

proof: Exercise. Hint: use the $\mathcal{I} \dashv \mathcal{N}$

adjunction and consider the maps

$$\mathcal{I}(\Lambda_k^n) \rightarrow \mathcal{I}(\Delta^n) \text{ for } 0 < k < n$$



Collections of morphisms of the form S^\square (or $S^{\square \square}$) have some remarkable "closure" properties.

def 5: Let $S \subseteq \text{Mor}(C)$. We say that:

* S is closed under pushouts if for every pushout diagram $A \xrightarrow{f} A'$, $g \in S \Rightarrow g' \in S$.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ g \downarrow & & \downarrow \\ B & \xrightarrow{g'} & B' \end{array}$$

* S is closed under retracts if for every retract diagram in C :

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & A & \xrightarrow{\quad} & A' & \xrightarrow{\quad} A \\
 g \downarrow & & g' \downarrow & & \downarrow g \\
 B & \xrightarrow{\quad} & B' & \xrightarrow{\quad} B & \\
 & & \text{id} & &
 \end{array}
 \quad (\text{expressing that } g' \text{ is a retract of } g)$$

we have $g \in S \Rightarrow g' \in S$.

* S is closed under transfinite compositions if for all ordinals α and a functor $F : [\alpha] \longrightarrow C$ with

$\forall 0 < \lambda \leq \alpha$ with λ limit ordinal,

$$F(\lambda) \simeq \operatorname{colim}_{\gamma < \lambda} F(\gamma)$$

(for all $\beta < \alpha$, $F(\beta < \beta + 1) \in S \Rightarrow F(0 < \alpha) \in S$)

[ex For $\alpha = \omega = (\mathbb{N}, \leq)$, then $[\omega] = \omega + 1 = \mathbb{N} \cup \{\infty\}$

and the condition says that if

$$F(\infty) \simeq \operatorname{colim}_{n \in \mathbb{N}} (F(0) \xrightarrow{\epsilon} F(n) \xrightarrow{\epsilon} F(2) \xrightarrow{\epsilon} \dots)$$

then $F(0) \longrightarrow F(\infty)$ is in S .]

* S is (weakly) saturated if it is stable under pushouts, retracts and transfinite compositions.

Rmk: • There is a related notion of "strongly
saturated" class which plays a role in the
theory of localizations of (∞ -) categories,
hence the terminology.

• The transfinite composition for $\alpha = 0$
(resp. $\alpha = 2$) means that S contains all

isos (resp. is stable by compositions).

Lemma 6: Let C admit arbitrary coproducts.

Then any saturated collection in C is stable under coproducts.

Proof: Let S be a saturated collection,

and $(g_\alpha : A_\alpha \rightarrow B_\alpha)$ be a set of morphisms such that the coproduct

$$\coprod g_\alpha : \coprod A_\alpha \rightarrow \coprod B_\alpha \text{ exists.}$$

We can construct $\coprod g_\alpha$ as a transfinite composition of pushouts. Ex:

$$\begin{array}{ccccc} A_0 & \longrightarrow & B_0 & \xrightarrow{\text{id}} & B_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_0 \amalg A_1 & \longrightarrow & A_1 \amalg B_0 & \longrightarrow & B_0 \amalg B_1 \\ \vdots & & & & \vdots \\ \coprod_{n \in \mathbb{N}} A_n & \xrightarrow{\coprod f_\alpha} & \coprod_{n \in \mathbb{N}} B_n & & \square \end{array}$$

This notion seems very artificial at first, but as we will see, it is in fact quite natural in view of Quillen's "small object argument".

First, we have:

Prop 7: Let $S \subseteq \text{Mor}(C)$ be any collection of morphisms. Then $\bigcap S$ is saturated.

Proof:

Stable under pushouts:

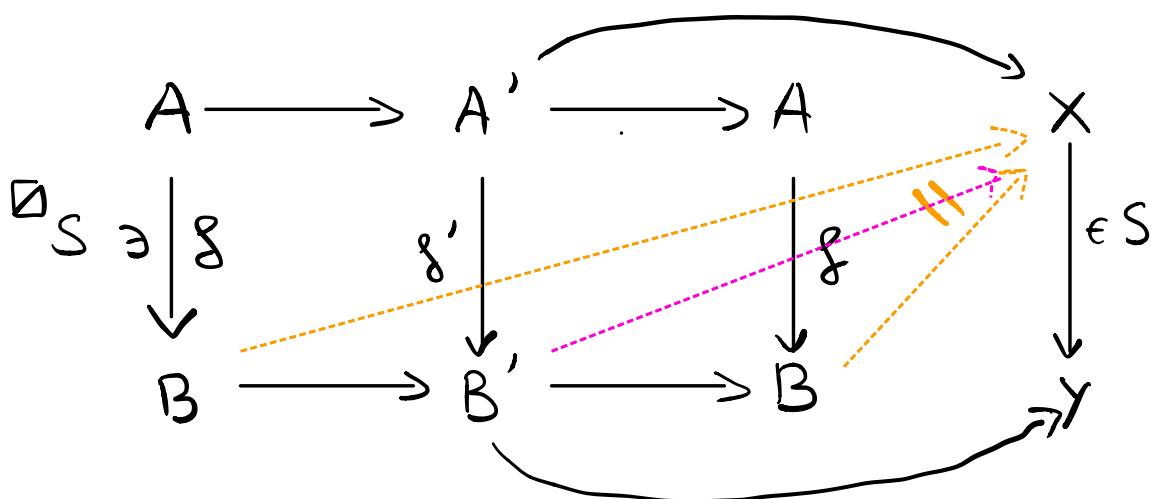
$$\begin{array}{ccccc}
 A & \longrightarrow & A' & \longrightarrow & X \\
 \bigcap S \ni f \downarrow & & \downarrow & & \downarrow \in S \\
 B & \xrightarrow{\quad} & B' & \xrightarrow{\quad} & Y
 \end{array}$$

