III Resolution of singularities of surfaces

<u>Motivation</u>: to get a minimal regular model, we need at least one! (regular model). Starting with some model (e.g a Weierstrass model) we need to resolve its singularities.

. The total space of a flat model of a curve is a 2-dimensional noetherian scheme. It is not completely arbitrary, but it is in general not of finite type over a field.

1) Resolution of singularities & quasi-excellence

This section can be almost ignored if you are not interested in hypergenerality or in making the theory pretty!

<u>def</u> X integral scheme.

- A <u>modification</u> of X is a proper birational morphism $\pi : X' \longrightarrow X$ with X' integral.
- A <u>resolution of singularities</u> of X is a modification $\pi : X' \to X$ with X' regular.

- Annoyingly, there are noetherian schemes which do not admit a resolution of sings, but most of those occurring in practice (are conjectured to) admit one.
- Let us look at the situation in dimension 1 & 2.

<u>Dimension 1</u>:

- Let R be an integral noetherian ring of dimension 1.

- R regular $\iff$ $\tilde{R}$ normal.

So *if* $R \hookrightarrow \tilde{R}$ normalization is proper ($\iff$ finite),

$\mathrm{Spec}(\tilde{R}) \longrightarrow \mathrm{Spec}(R)$ is a resolution of singularities.

- Two types of obstructions: one local and one global:

<u>local obstruction:</u>

<u>lemma:</u> R noetherian local ring of dim 1.
Then $R \hookrightarrow \tilde{R}$ finite $\iff$ $\hat{R}$ reduced.

<u>proof:</u> $\Rightarrow$ $\tilde{R}$ is then regular. This implies $\hat{\tilde{R}}$ (completion of a semilocal ring) is also regular. We have $\underset{\substack{\uparrow \\ \text{exactness of} \\ \text{completion in noeth.} \\ \text{setting}}}{\hat{R} \subseteq \hat{\tilde{R}}}$, which implies $\hat{R}$ reduced.

$\Leftarrow$ · Using flatness of $\hat{R}/R$, we have $\tilde{R} \otimes_R \hat{R} \hookrightarrow \mathrm{Frac}(R) \otimes_R \hat{R}$.

- Because $\hat{R}$ is reduced by assumption, we have $\hat{R} \hookrightarrow \mathrm{Frac}(\hat{R})$, hence also $\mathrm{Frac}(R) \otimes_R \hat{R} \hookrightarrow \mathrm{Frac}(R) \otimes_R \mathrm{Frac}(\hat{R}) = \mathrm{Frac}(\hat{R})$; in total $\tilde{R} \otimes_R \hat{R} \hookrightarrow \mathrm{Frac}(\hat{R})$.
- Integral morph. are stable by base change: $\tilde{R} \otimes_R \hat{R}$ is integral over $\hat{R}$.
- From $\hat{R} \underset{\text{integral}}{\hookrightarrow} \tilde{R} \otimes R \hookrightarrow \mathrm{Frac}(\hat{R})$ we deduce a factorisation $\hat{R} \hookrightarrow \tilde{R} \otimes_R \hat{R} \hookrightarrow \tilde{\hat{R}}$.
- We now use a result of Nagata, which is key to the rest of the story:

<u>thm:</u> Let A be a <u>complete</u> noetherian <u>integral</u> ring.
$A \hookrightarrow \tilde{A}$ is a finite ring map.

- We thus get $\hat{R} \hookrightarrow \tilde{\hat{R}}$ finite. Combined with the previous factorization we deduce that $\tilde{R} \otimes_R \hat{R}$ is a finite $\hat{R}$-module.

- By faithful flatness of $\hat{R}$ over R, we deduce that $\tilde{R}$ is a finite R-module, as needed. $\square$

<u>ex:</u> · There are examples of R not satisfying this:
- in char $p > 0$, finite extension of a DVR
- in char 0 (cannot then be f. ext of a DVR)

[St, 0BPB, 02JD]

c-ex: Let $k = \mathbb{F}_p(t_1, t_2, \cdots)$. We have $[k : k^p] = \infty$.

$$A = \left\{ \sum a_i x^i \in k[[x]] \;\middle|\; [k^p(a_0, a_1, \dots) : k^p] < \infty \right\}.$$

Then $k^p[[x]] \subset A \subsetneq k[[x]]$. For instance $\sum t_i x^i \notin A$.

- Can show that A is a DVR with res. field k and in fact $\hat{A} \simeq k[[x]]$.
- For $f \in k[[x]] \setminus A$, put $R = A[y]/_{y^p - f^p}$. We have $f^p \in k^p[[x]] \subseteq A$ so $A \longrightarrow R$ is finite. R is noetherian integral of dim 1 ($\Leftarrow$ $y^p - f^p$ irred.)
- Finally, we have $f^p = 0$ in $\hat{R}$, so that $\hat{R}$ is non-reduced.
- One can show by hand that $R \hookrightarrow \tilde{R}$ is not finite; see [Kollar_res, Claim 1.104]

global obstruction:

ex: There are examples of noetherian integral domains of dim 1 with infinitely many singular points. [Kollar, Ex. 1.105]

Such schemes clearly cannot have a resolution of singularities.

- These obstructions are actually the only ones:

prop: Let X be an integral noetherian scheme of dimension 1. Then X admits a resolution of singularities iff
1) the regular locus of X is open, and
2) $\forall x \in X$, $\widehat{\mathcal{O}}_{X,x}$ is reduced.

