Prop 21: Let X, Y be ∞-categories. Then $X * Y$ is also an ∞-category.

proof: Let $0 < i < n$ and

$$\sigma_0 : \Lambda^n_i \longrightarrow X * Y.$$

We must extend σ_0 to an n-simplex of $X * Y$.

- Let $J \subseteq [n]$ be the set of vertices such that $\sigma_0(j) \in X$ for $j \in J$.

Suppose that J is not an initial segment of $[n]$: then there is $j \in J$ with $j > 0$ and $j-1 \notin J$. Let $e = \{j-1, j\} \in (\Lambda^n_i)_1$.

Then $\sigma_0(e) \in (X * Y)_1$ has source in Y and target in X, which is impossible.

$\Rightarrow J = [i]$ is an initial segment $(-1 \leq i \leq n)$

- If $J = \emptyset$ or $J = [n]$, then σ_0 factors through either $Y \longrightarrow X * Y$ or $X \longrightarrow X * Y$ and we use the assumption that X, Y are ∞-categories to extend σ_0.

- If $0 \leq i < n$, the composite

$$\Delta^J \longrightarrow \Lambda^n_i \xrightarrow{\sigma_0} X * Y$$

factors through $X \rightsquigarrow$

$$\sigma_- : \Delta^J \longrightarrow X.$$

Similarly, $\Delta^{[n] \setminus J} \xrightarrow{\sigma_+} Y$,

and σ_0 admits a unique extension given by

$$\Delta^n \simeq \Delta^J * \Delta^{[n] \setminus J} \xrightarrow{\sigma_- * \sigma_+} X * Y. \square$$

Def 22: Let $X \in sSet$. The left cone (resp. right cone) $X^{\triangleleft}$ (resp. $X^{\triangleright}$) is:

$$\begin{cases} X^{\triangleleft} := \Delta^0 * X \\ X^{\triangleright} := X * \Delta^0 \end{cases}$$

Ex 23: (Outer horns as cones) $n \geq 0$.

$$\Lambda^{n+1}_0 \simeq (\partial\Delta^n)^{\triangleleft}, \quad \Lambda^{n+1}_{n+1} \simeq (\partial\Delta^n)^{\triangleright}.$$

Lemma 24: Let $i : A \to B$ be a monomorphism in sSet. Then $i * id : A * K \to B * K$ is also a mono. $\square$

Rmk 25: The functor $* : sSet \times sSet \to sSet$ does not commute with colimits, since

$\emptyset * X \simeq X$ is not an initial object for $X \neq \emptyset$.

We will see that $*$ still preserves some colimits if interpreted correctly.

- One could define another functor $*'$ by $\iota^*\left((\iota_! X) * (\iota_! Y)\right)$. The advantage of this new functor is that it would commute with all colimits. The defect is that $\iota_!$ does not send representables to representables, so we have $\Delta^m *' \Delta^n \neq \Delta^{m+1+n}$.

Lemma 26: Let $C, D \in \mathrm{Cat}$. There is an isomorphism of simplicial sets

$$N(C * D) \xrightarrow{\sim} N(C) * N(D).$$

proof:

$$N(C*D)_n \simeq \left\{ x_0 \xrightarrow{g_0} x_1 \to \cdots \xrightarrow{g_n} x_n \;\; \begin{array}{l}\text{composable} \\ \text{morphisms in } C*D\end{array} \right\}$$

we "jump" from C to D at some point.

$$= \coprod_{i+1+j=n} \left\{ x_0 \to \cdots \to x_i \text{ in } C \right\} \times \left\{ x_{i+1} \to \cdots \to x_n \text{ in } D \right\}$$

$$\cong (N(C) * N(D))_n .$$ $\square$

Lemma 27: Let $X \in sSet$. The join construction lifts to functors:

$$\begin{cases} X * - : sSet \longrightarrow sSet_{X/} \\ - * X : sSet \longrightarrow sSet_{X/} \end{cases}.$$

proof: The maps from X are given by

$$X \cong X * \emptyset \longrightarrow X * Y.$$ $\square$

Prop 28: The functors

$$\begin{cases} X * - : sSet \longrightarrow sSet_{X/} \\ - * X : sSet \longrightarrow sSet_{X/} \end{cases}$$

preserve colimits.

proof: . sSet is a presheaf category so colimits are computed objectwise.