The diagram shows a commutative square with vertices \$A\$, \$A'\$, \$B\$, and \$B'\$. There are two horizontal arrows: \$A \rightarrow A'\$ and \$A' \rightarrow X\$. There are also two vertical arrows: \$B \rightarrow B'\$ and \$B' \rightarrow Y\$. A dashed orange arrow \$f: A \rightarrow B\$ is shown, with the condition \$\bigcap S \ni f\$. A dashed yellow arrow \$g: A' \rightarrow B'\$ is also shown. A dashed pink arrow \$h: B' \rightarrow X\$ is shown, and a dashed magenta arrow \$k: B' \rightarrow Y\$ is shown. The bottom-right arrow \$B' \rightarrow Y\$ is highlighted in orange, indicating its existence due to \$f \in \bigcap S\$.

where \longrightarrow exists because $f \in \bigcap S$

→ exists because of universal property
of the pushout.

Stable under retracts:



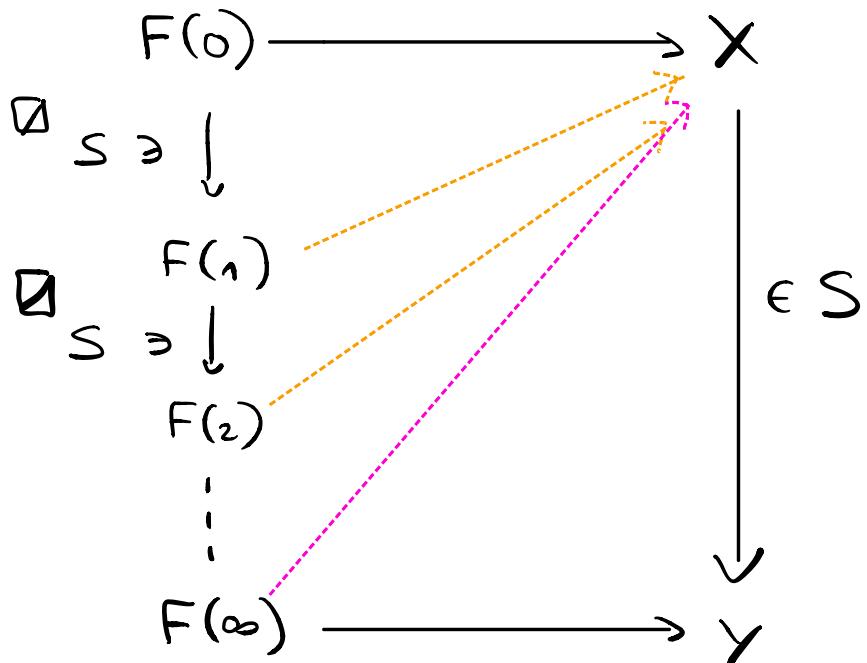
where the two (equal) → exist because
of $\square g \in S$, the → is just defined as
composition → → , and everything

commutes because of the retract

property $\begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{array}$.

Stable under transfinite compositions:

For notational simplicity I explain the case $\omega = (\mathbb{N}, \leq)$, the general argument is the same:



where: \rightarrow 's are constructed by induction
using that $F(i < i+1) \in \square^{\square} S$.

- \rightarrow is constructed from the universal property of colimits.



An intersection of saturated collections is saturated. This implies the following is well-defined:

Def 8: Let C be a category and S_0 be a collection of morphisms in C . There is a smallest saturated collection \bar{S}_0 which contains S_0 . We call it the **saturated collection generated by S_0** .

As a first example, we have:

prop 9: The collection of all monomorphisms in sSet is the saturated collection generated by $(\partial \Delta^n \hookrightarrow \Delta^n)_{n \geq 0}$.

proof: The fact that (monos) is saturated is easy and left as an exercise.

$$\Rightarrow \overline{(\partial \Delta^n \hookrightarrow \Delta^n)} \subseteq (\text{monos}).$$

• The fact that $(\text{monos}) \subseteq \overline{(\Delta^n \hookrightarrow \Delta^n)}$

Follows from the existence of
the skeletal filtration (Prop I.24)
for any monomorphism. \square

Cor 7.0: 1) Let $f: X \rightarrow Y$ in $sSet$. TFAE

a) f is a trivial fibration.

b) f has the right lifting property wrt monos.

2) Let $f: X \rightarrow Y$ be a trivial Kan fibration.

Then f admits a section $s: Y \rightarrow X$.

