

4) Differential graded categories

* Recall that a differential graded category (dg-category for short) is a category enriched over Cpl_R for some ring R .

- $\text{Ob}(\text{Cpl}_R) : (C_*, \delta)$ with $\begin{cases} C_n \in \text{Mod}_R \\ \delta_n : C_n \rightarrow C_{n-1} \end{cases}$ with $\delta_n \circ \delta_{n+1} = 0$
boundary operator
- $\text{Cpl}_R((C_*, \delta), (D_*, \delta)) = \left\{ g_n : C_n \rightarrow D_n \mid \begin{array}{c} C_n \xrightarrow{\delta_n} C_{n-1} \\ \downarrow g_n \\ D_n \xrightarrow{\delta_{n-1}} D_{n-2} \end{array} \right\}$

- $(C_* \otimes D_*)_n := \bigoplus_{n=n'+n''} C_{n'} \otimes D_{n''}$
 and $\delta \left(\begin{smallmatrix} c \\ \otimes \\ d \end{smallmatrix} \right) := \delta(c) \otimes d + (-1)^{n'} c \otimes \delta(d)$
 $C_{n'}, \quad D_{n''}$
- $H_n(C_*) = \frac{\ker(\delta_n : C_n \rightarrow C_{n-1})}{\text{Im}(\delta_{n+1} : C_{n+1} \rightarrow C_n)}$ homology groups

- $g : C_* \rightarrow D_*$ is a quasi-isomorphism if $\forall n \in \mathbb{Z}$,
 $H_n(g) : H_n(C_*) \xrightarrow{\sim} H_n(D_*)$.

* dg-categories are very common
 in homological algebra and algebraic geometry,
 in the same way that simplicial and
 topological categories are in algebraic topology.
 * We write $dg\mathbf{Cat}_R$ for the category of
 R -linear dg-categories.

Ex 24:

- * Let \mathcal{A} be an R -linear additive category,
 i.e. a category enriched over Mod_R and which
 admits all finite products. Then those finite
 products are automatically coproducts (see Ex. sheet 9)
 Hence \mathcal{A} has in particular a zero object
 (= initial + final). Using this, one can define
 the category $Ch(\mathcal{A})$ of chain complexes in \mathcal{A}

as above (so that $\text{Cpl}_R = \text{Gol}(\text{Mod}_R)$).

Then $\text{Cpl}(A)$ admits a natural structure of R -linear dg-category, with

mapping complex

$$\underline{\text{Hom}}_{\text{Cpl}(A)}(C_*, D_*)_n = \prod_{i \in \mathbb{Z}} \underline{\text{Hom}}_A(C_i, D_{i+n})^{\text{Mod}_R}$$

and boundary operator:

$$\partial \{g_i : C_i \rightarrow D_{i+n}\}_i = \left\{ \partial \circ g_j - (-1)^j g_{j-1} \circ \partial \right\}_j$$

(In particular, Cpl_R is canonically enriched over itself).

- * A dg- R -algebra A_* is a (non-nec. commutative) graded R -algebra + a differential $\partial : A_* \rightarrow A_{*-1}$ satisfying

- $\delta^2 = 0$
- $\delta(xy) = \delta(x)y + (-1)^{|x|} x\delta y$ (Leibniz rule)

Then $\left\{ \begin{array}{l} \text{dg-R-categories} \\ \text{with one object } X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{dg-R} \\ \text{algebras} \end{array} \right\}$

$$\subset \xrightarrow{\quad \quad \quad \quad \quad} \text{End}_C(X)_*$$

* As with Cat_{Δ} , dgCat_R is naturally connected with other types of enriched categories via lax-monoidal functors.

We have:

$$\begin{aligned} & \cdot \text{Cpl}_R(R[0], C_*) = Z_0(C_*) \\ & \qquad \qquad \qquad \text{O-cycles} \\ & \cdot \left(\text{B}_{\text{Cpl}_R} \right) = \left\{ x \in C_0 \mid \delta(x) = 0 \right\} \end{aligned}$$

$$\Rightarrow \cup : \text{dgCat}_R \longrightarrow \text{Cat}_R$$

underlying category functor

- The 0-th homology functor:

$$H_0 : \text{Cpl}_R \longrightarrow \text{Mod}_R$$
$$C_* \longmapsto H_0(C_*) := \frac{\text{Ker}(d_0 : H_0(C) \rightarrow H_1(C))}{\text{Im}(d_1 : H_1(C) \rightarrow H_0(C))}$$

H_0 has a natural lax-monoidal structure

↪ homotopy category functor

$$R = (H_0)_* : \text{dgCat}_R \longrightarrow \text{Cat}_R$$

Applied to $\text{Cpl}(A)$, this encodes nicely
the notion of chain homotopy of
morphisms of complexes which plays a
key role in homological algebra.

