# III Minimal regular models

**Setup** $S$ (integral) Dedekind scheme

$C/\eta$ smooth projective connected curve

By applying resolution of singularities to any proper flat model of $C$ (see next chapter) we get $X/S$ proper flat with $X$ regular connected. and $X_\eta \simeq C$.

· **Goal:** Modify $X$ to make it "minimal".

The resulting theory is very similar to the study of minimal models of smooth projective surfaces over an alg. closed field (see [Hartshorne, V]).

The two theories can be developped in parallel, and the fibered case is somewhat easier because vertical divisors play a distinguished role.

**def 1** | A fibered surface is a proper flat morphism $f : X \longrightarrow S$ with $\begin{cases} X \text{ 2-dim. noetherian} \\ S \text{ Dedekind scheme.} \end{cases}$

· Fibers of such a morphism are proper curves over general fields, which can be very singular. (For $X$ regular, they are at least l.c.i).

rmk $X$ curve. (over field)

$$\begin{array}{ccccc} X \text{ reduced} & \not\Longleftrightarrow & X \text{ l.c.i} & \Longrightarrow & \omega_{X/k} \text{ exists and is invertible} \\ \checkmark \Downarrow & (S1) & \Downarrow \text{[Liu 8.2.18]} & & \\ X \text{ has no embedded points} & \Longleftrightarrow & X \text{ Cohen-Macaulay} & \Longrightarrow & \omega_{X/k} \text{ exists.} \end{array}$$

## 1) Degree of divisors on singular curves:

· In this section, $X$ is a proper curve on a field $k$ (not necc. irred or reduced).

· Recall that if $A$ is a noetherian 1-dim ring and $f \in A$ is not a zero-divisor, then the length $\mathrm{len}_A(A/f)$ is finite, and that
$$\mathrm{len}_A(A/fg) = \mathrm{len}_A(A/f) + \mathrm{len}_A(A/g) \quad [\text{Liu, Lemma } 7.1.26].$$

**def 2** | Let $x \in X^{(0)}$ and $f \in \mathcal{O}_{X,x}$ non-zerodivisor. The multiplicity of $f$ at $x$ is
$$\mathrm{mult}_x(f) := \mathrm{len}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f) < \infty.$$

. Because of additivity, we can extend $\mathrm{mult}_x$ to the total ring of fractions.

**def 3** Let $D$ be a Cartier divisor on $X$. The *multiplicity of $D$ at $x$* is $\mathrm{mult}_x(D) := \mathrm{mult}_x(f_x) - \mathrm{mult}_x(g_x)$ for $f_x/g_x \in \mathrm{Frac}(\mathcal{O}_{X,x})$ local representative of $D$.

- The associated Weil divisor is then $\sum_{x \in X^{(0)}} \mathrm{mult}_x(D) \cdot [x] \in Z^1(X)$.

**def 4** The *(total) degree of $D$* is

$$\deg(D) := \sum_{x \in X^{(0)}} \mathrm{mult}_x(D) \cdot [\kappa(x) : k] \qquad (\triangle!\ \text{depends on } k)$$

- We have $\begin{cases} \deg(D_1 + D_2) = \deg(D_1) + \deg(D_2) \\ \deg(D) = \dim_k H^0(X, \mathcal{O}_D) \text{ if } D \text{ is effective.} \end{cases}$

- The basic fact of life, as in the case of smooth curves, is:

**thm 5** (Riemann-Roch)

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X).$$

The proof is the same as in the smooth case, by reduction to the effective case. [Liu, 7.3.17].

**cor 6** Let $f \in \kappa(X)$. Then $\deg(\mathrm{div}(f)) = 0$. (proof: $\mathcal{O}(\mathrm{div} f) \simeq \mathcal{O}$ via $f$)

**def 7** Let $\mathcal{L}$ be a line bundle on $X$. Its *(total) degree* is

$$\deg(\mathcal{L}) := \chi(\mathcal{L}) - \chi(\mathcal{O}_X).$$

**def 8** The *arithmetic genus of $X$* is $p_a(X) := 1 - \chi(\mathcal{O}_X)$.

- To go further in the study of RR, want to apply Serre duality.

**prop** (RR + duality)
$X$ proper CM curve

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \omega_{X/k}(-D)) = \deg D + 1 - p_a(X).$$

- **def** $X$ proper l.c.i curve. A Cartier divisor $K_{X/k}$ with $\mathcal{O}(K_{X/k}) \simeq \omega_{X/k}$ (which is invertible in this case) is called a *canonical divisor*.

**prop 10** [Liu, 7.3.31] $X$ proper l.c.i.
- $\deg(\omega_{X/k}) = 2(p_a - 1)$.
- $\dim_k H^0(X, \omega_{X/k}) = p_a$ if $X$ is geom. red. and geom. connected.

· Clearly, if $X$ is not irreducible, then the total degree is a rather weak invariant. However:

prop 11 | a) $X$ is projective.

b) Let $D$ be a Cartier divisor on $X$. Then

$$D \text{ ample} \iff \forall\, Y \text{ irred. comp of } X,\ \deg(D_{|Y}) > 0.$$

idea of proof [Liu, exc. 4.1.16, 7.5.3, 7.5.4]

· The order of proof is: (a) for normal curves) $\Rightarrow$ b) $\Rightarrow$ a).

- a) for normal curves is proved by patching embeddings of affine opens: If $C = \bigcup U_i$ affine open cover and $U_i \hookrightarrow Y_i$ with $Y_i/k$ projective, then the natural map $\bigcap U_i \to \prod Y_i$ extends to $C$ by normality and valuative criterion and gives a projective embedding.

- b): By Serre's criterion, enough to show that for all $\mathcal{F} \in \mathrm{Coh}(X)$ and $n >> 0$, $H^1(X, \mathcal{F} \otimes \mathcal{O}(nD)) = 0$.

· Let $\pi : X' \to X$ be the normalization. Then $\deg(\pi^*\mathcal{O}(nD)) = \deg(\mathcal{O}(nD))$

Hence (by the usual argument, since $X'$ is projective) $\pi^*\mathcal{O}(nD)$ is ample (to be precise, need to do this for each irred. comp. of $X'$)

We have a short exact sequence

$$(+)\quad 0 \to \pi_*\pi^!\mathcal{F} \to \mathcal{F} \to \underset{\substack{\uparrow \\ \text{skyscraper sheaf}}}{\mathcal{L}} \to 0 \qquad \text{with } \pi^!\mathcal{F} := \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{X'}, \mathcal{F})$$

equipped with its natural coherent $\mathcal{O}_{X'}$-module structure ($\pi$ finite)

and
$$H^1(X, (\pi_*\pi^!\mathcal{F}) \otimes \mathcal{O}(nD)) \underset{\text{projection formula}}{\simeq} H^1(X, \pi_*(\pi^!\mathcal{F} \otimes \pi^*\mathcal{O}(nD)))$$
$$\underset{\pi \text{ finite}}{\simeq} H^1(X', \pi^!\mathcal{F} \otimes \pi^*\mathcal{O}(nD))$$
$$\underset{\pi^*\mathcal{O}(nD) \text{ ample}}{\simeq} 0 \quad \text{for } n >> 0$$

· The result then follows from the LES of (+).

- a): it is then enough to construct an effective Cartier divisor which meets every irreducible component of $X$. But, given any locally noetherian scheme $X$ and any non-associated point $x$, there is an effective Cartier divisor on $X$ with support containing $x$. □

· Application to fibered surfaces:

thm 12 (Lichtenbaum [Liu, 8.3.16])
Let $f: X \to S$ be a regular fibered surface (i.e $X$ regular)
Then $f$ is locally projective.

proof: Let $\pi: Y \to T$ be a proper morphism of noetherian schemes, and $\mathcal{L}$ be a line on $Y$. We need the following classical facts:
- If $\mathcal{L}$ is ample, then $\pi$ is projective (in the sense of EGA)
- If for $t \in T$, $\mathcal{L}_t$ on $X_t$ is ample, then $\exists U \ni t$ open with $\mathcal{L}|_{\pi^{-1}(U)}$ ample.

