

(because CW-complexes $\subset \text{Top}_{\text{cof}} =$ retracts of cell complexes)

We now discuss the functoriality of model categories.

Def 53: Let M, N be model categories.

- A **Quillen adjunction** is an adjunction

$$F : M \rightleftarrows N : G$$

such that $\begin{cases} F(\text{Cof}) \subset \text{Cof} \\ G(\text{Fib}) \subset \text{Fib} \end{cases}$

Lemma 54: $F \dashv G$ if a Quillen adjunction



$$F(\text{Cof}) \subset \text{Cof} \quad \text{and} \quad F(\text{Cof} \cap W) \subset \text{Cof} \cap W$$



$$G(\text{Fib}) \subset \text{Fib} \quad \text{and} \quad G(\text{Fib} \cap W) \subset \text{Fib} \cap W.$$

Proof: Exercise.

Lemma 55: Let $F: M \rightleftarrows N: G$ be a Quillen adjunction. Then there is an induced derived adjunction

$$[LF : R_0(M) \rightleftarrows R_0(N) : RG]$$

such that

- $L F(x) \simeq F(X_{\text{cof}})$
- $R F(y) \simeq G(Y_{\text{fib}})$.

□

This, applied to model categories like $\mathbf{Cpl}_R^{(\pm)}$, is the source of derived functors in Homological algebra.

Def 56: Let $F: M \rightleftarrows N: G$ be a Quillen adjunt.

It is a Quillen equivalence if the derived adjunction $[LF \rightarrow RG]$ is an equivalence of categories.

(this is equivalent to: For $X \in M_{\text{cof}}$

and $Y \in N_{\text{fib}}$, then

$$F(x) \rightarrow Y \text{ in } W_M \Leftrightarrow X \rightarrow G(Y) \text{ in } W_N$$

Many functors we have seen in this course turn out to be part of Quillen adjunctions or equivalences.

Thm 57: (Quillen) The adjunction

$$\|-| : sSet \rightleftarrows Top : Sing$$

↑ ↑
(with Kan-Quillen) (with Quillen)

is a Quillen equivalence.

Sketch: • Quillen adjunction: We check

that $| \text{cof} | \subset \text{cof}$ and $\text{Sing}(\text{Fib}) \subset \text{Fib}$.

The first part, we know because of the (relative) skeletal filtration for monomorphisms of simplicial sets.

The second part is easy because of the following:

$$\begin{array}{ccc}
 \Delta_k^n & \longrightarrow & \text{Sing}(X) \\
 \downarrow & \nearrow ? & \downarrow \\
 \Delta^n & \longrightarrow & \text{Sing}(Y)
 \end{array} \qquad \Leftrightarrow \qquad
 \begin{array}{ccc}
 D^{n-1} = |\Delta_{k-1}^n| & \longrightarrow & X \\
 \downarrow & \nearrow ? & \downarrow \\
 D^n \times I = |\Delta^n| & \longrightarrow & Y
 \end{array}$$

which shows that $\text{Sing}(\text{Serre fibrat}^\circ) \subset \text{Kan fibrat}^\circ$.

- Quillen equivalence: This is a theorem of Milnor which we already mentioned in previous discussions of simplicial homotopy theory.

See [Kerodon, § 3.5].



This is some sense the final version of the representation of homotopy types by

simplicial sets.

Thm 58: (Lurie) The adjunction

$$\text{Path}[-]: \text{sSet} \rightleftarrows \text{Cat}_{\Delta}: N_{\Delta}$$

(with [↑]Joyal) (with [↑]Bergner)

is a Quillen equivalence. □

- This is the promised “equivalence of homotopy theories” between two models of $(\infty, 1)$ -categories.
- We now briefly mention simplicial model categories.

