II Weierstrass models of elliptic curves

- We start with elliptic curves. For this special case, there is a theory of models which is more elementary than both minimal regular models and Néron models

1) Weierstrass equations

<u>def 1</u> Let R be a ring. A <u>Weierstrass equation</u> with coefficients in R is an homogeneous equation of the form

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

- Our goal is to derive the main properties of such equations with as little computations as possible, and then forget about the equations!

(not recommended if you actually want to compute things)

<u>Lemma 2</u> Let k be a field, $W \subseteq \mathbb{P}_k$ given by a Weierstrass equation. Then W is geometrically integral, smooth at $[0:1:0]$.

<u>Proof</u>: – Writing the equation as $F(x,y,z) = 0$, we have

$$\frac{\partial F}{\partial z} = y^2 + a_1xy + 2a_3yz - a_2x^2 - 2a_4xz - 3a_6z^2$$

Hence $\frac{\partial F}{\partial z}([0:1:0]) = 1 \neq 0 \Rightarrow W$ is smooth at $[0:1:0]$.

- $W \cap V(z) = \{[0:1:0]\}$. Since $V(z)$ intersects all irreducible components of W (2 curves in $\mathbb{P}^2_k$ intersect!), and $[0:1:0]$ is a smooth point, we get W irreducible.
- W irreducible cubic curve. If W non-reduced, then necessarily $F = L^3$, and W everywhere non-reduced $\lightning$
- This argument applies over any field extension $\Rightarrow W$ geometrically integral. $\square$

prop 3 | Let (E, e) be an elliptic curve over a field k. There exists $x, y \in K(E)$ such that the birational map $\phi : E \longrightarrow \mathbb{P}^2_k$, $[x : y : 1]$ is a closed immersion whose image is cut out by a Weierstrass equation, and $\phi(e) = [0 : 1 : 0]$ unique pt in the line at ∞

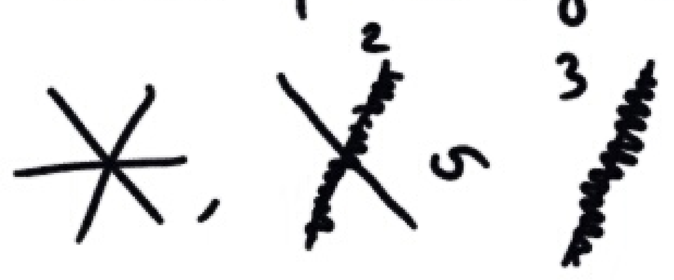
proof:
- RR $\Rightarrow \forall n \geq 1$, $\dim H^0(E, \mathcal{O}(ne)) = n$.
- Choose meromorphic functions x, y such that
  $\{1, x\}$ basis of $H^0(E, \mathcal{O}(2e))$,
  $\{1, x, y\}$ basis of $H^0(E, \mathcal{O}(3e))$.
- Then $\{1, x, y, x^2, xy, y^2, x^3\} \subseteq H^0(E, \mathcal{O}(6e))$

(Recall that if $k = \mathbb{C}$, $E^{an} = \mathbb{C}/\Lambda$ then can choose $x = \wp$, $y = \wp'$ with $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$ Weierstrass elliptic function.)