· We can assume $X$ connected $\Rightarrow X_\eta$ connected.

· Let $x \in X_\eta^{(0)}$ be a closed point. Then $D_0 = \overline{\{x\}}$ is a Weil = Cartier division on $X$. Since $\mathcal{O}(D_0)_\eta$ is ample by $X$ connected curve, there exists $U \subseteq S$ non-empty open with $(s \in U \Rightarrow (D_0)_s$ ample$)$

Let $S \setminus U = \{s_1, \cdots, s_n\}$.

lemma 13 There exists an effective division $D_i$ which meets all irreducible components of $X_{s_i}$.

proof: It is enough to construct, for any $\sigma \in S^{(0)}$ and $x \in X_\sigma^{(0)}$, an effective Weil division $D$ which contains $x$. Let $m_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $p_1, \cdots, p_n$ the prime ideals of $\mathcal{O}_{X,x}$ corresponding to the irred. components of $X_\sigma$ containing $x$. Then $m_x \not\subset \bigcup p_i$ (Prime avoidance lemma); indeed by induction $\exists f_i \in m_x \setminus \bigcup_{j \neq i} p_i$, and then $f_1 \cdots f_{n-1} + f_n \notin \bigcup p_i$ using induction and $p_n$ prime.
Pick $f \in m_x \setminus (\bigcup p_i)$, $U$ open with $g \in \mathcal{O}_X(U)$ representing $x$. Then $x \in V(g)$, so any irreducible component of $\overline{V(g)}$ passing through $x$ works. □

. Then $D := D_0 + D_1 + \cdots + D_n$ is a Cartier division such that $\forall s \in S$, $\mathcal{O}(D)_s$ is ample. □

rmk: for $f: X \to S$ smooth or $S$ "nice" (e.g quasi-excellent), $f$ has finitely many singular fibers and we can dispense with $D_0$.

rmk: - Regular proper surfaces over a field are projective.
See [Liu, Remark 9.3.5] for a proof; we will sketch it later.
- There exist normal proper non-projective surfaces [Schröer].
However normal proper surfaces over a finite field are projective! [Artin, 2.11]

## 2) Intersection theory on a regular fibered surface

- Intersection theory in general: try to define intersection pairing $CH^i(X) \times CH^j(X) \longrightarrow CH^{i+j}(X)$ on cycle groups up to rat. equivalence.
- On a surface, only interesting case is 2 divisors: $CH^1 \times CH^1 \to CH^2 = CH_0$
- <u>Pb 1</u> : $CH^2(X)$ is hopelessly complicated and not so interesting for us.

  Sol: only keep track of intersection degrees.

- <u>Pb 2</u> : without some form of properness, degrees of 0-cycles are not invariant under rational equivalence.

  Sol: only allow intersections with at least one divisor proper.

- Write $\mathrm{Div}(X)$ for the group of all Cartier divisors on $X$

  $\mathrm{Div}_h(X)$ for the subgroup of <u>horizontal</u> divisors ($f_{|D}$ surjective for $D$ prime, $\Updownarrow$ finite)

  for $s \in S^{(0)}$, $\mathrm{Div}_s(X)$ for the subgroup of divisors with support in $X_s$.

  $\mathrm{Div}_v(X) = \sum_{s \in S} \mathrm{Div}_s(X)$ for the subgroup of <u>vertical</u> divisors.

<u>def 1:</u> Let $D \in \mathrm{Div}(X)$, $E \in \mathrm{Div}_v(X)$.

Write $E = \sum n_\Gamma \cdot [\Gamma]$ with $\Gamma$ running through the irred. components of closed fibers.

Put $i(D,E) = \sum n_\Gamma \deg_{k(s)}\left(\mathcal{O}(D)_{|\Gamma}\right) \in \mathbb{Z}$.

(this makes sense because $\Gamma/_{k(s)}$ is a proper curve).

<u>thm 2</u>
(i) $i : \mathrm{Div}(X) \times \mathrm{Div}_v(X)$ is a bilinear form.

(ii) $i : \mathrm{Div}_v(X) \times \mathrm{Div}_v(X)$ is symmetric.

(iii) If $D \sim D'$ (i.e $D' - D = \mathrm{div}(F)$ for $F \in K(X)^\times$)

we have $i(D,E) = i(D',E)$.

(iv) If $D, E$ are effective, with no common component, we have:

$$i(D,E) = \sum_{x \in |D| \cap |E|} [k(x) : k(f(x))] \cdot \mathrm{len}_{\mathcal{O}_{X,x}}\left( \mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x \right) \geq 0$$

proof: (i): • Linearity in $E$ is by construction.
• Linearity in $D$ follows from additivity of degree.

(iii): $D \sim D' \Rightarrow \mathcal{O}_X(D) \simeq \mathcal{O}_X(D') \Rightarrow i(D,E) = i(D',E)$.

(iv): • First, the condition on supports imply that in a neighborhood of $x \in |D| \cap |E|$, the point $x$ is the only pt of intersection of the supports. This implies

$$m_x \subseteq \sqrt{\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x}$$

hence $\mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x$ is an artinian ring $\Rightarrow$ length $< \infty$.

• $D$ effective Cartier divisor in $X \Rightarrow D \hookrightarrow X$ l.c.i. Since $X$ is regular, we deduce that $D$ is Cohen-Macaulay, hence has no embedded points [Liu, section 8.2].
• By this together with the condition on supports, this implies: $\exists D_{|E}$ effective on $E$, with $\mathcal{O}_X(D)_{|E} \underset{(+)}{\simeq} \mathcal{O}_E(D_{|E})$ [Liu, lemma 7.1.29].

$$\Longrightarrow i(D,E) = \deg(\mathcal{O}_E(D_{|E})) = \sum_{x \in |E|} \mathrm{mult}_x(D_{|E}) \cdot [k(x):k(s)].$$

We have
$$\mathrm{mult}_x(D_{|E}) = \mathrm{len}\left( \mathcal{O}_{E,x} / \mathcal{O}_E(-D_{|E})_x \right)$$
$$\uparrow \wr \ (+)$$
$$\mathrm{len}\left( \mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x \right) \checkmark$$

(ii): Can reduce to $D = \Gamma_i$, $E = \Gamma_j$ for $\Gamma_i, \Gamma_j$ 2 components of a fiber $X_s$. Then it follows from symmetry in (iv) for $i \neq j$, and it is obvious for $i = j$.

$\square$

notat° | $D \cdot E := i(D,E)$
$E^2 := i(E,E)$

rmk: Compared with case of smooth proj surface [Hartshorne, V §1]:
- no Bertini thm to always reduce to transversality
- still a moving lemma [Liu, 9.1.10-11]
but not necessary for theory.

thm 3 (Hodge index theorem for fibered surfaces)

Let $f : X \to S$ be a regular fibered surface and $s \in S$ closed pt.

(i) For any $E \in \mathrm{Div}_s(X)$, $i(E, X_s) = 0$.

(ii) Let $\Gamma_i$ be the irreductible components of $X_s$, with multiplicities $d_i$. Then

$$\Gamma_i^2 = -\frac{1}{d_i} \sum_{j \neq i} d_j \Gamma_i \cdot \Gamma_j \leq 0$$

(iii) The bilinear form $i$ is negative semi-definite ($\otimes \mathbb{R}$)

If $X_s$ is connected, then the kernel of $i_{\mathbb{R}}$ is precisely $\mathbb{R} \cdot X_s$.

proof: (i): For $D \in \mathrm{Div}(X)$ and $E \in \mathrm{Div}(X_s)$ it is easy to see that $i(D, E) = i(D_{|X_{(s)}}, E_{|X_{(s)}})$ with $X_{(s)} \longrightarrow \mathrm{Spec}(\mathcal{O}_{S,s})$.