Def 59: A simplicial model category

M is both a simplicial category and

a model category such that:

- M is tensored and cotensored over $sSet$
- (SM7) for every cofibration $i: A \hookrightarrow B$ and fibration $p: X \rightarrow Y$,

the simplicial pullback-hom

$$\underset{C}{\text{Hom}}(B, X) \longrightarrow \underset{C}{\text{Hom}}(A, X) \times_{\underset{C}{\text{Hom}}(A, Y)} \underset{C}{\text{Hom}}(B, Y)$$

is a Kan fibration, which is

trivial if either i or p is.



- Ex 60:
- $sSet_{\text{Kan-Quillen}}$, with its self-enrichment coming from the cartesian closed structure, is a simplicial model category (this is proven with the same "lifting calculus" methods that we saw in Chapter III)
 - $CGHaus$ has a natural structure of

simplicial category with

$$\underline{\text{Hom}}_C(X, Y) = \text{Sing}(\underline{\text{Hom}}(X, Y))$$

and the Quillen-type model structure on

CGPlans is a simplicial model category.

-  $s\text{Set}_{\text{Joyal}}$ is not a simplicial

model category in this sense.,

because it has fibrations which are
not Kan fibrations.

Def 61: Let M be a simplicial model

category. Then, because of (SM7),

the full simplicial subcategory on M_{cf}

is locally Kan. The ∞ -category

associated to M is $N_{\Delta}(M_{cf})$. □

Thm 62: (Simpson, Dugger, Lurie)

Let C be an ∞ -category.

The following are equivalent:

- C is a presentable ∞ -category
(C admits all colimits in ∞ -categorical sense, and is generated under filtered colimits by "small" objects)
- There exists a combinatorial simplicial model category M such that $C \simeq N_{\Delta}(M_{cf})$.



This theorem explains a posteriori the success of the theory of model categories:

they "model" many interesting ∞ -categories.

V Joins, slices, (co)limits

1) The 1-categorical story

Let's start with the piece of category which we want to generalize to ∞ -categories.

Def 1: Let C be a category.

(1) Let X be an object of C . The slice category $C_{/X}$ is defined as the pull back

$$\begin{array}{ccc} C_{/X} & \longrightarrow & \text{Fun}([1], C) = \text{Ar}(C) \\ \downarrow & \lrcorner & \downarrow \text{ev}_1 \\ \{X\} & \longrightarrow & C \end{array}$$

- The coslice category $C_{X/}$ is defined dually (using ev_0).

Concretely, an object of $C_{/X}$ is a morphism $Y \rightarrow X$, and a morphism of $C_{/X}$ is a commutative triangle

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & \equiv & \downarrow \\ X & & \end{array}.$$

(2) Let \mathcal{J} be another category and $F : \mathcal{J} \rightarrow C$ a functor.

Let $C \xrightarrow{(-)} \text{Fun}(\mathcal{J}, C)$ be the “constant functor” functor.

The slice category $C_{/F}$ is defined as

$$C \times_{\text{Fun}(\mathcal{J}, C)} \left(\text{Fun}(\mathcal{J}, C) \right)_{/F}.$$

i.e. the pullback

$$\begin{array}{ccc}
 C_F & \longrightarrow & \text{Fun}(J, C)_F \\
 \downarrow & & \downarrow \\
 C & \xrightarrow{\cong} & \text{Fun}(J, C)
 \end{array}$$

i.e. its objects are the **cones** over F :

an object $X \in C$ together with a natural transformation $\underline{X} \xrightarrow{\lambda} F$, so a collection of morphisms $(X \xrightarrow{\lambda_i} F(i))_{i \in J}$ such that,

for every morphism $g: i \rightarrow j$, we have

$$X \begin{matrix} \xrightarrow{\lambda_i} & F(i) \\ \equiv & \downarrow Fg \\ \xrightarrow{\lambda_j} & F(j) \end{matrix}$$

The **coslice category** $C_{F/}$ is defined

dually as the category of **cocones** under F .

Def 2: Let C be a category and $X \in C$.