$\Longrightarrow \exists A_1, \cdots, A_7 \in k$, not all 0,

$A_1 + A_2 x + A_3 y + \cdots + A_7 x^3 = 0$

- By considering pole order at e and using $\dim H^0(E, \mathcal{O}(5e)) = 5$ see that $A_6 \neq 0$ and $A_7 \neq 0$.
- Subsitute $x \rightsquigarrow -A_6 A_7 x$, $y \rightsquigarrow A_6 A_7^2 y$ $\Longrightarrow$ $F = 0$ (Weierstrass equat°) $\cdot A_6^3 A_7^4$
- Get $\phi : E \dashrightarrow \mathbb{P}^2$. Since E is smooth of dim ≤ 1 and $\mathbb{P}^2$ is proper, ϕ is automatically a morphism. The image of ϕ is thus a closed subset of $\mathbb{P}^2$, not reduced to a point, and contained in the integral locus $F = 0$ $\Rightarrow \mathrm{Im}(\phi) = \{F = 0\}$.
- By composing with proj to $\mathbb{P}^1$ via $[x : 1]$, $[y : 1]$, we see that $\deg(\phi)$ divides 2 and 3 $\Rightarrow \deg(\phi) = 1$.
- To conclude, it remains to show that $\mathrm{Im}(\phi)$ is smooth. By Lemma 2, $\mathrm{Im}(\phi)$ is an integral cubic curve in $\mathbb{P}^2$.

• Assume that $\mathrm{Im}(\phi)$ is singular and let p be a singular point. Then p is of multiplicity exactly 2 (otherwise, $\mathrm{Im}(\phi)$ would be either , hence not integral). This implies that the projection map $\mathbb{P}^2 \setminus \{p\} \longrightarrow \mathbb{P}^1$ restricts to a degree 1 rational map

$\mathrm{Im}(\phi) \dashrightarrow \mathbb{P}^1$, which composed with $E \to \mathrm{Im}(\phi)$

gives $E \dashrightarrow \mathbb{P}^1$ of degree 1 $\lightning$ $\square$

<u>rmk:</u> If $\mathrm{char}(k) \neq 2, 3$, there are simplified forms of Weierstrass equations; since we try to develop a purely geometric theory this is not so important for us.

<u>prop 4:</u> Let $W \subseteq \mathbb{P}^2_k$ be defined by a Weierstrass equation. Then W smooth $\Leftrightarrow$ $(W, [0:1:0])$ elliptic curve.

<u>proof:</u>

$\Leftarrow$ is clear.

$\Rightarrow$ follows from Hurwitz's genus formula and $[0:1:0] \in W(k) \neq \emptyset$.

Alternative argument for $\Rightarrow$: from Weierstrass equation, can write down

$$\omega = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3x^2 + 2a_2 x + a_4} \in \Omega^1_{K(W)/k}$$

and show (assuming W smooth) that $\omega \in H^0(W, \Omega^1_{W/k})$ and that ω is everywhere non-vanishing.

This implies $\Omega^1_{W/k} \simeq \omega \cdot \mathcal{O}$, and that W is of genus 1 $\square$

Here are some further geometric properties of Weierstrass equations which are not strictly necessary for what follows and which we state without proofs.

<u>def 5</u>

- The <u>discriminant</u> Δ of a Weierstrass equation is
$$\Delta = 2^{-4}\, \mathrm{disc}_X\left(4\left(X^3 + a_2 X^2 + a_4 X + a_6\right) + \left(a_1 X + a_3\right)^2 \right) \in k.$$
- We also define
$$c_4 = \left(a_1^2 + 4a_2\right)^2 - 24\left(2a_4 + a_1 a_3\right) \in k$$

<u>prop 6</u>

Any two Weierstrass equation for the same curve are related by a change of variables of the form
$$\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases} \quad \text{with } u \in k^\times,\ r, s, t \in k.$$

- Such a change has the effect that $\begin{cases} c_4 \rightsquigarrow u^{-4} c_4 \\ \Delta \rightsquigarrow u^{-12} \Delta \end{cases}$.

<u>prop 7</u>

If $\mathrm{char}(k) \neq 2,3$, using such a change of variable, we can put the equation in the form: $y^2 = x^3 + Ax + B$

and then $\begin{cases} \Delta = -\left(4A^3 + 27B^2\right) \\ c_4 = AB. \end{cases}$

<u>prop 8</u>

Let k be a field and W given by a Weierstrass equation.

(i) W non-singular $\iff \Delta \neq 0$.

(ii) If $\Delta = 0$, $c_4 \neq 0$, then W has a unique geometric singularity which is a node defined over k.
(note that the branches at the node may not be defined over k.)