- There is a more precise statement:

def: Let X be a curve as in the proposition.
We construct the blow-up sequence of X as:
- $X_0 = X$
- $X_1 =$ blow-up of X_0 at the finitely many points of X_0
- $X_2 =$ " " " " " X_1
- $\circ\circ\circ$ (well defined because each step introduces only f. many sing pts)

- prop: For such a curve, X_n is regular for $n \gg 0$.
- c-ex: Let R be the ring from the previous c-ex, $X_0 = \operatorname{Spec} R$. Then X_1 contains as affine chart

$$\operatorname{Spec}\left(A\left[\tfrac{y}{x}\right] \Big/ \left(\tfrac{y}{x}\right)^p - \sum t_i^p x^{pi-p} \right) \underset{y_1 := \frac{y}{x} - t_1}{\simeq} \operatorname{Spec}\left(A[y_1] \Big/ y_1^p - \sum_{i \geq 1} x_{i+1}^p t^{pi} \right) \simeq X_0 \ (!)$$

Dimension 2: Here is the main result of this chapter:

thm (Lipman '78)
X noetherian 2-dimensional integral scheme (<u>surface</u> in this chapter)
Then X admits a resolution of singularities
$\iff$ · $\tilde{X} \longrightarrow X$ is finite.
· the regular locus of $\tilde{X}$ is open ($\iff$ finitely many singular points)
· $\forall x \in X$, $\widehat{\mathcal{O}}_{\tilde{X},x}$ is normal.

In fact, there is a more concrete result.

def X surface as in Lipman's theorem.
We construct the <u>normalized blow-up sequence</u> of X as
- $X_0 = X$
- X_1 = normalization of X_0 (finite$/X_0$ by hyp.)
- X_2 = normalization of the blow-up of X_1 at the finitely many singular points.
- ooo (need to show all new normalisations are finite)

thm: Let X surface as in Lipman's theorem.
Then X_n is regular for $n >> 0$.

· **rmk**: - Because normalizations are difficult to compute, in practice one prefers to resolve with a combination of blow-ups of smooth points and smooth curves: this is always possible over a field, by work of Hironaka.
- For rational singularities, only point blow-ups are needed: see later.

· The precise hyp of Lipman's are not so important. There are two classes of surfaces which satisfy them and which cover most applications:

- quasi-excellent surfaces
- normal fibered surfaces with smooth generic fiber.

- **Aside:** Quasi-excellent rings and schemes.

def
- R noeth ring is <u>quasi-excellent</u> if
  1) $\forall R'/R$ finite, the regular locus in $\mathrm{Spec}(R')$ is open.
  2) $\forall \mathfrak{p} \in \mathrm{Spec}(R)$, $\mathrm{Spec}(\hat{R}_\mathfrak{p}) \longrightarrow \mathrm{Spec}(R_\mathfrak{p})$ is regular, i.e it is flat (aut. by noetherianity) and its geometric fibers are regular.
- X noetherian scheme is <u>quasi-excellent</u> if it has an affine cover by spectra of q-exc rings.

rmk: Condition 2) implies that, for (P) = reduced, normal, CM, regular, ... $R_\mathfrak{p}$ has (P) $\iff$ $\hat{R}_\mathfrak{p}$ has (P).

thm R q-exc reduced $\Longrightarrow$ $R \hookrightarrow \tilde{R}$ finite.

rmk: The basic examples of q-exc rings are:
- fields
- Dedekind domains of generic char. 0
- <u>complete local noetherian rings</u> (most of the theory is built on red. to this case)
- any localisation of a finite type algebra over a q-exc ring.
- local rings of functions on complex analytic spaces.

C-ex: Let $k = \mathbb{F}_p(t_1, t_2, \cdots)$. We have $[k : k^p] = \infty$.

$$A = \left\{ \sum a_i x^i \in k[[x]] \;\middle|\; [k^p(a_0, a_1, \cdots) : k^p] < \infty \right\}.$$

Then A is a non quasi-excellent DVR; indeed, if it were, then $R = A\left[\sum t_i x^i\right]$ would be as well, but we have seen that $\hat{R}$ is non-reduced.

- The link between resolution of sings and quasi-excellence comes from the following theorem.

thm (Grothendieck) [EGA IV_2 7.9.5, Conrad-alt 2.3.6]
X noetherian scheme. If for all $U \subseteq X$ open and $Y \to U$ finite with Y integral, Y admits a resolution of singularities, then X is quasi-excellent.

<u>idea of proof:</u>

. We will show that the hypothesis implies the local rings of X satisfy: $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$ has geom. regular fibers. Can assume $X = \mathrm{Spec}(A)$. Let $p \in \mathrm{Spec}(A)$, $q \subseteq p$. Want to show $\kappa(q) \otimes_{A_p} \hat{A}_p$ geometrically regular$/\kappa(q)$.

. Can replace A by $(A/q)_p$ to get A local domain. Let $K = \mathrm{Frac}(A)$. Then we must show that for all K'/K finite, $K' \otimes_A \hat{A}$ is regular.

<u>Reduct° to $K = K'$:</u> Find $A \subseteq A' \subseteq K'$ finite over A with $K' = \mathrm{Frac}(A')$ ($\mathrm{Spec}(A') \leftrightarrow Y$ in hypothesis)

Then $K' \otimes_A \hat{A} \simeq K' \otimes_{A'} \left(\prod_m \hat{A'}_m \right)$ so we are reduced to $K = K'$.

<u>Case $K = K'$:</u> Applying hypothesis, let $Z \to X$ a res. of singularities.

