Reminder on Grothendieck topologies and descent:

- The theory of the Picard scheme requires an healthy dose of sheaf and descent theory. Here is a quick recap of the basic ideas; see [BLR, Chap 6, Chap 8] for a thorough introduction tailored for the theory of the Picard scheme.
- In algebraic geometry, the Zariski topology and Zariski sheaves are good enough for the study of coherent sheaves and their cohomology. To go beyond these, Grothendieck developed the notion of Grothendieck (pre)topology.

def: Let $\mathcal{C}$ be a category admitting fiber products (for us, $\mathcal{C} = (\text{Schemes})$ or $(\text{Schemes}/_S)$).
A Grothendieck pretopology on $\mathcal{C}$ is the data, for every object $X \in \mathcal{C}$, of a set of covering families $\{U_i \to X\}_{i\in I}$ such that:

(i) For any isomorphism $X' \xrightarrow{\sim} X$, $\{X' \to X\}$ is a covering family.

(ii) For any covering family $\{U_i \to X\}_{i\in I}$ and any morphism $Y \to X$, $\{U_i \times_X Y \to Y\}_{i\in I}$ is a covering family. } base change

(iii) For any covering family $\{U_i \to X\}_{i\in I}$ and covering families $\{V_{i,j} \to U_i\}_{j\in J_i}$, the set of all compositions $\{V_{i,j} \to U_i \to X\}$ is a covering family. } locality

- A category equipped with a Grothendieck pretopology is called a site.

ex: Let X be a scheme. A family $\{U_i \to X\}$ is covering for

* the Zariski pretopology if $U_i \to X$ open immersion and $\coprod U_i \to X$ surjective.
* the étale pretopology if $U_i \to X$ étale morphism " " " "
* the fppf pretopology if $U_i \to X$ is flat and locally of finite presentation " " " "

Clearly, an fppf covering is an étale covering, and an étale covering is a Zariski covering. We say that the fppf topology is finer than the étale topology, ...

def: Let $(\mathcal{C}, \tau)$ be a site and $F : \mathcal{C}^{op} \to \text{Sets}$ be a functor (i.e a presheaf on $\mathcal{C}$).
F is a sheaf (for τ) is for every covering family $\{U_i \xrightarrow{f_i} X\}$,

$$F(X) \xrightarrow{\prod f_i^*} \prod_i F(U_i) \underset{\prod (f_j \times id)^*}{\overset{\prod (id \times f_k)^*}{\rightrightarrows}} \prod_{j,k} F(U_j \times U_k)$$ is an equalizer diagram.

(i.e, "a section of F on X is the same thing as compatible sections on F on the U_i's."). If F is a presheaf of abelian groups, this is equivalent to saying that the induced sequence

$$0 \to F(X) \to \prod F(U_i) \longrightarrow \prod F(U_j \times_X U_k)$$ is exact.

- For sheaves of abelian groups on a site, one can then develop the theory of sheaf cohomology as for usual topological spaces.
- For descent theory, we refer to [BLR, Chap 6].

IV Picard functors and Picard schemes

- As we have already seen in this course, line bundles on varieties, including on singular varities and in families, play an important role in the study of models. We introduce a systematic, geometric way of understanding families of line bundles, the <u>Picard functor</u> and study its representability.
- The Picard functor is a useful tool for many things in algebraic geometry. For us, the main motivations are:
  - * duality theory for abelian varieties,
  - * the construction of Jacobians of smooth projective curves, and
  - * Raynaud's theorem on Néron models of Jacobians via the Picard functor.

1) Picard functors

- <u>def</u>: S scheme. $\mathrm{Pic}(S) := \{\text{line bundles on } S\}/\text{isomorphism}$.

  $\mathrm{PIC}(S) :=$ groupoid of line bundles on S + (iso)morphisms.; $\mathrm{Pic}(S) = \pi_0(\mathrm{PIC}(S))$

- The tensor product of line bundles makes $\mathrm{Pic}(S)$ into an abelian group with unit $\mathcal{O}_S$.

