

III Minimal regular models

Setup S (integral) Dedekind scheme

C/S smooth projective connected curve

By applying resolution of singularities to any proper flat model of C (see next chapter)

we get X/S proper flat with X regular connected.
and $X_S \cong C$.

Goal: Modify X to make it "minimal".

The resulting theory is very similar to the study of minimal models of smooth projective surfaces over an alg. closed field (see [Hartshorne, IV]).

The two theories can be developed in parallel, and the fibered case is somewhat easier because vertical divisors play a distinguished role.

def 1 A fibered surface is a proper flat morphism
 $g: X \rightarrow S$ with $\begin{cases} X \text{ 2-dim. noetherian} \\ S \text{ Dedekind scheme.} \end{cases}$

• Fibers of such a morphism are proper curves over general fields, which can be very singular. (For X regular, they are at least l.c.i).

mult X curve. X reduced $\Leftrightarrow X$ l.c.i $\Rightarrow \omega_{X/R}$ exists and
 $\begin{matrix} \text{(over field)} & \checkmark \Downarrow & (S1) & \Downarrow [\text{Lm 8.2.18}] & \text{is invertible.} \end{matrix}$
 X has no embedded $\Leftrightarrow X$ Cohen-Macaulay $\Rightarrow \omega_{X/R}$ exists.

1) Degree of divisors on singular curves:

• In this section, X is a proper curve on a field R (not nec. irreducible or reduced).

• Recall that if A is a noetherian 1-dim ring and $f \in A$ is not a zero-divisor, then the length $\text{len}_A(A/f)$ is finite, and that

$$\text{len}_A(A/fg) = \text{len}_A(A/f) + \text{len}_A(A/g) \quad [\text{Lm, Lemma 7.1.26}].$$

def 2 Let $x \in X^{(0)}$ and $f \in \mathcal{O}_{X,x}$ non-zero divisor. The multiplicity of f at x is
 $\text{mult}_x(f) := \text{len}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f) < \infty$.

• Because of additivity, we can extend mult_x to the total ring of fractions.

def 3 Let D be a Cartier divisor on X . The multiplicity of D at x is $\text{mult}_x(D) := \text{mult}_x(f_x) - \text{mult}_x(g_x)$ for $f_x/g_x \in \text{frac}(O_{X,x})$ local representative of D .

• The associated Weil divisor is then $\sum_{x \in X^{(0)}} \text{mult}_x(D) \cdot [x] \in \mathbb{Z}^1(X)$.

def 4 The (total) degree of D is $\deg(D) := \sum_{x \in X^{(0)}} \text{mult}_x(D) \cdot [\kappa(x) : k]$ (Δ depends on k)

• We have $\begin{cases} \deg(D_1 + D_2) = \deg(D_1) + \deg(D_2) \\ \deg(D) = \dim_k H^0(X, O_D) \text{ if } D \text{ is effective.} \end{cases}$

• The basic fact of life, as in the case of smooth curves, is:

thm 5 (Riemann-Roch) $\chi(O_X(D)) = \deg D + \chi(O_X)$.

The proof is the same as in the smooth case, by reduction to the effective case. [Liu, 7.3.17].

cor 6 Let $g \in \kappa(x)$. Then $\deg(\text{div}(g)) = 0$. (proof: $\mathcal{O}(\text{div } g) \cong \mathcal{O}$ via g)

def 7 Let \mathcal{L} be a line bundle on X . Its (total) degree is $\deg(\mathcal{L}) := \chi(\mathcal{L}) - \chi(O_X)$.

def 8 The arithmetic genus of X is $p_a(X) := 1 - \chi(O_X)$.

• To go further in the study of RR, want to apply Serre duality.

prop (RR + duality)
 X proper CM curve
 $\dim_k H^0(X, O_X(D)) - \dim_k H^0(X, \omega_{X/k}(-D)) = \deg D + 1 - p_a(X)$.

def X proper l.c.i. curve. A Cartier divisor $K_{X/k}$ with $\mathcal{O}(K_{X/k}) \cong \omega_{X/k}$ (which is invertible in this case) is called a canonical divisor.

X proper l.c.i.
 $\deg(\omega_{X/k}) = 2(p_a - 1)$.

prop 10 $\dim_k H^0(X, \omega_{X/k}) = p_a$ if X is geom. red. and geom. connected.
[Liu, 7.3.31]

- Clearly, if X is not irreducible, then the total degree is a rather weak invariant. However:

Prop 11 | a) X is projective.
 b) Let D be a Cartier divisor on X . Then
 D ample $\Leftrightarrow \forall Y$ irreduc. comp of X , $\deg(D_{|Y}) > 0$.

idea of proof [Liu, ex. 4.1.16, 7.5.3, 7.5.4]

- The order of proof is: (a) for normal curves \Rightarrow b) \Rightarrow a).
- a) for normal curves is proved by patching embeddings of affine opens:
 If $C = \bigcup U_i$ affine open cover and $U_i \hookrightarrow Y_i$ with Y_i/\mathbb{A} projective,
 then the natural map $\bigcap U_i \rightarrow \prod Y_i$ extends to C by normality and valuative criterion and gives a projective embedding.
- b): By Serre's criterion, enough to show that for all $\bar{j}^* \in \text{Coh}(X)$ and $n \gg 0$, $H^1(X, \bar{j}^* \otimes \mathcal{O}(nD)) = 0$.
- Let $\pi: X' \rightarrow X$ be the normalization. Then $\deg(\pi^*\mathcal{O}(nD)) = \deg(\mathcal{O}(nD))$
 Hence (by the usual argument, since X' is projective) $\pi^*\mathcal{O}(nD)$ is ample / to be precise, need to do this for each irreduc. comp of X' .