Then X is an initial (resp. terminal)
Final

object of C if the canonical functor

$C_{X/} \rightarrow C$ (resp. $C_{/X} \rightarrow C$)

is an equivalence of categories.

Exercise 3: Check this is equivalent to

your favourite definition of initial/terminal.

Def 4: Let $F : \mathcal{D} \rightarrow C$ be a functor.

A limit (resp. a colimit) of F is an
terminal (resp. terminal) object in C/F

(resp. C_{F_1}).

Exercise 5: Again, check that this

fits with your favourite definition of (ω)limit.

- Check using these definitions that an initial object is the same thing as
 - a colimit of the empty functor $\emptyset \rightarrow C$
 - a limit of the identity functor $C \rightarrow C$.

The charm of these definitions is that everything is built from slice categories. So if we can construct “slice ∞ -categories” we can hope to define limits and colimits.

• The problem is that in the ∞ -categorical context, we don't want the triangles

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ g \searrow & \cong & \swarrow g \\ & X & \end{array}$$

to commute on the nose
but up to coherent homotopy.

It turns out to be easier to define the left adjoint of the (ω) slice construction.

Def 6: Let C, D be categories. The join $C * D$ of C and D is the category defined as follows:

$$- \text{Ob}(C * D) = \text{Ob}(C) \amalg \text{Ob}(D)$$

$$- C * D (x, y) = \begin{cases} C(x, y), & x, y \in C \\ D(x, y), & x, y \in D \\ *, & x \in C, y \in D \\ \emptyset, & x \in D, y \in C \end{cases}$$

and composition is defined in the obvious way.

. The join defines a functor:

$$-\ast- : \text{Cat} \times \text{Cat} \longrightarrow \text{Cat}$$

. By construction, there are functors

$$C \xrightarrow{\iota_C} C \ast D \xleftarrow{\iota_D} D.$$

. We define the left cone $C^\triangleleft := [0] \ast C$
right cone $C^\triangleright := C \ast [0]$.

Rmk 7: \ast defines a (non-symmetric)

monoidal structure, with monoidal
unit \emptyset .

. $C \rightarrow C^\triangleleft$ is “the universal way
to add an initial object to C ”, as
we will see later.

Lemma 8: Let C, D be categories.

(1) The functor L_C factors uniquely as

$$C \xrightarrow{L_C} (C * D)_{/L_D} \longrightarrow C * D$$

(2) The functor L_D factors uniquely as

$$D \xrightarrow{L_D} (C * D)_{/L_C} \longrightarrow C * D$$

proof: By definition,

$$(C * D)_{/L_D} = (C * D) \times \text{Fun}([1] \times D, C * D) \times \{L_D\}$$
$$\quad \quad \quad \text{Fun}(D, C * D) \quad \quad \quad \text{Fun}(D, C * D)$$

so a factorisation as in (1) is the same thing as a functor

$$C \longrightarrow \text{Fun}([1] \times D, C * D)$$

satisfying some properties, and by adjunction the same as a functor

$$C \times D \times [r] \longrightarrow C * D,$$

that is a natural transformation
from

$$\iota_C \circ \pi_C : C \times D \longrightarrow C * D$$

$$\text{to } \iota_D \circ \pi_D : C \times D \longrightarrow C * D.$$

There is a unique such transformation ψ ,
given by $(x, y) \in C \times D \mapsto$ the
unique element of $C * D(x, y)$. □

Prop 9: Let $\begin{cases} \mathcal{C} \text{ be a category} \\ G : D \rightarrow E \text{ be a functor.} \end{cases}$

There is a bijection

$$\left\{ \begin{array}{c} \overset{\iota_D}{\curvearrowright} \overset{D}{\curvearrowright} \overset{G}{\curvearrowright} \\ \mathcal{C} * D \dashrightarrow E \end{array} \right\} \simeq \text{Cat}(\mathcal{C}, E/G) .$$

$$U \longmapsto \bar{F}(U)$$

given as follows:

