7) First applications of Joyal extension

Thm 85: Let C be an ∞-category.

Then TFAE:

(i) C is a quasigroupoid.

(ii) C is a Kan complex (i.e $C \to \Delta^0$ Kan fibration)

(iii) $C \to \Delta^0$ is a left fibration.

(iv) $C \to \Delta^0$ is a right fibration.

proof: • We know (ii) $\Rightarrow$ (i) $\wedge$ (iii) $\wedge$ (iv).

• It suffices to prove (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i) (and (iv) $\Rightarrow$ (i) which is dual).

(i) $\Rightarrow$ (ii): C is an ∞-category so it has the extension property for inner horns. Every morphism in C is an isomorphism, so by Joyal extension, C has the extension property for outer horns. Hence C is a Kan complex.

(iii) $\Rightarrow$ (i): This follows directly from the "easy" direction of the Joyal lifting theorem. $\square$

Cor 86: Let $f: X \to S$ be a left or right fibration of simplicial sets. Then for each $s \in S_0$, the fiber $X_s := X \times_S \{s\}$ is a Kan complex.

Proof: Left/right fibrations are stable under pullbacks (because the collection of left/right fibrat° is a right complement), so the result follows from the theorem. $\square$

Cor 87: Let $f: x \to y$ be an isomorphism in an ∞-category C, then

- $C_{x/}$ and $C_{y/}$ are categorically equivalent.
- $C_{/x}$ and $C_{/y}$ are categorically equivalent.

proof: We have a diagram

$$\mathcal{C}_{/x} \xleftarrow{r_0} \mathcal{C}_{/g} \xrightarrow{r_1} \mathcal{C}_{/y}$$ induced by

the inclusions $\{0\} \to \Delta^1$ and $\{1\} \to \Delta^1$.

Because $\{0\} \hookrightarrow \Delta^1$ is left-anodyne, r_0 is a trivial fibration (Corollary 39).

By the dual version of the Joyal extension theorem, r_1 is also a trivial fibration.

Since trivial fibrations are categorical equivalences, this finishes the proof. $\square$

- There are other interesting applications to initial / terminal objects, see [Rezk, 30.8-9].
- Another major application is the following characterisation of natural isomorphisms of functors (which was mentioned earlier at some point,

as a shortcut in the proof that trivial fibrations are categorical equivalences.)

Thm 88: Let C be an ∞-category, K a simplicial set and $F, F' : K \to C$ two diagrams. Let $u : F \to F'$ be a natural transformation (i.e $u \in \mathrm{Fun}(K,C)_1$)

Then: u is a natural iso. (iso in $\mathrm{Fun}(K,C)$)

$\iff \forall x \in K_0$, $ev_x(u) : F(x) \to F'(x)$ is an iso. $\square$

The proof is a little long and I am running out of time! I refer you to [Rezk, § 31] or [Kerodon, § 4.4].

- Let C be an ∞-category. We have defined the core $\mathrm{Core}(C) = C^{\simeq}$ of C as the largest sub-quasigroupoid of C.

The fact that quasigroupoids are exactly the Kan complexes implies:

Cor 89: Core(C) is a Kan complex, the largest sub-Kan complex of C. $\square$

As an application of the core, we finally construct the ∞-category of ∞-categories.

- Let $\widetilde{qCat}$ be the simplicial full subcategory of $\widetilde{sSet}$ (i.e $sSet$ with its self-enrichment) spanned by quasicategories. Since $Fun(K,C)$ is a quasicategory whenever C is, $\widetilde{qCat}$ is actually enriched in $qCat$.

- Core : $qCat \longrightarrow Kan$ is the right adjoint of $Kan \hookrightarrow qCat$, so it preserves products.

$\rightsquigarrow$ we have an endofunctor

$$Core_* : Cat_\Delta^{qCat} \longrightarrow Cat_\Delta^{Kan}$$

We get: $\mathrm{Core}_*(q\widehat{\mathrm{Cat}})$, whose mapping Kan complexes are $\mathrm{Map}(C, D) := \mathrm{Core}(\mathrm{Fun}(C, D))$.