We have a short exact sequence

$$(+) \quad 0 \rightarrow \pi_* \pi^! \bar{j}^* \rightarrow \bar{j}^* \rightarrow \begin{cases} 0 \\ \text{sheaf} \end{cases} \rightarrow 0 \quad \text{with } \pi^! \bar{j}^* := \mathcal{H}_{\text{om}}^0(\pi_* \mathcal{O}_{X'}, \bar{j}^*)$$

equipped with its natural $\mathcal{O}_{X'}$ -module structure (π finite)

$$\text{and } H^1(X, (\pi_* \pi^! \bar{j}^*) \otimes \mathcal{O}(nD)) \underset{\text{projection formula}}{\cong} H^1(X, \pi_*(\pi^! \bar{j}^* \otimes \pi^*\mathcal{O}(nD)))$$

$$\underset{\pi \text{ finite}}{\cong} H^1(X', \pi^! \bar{j}^* \otimes \pi^*\mathcal{O}(nD))$$

$$\underset{\pi^*\mathcal{O}(nD) \text{ ample}}{\cong} 0 \quad \text{for } n \gg 0$$

- The result then follows from the LES of (+).

- a): it is then enough to construct an effective Cartier divisor which meets every irreducible component of X . But, given any locally noetherian scheme X and any non-associated point x , there is an effective Cartier divisor on X with support containing x . \square

• Application to fibered surfaces:

Thm 12 | (Lichtenbaum [Lin, 8.3.16])

Let $f: X \rightarrow S$ be a regular fibered surface (i.e. X regular)

Then f is locally projective.

Proof: Let $\pi: Y \rightarrow T$ be a proper morphism of noetherian schemes, and L be a line on Y . We need the following classical facts:

- If L is ample, then π is projective (in the sense of EGA)
- If for $t \in T$, L_t on X_t is ample, then $\exists U \ni t$ open with $L|_{\pi^{-1}(U)}$ ample.

• We can assume X connected $\Rightarrow X_y$ connected.

• Let $x \in X_y^{(0)}$ be a closed point. Then $D_0 = \overline{\{x\}}$ is a

Weil = Cartier divisor on X . Since $(D_0)_y$ is ample by X connected, there exists $U \subseteq S$ non-empty open with ($s \in U \Rightarrow (D_0)_s$ ample)

Let $S \setminus U = \{s_1, \dots, s_n\}$.

Lemma 13 | There exists an effective divisor D_i which meets all irreducible components of X_{s_i} .

Proof: It is enough to construct, for any $g \in S^{(0)}$ and $x \in X_g^{(0)}$, an effective Weil divisor D which contains x . Let m_x be the maximal ideal of $\mathcal{O}_{X,x}$ and P_1, \dots, P_n the prime ideals of $\mathcal{O}_{X,x}$ corresponding to the irreduc. components of X_g containing x . Then $m_x \not\subseteq \bigcup P_i$ (Prime avoidance lemma); indeed by induction $\exists g_i \in m_x \setminus \bigcup_{j \neq i} P_j$, and then $g_1 \cdots g_{n-1} + g_n \notin \bigcup P_i$ using induction and P_n prime.

Pick $g \in m_x \setminus (\bigcup P_i)$, U open with $g \in \mathcal{O}_X(U)$ representing x . Then $x \in V(g)$, so any irreducible component of $V(g)$ passing through x works. \square

Then $D := D_0 + D_1 + \dots + D_n$ is a Cartier divisor

such that $\forall s \in S$, $(D)_s$ is ample. \square

rmk: for $f: X \rightarrow S$ smooth or S "nice" (e.g. quasi-excellent), f has finitely many singular fibers and we can dispense with D_0 .

rmk: - Regular proper surfaces over a field are projective.

See [Lin, Remark 9.3.5] for a proof; we will sketch it later.

- There exist normal proper non-projective surfaces [Schröer].

However normal proper surfaces over a finite field are projective! [Artin, 2.1]

2) Intersection theory on a regular fibered surface

- Intersection theory in general: try to define intersection pairing $CH^i(X) \times CH^j(X) \rightarrow CH^{i+j}(X)$ on cycle groups up to nat. equivalence.
- On a surface, only interesting case is 2 divisors: $CH^1 \times CH^1 \rightarrow CH^2 = CH_0$.
- Pb 1: $CH^2(X)$ is hopelessly complicated and not so interesting for us.

Sol: only keep track of intersection degrees.

- Pb 2: without some form of properness, degrees of 0-cycles are not invariant under rational equivalence.

Sol: only allow intersections with at least one divisor proper.

- Write $\text{Div}(X)$ for the group of all Cartier divisors on X ,
 $\text{Div}_h(X)$ for the subgroup of horizontal divisors ($g|_D$ finite, objective)
for $s \in S^{(0)}$, $\text{Div}_s(X)$ for the subgroup of divisors with support in X_s .

$$\text{Div}_v(X) = \sum_{s \in S} \text{Div}_s(X) \text{ for the subgroup of } \underline{\text{vertical}} \text{ divisors.}$$

def 1: Let $D \in \text{Div}(X)$, $E \in \text{Div}_v(X)$.

Write $E = \sum n_\Gamma \cdot [\Gamma]$ with Γ running through the irreduc. components of closed fibers.

$$\text{Put } i(D, E) = \sum n_\Gamma \deg_{R(s)}((\mathcal{O}(D))_{|\Gamma}) \in \mathbb{Z}.$$

(this makes sense because $\Gamma/R(s)$ is a proper curve).

Thm 2 (i) $i: \text{Div}(X) \times \text{Div}_v(X)$ is a bilinear form.

(ii) $i: \text{Div}_v(X) \times \text{Div}_v(X)$ is symmetric.

(iii) If $D \sim D'$ (i.e. $D' - D = \text{div}(F)$ for $F \in k(X)^\times$)

we have $i(D, E) = i(D', E)$.

(iv) If D, E are effective, with no common component, we have:

$$i(D, E) = \sum_{x \in |D| \cap |E|} [k(x):k(g(x))] \cdot \text{len}_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x}/(\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x) \right) \geq 0$$

Proof: (i): • Linearity in E is by construction.
 • Linearity in D follows from additivity of degree.

$$\text{(iii): } D \sim D' \Rightarrow \mathcal{O}_X(D) \simeq \mathcal{O}_X(D') \Rightarrow i(D, E) = i(D', E).$$

(iv): • First, the condition on supports imply that in a neighbourhood of $x \in |D| \cap |E|$, the point x is the only pt of intersection of the supports. This implies

$$m_x \leq \sqrt{\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x}$$

Hence $\mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x$ is an artinian ring $\Rightarrow \text{length} < \infty$.