. Put
$$\begin{tikzcd} Z' \arrow[r, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] \\ \mathrm{Spec}(\hat{A}) \arrow[r, "h"'] & \mathrm{Spec}(A) \end{tikzcd}$$
. Want to show that the gen. fiber of h is regular. By birationality of f, f', only need to show Z' is regular.

. $\hat{A}$ is q-excellent because complete $\Rightarrow Z'$ q-excellent $\Rightarrow \mathrm{Reg}(Z')$ open. Since $Z' \to \mathrm{Spec}(\hat{A})$ is proper, we only need to show that $\mathrm{Reg}(Z')$ contains the special fiber Z'_0. But for any $z' \in Z'_0$ we have $\hat{\mathcal{O}}_{Z',z'} \simeq \hat{\mathcal{O}}_{Z,h'(z)}$ by $Z' \simeq Z \otimes_{\mathrm{Spec}\,A} \mathrm{Spec}\,\hat{A}$. Hence we are done by the regularity of Z. $\square$

<u>conj:</u> (Grothendieck)

X noetherian quasi-excellent integral.

Then X admits a resolution of singularities.

<u>thm</u> Grothendieck's conjecture is true for X noeth. q-exc integral:

a) (Hironaka, Temkin) $\mathbb{Q}$-schemes

b) (Lipman) of dimension ≤ 2.

- Since not all Dedekind rings are q-excellent, not all fibered surfaces are q-excellent. Nonetheless, we have the following useful result.

<u>Prop:</u> Let S be a Dedekind scheme (not necc. excellent!), and $f : X \longrightarrow S$ be a proper flat morphism with $\begin{cases} X \text{ normal surface} \\ X_\eta/\eta \ \underline{\text{smooth}}. \end{cases}$

Then X satisfies Lipman's assumptions, hence has a resolution of singularities.

<u>Proof:</u> · By assumption X is normal, so the condition on $\tilde{X} \to X$ is automatic.

- From X_η/η smooth, we deduce that there exists an open set $U \subseteq S$ with X_U/U smooth. The singular locus of $X \to S$ is contained in the finitely many closed fibers above $S \setminus U$ and is closed in those fibers (which are just proper curves over a field!), hence $\mathrm{Reg}(X)$ is open.
- Let $x \in X_s$, $s \in S$. We want to show that $\hat{\mathcal{O}}_{X,x}$ is normal. Write $\hat{S} = \mathrm{Spec}(\hat{\mathcal{O}}_{S,s})$ and $\hat{X} = X \times_S \hat{S} \longrightarrow \hat{S}$ fibered surface (a priori not even normal). We have $\hat{X}_s \simeq X_s \ni x$, and $\hat{\mathcal{O}}_{\hat{X},x} \simeq \hat{\mathcal{O}}_{X,x}$ [Liu, 8.3.49(b)]
Now $\hat{S}$ is q-excellent because $\hat{\mathcal{O}}_{S,s}$ is complete. Hence $\hat{X}$ is also q-excellent. So the morphism $\mathcal{O}_{\hat{X},x} \longrightarrow \hat{\mathcal{O}}_{\hat{X},x}$ is regular, and to prove that $\hat{\mathcal{O}}_{\hat{X},x}$ is normal it is enough to show that $\mathcal{O}_{\hat{X},x}$ is normal. We show that in fact $\hat{X}$ is normal.
- Let $\tilde{\hat{X}} \to \hat{X}$ be the normalization morphism. It is finite because $\hat{X}$ is q-excellent. The key observation is that it is an iso above $\hat{X}_{\mathrm{Frac}(\hat{\mathcal{O}}_{S,s})}$. This is because by assumption and base change, $\hat{X}_{\mathrm{Frac}(\hat{\mathcal{O}}_{S,s})}$ is smooth over $\mathrm{Spec}(\mathrm{Frac}(\hat{\mathcal{O}}_{S,s}))$, hence regular, hence normal. We deduce that $\tilde{\hat{X}} \to \hat{X}$ is a finite morphism, iso over the generic fiber. As any such morphism, it can be obtained from a blow-up of a closed subscheme Z in $\hat{X}_s$. Since $\hat{X}_s \simeq X_s$, we can consider Z as a closed subscheme in X_s. Then we have
$$\tilde{\hat{X}} = Bl_Z \hat{X} \underset{\uparrow}{\simeq} (Bl_Z X) \times_S \hat{S}$$
$\llcorner$ flatness arg [Liu, 8.3.48]

; the morphism $Bl_Z X \to X$ is finite birational, which by normality of X forces it to be an iso. This means that Z is a Cartier divisor in X. It is then also Cartier in $\hat{X}$, so that $\tilde{\hat{X}} \xrightarrow{\sim} \hat{X}$ and we are done. $\square$

· Examples Before discussing the general theory let us look at some examples. We look at hypersurfaces in a regular threefold, and try to resolve them by pt blow-ups.

Ex 1: R DVR, $a \in R \setminus \{0\}$, $X = \mathrm{Spec}\left(R[x,y]/_{xy-a} \right) \longrightarrow S = \mathrm{Spec}(R)$.

$X \to S$ is flat, smooth everywhere if $a \in R^*$, smooth outside of $x = y = 0$.

If $v(a) = 1$, then $xy - a \in m \setminus m^2$ with m max ideal of $R[x,y]$ defining P

so that X is regular at P. Let $e = v(a) \geq 2$, write $a = \pi^e \cdot u$, $u \in R^\times$.