For every functor $U : \mathcal{C} * D \rightarrow E$ such

that $D \xrightarrow{\iota_D} \mathcal{C} * D \rightarrow E$ is equal to G ,

let $\bar{F}(U)$ denote the composite

$$\mathcal{C} \xrightarrow{\iota_C} \mathcal{C} * D \xrightarrow{\iota_D} E/U \circ \iota_D = E/G$$

Dually, we have:

$$\left\{ \begin{array}{c} \overset{\iota_C}{\curvearrowright} \overset{C}{\curvearrowright} \overset{F}{\curvearrowright} \\ \mathcal{C} * D \dashrightarrow E \end{array} \right\} \simeq \text{Cat}(D, E/F),$$

Proof: Omitted. See [Kerodon, Prop 4.3.2.10
and Prop 4.3.2.13]



Rmk: The proof shows that we have

in fact a pushout square in Cat :

$$\begin{array}{ccc} (C \times \{0\} \times D) \amalg (C \times \{1\} \times D) & \longrightarrow & C \times [1] \times D \\ \downarrow & & \downarrow v \\ (C \times \{0\}) \amalg (\{1\} \times D) & \longrightarrow & C * D . \end{array}$$

Cor 10: A colimit of $F : I \rightarrow C$ is

the same thing as

a functor $\hat{F} : I^D \rightarrow C$ extending F

and initial for this property.

proof: By Prop. , we have that

$$\text{Ob}(C_{F/}) \simeq \left\{ \begin{array}{c} I \\ I^{\Delta} = I * [0] \xrightarrow{F} C \end{array} \right\}$$

This actually lifts to an equivalence

of categories

$$C_{/F} \simeq \left\{ \begin{array}{c} I \\ I^{\Delta} = I * [0] \xrightarrow{F} C \end{array} \right\}$$

A colimit of F is an initial object

of $C_{/F}$; by looking at initial objects
on the right, the result follows.

□

Cor 11: • For any category D , the
functor

$$\text{Cat} \longrightarrow \text{Cat}_{D/}, C \mapsto C * D$$

is the left adjoint to

$$\text{Cat}_{D/} \longrightarrow \text{Cat}, (G : D \rightarrow E) \mapsto E_{G/}$$

- For any category C , the functor

$$\text{Cat} \longrightarrow \text{Cat}_{C/}, D \mapsto C * D$$

is the left adjoint to

$$\text{Cat}_{C/} \longrightarrow \text{Cat}, (F : C \rightarrow E) \mapsto E_{F/}.$$

proof: This is just a reformulation

of Prop. □

2) Joins of simplicial sets

Def 12: The augmented simplex category

Δ_+ is the full subcategory of PoSet

spanned by $[-1] = \emptyset, [0], [1], \dots$.

In other words, we add an initial object

to Δ . We can also write

$\Delta_+ = \Delta^\Delta$ using the notation from the previous section.

Def 13: The category of augmented

simplicial sets is $s\text{Set}_+ := \text{Fun}(\Delta_+^{\text{op}}, \text{Set})$.

We write $\Delta^n = y[n], n \geq -1$ for representables.

Lemma : Let $\iota : \Delta \rightarrow \Delta_+$ be the

canonical functor. The precomposition

functor $\iota^* : s\text{Set}_+ \longrightarrow s\text{Set}$

has $\begin{cases} \text{a left adjoint } L_! : s\text{Set} \rightarrow s\text{Set}_+ \\ \text{a right adjoint } L_* : s\text{Set} \rightarrow s\text{Set}_+ \end{cases}$.

proof: This is a special case of the functoriality of presheaf categories
 (see Exercise 2.2). □

