicategory model of $(\infty,1)$-categories.

- Another important theorem, which will be our main goal in the next section of the course, is

thm 21: Let $K \in sSet$, $C \in Cat^1_\infty$.
Then the simplicial set $Fun(K,C)$, with $Fun(K,C)_n = sSet(K \times \Delta^n, C)$, is an ∞-category. $\square$

In fact, the definition of natural isomorphisms and categorical equivalences can be reformulated as:

- $\begin{pmatrix} \alpha : F \Rightarrow G \\ \text{natural iso} \end{pmatrix} \iff \begin{matrix} \alpha : F \to G \text{ isomorphism} \\ \text{in } Fun(C,D) \end{matrix}$

- $\begin{pmatrix} C \xrightarrow{F} D \\ \text{categorical} \\ \text{equivalence} \end{pmatrix} \iff \begin{pmatrix} \exists\, G : D \to C, \\ F \circ G \text{ iso. to } id_D \text{ in } Fun(D,D) \\ G \circ F \text{ ——— } id_C \text{ in } Fun(C,C) \end{pmatrix}$

. Another application of the homotopy category is to the notion of subcategory:

def 22: A subcategory C' of an ∞-category C is a simplicial subset which satisfies moreover: $\forall n \geq 2$, $\forall \gamma \in C_n$,

$$\gamma \in C'_n \iff \gamma_{|I^n} \text{ has edges in } C'_1.$$

Hence C' is determined by C'_0 and C'_1.

C' is a full subcategory if

$\forall n \geq 1$, $\forall \gamma \in C_n$,

$$\gamma \in C'_n \iff \text{the vertices of } \gamma \text{ lie in } C'_0.$$

C' is then determined entirely by C'_0.

lemma 23: A subcategory C' of an ∞-category C an ∞-category.

proof: This follows immediately from $I^n \subseteq \Lambda^n_k$.

$\square$

Prop 25: Let C be an ∞-category and $\eta : C \longrightarrow NhC$ be the unit map.

Then a) If D is a subcategory of the 1-category hC, then the pull back $\eta^{-1}(ND) := ND \times_{NhC} C$ is a subcategory of C.

$$\begin{tikzcd}
\eta^{-1}(ND) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
NC' \arrow[r, hook] & NhC
\end{tikzcd}$$

b) This construction gives a bijection

$$\{\text{Subcategories of } hC\} \simeq \{\text{Subcategories of } C\}$$

which restricts to a bijection of full subcategories.

proof: We only do the case of subcategories, the full case is easier.

a)

Claim | If C is a 1-category and C' is a (full) subcategory of C, then NC' is a (full) subcategory of NC.

pf: obvious from the unique lifting property of nerves along $I^n \hookrightarrow \Delta^n$.

Claim: | the pullback of a subcategory is a subcategory.

pf: Let $C' \subseteq C$ be a subcategory of an ∞-cat. and $E \xrightarrow{f} C$ be any morphism. Then $E' := E \times_C C'$ is a simplicial subset of E (pullback of mono is mono in any category)

By definition, we have for $n \in \mathbb{N}$: $E'_n = E_n \times_{C_n} C'_n$, so for $z \in E_n$, we have:

$$
\begin{aligned}
z \in E'_n &\overset{\text{pullback}}{\iff} f(z) \in C'_n \\
&\overset{\text{subcategory}}{\iff} f(z)_{|I^n} \text{ has edges in } C'_1 \\
&\iff f(z_{|I^n}) \text{ ——— } C'_1 \\
&\overset{\text{pullback}}{\iff} z_{|I^n} \text{ ——— } E'_1
\end{aligned}
$$

so that E' is a subcategory.

Together the two claims imply a).

b) <u>Injectivity:</u>

The morphism $C \longrightarrow NhC$ induces

- a bijection $C_0 \xrightarrow{\sim} (NhC)_0$
- a surjection $C_1 \twoheadrightarrow (NhC)_1$.

So if D is a subcategory of hC, then

$\eta^{-1} h D$ determines $\begin{cases} D_0 = \eta\left((\eta^{-1} N D)_0\right) \\ D_1 = \eta\left((\eta^{-1} N D)_1\right) \end{cases}$

hence determines D.

<u>Surjectivity:</u> Let $C' \subseteq C$ be a subcategory.

The natural candidate for D is hC', so let's try that! We first check that $hC' \to hC$ is a subcategory. We have $Ob(hC') = C'_0 \hookrightarrow C_0 = Ob(hC)$.

<u>Q</u>: $Mor(hC') = C'_1/_{homotopy} \overset{?}{\hookrightarrow} C_1/_{homotopy} = Mor(hC)$

If $f, g \in C'_1$ are homotopic in C,

it means there exists
$$\begin{tikzcd} & y \arrow[dr, equal] \arrow[d, phantom, "t"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & {} & y \end{tikzcd} \in C_2$$

but $t \in C_2' \subset C_2$ because f, id_y lie in C_1'.
and C' is a subcategory.

$\Rightarrow$ hC' is a subcategory of hC.

It remains to show that
$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ NhC' \arrow[r] & NhC \end{tikzcd}$$

is a pullback square. On n-simplices, this says that

$z \in C_n' \overset{?}{\iff}$ "the edges in $z_{|I^n}$ are homotopic in C to morphisms in C'."

But as we saw, this last condition is $\iff$

"the edges of $z_{|I^n}$ are morphisms of C_1'"

which by definition of a subcategory is precisely $\iff z \in C_n'$ $\square$

- To finish this introductory section, we need to see at least one example which does not come from nerves or Kan complexes.