• D effective Cartier divisor in $X \Rightarrow D \hookrightarrow X$ l.c.i. Since X is regular, we deduce that D is Cohen-Macaulay, hence has no embedded points [Liu, section 8.2]. By this together with the condition on supports, this implies: $\exists D_{1,E}$ effective on E , with $\mathcal{O}_X(D)_{1,E} \underset{(+)}{\simeq} \mathcal{O}_E(D_{1,E})$ [Liu, lemma 7.1.29].

$$\Rightarrow i(D, E) = \deg(\mathcal{O}_E(D_{1,E})) = \sum_{x \in |E|} \text{mult}_x(D_{1,E}) \cdot [h(x) : h(0)].$$

$$\begin{aligned} \text{We have } \text{mult}_x(D_{1,E}) &= \text{len}\left(\mathcal{O}_{E,x} / \mathcal{O}_E(-D_{1,E})_x\right) \\ &\quad \text{len}\left(\mathcal{O}_{X,x} / \mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x\right) \checkmark \end{aligned}$$

(ii): Can reduce to $D = \Gamma_i$, $E = \Gamma_j$ for Γ_i, Γ_j 2 components of a fiber X_S . Then it follows from symmetry in (iv) for $i \neq j$, and it is obvious for $i = j$. □

$$\begin{array}{c|l} \text{notat}^o & D \cdot E := i(D, E) \\ & E^2 := i(E, E) \end{array}$$

rmk: Compared with case of smooth proj surface [Hartshorne, § 1]:
 - no Bertini thm to always reduce to transversality
 - still a moving lemma [Liu, 9.1.10.ii]
 but not necessary for theory.

Thm 3 (Hodge index theorem for fibered surfaces)

Let $g: X \rightarrow S$ be a regular fibered surface and $s \in S$ closed pt.

(i) For any $E \in \text{Div}_s(X)$, $i(E, X_s) = 0$.

(ii) Let Γ_i be the irreducible components of X_s , with multiplicities d_i . Then

$$\Gamma_s^2 = -\frac{1}{d_i} \sum_{j \neq i} d_j \Gamma_i \cdot \Gamma_j \leq 0$$

(iii) The bilinear form i is negative semi-definite ($\otimes \mathbb{R}$)

If X_s is connected, then the kernel of $i|_{\mathbb{R}}$ is precisely $\mathbb{R} \cdot X_s$.

Proof: (i): For $D \in \text{Div}(X)$ and $E \in \text{Div}(X_s)$ it is easy to see

that $i(D, E) = i(D|_{X_s}, E|_{X_s})$ with $X_s \rightarrow \text{Spec}(\mathcal{O}_{S,s})$.

$\mathcal{O}_{S,s}$ is a DVR $\Rightarrow s$ is a principal divisor on $\text{Spec}(\mathcal{O}_{S,s})$

$$\Rightarrow X_s \quad " \quad " \quad X_s$$

By pt (iii) in previous thm, we get $i(X_s, -) = 0$

(ii): This follows from $X_s = \sum d_i \Gamma_i$, bilinearity and $X_s^2 = 0$ by (i).

(iii): Put $a_{ij} = \Gamma_i \cdot \Gamma_j$ and $b_{ij} = d_i d_j a_{ij}$. We have

$\left\{ \begin{array}{l} b_{ij} \geq 0 \text{ if } i \neq j, \text{ and } b_{ij} > 0 \Leftrightarrow \Gamma_i \cap \Gamma_j \neq \emptyset \text{ (Thm 2(iv))} \\ \sum_j b_{ij} = X_s \cdot d_i \Gamma_i = 0 \text{ for all } i \text{ by (i), and } \sum_i b_{ij} = 0 \text{ by symmetry.} \end{array} \right.$

Let $v = \sum x_i \Gamma_i \in \text{Div}_s(X)|_{\mathbb{R}}$, $y_i := \frac{x_i}{d_i}$.

$$i(v, v) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j = \sum_{1 \leq i, j \leq n} b_{ij} y_i y_j = 2 \sum_{1 \leq i < j \leq n} b_{ij} y_i y_j + \sum_{1 \leq k \leq n} b_{kk} y_k^2$$

$$\left(\sum_i b_{ii} = 0 \right) \rightarrow = - \sum_{1 \leq i < j \leq n} b_{ij} (y_i - y_j)^2 \leq 0 \text{ with equality iff } (\Gamma_i \cap \Gamma_j \neq \emptyset \Rightarrow y_i = y_j) \quad \square$$

which implies the result.

Thm 4: Let $f: X \rightarrow Y$ be a birational morphism of fibered surfaces with X regular and Y normal. Let $y \in Y$ with $\dim X_y = 1$.

Write $\Gamma_1, \dots, \Gamma_n$ for the irreducible components of X_y .

The intersection form on $\bigoplus \mathbb{R} \cdot \Gamma_i$ is negative definite.

Proof: Let C be any effective Cartier divisor on Y with support

containing y . Write $\underbrace{f^*C}_{\Gamma} = \tilde{C} + D$ with \tilde{C} effective with no common comp. with X_y (exists by X, Y locally integral, f dominant [Liu, 7.1.33-34])

$$D = \sum d_i \Gamma_i, d_i \geq 0$$

Note that, to do this decomposition, we use "Weil = Cartier" on X .

Lemma: $\forall i, d_i > 0$ and $D \cdot \Gamma_i \leq 0$.

Moreover $\exists i_0, D \cdot \Gamma_{i_0} < 0$

Proof: Locally around y , we have $\mathcal{O}_y(C) = a^{-1}\mathcal{O}_y$ for some $a \in \mathcal{O}_y$.

Hence $\mathcal{O}_X(f^*C) = (f \circ a)^{-1}\mathcal{O}_X$, and $d_i = v_{\Gamma_i}(f \circ a) > 0$ since $a \in m_{Y,y} \cap \mathcal{O}_y$.