Lemma 14: There is an equivalence of

categories :

$$s\text{Set}_+ \xrightarrow{\sim} \left\{ (X, E, a) \middle| \begin{array}{l} X \in s\text{Set} \\ E \in \text{Set} \\ a: X \rightarrow cE \end{array} \right\}$$

$$\tilde{X} \xrightarrow{\sim} (L^* \tilde{X}, ev_{[-]} \tilde{X}, a_{\tilde{X}})$$

where $a_{\tilde{X}}$ is the collection of maps

$$(\tilde{X}([-] \rightarrow [i]))_{i \geq 0}.$$

• Via this equivalence, the functors
 $L_! + L^* + L_*$ are given by the formulas

$$\begin{cases} L^*(X, \mathbb{E}, a) = X \\ L_! X = (X, \pi_0(X), X \rightarrow c\pi_0(X)) \\ L_* X = (X, *, X \rightarrow \Delta^0 = c*) \end{cases}$$

proof: Exercise.

Lemma 15: The category Δ_+ has a

monoidal structure defined by ordinal sum:

$m, n \geq -1$, this is the join of categories
in the sense of the previous section

$$[m] * [n] = [m+1+n]$$

$$\{0 <_1 \dots < m\} \oplus \{\bar{0} < \bar{1} < \dots < \bar{n}\} = \{0 < 1 < \dots < m < \bar{0} < \bar{1} < \dots < \bar{n}\}$$

and monoidal unit $[-1] = \emptyset$.

Def 16: The join of augmented

simplicial sets is defined as the free cocompletion

$$*: \text{sSet}_+ \times \text{sSet}_+ \longrightarrow \text{sSet}_+$$

of

$$*: \Delta_+ \times \Delta_+ \longrightarrow \Delta_+ \xrightarrow{\delta} \text{sSet}_+$$

in both variables; i.e. the unique colimit preserving functor such that for all $m, n \geq -1$, we have

$$\Delta^m * \Delta^n = \Delta^{m+n}.$$

Rmk: This is a special case of an important general construction in category theory, the Day convolution: any monoidal structure on a category \mathcal{C} induces a monoidal structure on $\text{PSh}(\mathcal{C})$ in a canonical way.

Def 17: Let J be a totally ordered set.

The set of cuts $\text{Cut}(J)$ is the set of decompositions $J = J_1 \amalg J_2$ such that $x < y$ whenever $x \in J_1$ and $y \in J_2$.

Lemma 18: Let $\alpha: J \rightarrow J'$ be an order-preserving map, and $(J'_1, J'_2) \in \text{Cut}(J')$. There exists a unique $(J_1, J_2) \in \text{Cut}(J)$ such that α restricts to maps

$$\alpha_1: J_1 \rightarrow J'_1 \text{ and } \alpha_2: J_2 \rightarrow J'_2.$$

Hence $\text{Cut}(-)$ is a contravariant functor on totally ordered sets.

proof: Put $J_i = \alpha^{-1}(J'_i)$. This is a cut since α is order-preserving. □

Prop 19: Let $X, Y \in \text{sSet}_+$. There is

a canonical identification, for $n \geq -1$

$$(X * Y)_n = \coprod_{(J_1, J_2) \in \text{Cut}(\mathbb{E}_n)} X(J_1) \times Y(J_2)$$

$$= \coprod_{\substack{i+1+j=n \\ i, j \geq -1}} X_i \times Y_j$$

Moreover, if $\alpha: [m] \rightarrow [n]$ is a map
 in Δ_+ and we fix a decomposit^o $i+1+j=n$,
 there is an induced decomposit^o $i'+1+j'=m$
 such that $\alpha^{-1}([i]) \subseteq [i']$ and $\alpha^{-1}(i+1+[j]) = [j']$,
 and the induced map

$$(X * Y)_n \xrightarrow{\alpha^*} (X * Y)_m$$

is given by $\coprod_{i+1+j=n} \alpha_i^* \times \alpha_j^*$

with $\begin{cases} \alpha_i: [i'] \rightarrow [i] \\ \alpha_j: [j'] \rightarrow [j] \end{cases}$ induced by α .

as in Lemma .