Def 50: The ∞-category of ∞-categories, Cat_∞, is defined as the homotopy coherent nerve as the resulting locally Kan simplicial cat:

$$\mathrm{Cat}_\infty := N_\Delta(\mathrm{Core}_* \, q\widehat{\mathrm{Cat}}).$$

. By construction, we have:

- $\mathrm{Ob}(\mathrm{Cat}_\infty) = \infty$-categories
- $\mathrm{Mor}(\mathrm{Cat}_\infty) =$ functors between ∞-categories.
- $(\mathrm{Cat}_\infty)_2 =$ <u>invertible</u> natural transformations between functors.

(So that $h\,\mathrm{Cat}_\infty$ is the homotopy category we have already constructed).

- $\widehat{\mathrm{Kan}}$ is a simplicial full subcategory of $\mathrm{Core}_*(q\widehat{\mathrm{Cat}})$

which implies that the ∞-category of spaces Spc is a full subcategory of Cat_∞.
(in the same way that Set is a full subcat. of Cat)

- Finally, let's look at mapping spaces in a given ∞-category.

Def 91: Let C be an ∞-category and $x, y \in C_0$. The mapping space $\mathrm{map}_C(x,y)$ is the simplicial set defined by the pullback square:

$$\begin{tikzcd}
\mathrm{map}_C(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\
\{(x,y)\} \arrow[r] & C \times C \ (\simeq \mathrm{Fun}(\partial\Delta^1, C))
\end{tikzcd}$$

To study this, we need an intermediate step of independent interest.

Prop 92: Let

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ D' \arrow[r, "v"'] & D \end{tikzcd}$$

be a pullback square of ∞-categories with p an inner fibration.

A morphism $f \in C'_1$ is an isomorphism <u>iff</u> $u(f)$ and $q(f)$ are isomorphism.

Equivalently, $(C')^{\simeq} \xrightarrow{\sim} C^{\simeq} \times_{D^{\simeq}} (D')^{\simeq}$.

Proof: Inner fibrations are stable under pullback so q is an inner fibration.

We apply Joyal lifting to q: since $q(f)$ is an iso, to show that f is an isomorphism, we must solve the lifting problem in the left square of:

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & C' \arrow[r, "u"] \arrow[d] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & D' \arrow[r] & D \end{tikzcd}$$

Since $u(g)$ is an isomorphism and p is an inner fibration, there exists a lift .

Then the lift exists by pullback.

This finishes the proof. $\square$

Cor 93: a) Let $p : C \to D$ be a conservative inner fibration between quasicategories. Then for every $d \in D_0$, the fiber C_d is a quasigroupoid.

b) Let C be an ∞-category and K be a simplicial set. Then the fibers of $\mathrm{Fun}(K, C) \to \mathrm{Fun}(\mathrm{ck}_0, C)$ are quasigroupoids.

proof: Part a) follows from the previous proposition applied to the square

$$\begin{tikzcd} C_d \arrow[r, "u"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ \{d\} \arrow[r, "v"'] & D \end{tikzcd} ;$$

we have $\cdot\ q(g) = \mathrm{id}_d$ iso

$\cdot\ pu(\gamma) = vq(\gamma) = id_d \implies u(\gamma)$ iso
p conservative.

. Part b) follows from the criterion of Thm for the characterisation of natural equivalences, which is equivalent to saying that $Fun(K, C) \to Fun(cK_0, C)$ is conservative. $\square$

Prop 94: Let C be an ∞-category and $x, y \in C_0$. $map_C(x,y)$ is a Kan complex.

proof: This is the special case of the previous corollary, for $K = \Delta^1$. $\square$

Lemma 95: Let $C \in Cat_\infty$, $x, y \in C_0$.

Then $\pi_0\, map_C(x,y) \simeq Hom_{hC}(x,y)$

proof: Exercise. $\square$

- We can generalize to higher mapping spaces

$$\begin{tikzcd} \mathrm{map}_C(x_0,\dots,x_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^n, C) \arrow[d] \\ \{(x_0,\dots,x_n)\} \arrow[r] & C^{\times(n+1)} \end{tikzcd}$$

Lemma 96: The spine inclusion $I^n \subseteq \Delta^n$ induces a trivial fibration

$$\mathrm{map}_C(x_0,\dots,x_n) \longrightarrow \mathrm{map}_C(x_0,x_1) \times \dots \times \mathrm{map}_C(x_{n-1},x_n)$$

proof: This is a base change of $\mathrm{Fun}(\Delta^n,C) \to \mathrm{Fun}(I^n,C)$ which is a trivial fibration because $I^n \subseteq \Delta^n$ is inner anodyne. $\square$

- Using this, it is possible to define an enriched (or full) homotopy category $\mathcal{H}C$ which is a category enriched over the homotopy category $h\,\mathrm{Kan} \simeq h\,\mathrm{CW} \simeq h\,\mathrm{Spc}$, whose underlying category is hC.