Since $\mathcal{O}_X(f^*C)$ is free around $f^{-1}(y)$, we get $f^*C \cdot \Gamma_i = \deg(\mathcal{O}(f^*C)|_{\Gamma_i}) = 0$.

Since \tilde{C} and Γ_i have no common components, we have $\tilde{C} \cdot \Gamma_i \geq 0$.

Hence $D \cdot \Gamma_i = (f^*C - \tilde{C}) \cdot \Gamma_i \leq 0$.

• $|\tilde{C}| \cap X_y \neq \emptyset$ for topological reasons: $f(\tilde{C})$ is a closed subset of C containing $C \setminus \{y\}$.

Let Γ_{i_0} with $|\tilde{C}| \cap \Gamma_{i_0} \neq \emptyset$. Then $\tilde{C} \cdot \Gamma_{i_0} > 0, \text{ so } D \cdot \Gamma_{i_0} < 0 \quad \square$

• Let $x_1, \dots, x_n \in \mathbb{R}$. Put $b_{ij} = d_i d_j \Gamma_i \cdot \Gamma_j$, $y_i = \frac{x_i}{d_i}$.

$$i(\sum x_i \Gamma_i, \sum x_i \Gamma_i) = \sum_{i,j} b_{ij} y_i y_j = \sum_i \left(\sum_j b_{ij} \right) y_i^2 - \sum_{i < j} b_{ij} (y_i - y_j)^2 \leq 0$$

• If there is equality, then $\begin{cases} \sum_i (\sum_j b_{ij}) y_i^2 = 0 & (1) \\ \sum_{i < j} b_{ij} (y_i - y_j)^2 = 0 & (2) \end{cases}$ $D \cdot d_i \Gamma_i \leq 0$

We then see that $y_{i_0} = 0$, and then using (2) and connectedness

of X_y (which holds by normality of Y and Zariski's main thm)

We get $y_i = 0$ for all i . \square

rmk: $X_y \subseteq X_{g(y)}$ so Thm 3 impliesⁱ the intersection form is negative semi-definite; the new content is the "definite" part, and the argument can be reformulated as showing that X_y is not a connected component of $X_{g(y)}$.

- In the proof, we have established the following.

Lemma 5 $\left| \begin{array}{l} g: X \rightarrow Y \text{ dominant morphism of regular fibered surfaces} \\ \text{Let } E \text{ be a divisor on } X \text{ with } g(\text{Supp } E) \text{ finite} \\ D \text{ be any divisor on } Y \\ \text{Then } E \cdot g^* D = 0. \end{array} \right.$

- There is an important generalization of this.

Def 6 $\left| \begin{array}{l} X, Y \text{ noetherian integral schemes} \\ g: X \rightarrow Y \text{ proper, } Z \subseteq X \text{ prime Weil divisor.} \\ g_* Z := \begin{cases} [\kappa(Z) : \kappa(g(Z))] \cdot g(Z), & [\kappa(Z) : \kappa(g(Z))] < \infty \\ 0, & \text{otherwise} \end{cases} \\ \text{This extends by linearity to a map } (\text{Weil div on } X) \rightarrow (\text{Weil div on } Y) \end{array} \right.$

Thm 7 $\left| \begin{array}{l} g: X \rightarrow Y \text{ dominant morphism of regular fibered surfaces} \\ C \text{ (resp. } D\text{) divisor on } X \text{ (resp. } Y\text{).} \\ \text{a)} \quad g_* g^* D = [\kappa(X) : \kappa(Y)] \cdot D \\ \text{b)} \quad \text{Assume either } C \text{ or } D \text{ is vertical. Then} \\ \quad C \cdot g^* D = g_* C \cdot D \quad (\text{projection formula}) \\ \text{c)} \quad \text{If } D' \text{ is a vertical divisor on } Y, \text{ then } g^* D' \text{ is vertical and} \\ \quad g^* D \cdot g^* D' = [\kappa(X) : \kappa(Y)] C \cdot D. \end{array} \right.$

[Lim, Thm 9.2.12]

2) Birational maps, blow-ups and contractions

- The notion of minimal fibered surface is defined in terms of arbitrary birational maps, but it turns out that birational maps between regular surfaces have a very simple structure.

Thm 1 | Let $f: X \rightarrow Y$ be a (proper) birational morphism between regular fibered surfaces. Then f can be written as a composition of blow-ups at closed points. The number of such blow-ups is equal to the number of irreducible components of the exceptional locus of f .

Proof: Assume f is not an isomorphism. We first show that there exists $y \in Y^{(0)}$ with $\dim(X_y) \geq 1$. Assume the opposite: then f is quasi-finite and proper, hence finite. Since it is moreover birational, f is an iso by γ normal {

• Let $y \in Y^{(0)}$ be such a point. Because $\begin{cases} X \text{ is integral, locally} \\ \dim(X) = 2 \end{cases}$, $\dim(X_y) = 1$.

• By normality of Y and f proper birational, we have $\mathcal{O}_y \xrightarrow{\sim} f_* \mathcal{O}_X$.

In this situation, Zariski's connectedness theorem tells us that the fibers of f are (geometrically) connected.

In particular, X_y has no isolated points and is of pure dimension 1.

• Because X_y is of pure codimension 1 in X regular, \mathcal{I}_{X_y} is invertible.
 $(\Rightarrow \text{loc. factorial})$

• Let $\tilde{Y} = \text{Bl}_y Y$ be the blow-up of Y at the closed point y .

Blow-ups are characterized by universal property [Lin, Corollary 8.1.16].

Prop 2 | Let T be a noetherian scheme and \mathcal{I} a coherent sheaf of ideals.