Moreover, the composition $s \circ f: X \rightarrow X$ is

homotopic to id_X over Y : there exists

$$X \times \Delta^1 \xrightarrow{R} X \quad \text{with} \quad \begin{cases} R|_{X \times \{0\}} = s \circ f \\ R|_{X \times \{1\}} = \text{id}_X \end{cases}$$

$\downarrow \quad \swarrow$

proof: 1): b) \Rightarrow a) is clear.

a) \Rightarrow b): Let $S = \boxed{g}$. We know

that $(\partial\Delta^n \hookrightarrow \Delta^n) \subseteq S$ by a) and we want to prove that $(\text{monos}) \subseteq S$.

We know by $\begin{cases} \text{Prop 5} & \text{that } S \text{ is saturated.} \\ \text{Prop} & \text{that } (\text{monos}) = \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}. \end{cases}$

and we are done.

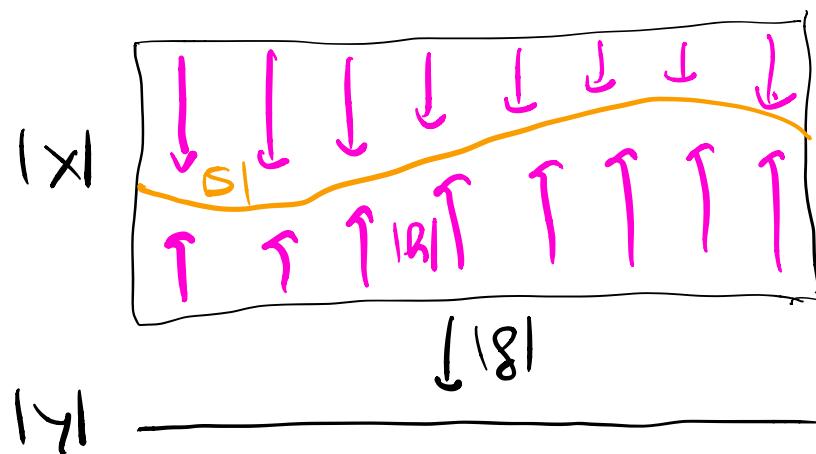
2) s (resp. R) is solution of:

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & X \\ \text{mono} \downarrow & \swarrow s & \downarrow g \\ Y & \xlongequal{\quad id \quad} & Y \end{array} \quad \begin{array}{ccc} \partial\Delta' \times X & \xrightarrow{(s \circ g, \text{id})} & X \\ \text{resp.} \downarrow & \swarrow R & \downarrow g \\ \Delta' \times X & \xrightarrow{\quad} & Y \end{array}$$

which exist by part 1). □

Rmk: Part 2) means that, not only are

the fibers of a trivial fibration contractible Kan complexes, but they can be contracted "simultaneously": after geometric realisation:



Our next objective is to get similar description of inner/left/right anodyne maps as saturated classes.

Prop 11: (Small object argument)

Let S be a set of morphisms in $sSet$.

Then every morphism g in $sSet$ admits a factorisation $g = p j$ with $j \in \bar{S}$ and $p \in S$. \square

Moreover this factorisation can be made
functional in f .

Proof: I give the proof under a stronger
assumption: $S = \{A_i \rightarrow B_i\}_{i \in I}$ with
 A_i has finitely non-degenerate simplices.

This assumption is satisfied in our applications
and means that we can replace transfinite
compositions with ω -indexed ones.

- We produce a first factorisation of $f: X \rightarrow Y$,

$$X \xrightarrow{Lg} Eg \xrightarrow{Rg} Y \quad \text{with } Lg \in \bar{S}$$

but not yet Rg in S . \square

Consider the set:

$$[S, g] = \left\{ (s_i, u, v) \mid s_i \in S, f_u = vs_i \right\} = \left\{ \begin{array}{c} A_i \xrightarrow{u} X \\ s_i \downarrow \\ B_i \xrightarrow{v} Y \end{array} \middle| g \right\}$$

We define Eg, Lg, Rg via the diagram:

$$\begin{array}{ccc}
 \varprojlim_{(s_i, u, v)} A_i & \xrightarrow{(\cup)} & X \\
 \downarrow & & \downarrow Lf \\
 \varprojlim_{(s_i, u, v)} B_i & \xrightarrow{\quad} & Ef \\
 & & \xrightarrow{Rf} Y
 \end{array}$$

By construction, $Lf \in \overline{S}$.
 Lemma 6

- We iterate the construction, applying it to Rf .

$$X \rightarrow Ef \rightarrow E^2 f \rightarrow \dots \rightarrow Y$$

Define $E^\infty f := \operatorname{colim}_{n \in \mathbb{N}} E^n f$ to get a factorisation $X \xrightarrow{j} E^\infty f \xrightarrow{p} Y$ of f .

Again by construction, we have $j \in \overline{S}$,

so all that remains is to show $p \in S \square$.

Consider a map $A_i \xrightarrow{f} B_i$ in S and a

lifting diagram

$$\begin{array}{ccc} A_i & \longrightarrow & E^\infty f \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & Y \end{array}$$

- By the assumption on A_i , as we have seen in the proof of Prop II.7, A_i is compact in $sSet$: the functor

$$sSet(A_i, -) : sSet \longrightarrow \text{Set}$$

commutes with filtered colimits.