. X is normal (see the proof we gave for Weierstrass models).

. Let $X_1 = Bl_P X = Bl_{(x,y,\pi)} X \longrightarrow X$.

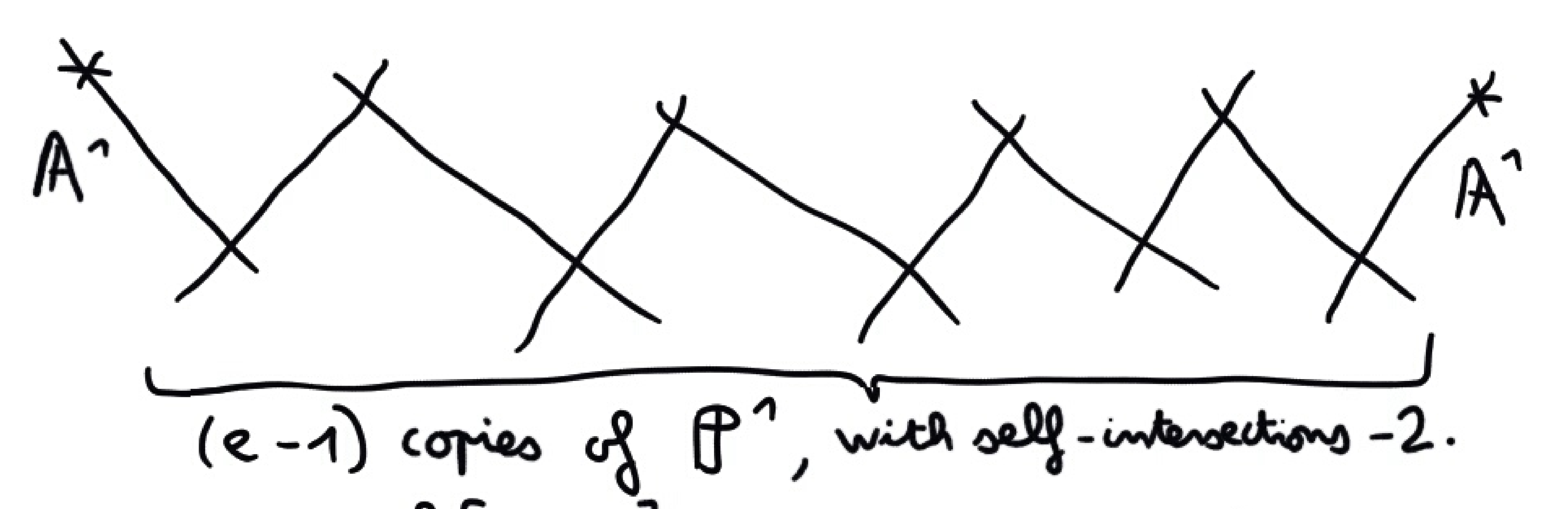
X_1 is covered by three affine charts $\mathrm{Spec}\, A_i$, $1 \leq i \leq 3$, with

$$A_1 = R\left[x, y, \frac{x}{\pi}, \frac{y}{\pi}\right]/_{xy-a} = R[x_1, y_1]/_{x_1 y_1 - \pi^{e-2} u}$$

$$\left.\begin{aligned} A_2 &= R\left[x, y, \frac{y}{x}, \frac{\pi}{x}\right]/_{xy-a} = R[x, \pi_2]/_{x\pi_2 - \pi} \\ A_3 &= R\left[x, y, \frac{x}{y}, \frac{\pi}{y}\right]/_{xy-a} = R[y, \pi_3]/_{y\pi_3 - \pi} \end{aligned}\right\} \text{regular since } v(\pi) = 1$$

· A_1 has a singularity of the same type as A, but with $e \rightsquigarrow e-2$.

. We deduce that, after $\lfloor \frac{e}{2} \rfloor$ blow-ups at reduced points, we get a resolution of singularities of X, with special fiber.



$(e-1)$ copies of $\mathbb{P}^1$, with self-intersections -2.

. These singularities are A_{e-1}-rational double points.

Ex 2: $R = k[x,y,z]/_{x^2+y^3+z^5}$ with k field of char $\neq 2, 3, 5$.

This is the E_8 rational double point singularity, or E_8 Du Val singularity. Such singularities play an important role in Lipman's approach, we will say more about them later. We follow the discussion in [Hauser, § Simplicity] for the resolution.

1st blowup: Only one singular affine chart, normal with isolated singular point.

$x = x_3 z_3$, $y = y_3 z_3$, $z = z_3 \rightsquigarrow x_3^2 + y_3^3 z_3 + z_3^3 = 0$ (E_7-singularity)

↑ eq. of strict transform of $\mathrm{Spec}(R)$.

2nd blow-up: only one singular affine chart, normal with isolated singularity:

$$x = xy_2,\ y = y_2,\ z = z_2 y_2 \rightsquigarrow x_2^2 + y_2^2 z_2 + y_2 z_2^3 = 0 \quad (\sim D_6\text{-singularity})$$

3rd blow-up: two singular charts:

(i) $x_2^2 + y_2 z_2 + y_2^2 z_2^3 = 0$; of type $y_1 z_1 = a$ with $v_x(a) = 2 \rightsquigarrow A_1$-sing. res by 1 extra blow-up.