$\mathcal{O}_{S,s}$ is a DVR $\Rightarrow$ $s$ is a principal divisor on $\mathrm{Spec}(\mathcal{O}_{S,s})$

$\Rightarrow X_s$ " " $X_{(s)}$

By pt (iii) in previous thm, we get $i(X_s, -) = 0$

(ii): This follows from $X_s = \sum d_i \Gamma_i$, bilinearity and $X_s^2 = 0$ by (i).

(iii): Put $a_{ij} = \Gamma_i \cdot \Gamma_j$ and $b_{ij} = d_i d_j a_{ij}$. We have

$$\begin{cases} b_{ij} \geq 0 \text{ if } i \neq j, \text{ and } b_{ij} > 0 \Leftrightarrow \Gamma_i \cap \Gamma_j \neq \emptyset \text{ (Thm 2 (iv))} \\ \sum_j b_{ij} = X_s \cdot d_i \Gamma_i = 0 \text{ for all } i \text{ by (i), and } \sum_i b_{ij} = 0 \text{ by symmetry.} \end{cases}$$

· Let $v = \sum x_i \Gamma_i \in \mathrm{Div}_s(X)_{\mathbb{R}}$, $y_i := \frac{x_i}{d_i}$.

$$i(v,v) = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j = \sum_{1 \leq i,j \leq n} b_{ij} y_i y_j = 2 \sum_{1 \leq i < j \leq n} b_{ij} y_i y_j + \sum_{1 \leq k \leq n} b_{kk} y_k^2$$

$$\left(\overset{\text{using}}{\sum_i b_{ij} = 0}\right) \searrow = -\sum_{1 \leq i < j \leq n} b_{ij} (y_i - y_j)^2 \leq 0$$ with equality iff $(\Gamma_i \cap \Gamma_j \neq \emptyset \Rightarrow y_i = y_j)$

which implies the result. $\square$

**thm 4:** Let $f : X \longrightarrow Y$ be a birational morphism of fibered surfaces with $X$ regular and $Y$ normal. Let $y \in Y$ with $\dim X_y = 1$. Write $\Gamma_1, \dots, \Gamma_r$ for the irreducible components of $X_y$. The intersection form on $\bigoplus \mathbb{R} \cdot \Gamma_i$ is negative definite.

**proof:** • Let $C$ be any effective Cartier divisor on $Y$ with support containing $y$. Write $\underbrace{f^* C}_{\Gamma} = \tilde{C} + D$ with • $\tilde{C}$ effective with no common comp. with $X_y$
(exists by $X, Y$ locally integral, $f$ dominant [Liu, 7.1.33-34])
• $D = \sum d_i \Gamma_i$, $d_i \geq 0$

Note that, to do this decomposition, we use "Weil = Cartier" on $X$.

**lemma:** $\forall i$, $d_i > 0$ and $D \cdot \Gamma_i \leq 0$.
Moreover $\exists i_0$, $D \cdot \Gamma_{i_0} < 0$

**proof:** • Locally around $y$, we have $\mathcal{O}_Y(C) = a^{-1} \mathcal{O}_Y$ for some $a \in \mathcal{O}_Y$.
Hence $\mathcal{O}_X(f^* C) = (f \circ a)^{-1} \mathcal{O}_X$, and $d_i = v_{\Gamma_i}(f \circ a) > 0$ since $a \in m_y \mathcal{O}_{Y,y}$

• Since $\mathcal{O}_X(f^* C)$ is free around $f^{-1}(y)$, we get $f^* C \cdot \Gamma_i = \deg(\mathcal{O}(f^* C)_{|\Gamma_i}) = 0$.
Since $\tilde{C}$ and $\Gamma_i$ have no common components, we have $\tilde{C} \cdot \Gamma_i \geq 0$.
Hence $D \cdot \Gamma_i = (f^* C - \tilde{C}) \cdot \Gamma_i \leq 0$.

• $|\tilde{C}| \cap X_y \neq \emptyset$ for topological reasons: $f(\tilde{C})$ is a closed subset of $C$ containing $C \setminus \{y\}$.

Let $\Gamma_{i_0}$ with $|\tilde{C}| \cap \Gamma_{i_0} \neq \emptyset$. Then $\tilde{C} \cdot \Gamma_{i_0} > 0$, so $D \cdot \Gamma_{i_0} < 0$ $\square$

• Let $x_1, \dots, x_n \in \mathbb{R}$. Put $b_{ij} = d_i d_j \Gamma_i \cdot \Gamma_j$, $y_i = \frac{x_i}{d_i}$.

$$i\left(\sum x_i \Gamma_i, \sum x_i \Gamma_i\right) = \sum_{i,j} b_{ij} y_i y_j = \sum_i \Big(\underbrace{\sum_j b_{ij}}_{= D \cdot d_i \Gamma_i \leq 0}\Big) y_i^2 - \sum_{i<j} b_{ij}(y_i - y_j)^2 \leq 0$$

• If there is equality, then
$$\begin{cases} \sum_i \left(\sum_j b_{ij}\right) y_i^2 = 0 & (1) \\ \sum_{i<j} b_{ij}(y_i - y_j)^2 = 0 & (2) \end{cases}$$

We then see that $y_{i_0} = 0$, and then using (2) and connectedness of $X_y$ (which holds by normality of $Y$ and Zariski's main thm) we get $y_i = 0$ for all $i$. $\square$

rmk: $X_y \subseteq X_{f(y)}$ so Thm 3 implies the intersection form is negative semidefinite ; the new content is the "definite" part, and the argument can be reformulated as showing that $X_y$ is not a connected component of $X_{f(y)}$.

· In the proof, we have established the following.

lemma 5 | $f : X \to Y$ dominant morphism of regular fibered surfaces
Let $\begin{cases} E \text{ be a divisor on } X \text{ with } f(\mathrm{Supp}\, E) \text{ finite} \\ D \text{ be any divisor on } Y \end{cases}$
Then $E \cdot f^* D = 0$.

· There is an important generalization of this.

def 6 | $X, Y$ noetherian integral schemes
$f : X \longrightarrow Y$ proper, $Z \subseteq X$ prime Weil divisor.

$$f_* Z := \begin{cases} [\kappa(Z) : \kappa(f(Z))] \cdot f(Z), & [\kappa(Z) : \kappa(f(Z))] < \infty \\ 0, & \text{otherwise} \end{cases}$$

This extends by linearity to a map $\begin{pmatrix} \text{Weil div} \\ \text{on } X \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Weil div} \\ \text{on } Y \end{pmatrix}$

thm 7 | $f : X \to Y$ dominant morphism of regular fibered surfaces
$C$ (resp. $D$) divisor on $X$ (resp. $Y$).

a) $f_* f^* D = [\kappa(X) : \kappa(Y)] \cdot D$

b) Assume either $C$ or $D$ is vertical. Then

$C \cdot f^* D = f_* C \cdot D$ (projection formula)

c) If $D'$ is a vertical divisor on $Y$, then $f^* D'$ is vertical and

$f^* D \cdot f^* D' = [\kappa(X) : \kappa(Y)]\, C \cdot D$.

[Liu, Thm 9.2.12]

## 2) Birational maps, blow-ups and contractions

- The notion of minimal fibered surface is defined in terms of arbitrary birational maps, but it turns out that birational maps between regular surfaces have a very simple structure.

**thm 1** Let $f : X \to Y$ be a (proper) birational morphism between regular fibered surfaces. Then $f$ can be written as a composition of blow-ups at closed points. The number of such blow-ups is equal to the number of irreducible components of the exceptional locus of $f$.