In particular,

$$\underset{n \in \mathbb{N}}{\text{colim}} sSet(A_i, E^n f) \hookrightarrow sSet(A_i, E^\infty f)$$

is a bijection. So we get $n \in \mathbb{N}$ and:

$$\begin{array}{ccccc} A_i & \longrightarrow & E^n f & \longrightarrow & E^\infty f \\ \downarrow & & \circledast & & \downarrow \\ B_i & \longrightarrow & Y & & \end{array}$$

So $\star \in [S, E^h g \rightarrow Y]$ and by definition of $E^{h+1} g$, there is a factorisation

$$\begin{array}{ccccc}
 A_j & \xrightarrow{\quad} & E^h g & \xrightarrow{\quad} & E^\infty g \\
 \downarrow & & \downarrow & & \downarrow \\
 B_j & \xrightarrow{\quad} & E^{h+1} g & \xrightarrow{\quad} & Y
 \end{array}$$

A dashed orange arrow goes from A_j to $E^{h+1} g$.
 A solid orange arrow goes from $E^h g$ to $E^{h+1} g$.
 A solid black arrow goes from $E^{h+1} g$ to Y .

and we are done. □

Rmk: As you can see, this argument is quite formal and does not use much about $sSet$. It is very useful in the context of constructing model categories, which is what we are doing piece by piece (working towards the so-called Joyal model structure on $sSet$).

- The small object argument relies on compactness and ultimately the behaviour of Set wrt filtered colimits. This is why saturated are much

more manageable than "cosaturated" ones.

Cor 12: Let S be any set of maps in $sSet$.

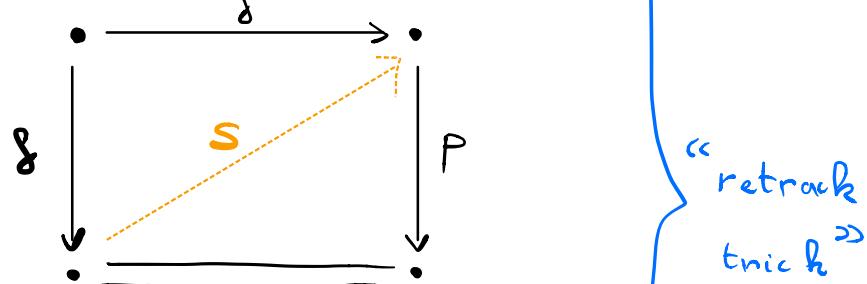
$$\text{Then } \bar{S} = {}^\square(S^\square).$$

Proof: We know by Lemma 2 and Proposition 7

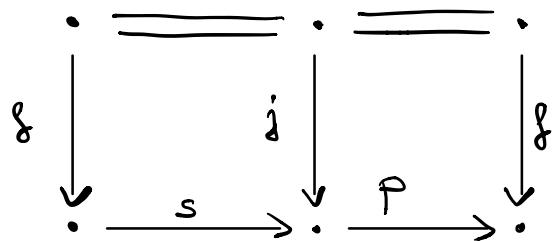
$$\left\{ \begin{array}{l} S \subseteq {}^\square(S^\square) \\ {}^\square(S^\square) \text{ is saturated} \end{array} \right. \Rightarrow \bar{S} \subseteq {}^\square(S^\square).$$

Conversely, let $g \in \bar{S}$ and apply Prop 11 to get $g = p_j$ with $j \in \bar{S}$ and $p \in S^\square$.

By $g \in p$, there is a lift in



Hence:



shows that g is a retract of j , hence in \overline{S}



def 13: A **weak factorisation system** in a category C is a pair (L, R) of classes of maps such that :

- every morphism g factors as $g = rl$ with $r \in R$ and $l \in L$.
- $L = {}^\perp R$ and $R = L^\perp$.

A weak factorisation where the factorisation is unique is an **orthogonal factorisation system**.

The small object argument tells you that

$(\overline{S}, {}^\perp(S^\perp))$ is a weak factorisation system.

Let's apply this to our anodyne classes.

Cor 14: We have

$$(\text{inner anodyne}) = \overline{(\Lambda_k^n \hookrightarrow \Delta^n, 0 < k < n)}$$

$$(\text{left anodyne}) = \overline{(\Lambda_k^n \hookrightarrow \Delta^n, 0 \leq k < n)}$$

$$(\text{right anodyne}) = \overline{(\Lambda_k^n \hookrightarrow \Delta^n, 0 < k \leq n)}$$

$$(\text{anodyne}) = \overline{(\Lambda_k^n \hookrightarrow \Delta^n, 0 \leq k \leq n)}$$

proof: Apply corollary 12 to each class of
horn inclusions. □

Let us discuss some concrete examples of
inner anodyne morphisms.