The idea is that, in the same way that the nerve N gives

$$N : \{1\text{-categories}\} \subset \{\infty\text{-categories}\}$$

there should exist a "fully faithful" functor:

$$N_2 : \{(2,1)\text{-categories}\} \subset \{\infty\text{-categories}\}$$

This does exist! However:

Pb: The definition of general (weak) (2,1)-categories is complicated!

Even strict (2,1)-categories require

ideas that I don't want to discuss (yet).

Sol: We know <u>one</u> example of a strict $(2,1)$-category Cat^2:

$$\begin{cases} \text{Ob } Cat^2 : \text{ small categories} \\ \text{Mor } Cat^2 : \text{ functors} \\ 2\text{-Mor } Cat^2 : \text{ natural isomorphisms.} \end{cases}$$

So the plan is to construct an ∞-category $N_2(Cat^2)$ from this; the recipe then generalizes to any weak $(2,1)$-category (N_2 is called the Duskin nerve).

Example: We define a simplicial set Cat^2 as follows: an n-dimensional element in $(Cat^2)_n$ is the datum of:

- $\forall\ 0 \leq i \leq n$, a small category C_i
- $\forall\ 0 \leq i \leq j \leq n$, a functor $F_{i,j} : C_i \longrightarrow C_j$

- $\forall\, 0 \le i \le j \le k \le n$, a natural iso. $\alpha_{i,j,k} : F_{i,k} \Rightarrow F_{j,k} F_{i,j}$

such that :
- $F_{i,i} = id_{C_i}$
- $\alpha_{i,i,j}$ and $\alpha_{i,j,j}$ are identities.
- $\forall\, 0 \le i \le j \le k \le l \le n$,

the diagram
$$\begin{tikzcd} F_{i,l} \arrow[r, Rightarrow, "\alpha_{i,j,l}"] \arrow[d, Rightarrow, "\alpha_{i,k,l}"'] \arrow[dr, phantom, "\circledast"] & F_{j,l} F_{i,j} \arrow[d, Rightarrow, "\alpha_{j,k,l}"] \\ F_{k,l} F_{i,k} \arrow[r, Rightarrow, "\alpha_{i,j,k}"] & F_{k,l} F_{j,k} F_{i,j} \end{tikzcd}$$
commutes.

- For $\delta : [m] \longrightarrow [n]$, we define

$$\delta^*(C_i, F_{i,j}, \alpha_{i,j,k}) = (C_{\delta(i)}, F_{\delta(i),\delta(j)}, \alpha_{\delta(i),\delta(j),\delta(k)})$$

Rmk: $N\,Cat$ is isomorphic to the simplicial subcomplex of Cat^2 on the elements where

$$F_{i,k} = F_{j,k} F_{i,j} \quad \text{and} \quad \alpha_{i,j,k} = id.$$

So we have added some new higher morphisms to $N\,Cat$, corresponding to non-id. natural isomorphisms.

prop 25: Cat^2 is an ∞-category.

proof: · The proof is similar to the proof that the nerve of a category is an ∞-category; we just need to go "one dimension higher".

· By definition of Cat^2, an n-simplex is determined by its restriction to $\mathrm{Sk}_3(\Delta^n)$ ("Cat^2 is 3-coskeletal"). Since we have

$$\mathrm{Sk}_3(\Lambda^n_k) = \mathrm{Sk}_3(\Delta^n) \quad \text{for all } n \geq 4,$$

we only need to check the inner horn liftings for $n \leq 3$.

n=2 A map $\Lambda^2_1 \longrightarrow \mathrm{Cat}^2$ is just a datum

$$\begin{tikzcd} & C_1 \arrow[dr, "F_{12}"] & \\ C_0 \arrow[ur, "F_{01}"] & & C_2 \end{tikzcd} \quad \text{which we fill up with} \quad \begin{tikzcd} & C_1 \arrow[dr, "F_{12}"] \arrow[d, phantom, "\parallel"] & \\ C_0 \arrow[ur, "F_{01}"] \arrow[rr, "F_{12}F_{01}"'] & {} & C_2 \end{tikzcd}$$

<ins>Note</ins>: this is not unique in general, we can take any natural iso $G \simeq F_{12}F_{01}$.

n = 3 A map $\Lambda^3_1 \longrightarrow \mathrm{Cat}^2$ is

a diagram:

$$
\begin{tikzcd}[column sep=large, row sep=large]
C_0 \arrow[dd, "F_{01}"'] \arrow[dr, "F_{02}"] \arrow[drr, bend left=40, "F_{03}"] \arrow[drr, phantom, bend left=15, "\Downarrow \alpha_{023}"] & & \\
{} \arrow[r, phantom, "\overset{\alpha_{012}}{\Rightarrow}"] & C_2 \arrow[r, "F_{23}"] & C_3 \\
C_1 \arrow[ur, "F_{12}"] \arrow[urr, bend right=40, "F_{13}"'] \arrow[urr, phantom, bend right=15, "\Uparrow \alpha_{123}"] & &
\end{tikzcd}
$$

By the condition ⊛ in the definition,

we must fill this in with

$$
\begin{tikzcd}[column sep=large, row sep=large]
C_0 \arrow[dd, "F_{01}"'] \arrow[dr, "F_{03}"] & \\
{} \arrow[r, phantom, "\Downarrow \alpha_{013}"] & C_3 \\
C_1 \arrow[ur, "F_{13}"'] &
\end{tikzcd}
$$

with $\alpha_{013} = \alpha_{123}^{-1} \circ \alpha_{012} \circ \alpha_{023}$

and it works. $\square$