  ($\mathrm{PIC}(S)$ is a "Picard groupoid", i.e "an abelian group object in the 2-cat. of groupoids")

<u>Lemma</u>: (i) $\mathrm{PIC}(S) \simeq$ (Principal $\mathbb{G}_m$-bundles for the Zariski topology)

$$\mathcal{L} \longmapsto \mathcal{L}^\times := \mathcal{L} \setminus 0(S) \text{ with } 0: S \to \mathcal{L} \text{ zero-section.}$$

$$\mathcal{S} \overset{\mathbb{A}^1}{\times} \mathbb{G}_m \longleftarrow \mathcal{S}$$

(ii) Let τ be one of the following topologies: Zariski, étale, fppf, fpqc. $\mathbb{G}_m$ is a τ-sheaf and $\mathrm{Pic}(S) \simeq H^1_\tau(S, \mathbb{G}_m)$.

<u>proof</u>: (i) is easy.

(ii) follows from (i) for $\tau = \mathrm{Zar}$, by the standard result on classification of G-torsors by $H^1(-, G)$. To prove it for the other topologies, enough to show that the fibered category $S \mapsto \mathrm{PIC}(S)$ satisfies descent for the topology τ, and this follows from fpqc descent for coherent sheaves, together with the additional fact that a locally free sheaf with a descent datum descends to a locally free sheaf of the same rank (in algebraic terms, this is fpqc descent for projective finite rank modules, which follows easily from the fact that a module over a ring is projective finite rank iff it is finitely presented & flat) $\square$

- For any morphism $g: T' \to T$, there is a pullback map / functor:

  $g^*: \mathrm{Pic}(T) \to \mathrm{Pic}(T')$ & $g^*: \mathrm{PIC}(T) \to \mathrm{PIC}(T')$.

def: Let $f : X \to S$ be a morphism of schemes. The <u>naive/absolute Picard functor</u> is $P_{X/S} : T \in (Sch/S)^{op} \longrightarrow \mathrm{Pic}(X \times_S T) \in Ab$

- The naive Picard functor, as its name suggests, is a bit too naive to be represented by a geometric object. Here are some obstructions.

- <u>Pb 1:</u> $\{$ Automorphisms coming from S

Let $\mathcal{L}$ be a line bundle on some scheme S. Then $\mathcal{O}(S)^\times \longrightarrow \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{L})$ is an isomorphism. The problem is then that we can glue line bundles using non-compatible elements of $\mathcal{O}^\times$ on overlaps. This leads to various obstructions.

- <u>Pb 1':</u> $\{$ $P_{X/S}$ is not a sheaf.

- We know that representable functors are sheaves for the fppf (hence also for étale, Zariski) topologies. However, $P_{X/S}$ is not a sheaf, even for Zariski.

<u>c-ex:</u> $X = S = \mathrm{Spec}(k)$. $T = \mathbb{P}^1 = \mathbb{A}^1 \amalg_{\mathbb{G}_m} \mathbb{A}^1 = U \amalg_W V$.

Then $P_{X/S}(T) \simeq \mathbb{Z}$ but $P_{X/S}(U) = P_{X/S}(V) = \{0\}$.

- There is an obstruction to be an étale sheaf of a more arithmetic nature, namely failure of Galois descent along field extensions: see the example at the end of this section.

- <u>Solution for 1/1':</u> Sheafify for a strong enough topology.

<u>rmk:</u> Alternative solution: introduce the groupoid-valued (lax 2-) functor $T \longmapsto \mathrm{PIC}(X \times_S T)$ and study its representability by the <u>Picard stack.</u> See the papers of Sylvain Brochard for much more on the topic.

