

Prop 55: Let $K \in \text{sSet}$, $C \in \text{Cat}_{\infty}^{\wedge}$ and

$$\tilde{p} : \begin{array}{c} K^{\Delta} \\ \downarrow \\ K^{\square} \end{array} \longrightarrow C. \text{ Write } p = \tilde{p}|_K.$$

Then \tilde{p} is a $\underset{\text{limit}}{\text{colimit}}$ diagram

$$\Leftrightarrow \begin{array}{ccc} C_{\tilde{p}/} & \longrightarrow & C_{p/} \\ C_{/\tilde{p}} & \longrightarrow & C_{/\bar{p}} \end{array} \text{ is a trivial fibration.}$$

proof: See exercise sheet 12, or [Rezk, 28.7]

□

Def 56: Let $C \in \text{Cat}_{\infty}^{\wedge}$ and $K \in \text{sSet}$

- * C admits all (∞)limits of shape K if every $p: K \rightarrow C$ has a (∞)limit.
- * C is (∞)complete if C admits colimits of shapes all (small) simplicial sets.
- * C admits finite (∞)limits if C admits (∞)limits of shapes all simplicial sets with

finitely many non-degenerate simplices.

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* Let $F: C \rightarrow D$ be a functor between ∞ -cat.

We say that F preserves limits of shape K

if for all $p: K \rightarrow C$, the functor

$C_{/p} \longrightarrow D_{/Fp}$ sends terminal objects to
terminal objects.

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Def 57: Let $g: K' \rightarrow K$ be a morphism in
 $sSet$. We say that g is final (resp. cofinal)

if for every functor $p: K \rightarrow C$ with C
 ∞ -category, the induced functor

$C_{/p} \longrightarrow C_{/pg}$
(resp. $C_{p/} \longrightarrow C_{pg/}$)

sends terminal objects to terminal objects
(resp. initial objects to initial objects).

It is possible to establish then many results parallel to the 1-categorical story. Here is a sample, without proofs for lack of time (also, some are probably easier to prove with other definitions of (co)limits).

Prop 58: [HTT, Cor 4.4.2.4] Let C be an ∞ -category. Then

Prop 4.4.2.6

- C admits finite colimits iff it admits limits

pushouts pullbacks	an initial object a terminal		pushouts pullbacks
- C is cocomplete iff it admits coproducts.

complete	products		pushouts pullbacks

 and arbitrary coproducts.

Prop 59: [HTT, Cor 4.4.2.5, Prop 4.4.2.7]

Let $F: C \rightarrow D$ be a functor between ∞ -categories

- Assume that C admits finite $\begin{cases} \text{colimits} \\ \text{limits} \end{cases}$

Then F preserves all finite $\begin{cases} \text{colimits} \\ \text{limits} \end{cases}$ iff

F preserves $\begin{cases} \text{pushouts and} \\ \text{pullbacks} \end{cases}$ initial objects.
 $\begin{cases} \\ \text{terminal} \end{cases}$

- Assume that C is $\begin{cases} \text{cocomplete} \\ \text{complete} \end{cases}$. Then F

preserves all $\begin{cases} \text{colimits} \\ \text{limits} \end{cases}$ iff F preserves $\begin{cases} \text{pushouts} \\ \text{pullbacks} \end{cases}$

and $\begin{cases} \text{coproducts} \\ \text{products} \end{cases}$.

Prop 60: [HTT, 4.2.3.14]

Let K be a simplicial set. There exists a 1-category I and a (co)final functor $p: N(I) \rightarrow K$.

In particular, let C be an ∞ -category.

Then C is (co)complete iff for all

small 1-categories I and functors $p: N(I) \rightarrow C$,

P admits a (co)limit.

Prop 61: [HTT, 4.1.10; ?]

Categorical equivalences are final, cofinal and preserve all limits and colimits.

- Let us briefly discuss the compatibility of this theory of (co)limits with the more classical theory of **homotopy (co)limits**. There are different formulation, let's focus on the simplest case.

Thm 62: Let M be a model category.
 I a small category.

Assume that M is combinatorial (technical condition, often satisfied). Then:

- $\text{Fun}(I, M)$ admits a projective model structure such that
 - weak equivalences are objectwise
 - cofibrations are objectwise.

which implies that the adjunction:

$$\text{colim} : \text{Fun}(I, M) \rightleftarrows M : \text{const}$$

is a Quillen adjunction.

- $\text{Fun}(I, M)$ admits an injective model structure

such that : . weak equivalences are objectwise
 . fibrations are objectwise

which implies that the adjunction:

$$\text{const}: \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{I}} : \lim$$

is a Quillen adjunction.

Def 63: With the same notations :

- The Homotopy colimit Functor is $\mathbb{L}\lim$
 (derived using the projective model structure).
- The Homotopy limit functor is $\mathbb{R}\lim$
 (derived using the injective model structure) .

Rmk 64 In particular cases, other model structures
 with more control on (co)fibrations can be
 used. In particular, when \mathcal{I} is a Reedy category
 (like \downarrow^{\rightarrow} , $\cdot \rightrightarrows \cdot$, \mathbb{N} , \mathbb{N}^{op} , Δ , Δ^{op} ...)
 there are Reedy model structures on $\text{Fun}(\mathcal{I}, \mathcal{M})$.