(iii) If $\Delta = 0$, $c_4 = 0$, then W has a unique geometric singularity which is a cusp. It is defined over k unless $\mathrm{char}(k) \in \{2,3\}$ and k is imperfect.

- All this can be found in [Silverman, chap III] except the discussion of non-rational cusps on non-perfect fields which is nicely explained in [Conrad_models, p.15].

2) Weierstrass models

We now turn to models. For simplicity, we start with $S = \mathrm{Spec}(R)$, R discrete valuation ring,

In this case, it turns out that Weierstrass equations provide rather good models of elliptic curves.

<u>lemma 1</u>: Let $W/_K$ be defined by a Weierstrass equation. Then W is isomorphic to a curve defined by a Weierstrass equation with coeffs in R.

<u>proof</u>: Let $u \in K$. One checks that the substitution

$$\begin{cases} x = u^{-2} x' \\ y = u^{-3} y' \end{cases}$$

acts on coefficients by $a_i' = u^i a$

Hence by taking $v_K(u) >> 0$, we can ensure that $a_i \in R$.

$\square$

def 2 Let (E, e) be an elliptic curve over K.
A <u>planar Weierstrass model</u> (PWM) of (E,e) is a pair (W, i) with $W \subseteq \mathbb{P}^2_R$ defined by a Weierstrass equation and

$$i : W_K \xrightarrow{\sim} E$$
$$[0:1:0] \mapsto e.$$

prop 3 Let $W/_S$ be a planar Weierstrass model.
(i) W is proper flat over S.
(ii) W_σ has geometrically integral fibers, and is smooth at $[0:1:0]$. Equivalently, W_σ is smooth at $\varepsilon(\sigma)$ for $\varepsilon \in W(R)$ the unique section extending $i^{-1}(e)$.
(iii) W is normal.

proof: <u>(i)</u> - properness $\Leftarrow$ projectivity.

- flatness $/_S$ $\iff$ $\mathcal{O}_W$ torsion-free over R; can be checked on affine patches; have to check π unif of R does not divide F in the factorial ring $R[x,y]$; follows from monicity of Weierstrass eq.

<u>(ii)</u> follows from previous prop.

<u>(iii)</u> Want to apply Serre's criterion.

<u>Recall</u> (R, m) local ring. A <u>regular sequence</u> $r_1, \dots, r_k \in m$ satisfies r_i non zero-divisor in $R/(r_1, \dots, r_{i-1})$.
The <u>depth</u> of R is the maximal length of a regular sequence.

<u>Serre's criteria:</u>

[St 0310] X locally noetherian scheme. Then X $\overset{\text{reduced}}{\text{normal}}$ iff

$\overset{R0}{(R1)}$ $\forall x \in X$, $\dim(\mathcal{O}_{X,x}) \leq \overset{0}{1} \Rightarrow \mathcal{O}_{X,x}$ regular

$\overset{S1}{(S2)}$ $\forall x \in X$, $\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min\{\overset{1}{2}, \dim(\mathcal{O}_{X,x})\}$.

- W/S is flat, W_η/η is smooth, W_σ/σ is gen. smooth ($\Leftarrow W_\sigma/\sigma$ is geom. reduced)

$\Rightarrow W \setminus \underbrace{W^{sm}}_{S\text{-smooth locus}}$ is a finite set of points in W_σ

$\Rightarrow W$ is (R1).

- W is reduced, hence S1, so it remains to show that for all x codimension 2 pt, we have $\operatorname{depth}(\mathcal{O}_{W,x}) = 2$. Such an x lies in W_σ.

- Lemma [St 090R] | (R,m) noeth. local ring, $\pi \in m$. π non-zero divisor $\Rightarrow \operatorname{depth}(R/(\pi)) = \operatorname{depth}(R) - 1$.