- We want to construct interesting ∞ -categories using dg-categories.

Here is a procedure to do it:

Def 25: Let C be in dgCat_R . The

differential graded nerve $N^{\text{dg}}(C)$ is the simplicial set defined as follows:

- $N^{\text{dg}}(C)_n$ is the collection of tuples

$((X_i)_{0 \leq i \leq n}, \{\delta_I\})$ with $X_i \in \text{Ob}(C)$

and, for every $I = \{i_0 > i_1 \dots > i_k\} \subseteq [n]^{\neq}$

with $|I| \geq 2$, we have $\delta_I \in \text{Hom}_C(X_{i_k}, X_{i_0})_{k-1}$

which satisfy

R_{k-1}

$$\delta \delta_I = \sum_{a=1}^{R_{k-1}} (-1)^a \left(\delta_{\{i_0 > i_1 \dots > i_a\}} \circ \delta_{\{i_a > \dots > i_k\}} - \delta_{I - \{i_a\}} \right)$$

(with the convention that $\delta_I = 0$ if $|I| < 2$)

- The simplicial structure is given by

$$\alpha : [n] \rightarrow [m],$$

$$\alpha^* (\{x_i\}, \{\delta_I\}) = (\{x_{\alpha(i)}\}, \{\delta_J\})$$

with $\delta_J = \begin{cases} \delta_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective} \\ id_{x_i} & \text{if } J = \{j_0 > j_1\} \text{ with } \begin{matrix} \alpha(j_0) \\ \alpha(j_1) \end{matrix} \\ 0 & \text{otherwise} \end{cases}$

□

Ex 26: Rather than getting lost in

a mess of indices, let's look concretely at the n -simplices for $n \leq 2$:

- $N^{dg}(C)_0 = Ob(C)$
- $N^{dg}(C)_1 = Mor(\cup C)$ ($= 0$ -cycles in $Hom_C(x,y)$)
- $N^{dg}(C)_2 = \{ (x_0, x_1, x_2), \delta_{01} \in \mathcal{Z}_0(x_0, x_1), \delta_{02} \in \mathcal{Z}_0(x_0, x_2), \delta_{12} \in \mathcal{Z}_0(x_1, x_2) \}$

and $\delta_{012} \in \text{Hom}_C(X_0, X_2)_1$ such
 that $\partial \delta_{012} = \delta_{02} - \delta_{12} \circ \delta_{01}$

- Like in the homotopy coherent nerve, we can think of this datum as

$$\begin{array}{ccc} & X_1 & \\ \delta_{01} \nearrow & & \searrow \delta_{12} \\ X_0 & \xrightarrow{\quad \delta_{02} \quad} & X_2 \\ & \parallel \delta_{012} & \end{array}$$

with δ_{012} witness of an equality of homology classes: $[\delta_{02}] = [\delta_{12} \circ \delta_{01}]$ in $H_0(\text{Hom}_C(X_0, X_2))$.

Thm 27: Let C be a dg-category.

Then $N^{\text{dg}}(C)$ is an ∞ -category. □

The proof is a straight forward

but lengthy computation.

* Thm 27, combined with standard construction from homological algebra, produces many interesting ∞ -categories.

⚠ Unlike in the simplicial case, not every ∞ -category arises this way, even up to categorical equivalence! One should think of the essential image as the "R-linear ∞ -categories".

* For instance, let \mathcal{A} be an abelian category with enough injectives (e.g $\mathcal{A} = \text{Mod}_R$, or $\mathcal{A} = \mathbb{Q}\text{Coh}_X$ for a scheme X)
Let $\text{Cpl}^+(\mathcal{A}_{\text{inj}})$ be the full subcategory of $\text{Cpl}(\mathcal{A})$ formed by the C_*

with : - C_n injective for all $n \in \mathbb{Z}$

- $C_n = 0$ for $n >> 0$

(bounded above^{homologically} complexes of injectives)

Then $\text{Cpl}^+(\mathcal{A}_{\text{inj}})$ "inherits" a dg-enrichment from $\text{Cpl}(\mathcal{A})$.

def 28: The (bounded-above) derived

∞ -category $\mathcal{D}^+(\mathcal{A})$ is defined as

$N^{\text{dg}}(\text{Cpl}^+(\mathcal{A}_{\text{inj}}))$.