- If M is also a simplicial model category and I a simplicial category, then $\text{Fun}_{\text{sset}}(I, M)$ can also be equipped with appropriate model structures.
-

- Here is the comparison theorem.

Thm 65: (special case of [HTT, Thm 4.2.4.1])
Cor 4.2.4.8

Let M be a simplicial combinatorial model category.

- 1) Let I be a small category.

Let $F : I \rightarrow M$ be a functor.

Hyp: F factors through M_{cf} .

(this can be arranged up to objectwise weak equivalence in $\text{Fun}(I, M)$)

Let $\tilde{F} : N(I) \longrightarrow N_{\Delta}(M_{cf})$

be the induced functor of ∞ -categories.

Then \tilde{F} admits a colimit and a limit
(in the ∞ -categorical sense) and we have

$$\begin{cases} \text{colim}(\tilde{F}) \simeq L \text{colim}(F) \\ \lim(\tilde{F}) \simeq R \lim(F) . \end{cases}$$

2) The ∞ -category $N_{\Delta}(M_{cf})$ is complete and cocomplete. □

Rmk 66 Part 2) follows from 1) together with Prop and a rectification result: any diagram $N(I) \rightarrow N_{\Delta}(M_{cf})$ can be “rectified”, up to isomorphism in $\text{Fun}(N(I), N_{\Delta}(M_{cf}))$, to a functor $I \rightarrow M_{cf}$.

Example 67:

- Let M be a simplicial combinatorial model category. Then the associated ∞ -category $N_{\Delta}(M_{cf})$ is complete and cocomplete.

In particular, Spc , $\mathcal{D}^+(\mathbf{A})$ (for \mathbf{A} abelian cat. with enough injectives) and many other important examples are complete and cocomplete.

- Finally, we try to compute some limits and colimits in Spc "by hand", using simplicial homotopy theory.
- First, some generalities about (∞)limits in homotopy coherent nerves. Let \mathcal{C} be a locally Kan simplicial category, K a simplicial set and $p: K \rightarrow N_\Delta(\mathcal{C})$ a functor.

Recall the adjunction

$$\text{Path}[-]: \text{sSet} \rightleftarrows \text{Cat}_\Delta : N_\Delta$$

- Using this, we see that a colimit of p is eq. to a functor $\hat{q}: \text{Path}[K^\Delta] \rightarrow \mathcal{C}$ in Cat_Δ .

extending the adjoint $\text{Path}[K] \xrightarrow{q} \mathcal{C}$ of p , and such that for all $n \geq 1$, a lift exists in

$$\begin{array}{ccccc}
 & & \hat{q} & & \\
 & \swarrow & & \searrow & \\
 \text{Path}[K^D] & \longrightarrow & \text{Path}[K * \partial\Delta^n] & \longrightarrow & \mathcal{C} \\
 & \downarrow & & & \nearrow \\
 & & \text{Path}[K * \Delta^n] & &
 \end{array}$$

So it is a matter of understanding those simplicial path categories in special cases.

- Recall that we already determined the initial and terminal objects of Spc .
(Prop 50).

Products and coproducts:

Let S be a set and $p: cS \longrightarrow \text{Spc}$,

which is equivalent to giving an S -indexed family $\{X_s\}_{s \in S}$ of Kan complexes.

By definition, $\left\{ \begin{array}{l} \text{a coproduct of } \{X_s\}_{s \in S} \text{ is a colimit of } p \\ \text{a product of } \{X_s\}_{s \in S} \text{ is a limit of } p. \end{array} \right.$

Prop 68: The (co)product in the 1-category

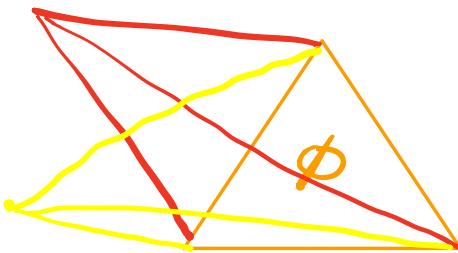
of Kan complexes is a (co)product of $\{X_s\}_{s \in S}$.

proof: We have $cS * \partial\Delta^n = \left(\coprod_{s \in S} \Delta^0 \right) * \partial\Delta^n$

in sSet

$$\begin{aligned} & \xrightarrow{\partial\Delta^n /} \\ &= \coprod_{s \in S} (\partial\Delta^n \rightarrow \Delta^0 * \partial\Delta^n) \end{aligned}$$

which as a simplicial set is a pushout of copies
of $\Delta^0 * \partial\Delta^n = \Delta_0^{n+1}$ glued along $\partial\Delta^{\{1, \dots, n\}}$



and similarly $cS * \Delta^n$ is a pushout of copies
of Δ^{n+1} glued along $\Delta^{\{1, \dots, n\}}$.

- Since $\text{Path}[-]$ is colimit-preserving, we can also describe $\text{Path}[cS * \partial\Delta^n]$ and $\text{Path}[cS * \Delta^n]$ quite concretely in terms of $\text{Path}[\Delta_0^{n+1}]$ and $\text{Path}[\Delta^n]$, which we already know quite well.