This reduces us to show $\operatorname{depth}(\mathcal{O}_{W_\sigma,x}) = 1$.
W_σ is reduced, hence (S1), and we are done.

[Alternative argument for (S2): W is an hypersurface in $\mathbb{P}^2_S$, hence local complete intersection, hence Cohen-Macaulay, hence (S_k) $\forall k$.]

$\square$

<u>def 4</u> | Let $(E,e)/\eta$ be an elliptic curve.
An <u>abstract Weierstrass model</u> (AWM) of (E,e) is a pair (W,i) with
- W/S proper flat with W normal and geometrically integral fibers.
- $i: W_K \xrightarrow{\sim} E$
- W_σ smooth at $\varepsilon(\sigma)$ for $\varepsilon \in W(R)$ unique section extending $i^{-1}(e)$.

- By the above, any planar Weierstrass model is an abstract Weierstrass model. Conversely:

<u>thm 5</u> | Every abstract Weierstrass model is isomorphic to a planar one.

<u>proof</u> Follows the same strategy as over a field with RR replaced by Serre duality for W_σ.

• Serre duality: (all we need for this section is in Hartshorne!)

def 6 | Let k be a field, X/k projective equidimensional of dim. $n \geq 0$.
A dualizing sheaf for X is a pair (ω_X, t)
with
- ω_X coherent sheaf on X
- $t : H^n(X, \omega_X) \xrightarrow{\sim} k$ (trace map)

such that for any coherent sheaf $\mathcal{F}$ on X and $i \in \mathbb{N}$,
the natural pairing

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \to H^n(X, \omega_X) \underset{\sim}{\xrightarrow{t}} k$$

Ext cup-product

is perfect.

rmk ω_X is unique up to a unique iso if it exists.
[Hartshorne, Proposition III.7.2]

thm 7 ("concrete cases")
1) Let X/k be smooth projective of dimension n.
Then $\Omega^n_{X/k} := \bigwedge^n \Omega^1_{X/k}$ is a dualizing sheaf

2) Let $X \overset{i}{\hookrightarrow} Y$ be a closed embedding of proj. equidimensional k-schemes, of dim. n and N.
Assume
- Y is smooth (hence by 1) has a dualizing sheaf)
- X is Cohen-Macaulay.

Then X has a dualizing sheaf given by

$$\omega_X \simeq i^* \mathcal{E}xt^{N-n}(i_* \mathcal{O}_X, \omega_Y).$$

ref: [Hartshorne, Cor. III.7.12 + Prop III.7.5 and its proof + Theorem III.7.6]

- If we already knew that W was a planar Weierstrass model, we would see that $W_6 \overset{i}{\hookrightarrow} \mathbb{P}^2_6$ has a dualizing sheaf $\omega_6 := \omega_{W_6/6}$ given by $\omega_6 \simeq i^* \mathcal{E}xt^1\left(i_* \mathcal{O}_{W_6}, \mathcal{O}_{\mathbb{P}^2}(-3)\right)$.

<u>thm 8</u> X/k projective equidimensional of dim n. If X is Cohen-Macaulay, then X has a dualizing sheaf.

<u>proof:</u> Apply thm 7 to a projective embedding.

<u>cor 9</u> Any reduced proper curve$/k$ has a dualizing sheaf.

<u>proof:</u> - A proper curve over a field is projective (will review later)

- A reduced curve over a field is Cohen-Macaulay (because it is (S1)).

- So in particular W_6 has a dualizing sheaf ω_6.

<u>lemma 10</u> i) ω_6 is torsion-free.

ii) ω_6 is generically invertible.

<u>proof:</u> i) If ω_6 has torsion, it would contain a torsion subsheaf $\mathcal{C} \neq 0$. By duality:

$$\operatorname{Hom}(\mathcal{C}, \omega_6) \simeq H^1(W_6, \mathcal{C})^\vee \underset{\uparrow}{\simeq} 0$$

$\mathcal{C}$ torsion on a curve

ii) Pick a projective embedding $W_6 \overset{i}{\hookrightarrow} \mathbb{P}^N_k$

By Thm 7.2), we have $\omega_6 \simeq i^* \mathcal{E}xt^{N-1}(i_* \mathcal{O}_{W_6}, \mathcal{O}_{\mathbb{P}^N}(-N-1))$.