This provides another illustration of the Grothendieck Homotopy Hypothesis:

$$\begin{array}{ccc} \infty\text{-categories} \simeq \text{" cat coherently enriched over } h\,\mathrm{Kan}\text{"} & & \mathcal{C} \\ \big\downarrow & & \big\downarrow \\ \text{cat. enriched over } h\,\mathrm{Kan} & & h\mathcal{C}. \end{array}$$

It is also possible to record the informat° of all the $\mathrm{map}_{\mathcal{C}}(x_0, \dots, x_n)$ into a Segal category, another model of ∞-categories (see [Rezk, § 33.11]).

Def 97: Let $F : \mathcal{C} \longrightarrow D$ be a functor between ∞-categories and $x, y \in \mathcal{C}_0$.

There is a morphism

$$F_{x,y} : \mathrm{map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{map}_D(F(x), F(y))$$

constructed as follows:

$$\begin{tikzcd}
\mathrm{map}_{C}(x,y) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & \mathrm{Fun}(\Delta^1, C) \arrow[dr, "\mathrm{Fun}(\Delta^1, F)"] \arrow[dd] & \\
& \mathrm{map}_{D}(F(x),F(y)) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, D) \arrow[dd] \\
\{(x,y)\} \arrow[rr] \arrow[dr] & & C \times C \arrow[dr, "F \times F"] & \\
& \{(F(x),F(y))\} \arrow[rr] & & D \times D
\end{tikzcd}$$

- We say that F is **fully faithful** if for all $x, y \in C_0$, this map is an homotopy equivalence of Kan complexes.
- We say that F is **essentially surjective** if the induced functor $hF : hC \longrightarrow hD$ is essentially surjective.

Thm 98: A functor F is a categorical equivalence iff F is fully faithful and essentially surjective. $\square$

Unfortunately, this is a rather difficult theorem, and it would take two extra lectures to give a proper treatment.

- For more interesting material around the core construction, see [Kerodon, § 4.4.3].

VI The Grothendieck Construction

1) Generalities

- The Grothendieck construction is a general categorical pattern, with many different incarnations. To organise the discussion, it is useful to use the informal terminology of (n,k)-categories.

"Def" 1: Let $0 \leq k \leq n \leq \infty$.

An (n,k)-category is a structure with
- objects ($= 0$-morphisms)
- 1-morphisms
- ...
- n-morphisms

with various notions of compositions,

associative (up to higher morphisms), and such that all i-morphisms for $k \leq i \leq n$ are invertible (up to higher morphisms).

Ex 2: We already know many examples:

- $(0,0)$: sets
- $(1,0)$: groupoids
- $(1,1)$: $(1-)$ categories
- $(2,2)$: bicategories
- $(2,1)$: bicategories with invertible 2-morphisms.
- $(\infty, 0)$: ∞-groupoids (= Kan complexes)
- $(\infty, 1)$: ∞-categories (= quasicategories)
- Let $n < \infty$:
  - $(n,0)$: n-groupoids.

  One model is given by Kan complexes

with $\pi_i = 0$ for $i > n$. This is "equivalent" to sets for $n=0$ and to groupoids for $n=1$.

- $(n,1)$: n-categories.

Def 3 : An ∞-category C is an n-category iff for every $m > n$ and $0 < i < m$, there exists a unique lift in every diagram:

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d, hook] & C \\
\Delta^m \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

$\square$

(For more on this notion, see [HTT, § 2.3.4]
NB: I have used an equivalent definition, see [HTT, Prop 2.3.4.9])

- For $n \leq 2$, this notion is "equivalent" to the above.

⚠ This notion of n-cat. is not stable under categorical equivalence. But:

Prop 4 [HTT, Prop 2.3.4.18]
Let C be an ∞-category. TFAE:

(i) C is categorically equivalent to an n-category.

(ii) $\forall x, y \in C_0$, the Kan complex $\mathrm{map}_C(x, y)$ is $(n-1)$-truncated.

- This leaves open a wide world of higher category theory. For instance, for a model of $(\infty, 2)$-categories close in spirit to quasicategories, see [Kerodon, § 5.4].