The same discussion applies dually to

$$\text{Path}[\partial\Delta^n * cS] \subseteq \text{Path}[\Delta^n * cS].$$

- Even more simply, $\text{Path}[(cS)^\Delta]$ and $\text{Path}[(cS)^\square]$ are constant simplicial categories

$$\left\{ \begin{array}{l} \bullet \xrightarrow{\quad} \bullet \\ \bullet \xrightarrow{\quad} \bullet \\ \bullet \xrightarrow{\quad} \bullet \end{array} \right\} S \text{ and } S \left\{ \begin{array}{l} \bullet \xrightarrow{\quad} \bullet \\ \bullet \xrightarrow{\quad} \bullet \\ \bullet \xrightarrow{\quad} \bullet \end{array} \right\} .$$

Products: Let's check that $\prod X_s$ (in $sSet$) is a product of $\{X_s\}$ in the ∞ -cat. sense.

$$\text{Path}[(cS)^\Delta] \xrightarrow{\hat{q}} \widetilde{\text{Kan}}$$

is simply given by the collection of maps.

$$\left\{ \pi_{X_s} : X_s \rightarrow X_s \right\}_{s \in S} . \text{ A diagram}$$

$$\begin{array}{ccc} & \hat{\pi} & \\ \text{Path}[(cs)^\Delta] & \xrightarrow{\quad} & \text{Path}[\Delta^n * cs] \xrightarrow{\circledast} \tilde{\text{Kan}} \\ \downarrow & & \nearrow \\ & \text{Path}[\Delta^n * cs] & \end{array}$$

is determined by the map \circledast , which

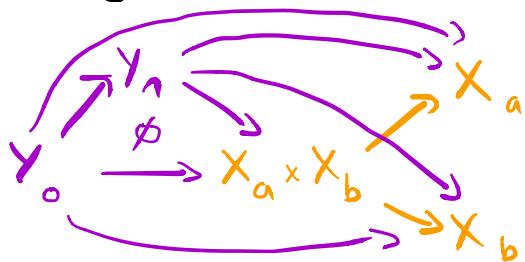
$$\text{is given by a collection } \left\{ \text{Path}[\Lambda_{n+1}^{n+1}] \rightarrow \tilde{\text{Kan}} \right\}_{S \in S}$$

which all send $\{n, n+1\}$ to the projection

$$\pi_{X_s} : X_s \rightarrow X_s \text{ and coincide on } \text{Path}[\Delta^{\{0, \dots, n\}}]$$

$\text{Path}[\Delta^{\{0, \dots, n\}}]$. Ex for $S = \{a, b\}$:

$$n = 2$$



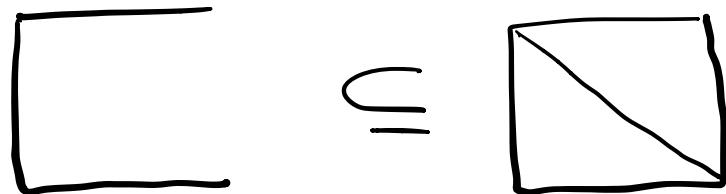
We have to extend to $\{\text{Path}[\Delta^{n+1}] \rightarrow \tilde{\text{Kan}}\}_{S\in S}$

with the same properties. The point is that we can do this separately for each S , and

the only difference between the two

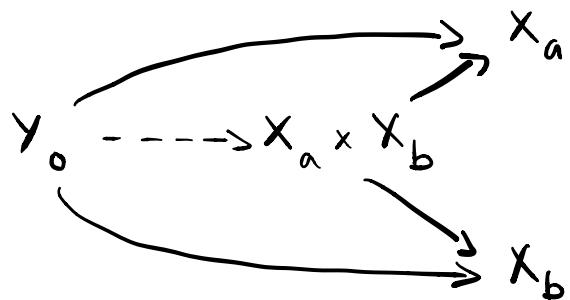
simplicial categories $\text{Path}[\Lambda_{n+1}^{n+1}]$ and $\text{Path}[\Delta^{n+1}]$

is the mapping space $0 \rightarrow n+1$



For $n \geq 2$, we can fill in as we like.

For $n = 1$, we have



and we fill in with the product map

given by the universal property of the product in sSet.

Coproducts The argument is the same,
dualized. □

Pushouts and pullbacks

Prop 69: Let $p: \Lambda_0^2 \rightarrow \text{Spc}$ be

a diagram: $\begin{array}{ccc} X & \xrightarrow{g} & Z \\ g \downarrow & & \\ Y & & \end{array}$

Assume that either of

g or g is a monomorphism.

(this can be achieved up to homotopy, by
using the WFS $\xrightarrow{\text{mono}} \cdot \xrightarrow{\sim} \xrightarrow{\text{trivial fib.}}$ in $s\text{Set}$)

Then the pushout diagram $\begin{array}{ccc} X & \xrightarrow{g} & Z \\ g \downarrow & \lrcorner & \downarrow \tilde{g} \\ Y & \xrightarrow{\tilde{g}} & Y \cup Z \\ & & X \end{array}$

in the 1-category $s\text{Set}$ is a pushout

in the ∞ -category Spc .