- Colimits in a coslice category like $sSet_{X/}$ are easy to compute:

$F : I \longrightarrow sSet_{X/}$ determines a diagram $\hat{F} : I^{\triangleleft} \longrightarrow sSet$ extending F by Prop 9. and $\operatorname{colim} F \simeq \operatorname{colim} \hat{F}$ together with its canonical map $X \simeq \hat{F}([0]) \longrightarrow \operatorname{colim} \hat{F}$.

In particular, to compute colimits in $sSet_{X/}$, we are reduced to compute colimits in $Set_{X_n/}$ for each $n \geq 0$.

Now

$$(X * Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg Y_n$$

Let us see this formula implies that.

$$sSet \longrightarrow Set_{X_n/}, \quad Y \longmapsto (X * Y)_n$$

commutes with colimits.

Let $F : I \longrightarrow sSet$ be a functor.

Then

$$\operatorname*{colim}_{i \in I} (X * F(i))_n = \operatorname*{colim}_{i \in I} X_n \amalg (X_{n-1} \times F(i)_0) \amalg \dots \amalg F(i)_n$$

Set

$$= X_n \amalg \operatorname*{colim}_{i \in I} \left((X_{n-1} \times F(i)_0) \amalg \dots \amalg F(i)_n \right)$$

$\amalg$, $\times$ commute with colimits in Set

$$= X_n \amalg \left(X_{n-1} \times \operatorname*{colim}_{i \in I} (F(i))_0\right) \amalg \dots$$

colims in sSet objectwise

$$= X_n \amalg \left(X_{n-1} \times \left(\operatorname*{colim}_{i \in I} F(i)\right)_0\right) \amalg \dots$$

$\square$

Rmk 29 The functor $sSet_{X/} \longrightarrow sSet$ does <u>not</u> preserves colimits (id_X is initial in the source, but X is not initial in $sSet$ unless $X = \emptyset$).

It does preserve all connected colimits, those indexed by a category with $|N(I)|$ connected. In particular, it preserves pushouts and filtered colimits.

Hence the join $X * - : sSet \to sSet$ also preserves connected colimits.

3) Slices of simplicial sets.

Prop 30: The functors.

$$\begin{cases} -* S : sSet \longrightarrow sSet_{S/} \\ T * - : sSet \longrightarrow sSet_{T/} \end{cases}$$

admit right adjoints, the slice / costlice functors:

$$sSet_{S/} \longrightarrow sSet, \ (p : S \to X) \longmapsto X_{/p}$$

$$sSet_{T/} \longrightarrow sSet, \ (q : T \to Y) \longmapsto Y_{q/}.$$

Explicitely, we have

$$(X_{/p})_n = sSet_{S/}(\Delta^n * S, X)$$

$$(Y_{q/})_n = sSet_{T/}(T * \Delta^n, Y)$$

Proof:

This is the case of any colimit-preserving functor out of sSet, as we have seen in the first lecture. $\square$

Ex 31: For $x \in X_0 \iff x : \Delta^0 \longrightarrow X$,

$$\mathrm{sSet}(K, X_{/x}) = \mathrm{sSet}_*((K^{\triangleright}, v), (X, x))$$

where $\mathrm{sSet}_* = \mathrm{sSet}_{\Delta^0/}$ category of pointed simplicial sets, and $v \in (K^{\triangleright})_0$ is the cone point.

Similarly, we have

$$\mathrm{sSet}(K, X_{x/}) = \mathrm{sSet}_*((K^{\triangleleft}, v), (X, x))$$

Rmk 32: Let $p : S \to X$ be a morphism of simplicial sets. The adjunction produces a morphism

$$c : X_{/p} * S \longrightarrow X .$$

the *slice contraction morphism.*

Similarly, there is a morphism

$$c : S * X_{p/} \longrightarrow X$$

the *coslice contraction morphism.*

Prop 33: Let $p : I \to C$ be a functor between 1-categories. Then we have

$$\begin{cases} N(C_{p/}) \simeq N(C)_{Np/} \\ N(C_{/p}) \simeq N(C)_{/Np} . \end{cases}$$

proof: Let's prove the first one.