For $I \subseteq [n]$, we have the I -horn:

$$\Delta_I^n = \bigcup_{i \notin I} \Delta^{[n] \setminus i} \hookrightarrow \Delta^n$$

Lemma 15: a) Let $\emptyset \neq I \subseteq [n]$. Then

$$\Delta_I^n \hookrightarrow \Delta^n \text{ is:}$$

- anodyne if $I = [n]$
- left anodyne if $n \notin I$
- right anodyne if $0 \notin I$
- inner anodyne, if I is not the complement of an interval.

b) The map $I^n \hookrightarrow \Delta^n$ (spine inclusion)

is inner anodyne.

proof: a) Let $\emptyset \neq I \ni i$ and $I' := I \setminus \{i\}$.

There is a pushout diagram:

$$\begin{array}{ccccc}
 \Delta^{[n]-i} & \cap & \wedge^n_{\mathbb{I}} & \longrightarrow & \Delta^{[n]-i} \\
 \downarrow & & & & \downarrow \\
 (\square) & & & & \\
 & & & & \\
 \wedge^n_{\mathbb{I}'} & \xrightarrow{\Gamma} & \wedge^n_{\mathbb{I}'} & \longrightarrow & \Delta^n
 \end{array}$$

The top horizontal map is \simeq to the map $\wedge^{[n]-i}_{\mathbb{I}'} \hookrightarrow \Delta^{[n]-i}$.

Using this, it is easy to show a) by induction on $|I'|$ (choosing i carefully in the inner anodyne case).

b) $I^n \hookrightarrow \Delta^n$ can be factored as

$$I^n \xrightarrow{f_n} \Delta^{[n]-0} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on n that both f_n and g_n are inner anodyne.

It is easy to check for $n \leq 2$.

For $n \geq 3$, we have a pushout diagram

$$\begin{array}{ccc} I^{[n] \setminus 0} & \longrightarrow & \Delta^{[n] \setminus 0} \\ \downarrow & & \downarrow \\ I^n & \xrightarrow{\delta_n} & \Delta^{[n] \setminus 0} \cup I^n \end{array}$$

So δ_n is a pushout of a smaller spine inclusion.

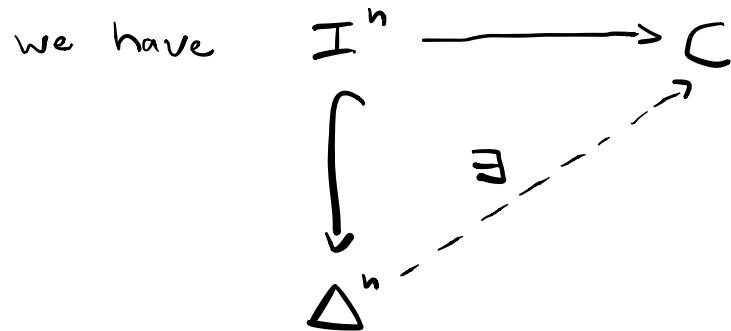
$\Rightarrow \delta_n$ inn. anodyne. Consider the pushout diagram:

$$\begin{array}{ccc} \Delta^{[n] \setminus \{0,n\}} \cup I^{[n] \setminus n} & \xrightarrow{g_{n-1}} & \Delta^{[n] \setminus n} \\ \downarrow & & \downarrow \\ \Delta^{[n] \setminus 0} \cup I^n & \xrightarrow{\quad} & \Delta^{[n] \setminus 0} \cup \Delta^{[n] \setminus n} \xrightarrow{\oplus} \Delta^n \\ \text{---} & & \text{---} \\ & & g_n \end{array}$$

\oplus is $\Delta^{[n] \setminus \{0,n\}} \hookrightarrow \Delta^n$ which is inner anodyne

by part a). By induction, g_n is inner anodyne. \square

cor 16: Let C be an ∞ -category. Then



“Infinity-categories admit compositions.”

2) Pushout-products and pullback-Roms

Our goal is to prove that $\text{Fun}(K, C)$ is an ∞ -category whenever K is, and also to prove a beautiful alternative characterisation of ∞ -categories.

def 17: Let C be a cartesian closed category

which admits pushouts (e.g $C = \text{sSet}$).
pullbacks

Let $f: X \rightarrow Y$ and $i: A \rightarrow B$ two morphisms.

We define the pushout-product of f and i by

$$g \otimes i : (A \times Y) \amalg (B \times X) \xrightarrow{A \times X} B \times Y$$

through the diagram:

$$\begin{array}{ccc}
 A \times X & \xrightarrow{g \times id} & B \times X \\
 \downarrow id \times i & & \downarrow \\
 A \times Y & \xrightarrow{\quad} & (A \times Y) \amalg (B \times X) \\
 & \searrow g \otimes i & \downarrow id \times i \\
 & & B \times Y
 \end{array}$$

- Dually, we define the pullback-Hom

$$\langle g, i \rangle : \underline{\text{Hom}}(B, X) \longrightarrow \underline{\text{Hom}}(A, X) \times \underline{\text{Hom}}(B, Y)$$

$$\underline{\text{Hom}}(A, Y)$$

through the diagram:

$$\begin{array}{ccccc}
 \underline{\text{Hom}}(B, X) & \xrightarrow{i^*} & & & \\
 \downarrow & \searrow \langle g, i \rangle & & & \\
 \underline{\text{Hom}}(A, X) \times \underline{\text{Hom}}(B, Y) & \xrightarrow{\quad} & \underline{\text{Hom}}(A, X) & & \\
 \downarrow \underline{\text{Hom}}(A, Y) & & & & \downarrow g^* \\
 & & & & \\
 \underline{\text{Hom}}(B, Y) & \xrightarrow{i^*} & \underline{\text{Hom}}(A, Y) & & \square
 \end{array}$$

- Rmk:
- \boxtimes is symmetric, unlike $<,>$.
 - The pullback-hom can be thought of as an “enriched” version of a lifting problem; for instance in $s\text{Set}$, we have

$i \boxtimes g \iff \langle g, i \rangle$ is surjective on vertices.