(ii) $x_3^2 + y_3^2 z_3 + y_3 z_3^2 = 0$; D_4-singularity

4th blow-up on chart (ii): two singular charts:

(a) $x_1^2 + y_1 z_1 + y_1 z_1^3 = 0 \rightsquigarrow$ type A_1, res by 1 extra blow-up

(b) $x_3^2 + y_3^3 z_3 + y_3 z_3 = 0 \rightsquigarrow$ type A_1, res by 1 extra blow-up

⚠ There is an additional singularity, namely $(x_1, y_1, z_1) = (0,0,-1)$

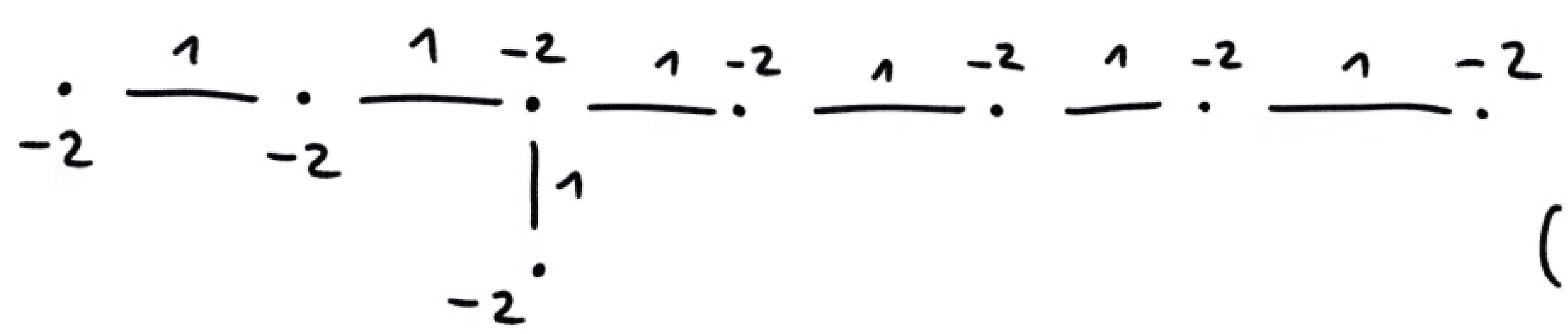
$$\Updownarrow$$

$$(x_3, y_3, z_3) = (0,-1,0)$$

It is also of type $A_1 \rightsquigarrow$ resolved by 1 blow-up.

. Summarizing: R resolved by 8 pt blow-ups at k-rational points.

Exceptional divisors are $\simeq \mathbb{P}^1$, and they have the configuration with int. numbers:



$\simeq$ the Dynkin diagram E_8! (uses hyp on char(k))

. We have no (-1)-curves, hence the resolution is minimal. The fact that we only needed blow-ups (no normalization) and that we directly obtained the minimal res. this way are features of rational singularities.

Ex 3: $R = \operatorname{Spec}\left(k[x,y,z]/_{x^2+y^3+z^6} \right)$, k field of char $\neq 2, 3$.

. This is a non-rational singularity. We try to resolve by pt blow-ups.

. 1st blow-up: only 1 singular chart, normal with one singular point.

Equation : $x_3^2 + y_3^3 z_3 + z_3^4 = 0$

. 2nd blow-up: two singular charts, non-normal !

(i) $x_2^2 + y_2^2 z_2 + y_2^2 z_2^4 = 0 \rightsquigarrow$ ring $R_{32} = k[x_2, y_2, z_2]/.$

(ii) $x_3^2 + y_3^3 z_3^2 + z_3^2 = 0 \rightsquigarrow$ ring $R_{33} = k[x_3, y_3, z_3]/.$

- In fact, $\left(\frac{x_2}{y_2}\right)^2 + z_2 + y_2 z_2^4 = 0$ shows that $\frac{x_2}{y_2} \in \mathrm{Frac}(R_{32})$ is integral over R_{32}. Moreover, $R_{32}\left[\frac{x_2}{y_2}\right] \simeq k[a,b,c]/_{a^2+b+b^4c}$ is smooth $\rightsquigarrow$ this is the normalization
- Similarly, $R_{33} \longrightarrow R_{33}\left[\frac{x_3}{z_3}\right] \simeq k[d,e,f]/_{d^2+e^3+1}$ is smooth $\rightsquigarrow$ this is the normalization.

- By normalizing, we get a resolution of singularities $\tilde{X} \to \mathrm{Spec}(R)$.
- We have two exceptional divisors C_1 and C_2 from the two blow-ups.

One can show : - $C_1 \simeq \mathbb{P}^1$, C_2 elliptic curve. (cf [Kollar, 2.2.4])

- $C_1^2 = -1$, $C_1 \cdot C_2 = 1$, $C_2^2 = -2$.

Hence C_1 is an exceptional curve (Castelnuovo) and can be contracted.

We get $\tilde{X} \longrightarrow \bar{X} \longrightarrow \mathrm{Spec}(R)$ with $\bar{X} \to \mathrm{Spec}(R)$ minimal resolution.

with one elliptic curve in the fiber.

2) Lipman's proof [Artin_res, Lipman_res, Stacks Chap 0ADW]

def | X normal surface, $x \in X$ closed.

- x is a <u>rational singularity</u> if $\forall f : X' \to X$ modification $\left(R^1 f_* \mathcal{O}_{X'}\right)_x = 0$.
- x is a <u>rational double point</u> if x is a rational singularity and the multiplicity of X at x is ≤ 2.

$\mu_x(X) = \mu(\mathcal{O}_{X,x})$ def by $H(\mathcal{O}_{X,x}, n) = \dfrac{\mu \cdot n^{(d-1)}}{(d-1)!}$ + lower-order terms.