(there are variants for bounded, bounded below and unbounded complexes as well).

* Essentially by construct*, the homotopy category of $\mathcal{D}^+(\mathcal{A})$ is

$$R\mathcal{D}^+(\mathcal{A}) = K^+(\mathcal{A}_{inj})$$

$$= \mathcal{D}^+(\mathcal{A}) \quad \begin{matrix} \leftarrow \\ \text{usual} \\ \text{derived} \\ \text{category} \end{matrix}$$

Natural questions:

- + Is the triangulated structure on $\mathcal{D}^+(\mathcal{A})$ somehow encoded in $\mathcal{D}^+(\mathcal{A})$?

Yes! $\mathcal{D}^+(\mathcal{A})$ is a **stable ∞ -category** and stable ∞ -categories have a canonical triangulated structure on their homotopy categories.

- + What is the advantage of working with $\mathcal{D}^+(\mathcal{A})$ rather than $\mathcal{D}^+(\mathcal{A})$?

Homotopy coherence! Makes it possible to talk about $\begin{cases} \text{sheaves with values in } \mathcal{D}^+(\mathcal{A}) \\ \text{higher algebraic structures} \end{cases}$ \rightsquigarrow derived algebraic geometry.

5) Dold - Kan correspondence

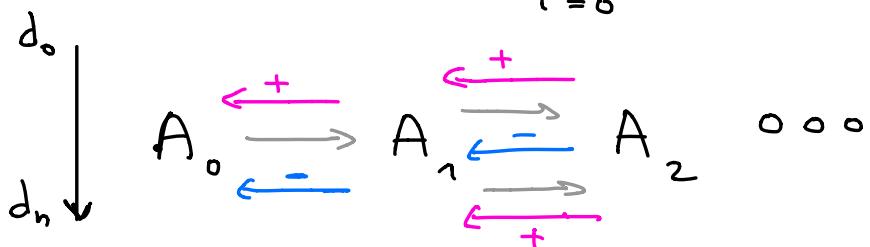
- There is a fundamental relationship between chain complexes and simplicial abelian groups / R -modules.

def 25: Let $A_{\cdot} \in s\text{Mod}_R$ be a simplicial R -module. Its Moore complex

is $C_*(A_{\cdot}) \in \text{Cpl}_R$ defined by

$$C_n(A_{\cdot}) = \begin{cases} A_n, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

and $\delta: A_n \rightarrow A_{n-1}$
 $\sigma \mapsto \sum_{i=0}^n (-1)^i d_i(\sigma)$



(Exercise: show $\partial^2 = 0$)

def 26: The free R -module functor

is $s\text{Set} \xrightarrow{R[-]} s\text{Mod}_R$
 $(S: \Delta^{\text{op}}\text{-Set}) \mapsto R[S]: \Delta^{\text{op}} \xrightarrow{R(-)} \text{Mod}_R$

(NB: this is strong monoidal: $R[S \times T] \simeq R[S] \otimes R[T]$)

def 27: Let $S \in s\text{Set}$. The chain

complex of X . (or associated to X .) is

$$C_*(S, R) := C_*(R[S]).$$

The homology R -modules of X are

$$H_n(S, R) := H_n(C_*(S, R)).$$

Ex 28:

Let $X \in \text{Top}$. Then $C_*(\text{Sing}(X), R)$

is the usual singular chain complex $C_*^{\text{sing}}(X, R)$

and $H_*(X, R)$ is singular homology.