If $x \in W_6$ is a smooth point of W_6, i is a regular immersion in a neighborhood U of x, and a local computation then shows that $\omega_{6|U}$ is invertible (see proof of [Hartshorne, Theorem III.7.11]) $\square$

• We can now start the proof in earnest.

By hypothesis, $\mathcal{E}$ lies in $W^{om}(R)$, hence $\mathcal{J}_{\mathcal{E}}$ is invertible.

Write $\mathcal{O}(\mathcal{E}) = \mathcal{J}_{\mathcal{E}}^{-1}$, $\mathcal{O}(n\mathcal{E}) = \mathcal{O}(\mathcal{E})^{\otimes n}$.

Write $\bar{\mathcal{E}} \in W_6^{om}(k)$ for the reduction, and $\mathcal{O}(\bar{\mathcal{E}})$, $\mathcal{O}(n\bar{\mathcal{E}})$ ...

Lemma 11 | $\forall n \geq 1$, $H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}})) = 0$

proof: • By duality:

$$H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}}))^\vee \simeq \mathrm{Hom}(\mathcal{O}(n\bar{\mathcal{E}}), \omega_6)$$
$$\simeq \mathrm{Hom}(\mathcal{O}, \omega_6(-n\bar{\mathcal{E}})) \quad (\mathcal{O}(n\bar{\mathcal{E}}) \text{ invertible})$$

• Assume there is a non-zero map $\mathcal{O} \xrightarrow{f} \omega_6(-n\bar{\mathcal{E}})$. Since $\omega_6(-n\bar{\mathcal{E}})$ is torsion-free and generically invertible, f is injective and its cokernel is torsion.

Write $0 \to \mathcal{O}_{W_6} \xrightarrow{f} \omega_6(-n\bar{\mathcal{E}}) \to \mathcal{C} \to 0$

• We have $\chi(\omega_6(-n\bar{\mathcal{E}})) = \chi(\mathcal{O}_{W_6}) + \chi(\mathcal{C}) = \chi(\mathcal{C}) \geq 0$

$$\underbrace{\chi(\mathcal{O}_{W_6})}_{\substack{\| \leftarrow \text{flatness} \\ \chi(\mathcal{O}_{W_\eta}) \underset{\text{ell. curve}}{=} 0}}$$

• By duality, $\chi(\omega_6(-n\bar{\mathcal{E}})) = -\chi(\mathcal{O}(n\bar{\mathcal{E}}))$

$$= -\deg(n\bar{\mathcal{E}}) - \underbrace{\chi(\mathcal{O}_{W_6})}_{= 0}$$

(Riemann-Roch for singular curves: will review later)

$$\underset{(*)}{=} -n < 0 \quad ⚡ \qquad \square$$

• Consider the R-module $H^0(W, \mathcal{O}(n\mathcal{E}))$.

It is of finite type over a DVR, hence direct sum of a free module and a torsion module. We want to show the torsion part vanishes.

$$\begin{cases} \text{We have seen that } H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}})) = 0 \\ \text{We also have } \quad H^1(W_\eta, \mathcal{O}(n\mathcal{E}_\eta)) = 0 \end{cases}$$

By the standard result on cohomology and base change (e.g. [Mumford, II.5. Cor 2]):

the formation of $H^0(W, \mathcal{O}(n\varepsilon))$ commutes with arbitrary base change. In particular

$$\begin{cases} H^0(W,\mathcal{O}(n\varepsilon)) \otimes_R k \simeq H^0(W_6, \mathcal{O}(n\bar{\varepsilon})) \text{ of dimension } n \text{ by lemma 11 and its proof } (*). \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \\ H^0(W,\mathcal{O}(n\varepsilon)) \otimes_R K \simeq H^0(W_\eta, \mathcal{O}(n\varepsilon_\eta)) \text{ of dimension } n \text{ by the elliptic curve case.} \end{cases}$$

So $H^0(W, \mathcal{O}(n\varepsilon))$ is a free R-module of rank n.