<u>rmk:</u> Other alternative solution: when f has a section ε, can define the <u>rigidified Picard functor</u>

$$\mathrm{Pic}_{X/S,\varepsilon} : (Sch/S)^{op} \longrightarrow Ab$$
$$T \longmapsto \left\{ (\mathcal{L}, \alpha) \;\middle|\; \begin{array}{l} \cdot\ \mathcal{L} \text{ line bundle on } X \times_S T \\ \cdot\ \alpha : \mathcal{O}_T \xrightarrow{\sim} \varepsilon_T^*(\mathcal{L}) \end{array} \right\}$$

<u>lemma:</u> If f is cohomologically flat in degree 0 (see def. below) and has a section ε, then $\mathrm{Pic}_{X/S,\varepsilon}$ is a sheaf in the fppf topology (and is in fact isomorphic to the Picard functor defined below).

<u>proof:</u> see [BLR, Chap 8 p 204]. $\square$

- This definition is particularly useful when there is a distinguished section, for instance for Picard schemes of abelian varieties.
- It is also useful because, using fppf descent, we can often assume f has a section.

- Pb 2: Finiteness issues.

- **Lemma:** X/k variety over a field. Then $\mathrm{Ker}\left(P_{X/k}\left(k[\varepsilon]/(\varepsilon^2)\right) \to P_{X/k}(k)\right) \simeq H^1(X, \mathcal{O}_X)$.

  proof: Exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}^*_{X[\varepsilon]} \rightleftarrows \mathcal{O}^*_X \to 0$ in the étale top (?)

  $f \longmapsto 1 + f\cdot\varepsilon$ split exactness

  $$\Rightarrow 0 \to \underset{\substack{\text{IS} \\ H^1(X, \mathcal{O}_X)}}{H^1(X_{\text{ét}}, \mathcal{O}_X)} \longrightarrow \underset{\substack{\text{IS} \\ \mathrm{Pic}(X[\varepsilon])}}{H^1(X[\varepsilon]_{\text{ét}}, \mathcal{O}^*_{X[\varepsilon]})} \longrightarrow \underset{\substack{\text{IS} \\ \mathrm{Pic}(X)}}{H^1(X_{\text{ét}}, \mathcal{O}^*_X)} \longrightarrow 0$$ $\square$

- So if $P_{X/k}$ (or indeed its étale sheafification) is representable by a group scheme $\mathcal{G}/k$ we would have $\mathrm{Lie}(\mathcal{G}/k) \simeq H^1(X, \mathcal{O}_X)$. But $H^1(X, \mathcal{O}_X)$ can be infinite-dimensional.

Solution to Pb 2: Restrictions on f, e.g properness.

- There are other issues but this is enough for now. Given the above, we put:

- **def:** The Picard functor $\mathrm{Pic}_{X/S}$ is the fppf sheafification of $P_{X/S}$.

i.e, an element in $\mathrm{Pic}_{X/S}(T)$ is represented by a collection $\{\mathcal{L}_i \in \mathrm{Pic}(X \times_S U_i)\}$ for $\{U_i \to T\}_i$ an fppf cover of T.

- **Lemma:** $\mathrm{Pic}_{X/S}(T) = H^0(T, R^1 f_* \mathbb{G}_m)$, with the higher direct image $R^1 f_*$ being computed in the fppf topology.

proof: This follows from the general formula $\mathrm{Pic}(T) = H^1(T_{\text{fppf}}, \mathbb{G}_m)$ and the fact that $R^1 f_* \mathbb{G}_m$ is the fppf-sheafification of the presheaf $U \longmapsto H^1((X \times_S U)_{\text{fppf}}, \mathbb{G}_m)$. $\square$

- We have a useful base change property.