- Let $S \in s\text{Set}$. We have a unit map

$$S \longrightarrow \text{Sing}|S|$$

which induces a quasi-isomorphism.

$$C_*(S, R) \xrightarrow{\quad} C_*^{\text{sing}}(|S|, R)$$

In other words, $C_*(S, R)$ is a simplicial way to capture the singular homology of $|S|$.

def 29: Let $A_\cdot \in s\text{Mod}_R$. The

degenerate sub complex $D_*(A) \subseteq C_*(A)$

$$\text{is } D_n(A) = \langle s_i : A_{n-i} \rightarrow A_n \mid 0 \leq i \leq n-1 \rangle$$

(Exercise: it is indeed a subcomplex, preserved by ∂)

The normalised Moore complex $N_*(A)$ is

$$N_*(A) := \frac{C_*(A)}{D_*(A)}$$

Prop 30: The quotient map $C_*(A) \rightarrow N_*(A)$

is a quasi-isomorphism. □

Ex 31: We write $N_*(-, R)$ for the composition

$$N_*(-, R) : s\text{-Set} \xrightarrow{R[-]} s\text{-Mod}_R \xrightarrow{N_*} \text{Cpl}_R$$

By construction, $N_*(\Delta^\bullet, R)$ is then
a cosimplicial object in Cpl_R .

It is in fact relatively simple

$$\begin{aligned} N_*(\Delta^n, R)_i &\simeq R \left[\begin{array}{l} \text{non-degenerate } i\text{-th simplices} \\ \text{of } \Delta^n \end{array} \right] \\ &\simeq R^{\binom{n}{i}} \end{aligned}$$

and $\partial : N_i(\Delta^n) \longrightarrow N_{i-1}(\Delta^n)$

$$\sigma \longmapsto \sum_{i=0}^n (-)^i \begin{cases} d_i(\sigma) & \text{if non-deg.} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{cases} N_*(\Delta^\circ) : \cdots \xrightarrow{\quad 0 \quad} \overset{\circ}{R} \xrightarrow{\quad 0 \quad} \overset{\circ}{R} \xrightarrow{\quad 0 \quad} \cdots \\ N_*(\Delta^\circ) : \cdots \xrightarrow{\quad 0 \quad} R \oplus R \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} R \xrightarrow{\quad 0 \quad} \cdots \end{cases}$$

We use this cosimplicial object to construct:

Def 32: The Eilenberg-MacLane Functor

$K : \text{Cpl}_R \longrightarrow s\text{Mod}_R$ is defined by

$$K(C_*)_n := \text{Cpl}_R(N_*(\Delta^n, R), C_*)$$

Rmk 33: * As with every simplicial

group , the underlying simplicial set of $K(C_*)$ is always a Kan complex.

(see exercise 3.8)

* If $C_* = A[n]$, $n \geq 1$ with $A \in \text{Ab}$,

$|K(A[n])|$ is an Eilenberg-MacLane

space in the sense of algebraic topology:

$$\pi_i(|K(A[n])|) \cong \begin{cases} * & , i=0 \\ A & , i=n \\ 0 & , \text{otherwise.} \end{cases}$$

Prop 34: K is the right adjoint of N_* .

Thm 35 (Dold-Kan correspondence)

$N_* \dashv K$ restricts to an adjoint equivalence

$$N_* : s\text{Mod}_R \rightleftarrows \text{Cpl}_{R, \geq 0} : K \quad \square$$

This is surprisingly (to me!) difficult to

prove for such an "abstract" result.

(see [Kerodon, 2.5.6])

- The interaction with the symmetric monoidal structures on both sides is also a surprisingly complicated story, which we don't have time to discuss. We just extract the result we need:

Thm 36 (some monoidal aspects of Dold-Kan)

The functors N_* and K both admit natural lax-monoidal structures.

(called the Eilenberg-Zilber and Alexander-Whitney map). □

Using those, we can finally move between simplicial and dg-categories: we get an adjunction

$$R[-]: \left(N_* \circ R[-] \right)_*: \text{Cat}_{\Delta} \rightleftarrows \text{dgCat}_R: (\text{forget} \circ K)_*$$

(! This is not at all an equivalence !)

Note that for any $C \in \text{dgCat}_R$,

the simplicial category $\cup C$ is locally Kan.

Combined with the homotopy coherent nerve,
this gives a functor

$$N_{\Delta} \circ : \text{dgCat}_R \longrightarrow \text{Cat}_{\infty}.$$

Thm : For any $C \in \text{dgCat}_R$, there
is a functor

$$N_{\Delta}(\circ C) \longrightarrow N^{\text{dg}}(C)$$

which is a trivial fibration, and thus
an equivalence of ∞ -categories. □

This shows that the two “natural”
ways to make ∞ -categories out of a
dg-category coincide.