- By the same argument as in the field case, we construct

$$W \longrightarrow \operatorname{Proj}_S\left(\operatorname{Sym}\left(H^0(W, \mathcal{O}(3\varepsilon))\right)\right) \simeq \mathbb{P}^2_S$$

with image a planar Weierstrass model W' of E.

The following lemma then finishes the proof.

<u>lemma 12</u> | Let $\phi : W \to W'$ be a morphism of abstract Weierstrass models. Then ϕ is an isomorphism.

<u>proof</u>:

- The map $\phi : W \longrightarrow W'$ is proper birational. (since ϕ_η is an iso. by hypothesis)

The map ϕ_6 is a non-constant map between <u>integral</u> curves, hence it is finite.

So ϕ is finite birational. Since W' is normal, we deduce that ϕ is an isomorphism. $\square$

<u>lemma 13</u> | Let W, W' be planar Weierstrass model. Then (iso)morphisms between W and W' as models are given by linear changes of coordinates

$$\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases} \quad \text{with } u \in R^\times,\ r, s, t \in R.$$

There is in fact at most one such isomorphism.

<u>proof</u> Follows from the intrinsic construction of the embedding $W \hookrightarrow \mathbb{P}^2_R$ described in previous proof + explicit manipulation of equations as in field case. See [Conrad-models, Cor 2.9] $\square$

Over a more general Dedekind scheme:

- Let S be a Dedekind scheme and E/η an elliptic curve. We can define AWMs for E in the same way, and, when S is affine, PWMs. (one should really allow coefficients in a line bundle on S, which would allow "PWM" beyond S affine, but I could not find a good reference).

Then any PWM is an AWM (same proof) and one can ask when the converse holds. This is not always true (ex. later!)

- Let $S = \mathrm{Spec}(R)$ affine Dedekind. The criterion for when an AWM is a PWM involves the relative dualizing sheaf $\omega_{W/S}$. We do not want to introduce too much machinery this early in the course, so let us just say that one can prove that $W/_S$ is a local complete intersection and that this implies that there exists $\omega_{W/S}$ invertible sheaf on W with fibers the dualizing sheaves $\omega_{W_s/s}$ for all $s \in S$.

thm 14 Let $\pi : W \longrightarrow S$ be an AWM. Then

(i) $\pi_* \omega_{W/S} \simeq (R^1 \pi_* \mathcal{O}_W)^\vee$ (instance of relative duality)

(ii) $R^1 \pi_* \mathcal{O}_W$ and $\pi_* \omega_{W/S}$ are locally free.

(iii) W is a PWM iff $R^1 \pi_* \mathcal{O}_W$ is free iff $\pi_* \omega_{W/S}$ is free.

cor 15: S affine + $\mathrm{Pic}(S) = 0 \Rightarrow$ Every AWM is a PWM.

rmk: The criterion in terms of $\pi_* \omega_{W/S}$ looks more complicated than the one for $R^1 \pi_* \mathcal{O}_W$ but it is more directly related to computations with diff. forms.

3) Minimal Weierstrass models (S trait again)

- Given an elliptic curve E, there are many non-isomorphic Weierstrass models, obtained by suitable transformations of x, y.

It turns out there is a "best" one, the minimal Weierstrass model $W^{\min}(E)$.

There are several ways to pin it down:

a) From a Weierstrass equation, one can extract the discriminant

$$\Delta := 2^{-4} \operatorname{disc}\left(4\left(X^3 + a_2 X^2 + a_4 X + a_6\right) + \left(a_1 X + a_3\right)^2 \right) \in R$$

Then $v(\Delta) \in \mathbb{N}$ is an invariant of the model, and $W^{\min}(E)$ is the unique model with minimal $v(\Delta)$.