We proceed by adjunction. For $K \in sSet$,

$$sSet(K, N(C_{p/})) \simeq Cat(\mathfrak{C}(K), C_{p/})$$

$$\simeq Cat_{A/}(A * \mathfrak{C}(K), C)$$

$$\overset{*}{\simeq} Cat_{A/}(\mathfrak{C}(N(A) * K), C)$$

$$\simeq Cat_{A/}(N(A) * K, N(C))$$

$$\simeq \mathrm{Cat}\left(K, N(C)_{N_{P/}}\right)$$

The only non-formal step is $\circledast$. To compute $\mathfrak{R}$, it suffices to determine the 0-, 1- and 2-simplices.

$$\begin{cases} (N(A) * K)_0 = N(A)_0 \amalg K_0 = \mathrm{Ob}(A * \mathfrak{R}(K)) \\ (N(A) * K)_1 = N(A)_1 \amalg (N(A)_0 \times K_0) \amalg K_1 \end{cases}$$

$$\mathrm{Mor}(N(A) * \mathfrak{R}K) = \mathrm{Mor}(A) \amalg (N(A)_0 \times K_0) \amalg \mathrm{Mor}(\mathfrak{R}(K))$$

and K_1 generates $\mathrm{Mor}(\mathfrak{R}(K))$.

The 2-simplices gives relations, and one checks they match up.

$$\Rightarrow A * \mathfrak{R}K \simeq \mathfrak{R}(N(A) * K)$$

$\square$

Exercise: Prove this in a different way by computing $\left(N(\mathcal{C})_{N_{p/}}\right)_n$ using the fact that, for A, B categories, one has $N(A * B) \simeq N(A) * N(B)$.

(see [Kerodon, Example 4.3.5.7].)

Def 34: Let $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{q} Y$ be a diagram of simplicial sets. We are going to construct commutative diagrams

$$\begin{tikzcd}
X_{/p} \arrow[r] \arrow[d] & Y_{/qp} \arrow[d] \\
X_{/pj} \arrow[r] & Y_{/qpj}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
X_{p/} \arrow[r] \arrow[d] & Y_{qp/} \arrow[d] \\
X_{pj/} \arrow[r] & Y_{/qpj}
\end{tikzcd}$$

(As noted by Rezk, "There seems to be no decent notation for the maps in [this] diagram.

The whole business of joins and slices can get pretty confusing because of this.")

. Let us do the case of slices. In fact, it suffices to construct the diagonal map $X_{/p} \longrightarrow Y_{/gpj}$; the others are special cases with $g = id$ or $j = id$.

We construct it by adjunction using joins.

$u : K \longrightarrow X_{/p}$ correspond to the map $\tilde{u}$ in

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d] & S \arrow[r, "p"] \arrow[d] & X \arrow[r, "g"] & Y \\ K * T \arrow[r, "K * j"] & K * S \arrow[ur, dashed, "\tilde{u}"'] & & \end{tikzcd}$$

and the map $K \longrightarrow X_{/p} \longrightarrow Y_{/gpj}$ corresponds to $g \circ \tilde{u} \circ (K * j)$.

Ex 35: . $\emptyset \longrightarrow S \xrightarrow{p} X = X$ yields restriction functors $\begin{cases} X_{/p} \longrightarrow X \\ X_{p/} \longrightarrow X \end{cases}$.

Def 36: Let $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{g} Y$ in sSet.

- From the commutative diagram of the previous definition, we get the

pullback-slice maps

$$\begin{cases} g^{\boxed{*}}p^{j} : X_{/p} \longrightarrow X_{/pj} \underset{Y_{/gpj}}{\times} Y_{/gp} \\ g^{j\boxed{*}}p : X_{p/} \longrightarrow X_{pj/} \underset{Y_{gpj/}}{\times} Y_{gp/} \end{cases}$$

Again, special cases (with $Y = *$ or $T = \emptyset$) recover the functoriality of the previous definit°.

Prop 37: Let C be an ∞-category, and $x \in C_0$. The map

$$\begin{cases} C_{x/} \longrightarrow C \\ C_{/x} \longrightarrow C \end{cases} \text{ is a } \begin{cases} \text{left fibration.} \\ \text{right fibration.} \end{cases}$$

In particular, $C_{x/}$ and $C_{/x}$ are ∞-categories.

proof: Let us check $C_{/x} \xrightarrow{\pi} C$ is a right fibration.