**proof:** Assume $f$ is not an isomorphism. We first show that there exists $y \in Y^{(0)}$ with $\dim(X_y) \geq 1$. Assume the opposite: then $f$ is quasi-finite and proper, hence finite. Since it is moreover birational, $f$ is an iso by $Y$ normal $\lightning$

- Let $y \in Y^{(0)}$ be such a point. Because $\begin{cases} X \text{ is locally integral} \\ \dim(X) = 2 \end{cases}$, $\dim(X_y) = 1$.
- By normality of $Y$ and $f$ proper birational, we have $\mathcal{O}_Y \xrightarrow{\sim} f_* \mathcal{O}_X$.

In this situation, Zariski's connexity theorem tells us that the fibers of $f$ are (geometrically) connected.
In particular, $X_y$ has no isolated points and is of pure dimension 1.

- Because $X_y$ is of pure codimension 1 in $X$ regular $(\Rightarrow$ loc. factorial$)$, $\mathcal{I}_{X_y}$ is invertible.
- Let $\tilde{Y} = Bl_y Y$ be the blow-up of $Y$ at the closed point $y$.

Blow-ups are characterized by a universal property [Liu, Corollary 8.1.16].

**prop 2** Let $T$ be a noetherian scheme and $\mathcal{I}$ a coherent sheaf of ideals. Then $\pi : \tilde{T} = Bl_{V(\mathcal{I})} T \longrightarrow T$ is characterized by the following property: for any $p : W \longrightarrow T$ such that $(\theta^{-1}\mathcal{I})\mathcal{O}_W$ is invertible. $\exists !\, q : W \to \tilde{T}$ with
$$\begin{tikzcd} W \arrow[r, "q"] \arrow[dr, "p"'] \arrow[dr, phantom, shift left=3, "\equiv"] & \tilde{T} \arrow[d, "\pi"] \\ & T \end{tikzcd}$$.

- By the above, $f$ factors as $f : X \to \tilde{Y} \to Y$ and we win $\square$

**rmk:** - In particular, $f$ is (locally) projective, which we already knew by Thm 1.12). (Recall that if $g \circ h$ is locally projective and $g$ is locally projective then $h$ is locally projective)

# Recollection on blow-ups and sketch of universal property

**def 3** | $X$ noetherian, $\mathcal{J}$ coherent sheaf of ideals.
Then $\bigoplus_{n\geq 0} \mathcal{J}^n$ is a graded $\mathcal{O}_X$-algebra.
We put $Bl_{\mathcal{J}} X := \mathrm{Proj}_X\left(\bigoplus_{n\geq 0} \mathcal{J}^n\right) \longrightarrow X$
$Bl_{\mathcal{J}} X$ has a natural invertible sheaf $\mathcal{O}(1)$ which corresponds to the graded ideal sheaf $\bigoplus_{n\geq 0} \mathcal{J}^{n+1}$.

- We have that $(\pi^{-1}\mathcal{J})\mathcal{O}_{Bl_{\mathcal{J}}X} \simeq \mathcal{O}(1)$ hence is invertible.

- Let us sketch the proof of the universal property of the blow-up. Let $p : W \to X$ such that $(p^{-1}\mathcal{J})\mathcal{O}_W$ is invertible. Then

$$
\begin{aligned}
Bl_{(p^{-1}\mathcal{J})\mathcal{O}_W} W &= \mathrm{Proj}_{\mathcal{O}_W}\left(\bigoplus_{n\geq 0}((p^{-1}\mathcal{J})\mathcal{O}_W)^n\right) \\
&\simeq \mathrm{Proj}_{\mathcal{O}_W}\left(\bigoplus_{n\geq 0} \mathcal{O}_W^n\right) \quad \left(\text{multiply by } (f^{-n})_{n\geq 0} \text{ with } f \text{ local generator of the invertible ideal}\right) \\
&= W.
\end{aligned}
$$

So it is enough to show that, without assuming $(p^{-1}\mathcal{J})\mathcal{O}_W$ invertible, there is a unique morphism which fits in a commutative diagram.

$$
\begin{tikzcd}
Bl_{(p^{-1}\mathcal{J})\mathcal{O}_W} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\equiv"] & Bl_{\mathcal{J}} X \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}
$$

The construction of the map simply comes from the functoriality of the Proj construction for the morphism $\bigoplus_{n\geq 0} \mathcal{J}^n \longrightarrow \bigoplus_{n\geq 0}((p^{-1}\mathcal{J})\mathcal{O}_W)^n$ induced by pullback of functions.

The uniqueness is not essential for us and will be omitted.

**Cor:** | $X, Y$ regular fibered surfaces, $f : X \dashrightarrow Y$ birational map. There exists $Z$ regular fibered and a commutative diagram
$$
\begin{tikzcd}
& Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "f"'] & & Y
\end{tikzcd}
$$
with $p, q$ composition of blow-ups at closed points.

**proof:** Put $\Gamma$ = closure of the graph of a representative of $f$ in $X \times Y$. Then $\Gamma_\eta \xrightarrow{\sim} X_\eta$ by normality of $X_\eta$, hence $\Gamma$ has a resolution of singularities $Z \to \Gamma$ (see later chapter). We then apply the previous theorem to
$$
\begin{tikzcd}
Z \arrow[r] \arrow[rr, bend left] \arrow[drr, bend right] & \Gamma \arrow[r] \arrow[dr] & X \\
& & Y
\end{tikzcd}
$$
$\square$

· This justifies a closer look at the intersection theory on blow-ups.

<u>def 4</u> Let $\pi : \tilde{X} \to X$ be a projective birational morphism of regular integral schemes. Let $D$ be an effective Cartier divisor on $X$. Let $F = \{x \in X \mid \dim \tilde{X}_x \geq 1\}$. The <u>strict transform</u> $\tilde{D}$ of $D$ is the scheme-theoretic closure of $\pi^{-1}(D \setminus F)$ in $\tilde{X}$.

<u>prop 5</u> Let $X$ be a regular fibered surface, $\pi : \tilde{X} \to X$ the blow-up at a closed point $x$. Let $E \subseteq \tilde{X}$ be the exceptional divisor. Then

$$\pi^* D = \tilde{D} + \mu_x(D)\cdot E \quad \text{with } \mu_x(D) = \max\{ n \geq 0 \mid f \in m^n \text{ for } f \text{ local eq. of } D \text{ at } x\}$$

<u>proof</u>: · $\pi^* D - \tilde{D}$ is supported on $E \Rightarrow \exists\, n \in \mathbb{Z},\ \pi^* D = \tilde{D} + nE$.
We have $n = \mathrm{mult}_\xi(F)$ for $\xi$ gen point of $E$, $F$ local eq. of $\pi^* D$.

· Let $U = \operatorname{Spec} A$ be an affine open neighbourhood of $x$ in $X$ such that $m_x = (a, b)$ and $\mathcal{O}_U(-D_U) = (f)$.

Then $\pi^{-1}(U) = \operatorname{Proj}\left( A[s,t]/_{sb - ta}\right)$ with $w = \frac{v}{u}$, $w' = \frac{u}{v}$

$$= \underbrace{\operatorname{Spec}\left(A\left[\tfrac{t}{s}\right]\right)}_{W} \cup \underbrace{\operatorname{Spec}\left(A\left[\tfrac{s}{t}\right]\right)}_{W'}$$

· $\mathcal{O}_W(-E_{|W}) = (a)$

Write $f = P(a,b) + Q$ with $P(a,b) \neq 0$ homogeneous of degree $\mu_x(D)$ and $Q \in m_x^{\mu_x(D)+1}$.

Then $F = \pi^* f = a^{\mu}\cdot P(1, \frac{t}{s}) + a^{\mu+1} g$, $g \in \mathcal{O}_{\tilde{X}}(W)$ on the blow-up

$\mathcal{O}_{\tilde{X}}(W)/_{(a)} \simeq k(x)\left[\frac{t}{s}\right] \Rightarrow P(1, w) \notin u\,\mathcal{O}_{\tilde{X}}(W) = \mathcal{O}_{\tilde{X}}(-E)(W)$

$\Rightarrow \mathrm{mult}_\xi(\pi^* D) = \mathrm{mult}_\xi(f) = \mu$ with $\xi$ generic point of $W$. " " $E$

Moreover, $\tilde{D} \cap W = (P(1, \frac{t}{s}) + ag)$. $\square$

<u>prop 6</u> a) $E$ is isomorphic to $\mathbb{P}^1_{k(x)}$.

b) $E^2 = -[k(x) : k(s)]$.