Then $\pi: \tilde{T} = \text{Bl}_{\mathcal{I}(T)} T \rightarrow T$ is characterized by the following

property: for any $p: W \rightarrow T$ such that $(\theta^{-1}\mathcal{I})|_{W_W}$ is invertible.

$$\exists! q: W \rightarrow \tilde{T} \text{ with } W \xrightarrow{q} \tilde{T} \quad \begin{array}{c} \uparrow \pi \\ p \searrow \end{array} .$$

• By the above, f factors as $f: X \rightarrow \tilde{Y} \rightarrow Y$ and we win \square

rmk: - In particular, f is ^(locally) projective, which we already knew by Thm 1.12).
 (Recall that if $g \circ h$ is ^{locally} projective and g is ^{locally} projective then h is ^{locally} projective)

Recollection on blow-ups and sketch of universal property

def 3 X noetherian, \mathcal{I} coherent sheaf of ideals.

Then $\bigoplus_{n \geq 0} \mathcal{I}^n$ is a graded \mathcal{O}_X -algebra.

We put $\text{Bl}_{\mathcal{I}} X := \text{Proj}_{X^n \geq 0} (\bigoplus_{n \geq 0} \mathcal{I}^n) \longrightarrow X$

$\text{Bl}_{\mathcal{I}} X$ has a natural invertible sheaf $\mathcal{O}(1)$ which corresponds to the graded ideal sheaf $\bigoplus_{n \geq 0} \mathcal{I}^{n+1}$.

• We have that $(\pi^{-1}) \mathcal{O}_{\text{Bl}_{\mathcal{I}} X} \simeq \mathcal{O}(1)$ hence is invertible.

• Let us sketch the proof of the universal property of the blow-up. Let $p: W \rightarrow X$ such that $(p^{-1}) \mathcal{O}_W$ is invertible. Then

$$\begin{aligned} \text{Bl}_{(p^{-1}) \mathcal{O}_W} W &= \text{Proj}_{\mathcal{O}_W} \left(\bigoplus_{n \geq 0} ((p^{-1}) \mathcal{O}_W)^n \right) \\ &\simeq \text{Proj}_{\mathcal{O}_W} \left(\bigoplus_{n \geq 0} \mathcal{O}_W^n \right) \quad (\text{multiply by } \left(g^{-n} \right)_{n \geq 0} \text{ with } g \\ &\quad \text{local generator of the invertible ideal}) \\ &= W. \end{aligned}$$

So it is enough to show that, without assuming $(p^{-1}) \mathcal{O}_W$ invertible, there is a unique morphism which fits in a commutative diagram.

$$\begin{array}{ccc} \text{Bl}_{(p^{-1}) \mathcal{O}_W} W & \longrightarrow & \text{Bl}_{\mathcal{I}} X \\ \downarrow & = & \downarrow \\ W & \longrightarrow & X \end{array}$$

The construction of the map simply comes from the functoriality of the Proj construction for the morphism $\bigoplus_{n \geq 0} \mathcal{I}^n \longrightarrow \bigoplus_{n \geq 0} ((p^{-1}) \mathcal{O}_W)^n$ induced by pullback of functions.

The uniqueness is not essential for us and will be omitted.

con: X, Y regular fibered surfaces, $g: X \dashrightarrow Y$ birational map. There exists Z regular fibered and a commutative diagram $\begin{array}{ccc} & Z & \\ p \swarrow & \downarrow q & \\ X & \dashrightarrow & Y \\ & \searrow g & \end{array}$ with p, q composition of blow-ups at closed points.

mod: Put $\Gamma = \text{closure of the graph of a representative of } g \text{ in } X \times Y$. Then $\Gamma_y \hookrightarrow X_y$ by normality of X_y , hence Γ has a resolution of singularities $Z \rightarrow \Gamma$ (see later chapter). We then apply the previous theorem to $Z \xrightarrow{\Gamma} X \xrightarrow{g} Y$ \square

This justifies a closer look at the intersection theory on blow-ups.

def 4 Let $\pi: \tilde{X} \rightarrow X$ be a projective birational morphism of regular integral schemes. Let D be an effective Cartier divisor on X .
 Let $F = \{x \in X \mid \dim \tilde{X}_x \geq 1\}$. The strict transform \tilde{D} of D is the scheme-theoretic closure of $\pi^{-1}(D \setminus F)$ in \tilde{X} .

Prop 5 Let X be a regular fibered surface, $\pi: \tilde{X} \rightarrow X$ the blow-up at a closed point x . Let $E \subseteq \tilde{X}$ be the exceptional divisor. Then

$$\pi^* D = \tilde{D} + \nu_x(D) \cdot E \quad \text{with } \nu_x(D) = \max \left\{ n \geq 0 \mid g \in m_x^n \text{ for } g \text{ local eq. of } D \text{ at } x \right\}$$

Proof: $\pi^* D - \tilde{D}$ is supported on $E \Rightarrow \exists n \in \mathbb{Z}, \pi^* D = \tilde{D} + nE$.

We have $n = \text{mult}_{\tilde{x}}(F)$ for \tilde{x} gen. point of E , F local eq. of $\pi^* D$.

Let $U = \text{Spec } A$ be an affine open neighbourhood of x in X such that $m_x = (a, b)$ and $\mathcal{O}_U(-D_U) = (g)$.

$$\begin{aligned} \text{Then } \pi^{-1}(U) &= \text{Proj} \left(A[\frac{a}{t}] /_{(a)} \right) \quad \text{with } w = \frac{v}{u}, w' = \frac{u}{v} \\ &= \underbrace{\text{Spec} \left(A \left[\frac{t}{a} \right] \right)}_w \cup \underbrace{\text{Spec} \left(A \left[\frac{a}{t} \right] \right)}_{w'} \end{aligned}$$

$$\mathcal{O}_w(-E|_W) = (a)$$

Write $g = P(a, b) + Q$ with $P(a, b) \neq 0$ homogeneous of degree $\nu_x(D)$ and $Q \in m_x^{\nu_x(D)+1}$.