Proof: defines an augmented simplicial set and

- The RHS^v commutes with colimits in both X and Y because colimits in presheaf categories are computed objectwise, so it suffices to show this for representables

- The point is then precisely that

$$(\Delta^p * \Delta^q)_n = (\Delta^{p+q})_n$$

$$= \Delta_+([n], [p+q])$$

$$= \coprod_{i+j=n} \Delta_+([i], [p]) \times \Delta_+([j], [q])$$

where $[i]$ is the preimage of $[p]$ under
 a map $[n] \rightarrow [p+q]$.

$$= \coprod_{i+j} \Delta_i^P \times \Delta_j^Q$$

This finishes the proof. □

Def 20: The join of simplicial sets is

defined as the functor

$$-\ast- : \text{sSet} \times \text{sSet} \longrightarrow \text{sSet}$$

$$(X, Y) \longmapsto \iota^*((\iota_* X) \ast (\iota_* Y))$$

• Note that ι_* preserves representables, $\iota_* \Delta^n = \Delta^n$:

For all $k \geq -1$,

$$\text{sSet}_+ (\Delta^k, \iota_* \Delta^n) = \text{sSet} (\iota^* \Delta^k, \Delta^n)$$

$$= \begin{cases} \text{sSet} (\emptyset, \Delta^n), & k = -1 \\ \text{sSet} (\Delta^k, \Delta^n), & k \geq 0 \end{cases}$$

$$= \Delta_+([k], [n])$$

$$= sSet_+(\Delta^k, \Delta^n).$$

so $\Delta^m * \Delta^n = \Delta^{m+n}$ both in $sSet$ and $sSet_+$.

Also, the (non-representable) empty simp.set \emptyset

satisfies $\Delta_* \emptyset = [-1]$.

- By combining the formulas above, one gets

that we still have a formula:

$$(X * Y)_n = \coprod_{\substack{i+j=n \\ i, j \geq -1}} X_i \times Y_j$$

with by convention $X_{-1} = Y_{-1} = \text{pt.}$,

or alternatively as

$$(X * Y)_n = X_n \amalg \coprod_{\substack{i+j=n \\ i, j \geq 0}} (X_i \times Y_j) \amalg Y_n$$

Prop 21: Let X, Y be ∞ -categories.

Then $X * Y$ is also an ∞ -category.

proof: Let $0 < i < n$ and

$$\epsilon_0 : \Delta_i^n \longrightarrow X * Y.$$

We must extend ϵ_0 to an n -simplex of $X * Y$.

- Let $J \subseteq [n]$ be the set of vertices such that $\epsilon_0(j) \in X$ for $j \in J$.

Suppose that J is not an initial segment of $[n]$: then there is $j \in J$ with $j > 0$ and $j-1 \notin J$. Let $e = \{j-1, j\} \in (\Delta_i^n)_n$.

Then $\epsilon_0(e) \in (X * Y)_n$ has source in Y and target in X , which is impossible.

$\Rightarrow J = [i]$ is an initial segment ($-1 \leq i \leq n$)

- If $J = \emptyset$ or $J = [n]$, then σ_0 .
 Factors through either $Y \rightarrow X * Y$
 or $X \rightarrow X * Y$ and we use the
 assumption that X, Y are ∞ -categories
 to extend σ_0 .