<u>proof</u>: · <u>a)</u> is well known. <u>b)</u>: Pick $D$ on $X$ with a regular point at $x$. Then $\mu_x(D) = 1$

$$0 = D\cdot \underbrace{f_* E}_{pt} \underset{\text{proj formula}}{=} f^* D \cdot E = (\tilde{D}\cdot E + E)\cdot E$$

But $n = 1 \Rightarrow P(1, \frac{t}{s})$ linear form $\Rightarrow \tilde{D}, E$ intersect transversally at a point of residue field $k(x)$

$\square$

<u>Alternatively</u>: We have $E^2 = \deg_{k(s)}\left(\mathcal{O}_X(E)_{|E}\right)$ by definition.

- For the effective Cartier divisor $E$ with ideal sheaf $\mathcal{J} = \mathcal{O}_X(-E)$, we have
$$\mathcal{O}_X(E)_{|E} \simeq \mathcal{J}^\vee \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_E \simeq i^*\left(\mathcal{J}/\mathcal{J}^2\right)^\vee \quad (\text{with } i : E \hookrightarrow X).$$
- By a local computation on the blow-up, we have $i^*\left(\mathcal{J}/\mathcal{J}^2\right) \simeq \mathcal{O}_{\mathbb{P}^1_k}(1)$ [Liu, Thm 1.19 (c)] so $\deg_{k(s)}\left(\mathcal{O}_X(E)_{|E}\right) = \deg_{k(s)}\left(\mathcal{O}_{\mathbb{P}^1_{k(x)}}(-1)\right) = -[k(x):k(s)]$.

**def 7**

- $X \to S$ normal fibered surface, $\mathcal{E}$ set of integral vertical curves in $X$. A <u>contraction</u> of $\mathcal{E}$ is a projective birational morphism $f : X \to Y$ to a normal fibered surface $/S$ such that for any $E$ integral vertical curve on $X$, we have
$$E \in \mathcal{E} \iff f(E) \text{ point}.$$
- Assume $X$ regular. An <u>exceptional curve</u> is an integral vertical curve $E$ such that there exists a contraction of $E$ with <u>regular</u> target.

**prop 8** Contractions are unique up to unique isomorphism when they exist.

<u>proof</u>: Let $\pi : X \to Y$ be a contraction. Then $Y$ is uniquely determined:
- as a set : $Y = \left(X \setminus \bigcup_{E \in \mathcal{E}} E\right) \cup \left(1 \text{ pt for each conn. comp of } \bigcup E\right)$.
- as a topological space : $\pi$ surjective and proper $\Rightarrow \forall Z \subseteq Y$, $Z$ closed $\Leftrightarrow \pi^{-1}(Z)$ closed.
- as a locally ringed space : $\pi$ proper birational $+$ $Y$ normal $\Rightarrow \mathcal{O}_Y \simeq \pi_*\mathcal{O}_X$. □

**prop 9** Let $X \to S$ normal fibered surface, $\mathcal{E}$ set of integral vertical curves.

(i) The contraction of $\mathcal{E}$ exists.

$\Updownarrow$

(ii) There exists a Cartier divisor $D$ on $X$ such that
- $\deg(D_\eta) > 0$,
- $\mathcal{O}_X(D)$ is generated by global sections, and
- $\forall E$ int. vert., $\mathcal{O}_X(D)_{|E}$ trivial $\iff E \in \mathcal{E}$.

<u>proof</u>: We only treat $\Uparrow$.
- $X_\eta$ is integral proj, so by assumption $D_\eta$ is ample. By replacing by a multiple, can assume $D_\eta$ very ample.
- Assume $S = \mathrm{Spec}(A)$ affine for simplicity. The general case follows using uniqueness.

Because $\mathcal{L} \simeq \mathcal{O}_X(D)$ is generated by global sections we get

$f : X \longrightarrow \mathbb{P}^N_A$. Put $Z = f(X)$ with reduced scheme structure.

- $f : X \to Z$ is projective and birational, it factors through the normalization $\tilde{Z} \to Z$ $\left(\begin{array}{l}\text{which is also finite because} \\ \tilde{Z} = \mathrm{Spec}_Z(\underbrace{f_* \mathcal{O}_X}_{Z\text{ coherent}})\end{array}\right)$ as $X \xrightarrow{g} \tilde{Z} \xrightarrow{\pi} Z$
- We claim that $g$ is the desired contraction.

- Because $\pi$ is finite, a curve is contracted by $f$ iff it is contracted by $g$, so we have to check that $f$ contracts the right curves.
- More generally, let us show that for $Z \subseteq X$ closed and connected,

$f(Z)$ point $\iff$ $\mathcal{L}_{|Z} \simeq \mathcal{O}_Z$.

Write $s_0, \cdots, s_N$ for sections generating $\mathcal{L}$ and $t_0, \cdots, t_N$ for the corresponding coordinates on $\mathbb{P}^N$.

Recall that $X = \bigcup X_{s_i}$ with $X_{s_i} = \{ x \in X \mid \mathcal{L}_x = s_{i,x} \mathcal{O}_{X,x} \}$

- Suppose that $f(Z) = \{y\}$. Then $y \in D(t_i)$ for some $i$, hence

$Z \subseteq f^{-1}(D(t_i)) = X_{s_i}$ and $\mathcal{L}_{|Z} = s_i \mathcal{O}_Z \simeq \mathcal{O}_Z$.

- Conversely, suppose that $\mathcal{L}_{|Z} = e\,\mathcal{O}_Z$. Write $B = H^0(Z, \mathcal{O}_Z)$ and $Y = \mathrm{Spec}\, B$. Then $Y$ is finite over $\mathrm{Spec}(k)$.

Choose $b_i \in B$ such that $s_i = e\, b_i$. Because the $s_i$ generate $\mathcal{L}$, we have that the $b_i$ generate $\mathcal{O}_Y$. Let $g : Y \to \mathbb{P}^N_k$ the associated morphism. We have

$$\begin{tikzcd} Z \arrow[r] \arrow[rr, bend right, "f_{|Z}"'] & Y \arrow[r, "g"] & \mathbb{P}^N \end{tikzcd} \quad \equiv$$

, which by finiteness of $Y$ and connectedness of $Z$ implies $f(Z) = \{pt\}$. $\square$

- By Thm 1, if $E \subset X$ is an exceptional curve in a regular surface $X$, with contraction $\pi : X \to Y$, then $\pi$ is the blow-up of $Y$ at the point $\pi(E)$.

Hence $\begin{cases} E \simeq \mathbb{P}^1_{k'}, \text{ for some } k' \text{ finite extension of } k(o). \\ E^2 = -[k' ; k(o)]. \end{cases}$

We are interested in the converse.

**thm 10** Let $X \to S$ be a regular fibered surface and let $E \subset X_\sigma$ be a vertical prime divisor with
$\cdot\ E \simeq \mathbb{P}^1_{k'}$, $(k' = H^0(E, \mathcal{O}_E))$

$\cdot\ E^2 < 0$ ; write $d = -\dfrac{E^2}{[k' ; k(\sigma)]} = \deg_{k'}\left(\mathcal{O}_X(-E)_{|E}\right)$.

(i) There exists a contraction $\pi : X \to Y$.