Then $F = \pi^* g = a^{\nu} \cdot P(1, \frac{b}{a}) + a^{\nu+1} g$, $g \in \mathcal{O}_{\tilde{X}}(w)$ on the blow-up

$$\mathcal{O}_{\tilde{X}}(w) /_{(a)} \simeq k(x) \left[\frac{t}{a} \right] \Rightarrow P(1, w) \notin \cup \mathcal{O}_{\tilde{X}}(w) = \mathcal{O}_{\tilde{X}}(-E)(w)$$

$$\Rightarrow \text{mult}_{\tilde{x}}(\pi^* D) = \text{mult}_{\tilde{x}}(g) = \nu \text{ with } \tilde{x} \mid \text{generic point of } W.$$

$$\text{Moreover, } \tilde{D} \cap W = (P(1, \frac{b}{a}) + ag).$$

□

Prop 6 a) E is isomorphic to $\mathbb{P}_{k(x)}^1$.

$$b) E^2 = - \left[k(x) : k(s) \right].$$

Proof: a) is well known. b): Pick D on X with a regular point at x . Then $\nu_x(D) = 1$

$$0 = D \cdot \underbrace{g^* E}_{\substack{\text{pt} \\ \text{proj formula}}} = g^* D \cdot E = (\tilde{D} \cdot E + E) \cdot E$$

But $n=1 \Rightarrow P(1, \frac{b}{a})$ linear form $\Rightarrow \tilde{D}, E$ intersect transversally at a point of residue field $k(x)$

□

Alternatively: We have $E^2 = \deg_{R(s)}(\mathcal{O}_X(E)|_E)$ by definition.

• For the effective Cartier divisor E with ideal sheaf $\mathcal{I} = \mathcal{O}_X(-E)$, we have

$$\mathcal{O}_X(E)|_E \simeq \mathcal{I}^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_E \simeq i^*(\mathcal{I}/\mathcal{I}^2)^\vee \quad (\text{with } i: E \hookrightarrow X).$$

• By a local computation on the blow-up, we have $i^*(\mathcal{I}/\mathcal{I}^2) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$

$$[\text{[Liu, Thm 1.19 (c)] so } \deg_{R(s)}(\mathcal{O}_X(E)|_E) = \deg_{R(s)}(\mathcal{O}_{\mathbb{P}^1}(-1)) = -[R(t), R(s)].]$$

def 7

- $X \rightarrow S$ normal fibered surface, \mathcal{E} set of integral vertical curves in X . A contraction of \mathcal{E} is a projective birational morphism $g: X \rightarrow Y$ to a normal fibered surface $/S$ such that for any E integral vertical curve on X , we have $E \in \mathcal{E} \iff g(E)$ point.

- Assume X regular. An exceptional curve is an integral vertical curve E such that there exists a contraction of E with regular target.

prop 8

Contractions are unique up to unique isomorphism when they exist.

proof: Let $\pi: X \rightarrow Y$ be a contraction. Then Y is uniquely determined:

- as a set: $Y = (X - \bigcup_{E \in \mathcal{E}} E) \cup (1 \text{ pt for each conn. comp of } \bigcup E)$.

- as a topological space: π surjective and proper $\Rightarrow \forall Z \subseteq Y$, Z closed $\Leftrightarrow \pi^{-1}(Z)$ closed.

- as a locally ringed space: π proper birational + Y normal $\Rightarrow \mathcal{O}_Y \simeq \pi_* \mathcal{O}_X$. \square

prop 9

Let $X \rightarrow S$ normal fibered surface, \mathcal{E} set of integral vertical curves.

(i) The contraction of \mathcal{E} exists.



(ii) There exists a Cartier divisor D on X such that

• $\deg(D_Y) > 0$,

• $\mathcal{O}_X(D)$ is generated by global sections, and

• $\forall E \text{ int. vert.}, \mathcal{O}_X(D)|_E \text{ trivial} \iff E \in \mathcal{E}$.

proof: We only treat \uparrow .

• X_Y is integral proj, so by assumption D_Y is ample. By replacing by a multiple, can assume D_Y very ample.

• Assume $S = \text{Spec}(A)$ affine for simplicity. The general case follows using uniqueness,

Because $\mathcal{L} \simeq \mathcal{O}_X(D)$ is generated by global sections we get

$g: X \rightarrow \mathbb{P}_A^N$. Put $Z = g(X)$ with reduced scheme structure.

- $g: X \rightarrow Z$ is projective and birational, it factors through the normalization $\tilde{Z} \rightarrow Z$ (which is an finite because $\tilde{Z} = \text{Spec}_Z(g_* \mathcal{O}_X)$) as $X \xrightarrow{g} \tilde{Z} \xrightarrow{\pi} Z$
- We claim that g is the desired contraction.
- Because π is finite, a curve is contracted by g iff it is contracted by g , so we have to check that g contracts the right curves.
- More generally, let us show that for $Z \subseteq X$ closed and connected, $g(Z)$ point $\Leftrightarrow \mathcal{L}|_Z \simeq \mathcal{O}_Z$.
- Write s_0, \dots, s_N for sections generating \mathcal{L} and t_0, \dots, t_N for the corresponding coordinates on \mathbb{P}^N .

Recall that $X = \bigcup X_{s_i}$ with $X_{s_i} = \{x \in X \mid \mathcal{L}_x = s_i \mathcal{O}_{X,x}\}$

- Suppose that $g(Z) = \{y\}$. Then $y \in D(t_i)$ for some i , hence $Z \subseteq g^{-1}(D(t_i)) = X_{s_i}$ and $\mathcal{L}|_Z = s_i \mathcal{O}_Z \simeq \mathcal{O}_Z$.
 - Conversely, suppose that $\mathcal{L}|_Z = e \mathcal{O}_Z$. Write $B = H^0(Z, \mathcal{O}_Z)$ and $Y = \text{Spec } B$. Then Y is finite over $\text{Spec}(k)$.
- Choose $b_i \in B$ such that $s_i = e b_i$. Because the s_i generate \mathcal{L} , we have that the b_i generate \mathcal{O}_Y . Let $g: Y \rightarrow \mathbb{P}_k^N$ the associated morphism. We have $Z \xrightarrow{\cong} Y \xrightarrow{g} \mathbb{P}^N$, which by finiteness of Y and connectedness of Z implies $g(Z) = \{pt\}$. \square

- By Thm 1, if $E \subset X$ is an exceptional curve in a regular surface X , with contraction $\pi: X \rightarrow Y$, then π is the blow-up of Y at the point $\pi(E)$.