Explicitely, this sends $a : \Delta^n \longrightarrow C_{/x}$ to $\breve{a}|(\Delta^n * \emptyset)$, where $\breve{a} : \Delta^n * \Delta^0 \longrightarrow C$ corresponds to a. Let $0 < j \leq n$. There is a equivalence of lifting problems:

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & C_{/x} \arrow[d, "\pi"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists ?"] & C
\end{tikzcd} \quad \Longleftrightarrow$$

$$\begin{tikzcd}
\emptyset * \Delta^0 \arrow[r] \arrow[rr, bend left, "x"] & (\Lambda^n_j * \Delta^0) \cup_{\Lambda^n_j * \emptyset} (\Delta^n * \emptyset) \arrow[r] \arrow[d] & C \\
& \Delta^n * \Delta^0 \arrow[ur, dashed, "\exists ?"'] &
\end{tikzcd}$$

The isomorphism $\Delta^n * \Delta^0 \simeq \Delta^{n+1}$ identifies, for any $S \subset [n]$, the simplicial subset $\Delta^S * \Delta^0$ with $\Delta^{S \cup \{n+1\}} \subset \Delta^{n+1}$, and the simplicial subset $\Delta^S * \emptyset$ with $\Delta^S \subset \Delta^{n+1}$.

Since $\Lambda^n_j = \bigcup_{k \in [n] \setminus j} \Delta^{[n] \setminus k}$ as a colimit,

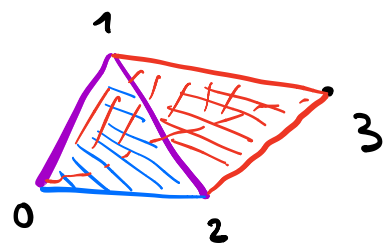
and the join commutes with pushouts, we get:

(1) The sub set

$$\left(\Lambda^n_j * \Delta^0\right) \underset{(\Lambda^n_j * \emptyset)}{\cup} \left(\Delta^n * \emptyset\right) \text{ of } \Delta^n * \Delta^0$$

is :

$$\left(\bigcup_{k \in [n] \setminus j} \left(\Delta^{[n] \setminus k} * \Delta^0\right)\right) \underset{\Lambda^n_j * \emptyset}{\cup} \left(\Delta^n * \emptyset\right)$$



$$= \left(\bigcup_{k \in [n] \setminus j} \Delta^{[n+1] \setminus k}\right) \underset{\Lambda^n_j * \emptyset}{\cup} \Delta^n$$

$$= \bigcup_{k \in [n+1] \setminus j} \Delta^{[n+1] \setminus k}$$

$$= \Lambda^{n+1}_j \subset \Delta^{n+1}.$$

(2) The sub set $\emptyset * \Delta^0$ of $\Delta^{n+1} * \Delta^0$ is $\{n+1\}$.

So the lifting problem is isomorphic to

$$\begin{tikzcd} \{n+1\} \arrow[r] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_j \arrow[r] \arrow[d] & C \\ & \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

which has a solution since $0 < j \leq n < n+1$.

This finishes the proof that $C_{/x} \to C$ is a right fibration. In particular, it is an inner fibration, so $C_{/x} \to C \to \Delta^0$ is as well and $C_{/x}$ is an ∞-category. $\square$

This result is true a lot more generally.

Thm 38: Let $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$ in sSet.

Consider the pullback-slice maps:

$$f^{\boxed{*}_p j} : X_{/p} \longrightarrow X_{/pj} \underset{Y_{/fpj}}{\times} Y_{/fp}$$

$$g^{j \boxed{*}_p} : X_{p/} \longrightarrow X_{pj/} \underset{Y_{gpj/}}{\times} Y_{gp/}$$

Assume that j is a monomorphism.

Then we have the following:

(1) g inner fibration $\Rightarrow \begin{cases} g^{j \boxed{*}_p} \text{ is a left fibration.} \\ g^{\boxed{*}_p j} \text{ is a right fibration.} \end{cases}$

(2) g trivial fibration $\Rightarrow g^{\boxed{*}_p j}, g^{j \boxed{*}_p}$ trivial fibration.

(3)

$j \begin{cases} \text{left anodyne} + g \text{ inner fibration} \\ \text{right anodyne} \end{cases} \Rightarrow \begin{cases} g^{j \boxed{*}_p} \\ g^{\boxed{*}_p j} \end{cases}$ trivial fib.

$\square$

Cor 39: With same notations:

j monomorphism $\Rightarrow \begin{cases} C_{p/} \longrightarrow C_{pj/} \text{ left fibration} \\ C_{/p} \longrightarrow C_{/pj} \text{ right fibration.} \end{cases}$

j left anodyne $\Rightarrow$ $C_{p/} \longrightarrow C_{pj/}$ trivial fibration.

j right anodyne $\Rightarrow$ $C_{/p} \longrightarrow C_{/pj}$ trivial fibration.