(ii) Write $y = \pi(E) \in Y$. Then $k(y) = k'$ and

$$\dim_{k(y)} T_{Y,y} = d + 1 .$$

(iii) (Castelnuovo's criterion)

$E$ is an exceptional curve iff $d = 1$.

rmk When $k' = k(\sigma)$, $E$ exceptional $\Leftrightarrow E^2 = -1$, hence the classical terminology of "$(-1)$-curves".

proof:

(i): We can assume $S$ affine. Since $X$ is projective over $S$ (Theorem 1.12)) we can choose an ample Cartier divisor $H$ on $X$. Replacing $H$ by a multiple, we can assume that : $\forall\, n \geq 1,\ H^1(X, \mathcal{O}(nH)) = 0$.

$\cdot$ For any $\Gamma$ vertical prime divisor, and in particular for $E$, we have $\mathcal{O}_X(H)_{|\Gamma}$ ample, hence $H \cdot \Gamma = \deg(\mathcal{O}(H)_{|\Gamma}) > 0$. Let $m = -E^2 > 0$, $r = H \cdot E > 0$ and put $D = mH + rE$.

lemma: 1) $\deg(D_\eta) > 0$.

2) For all $\Gamma \neq E$, $D \cdot \Gamma > 0$, hence $\mathcal{O}(D)_{|\Gamma} \simeq \mathcal{O}_\Gamma$.

3) $\mathcal{O}(D)_{|E} \simeq \mathcal{O}_E$.

4) $\mathcal{O}(D)$ is generated by global sections.

proof 1) $D_\eta = mH_\eta$ ample ✓

2) $D \cdot \Gamma = mH\cdot\Gamma + rE\cdot\Gamma \underset{\Gamma \neq E}{\geq} mH\cdot\Gamma \underset{H \text{ ample}}{>} 0$

3) $D \cdot E = mH\cdot E + rE^2 = mr - rm = 0$, hence $\mathcal{O}(D)_{|E}$ is a degree 0 line bundle on $E \simeq \mathbb{P}^1_{k'}$ $\Rightarrow \mathcal{O}(D)_{|E} \simeq \mathcal{O}_E$.

4) Let $0 \leq i \leq r-1$. Then

$(mH + (i+1)E) \cdot E = m(r - (i+1)) \geq 0$

Hence $H^1(E, \mathcal{O}(mH + (i-1)E)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(m(r-(i+1)))) = 0$.

- Using this and an induction starting from $H^1(X, \mathcal{O}(mH)) = 0$ we get

$H^1(X, \mathcal{O}(mH + (k-1)E)) = 0$, hence

$H^0(X, \mathcal{O}(D)) \twoheadrightarrow H^0(E, \mathcal{O}(D)_{|E}) \simeq H^0(\mathbb{P}^1, \mathcal{O}) = k'$.

This implies that $\mathcal{O}(D)$ is generated by global sections at points of $E$ (by Nakayama)

But $\mathcal{O}(D)_{|X \setminus E} \simeq \mathcal{O}(mH)_{|X \setminus E}$ is generated by global sections since $H$ is ample. $\square$

- By the lemma, we can apply Proposition 5 and deduce the existence of the contraction $\pi$.

<u>(ii):</u> - $\pi$ proper birational, $Y$ normal $\Rightarrow \mathcal{O}_Y \xrightarrow{\sim} \pi_* \mathcal{O}_X$, so $\hat{\mathcal{O}}_{Y,m} \simeq (\pi_* \mathcal{O}_X)^\wedge_m$.

By the theorem on formal functions: $(\pi_* \mathcal{O}_X)^\wedge_m \simeq \varprojlim_k H^0(X, \mathcal{O}_X/m^k\mathcal{O}_X)$.

Let $\mathcal{J}$ be the ideal sheaf of $E$ on $X$. We have $m\mathcal{O}_X \subseteq \mathcal{J}$, and because $\mathcal{J}$ and $m\mathcal{O}_X$ define the same reduced scheme $E$, we have $\sqrt{m\mathcal{O}_X} = \mathcal{J}$. This implies

$$\varprojlim_k H^0(X, \mathcal{O}_X/m^k\mathcal{O}_X) = \varprojlim_k \overbrace{H^0(X, \mathcal{O}_X/\mathcal{J}^k)}^{A_k}$$ (functions on infinitesimal neighborhoods of $E$)

- Now $H^1(X, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^1(E, \mathcal{O}(nd)) = 0$ and an induction yields $H^1(X, \mathcal{J}^n/\mathcal{J}^m) = 0$ for all $m \geq n+1$, hence $A_k \twoheadrightarrow A_{k-1}$.

- This gives exact sequences $0 \to B_k \to \hat{\mathcal{O}}_{Y,y} \to A_k \to 0$.

We have $A_1 = H^0(E, \mathcal{O}_E) = k'$ is a field, hence $B_1 = m$ (and $k(y) = k'$).

- <u>lemma:</u> | $B_1^2 = B_2$ (as ideals of $\hat{\mathcal{O}}_{Y,y}$).

<u>proof:</u> - It suffices to prove this mod $B_k$ for all $k$, as $\bigcap B_k = 0$.

- For $k \leq \ell$ we have $B_k/B_\ell \simeq \mathrm{Ker}(A_\ell \to A_k) \simeq H^0(X, \mathcal{J}^k/\mathcal{J}^\ell)$, so we have to prove $H^0(X, \mathcal{J}/\mathcal{J}^n)^2 = H^0(X, \mathcal{J}^2/\mathcal{J}^n)$ (as ideals in $H^0(X, \mathcal{O}_X/\mathcal{J}^n)$)

The inclusion $\subseteq$ is ok, we prove the other by induction on $n$ (ok for $n=2$: $0=0$)

We have that $\begin{cases} H^0(X, \mathcal{J}/\mathcal{J}^{n+1}) \twoheadrightarrow H^0(X, \mathcal{J}/\mathcal{J}^2) \text{ (by } H^1(X, \mathcal{J}^2/\mathcal{J}^{n+1}) = 0) \\ H^0(X, \mathcal{J}^2/\mathcal{J}^{n+1})^n = 0 \end{cases}$

So $H^0(X, \mathcal{J}/\mathcal{J}^2)^n = H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^n$.

and $H^0(X, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^0(E, \mathcal{O}(nd)) \underset{E = \mathbb{P}^1}{=} H^0(E, \mathcal{O}(d))^n = H^0(X, \mathcal{J}/\mathcal{J}^2)^n = H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^n$.

Hence $H^0(X, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^n \underset{n\geq 2}{\subseteq} H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^2$ $(*)$

- We have the exact sequence

$$0 \to H^0(X, \mathcal{J}^n/\mathcal{J}^{n+1}) \to H^0(X, \mathcal{J}^2/\mathcal{J}^{n+1}) \to H^0(X, \mathcal{J}^2/\mathcal{J}^n) \to 0 = H^1(X, \mathcal{J}^n/\mathcal{J}^{n+1})$$

$$\underset{(*)}{\cap\!\!|} \qquad\qquad \overset{\text{induction}}{\|}$$

$$H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^2 \qquad\qquad H^0(X, \mathcal{J}/\mathcal{J}^n)^2 \subseteq H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^2$$

Hence $H^0(X, \mathcal{J}^2/\mathcal{J}^{n+1}) \subseteq H^0(X, \mathcal{J}/\mathcal{J}^{n+1})^2$ as needed $\square$

We deduce that $m/m^2 = B_1/B_1^2 \underset{\text{lemma}}{=} B_1/B_2 = H^0(X, \mathcal{J}/\mathcal{J}^2) = H^0(E, \mathcal{O}(d))$

is of dimension $d+1$. $\square$

<u>Alternative argument for (ii):</u>

- We have $H^0(X, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^0(E, \mathcal{O}(-nE)_{|E}) = H^0(\mathbb{P}^1_k, \mathcal{O}(nd))$.

so to conclude, it is enough to show:

<u>Lemma:</u> 1) For $n \leq 1$, $H^1(X, \mathcal{J}^n) = 0$.

2) For $n \leq 2$, $H^0(X, \mathcal{J}^n) \simeq m^n$.

<u>proof:</u> We show both for all $n$.