Hence $\begin{cases} E \simeq \mathbb{P}_k^1, \text{ for some } k' \text{ finite extension of } k(0). \\ E^2 = -[k'; k(0)]. \end{cases}$

We are interested in the converse.

Thm 10 Let $X \rightarrow S$ be a regular fibered surface and let $E \subset X_S$ be a vertical prime divisor with $E \cong \mathbb{P}_{R'}^1$, ($R' = H^0(E, \mathcal{O}_E)$)
 $E^2 < 0$; write $d = -\frac{E^2}{[R'; R(b)]} = \deg_{R'}(\mathcal{O}_X(-E)|_E)$.

(i) There exists a contraction $\pi: X \rightarrow Y$.

(ii) Write $y = \pi(E) \in Y$. Then $R(y) = R'$ and

$$\dim_{R(y)} T_{Y, y} = d + 1.$$

(iii) (Castelnuovo's criterion)

E is an exceptional curve iff $d = 1$.

rmk When $R' = R(0)$, E exceptional $\Leftrightarrow E^2 = -1$, hence the classical terminology of " (-1) -curves".

Proof:

(i): We can assume S affine. Since X is projective over S (Theorem 1.12)) we can choose an ample Cartier divisor H on X . Replacing H by a multiple, we can assume that: $\forall n \geq 1, H^n(X, \mathcal{O}(nH)) = 0$.

• For any Γ vertical prime divisor, and in particular for E , we have $\mathcal{O}_X(H)|_\Gamma$ ample, hence $H \cdot \Gamma = \deg(\mathcal{O}(n)|_\Gamma) > 0$. Let $m = -E^2 > 0$, $n = H \cdot E > 0$

and put $D = mH + nE$.

Lemma: 1) $\deg(D_\gamma) > 0$.
 2) For all $\Gamma \neq E$, $D \cdot \Gamma > 0$, hence $\mathcal{O}(D)|_\Gamma \cong \mathcal{O}_\Gamma$.
 3) $\mathcal{O}(D)|_E \cong \mathcal{O}_E$.
 4) $\mathcal{O}(D)$ is generated by global sections.

Proof 1) $D_\gamma = mH_\gamma$ ample \checkmark

$$2) D \cdot \Gamma = m H \cdot \Gamma + n E \cdot \Gamma \underset{\Gamma \neq E}{\geq} m H \cdot \Gamma > 0$$

3) $D \cdot E = m H \cdot E + n E^2 = mn - nm = 0$, hence $\mathcal{O}(D)|_E$ is a degree 0 line bundle on $E \cong \mathbb{P}_{R'}^1$, $\Rightarrow \mathcal{O}(D)|_E \cong \mathcal{O}_E$.

4) Let $0 \leq i \leq n-1$. Then

$$(mH + (i+1)E) \cdot E = m(n-(i+1)) \geq 0$$

$$\text{Hence } H^*(E, \mathcal{O}(mH + (i+1)E)) \cong H^*(\mathbb{P}^1, \mathcal{O}(m(n-(i+1)))) = 0.$$

Using this and an induction starting from $H^0(X, \mathcal{O}(mH)) = 0$ we get

$$H^1(X, \mathcal{O}(mH + (n-1)E)) = 0, \text{ hence}$$

$$H^0(X, \mathcal{O}(D)) \rightarrow H^0(E, \mathcal{O}(D)|_E) \cong H^0(\mathbb{P}^1, \mathcal{O}) \cong k'.$$

This implies that $\mathcal{O}(D)$ is generated by global sections at points of E (by Nakayama)

But $(\mathcal{O}(D))_{|X-E} \cong (\mathcal{O}(mH))_{|X-E}$ is generated by global sections since H is ample. \square

By the lemma, we can apply Proposition 5 and deduce the existence of the contraction π .

(ii): π proper birational, y normal $\Rightarrow \mathcal{O}_y \xrightarrow{\sim} \pi_* \mathcal{O}_X, \text{ so } \hat{\mathcal{O}}_{y,m} = (\pi_* \mathcal{O}_X)_m^\wedge$.

By the theorem on formal functions: $(\pi_* \mathcal{O}_X)_m^\wedge \cong \lim_{R \rightarrow 0} H^0(X, \mathcal{O}_{X/\mathbb{A}^1_R} / m^R \mathcal{O}_X)$.

Let \mathcal{J} be the ideal sheaf of E on X . We have $m \mathcal{O}_X \subseteq \mathcal{J}$, and because \mathcal{J} and $m \mathcal{O}_X$ define the same reduced scheme E , we have $\sqrt{m \mathcal{O}_X} = \mathcal{J}$. This implies

$$\lim_{R \rightarrow 0} H^0(X, \mathcal{O}_{X/\mathbb{A}^1_R} / m^R \mathcal{O}_X) = \lim_{R \rightarrow 0} H^0(X, \mathcal{O}_{X/\mathcal{J}^R}) \quad \begin{matrix} \text{functions on infinitesimal} \\ \text{neighbourhoods of } E. \end{matrix}$$

Now $H^0(X, \mathcal{J}^n / \mathcal{J}^{n+1}) = H^0(E, \mathcal{O}(nd)) = 0$ and an induction yields $H^0(X, \mathcal{J}^n / \mathcal{J}^m) = 0$ for all $m \geq n+1$, hence $A_k \rightarrow A_{k-1}$.

This gives exact sequences $0 \rightarrow B_k \rightarrow \hat{\mathcal{O}}_{y,y} \rightarrow A_k \rightarrow 0$.

We have $A_1 = H^0(E, \mathcal{O}_E) = k'$ is a field, hence $B_1 = m$ (and $k(y) = k'$).

Lemma: $|B_1|^2 = B_2$

Proof: It suffices to prove this mod B_k for all k .