“The target of $\langle g, i \rangle$ parametrizes families of commutative squares $\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & \nearrow & \downarrow g \\ \bullet & \xrightarrow{\quad} & \bullet \end{array}$ while the source parametrizes families of

liftings $i \begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & \nearrow & \downarrow g \\ \bullet & \xrightarrow{\quad} & \bullet \end{array} .$

Lemma 18:

a) Let $F : C \rightleftarrows D : G$ be an adjunction.

Then we have an equivalence of lifting problems:

$$\begin{array}{ccc} FA & \longrightarrow & C \\ \downarrow & \swarrow ? \quad \searrow ? & \downarrow \\ FB & \longrightarrow & D \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \longrightarrow & GC \\ \downarrow & & \downarrow \\ B & \longrightarrow & GD \end{array}$$

b) With the notations of def 17, if we have

morphisms $\begin{cases} f : X \rightarrow Y \\ i : A \rightarrow B \\ g : S \rightarrow T \end{cases} \quad \text{in } C$

we have an equivalence of lifting problems

$$\begin{array}{ccc} S & \xrightarrow{\underline{\text{Hom}}(B, X)} & (A \times T) \underset{A \times S}{\amalg} (B \times T) \longrightarrow X \\ g \downarrow & \swarrow ? \quad \searrow ? & \downarrow i \otimes g \\ T & \xrightarrow{\underline{\text{Hom}}(A, X) \times \underline{\text{Hom}}(B, Y)} & B \times T \longrightarrow Y \end{array}$$

In other words, $g \square \langle f, i \rangle \Leftrightarrow (i \otimes g) \square f$.

Proof: a) is a direct application of the natural isomorphism $D(F(-), -) \simeq C(-, G(-))$.

b) follows from a) applied to the adjunction

$B \times - \dashv \underline{\text{Hom}}(B, -)$ and the definition of the maps $\langle f, i \rangle$ and $i \otimes g$.



Here is the technical heart of this section:

Prop 19: Let i, g be monomorphisms in $sSet$. Assume that one of i or g is inner anodyne (resp. left anodyne, right anodyne, anodyne).

Then so is $i \otimes g$.



The proof is quite involved. We start with:

Lemma 20: Let S, T be two sets of maps in $sSet$. Then

$$\overline{S \boxtimes T} \subseteq \overline{S \boxtimes T}$$

proof: Let $F = (S \boxtimes T)^\square$.

By the small object argument, we have

$$\overline{S \boxtimes T} = \square F, \text{ so we have to prove}$$

$(\overline{S \boxtimes T}) \boxtimes F$. We first show

$(\overline{S \boxtimes T}) \boxtimes F$. Consider

$$A = \{ a \mid (a \boxtimes T) \boxtimes F \}$$

By Lemma 18.b), we have

$$A = \{ a \mid a \boxtimes \langle F, T \rangle \}$$

So \mathcal{A} is a left-complement, and in particular is saturated by Prop 7.

$S \subseteq \mathcal{A}$, hence $\bar{S} \subseteq \mathcal{A}$ and we have

$$(\bar{S} \boxtimes T) \boxtimes F.$$

A similar argument with

$$\begin{aligned}\mathcal{B} &= \{b \mid (\bar{S} \boxtimes b) \boxtimes F\} \\ &= \{b \mid b \boxtimes \langle F, \bar{S} \rangle\}\end{aligned}$$