6) Model categories

Model categories are yet another way to present homotopical ∞ -categorical information with 1-categories.

def : (Quillen) A model category is a category M together with 3 collections of morphisms:

- the weak equivalences W
- the cofibrations Cof
- the fibrations Fib

such that the following axioms hold:

(not essential, could ask for finite (co)limits)

- M is complete and cocomplete.
- W satisfies the 2-out-of-3 property;
if f, g are composable and 2 out of 3

among g , g , $g \circ g$ are in W , so is the third.

trivial fibrations trivial cofibrations

\downarrow \checkmark

- $(\text{Cof}, \text{Fib} \cap W)$ and $(\text{Cof} \cap W, \text{Fib})$
are weak factorisation systems:

every morphism $g: X \rightarrow Y$ in M factors

(non-uniquely) as $\left\{ \begin{array}{l} g: X \xrightarrow[\text{Cof}]{} P \xrightarrow[\text{Fib} \cap W]{} Y \\ g: X \xrightarrow[\text{Cof} \cap W]{} C \xrightarrow[\text{Fib}]{} Y \end{array} \right.$

and we have lifting properties:

$$\left\{ \begin{array}{ll} \text{Cof} = \square (\text{Fib} \cap W) & \text{Fib} \cap W = \text{Cof}^{\square} \\ & \Leftrightarrow \\ \text{Cof} \cap W = \square \text{ Fib} & \text{Fib} = (\text{Cof} \cap W)^{\square}. \end{array} \right.$$

\square

Remarks :

* The definition is self-dual:

$(M, W, \text{cof}, \text{Fib})$ is a model category



$(M^{\text{op}}, W^{\text{op}}, \text{Fib}^{\text{op}}, \text{CoF}^{\text{op}})$ —————.

* X is cofibrant (resp. fibrant)

if the unique map $\emptyset \rightarrow X$ (resp. $X \rightarrow *$) is a cofibration (resp. a fibration).

* Using the factorisations, we see that

any object $X \in M$ admits:

. $X_{\text{cof}} \xrightarrow{\sim} X$ with X_{cof} cofibrant

(a cofibrant replacement of X)

. $X \hookrightarrow X_{\text{fib}}$ with X_{fib} fibrant.

(a fibrant replacement of X).

By combining the two, we get that

any object X is weak equivalent to a

bifibrant (= fibrant + cofibrant) object:

$$X \rightleftarrows X_{\text{cof}} \hookrightarrow (X_{\text{cof}})_{\text{fib}}$$

$$\left(\text{or } X \hookrightarrow X_{\text{fib}} \rightleftharpoons (X_{\text{fib}})_{\text{cof}} \right)$$

Let us write M_{cof} , M_{fib} and M_{cf}

for the full subcategories of M spanned
by cofibrant, fibrant and bifibrant objects.

* As we will see, the main point of model structures is to give tools to work with the localisation $M[W^{-1}]$. In particular, (co)fibrations are less important than weak equivalences and are a technical tool to access $M[W^{-1}]$.

* By the lifting properties, the 3 classes of maps are determined by either:

- W and Cof

- W and Fib
- Cof and Fib (Exercise)

So to "construct a model structure", one usually proceeds by choosing W and one of Cof or Fib, define Fib or Cof by lifting, and check the axioms. This can be quite difficult, and almost always relies on the small object argument.

Sometimes with additional work, one can get concrete descriptions of both Cof and Fib, or at least of Cof and all fibrant objects (for instance).

* In fact the structure is determined by even less:

prop : (Joyal) Let M be a model category .

The whole model structure is determined by the cofibrations and the fibrant objects (or by the fibrations and the cofibrant objects).

□

* The factorisations in the WFS can usually be chosen to be functorial, because they are usually constructed with the small object argument. This makes a few proofs simpler but does not really affect the theory, and it does not make any difference in practice.

* The axioms of model categories were abstracted by Quillen from properties of the homotopy theory of three central examples:

- topological spaces
- simplicial sets
- chain complexes.

I will discuss very briefly all three.

Prop : In $s\text{Set}$, we have two weak factorisation systems:

- (Monomorphisms, Trivial fibrations)
- (Anodyne morphisms, Kan fibrations).

proof. We have already proved this as an application of the small object argument! See Corollary III.1.15.
T
developed by (Grothendieck and) Quillen for this kind of purpose.