"proof" Apply the comparison Theorem

above to $M = s\text{Set}_{\text{Kan-Quillen}}$, combined

with [HTT, § A.2.4]. □

Dually, with the same proof:

Prop 70: Let $p: \Delta^2 \rightarrow \text{Spc}$ be a diagram

$$\begin{array}{ccc} & z & \\ & \downarrow g & \\ y & \xrightarrow{g} & x \end{array} . \quad \begin{array}{l} \text{Assume that either} \\ f \text{ or } g \text{ is a Kan fibration.} \end{array}$$

(this can be achieved up to homotopy by
using the WFS $\hookrightarrow \rightarrowtail$ in $s\text{Set}$)
anodyne Kan fib.

Then the pullback diagram

$$\begin{array}{ccc} Y \times_Z \tilde{z} & \xrightarrow{\tilde{g}} & Z \\ \tilde{g} \downarrow & \nearrow & \downarrow g \\ Y & \xrightarrow{g} & X \end{array}$$

in the 1-category $sSet$ is a pullback

in the ∞ -category Spc . □

These results also apply in the

∞ -category of pointed spaces

$$Spc_* := Spc_{*/} \simeq N_{\Delta}(\tilde{Kan}_{*/}).$$

There, we have two fundamental examples.

Example 71. Let $p: \Lambda^2_0 \rightarrow Spc_*$ given by

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

We have another diagram

$$\cdot \quad q: \Lambda^2_0 \rightarrow Spc_* \text{ given by}$$

by $\begin{array}{ccc} X & \longrightarrow & X^\Delta \\ \downarrow & & \\ X^\nabla & & \end{array}$ with X^Δ the cone
(in Spc , $X^\Delta \simeq X^\nabla$!)

p and q are naturally equivalent,

because X^Δ is a contractible Kan complex,

“so” they have the same colimit.

$X \rightarrow X^\Delta$ is mono, so now we apply Prop and see that

$$\ast \coprod_X \ast \simeq \sum X \text{ (Suspension of } X\text{)}$$

- By a similar argument, one gets
 $\ast \times_X \ast \simeq \Omega X \text{ (loop space of } X\text{).}$

6) Isomorphisms and the Joyal

extension and lifting theorems

- Let C be an ∞ -category and $g: x \rightarrow y$ a morphism.
Recall that g is an isomorphism iff $R(g) \in R(C(x,y))$ is.
Recall the core $\text{Core}(C) = C^{\leq}$ is the (non-full)
subcategory spanned by isomorphisms in C .
- We are going to study in more details
isomorphisms in ∞ -categories, and prove
a fundamental characterisation of them
due to Joyal. This will in particular imply
that quasigroupoids are Kan complexes.
- We first need some preliminaries about
conservative functors and isofibrations.

Def 72: Let C, D be ∞ -categories.

A functor $F: C \rightarrow D$ is **conservative** if

for all $g \in C_1$, $F(g)$ iso. $\Leftrightarrow g$ iso.

Rmk: The corresponding notion in

1-category theory is also very useful.

- This implies in particular that, for any $d \in D$, the subcategory of C whose morphisms are sent to id_d is a quasigroupoid. What about the subcategory whose objects are sent to d (the fiber C_d)? We need some extra assumptions.

Prop 73: Left/right Fibrations between ∞ -categories are conservative.

proof: Let $F: C \rightarrow D$ be a right fibration between ∞ -categories.

- Let $g: x \rightarrow y$ be a morphism in C such that $F(g)$ is an isomorphism.

Let $a: \Delta^2 \rightarrow C$ with $a_{12} = g$ and $a_{02} = \text{id}_y$.

Since $F(g)$ is an isomorphism, there exists

$b: \Delta^2 \rightarrow C$ such that $b_{12} = F(g)$ and $b_{02} = \text{id}_{F(y)}$.

We get a commutative diagram:

$$\begin{array}{ccc}
 \Delta^2 & \xrightarrow{a} & C \\
 \downarrow & \nearrow s & \downarrow F \\
 \Delta^2 & \xrightarrow{b} & D
 \end{array}
 \quad \text{and we have a lift } s \text{ since } F \text{ is a right fibration.}$$

in hC

Then $g = s|_{\Delta^2 \setminus \{(0,1)\}}$ is such that $[g] \circ [g]^{-1} = [\text{id}_{\Delta^2}]$

$F(g)$, being an inverse of $F(g)$ is an iso;

by the same argument, there exists $h : x \rightarrow y$

such that $[g] \circ [h] = [\text{id}_x]$. This

implies that $[g] = [h]$ and $[g]$ are isos,

Hence g is an iso. □

Cor 74: (Slices of quasigroupoids are quasigroupoids)

Let C be a quasigroupoid and $x \in C_0$.

Then $C_{x/}$ and $C_{/x}$ are also quasigroupoids.

proof: The restriction map $C_{x_1} \rightarrow C$ (resp. $C_{x_2} \rightarrow C$) is a left (resp. right) fibration, hence conservative by Prop . When $F: D \rightarrow D'$ is a conservative functor with target a quasi groupoid, then D is also a quasi-groupoid.



Def 75: Let C, D be ∞ -categories.

