

Abelian varieties and line bundles

- We do a quick introduction to abelian varieties, including duality theory.

1) Definition and elementary properties

def: Let S be a base scheme. An abelian scheme over S is an S -group scheme which is proper smooth with connected fibers. If $S = \text{Spec}(\mathbb{R})$, we say that A is an abelian variety over \mathbb{R} .

mh: A scheme X locally of finite type over a field \mathbb{R} which is connected and such that $X(\mathbb{R}) \neq \emptyset$ is geometrically connected [Stack, $\phi^4 KV$]. This applies to any group schemes, so the fibers are geometrically connected.

ex: (Jacobian of a smooth projective curve). Let C/\mathbb{R} be a smooth projective curve over a field \mathbb{R} . Consider $\text{Pic}^\circ_{C/\mathbb{R}}$. By the results of the previous chapter:

- $\text{Pic}^\circ_{C/\mathbb{R}}$ is a (geometrically) connected group scheme of finite type over \mathbb{R} .

[mh: a connected, locally of finite type scheme over a field is of finite type]

- $\text{Pic}^\circ_{C/\mathbb{R}}$ is smooth ($\leq \dim C = 1$), of dimension $= \dim_{\mathbb{R}} T_0 \text{Pic}^\circ_{C/\mathbb{R}} = \dim_{\mathbb{R}} H^1(C, \mathcal{O}_C)$
= genus of C .

- $\text{Pic}^\circ_{C/\mathbb{R}}$ is proper ($\leq C/\mathbb{R}$ is smooth).

So $\text{Pic}^\circ_{C/\mathbb{R}}$ is an abelian variety, called the Jacobian of C . Note that, when C is singular, $\text{Pic}^\circ_{C/\mathbb{R}}$ is still a smooth commutative algebraic group over \mathbb{R} , whose structure can be described in relatively concrete terms from the singularities of C , see [BLR, Chap 9].

Jacobs are clearly commutative, and projective by Grothendieck's theorem. We are going to see that those properties hold for all abelian varieties.

For commutativity (and many other results about abelian vars) the key result is:

thm (rigidity) X, Y geometrically integral finite type/ \mathbb{R} , Z separated/ \mathbb{R} .

Let $g: X \times Y \rightarrow Z$ morphism. Assume

1) X proper

2) $\exists K/\mathbb{R}$ and $y_0 \in Y(K)$ such that $g_{y_0}: X_K \rightarrow Z_K$ is constant.

Then g factors through the second projection $X \times Y \rightarrow Y$.

i.e "constant maps from a proper variety to a separated variety are rigid".

proof: There is a preliminary reduction to \mathbb{R} alg. closed and y_0 rational point.

Let $\{z_0\} = g(X \times \{y_0\})$ and $U \subseteq Z$ affine neighbourhood of z_0 . By closedness of $X \times Y \rightarrow X$, there exists $V \subset Y$ such that $X \times V \subset g^{-1}(U)$. Because U is affine and X is proper, we have: $\forall (x, y) \in (X \times V)(\mathbb{R})$, $g(x, y) = g(x_0, y)$.

Since Z is separated, the equality scheme $(g, g(x_0, -))^{-1}(\Delta_{Z/\mathbb{R}}) \subset X \times Y$ is a closed subscheme of $X \times Y$, which contains $(X \times V)(\mathbb{R})$. Since $X \times Y$ is reduced and $(X \times V)(\mathbb{R})$ is Zariski-dense (by $\mathbb{R} = \bar{\mathbb{R}}$ and Y irreduc.) we have $g = g(x_0, -)$ \square

- cor:
- (i) A, A' ab. var/ \mathbb{R} . Then any morphism $A \xrightarrow{f} A'$ with $f(e_A) = e_{A'}$ is an isomorphism.
 - (ii) Any abelian var over a field is commutative.
 - (iii) The multiplication map of an abelian variety is completely determined by e .

mod: (i) \Rightarrow (ii): apply to the inversion map.

(i) \Rightarrow (iii): apply to the identity map.