Thm : There is a unique model structure on $s\text{Set}$, the Kan-Quillen model structure, such that:

- the cofibrations are the monomorphisms.
- the fibrations are the Kan fibrations.
- the weak equivalences are the maps f such that, for all K Kan complex,

$$[Y, K] \xrightarrow[\text{bij.}]{} [X, K]. \quad (\Leftrightarrow \text{1st homotopy equivalence})$$

- In the Kan-Quillen model structure, every object is cofibrant and the fibrant objects are the Kan complexes.

Sketch:

- The weak equivalences clearly satisfy the 2-out-of-3 property.
- By the previous proposition, we have two WFS. To finish the proof, it suffices to show that:

I) A morphism is a trivial Kan fibration
iff it is a Kan fibration
and a weak equivalence.

II) A morphism is anodyne
iff it is a monomorphism
and a weak equivalence.

(One nice, purely simplicial fact of I)
and II) relies on Kan's " Ex^∞ " functor.
which is an explicit fibrant replacement
functor with good properties.

Here are references:

- I) : [Kerodon, Cor 3.3.7.4]

- II) : [Kerodon, Cor. 3.3.7.5]



Thm : There is a unique model

structure on Top , the Quillen model
structure , such that :

- the weak equivalences are ... the
weak (homotopy) equivalences !
- the fibrations are the Serre

Fibrations $(D^n \hookrightarrow D^n \times I \mid n \geq 0)^{\square}$.

- the cofibrations are $(S^{n-1} \hookrightarrow D^n \mid n \geq 0)$
(the "retracts of relative cell-complexes")
- In the Quillen model structure,
every object is fibrant, and the
cofibrant objects are the retracts

of cell complexes.

Sketch:

- Weak homotopy equivalences in Top satisfy the 2-out-of-3.
- The small object argument produces WFS:

$$- \left(\left(\overline{S^{n-1} \hookrightarrow D^n} \right), (S^{n-1} \hookrightarrow D^n)^{\square} \right)$$

↑

Cofibrations

$$- \left(\left(\overline{D^n \hookrightarrow D^n \times I} \right), (D^n \hookrightarrow D^n \times I)^{\square} \right)$$

↑

fibrations

- Again to finish the proof one must show compatibilities between these. The key case here is:

$$(D^n \hookrightarrow D^n \times I)^{\square} = (S^{n-1} \hookrightarrow D^n)^{\square} \cap W$$

which turns out to follow from standard arguments from the study of CW-complexes (what Peter May calls the HELP lemma: "Homotopy extension and lifting properties"; see refs in the summary of the lecture).



Rmk: There are other model structures on Top with the same weak equivalences; in particular if you prefer Hurewicz fibrations, you can have a look at [May-Ponto, § 17.4].

Thm : Let R be a commutative ring.

(can do a lot more general abelian categories)

1) There is a unique model structure on
 $Cpl_R^{\geq 0}$, the injective model structure,

such that:

- the weak equivalences are the quasi-isomorphisms.
- the cofibrations are the levelwise monomorphisms. ($A_\cdot \rightarrow B_\cdot$ with $A_n \hookrightarrow B_n$)
- the fibrations are the levelwise epimorphisms with injective kernels
($A_\cdot \rightarrow B_\cdot$ with $0 \rightarrow I_n \xrightarrow{\text{injective}} A_n \rightarrow B_n \rightarrow 0$)

1) There is a unique model structure on
 $Cpl_R^{\leq 0}$, the projective model structure,

such that:

- the weak equivalences are the quasi-isomorphisms.
- the fibrations are the levelwise epimorphisms . ($A_\cdot \rightarrow B_\cdot$ with $A_n \rightarrow B_n$)
- the cofibrations are the levelwise monomorphisms with injective kernels
 $\xrightarrow{\text{projective}}$
 $(A_\cdot \rightarrow B_\cdot \text{ with } 0 \rightarrow A_n \rightarrow B_n \rightarrow P_n \rightarrow 0)$



- * There are also model structures for unbounded complexes but they are a little more subtle.
- * These model structures on complexes are very useful to understand what is going on in homological algebra.

- One first use of model categories for ∞ -categories is to organise the comparison between different models.

Thm : (Joyal) There exists a unique model structure on $sSet$, the Joyal model structure, such that:

- the weak equivalences are the categorical equivalences.
- the cofibrations are the monomorphisms.
- the fibrant objects are the ∞ -categories. \square

One can describe the structure in some more details, in particular the fibrations between fibrant objects (= isoFibrations). We may come back to this - or not.