"Def" 5: Let $0 \leq k \leq n \leq \infty$. The collection of all (small) (n,k)-categories forms an $(n+1, k+1)$-category $\mathrm{Cat}_{n,k}$.

- Note that, for every $m \leq n$ and $\ell \leq k$, there is an "adjunction"

$$\iota : \mathrm{Cat}_{m,\ell} \rightleftarrows \mathrm{Cat}_{n,k} : \mathrm{Core}$$

where
- ι adds identities and forgets that some morphisms are invertible
- Core throws aways morphisms as needed.

⚠ This leaves undefined the precise notion of functors of (n,k)-categories ($=1$-morphisms in $\mathrm{Cat}_{n,k}$)

as well as the higher morphisms in $\mathrm{Cat}_{n,k}$. When $k \geq 2$, there are many choices, roughly corresponding to direct° of certain arrows / natural transformat°. For instance, for $(2,2)$-categories, there are lax and oplax functors:

$$\text{lax}: \quad F(f) \circ F(g) \longrightarrow F(f \circ g)$$

$$\text{oplax}: \quad F(f \circ g) \longrightarrow F(f) \circ F(g)$$

This will not play a big role so we stay vague!

Ex 6:

- $\mathrm{Cat}_{(0,0)} = \mathrm{Set}$
- $\mathrm{Cat}_{(1,0)} = \mathrm{Groupoids}$
- $\mathrm{Cat}_{(1,1)} = \mathrm{Cat}$

- $\mathrm{Cat}_{(\infty,0)} = \mathrm{Spc}$ $\quad (\infty+1=\infty\ !)$

- $\mathrm{Cat}_{(\infty,1)}$ should be an $(\infty,2)$-category.

  (with 3 variants : strict, lax , oplax)

  We have constructed Cat_∞, its "$(\infty,1)$-core".

The general idea of the Grothendieck construction is the following.

Let $\begin{cases} \mathcal{C} \text{ be an } (n+1, k+1)\text{-category} \\ F : \mathcal{C}^{op} \longrightarrow \mathrm{Cat}_{n,k} \text{ a (lax)functor} \end{cases}$

also dual version without op, see below

We want to describe an $(n+1, k+1)$-category over $\mathcal{C}$:

$$\int F \xrightarrow{p_F} \mathcal{C}$$

whose objects are pairs $(c \in C, x \in \mathrm{Ob}\, F(c))$
(so we "integrate all the values of F")

and from which we can completely recover F in the following sense:

$\int$ can be made into a "fully faithful" functor of $(n+1, k+1)$-categories:

$$\mathrm{Fun}(C^{op}, \mathrm{Cat}_{n,k}) \xrightarrow{\int \, = \int_{n,k}} \mathrm{Cat}_{n+1,k+1/C}$$

whose essential image is chararacterised as a certain class of Grothendieck fibrations

so that we get an "equivalence" of $(n+1, k+1)$-cats

$$\mathrm{Fun}^{lax}(C^{op}, \mathrm{Cat}_{n,k}) \xrightarrow[\sim]{\int} \mathrm{Grothfib}(C)$$

Moreover, $\int$ is obtained by pulling back along F^{op} the universal Grothendieck fibration

$$\begin{array}{c} \widehat{\mathrm{Cat}}^{op}_{n,k} \\ \Big\downarrow {\scriptstyle p_{univ}} \\ \mathrm{Cat}^{op}_{n,k} \end{array} \quad \left(= \mathrm{Cat}^{op}_{n,k,*,oplax}\right)$$

"oplax-pointed (n, k)-categories"

We say that F is classified by the fibration $\int F$.

- The functor in the other direction

$$\mathrm{Grothfib}(C) \longrightarrow \mathrm{Fun}^{lax}(C^{op}, \mathrm{Cat}_{n,k})$$

is less canonical and typically involves a lot of choices.

- There is an equally important dual version:

$$\mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C}, \mathrm{Cat}_{n,k}) \xrightarrow[\sim]{\int} \mathrm{Grothopfib}(\mathcal{C})$$

Grothendieck opfibrations

- Now assume $\mathcal{C}$ is only an (n,k)-category and we have

$$F : \iota\,\mathcal{C} \longrightarrow \mathrm{Cat}_{n,k}$$

(still an $(n+1,k+1)$-Functor!)