For $k \leq 1$ we have $B_k / B_1 = \text{Ker}(A_1 \rightarrow A_k) \cong H^0(X, \mathcal{J}^k / \mathcal{J}^1)$, so we have to prove $H^0(X, \mathcal{J} / \mathcal{J}^n)^2 = H^0(X, \mathcal{J}^2 / \mathcal{J}^n)$ as ideals in $\mathcal{O}_{X/\mathcal{J}^n}$.

The inclusion \subseteq is ok, we prove the other by induction on n (ok for $n=2$:

We have that $\begin{cases} H^0(X, \mathcal{J} / \mathcal{J}^{n+1}) \rightarrow H^0(X, \mathcal{J} / \mathcal{J}^2) \\ 0 = 0 \end{cases}$

$$H^0(X, \mathcal{J} / \mathcal{J}^{n+1})^2 = H^0(E, \mathcal{O}(nd)) = H^0(E, \mathcal{O}(d))^n = H^0(X, \mathcal{J} / \mathcal{J}^2)^n$$

$$\text{Hence } H^0(X, \mathcal{J} / \mathcal{J}^{n+1})^2 \supseteq H^0(X, \mathcal{J} / \mathcal{J}^{n+1})^n = H^0(X, \mathcal{J} / \mathcal{J}^{n+1})^n = H^0(X, \mathcal{J} / \mathcal{J}^{n+1})^n.$$

• We have the exact sequence

$$0 \rightarrow H^0(X, \mathcal{I}/\mathcal{J}^{n+1}) \rightarrow H^0(X, \mathcal{I}/\mathcal{J}^n) \rightarrow H^0(X, \mathcal{I}/\mathcal{J}^n) \rightarrow 0$$

$$\text{H}^0(X, \mathcal{I}/\mathcal{J}^n)^2 \quad \text{H}^0(X, \mathcal{I}/\mathcal{J}^n)^2$$

$$\text{H}^0(X, \mathcal{I}/\mathcal{J}^{n+1})^2$$

Hence $H^0(X, \mathcal{I}/\mathcal{J}^{n+1}) \subseteq H^0(X, \mathcal{I}/\mathcal{J}^{n+1})^2$ as needed \square

We deduce that $m/m^2 = B_1/B_1^2 = B_1/B_2 = H^0(X, \mathcal{I}/\mathcal{J}^2) = H^0(E, \mathcal{O}(d))$
 is of dimension $d+1$. \square

Alternative argument for (ii):

• We have $H^0(X, \mathcal{I}/\mathcal{J}^{n+1}) = H^0(E, \mathcal{O}(-nE)|_E) = H^0(\mathbb{P}_k^1, \mathcal{O}(nd))$.

So to conclude, it is enough to show:

Lemma: $\begin{cases} 1) \text{ For } n \leq 1, H^0(X, \mathcal{I}^n) = 0. \\ 2) \text{ For } n \leq 2, H^0(X, \mathcal{I}^n) \cong m^n. \end{cases}$

Proof: We show both for all n .

1) By the theorem on formal functions [Hartshorne, III. 11.1] we have

$$H^0(X, \mathcal{I}^n) \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y} \cong \lim_{R} H^0(X, \mathcal{I}/m^R \mathcal{J}^n).$$

• Since $|E| = |\pi^{-1}(y)|$ we have $\sqrt{m \mathcal{O}_X} = \mathcal{I}$, hence $\exists n, \mathcal{I}^n \subseteq m \subseteq \mathcal{I}$, so

$$\lim_{R} H^0(X, \mathcal{I}/m^R \mathcal{J}^n) = \lim_{m \geq n} H^0(X, \mathcal{I}/\mathcal{J}^m).$$

Now $H^0(X, \mathcal{I}/\mathcal{J}^{n+1}) = H^0(E, \mathcal{O}(nd)) = 0$ and an induction

yields $H^0(X, \mathcal{I}/\mathcal{J}^m) = 0$ for all $m \geq n+1$, hence

$$H^0(X, \mathcal{I}^n) \otimes_{\mathcal{O}_{Y,y}} \hat{\mathcal{O}}_{Y,y} = 0.$$

• $(\mathcal{O}_{Y,y}, m)$ noetherian local ring $\Rightarrow \mathcal{O}_{Y,y} \rightarrow \hat{\mathcal{O}}_{Y,y}$ faithfully flat
 $\Rightarrow H^0(X, \mathcal{I}^n) = 0$.

2). As in the proof of (i), we see that $1) \Rightarrow \mathcal{I}^n$ generated by global sections.

. We have $H^0(X, \mathcal{I}) = m$ as $\mathcal{O}_{Y, y}$ -modules. ?

Hence $H^0(X, \mathcal{I})^{\otimes n} = m^{\otimes n}$. We have $H^0(X, \mathcal{I}^n) \subseteq H^0(X, \mathcal{I}^n)$ and we want to show the opposite inclusion. This follows from:

Lemma: Let X be a noetherian scheme such that

[dipmam, Lemma 7.3] $\left. \begin{array}{l} a) H^1(X, \mathcal{O}_X) = 0 \\ b) H^2(X, \mathcal{L}) = 0 \text{ for all ideal sheaves } \mathcal{L} \end{array} \right\}$ blow-up of affine at a smooth point satisfies this.

Then for any two coherent sheaves \mathcal{J}^i and \mathcal{L} generated by their global section, the map $H^0(X, \mathcal{J}^i) \otimes H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{J}^i \otimes \mathcal{L})$ is surjective. \square

rk: One can ask more generally when the contraction of an effective divisor which is not irreducible exists, and when it is regular.

This can be studied by similar methods [Lin, 9.4.1-2].

prop: A regular proper surface over a field is projective.

prop: Let X/k be such a surface.

. By Chow's lemma [EGA II, 5.6.1] there exists Z/k projective and $Z \rightarrow X$ projective birational. By res. of singularities of surfaces, we can assume Z is regular. The morphism $Z \rightarrow X$ is a composition of blow-ups of closed points (same proof as in the fibered regular case). By Castelnuovo, we see that Z projective \Rightarrow all contractions of (-1) -curves are projective, hence X is projective. \square