shows that $(\bar{S} \boxtimes \bar{T}) \boxtimes F$. □

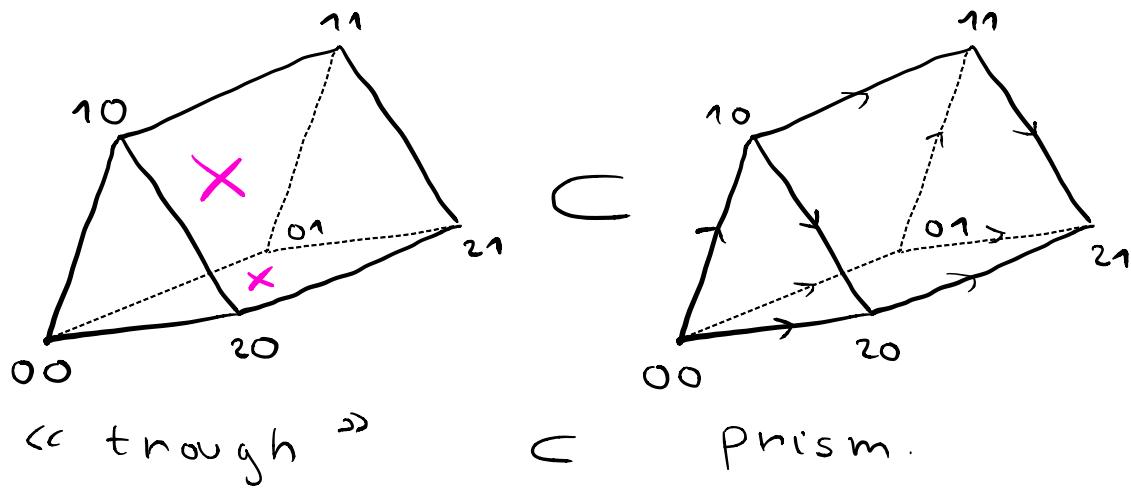
Lemma 21: We have

$$\begin{aligned}(\Lambda_k^n \hookrightarrow \Delta^n \mid 0 < k < n) \boxtimes (\delta \Delta^n \hookrightarrow \Delta^n) \\ \cap \\ (\text{inner anodyne maps})\end{aligned}$$

Proof: This is really difficult combinatorics!

Following Rezk, I will just explain a special case: $(\Lambda_1^2 \subset \Delta^2) \boxtimes (\partial \Delta^1 \subset \Delta^1)$.

$$(\Lambda_1^2 \times \Delta^1) \amalg (\Delta^2 \times \partial \Delta^1) \subset \Delta^2 \times \Delta^1$$
$$\Lambda_1^2 \times \partial \Delta^1$$

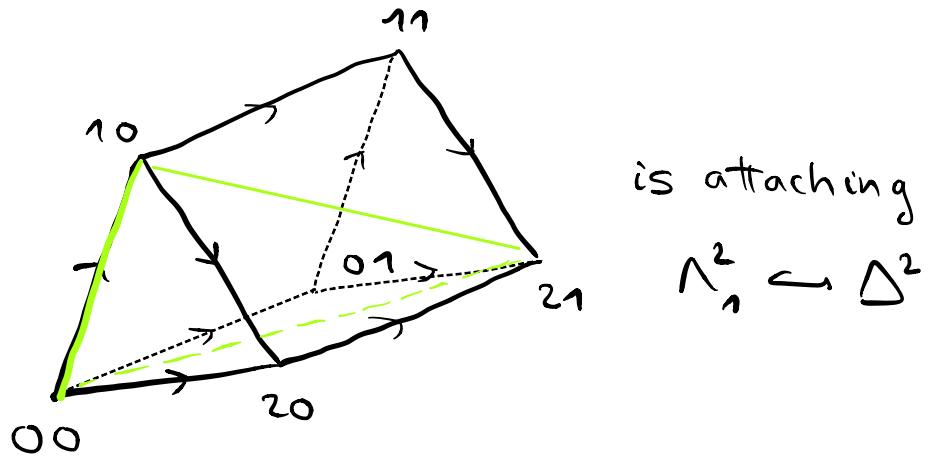


We attach the following simplices in order :

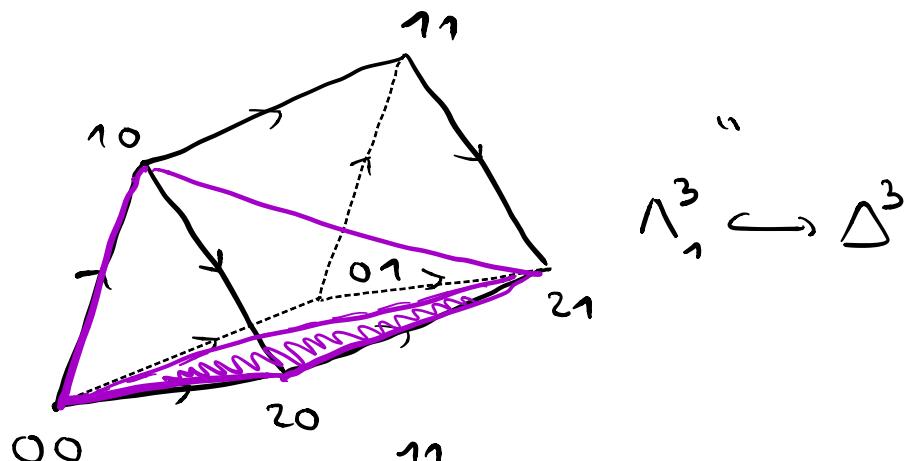
- ① $\langle 00, 10, 21 \rangle$
- ② $\langle 00, 10, 20, 21 \rangle$
- ③ $\langle 00, 10, 11, 21 \rangle$
- ④ $\langle 00, 01, 11, 21 \rangle$

In each case, this is a pushout along an inner horn. This can be seen geometrically with a little patience!