$\square$

Cor 40: $C_{p/}$ and $C_{/p}$ are ∞-categories $\square$

The proof of all of this relies on a similar study of the left adjoints.

Def 41: Let $\begin{cases} i : A \to B \\ j : K \to L \end{cases}$ in sSet.

The pushout-join $i \boxed{*} j$ is the map

$$i \boxed{*} j : (A * L) \coprod_{(A * K)} (B * K) \xrightarrow{(i * L, B * j)} B * L$$

The previous results then follow by adjunction

From:

Prop 42:
- Monos $\boxed{\ast}$ Monos $\subseteq$ Monos
- (right anodynes) $\boxed{\ast}$ Monos $\subseteq$ (inner anodyne)
- Monos $\boxed{\ast}$ (left anodyne) $\subseteq$ (inner anodyne)
- (anodyne) $\boxed{\ast}$ Monos $\subseteq$ (left anodyne)
- Monos $\boxed{\ast}$ (anodyne) $\subseteq$ (right anodyne)

"proof": We have seen in the proof of Prop that

$$(\Lambda^n_j \subset \Delta^n) \boxed{\ast} (\varnothing \subset \Delta^0) \simeq (\Lambda^{n+1}_j \subset \Delta^{n+1}).$$

The proof consists in computing similar pushout-joins of horn and boundary inclusions, and also proving that

$$\overline{S} \boxed{\ast} \overline{T} \subset \overline{S \boxed{\ast} T}.$$

$\square$

4) Initial and terminal objects

Def 43: An object x in an ∞-category C is **initial** if for every $f : \partial\Delta^n \longrightarrow C$ with $f(0) = x$, there exists an extension $f' : \Delta^n \to C$.

An object y in C is **terminal** if for every $f : \partial\Delta^n \longrightarrow C$ with $f(n) = y$, there exists an extension $f' : \Delta^n \to C$.

Rmk 44: • Let's look at the "initial" condition for small n:

• $n = 1$: for every $c \in C$, there is a morphism $x \to c$.

• $n = 2$: for every triple of maps
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g"'] & c \arrow[d, "h"] \\ & c' \end{tikzcd},$$
we have a filling triangle, hence $[g] = [h][f]$ in hC.

Lemma 45: Let C be a 1-category. Then

$x \in C$ is $\left|\begin{array}{l}\text{initial}\\ \text{terminal}\end{array}\right.$ iff $x \in N(C)_0$ is $\left|\begin{array}{l}\text{initial.}\\ \text{terminal}\end{array}\right.$

proof: $\Rightarrow$ By the previous remark, the conditions hold for $n \leq 2$. For any $n \geq 3$, we have $\mathrm{sSet}(\Delta^n, N(C)) \xrightarrow{\sim} \mathrm{sSet}(\partial\Delta^n, N(C))$ (because $sk_2(\partial\Delta^n) \simeq sk_2(\Delta^n)$ and $N(C)$ is 2-coskeletal.) so the result holds.

$\Leftarrow$: Also follows from the previous remark and $hN(C) \simeq C$. $\square$

Lemma 46: Let $x \in C$ be $\left|\begin{array}{l}\text{initial}\\\text{terminal}\end{array}\right.$. Then $x \in hC$ is $\left|\begin{array}{l}\text{initial}\\\text{terminal}\end{array}\right.$.

proof: This also follows for the conditions for $n \leq 2$. $\square$

- The converse is not true; there are "higher coherence" conditions.

Prop 47: Let C be an ∞-category, and $x \in C$.

Then x is $\begin{cases} \text{initial} \\ \text{terminal} \end{cases}$ iff $\begin{cases} C_{x/} \to C \\ C_{/x} \to C \end{cases}$ is a trivial fibration.

proof: There is an equivalence of lifting problems:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C_{x/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & C \end{tikzcd} \iff \begin{tikzcd} \Delta^0 * \emptyset \arrow[r] \arrow[rr, bend left, "x"] & (\Delta^0 * \partial\Delta^n) \amalg_{\emptyset * \partial\Delta^n} (\emptyset * \Delta^n) \arrow[r] \arrow[d] & C \\ & \Delta^0 * \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

and the right side is isomorphic to

$$\begin{tikzcd} \{0\} \arrow[r] & \partial\Delta^{n+1} \arrow[r] \arrow[d] & C \\ & \Delta^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

so $C_{x/} \to C$ is a trivial fibration iff x is an initial object. $\square$

Rmk: It is true, but outside our reach at this point, that

x initial $\iff C_{x/} \to C$ categorical equivalence.