Thm : (Bergner) There exists a unique

model structure on Cat_{Δ} , the Bergner model structure, such that:

- the weak equivalences are Dwyer-Kan equivalences.

- the fibrations are functors $F: C \rightarrow D$ such

that $\forall x, y \in C, \text{Hom}_C(x, y) \xrightarrow{\sim} \text{Hom}_D(F(x), F(y))$

is a Kan fibration.

$\forall x_1 \in C, y \in D$, and homotopy eq. $e: F(x_1) \rightarrow y$,

there exists an homotopy eq. $d: x_1 \rightarrow x_2$ such

that $F(d) = e$.

In particular, the fibrant objects are

Precisely the locally Kan simplicial categories.



* Let's come back to the general theory.

What does having a model structure give you?

Quillen discovered that there is an intrinsic notion of homotopy between morphisms in a model category, which generalizes the notions in Top and sSet, and also chain homotopies in Cpl_R .

(In practice, this intrinsic notion is not essential because one uses enriched model categories, as we will see soon.)

* Nevertheless, Quillen defines, for

$C \in M_{\text{cof}}$ and $F \in M_{\text{fib}}$:

$$[C, F] := M(C, F) / \text{homotopy}.$$

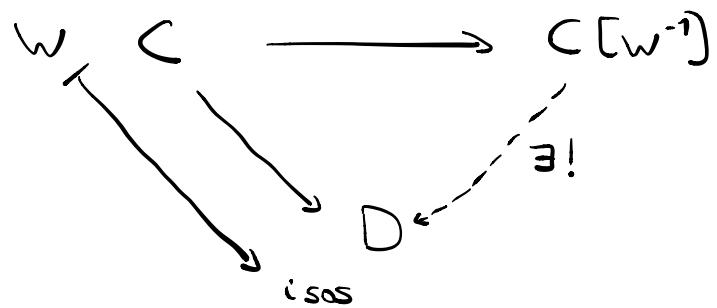
and the homotopy category $\text{Ho}(M)$:

- $\text{Ob } R_0(M) = \text{Ob}(M_{\text{cf}})$

- $R_0(M)(x, y) = [x, y]$

(He also proves that composition is well-defined up to homotopy, etc.)

- * Recall also the notion of localisation of a category C at a class of maps W : it is the universal category $C[W^{-1}]$ equipped with a functor $C \rightarrow C[W^{-1}]$ which sends morphisms in W to isomorphisms:



We can then state:

Thm. : (Quillen's fundamental theorem)

M model category.

1) Let $C \in M_{\text{cof}}$, $F \in M_{\text{fib}}$.

Then $[C, F] \xrightarrow{\sim} M[w^{-1}](C, F)$

2) The inclusion $M_{\text{cf}} \hookrightarrow M$ induces

an equivalence of categories $R_0(M) \xrightarrow{\sim} M[w^{-1}]$.

□

Since every object admits (co)fibrant replacements,
this gives in principle a way to compute in $M[w^{-1}]$.

Ex:

- For Top with the Quillen model structure,
this is closely related with two well-known
foundational facts in algebraic topology:
 CW -approximation and the Whitehead theorem.

(because CW-complexes $\subset \text{Top}_{\text{cof}} =$ retracts of cell complexes)

We now discuss the functoriality of model categories.

Def : Let M, N be model categories.

- A **Quillen adjunction** is an adjunction

$$F : M \rightleftarrows N : G$$

such that $\begin{cases} F(\text{Cof}) \subset \text{Cof} \\ G(\text{Fib}) \subset \text{Fib} \end{cases}$

Lemma : $F \dashv G$ if a Quillen adjunction



$$F(\text{Cof}) \subset \text{Cof} \quad \text{and} \quad F(\text{Cof} \cap W) \subset \text{Cof} \cap W$$



$$G(\text{Fib}) \subset \text{Fib} \quad \text{and} \quad G(\text{Fib} \cap W) \subset \text{Fib} \cap W.$$

Proof : Exercise.

Lemma : Let $F : M \rightleftarrows N : G$ be a Quillen adjunction. Then there is an induced derived adjunction

$$[LF : R_0(M) \rightleftarrows R_0(N) : RG]$$

such that

- $L_0 F(X) \simeq F(X_{\text{cof}})$
- $R_0 F(Y) \simeq G(Y_{\text{fib}})$.



This, applied to model categories like $Cpl_R^{(\pm)}$, is the source of derived functors in Homological algebra.