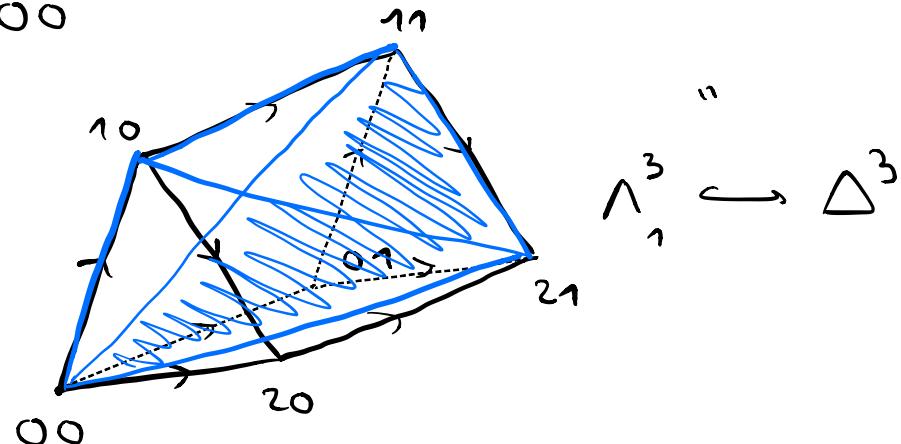
(1)



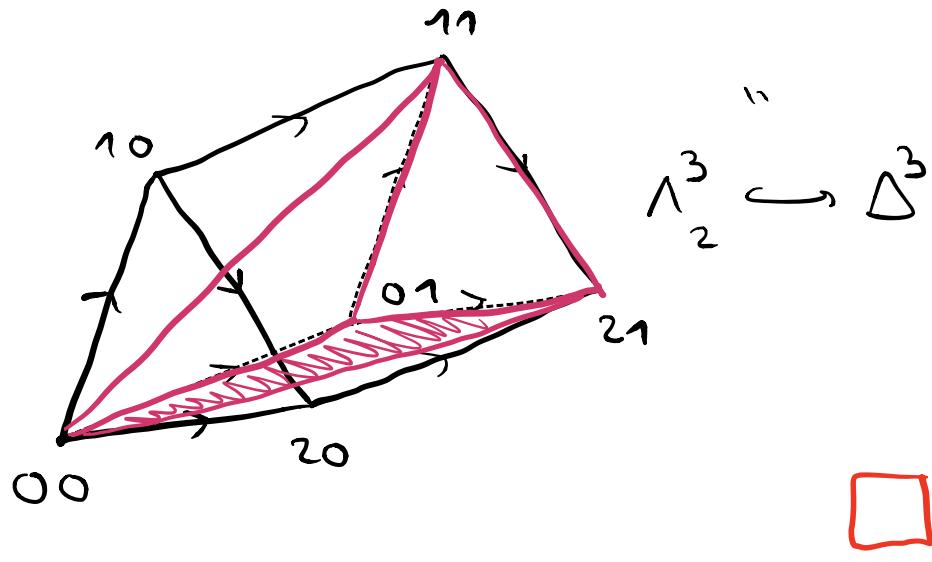
(2)



(3)



(4)



proof of 1g:

I only explain the case of inner anodyne maps. The others use variants of Lemma 20 which I don't want to get into; moreover for the application in this section, this is the case we need.

We know that

$$(\text{monos}) = \overline{(\delta \Delta^n \hookrightarrow \Delta^n)}$$

by Prop 9.

We see that

$$(\text{inner anodyne}) \otimes (\text{monos}) \\ = \overline{(\Lambda_h^n \hookrightarrow \Delta^n | 0 < h < n)} \otimes \overline{(\partial \Delta^n \hookrightarrow \Delta^n)}$$

Lemma 20

$$\subseteq \overline{(\Lambda_h^n \hookrightarrow \Delta^n | 0 < h < n)} \otimes \overline{(\partial \Delta^n \hookrightarrow \Delta^n)}$$

Lemma 21

$$\subseteq (\text{inner anodyne})$$



Now the hard work is done, and we get:

Prop 22: Let $f: X \rightarrow Y$ be an

(inner, left, right) fibration and $i: A \rightarrow B$

a monomorphism. Then

1) $\langle f, i \rangle$ is an (inner, left, right) fibration.

2) If i is moreover (inner, left, right) anodyne,
then $\langle f, i \rangle$ is a trivial fibration.

proof: This follows directly from Lemma 18.b) and Prop 19. □

There are many interesting special cases :

cor 23:

a) if $i: A \rightarrow B$ is inner anodyne, so

is $i \times id: A \times K \rightarrow B \times K$

b) if $p: X \rightarrow Y$ is an inner fibration,

so if $p_*: \text{Fun}(Z, X) \rightarrow \text{Fun}(Z, Y)$

c) if $i: A \rightarrow B$ is a mono and C

is an ∞ -category, then

$i^*: \text{Fun}(B, C) \rightarrow \text{Fun}(A, C)$

is an inner fibration.

d) If $i: A \rightarrow B$ is inner anodyne and

C is an ∞ -category, then

$$j^*: \text{Fun}(B, C) \rightarrow \text{Fun}(A, C)$$

is a trivial fibration.

e) If $\begin{cases} K \text{ is any simplicial set} \\ C \text{ is an } \infty\text{-category} \end{cases}$

then C is an ∞ -category

proof: They are all obtained by applying

Prop 22 to special cases with $A = \emptyset$

or $\gamma = \Delta^\circ$. For instance case b)

follows from 22. a) with $A = \emptyset$.

and case e) follows from b) with

$$\gamma = \Delta^\circ.$$