Lemma: For any morphism $S' \to S$ we have

$\mathrm{Pic}_{X/S}|_{(Sch/S')^{op}} \simeq \mathrm{Pic}_{X \times_S S'/S}$, i.e

for any $T \to S'$ we have

$\mathrm{Pic}_{X/S}(T) \simeq \mathrm{Pic}_{X \times_S S'/S'}(T)$.

proof: We have $(X \times_S S') \times_{S'} T \simeq X \times_S T$, hence $P_{X \times_S S'/S'}(T) = P_{X/S}(T)$.
The result still holds after sheafification $\square$

2) Analysis of the fppf sheaffification

- In general, computing an fppf sheaffification is very difficult. The goal of this section is to show the specific case at hand is not so bad.
- We have the Leray spectral sequence for f and $\mathbb{G}_m$ in the fppf topology:

$$E_2^{p,q} = H^p(T_{fppf}, R^q(f_T)_* \mathbb{G}_m)$$

This is a first quadrant spectral sequence, and it has an exact sequence of low-degree terms:

$$0 \to E_2^{1,0} \to E_\infty^1 \to E_2^{0,1} \to E_2^{2,0} \to E_\infty^2$$

$$(*)\ 0 \to H^1(T, f_{T*}\mathbb{G}_m) \to \mathrm{Pic}(X_T) \to H^0(T, R^1 f_{T*}\mathbb{G}_m) \to H^2(T, f_{T*}\mathbb{G}_m) \to H^2(X_T, \mathbb{G}_m)$$

<u>Analysis of (*):</u>

- We see that the direct images $f_{T*}\mathbb{G}_m$ play an important role.

Note that $f_{T*}\mathcal{O}$ is an $\mathcal{O}_S$-algebra and that we have $(f_T)_*\mathbb{G}_m \simeq [(f_T)_*\mathcal{O}]^\times$.

- <u>def</u>:
  - We say that f is <u>$\mathcal{O}$-connected (resp. universally $\mathcal{O}$-connected)</u> if $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ (resp. $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$ for all $T \to S$).
  - We say that f is <u>cohomologically flat (in degree 0)</u> if for all $T \to S$ the base change map $(f_*\mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T \longrightarrow f_{T*}\mathcal{O}_T$ is an isomorphism.

<u>rmk</u>: Universally $\mathcal{O}$-connected $\Rightarrow$ cohomologically flat in degree 0.
One can often make reductions in the other direction when f is proper, using the Stein factorization.

<u>thm</u>: [EGA III.2, 7.8.6]
A proper flat finite morphism with geometrically reduced fibers is cohomologically flat in degree 0.

We now turn to the H^2 terms.

- <u>prop</u> (Grothendieck) For all $i \geq 0$, we have $H^i(X_{fppf}, \mathbb{G}_m) \simeq H^i(X_{\acute{e}t}, \mathbb{G}_m)$.
- The group $H^2(X, \mathbb{G}_m)$ is sometimes called the <u>cohomological Brauer group</u> (sometimes this name is reserved for its torsion subgroup).
- Suppose now that f has a section $\varepsilon : S \to X$. Then ε induces a retraction $\varepsilon^* : H^2(X, \mathbb{G}_m) \to H^2(S, \mathbb{G}_m)$ of $f^* : H^2(S, \mathbb{G}_m) \to H^2(X, \mathbb{G}_m)$.

· In conclusion we get:

<u>Prop</u>: · Assume f universally $\mathcal{O}$-connected or proper; then $\mathrm{Pic}_{X/S}$ can be computed in the étale topology.

· Assume f universally $\mathcal{O}$-connected. Then the exact sequence (*) yields:

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T) \to H^2(T, \mathbb{G}_m) \to H^2(X_T, \mathbb{G}_m).$$

· Assume moreover that f has a section. Then

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{X/S}(T) \to 0.$$

· Let us show that, without a section, there can really be a Brauer obstruction.

<u>ex</u>: Let k be a field and X be a conic over k without a rational point.