Def : Let $F : M \rightleftarrows N : G$ be a Quillen adjunt.

It is a Quillen equivalence if the derived adjunction $[LF \rightarrow RG]$ is an equivalence of categories.

(this is equivalent to: For $X \in M_{\text{cof}}$

and $Y \in N_{\text{fib}}$, then

$$F(X) \rightarrow Y \text{ in } W_M \Leftrightarrow X \rightarrow G(Y) \text{ in } W_N$$

Many functors we have seen in this course turn out to be part of Quillen adjunctions or equivalences.

Thm : (Quillen) The adjunction

$$\mathbf{I} : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathbf{Sing}$$

↑
(with Kan-Quillen) ↑
(with Quillen)

is a Quillen equivalence.

Sketch : • Quillen adjunction: We check

that $| \text{cof} | \subset \text{cof}$ and $\text{Sing}(\text{Fib}) \subset \text{Fib}$.

The first part, we know because of the (relative) skeletal filtration for monomorphisms of simplicial sets.

The second part is easy because of the following:

$$\begin{array}{ccc}
 \Delta_k^n & \longrightarrow & \text{Sing}(X) \\
 \downarrow & \nearrow ? & \downarrow \\
 \Delta^n & \longrightarrow & \text{Sing}(Y)
 \end{array} \qquad \Leftrightarrow \qquad
 \begin{array}{ccc}
 D^{n-1} = |\Delta_{k-1}^n| & \longrightarrow & X \\
 \downarrow & \nearrow ? & \downarrow \\
 D^n \times I = |\Delta^n| & \longrightarrow & Y
 \end{array}$$

which shows that $\text{Sing}(\text{Serre fibrat}^\circ) \subset \text{Kan fibrat}^\circ$.

- Quillen equivalence: This is a theorem of Milnor which we already mentioned in previous discussions of simplicial homotopy theory.

See [Kerodon, § 3.5].



This is some sense the final version of the representation of homotopy types by

Simplicial sets.

Thm : (Lurie) The adjunction

$$\text{Path}[-]: \text{sSet} \rightleftarrows \text{Cat}_{\Delta}: N_{\Delta}$$

(with [↑]Joyal) (with [↑]Bergner)

is a Quillen equivalence.

- This is the promised “equivalence of homotopy theories” between two models of $(\infty, 1)$ -categories.
 - We now briefly mention simplicial and dg-model categories.

Def : A simplicial model category

M is both a simplicial category and

a model category such that:

- M is tensored and cotensored over $sSet$
- (SM7) for every cofibration $i: A \hookrightarrow B$ and fibration $p: X \rightarrow Y$,

the simplicial pullback-hom

$$\underset{C}{\text{Hom}}(B, X) \longrightarrow \underset{C}{\text{Hom}}(A, X) \times_{\underset{C}{\text{Hom}}(A, Y)} \underset{C}{\text{Hom}}(B, Y)$$

is a Kan fibration, which is

trivial if either i or p is.



Ex : • $sSet_{\text{Kan-Quillen}}$, with its self-enrichment coming from the cartesian closed structure, is a simplicial model category (this is proven with the same "lifting calculus" methods that we saw in Chapter III)

- $CGHaus$ has a natural structure of

simplicial category with

$$\underline{\text{Hom}}_C(X, Y) = \text{Sing}(\underline{\text{Hom}}(X, Y))$$

and the Quillen-type model structure on

CGNaus is a simplicial model category.

-  $s\text{Set}_{\text{Joyal}}$ is not a simplicial

model category in this sense.,

because it has fibrations which are
not Kan fibrations.

Def : Let M be a simplicial model

category. Then, because of (SM7),

the full simplicial subcategory on M_{cf}

is locally Kan. The ∞ -category

associated to M is $N_{\Delta}(M_{cf})$. □

Thm : (Simpson, Dugger, Lurie)

Let C be an ∞ -category.

The following are equivalent:

- C is a presentable ∞ -category
(C admits all colimits in ∞ -categorical sense, and is generated under filtered colimits by "small" objects)
- There exists a combinatorial simplicial model category M such that $C \simeq N_{\Delta}(M_{cf})$.



This theorem explains a posteriori the success of the theory of model categories:

they "model" many interesting ∞ -categories.

(eg any small ∞ -category can be embedded as a full subcategory of a presentable ∞ -category by the ∞ -version of the Yoneda lemma).