(because trivial fib $\iff$ left or righ fib + cat. equivalence)

Prop 48: Let C be an ∞-category.

Write C^{init} (resp. C^{term}) denote respectively the full subcategory spanned by initial (resp. terminal) objects. Then each of them is either empty, or categorically equivalent to Δ^0.

proof: Suppose C^{init} is $\neq \emptyset$.

Then $C^{init} \to \Delta^0$ satisfies the right lifting condition with respect to $\partial\Delta^n \subset \Delta^n$ for

- $n \geq 1$ because of the initial property
- $n = 0$ because $C^{init} \neq \emptyset$.

So $C^{init} \to \Delta^0$ is a trivial fibration, and in particular a categorical equivalence. $\square$

Lemma 49: Let C be an ∞-category and $x \in C_0$. Then x is $\left|\begin{array}{l}\text{initial}\\ \text{terminal}\end{array}\right.$ <u>iff</u> $x : \Delta^0 \to C$ is $\left|\begin{array}{l}\text{left}\\ \text{right}\end{array}\right.$ anodyne.

proof: We prove the result for initial objects.

$\Longrightarrow$ $C_{x/} \longrightarrow C$ is a trivial fibration by Prop . So by Corollary III.1.10, it admits a section $s : C \longrightarrow C_{x/}$.

Consider the diagram

$$
\begin{tikzcd}
\Delta^0 \arrow[r, "\text{first fact}"] \arrow[d, "x"'] & \Delta^0 * \Delta^0 \arrow[r, "\text{proj.}"] \arrow[d, "\Delta^0 * x"] & \Delta^0 \arrow[d] \\
\Delta^0 \arrow[r, "\text{first.}"] & \Delta^0 * C \arrow[r, "\hat{s}"] & C
\end{tikzcd}
$$

$\Longrightarrow$ $\Delta^0 \xrightarrow{x} C$ is a retract of $\Delta^0 * x$.

But $\Delta^0 * x$ is left anodyne because id_{Δ^0} is anodyne and x is mono by Prop .

$\underline{\Leftarrow :}$ $C_{x/} \to C$ is always a left fibration by Prop   , so if $\Delta^0 \xrightarrow{x} C$ is left anodyne we can solve

$$\begin{tikzcd}
\Delta^0 \arrow[r, "id_x"] \arrow[d] & C_{x/} \arrow[d] \\
C \arrow[r, "id_C"'] \arrow[ur, dashed, "t"] & C
\end{tikzcd}$$

.

We can then use t to prove initiality.

Consider a diagram:

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & C \arrow[r, "t"] & C_{x/} \arrow[r] & C \\
& \Delta^n \arrow[ur, dashed] \arrow[urr, dashed] \arrow[urrr, dashed] & & &
\end{tikzcd}$$

$t(x) = id_x$ is an initial object of $C_{x/}$ (see Exercise sheet) so the red arrow exists. We define the purple arrow by composition and it solves the problem.

$\square$

Prop 50: Let $Spc = N_\Delta(\widetilde{Kan})$ be the ∞-category of spaces. Let K be a Kan complex. Then

- K is an initial object in $Spc \iff K = \emptyset$.
- K is a terminal object in $Spc \iff |K|$ is contractible.

proof:

- We first do the $\Rightarrow$ directions. If K is initial/terminal in Spc, then it is initial/terminal in $hSpc$, which is equivalent to the usual homotopy category of CW-complexes via $|-|$. It is then easy to show that K is empty/contractible.

- Consider a diagram:

$$\begin{tikzcd}
\{0\} \text{ or } \{n\} \arrow[r] \arrow[rr, bend left, "K"] & \partial\Delta^n \arrow[r] \arrow[d, hook] & Spc \\
& \Delta^n &
\end{tikzcd}$$

The arrow $\partial\Delta^n \to \mathrm{Spc} = N_\Delta(\widetilde{\mathrm{Kan}})$ corresponds to a simplicial functor

$$\mathrm{Path}[\partial\Delta^n] \longrightarrow \widetilde{\mathrm{Kan}}$$

To fill it in, we have to understand the difference between $\mathrm{Path}[\partial\Delta^n]$ and $\mathrm{Path}[\Delta^n]$. This is similar to the study of $\mathrm{Path}[\Lambda^n_j]$ we did to prove that $N_\Delta(\mathrm{Col}\cdot\mathrm{Kan})$ is a quasi category. The outcome is as follows:

$\mathrm{Path}[\partial\Delta^n]$ and $\mathrm{Path}[\Delta^n]$ have the same objects and the same Hom-simplicial sets, except for

$$\mathrm{Path}[\partial\Delta^n](0,n) \subseteq \mathrm{Path}[\Delta^n](0,n) \simeq (\Delta^1)^n$$

which is the hollow n-cube $\overset{\circ}{(\Delta^1)^n}$ with the interior removed.