- If $0 \leq i < n$, the composite

$$\Delta^J \rightarrow \Lambda_i^n \xrightarrow{\sigma_0} X * Y$$

factors through X via

$$\sigma_- : \Delta^J \longrightarrow X.$$

Similarly, $\Delta^{[n] \setminus J} \xrightarrow{\sigma_+} Y$,

and σ_0 admits a unique extension

given by

$$\Delta^n \simeq \Delta^J * \Delta^{[n] \setminus J} \xrightarrow{\sigma_- * \sigma_+} X * Y. \quad \square$$

Def 22: Let $X \in \text{sSet}$. The left cone

(resp. right cone) X^{Δ° (resp. X^{Δ}) is :

$$\begin{cases} X^{\Delta^\circ} := \Delta^\circ * X \\ X^{\Delta} := X * \Delta \end{cases}$$

Ex 23: (Outer horns as cones) $n \geq 0$.

$$\Lambda_n^{n+1} \simeq (\partial \Delta^n)^{\Delta^\circ}, \quad \Lambda_{n+1}^{n+1} \simeq (\partial \Delta^n)^{\Delta}.$$

Rmk: The functor $* : \text{sSet} \times \text{sSet} \rightarrow \text{sSet}$ does not commute with colimits, since

$$\emptyset * X \simeq X \text{ is not an initial object for } X \neq \emptyset.$$

We will see that $*$ still preserves some colimits

if interpreted correctly.

- One could define another functor \ast' by $\iota^*((\iota_! X) \ast (\iota_! Y))$. The advantage of this new functor is that it would commute with all colimits. The defect is that $\iota_!$ does not send representables to representables, so we have $\Delta^m \ast' \Delta^n \neq \Delta^{m+n}$.

Lemma 24: Let $C, D \in \text{Cat}$. There is an

isomorphism of simplicial sets

$$N(C \ast D) \xrightarrow{\sim} N(C) \ast N(D).$$

proof:

$$N(C \ast D)_n \cong \left\{ x_0 \xrightarrow{\delta_0} x_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_n} x_n \begin{array}{l} \text{composable} \\ \text{morphisms in } C \ast D \end{array} \right\}$$

we "jump" from
 C to D at
some point. $\xrightarrow{i+j=n}$

$$= \coprod_{i+j=n} \{x_0 \rightarrow \cdots \rightarrow x_i \text{ in } C\} \times \{x_{i+1} \rightarrow \cdots \rightarrow x_n \text{ in } D\}$$

$$= (N(c) * N(D))_n.$$



Lemma 25: Let $X \in \text{sSet}$. The join

construction lifts to functors:

$$\begin{cases} X * - : \text{sSet} \longrightarrow \text{sSet}_{X/} \\ - * X : \text{sSet} \longrightarrow \text{sSet}_{X/} \end{cases}$$

Proof: The maps from X are given by

$$X \curvearrowright X * \emptyset \longrightarrow X * Y.$$



Prop 26: The functors

$$\begin{cases} X * - : \text{sSet} \longrightarrow \text{sSet}_{X/} \\ - * X : \text{sSet} \longrightarrow \text{sSet}_{X/} \end{cases}$$

preserve colimits.

Proof: sSet is a presheaf category

so colimits are computed object wise.

• Colimits in a coslice category like

$sSet_{X_1}$ are easy to compute :

$$F : I \longrightarrow sSet_{X_1}, \quad \pi : sSet_{X_1} \rightarrow sSet$$

Then $\text{colim } F = (X \xrightarrow{\text{given by univ. property of}} \text{colim } \pi \circ F).$

In particular, to compute colimits in $sSet_{X_1}$, we are reduced to compute colimits in Set_{X_n} for each $n \geq 0$.

Now

$$(X * Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \dots \amalg Y_n$$

and the functor

$$Y \mapsto (X * Y)_n, \quad sSet \rightarrow \text{Set}_{X_n}$$

commutes with colimits (as it is essentially a coproduct).

This finishes the proof. \square

Rmk: The functor $sSet_{X/} \rightarrow sSet$ does not preserves colimits (id_X is initial in the source, but X is not initial in $sSet$ unless $X = \emptyset$).

It does preserve all connected colimits, those indexed by a category with $|N(I)|$ connected. In particular, it preserves pushouts and filtered colimits.

Prop²⁷: The functors.