A functor $F: C \rightarrow D$ is an **isofibration** if it is an inner fibration and F "lifts isomorphisms": For any $c \in C_0$ and $g: F(c) \xrightarrow{\sim} d'$ isomorphism, there exists $c' \in C_0$ and $g: c \xrightarrow{\sim} c'$ such that $F(g) = g$

Equivalently, in terms of cores, we can solve the lifting problem:

$$\begin{array}{ccccc}
 \{0\} & \longrightarrow & \text{Core}(C) & \longrightarrow & C \\
 \downarrow & \nearrow g & \downarrow \text{Core}(f) & & \downarrow F \\
 \Delta^1 & \xrightarrow{g} & \text{Core}(D) & \longrightarrow & D
 \end{array}$$

- Isofibrations of 1-categories are defined similarly.

Prop 76: Let $F: C \rightarrow D$ be an inner fibration of simplicial sets. Then

F is an isofibration $\Leftrightarrow RF$ is an isofibration.
 $(\Leftrightarrow NRF \quad \underline{\hspace{1cm}})$

proof: \Rightarrow is clear since isomorphisms are detected by $h(-)$.

\Leftarrow : Let $g: F(c) \rightarrow d'$ be an isomorphism in D

Since RF is an isofibration, there exists

an isomorphism $g': c \rightarrow c'$ such that

$[F(g')] = [g]$ Choose $b: \Delta^2 \rightarrow D$ which

exhibits this as a right homotopy:

$$\begin{array}{ccc} F(g') & \xrightarrow{d'} & \\ b \swarrow & \parallel & \\ F(c) & \xrightarrow{g} & d' \end{array}$$

Let $a: \Lambda_1^2 \rightarrow C$ be the map with $a_{01} = g'$ and $a_{12} = id_C$.

We have a commutative diagram:

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{a} & C \\ \downarrow & \nearrow s & \downarrow F \\ \Delta^2 & \xrightarrow{b} & D \end{array}$$

where a lift s exists since F is an inner fibration.

Then $\tilde{g} := s_{02}$ is a lift of g , and is

an iso because $[g] = [g']$. □

This implies the notion of isofibration is more symmetrical than it seems:

Cor 77: Let $F: C \rightarrow D$ be an isofibration

between ∞ -categories. Then we have:

for any $c \in C_0$ and $g': d' \xrightarrow{\sim} F(c)$,

there exists $c' \in C_0$ and $g': c' \xrightarrow{\sim} c$ such that $F(g') = g'$.

Proof: This is clear for isofibrations between 1-categories, and the result then follows from the previous Proposition. □

Prop 78: Left/right fibrations between ∞ -categories are isofibrations.

proof: Let $F: C \rightarrow D$ be a right fibration with C, D ∞ -categories. Let g be an iso in D .

$$\begin{array}{ccc} \{1\} & \longrightarrow & C \\ \downarrow & \nearrow g & \downarrow F \\ \Delta^1 & \xrightarrow{g} & D \end{array}$$

The lift exists because F is a right fibration.

Now F is a lift of g , and since F is conservative by Prop, F is also an iso. \square

Rmk 7g Remember that the Joyal model structure on $sSet$ is the unique model structure with:

- W = categorical equivalences.
- Cof = monomorphisms.

It is not too difficult to construct using the small object argument (see [Rezk, first part of Thm 45.8]). What is more difficult is to prove that

- fibrant objects are quasicategories.
- fibrations between quasicategories are precisely the isofibrations.

The proof (see [Rezk, second part of Thm 45.8])
Cisinski, Thm 3.6.1

is a little involved.

Now we can state and prove:

Thm 80: (Joyal extension theorem)

Let C be an ∞ -category and $f: x \rightarrow y$ be a morphism in C . Then TFAE:

(i) f is an isomorphism.

(ii) Every morphism $a: \Delta_0^n \rightarrow C$ with $a_{01} = f$ and $n \geq 2$ extends to Δ^n .

(iii) Every morphism $b: \Delta_n^n \rightarrow C$ with $b_{n-1,n} = f$ and $n \geq 2$ extends to Δ^n .

proof: We prove $(i) \Leftrightarrow (ii)$ ($(i) \Leftrightarrow (iii)$ is dual).

$(ii) \Rightarrow (i)$: This is the "easy" direction,

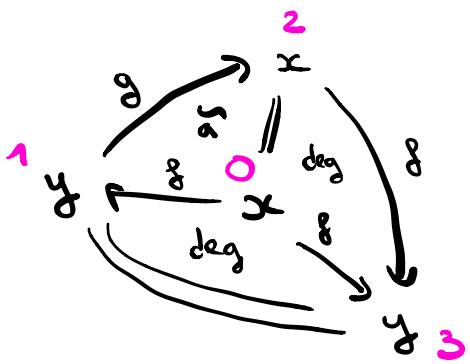
and an argument we have already seen before when proving that Kan complexes are quasigroupoids.

- Take $n=2$ and a given by $\frac{y}{x} = x$.

By (ii), there exists an extension $\tilde{a} : \Delta^2 \rightarrow C$

Put $g = \tilde{a}_{\gamma_2}$. Then $[g][g] = \text{id}_x$ in RC .

- Take $n = 3$ and $b : \Lambda_0^3 \rightarrow C$ given by



By (ii), there is an extension $\tilde{b}: \tilde{\Delta}^3 \rightarrow C$

$\Rightarrow g$ is iso with inverse g^{-1} .

(i) \Rightarrow (ii) :

Let $a: \bigwedge^n_{\mathbb{Q}} \rightarrow C$ with $f := a|_{\mathbb{Q}^n}$ isomorphism.