$\left(v(\Delta_{\min}) = 0 \iff E \text{ has good reduct}^{\circ} \iff W^{\min}(E) \text{ smooth} \right)$

- Using the explicit form of the changes of variables, we see that $\begin{cases} \Delta' = u^{-12} \Delta \\ c_4' = u^{-4} c_4 \end{cases}$

Lemma 1 | $v(\Delta) < 12$ or $v(c_4) < 4 \implies W$ minimal.

$\Longleftarrow$ $(\operatorname{char}(k) \neq 2, 3)$ [Silverman, ex. VII.7.1]

def 2 | Let E be an elliptic curve over K with minimal Weierstrass model W. We say that E

- has good reduction if W_k is smooth $(\iff v(\Delta) = 0)$
- has multiplicative reduction if W_k has a node $\left(\iff \begin{array}{l} v(\Delta) > 0 \\ v(c_4) = 0 \end{array}\right)$
- has additive reduction if W_k has a cusp $\left(\iff \begin{array}{l} v(\Delta) > 0 \\ v(c_4) > 0 \end{array}\right)$

- For this approach see [Silverman, chap 7].

b) For any Weierstrass model $W/_S$, one can prove that $H^0(W^{sm}, \Omega^1_{W^{sm}/R})$ is a free rk 1 R-submodule of $H^0(E, \Omega^1)$. [Conrad_models, thm 2.6] Then $W^{min}(E)$ is the unique WM with maximal $H^0(W^{sm}, \Omega^1)$. [Conrad_models, cor 2.10]

c) $W^{min}(E)$ is the only Weierstrass model of E with rational singularities. [Conrad_models, Cor 8.4]

- Pt of view a) is classical and suitable for computations (incl. with a computer).

- Pt of view b) is useful to relate Weierstrass models with the main objects of the course. (see later)

- Pt of view c) is nice because of the distinguished role of rational singularities in resolution of singularities of surfaces (see later).

. The relation between a) and b) is based on the direct computation:

Lemma 3 | Let $E/_K$ be an elliptic curve with a fixed Weierstrass equation. We have $\begin{cases} \Delta \in k^\times \\ \omega = \dfrac{dx}{2y + a_1 x + a_3)} \in H^0(E, \omega_{E/K}) \end{cases}$.

Then $\Delta\, \omega^{\otimes 12} \in (H^0(E, \omega_{E/K}))^{\otimes 12}$ is independent of the choice of the equation.

i.e. "Δ is a weight 12 modular form" !

Semistable reduction for elliptic curves

- We look at the first instance where the properties of a model can be improved by passing to an extension of K.

thm 4: Let $E/_K$ be an elliptic curve. There exists a finite separable extension $L/_K$ such that E_L has good or multiplicative reduction.

- Unfortunately I could not find an (elementary) equation-free proof of this in the litterature, so I refer you to [Silverman, Prop 5.4.(c)].

Situation over a Dedekind scheme:

- Let S be a general Dedekind scheme, and $(E,e)/_\eta$ be an elliptic curve. Then E admits a <u>minimal AWM</u> $W^{min}(E)/_S$ i.e., an AWM such that for every $s \in S^{(0)}$, $W \times_S \mathrm{Spec}(\mathcal{O}_{S,s})/_{\mathrm{Spec}(\mathcal{O}_{S,s})}$ is minimal.
- Assume S affine. Then $W^{min}(E)$ is planar

  <u>iff</u> $\pi_* \omega_{W^{min}(E)/_S}$ is free

  <u>iff</u> the "<u>Weierstrass ideal</u>" $\mathfrak{a}_{E/K} \subseteq R$ is principal.