Then we expect $\int F$ to be itself ι of an (n,k)-category and to get an equivalence of $(n+1,k+1)$-categories

$$\mathrm{Fun}^{\mathrm{lax}}(\iota\,\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_{n,k}) \xrightarrow{\sim} \mathrm{Grothfib}(\mathcal{C})$$

where the Grothendieck fibrations over C are $(n, \underline{\underline{k}})$-categories.

- Same deal for C an $\begin{cases} (n+1,k)\text{-cat} \\ (n,k+1)\text{-cat} \end{cases}$

  when this makes sense.

- This is the sense in which the Grothendieck construction (sometimes) lowers the categorical degree of a situation.

- One should then investigate the functoriality in C! Lots of fun awaits.

- Finally, variants should exists for monoidal categories, enriched categories, etc.

None of this is precise mathematics! Indeed, making this scheme rigourous can be difficult ; this is an active area of research in (higher) category theory.

2) Classical examples

Let us look at some examples before going to ∞-categories.

- $\underline{(n,k)=(0,0)}$:

Let $\begin{cases} C \text{ be a } (1,1)\text{-category.} \\ F : C^{op} \longrightarrow \mathrm{Set} \text{ a presheaf} \end{cases}$

Then we have constructed in Lecture 1 the category of elements $\int F$ of F

Ob : $(c \in \mathrm{Ob}(C), x \in F(c))$

$$\mathrm{Mor}\left((c,x),(d,y)\right) = \left\{ f : c \to d \;\middle|\; f^*(y) = x \right\}$$

- Alternatively, there is a pullback diagram in $\mathrm{Cat}_{(1,1)}$:

$$\begin{tikzcd} \int F \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Set}_*)^{op} \arrow[d] \\ C \arrow[r, "F^{op}"] & \mathrm{Set}^{op} \end{tikzcd}$$

Def 7: A functor $D \xrightarrow{p} C$ in Cat is a discrete (Grothendieck) fibration / fibration in sets if for every $d \in D$ and $\bar{\delta} : c \to p(d)$, there exists a unique lift $\delta : d' \to d$ of $\bar{\delta}$.

Prop 8: The category of elements construction gives rise to an equivalence of categories

$$\mathrm{Fun}(C^{op}, \mathrm{Set}) \xrightarrow[\sim]{\int} \mathrm{Discfib}(C)$$

(so that $\mathrm{Set}_*^{op} \to \mathrm{Set}$ is the universal discrete fibration)

Exercise: What happens when C is a groupoid or a set?

- $(n,k) = (1,0)$ and $(1,1)$

- This is the original case studied by Grothendieck in the context of the study of the étale fundamental group of schemes. and more generally of descent problems in sheaf theory.

- It is also still the case most applied outside of higher category theory, in the form of the theory of stacks (of groupoids) in algebraic geometry.

(It is the "trick" by which most algebraic geometers avoid thinking about 2-categorical structures.)

could be (2,1) or (2,2)

· Let $\begin{cases} \mathcal{C} \text{ be a } (1,1)\text{-category.} \\ F : \mathcal{C}^{op} \longrightarrow \begin{cases} \mathrm{Cat} \\ \mathrm{Groupoids} \end{cases} \text{ a lax } \begin{vmatrix} (2,2) \\ (2,1) \end{vmatrix}\text{-functor} \end{cases}$

(F is sometimes called a pseudofunctor.

In particular, F could be an ordinary functor into $\begin{vmatrix} \mathrm{Cat} \\ \mathrm{Groupoids} \end{vmatrix}$ seen as a 1-category.)

Then the Grothendieck construction $\int F$ is the category with

$$\mathrm{Ob} : \big(c \in \mathrm{Ob}(\mathcal{C}),\ x \in \mathrm{Ob}\, F(x) \big)$$

$$\mathrm{Mor}\big((c,x),(d,y)\big) =$$

$$\left\{ \begin{array}{l} f : c \longrightarrow d \ \text{ in } \mathcal{C} \\ u : x \longrightarrow F(f)(y) \ \text{ in } F(c) \end{array} \right\}$$

The composition of morphisms in $\int F$

uses the structure of F as a lax 2-functor.

In particular if F is an honest 1-cat functor it is easy to define.

Let now assume $(n,k) = (1,0)$ for the moment.