(i): Apply rigidity to $A \times A \rightarrow A'$

$$(a_1, a_2) \mapsto f(a_1 a_2) f(a_2)^{-1} f(a_1)^{-1}$$

□

2) The dual and the Mumford bundle

- We are interested in the Picard scheme of A/\mathbb{R} . Since we have a distinguished rational point given by $e_A \in A(\mathbb{R})$, we always use the Picard functor rigidified at e_A :

$$\text{Pic}_{A/\mathbb{R}}(S) = \left\{ (\mathcal{L}, \alpha) \mid \begin{array}{l} \mathcal{L} \text{ line bundle on } A \times_S \mathbb{R} \\ \alpha: e_{A_S}^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_S \end{array} \right\}.$$

We know that $\text{Pic}_{A/\mathbb{R}}$ is representable by a group scheme locally of finite type, and that its identity component $\text{Pic}_{A/\mathbb{R}}^\circ$ is proper (Mumford for the representability, smoothness of A for the properness). So if we know that $\text{Pic}_{A/\mathbb{R}}^\circ$ is smooth, it would be an abelian variety. This is true, but not so easy to prove, we need some preliminaries.

- By definition, there is a universal rigidified line bundle $(\mathcal{P}, \alpha) \in \text{Pic}_{A/\mathbb{R}}^\circ(\text{Pic}_{A/\mathbb{R}})$ the Poincaré bundle. We also write \mathcal{P} for its restriction to $A \times \text{Pic}_{A/\mathbb{R}}^\circ$.

def: Let \mathcal{L} be a line bundle on A . The Mumford bundle $\Lambda(\mathcal{L})$ is

$$\Lambda(\mathcal{L}) := m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1} \text{ line bundle on } A \times A.$$

together with its canonical rigidification on $\{e_A\} \times A$ given by

$$(e_A \times \text{id}_A)^* \Lambda(\mathcal{L}) \simeq (e_A \times \text{id})^* [m^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}]$$

$$\simeq \mathcal{L} \otimes \mathcal{O}_A \otimes \mathcal{L}^{-1}$$

$$\simeq \mathcal{O}_A.$$

mk: $\Lambda(\mathcal{L})(\alpha_{\mathcal{L}})$ measures the failure of \mathcal{L} to be a character sheaf.

- By construction, we get a morphism $A \xrightarrow{\phi_{\mathcal{L}}} \text{Pic}_{A/\mathbb{R}}^\circ$, which on \mathbb{R} -points can be

described as $x \in A(\mathbb{R}) \mapsto (t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}, \text{can. rigidification}) \in \text{Pic}_{A/\mathbb{R}}^\circ(\mathbb{R})$.

- lemma: The morphism $\phi_{\mathcal{L}}$ factors through $\text{Pic}_{A/\mathbb{R}}^\circ$.

mod: A is connected and $\phi_{\mathcal{L}}(0) = 0$. □

- At this point, if we knew that $\text{Pic}_{A/\mathbb{R}}^\circ$ was reduced (hence an abelian variety) we would get $\phi_{\mathcal{L}}$ homomorphism by the previous section! We have to prove this in a different way.

This will follow from a rigidity theorem for line bundles:

thm: ("Theorem of the cube")

$X, Y, Z/k$ with:

- X, Z proper, geometrically integral
- Y separated, finite type and geometrically integral.

Let $(x_0, y_0, z_0) \in (X \times Y \times Z)(k)$ and \mathcal{L} line bundle on $X \times Y \times Z$ such that \mathcal{L} restricted to $x_0 \times Y \times Z$, $X \times y_0 \times Z$ and $X \times Y \times z_0$ is trivial. Then \mathcal{L} is trivial.

proof: The rigidified Picard functor $\text{Pic}_{Z/k, z_0}$ is representable by a separated, loc flt scheme $/k$.

Using $\mathcal{L}_{z_0} \simeq \mathcal{O}_{X \times Y}$ we get a map $X \times Y \xrightarrow{g} \text{Pic}_{Z/k, z_0}$. We have that

$g|_{X \times y_0}$ is constant by \mathcal{L}_{y_0} trivial. By the rigidity theorem of i), we deduce

that $g(x, y) = g(x_0, y)$. But $g(x_0, -) = 0$ because \mathcal{L}_{x_0} is trivial. Hence

g is identically 0, hence \mathcal{L} is trivial. \square

This general fact implies a general relation for line bundles on abelian varieties:

thm ("All line bundles on an abelian variety have a natural cubical structure.")