Let us compare the two sequences above for $T = \mathrm{Spec}(k)$ and $T = \mathrm{Spec}(\bar{k})$.

$$1 \longmapsto [Q_X]$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(X) \simeq \mathbb{Z} \arrow[r] \arrow[d, "\times 2"] & \mathrm{Pic}_{X/k}(k) \arrow[r, "\delta"] \arrow[d, hook, "s"] & \mathrm{Br}(k) \arrow[r] \arrow[d] & \mathrm{Br}(X) \\
0 \arrow[r] & \mathrm{Pic}(X_{\bar{k}}) \simeq \mathbb{Z} \arrow[r, "\sim"] & \mathrm{Pic}_{X/k}(\bar{k}) \arrow[r] & 0 &
\end{tikzcd}$$

$$\begin{cases} \mathrm{Pic}_{X/k}(\bar{k})^{G_k} \simeq \mathrm{Pic}_{X/k}(k) \\ \quad \cap \\ \mathrm{Pic}_{X/k}(\bar{k}) \end{cases}$$

There is an "extra class" in $\mathrm{Pic}_{X/k}(k)$ not coming from $\mathrm{Pic}(X)$, whose image in $\mathrm{Br}(k)$ is the class of the quaternion algebra Q_X associated to the conic X: recall that any conic over k can be written as

$V(ax^2 + by^2 = z^2) \subseteq \mathbb{P}^2_k$ with $(a,b) \in (k^\times)^2$, and then

$Q_X := \langle 1, i, j, ij \rangle_k / i^2 = a,\ j^2 = b,\ ij = -ji$ is a quaternion algebra, well-defined up to isomorphism. Any quaternion algebra has a well-defined class in $\mathrm{Br}(k) \simeq H^2(k_{ét}, \mathbb{G}_m)$, which is a 2-torsion class.

This claim: $\delta(1) = [Q_X]$ is a special case of a thm of Lichtenbaum on Severi-Brauer varieties, explained well in [Gille-Szamuely, Thm 5.4.5].

c

3) Representability

- There are two major approaches to representability theorems in alg. geometry.

(i) The projective approach (Grothendieck): using auxiliary projective embeddings and reducing to the representability of the appropriate Hilbert / Quot scheme.

(ii) The algebraization approach (Artin): using deformation theory and algebraization results.

- Advantages and disadvantages:

(i): A: + Older approach, less technology.
+ Provides stronger results: the resulting objects are separated, have quasi-projective components.

D: + Require stronger hypotheses, often not satisfied in practice.
+ Not as conceptual / requires extra ingenuity.

(ii): A: + More general, requires less hypothesis.
+ Applies also to set-up of representability of groupoid-valued functors by algebraic stacks.
+ More conceptual and systematic, gives necessary and sufficient conditions for representability.

D: + More technical
+ the resulting objects are algebraic spaces, not necc. separated.

Projective approach to representability:

thm: Let $f: X \to S$ be projective flat, of finite presentation, whose fibers are geometrically integral.
Then $\mathrm{Pic}_{X/S}$ is representable by a separated S-scheme, locally of finite presentation.

sketch of proof: (see [BLR § 8.2, Kleiman - Picard])

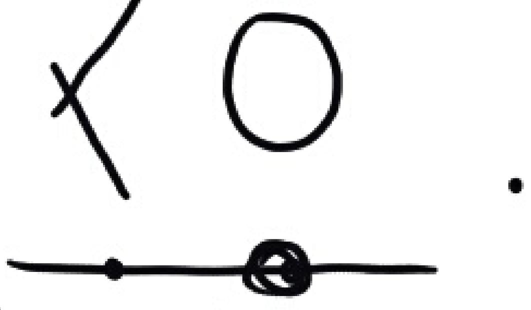
- We have f universally $\mathcal{O}$-connected, so the analysis before applies.
- The problem is local on S for Zariski topology $\rightsquigarrow$ OPS $f: X \hookrightarrow \mathbb{P}^n_S \to S$.
- For $\Phi \in \mathbb{Q}[T]$, put $\mathrm{Pic}^{\Phi}_{X/S}(T) \subseteq \mathrm{Pic}_{X/S}(T)$ for the classes represented (on some fppf cover) by a line bundle with Hilbert polynomial Φ.
- Then one proves that $\mathrm{Pic}^{\Phi}_{X/S}$ is an open and closed subfunctor of $\mathrm{Pic}_{X/S}$ and that $\mathrm{Pic}_{X/S} = \coprod_{\Phi} \mathrm{Pic}^{\Phi}_{X/S}$. The hard work is then in proving:

- Claim | $\mathrm{Pic}^{\Phi}_{X/S}$ is represented by a quasi-projective S-scheme.
- This is proved by:
  - Relating $\mathrm{Pic}_{X/S}$ to a functor $\mathrm{Div}_{X/S}$ parametrizing relative effective Cartier divisors.
  - Proving that $\mathrm{Div}_{X/S}$ is representable as an open subfunctor of the Hilbert scheme $\mathrm{Hilb}_{X/S}$.
  - Pick Φ such that line bundles "are well-parametrized by divisors" (i.e $\mathrm{Div}^{\Phi}_{X/S} \twoheadrightarrow \mathrm{Pic}^{\Phi}_{X/S}$ + some other technical conditions)

    Then, by writing $\mathrm{Pic}^{\Phi}_{X/S}$ as a quotient of $\mathrm{Div}^{\Phi}_{X/S}$ by a certain proper flat equivalence relation, conclude that $\mathrm{Pic}^{\Phi}_{X/S}$ is representable.
  - Reduce the general case to this, using the theory of bounded families of coherent sheaves. $\square$

- The most restrictive assumption in this theorem is not the projectivity of f, but the geometric irreductibility of the fibers.

ex: (Mumford) $S = \mathrm{Spec}(\mathbb{R}[[t]])$, $X = V(x^2+y^2=tz^2) \subseteq \mathbb{P}^2_S$.

$S' = \mathrm{Spec}(\mathbb{C}[[t]])$, $X' = X_{S'} =$  .

Then $\mathrm{Pic}_{X'/S'} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_{X'/S'}$, line bundles of total degree d.

Each $\mathrm{Pic}^d_{X'/S'}$ is obtained by gluing copies $S'_{a,b} \simeq S'$ for every decomposition $a+b=d$ <~> partial degrees of line bundles on the two components of the special fibers. So $\mathrm{Pic}_{X'/S'}$ is represented by a scheme P', but it is highly non-separated. If $\mathrm{Pic}_{X/S}$ was represented by a scheme P, we would have $P' \simeq P \times_S S'$ by base change, and P would be obtained by quotienting by the action of $\mathrm{Gal}(S'/S) \simeq \mathbb{Z}/2\mathbb{Z}$. However this quotient does not exist as a scheme, but as a (non-separated) algebraic space.

Algebraization approach to representability:

- To get meaningful geometry in that situation, we turn to Artin's result. For this, we need to define algebraic spaces. We follow the treatment of [BLR].

def: Let S be a scheme. A (locally separated) algebraic space X over S is a presheaf $X : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Sets}$ with the following properties:

- X is a sheaf with respect to the étale topology.
- $\exists$ a morphism $U \xrightarrow{\tau} X$ with U an S-scheme locally of finite pres., such that for all $V \longrightarrow X$ with V S-scheme, $U \times_S V \longrightarrow V$ is étale surjective.
- (technical) $U \times_X U \hookrightarrow U \times_S U$ is a quasi-compact immersion.

- Algebraic spaces are quite similar to schemes. The slogan is "a scheme is glued from spectra of rings in the Zariski topology, an algebraic space is glued from (étale) spectra of rings in the étale topology"; indeed, if you replace "étale surjective" by "Zariski covering" and assume U affine, you can get with some effort to a working definition of schemes from rings.
- Another perspective is that "$X = U/R$" with R the "étale equivalence relation" $U \times_X U \subseteq U \times_S U$.
- Most of EGA-style algebraic geometry can be adapted to algebraic spaces: the main difference is the absence of affine open neighbourhoods.

Here is the main representability result.

thm: Let $f : X \to S$ be a morphism of algebraic spaces, which is proper flat and finitely presented. Assume f cohomologically flat in degree 0 (i.e. f has geometrically reduced fibers).