Def 9: Let $p : D \to C$ be a functor of 1-categories. We say that p is a Grothendieck fibration in groupoids if:

1) for every object $d \in D$ and morphism $f : d \to p(c)$, there exists a lift $\tilde{f}$.

2) For every $f : d \to d'$ and object d'' in D, the map

$$\mathrm{Hom}(d'', d) \longrightarrow \mathrm{Hom}(d'', d') \times_{\mathrm{Hom}(p(d''), p(d'))} \mathrm{Hom}(p(d''), p(d))$$

is a bijection.

- The natural forgetful functor

$p_F : \int F \longrightarrow C$ is a

Grothendieck fibration in groupoids.

Thm 10 $\mathrm{Fun}^{lax}(C^{op}, \mathrm{Grpd}) \xrightarrow[\sim]{\int} \mathrm{FibGrpd}(C)$

This thm underlies the definition of stacks as categories fibered in groupoids satisfying some descent conditions.

- There is a characterisation of the image in the $(1,1)$-case as well in terms of cartesian fibrations in categories; we will come back to it later.

3) The $(\infty, 0)$-Grothendieck construction

Ref: Barwick-Shah, Fibrations in ∞-category theory

(gives a great survey of the topic)

- Let C be an ∞-category. We want to describe functors $C^{(op)} \longrightarrow \mathrm{Spc}$, and more generally the functor ∞-category $\mathrm{Fun}(C^{(op)}, \mathrm{Spc})$ in terms of certain fibrations. It turns out we already know the appropriate class: left / right fibrations!

- First, let's do a sanity check.

Lemma 11: Let $p: D \rightarrow C$ be a functor between 1-categories. TFAE:

(i) p is $\left|\begin{array}{l}\text{a fibration in groupoids.}\\ \text{an opfibration}\end{array}\right.$

(ii) $N(p)$ is a $\left|\begin{array}{l}\text{right fibration.}\\ \text{left}\end{array}\right.$

proof: $N(p)$ is always an inner fibration, so (ii)$_{\text{right}}$ is equivalent to having the RLP against $\Lambda^n_n \subseteq \Delta^n$.

$n > 3$: automatic

$\underline{n=1}$: equivalent to part 1) of Def

$\underline{n=2}$ equivalent to surjectivity in part 2) of Def .

$\underline{n=3}$ equivalent to injectivity in part 2) of Def $\square$

- The fibers of left / right fibrations are Kan complexes. This has some further consequences.

Recall that $s\widetilde{\mathrm{Set}}$ is a simplicial category. Let C be an ∞-category. The slice category $s\widetilde{\mathrm{Set}}_{/C}$ has itself a structure of simplicial category.

Prop 12: Let $\begin{cases} p : D \to C \\ q : D' \to C \end{cases}$ be left fibrations. between ∞-categories.

The simplicial set $\mathrm{Fun}_{/C}(p, q)$ of $s\widetilde{\mathrm{Set}}_{/C}$ is

$\cdot c$

a Kan complex.

Cor 13: Let C be an ∞-category. The simplicial subcategory $\widetilde{\mathrm{LFib}}(C)$ of $\widetilde{\mathrm{sSet}}_{/C}$ spanned by left fibrations is locally Kan.

Def 14: The ∞-category of left fibrations $\mathrm{LFib}(C)$ is defined as $N_\Delta(\widetilde{\mathrm{LFib}}(C))$.

- Let $\mathrm{Spc}_* := \mathrm{Spc}_{*/}$ be the ∞-category of pointed spaces. The canonical functor $\mathrm{Spc}_* \longrightarrow \mathrm{Spc}$ is a left fibration.

- We can now state the main theorem of this section:

Thm 15 : (Joyal)

There is an equivalence of ∞-categories

$$\int : \mathrm{Fun}(C, \mathrm{Spc}) \xrightarrow{\sim} \mathrm{LFib}(C)$$

which on objects sends a functor $F : C \longrightarrow \mathrm{Spc}$ to the pullback :

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spc}_* \arrow[d] \\ C \arrow[r, "F"] & \mathrm{Spc} \end{tikzcd}$$

Dually, there is an equivalence of ∞-cat.

$$\int : \mathrm{Fun}(C^{op}, \mathrm{Spc}) \xrightarrow{\sim} \mathrm{RFib}(C)$$

which on objects sends a functor F to

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spc}^{op}_* \arrow[d] \\ C \arrow[r, "F^{op}"] & \mathrm{Spc}^{op} \end{tikzcd}$$

Proofs:

- This theorem has a number of different proofs in the litterature; among them:

I) [HTT, Thm 2.2.1.2]

II) [Cisinski, Thm 5.4.5 and 7.8.9] (~)

III) Heuts and Moerdijk, Left fibrations and homotopy colimits II — the simplest?