- Process:
- Step 0:
  - any modification is dominated by the normalized blow-up sequence.
  - reduction to complete local ($\Rightarrow$ q-e) case.

- $$\begin{tikzcd} X & X' \arrow[l, "\text{Step 1}"'] & X'' \arrow[l, "\text{Step 2}"'] & X''' \arrow[l, "\text{Step 3}"'] \end{tikzcd}$$

Step 1: any mod. with maximal $\mathrm{len}(R^1 f_* \mathcal{O})$

X': rational sings.

Step 2: suitable blow-up to make ω_X locally free

X'': rational double points

Step 3: explicit resolution by blow-ups.

X''': regular

- Step 0:

- **prop** | X surface satisfying Lipman's assumptions.
  Any modification of X is dominated by some X_n in the normalized blow-up sequence.

- Let $x \in X$ be a singular point. Then one can show that the normalized blow-up sequence of $\hat{X}_x$ is obtained from that of X by localization and completion. Since we have finitely many singular points, a patching argument reduces us to the complete local case.

Step 1:

- The key tool to control $R^1 f_* \mathcal{O}$ and get to rational singularities is the dualizing sheaf.
- Let X be a normal surface. It is then Cohen-Macaulay (recall normal $\Rightarrow$ (S2), and CM $\Leftrightarrow$ ($S_{\dim}$)).

Hyp: | X has a dualizing sheaf ω_X.

(This is unfortunately not automatic even for X q-ex CM surface. However it is ok if one assume moreover X local complete, and this is enough for Lipman's proof [St 0BFR])

- ω_X is a generalization of the more usual $\omega_{X/k}$ for X proj CM over a field.

Relative duality

- Let $f : Y \longrightarrow X$ proper morphism, with Y also assumed CM. Then Y also has a dualizing complex (in Grothendieck's 6 ops formalism, $\omega_Y = f^! \omega_X$).

thm: (relative duality) | Let $F \in D^b_c(\mathcal{O}_X)$. Then
$$R f_* R\mathcal{H}om(F, \omega_Y) \simeq R\mathcal{H}om(R f_* F, \omega_X).$$

- We apply this to f modification and $F = \mathcal{O}_Y$:

$$R f_* \omega_Y \simeq R\mathcal{H}om(R f_* \mathcal{O}_Y, \omega_X)$$

lemma: |
a) $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$. ($\Leftarrow$ normality)
b) $R^1 f_* \mathcal{O}_Y$ is a finite length $\mathcal{O}_X$-module.
c) $R^i f_* \mathcal{O}_Y = 0$ for $i > 1$. ($\Leftarrow$ dim fibers ≤ 1)
d) $\mathcal{E}xt^i(\mathcal{O}_Y, \omega_Y) = 0$ for $i \geq 1$ ($\Leftarrow$ $\mathcal{O}_Y$ locally free)

- Applying this lemma to the isomorphism in the derived category we get a short exact sequence of sheaves on X:

$$\mathcal{E}xt^1(R^1 f_*\mathcal{O}_Y, \omega_X) \to f_*\omega_Y \to \omega_X \to \mathcal{E}xt^2(R^1 f_*\mathcal{O}_Y, \omega_X) \to R^1 f_*\omega_Y$$

- To go further, we apply another property of ω_X, local duality (applied at the finite support of $R^1 f_*\mathcal{O}_Y$):

<u>thm:</u> (Local Grothendieck duality)
E coherent sheaf on X with finite support.
Then for all $i \neq 2$, $\mathcal{E}xt^i(E, \omega_X) = 0$,
$E^D := \mathcal{E}xt^2(E, \omega_X)$ is a coh. sheaf with the same support.
Moreover, $E \simeq (E^D)^D$, and the functor $(-)^D$ is exact.

<u>rmk:</u> - For X smooth over k of dim d, $\omega_X = \omega_{X/k} \simeq \Omega^2_{X/k}$ locally free
- Let $x \in X$, (t_1, t_2) local parameters at x, $E = x_*(k)$ skyscraper sheaf.
We have, locally around x, the Koszul resolution:

$$0 \to \mathcal{O} \xrightarrow{\binom{t_2}{-t_1}} \mathcal{O} \oplus \mathcal{O} \xrightarrow{(t_1, t_2)} \mathcal{O} \to x_* k \to 0$$

which implies: $\mathcal{E}xt^i(x_* k, \Omega^2_{X/k}) \simeq H^i(0 \to \Omega^2 \to (\Omega^2)^{\oplus 2} \to (\Omega^2) \to 0)$
$\simeq 0$ unless $i = 2$, in which case it is a skyscraper sheaf, generated by $dt_1 \wedge dt_2$.

- Plugging this in, we get:

$$0 \to f_*\omega_Y \to \omega_X \to (R^1 f_*\mathcal{O}_Y)^D \to R^1 f_*\omega_Y$$

<u>cor:</u> X has rational sings $\Rightarrow f_*\omega_Y \xrightarrow{\sim} \omega_X$ for any mod. f.

lemma | Let $\begin{tikzcd} Z \arrow[rr, "\pi"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$ be a diagram of modifications of X.

(i) There is an exact sequence

$$0 \to R^1 f_* \mathcal{O}_Y \to R^1 g_* \mathcal{O}_Z \to f_* R^1 \pi_* \mathcal{O}_Z \to 0$$

(ii) If X has rational sings, then so do Y (and Z).

proof: · (i): follows from the low-degree exact sequence from the SS

$$E_2^{p,q} = R^p f_* R^q \pi_* \mathcal{O} \Rightarrow R^{p+q} g_* \mathcal{O}$$

and the previous lemma.