So as in the proof of "N_Δ quasicategory", we only have to consider one simplicial hom-set at a time.

$\emptyset$ is initial:

$\Leftarrow$ Let $g : \partial\Delta^n \longrightarrow Spc$ with $g(0) = \emptyset$.

We have thus a map $(\mathring{\Delta}^1)^n \longrightarrow \widetilde{Kan}(g(0), g(n))$

Since $g(0) = \emptyset$ and $\emptyset$ is initial in Kan, $\widetilde{Kan}(g(0), g(n))$ is a singleton, so the map is constant, and we extend it to a constant map $(\Delta^1)^n \to Kan(g(0), g(n))$ $\rightsquigarrow$ this provides the extension $\tilde{g} : \Delta^n \to Spc$.

Contractible Kan complexes are terminal:

- Let K be a contractible Kan complex.
- Let $g : \partial\Delta^n \longrightarrow Spc$ such that $g(n) = K$.

We have a map $(\mathring{\Delta}^1)^n \longrightarrow \widetilde{Kan}(g(0), K)$

Now K contractible $\Rightarrow \widetilde{\mathrm{Kan}}(X, K)$ contractible for all X.

$\rightsquigarrow$ the map $(\mathring{\Delta}^1)^n \longrightarrow \widetilde{\mathrm{Kan}}(g(0), K)$ extends to $(\Delta^1)^n$.

This provides the extension $g : \Delta^n \to \mathrm{Spc}$.

$\square$

Exercise: Let $\mathcal{A}$ be an abelian category with enough injectives. Prove that $K_\bullet \in Ch^+(\mathcal{A}_{inj})$ be a complex of injectives. Recall $\mathcal{D}^+(\mathcal{A}) := N^{dg}(Ch^+(\mathcal{A}_{inj}))$

$$K \text{ is an initial object in } \mathcal{D}^+(\mathcal{A})$$
$$\Updownarrow$$
$$K \text{ is a terminal object in } \mathcal{D}^+(\mathcal{A})$$
$$\Updownarrow$$
$$K \text{ is acyclic: } H_*(K) = 0.$$

5) Limits and colimits

Def 51: Let $K \in sSet$, $C \in \widehat{Cat}_\infty$ and $p : K \to C$. A limit (resp. colimit) of p is a terminal object of $C_{/p}$ (resp. an initial object of $C_{p/}$).

Explicitly, a limit of p is a limit cone $\hat{p} : \Delta^0 * K = K^\triangleleft \to C$ extending p and such that for $n \geq 1$, we have lifts in any diagram of the form:

$$\begin{tikzcd}
\{n\} * K \arrow[r] \arrow[rr, bend left, "\hat{p}"] & \partial\Delta^n * K \arrow[r] \arrow[d] & C \\
 & \Delta^n * K \arrow[ur, dashed] &
\end{tikzcd}$$

Ex 52: • A colimit of $\emptyset \to C$ is an initial object, a limit of $\emptyset \to C$ is a final object.

- Let's look at the condition for $n=1$:

$$\begin{tikzcd}
K^{\triangleright} \arrow[r] \arrow[rr, bend left, "\hat{p}"] & K^{\triangleright} \amalg_K K^{\triangleright} \arrow[r] \arrow[d] & C \\
& \Delta^1 * K \arrow[ur, dashed] &
\end{tikzcd}$$

so we are given another $\hat{q} : K^{\triangleright} \longrightarrow C$ extending p

The existence of $\dashrightarrow$ means we have a map

$$\hat{q}(1) \longrightarrow \lim p = \hat{p}(\{1\})$$

which "makes the diagram commute."

It doesn't say anything about uniqueness of this map, unlike in the 1-cat. case!

Indeed, the condition for $n=2$ means roughly that this map is well-defined in $ho(C)$ and the conditions for $n \geq 3$ are higher coh.

conditions.

⚠ Unlike with initial/terminal object, even if $K = N(I)$ is the nerve of a 1-category, the induced functor

$$I^{\triangleright} = \mathfrak{h}N(I)^{\triangleright} \xrightarrow{\mathfrak{h}\hat{p}} \mathfrak{h}C$$

is **not** a limit cone in $\mathfrak{h}C$ in general. We will see an example later.