- All of them require presenting the equivalence using model structures. The proofs I) and III) also rely on the comparison with simplicial categories as model for ∞-categories.

The equivalence of the theorem is

obtained by "deriving" a Quillen equivalence between simplicial model categories.

Rmk 16: This formulation of the theorem is unsatisfying because it only describes what happens on objects. It would be better to have a purely ∞-categorical caracterisation of the functor, but I don't know of one!

Terminology:

Lurie calls this equivalence the "straightening / unstraightening" equivalence:

- unstraightening : $\mathrm{Fun}(\mathcal{C}, \mathrm{Spc}) \longrightarrow \mathrm{LFib}(\mathcal{C})$
- straightening : $\mathrm{LFib}(\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Spc})$.

- Let us explore a little the "straightening" process. We start with a left fibration $p : \mathcal{D} \longrightarrow \mathcal{C}$. We won't quite produce a functor $\mathcal{C} \to \mathrm{Spc}$, but a functor $h\mathcal{C} \longrightarrow h\mathrm{Spc} \simeq h\mathrm{Kan}$.

- For $c \in \mathcal{C}$, the fiber $\mathcal{D}_c$ is a Kan complex.

- Let $f : c \to c'$ be a morphism in $\mathcal{C}$. The inclusion $\mathcal{D}_c \simeq \mathcal{D}_c \times \{0\} \hookrightarrow \mathcal{D}_c \times \Delta^1$ is left anodyne because $\{0\} \hookrightarrow \Delta^1$ is.

So we can solve the lifting problem:

$$\begin{tikzcd} D_c \arrow[r, hook] \arrow[d, hook] & D \arrow[d, "p"] \\ D_c \times \Delta^1 \arrow[r] \arrow[ur, dashed] & \Delta^1 \xrightarrow{f} C \end{tikzcd}$$

Restricting $\dashrightarrow$ to $D_c \times \{1\}$, we get

$$f_! : D_c \longrightarrow D_{c'} .$$

$f_!$ is not uniquely determined. However it is determined up to homotopy, and:

Lemma 17: The assignement

$$c \rightsquigarrow D_c \quad , \quad f \rightsquigarrow f_!$$

determines a functor $hC \longrightarrow h\,\mathrm{Kan}$

proof: Let $\sigma : \Delta^2 \longrightarrow C$ be a 2-simplex

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "h"'] & & W \end{tikzcd}$$

. Choose $f_!, g_!, h_!$

Using the fact that $D_c \times \{0\} \hookrightarrow D_c \times \Delta^2$ is also left anodyne, one shows that

$h_! = g_! \circ f_!$ in $h\mathrm{Kan}$.

(see [HTT, Lemma 2.1.1.4]) $\square$

4) Applications

. <u>Representable and corepresentable functors</u>

Let C be an ∞-category and $x \in C$.

We have:

- a left fibration $C_{x/} \longrightarrow C$
- a right fibration $C_{/x} \longrightarrow C$

The functors classified by them are:

- the corepresentable functor

$$R^x : C \longrightarrow Spc$$

- the representable functor

$$R_x : C \longrightarrow Spc$$

Let us compute their values on objects: $R^x(y)$ is by definition isomorphic in Spc to $(C_{x/})_y$ and $R_x(y)$ is iso. to $(C_{/x})_y$.

These simplicial sets are variants of the mapping spaces $\mathrm{map}_C(x,y)$.