1) By the theorem on formal functions [Hartshorne, III.11.1] we have

$$H^1(X, \mathcal{J}^n) \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y} \simeq \lim_k H^1(X, \mathcal{J}^n/m^k\mathcal{J}^n).$$

- Since $|E| = |\pi^{-1}(y)|$ we have $\sqrt{m\mathcal{O}_X} = \mathcal{J}$, hence $\exists r$, $\mathcal{J}^r \subseteq m \subseteq \mathcal{J}$, so

$$\lim_k H^1(X, \mathcal{J}^n/m^k\mathcal{J}^n) = \lim_{m\geq n} H^1(X, \mathcal{J}^n/\mathcal{J}^m).$$

Now $H^1(X, \mathcal{J}^n/\mathcal{J}^{n+1}) = H^1(E, \mathcal{O}(nd)) = 0$ and an induction

yields $H^1(X, \mathcal{J}^n/\mathcal{J}^m) = 0$ for all $m \geq n+1$, hence

$$H^1(X, \mathcal{J}^n) \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,y} = 0.$$

- $(\mathcal{O}_{Y,y}, m)$ noetherian local ring $\Rightarrow$ $\mathcal{O}_{Y,y} \to \hat{\mathcal{O}}_{Y,y}$ faithfully flat

$\Rightarrow H^1(X, \mathcal{J}^n) = 0$.

2) · As in the proof of (i), we see that 1) $\Rightarrow \mathcal{I}^n$ generated by global sections.

· We have $H^0(X,\mathcal{I}) = \mathfrak{m}$ as $\mathcal{O}_{Y,y}$-modules.

Hence $H^0(X,\mathcal{I})^{\otimes n} = \mathfrak{m}^{\otimes n}$. We have $H^0(X,\mathcal{I})^{\otimes n} \subseteq H^0(X,\mathcal{I}^n)$ and we want to show the opposite inclusion. This follows from:

<u>lemma</u>: [Lipman, Lemma 7.3] Let $X$ be a noetherian scheme such that

a) $H^1(X,\mathcal{O}_X) = 0$

b) $H^2(X,\mathcal{J}) = 0$ for all ideal sheaves $\mathcal{J}$

blow-up of affine at a smooth point satisfies this.

Then for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ generated by their global section, the map $H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{G}) \to H^0(X,\mathcal{F}\otimes\mathcal{G})$ is surjective.

$\square$

<u>rmk</u> · One can ask more generally when the contraction of an effective divisor which is not irreducible exists, and when it is regular. This can be studied by similar methods [Liu, 9.4.1-2].

<u>prop</u>: A regular proper surface over a field is projective.

<u>proof</u>: Let $X/_k$ be such a surface.

· By Chow's lemma [EGA II, 5.6.1] there exists $Z/_k$ projective and $Z \longrightarrow X$ projective birational. By res. of singularities of surfaces, we can assume $Z$ is regular. The morphism $Z \to X$ is a composition of blow-ups of closed points (same proof as in the fibered regular case). By Castelnuovo, we see that $Z$ projective $\Rightarrow$ all contractions of $(-1)$-curves are projective, hence $X$ is projective.

$\square$

<u>exc</u>: <u>Elementary transforms</u>

Let $X \xrightarrow{f} S$ be a regular fibered surface and $s \in S^{(0)}$ such that $X_s \simeq \mathbb{P}^1_{k(s)}$.

Let $x \in X_s(k(s)) \simeq \mathbb{P}^1_{k(s)}$.

Let $\tilde{X} = Bl_x X$. Then $\tilde{X}_s = \tilde{D} \cup E$ with $\begin{cases} \tilde{D} \text{ strict transform of fiber} \\ E \text{ exceptional divisor.} \end{cases}$

We have:
* $E^2 = -1$
* $\pi^*(X_s) = \tilde{D} + E$ (by $\mu_x(X_s) = 1$) $\Rightarrow (\tilde{D}+E)\cdot E = 0 \Rightarrow \tilde{D}\cdot E = +1$
* $\tilde{D}^2 = (\tilde{D}+E)^2 - E^2 - 2\cdot\tilde{D}\cdot E = 0 + 1 - 2 = -1$

- But $\breve{D} \simeq \mathbb{P}^1$ (because $\breve{D} \setminus (\breve{D} \cap E) \xrightarrow{\sim} X_\sigma \setminus x \simeq \mathbb{P}^1 \setminus x$), so that $\breve{D}$ is a $(-1)$-curve on $\breve{X}$.
- By Castelnuovo, there exists a contraction $\breve{X} \to \overline{X}$ of $\breve{D}$ with $\overline{X}$ regular.

The fibered surface $\overline{X}$ is called the <u>elementary transform</u> of $X$ at $x$.

<u>rmk</u>:

- In the classical setting of smooth projective surfaces, this construction is very important in the study of geometrically ruled surfaces.

  In particular, there is the Enriques-Noether theorem: if $S$ is a smooth complex curve and $f : X \to S$ is smooth along $X_\sigma \simeq \mathbb{P}^1$, then there exists $\sigma \in U \subseteq S$ such that $f^{-1}(U) \simeq \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ rank 2 vector bundles over $U$, and elementary transforms can then be interpreted in terms of $\mathcal{E}$.
- In the arithmetic setting, the situation is more complicated because there can exist non-trivial conics over $k(\eta)$, which have models over $S$ and fibers $\simeq \mathbb{P}^1_\sigma$. However the situation for models of $\mathbb{P}^1_\eta$ should be similar to the geometric case; I do not have a good reference for this.

# 3) Minimal models

**def** | Let $\pi : X \to S$ be a regular fibered surface.
It is called <u>relatively minimal</u> if any proper birational morphism $X \to Z$ with $Z$ regular fibered surface is an isomorphism.
It is called <u>minimal</u> if every birational map $Y \dashrightarrow X$ with $Y$ regular fibered surface is a morphism.

**rmk**: A minimal fibered surface is a terminal object in the category of regular model of its generic fiber, hence is unique up to unique iso. of models.

**prop**: | 1) $X$ relatively minimal $\iff$ $X$ does not contain exceptional curves.
2) Any regular fibered surface $X$ admits a birational morphism $X \to X'$ with $X'$ relatively minimal.

**proof**: 1) This follows immediately from Thm 2.1) and the def. of exceptional curves.

2) **lemma**: $X \to S$ has finitely many exceptional curves.

**proof**: • This is very easy if $X_\eta/\eta$ is smooth as there is then an open subset of $S$ over which $X$ is smooth, and then the exceptional fibers live above the complement. We refer to [Liu, Lemma 9.3.17, Remark 9.3.18 and Prop. 8.3.8] for two proofs in the general case. The difficulty is that $X \to S$ can have infinitely many singular fibers if $S$ is a "weird" (non-excellent) Dedekind scheme. $\square$

The result then follows immediately from part 1). $\square$

**lemma**: | $X$ minimal $\Rightarrow$ $X$ relatively minimal and all relatively minimal models of its generic fiber are isomorphic (as models).

**proof**:

• Assume $X$ minimal. Let $X \to Z$ be proper birational. Then its inverse $Z \dashrightarrow X$ is a morphism by minimality. This implies in particular that $X \to Z$ is a proper bij. morphism, hence (by normality) an isomorphism.

• Let $X'$ be a relatively minimal model of $X_\eta$. In particular, we a birational iso $X' \overset{\pi}{\dashrightarrow} X$. By minimality of $X$, $\pi$ is a birational morphism. By relative minimality of $X'$, $\pi$ is an isomorphism of models. $\square$

· From this we can deduce that some curves have no minimal model.

<u>ex</u>: Let $X_1 \to S$ be a rational fibered surface and $\sigma \in S^{(0)}$ with $X_{1,\sigma} \simeq \mathbb{P}^1_{k(\sigma)}$.

Let $X_2$ be the elementary transform of $X_1$ at any rational point of $X_{1,\sigma}$.