· (i) $\Rightarrow$ (ii) $\square$

. So to complete step 1, it suffices to show that

$$\left\{ \operatorname{len}\left(R^1 f_* \mathcal{O}_Y \right) \middle| f \right\} \text{ is bounded.}$$

rmk: Lipman makes the "tantalizing reformulation" that this is equivalent to $H^1(\mathcal{O}_{RZ(X)})$ being finite-dim, with $RZ(X)$ the Riemann-Zariski space of X.

thm | (Grauert - Riemenschneider)

X normal surface with a dualizing sheaf.

$f : Y \to X$ modification.

Then $R^1 f_* \omega_Y = 0$

rmk: | For X, Y smooth over a field, this is still an interesting result, and the proof may seem more transparent.

proof: We can assume f iso. away from $x \in X$.

· One basic idea is that, on the normal surface Y, there is a kind of intersection theory, which satisfies analogous properties to the one on a regular surface. Namely, given a Weil divisor Z, we can consider the sheaf $\mathcal{O}_Y(Z)$ of rational functions with poles $\leq Z$. Let C be an irreducible component of $f^{-1}(p)$. and $\tilde{C} \xrightarrow{f} C$ its normalization. Put

$$\mathcal{O}_{\tilde{C}}(Z) := \left[f^*\left(\mathcal{O}_Y(Z) \otimes_{\mathcal{O}_Y} \mathcal{O}_C \right) \right] / \text{torsion}$$

; this is a torsion-free rk 1 sheaf on a regular curve $\Rightarrow$ invertible sheaf.

• Now put $Z \cdot C = \deg_{k(p)} \mathcal{O}_{\bar{C}}(Z)$.

These intersection numbers satisfy the same key negativity property of the fiber as in the case X' regular.

<u>prop</u>: Let $Z = \sum a_i C_i \geq 0$ with C_i comp. of $f^{-1}(p)$, $a_i \in \mathbb{N}$, $Z \neq 0$.

Then there exists j such that $Z \cdot C_j < 0$, which implies

$H^0(\bar{C}_j, \mathcal{O}_{\bar{C}_j}(Z))$, hence also $H^0(C_j, \mathcal{O}_{C_j}(Z)) = 0$.

• Note also that $Z \cdot C < 0 \Rightarrow Z - C \geq 0$.

• This negativity property is used through:

<u>Lemma</u>: Let Z be as above. Then

$$R^1 f_* \mathcal{O}_Y \hookrightarrow R^1 f_* \mathcal{O}_Y(Z).$$

<u>proof</u>: We have $H^0(C_j, \mathcal{O}_{C_j}(Z)) = 0$. This implies

$$H^1(Y, \mathcal{O}_Y(Z - C_j)) \hookrightarrow H^1(Y, \mathcal{O}_Y(Z)).$$

Since $Z - C_j \geq 0$, unless $Z = C_j$, we can find C_k with $(Z - C_j) \cdot C_k < 0$

... By induction, we find:

$$H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(Y, \mathcal{O}_Y(Z)).$$

This computation applies with Y replaced by $f^{-1}(U)$ for any open neighborhood of p. The result follows. $\square$

• Since $R^1 f_* \omega_{X'}$ is supported at p, we deduce from the theorem on formal functions that: $R^1 f_* \omega_{X'} \simeq \lim_Y R^1 f_* (\omega_{X'} \otimes \mathcal{O}_Y)$.

Since $R^2 f_*(-)$ vanishes, the transition maps in that projective system are surjective, hence it has to stabilize. This implies in turn that for Y big enough, we have

$$R^1 f_* \mathcal{O}_{X'}(-Y) \xrightarrow[= 0]{\phi_Y} R^1 f_* \mathcal{O}_{X'}.$$

· Now we have, using duality, a diagram:

$$\begin{tikzcd} 0 \arrow[r] & f_* \omega_{X'} \arrow[r] & \omega_X \arrow[r] & (R^1 f_* \mathcal{O}_{X'})^D \arrow[r] & R^1 f_* \omega_{X'} \arrow[r] & 0 \\ 0 \arrow[r] & f_* \omega_{X'}(-y) \arrow[r] \arrow[u] & \omega_X \arrow[r] \arrow[u, equal] & (R^1 f_* \mathcal{O}_{X'}(y))^D \arrow[r] \arrow[u, "\psi_y"'] & R^1 f_* \omega_{X'} \arrow[r] \arrow[u, "\phi_y"'] & 0 \end{tikzcd}$$

$\left\{\begin{array}{l} \text{The map } \psi_y \text{ is the dual of the map in the lemma} \Rightarrow \psi_y \text{ surjective.} \\ \text{The map } \phi_y \text{ is } 0 \text{ for } y >> 0 \end{array}\right.$

This implies by a diagram chase that $R^1 f_* \omega_{X'} = 0$ □

· We thus get an exact sequence:

$$0 \to f_* \omega_{X'} \to \omega_X \to (R^1 f_* \mathcal{O}_{X'})^D \to 0$$

<u>cor:</u> | p is a rational singularity iff $\forall f : X' \to X$ mod., $f_* \omega_{X'} \xrightarrow{\sim} \omega_X$.