- We are indexing limits and colimits by arbitrary simplicial sets. This can be useful but the case $K = N(I)$ is already very interesting (and in some sense the general case reduces to it, see [HTT, Prop 4.2.3.14 and 4.1.1.8])
- In particular it is still true that a colimit of $id: C \to C$ is a terminal object.

At this point we can prove one direct', see Exercise Sheet.

· Consider $K = \Lambda^2_0 = N\left(1 \leftarrow 0 \rightarrow 2\right)$.

Then $(\Lambda^2_0)^{\triangleright} \simeq \Delta^1 \times \Delta^1 = N([1] \times [1])$.

A colimit cone $(\Lambda^2_0)^{\triangleright} \longrightarrow C$ is called a pushout diagram in C. Let's try to make the definition explicit in this case.

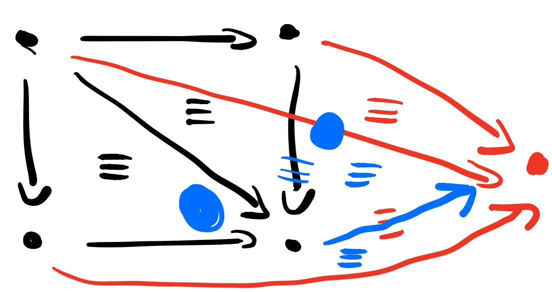
The diagram $\hat{p} : (\Lambda^2_0)^{\triangleright} \to C$ is

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, "\equiv", "\equiv"'] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd},$$

Hence an "homotopy commutative square with prescribed homotopies."

<u>$n=1$:</u> $K * \Delta^1 = N\left((1 \leftarrow 0 \rightarrow 2) * [1] \right)$

so we are given $\hat{q}$ extending p -- see above.



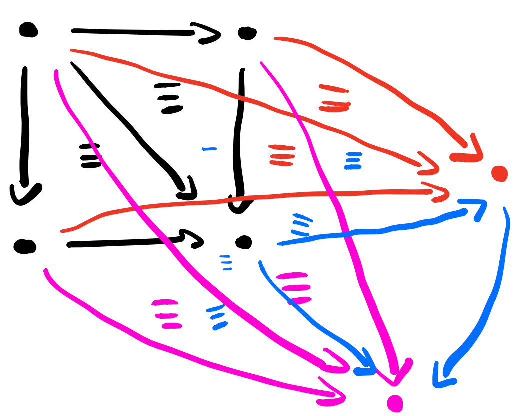
$$\underline{n=2:}\quad \begin{cases} K * \Delta^2 = N\left(1 \leftarrow 0 \rightarrow 2 \; * \; [2] \right) \\ K * \partial\Delta^2 \text{ is not a nerve.} \end{cases}$$

So we are given $\hat{q}, \hat{r} : K^{\triangleright} \to C$

together with three maps $\begin{cases} \hat{p} \to \hat{q} \\ \hat{p} \to \hat{r} \\ \hat{q} \to \hat{r} \end{cases}$

and we are asking for filling simplexes;

...



It is clear that in general (co)limits in ∞-categories are very complicated beasts!

We need various tools to compute and handle them without too much simplicial combinatorics. We can't go very far in this course; a lot of [HTT] is devoted to this problem.

Let's start with a reassuring fact.

Lemma 53: Let C be a 1-category and $p : K \to N(C)$ be a diagram.

Then p admits a limit / colimit in $N(C)$

$\iff$ the induced diagram $\mathfrak{h}p : \mathfrak{h}K \to C$ has a limit / colimit in $N(C)$.

proof: We have

- $$N(C)_{p/} = N(C)_{N\mathfrak{h}p/} \simeq N(C_{\mathfrak{h}p/})$$

and an object is initial in $C_{\mathfrak{h}p/}$ <u>iff</u> it is initial in $N(C_{\mathfrak{h}p/})$. This proves the result. $\square$

Prop 54: Let $p: K \longrightarrow C$ be a diagram.

Let $\left(C_{p/}\right)^{\mathrm{colim}}$ | $\left(C_{/p}\right)^{\mathrm{lim}}$ $\subseteq C_{p/}$ | $C_{/p}$ be the full subcategory spaned by colimit | limit cones. Then it is either empty or categorically equivalent to Δ^0. $\square$