Then $\mathrm{Pic}_{X/S}$ is represented by an algebraic space (locally of finite presentation $/S$)

- The proof of this theorem relies on a general criterion of Artin for representability of a functor by an algebraic space. One just has to check a list of axioms. Most of them are related to the deformation theory of line bundles, which is relatively elementary.

- In the specific context of group objects, the difference between schemes and algebraic spaces is particularly small, because of the following result.

prop (Artin) A group object in the category of algebraic spaces locally of finite type is in fact a scheme.

- So in particular, we get the following result from Artin's theorem, which had been obtained previously by projective methods.

thm (Murre, Oort) Let X/k be a proper scheme over a field. Then $\mathrm{Pic}_{X/k}$ is represented by a group scheme locally of finite type over k.

- Here is another cool result of Raynaud in the same vein.

thm: (Raynaud) A group object in the category of smooth proper S-algebraic spaces with geometrically connected fibers is a group scheme (hence an abelian scheme.

4) Smoothness and finiteness

- <u>Hyp</u>: $f : X \to S$ proper flat finitely presented coh. flat in degree 0.
  $\Rightarrow \mathrm{Pic}_{X/S}$ exists at least as an algebraic space. We pretend it is a scheme for simplicity.

- We first explain the structure of the Lie algebra. By the same computat° as in part 1):

<u>thm</u>: Let $f : X \to S$ be

a) $\mathrm{Lie}(\mathrm{Pic}_{X/S}) \xrightarrow{\sim} R^1 f_* \mathcal{O}_X$.

b) For $S = \mathrm{Spec}(k)$, then
$$\dim_k \mathrm{Pic}_{X/k} \leq \dim_k H^1(X, \mathcal{O}_X)$$
with equality iff $\mathrm{Pic}_{X/k}$ is smooth over k.

- Remember the important result:

<u>prop</u>: (Cartier) Let k be a field of characteristic 0, and G be a group scheme over k which is locally of finite type. Then G is smooth over k.

- As a consequence, the theory of the Picard functor is quite a bit simpler in char 0. However there are still pitfalls, as $\mathrm{Pic}_{X/S}$ can fail to be flat in a number of ways, and can also be highly non-separated. One positive result is:

- <u>prop</u>: Let $s \in S$ be a point such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$.
  Then there exists a neighbourhood $U \subseteq S$ of s such that $\mathrm{Pic}_{X/S}$ is smooth over U ($\Leftrightarrow \mathrm{Pic}_{X_U/U}$ smooth).

<u>proof sketch</u>: - By assumption, $\mathrm{Pic}_{X/S} \to S$ is locally of finite presentation. So to prove smoothness, it is enough to prove formal smoothness.
By the semi-continuity theory on coherent cohomology, there exists $s \in U$ such that for all $t \in U$, we have $H^2(X_t, \mathcal{O}) = 0$. By the base change property of $\mathrm{Pic}_{X/S}$, we can assume $S = U$. Let $Z \to S$ with Z affine and $Z_0 \subseteq Z$ defined by an ideal sheaf $\mathcal{N}$ with $\mathcal{N}^2$. To prove formal smoothness, we have to show
$$R^1(f \times_S Z)_* \mathbb{G}_m \longrightarrow R^1(f \times_S Z_0)_* \mathbb{G}_m \text{ is surjective.}$$
Because $\mathcal{N}^2 = 0$, we have a s.e.s $0 \to (f \times_S Z)^* \mathcal{N} \to \mathbb{G}_{m, X \times_S Z} \to \mathbb{G}_{m, X \times_S Z_0} \to 0$ (not necc split!). Hence it is enough to show $R^2(f \times_S Z)_* (f \times_S Z)^* \mathcal{N} = 0$. It turns out that this follows from $H^2(X_t, \mathcal{O}) = 0$ for all t by coh. and base change results $\square$

- <u>cor</u>: If the relative dimension of X/S is at most 1, then $\mathrm{Pic}_{X/S}$ is smooth.