We see that, to complete step 1), it is enough to show that the length of the finite $\mathcal{O}_X$-module $\left(\dfrac{\omega_X}{f_* \omega_{X'}}\right)$ is bounded indpt of f.

· This is the step of the proof which is genuinely harder in the arithmetic setting. We only present the simplest case and refer to [Artin_res, Lipman_res] for the general case.

Namely:

<u>Lemma:</u> | Assume X is of finite type over a perfect field k.
Then there is a canonical map $\Omega^2_{X/k} \to \omega_{X/k} \simeq \omega_X$
and it factors through $f_* \omega_{X'}$.
Hence $\operatorname{len}\left(\dfrac{\omega_X}{f_* \omega_{X'}}\right) \leq \operatorname{len}\left(\dfrac{\omega_X}{\operatorname{Im} \Omega^2_{X/k}}\right)$ is bounded indpt of f.

<u>proof:</u> This follows from the fact that Ω^2 has the opposite functoriality to ω:

$$\begin{tikzcd} \Omega^2_X \arrow[r] \arrow[rrr, bend left] & f_* \Omega^2_{X'} \arrow[r] & f_* \omega_{X'} \arrow[r, hook] & \omega_X \end{tikzcd}$$ commutes. □

· **Step 2** Assume X has rational singularities.

Want to reduce to the case of rational double points. This relies on the following facts.

· **Fact:** · For a rational singularity, multiplicity = embedding dimension:

$$\mu_x(X) = \dim_{k(x)}\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) - 1. \quad (\text{in general only} \leq)$$

· In fact, the whole Hilbert function is determined by μ:

$$\dim_k \mathfrak{m}^r/\mathfrak{m}^{r+1} = r\mu + 1.$$

Fact: The blow-up of a rational singular point is normal, and

$$\omega_{Bl_x X} \simeq f^* \omega_X.$$

It is at least plausible that we understand the blow-up since we know the Hilbert function.

Prop: Let p be a rational singularity of multiplicity μ. If ω_X is locally free at p, then $\mu \leq 2$.

proof: Let x, y be a regular sequence in $\mathfrak{m}_p$. The multiplicity μ is the length of the artinian ring $\bar{\mathcal{O}} = \mathcal{O}_{X,p}/(x,y)$.

Let $\bar{\mathfrak{m}}$ be the maximal ideal of $\bar{\mathcal{O}}$. We have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \mu + 1$, so $\dim \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 \geq \mu - 1$. So $\bar{\mathfrak{m}}^2 = 0$ and $\dim \bar{\mathfrak{m}} = \mu - 1$ (since $\dim_k \bar{\mathcal{O}} = 1 + \dim \bar{\mathfrak{m}} = \mu$)

. The dualizing module $\bar{\omega}$ of $\bar{\mathcal{O}}$ is isomorphic to $\omega/(x,y)\omega$.

Since $\bar{\mathcal{O}}$ is zero-dim, $N \rightsquigarrow \mathrm{Hom}_{\bar{\mathcal{O}}}(N, \bar{\omega})$ is a perfect duality on finite length $\mathcal{O}$-module. From $0 \to \bar{\mathfrak{m}} \to \bar{\mathcal{O}} \to k \to 0$, we thus deduce

$$0 \to k \to \bar{\omega} \to \underbrace{\mathrm{Hom}_{\bar{\mathcal{O}}}(\bar{\omega}, \bar{\mathfrak{m}})}_{\dim\ \mu - 1} \to 0.$$

Since we assumed ω locally free, this forces $\dim \bar{\omega} \leq 2$, hence $\mu \leq 2$ □

- With all these facts at hand, we can finish the proof of Step 2.
- Pick a modification $f : X' \to X$ with $f^* \omega_X$ locally free. This is possible because we can blow-up the module ω_X (which is locally an ideal because it is generically free [Stacks, ØBBV].
- This is dominated by a sequence of normalized blow-ups. Since X is rational, it is in fact dominated by a sequence of blow-ups. We then get a sequence of blow-ups $X'' \xrightarrow{g} X$ with $g^* \omega_X \simeq \omega_{X''}$ locally free.

<u>Step 3</u>: X normal surface with rational double point singularities.

- There is now a relatively simple proof that blowing singular points (which still produces normal rational double points) eventually results in a regular surface. See [Artin_res, § 6].
- In fact RDPs can be classified to a great extent! We only explain the situation for an algebraically closed residue field of char $\neq 2,3,5$. The general case is in [Lipman_rat, Part VI].

<u>thm</u>: Let R be a 2-dimension noetherian local ring with algebraically closed residue field of char $\neq 2,3,5$. Assume that the closed point of Spec(R) is an RDP. Then there exists a complete 3-dim regular local ring S and $x, y, z \in m_S \setminus m_S^2$ such that $\hat{R}$ is one of the following:

(i) $S/(x^2+y^2+z^{n+1})$, $n \geq 1$ (type A_n)

(ii) $S/(x^2+z(y^2+z^{n-2}))$, $n \geq 4$ (type D_n)

(iii) $S/(x^2+y^3+y^4)$ (type E_6)

(iv) $S/(x^2+y(y^2+z^3))$ (type E_7)

(v) $S/(x^2+y^3+z^5)$ (type E_8; we have seen it before)

- In each case, the fiber of the minimal resolution of Spec$(\hat{R})$ is a configuration of rational (-2)-curves which form a Dynkin diagram of the corresponding type.

<u>rmk</u>: In complex geometry, RDPs are often called Du Val singularities or Kleinian singularities or ADE singularities. They play an important role in enumerative geometry and representation theory.