Def 18: Let C be an ∞-category and $x, y \in C_0$. The left(-pinched) mapping space $\mathrm{map}^L_C(x,y)$ is defined as

$$\mathrm{map}^L_C(x,y) := (C_{x/})_y .$$

More concretely, for any $n \geq 0$, we have

$$\mathrm{map}^L_C(x,y)_n = \left\{ \sigma \in C_{n+1} \;\middle|\; \begin{array}{l} \sigma(0) = x,\ d_0(\sigma) \\ \text{constant map } \Delta^n \to \{y\} \to C \end{array} \right\}$$

Dually, the right(-pinched) mapping space

$$\mathrm{map}^R_C(x,y) := (C_{/x})_y$$

More concretely, for any $n \geq 0$, we have

$$\mathrm{map}^R_C(x,y)_n = \left\{ \sigma \in C_{n+1} \;\middle|\; \begin{array}{l} d_{n+1}(\sigma) \text{ constant map at } x \\ \sigma(n+1) = y \end{array} \right\}$$

$\square$

As fibers of left/right fibrations, those are Kan complexes. Moreover:

Prop 19: There are natural monomorphisms

$$\mathrm{map}^L_C(x,y) \hookrightarrow \mathrm{map}_C(x,y) \hookleftarrow \mathrm{map}^R_C(x,y)$$

which are homotopy equivalences.

See [Kerodon, Prop 4.6.6.9].

- There is a recognition principle for (co)representable functors as in 1-cat. theory:

Lemma 20: Let $F : C^{op} \longrightarrow Spc$ be a functor. TFAE:

(i) F is isomorphic to a representable functor h_c in $Fun(C^{op}, Spc)$.

(ii) The ∞-category $\int F$ has a terminal object $\hat{c}$.

In that situation, $p_F(\hat{c}) = c$. $\square$

Mapping spaces

We have treated the functoriality of mapping spaces separately in each variable. We also expect a functor

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Spc}$$

. For this, we should construct either a left or right fibration to $\mathcal{C}^{op} \times \mathcal{C}$ (since $(\mathcal{C}^{op} \times \mathcal{C})^{op} \simeq \mathcal{C}^{op} \times \mathcal{C}$ canonically).

Def 21: Let $\mathcal{C}$ be an ∞-category. The *twisted arrow ∞-category* $\tilde{\mathcal{O}}(\mathcal{C})$ is the simplicial set

$$\tilde{\mathcal{O}}(\mathcal{C})_n := \mathrm{sSet}(\Delta^{n,op} \ast \Delta^n, \mathcal{C})$$

There are morphisms

$$\tilde{\mathcal{O}}(\mathcal{C}) \xrightarrow{s} \mathcal{C}^{op} \quad \text{and} \quad \tilde{\mathcal{O}}(\mathcal{C}) \xrightarrow{t} \mathcal{C}$$

induced by the inclusions

$$\Delta^{n,op} \hookrightarrow \Delta^{n,op} * \Delta^n \hookleftarrow \Delta^n.$$

Rmk 22: This is a simplicial variant of the twisted arrow category $\tilde{\mathcal{O}}(C)$ of a 1-category C, which is the category of elements of $\mathrm{Hom}_C : C^{op} \times C \longrightarrow \mathrm{Set}$.

$\mathrm{Ob}\, \tilde{\mathcal{O}}(C)$: morphisms in C

$\mathrm{Mor}_{\tilde{\mathcal{O}}(C)}(f, g)$: diagrams
$$\begin{tikzcd} x \arrow[d, "f"'] & z \arrow[l, "p"'] \arrow[d, "g"] \\ y \arrow[r, "q"'] & t \end{tikzcd}$$

Exercise: Let C be a 1-category.

Prove that $\tilde{\mathcal{O}}(N(C)) \simeq N(\tilde{\mathcal{O}}(C))$

Prop 23: The morphism
$$(s,t) : \tilde{\mathcal{O}}(C) \longrightarrow C^{op} \times C$$
is a left fibration; in particular,

$\breve{\mathcal{O}}(C)$ is an ∞-category. $\square$

There is a simple proof of this in Barwick-Glasman, on <u>the fibrewise effective Burnside ∞-category.</u>, § 1.

TODO

- (n,k)-cats :
  - $(n,0)$ and $(n,1)$ cats via qcats
  - $\mathrm{Cat}_{n,k}$ $(n+1, k+1)$-category
- Cat of elements of a presheaf ; twisted arrow cat.
- $(2,1)$-Groth. construction; $\pi_1^{ét}$, stacks ...
- Statement of the $(\infty, 0)$-groth construction.
- action on objects
- Explain informal / incoherent constructº in the other direction.
- Applications :
  - representables via slices,

    an left fibratº is repr. <u>iff</u> initial object.
  - $\mathrm{Map}(-,-)$ via the twisted arrow ∞-cat.
  - Kan fibrations ⟵ Yoneda lemma.
  - adjoint functors via this POV

- limits / colimits