We have a pushout-join computation:

$$(\Delta^{\{0\}} \hookrightarrow \Delta^1) \boxtimes (\partial\Delta^{\{2, \dots, n\}} \hookrightarrow \Delta^{\{2, \dots, n\}})$$

$$:= (\Delta^1 * \partial\Delta^{\{2, \dots, n\}}) \cup_{\Delta^{\{0\}} * \partial\Delta^{\{2, \dots, n\}}} (\Delta^{\{0\}} * \Delta^{\{2, \dots, n\}}) \hookrightarrow \Delta^1 * \Delta^{\{2, \dots, n\}}$$

$$\begin{aligned}
&= \left(\Delta * \left(\bigcup_{i=2}^n \Delta^{\{2, \dots, i-1\}} \right) \cup \Delta^{\{0, 2, \dots, n\}} \right) \hookrightarrow \Delta^n \\
&= \bigcup_{i=2}^n \Delta^{\{1, 2, \dots, n\} \setminus i} \cup \Delta^{\{0, 2, \dots, n\}} \hookrightarrow \Delta^n \\
&= \bigwedge_0^n \hookrightarrow \Delta^n.
\end{aligned}$$

Using this and the join-slice adjunction, we

get an equivalence of lifting problems

$$\begin{array}{ccc}
\Delta^{\{0, 1\}} & \xrightarrow{g} & \Delta^{\{0\}} \\
\hookrightarrow \bigwedge_0^n & \xrightarrow{a} & \hookrightarrow C / a | \Delta^{\{2, \dots, n\}} \\
\downarrow & \nearrow & \downarrow q \\
\Delta^{\{0, 1\}} & \xrightarrow{g} & C / a | \Delta^{\{2, \dots, n\}} \\
\downarrow p & & \downarrow p \\
C & &
\end{array}$$

C is an ∞ -category and $\emptyset \subset \partial \Delta^{\{2, \dots, n\}} \subset \Delta^{\{2, \dots, n\}}$.

By Corollary 39, this implies that p and q

are right fibrations.

By Props and , p and q are thus conservative isofibrations.

Since g is an iso and p is conservative, g is an iso.

Since q is an isofibration, there exists a lift.

This concludes the proof. □

Rmk 81 This result can be interpreted as an homotopy coherence result ; it tells you that an isomorphism , which is defined in terms of HC and so depends only on low-dimensional data, always comes with higher-dimensional coherence conditions.

This is remarkable, and forms the basis

of a lot of further developments of the theory of quasi-categories.

The theorem admits a relative version.

Thm 82 (Joyal lifting theorem)

Let $p: C \rightarrow D$ be an inner fibration between ∞ -categories and $g \in C_1$, such that $p(g)$ is an isomorphism in D . TFAE:

(i) g is an isomorphism in C

(ii) For all $n \geq 2$, there is a lift in any diagram of the form:

$$\begin{array}{ccccc} \Delta^{\{0,1\}} & \xrightarrow{\quad g \quad} & \wedge^n_0 & \xrightarrow{\quad} & C \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\quad} & D & & \end{array}$$

(iii) For all $n \geq 2$, there is a lift in any

diagram of the form:

$$\begin{array}{ccccc} \Delta^{\{m,n\}} & \xrightarrow{\quad g \quad} & \wedge_n & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\quad \quad \quad} & D & & \end{array}$$



We omit the proof, which is a variant of the previous one.

There are dual formulations of Joyal extension and liftings in terms of (∞)slices.

Prop 83: Let C be an ∞ -category.

(a) Let $g: x \rightarrow y$ be a morphism in C . TFAE:

(i) g is an isomorphism.

(ii) $C_{g/} \rightarrow C_{x/}$ trivial fibration

(iii) $C_{/g} \rightarrow C_{/y}$ trivial fibration.

(b) Let D be another ∞ -cat. and $F : C \rightarrow D$ an inner fibration. Let $g : x \rightarrow y$ be a morphism in C such that $F(g)$ is an iso. TFAE

(i) g is an isomorphism.

$$(ii) C_{g/} \longrightarrow C_{x/} \times_{D_{F(x)/}}^D F(g)/$$

is a trivial fibration.

pullback-slices

$$(iii) C_{/g} \longrightarrow C_{/y} \times_{D_{/F(y)}}^D D_{/F(g)}$$

is a trivial fibration.

Proof: We do a) (i) \Leftrightarrow (ii). As before, the key is an equivalence of lifting problems:

for $n \geq 0$, we have:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & C_{g/} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & C_{x/} \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} \overset{g}{\curvearrowright} & & \Delta^* \times \varnothing \rightarrow (\Delta^* \times \partial\Delta^n) \cup (\{0\} \times \Delta^n) \rightarrow C \\ & & \downarrow \\ & & \Delta^* \times \Delta^n \end{array}$$

The RHS is

$$\begin{array}{ccccc} \Delta' & \xrightarrow{\quad} & \Lambda_0^{1+1+n} & \xrightarrow{\quad} & C \\ & & \downarrow & & \nearrow \\ & & \Delta^{1+1+n} & & \end{array}$$

which is precisely the lifting problem in
the Joyal extension theorem. So the
Joyal extension theorem is in fact equivalent
to a) (i) \Leftrightarrow (ii) (together with a) (i) \Leftrightarrow (iii))



Rmk 84: The Joyal lifting theorem

and its dual version motivate the
definition of the last - but not least -
major class of fibrations in the
theory of quasicategories: Cartesian
and Cocartesian fibrations.