Given that we are interested in models of curves, this is a fundamental result.

- One major source of complication, preventing flatness of $\mathrm{Pic}_{X/S}$ in general, is that the group of connected components of $\mathrm{Pic}_{X_s/s}$ vary wildly in $s \in S$.

<u>def</u>: $\mathrm{Pic}^{\mathring{\mid}\tau}_{X/S}(T) := \{ \alpha \in \mathrm{Pic}_{X/S}(T) \mid \forall s \in S,\ s^*(\alpha) \in \mathrm{Pic}^{\mathring{\mid}\tau}_{X_s/s}(T) \}$

which makes sense because, by the above, $\mathrm{Pic}_{X_s/s}$ is a group scheme locally of finite type over a field and hence has a | neutral / torsion component of the identity whose construction commutes with base change $\Rightarrow$ $\mathrm{Pic}^{\mathring{\mid}\tau}_{X/S}$ subfunctor of $\mathrm{Pic}_{X/S}$.

- If $S = \mathrm{Spec}(k)$, then $\mathrm{Pic}^{\mathring{\mid}\tau}_{X/k}$ is a quasi-projective group scheme, which can be fairly complicated, close to an arbitrary such gadget [Brion–Picard].

<u>thm</u>: Assume f is projective with geometrically integral fibers.

[Kleiman–Picard, Thm 6.16] Then $\mathrm{Pic}^{\tau}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$ is an open and closed subscheme of finite type.

- In general, $\mathrm{Pic}^{\tau}_{X/S}$ has a chance to be | smooth even when $\mathrm{Pic}_{X/S}$ is not. / of finite presentation / proper

- In particular, we can ask what the regularity of f gives you.

<u>prop</u>: (i) Assume f has geometrically integral fibers. Then $\mathrm{Pic}_{X/S}$ is separated over S.

(ii) Assume f is smooth. Then $\mathrm{Pic}_{X/S}$ satisfies the valuative criterion of properness.

(iii) In (ii), assume f is moreover projective. Then $\mathrm{Pic}^{\tau}_{X/S}$ is proper over S.

<u>sketch of proof</u>: - (iii) follows from (ii) and the previous theorem.

- In (i) and (ii), the key point is to check the valuative criterion. So we can assume that $S = \mathrm{Spec}(R)$ with R DVR. By fppf descent we can assume that f has a section, so that $\mathrm{Pic}_{X/S}(S) \simeq \mathrm{Pic}(X)$ (since $\mathrm{Pic}(R) = 0$ by R local)

(i) We have to show that a line bundle $\mathcal{L}$ on X which is trivial on the generic fiber is trivial. Let $f \in \Gamma(X, \mathcal{L})$ which generates $\mathcal{L}$ on X_K. We have to show that we can pick f which does not vanish anywhere. It suffices to prove non-vanishing at the generic point η of X_σ. But we have $R \longrightarrow \mathcal{O}_{X,\eta}$ ext. of DVR's of ramification index 1, so by dividing by the appropriate power of a uniformizer, we get f non-vanishing at η. This finishes the proof.

(ii) It remains to show the existence part of the valuative criterion, i.e., we have to show that any line bundle on X_K can be extended to a line bundle on X. Since f is smooth, X is regular. Hence Cartier divisors and Weil divisors coincide on both X and X_K. However, any Weil divisor on X_K extends to a Weil divisor on X: given a prime Weil divisor D on X_K extends to its Zariski closure in X. $\square$

- Finally, here is the best possible world, which is enough for some geom. applications.

<u>prop:</u> [Kleiman_Picard, 5.2?] S scheme of char. 0, f smooth locally projective. Then $\mathrm{Pic}^0_{X/S}$ is a smooth proper group scheme with geometrically connected fibers, i.e. an abelian scheme.

- There is one last major result of Raynaud [BLR, 9.4/Theorem 2] about curves over DVRs, which we will discuss when talking about stable reduction.