Suppose $X_1$ relatively minimal (for example $X_1 = \mathbb{P}^1_S$). Then $X_2$ is also relatively minimal: any new exceptional curve would be in $X_{2,\sigma}$ which is irreducible.

Then $X_1, X_2$ are two relatively minimal models of their common generic fiber,

Let us show that $X_1 \not\simeq X_2$ as models. Let $f : X_1 \xrightarrow{\sim} X_2$ be such an iso.

Then we would have a commutative triangle

$$\begin{tikzcd} & \mathrm{Bl}_x X_1 \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr, "f"'] & & X_2 \end{tikzcd}$$

which would force $f(X_{1,\sigma})$ to be a point.

· Here is the key proposition which shows the previous situation is essentially the only thing that can go wrong.

<u>prop:</u> Let $X \xrightarrow{f} S$ be regular fibered, and $E \neq C$ be exceptional curves on $X$. Write $\pi : X \to Y$ for the contraction of $E$.

1) If $p_a(X_\eta) > 0$, then $\pi(C)$ is an exceptional curve disjoint from $\pi(E)$

2) If $C \cap E \neq \emptyset$, then $p_a(X_\eta) \leq 0$ and $C \cup E$ is a fiber of $X$.

<u>proof:</u> Write $k_C = H^0(C, \mathcal{O}_C)$ and $k_E = H^0(E, \mathcal{O}_E)$. Put $\bar{C} = \pi(C)$.

· $\bar{C}$ is a curve on $Y$, birational to $C$. If $\bar{C}$ is disjoint from $\pi(E)$ then $\bar{C} \simeq C \simeq \mathbb{P}^1_{k_C}$, and we have $\bar{C}^2 = C^2 = -[k_C : k(s)]$, so that $\bar{C}$ is an exceptional curve.

· Assume now that $\pi(E) \in \bar{C}$ with some multiplicity $\mu > 0$. Note that $C$ is the strict transform of $\bar{C}$, so that $\pi^* \bar{C} = C + \mu E$.

We have $\bar{C}^2 = (\pi^* \bar{C})^2$ ($\pi$ birational)

$$= \pi^* \bar{C} \cdot (C + \mu E)$$
$$= \pi^* \bar{C} \cdot C \qquad (\pi^* \bar{C} \cdot E = \bar{C} \cdot \pi_* E = 0)$$
$$= C^2 + \mu\, C \cdot E$$

We have $C^2 = -[k_C : k]$. Because $C$ is integral, for any $x \in C \cap E$, $k(x)$ is an extension of $k_C$, hence $C \cdot E \geq \mu \cdot [k_C : k]$.

So $\bar{C}^2 \geq (\mu - 1) [k_C : k] \geq 0$

By the known properties of the intersection product, we deduce that:

* $\bar{C}^2 = 0 \Rightarrow \mu = 1$.
* $|\bar{C}|$ is the fiber $Y_s$ of $Y$ $\Rightarrow |C \cup E|$ is the fiber $X_s$ of $X$.

We can assume that $S$ is affine and that $X_s$ is principal, given by $\pi = 0$. Then we see that there exists $m > 0$ such that $\mathcal{I}_{\bar{C}}^m = (\pi)$.

· Because $\mu = 1$, we deduce that $C$ and $E$ intersect at exactly one $k_C$-point, and that $C \to \bar{C}$ is an isomorphism. So $\bar{C} \simeq \mathbb{P}^1_{k_C}$.

· It remains to show that $p_a(X_\eta) \leq 0$. It is equivalent to $H^1(X_\eta, \mathcal{O}_{X_\eta}) = 0$.

By $\bar{C} \simeq \mathbb{P}^1_{k_C}$ and the sequence $0 \to \mathcal{I}_{\bar{C}}^n / \mathcal{I}_{\bar{C}}^{n+1} \to \mathcal{O}_Y / \mathcal{I}_{\bar{C}}^{n+1} \to \mathcal{O}_Y / \mathcal{I}_{\bar{C}}^n \to 0$

we get $H^1(Y, \mathcal{O}_Y / \mathcal{I}_{\bar{C}}^m) = 0$. This implies $H^1(Y, \mathcal{O}_Y) \xrightarrow{\times \pi} H^1(Y, \mathcal{O}_Y)$, which implies that $H^1(X_\eta, \mathcal{O}_{X_\eta}) = H^1(Y_\eta, \mathcal{O}_{Y_\eta}) = 0$ $\square$

**thm:** Let $X \to S$ be a regular fibered surface. Assume that $X_\eta$ is a regular curve of arithmetic genus $\geq 1$.
Then $X$ admits a minimal model $X^{reg}$ ( its unique relative minimal model)

**proof:** We will show that any relatively minimal model of $X$ is minimal. Let $X_0$ be such a model and $Y \overset{g}{\dashrightarrow} X_0$ be a birational map. By a previous result, we can find $Z$ regular fibered and a diagram
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y \arrow[rr, dashed, "g"] & & X_0 \end{tikzcd}$$
with $p, q$ birational.
Assume that $g$ is not a morphism, say, it is not defined at a point $y \in Y$. Then $p^{-1}(y)$ has to have dimension $\geq 1$. Because $p$ is a composition of blow-up it is clear that $p^{-1}(y)$ must contain some exceptional curve $E$. Since $g$ is not defined at $y$, $q$ does not contract $E$. We can write $q$ as
$Z = Z_n \xrightarrow{q_n} Z_{n-1} \to \cdots \xrightarrow{q_1} Z_0 = X_0$ composition of blow-ups, with $\Gamma_i$ the exceptional curve in $Z_i$ contracted in $Z_{i-1}$.

· Since $q$ does not contract $E$, we have $E \neq \Gamma_n$. From $p_a(Z_\eta) = p_a(X_\eta) \geq 1$ and the previous prop, we deduce that $E \cap \Gamma_n = \emptyset$ and that $q_n(E)$ is an exceptional curve in $Z_{n-1}$. Arguing inductively, we conclude that $q(E)$ is an exceptional curve in $X_0$, contradicting the relative minimality of $X_0$ $\square$

. We now state two closely related results: the existence of minimal regular models with normal crossings, and of minimal resolutions of singularities.

**thm:** Let $X \to S$ be a regular fibered surface with finitely many singular fibers.

[Liu, Prop 9.3.36]

1) There exists a projective birational morphism $X' \to X$ with $X'$ regular with normal crossings fibers.
2) If $p_a(X) \geq 1$, then there exists a regular model with normal crossings $X^{nc}$ of $X$ which is minimal for this property, i.e for any other such model $Y$ there exists a morphism $Y \to X^{nc}$.

**proof:** · Besides the methods used for the previous theorem, relies on two ingredients
* embedded resolution of curves on surfaces [Liu, Cor. 9.2.30]
* study of when a contraction still has normal crossings [Liu, Lemma 9.3.35]

$\square$

· **thm:** [Liu, Prop 3.32] Let $Y \to S$ be a normal fibered surface which admits a resolution of sing. Then it admits a minimal resolution of singularities $Y^{min}$ which is characterized by the fact that the exceptional locus of $Y^{min} \to Y$ does not contain an exceptional curve.

**proof:** Same method of proof as for minimal models $\square$

**rmk:** There are curves of arithmetic genus 0 with a minimal model, with a precise criterion [Liu, Exercise 9.3.1].

**rmk:** By construction, the minimal model is only functorial with respect to birational maps. Let $X \to Y$ be proper and generically finite; then in general there is no morphism $X^{reg} \to Y^{reg}$ extending $X_\eta \to Y_\eta$. An explicit example, with modular curves, is $X_1(p) \to X_0(p)$ for $p$ prime $\neq 2,3,5,7,13$. [CES].

Canonical sheaf and minimal model

We want to rephrase some of the preceding results in terms of the relative canonical bundle. This point of view cannot be avoided in the analogous theory of surfaces over a field, and will also be important for the study of stable curves.

**def:** Let $Y$ be locally noetherian