($\mathfrak{a}_{E/K}$ is defined as a product of local terms obtained from turning a fixed PWM into a minimal one at that point, see [Silverman, Chap $\overline{\text{VIII}}$]; $\mathfrak{a}_{E/K} \sim \frac{1}{12}(\Delta \text{ of any PWM of } E)$)

Relationship with general theory:

- $W^{min}(E)$ is neither the minimal regular model (it's not regular in general) nor the Néron model (it's not smooth in general.
- Let us sketch the relationship with the introduction. Write
  - $\mathcal{E}^{reg}$ for the minimal regular model of E over S.
  - $\mathcal{N}(E)$ for the Néron model of E over S.

Then:

- There is a canonical morphism $\mathcal{E}^{reg} \longrightarrow W^{min}(E)$ of models that contracts all the irreducible components of the special fibers not meeting the unique section $\varepsilon \in \mathcal{E}^{reg}(S)$ which extends $e \in E(\eta)$.
- There is a canonical isomorphism $(\mathcal{E}^{reg})^{sm} \simeq \mathcal{N}(E)^{0}$ with $(\mathcal{E}^{reg})^{sm}$ the S-smooth locus and $\mathcal{N}(E)^{0}$ is the union of the identity components of the fibers of the group scheme $\mathcal{N}(E)$.

- These results and others will show up in the next instances of:

Elliptic curves, the running example

- The other moral is that equations can fail us, already in the case of elliptic curves over global fields. Needless to say, for higher genus curves / higher dimensional abelian varieties, the situation will not improve and "abstract" algebraic geometry will be essential.

Examples:

- $\underline{S = \operatorname{Spec} \mathbb{Z}}$: the situation is especially simple: $\mathbb{Z}$ PID $\Rightarrow$ minimal PWM exists.

+ For any $E/_{\mathbb{Q}}$, there exists a unique MPW equation with $a_1, a_3 \in \{0,1\}$, $a_2 \in \{-1,0,1\}$. ("reduced min equat°")

- Here is a Weierstrass equation $/\mathbb{Z}$: $W : y^2 = x^3 - 432$

We have $\Delta_W = -2^{12} \cdot 3^9$, and $W_{\mathbb{F}_2} : y^2 = x^3$ is singular.

On the other hand, if we put $x = 4x'$, $y = 8y' - 4$, we get

$W' : y'^2 - y' = x'^3 - 7$ with $\Delta_{W'} = -3^9$.

- So $\begin{cases} W & \text{is not minimal at } 2, \\ W' & \text{is minimal at } 2 \ (\Leftarrow \text{good red at } 2). \end{cases}$

- What about at 3? By looking at the possible changes of variables of Weierstrass equation, one sees that $v_p(\Delta)$ at a prime p changes by multiples of 12. So $v_3(\Delta_W) = v_3(\Delta_{W'}) = 9 < 12$ implies that both W and W' are minimal at 3; $c_4(W) = 0 \Rightarrow$ additive reduction.

- In conclusion W' is the reduced MPWE of its generic fiber E.

- E is a very cool elliptic curve:
  - it is isomorphic to the Fermat cubic $X^3 + Y^3 + Z^3 = 0$
  - it has complex multiplication by $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$
  - it is the modular curve $X_0(27)$.

- E acquires semistable reduction after an extension of degree 12: $\mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{3})$

- $\underline{\operatorname{Pic}(S) \neq 0}$: Let us describe an example of elliptic curve with no MPWM. We need $\operatorname{Pic}(S) \neq 0$.

Given any number field K with class number > 1, a result of Silverman states that there exists an elliptic curve with no MPWM. For instance, for $K = \mathbb{Q}(\sqrt{-10})$:

$\operatorname{Pic}(\mathcal{O}_K) = \langle (5, \sqrt{-10}) \rangle \simeq \mathbb{Z}/_{2\mathbb{Z}}$.

$E : y^2 = x^3 + 125$

has Weierstrass ideal equal to $(5, \sqrt{-10})$, so

E has no MPWM.