7) First applications of Joyal extension

Thm 85: Let C be an ∞ -category.

Then TFAE :

- (i) C is a quasigroupoid.
- (ii) C is a Kan complex (i.e $C \rightarrow \Delta^0$ Kan fibration)
- (iii) $C \rightarrow \Delta^0$ is a left fibration.
- (iv) $C \rightarrow \Delta^0$ is a right fibration.

proof: We know (ii) \Rightarrow (i) \wedge (iii) \wedge (iv).

It suffices to prove (i) \Rightarrow (ii), (iii) \Rightarrow (i)
(and (iv) \Rightarrow (i) which is dual).

(i) \Rightarrow (ii): C is an ∞ -category so it has
the extension property for inner horns. Every
morphism in C is an isomorphism, so by
Joyal extension, C has the extension property

for outer horns. Hence C is a Kan complex.

(iii) \Rightarrow (i): This follows directly from the “easy” direction of the Joyal lifting theorem. \square

Cor 86: Let $g: X \rightarrow S$ be a left or right fibration of simplicial sets. Then for each $s \in S_0$, the fiber $X_s := \underset{S}{\times} \{s\}$ is a Kan complex.

Proof: Left/right fibrations are stable under pullbacks (because the collection of left/right fibrat° is a right complement), so the result follows from the theorem. \square

Cor 87: Let $g: x \rightarrow y$ be an isomorphism in an ∞ -category C , then

- C_x and $C_{y/}$ are categorically equivalent.

- $C_{/x}$ and $C_{/y}$ are categorically equivalent.

proof: We have a diagram

$$C_{/\underline{x}} \xleftarrow{r_0} C_{/\underline{y}} \xrightarrow{r_1} C_{/\underline{y}} \text{ induced by}$$

the inclusions $\{0\} \hookrightarrow \Delta^1$ and $\{1\} \hookrightarrow \Delta^1$.

Because $\{0\} \hookrightarrow \Delta^1$ is left-anodyne,

r_0 is a trivial fibration (Corollary 39).

By the dual version of the Joyal extension

theorem, r_1 is also a trivial fibration.

Since trivial fibrations are categorical equivalences,

this finishes the proof. □

- There are other interesting applications to initial/terminal objects, see [Rezk, 30.8-9].
- Another major application is the following characterisation of natural isomorphisms of functors (which was mentioned earlier at some point),

as a shortcut in the proof that trivial fibrations
are categorical equivalences.)

Thm 88: Let C be an ∞ -category,
 K a simplicial set and $F, F': K \rightarrow C$
two diagrams. Let $v: F \rightarrow F'$ be a
natural transformation (i.e. $v \in \text{Fun}(K, C)_1$)

Then: v is a natural iso. (iso in $\text{Fun}(K, C)$)

$\Leftrightarrow \forall x \in K_0, ev_x(v): F(x) \xrightarrow{\sim} F'(x)$ is an iso.



The proof is a little long and I am
running out of time! I refer you to
[Rezk, § 3.1] or [Kerodon, § 4.4].

• Let C be an ∞ -category. We have
defined the core $\text{Core}(C) = \overset{\cong}{C}$ of C
as the largest sub-quasigroupoid of C .

The fact that quasigroupoids are exactly the Kan complexes implies:

Cor 89: $\text{Core}(C)$ is a Kan complex,
the largest sub-Kan complex of C . □

As an application of the core, we finally construct the ∞ -category of ∞ -categories.

- Let $\widetilde{\text{qCat}}$ be the simplicial Full subcategory of $\widetilde{s\text{Set}}$ (i.e $s\text{Set}$ with its self-enrichment) spanned by quasicategories. Since $\text{Fun}(K, C)$ is a quasicategory whenever C is, $\widetilde{\text{qCat}}$ is actually enriched in qCat .
- $\text{Core} : \text{qCat} \rightarrow \text{Kan}$ is the right adjoint of $\text{Kan} \hookrightarrow \text{qCat}$, so it preserves products.
 \rightsquigarrow we have an endofunctor

$$\text{Core}_* : \text{Cat}_{\Delta}^{\text{qCat}} \longrightarrow \text{Cat}_{\Delta}^{\text{Kan}}$$

We get $\text{Core}_*(\widetilde{\text{qCat}})$, whose mapping Kan complexes are $\text{Map}(C, D) := \text{Core}(\text{Fun}(C, D))$.

Def 50: The ∞ -category of ∞ -categories,

Cat_∞ , is defined as the homotopy coherent nerve as the resulting locally Kan simplicial cat:

$$\text{Cat}_\infty := N_\Delta(\text{Core}_*\widetilde{\text{qCat}}).$$

• By construction, we have:

- $\text{Ob}(\text{Cat}_\infty) = \infty$ -categories
- $\text{Mor}(\text{Cat}_\infty) = \text{functors between } \infty\text{-categories.}$
- $(\text{Cat}_\infty)_2 = \underline{\text{invertible}}$ natural transformations
between functors.

(So that $R\text{Cat}_\infty$ is the homotopy category we have already constructed).

- $\overset{\curvearrowleft}{\text{Kan}}$ is a simplicial $\overset{\text{full}}{\vee}$ subcategory of $\text{Core}_*(\widetilde{\text{qCat}})$

which implies that the ∞ -category of spaces Spc is a full subcategory of Cat_∞ .

- Finally, let's look at mapping spaces in a given ∞ -category.

Def 91: Let C be an ∞ -category and $x, y \in C_0$. The mapping space $\text{map}_C(x, y)$ is the simplicial set defined by the pullback square:

$$\begin{array}{ccc} \text{map}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & C \times C (\cong \text{Fun}(\partial\Delta^1, C)) \end{array}$$

To study this, we need an intermediate step of independent interest.

Prop 92: Let $\begin{array}{ccc} C' & \xrightarrow{u} & C \\ q \downarrow & \lrcorner & \downarrow p \\ D' & \xrightarrow{v} & D \end{array}$ be a pullback square of ∞ -categories

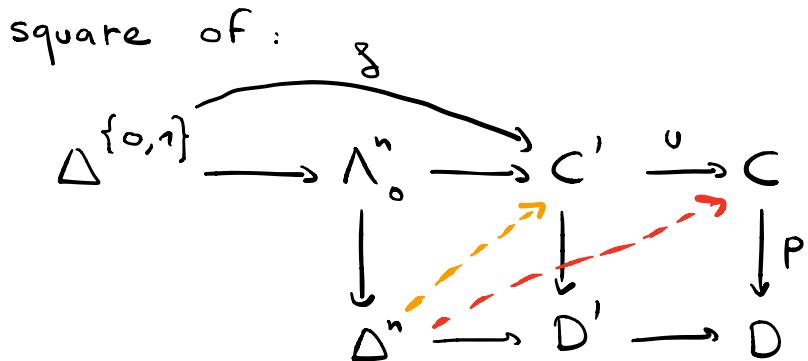
with p an inner fibration.

A morphism $g \in C'$, is an isomorphism iff $u(g)$ and $q(g)$ are isomorphisms.

Equivalently, $(C')^{\simeq} \xrightarrow{\sim} \underset{D^{\simeq}}{\simeq} C^{\simeq} \times (D')^{\simeq}$.

Proof: Inner fibrations are stable under pullback so q is an inner fibration.

We apply Joyal lifting to q : since $q(g)$ is an iso, to show that g is an isomorphism, we must solve the lifting problem  in the left square of:



Since $u(g)$ is an isomorphism and p is an inner fibration, there exists a lift .

Then the lift  exists by pullback.

This finishes the proof. 

Cor 93: a) Let $p: C \rightarrow D$ be a conservative inner fibration between quasicategories.

Then for every $d \in D_0$, the fiber C_d is a quasigroupoid.

b) Let C be an ∞ -category and K be a simplicial set. Then the fibers of $\text{Fun}(K, C) \rightarrow \text{Fun}(ck_0, C)$ are quasigroupoids.

proof: Part a) follows from the previous proposition

applied to the square

$$\begin{array}{ccc} C_d & \xrightarrow{u} & C \\ q \downarrow & \lrcorner & \downarrow p \\ \{d\} & \xrightarrow{\quad} & D \end{array}$$

we have $u \circ q(g) = \text{id}_d$ iso

• $p \circ u(g) = u \circ q(g) = \text{id}_J \Rightarrow u(g)$ is
 p conservative.

• Part b) follows from the criterion of Thm for the characterisation of natural equivalences, which is equivalent to saying that $\text{Fun}(K, C) \rightarrow \text{Fun}(cK, C)$ is conservative. □

Prop 94: Let C be an ∞ -category and $x, y \in C_0$. $\text{map}_C(x, y)$ is a Kan complex.

proof: This is the special case of the previous corollary, for $K = \Delta^1$. □

• For more interesting material around the core construction, see [Kerodon, § 4.4.3].

- We can generalize to higher mapping spaces

$$\begin{array}{ccc} \text{map}_C(x_0, \dots, x_n) & \longrightarrow & \text{Fun}(\Delta^n, C) \\ \downarrow & \lrcorner & \downarrow \\ \{(x_0, \dots, x_n)\} & \longrightarrow & C^{\times(n+1)} \end{array}$$

Lemma 95: The spine inclusion $I^n \subseteq \Delta^n$

induces a trivial fibration

$$\text{map}_C(x_0, \dots, x_n) \longrightarrow \text{map}_C(x_0, x_1) \times \dots \times \text{map}_C(x_{n-1}, x_n)$$

Proof: This is a base change of $\text{Fun}(\Delta^n, C) - \text{Fun}(I^n, C)$

which is a trivial fibration because $I^n \subseteq \Delta^n$ is
inner anodyne. □

- Using this, it is possible to define an enriched (or full) Homotopy category \mathcal{HPC} which is a category enriched over the Homotopy category $R\text{Kan} \simeq R\text{CW} \simeq R\text{Spc}$.

This provides another illustration of the Grothendieck Homotopy Hypothesis:

∞ -categories \simeq "cat coherently enriched over R Kan"

C
↓
cat. enriched over R Kan

It is also possible to record the information of all the maps (x_0, \dots, x_n) into a Segal category, another model of ∞ -categories (see [Rezk, § 33.11]).

VI The Grothendieck Construction