Let \mathcal{L} be a line bundle on an abelian variety. Then

$$\Theta(\mathcal{L}) := m_{123}^* \mathcal{L} \otimes (m_{12}^* \mathcal{L}^{-1} \otimes m_{13}^* \mathcal{L}^{-1} \otimes m_{23}^* \mathcal{L}^{-1}) \otimes (m_1^* \mathcal{L} \otimes m_2^* \mathcal{L} \otimes m_3^* \mathcal{L}) \otimes e^* \mathcal{L}$$

(with m_I the sum of the corresponding components) on $A \times A \times A$ is canonically trivial.

rk: We have $\Theta(\mathcal{L}) = (1 \times m)^* \Lambda(\mathcal{L}) \otimes P_{12}^* \Lambda(\mathcal{L})^{-1} \otimes P_{13}^* \Lambda(\mathcal{L})^{-1}$

(and similar iso by permutation of variables). Hence " \mathcal{L} is a 2nd order character sheaf". The canonical section to $\Theta(\mathcal{L})$ has a number of interesting properties [Breen].

proof: We just prove that $\Theta(\mathcal{L})$ is trivial, by applying the previous result.

By symmetry it is enough to show that $\Theta(\mathcal{L})|_{e \times A \times A}$ is trivial.

$$\begin{aligned} \Theta(\mathcal{L})|_{e \times A \times A} &= m^* \mathcal{L} \otimes P_1^* \mathcal{L}^{-1} \otimes P_2^* \mathcal{L}^{-1} \otimes m^* \mathcal{L}^{-1} \otimes (e^* \mathcal{L})_{A \times A} \otimes P_1^* \mathcal{L} \otimes P_2^* \mathcal{L} \otimes (e^* \mathcal{L}^{-1})_{A \times A} \\ &\simeq \mathcal{O}_{A \times A} \end{aligned} \quad \square$$

cor: ("Theorem of the square") For all \mathcal{L} , $\phi_{\mathcal{L}}$ is a homomorphism.

proof 1: We have to show that, for any K/k field ext. and $x, y \in A(K)$, we

have $\phi_{\mathcal{L}}(x+y) = \phi_{\mathcal{L}}(x) + \phi_{\mathcal{L}}(y)$ in $\text{Pic}_{A/k}^\circ(K)$.

i.e. we have to show: $t_{x+y}^*(\mathcal{L}_K) \otimes \mathcal{L}_K^{-1} \simeq (t_x^* \mathcal{L}_K \otimes \mathcal{L}_K^{-1}) \otimes (t_y^* \mathcal{L}_K \otimes \mathcal{L}_K^{-1})$.

We apply the previous theorem, pulled back to $T \times A$ via the map

$K \times A \longrightarrow A \times A \times A$ whose components are:

* $K \times A \rightarrow A$ projection

* $[x]: T \times A \rightarrow K \xrightarrow{\phi_L} A$

* $[y]: T \times A \rightarrow K \xrightarrow{\phi_L} A$

We get after a small computation (using $\text{Pic}(\text{Spec } K) \cong \{0\}$):

$$t_{x+y}^* \mathcal{L}_K \otimes (0 \otimes t_x^* \mathcal{L}_K^{-1} \otimes t_y^* \mathcal{L}_K^{-1}) \otimes (\mathcal{L}_K \otimes 0 \otimes 0) \otimes 0 \cong 0$$

which is exactly the required property. \square

Proof 2: (More functorial) We check directly that the diagram

$A \times A \xrightarrow{m} A \xrightarrow{\phi_L} \text{Pic}_{A/R}$ commutes. A map $A \times A \rightarrow \text{Pic}_{A/R}$ classifies a rigidified line bundle on $A \times A \times A$.

$\phi_L \times \phi_L \downarrow$ m One checks that the top path classifies $(\text{id} \times m)^* \Lambda(\mathcal{L})$, while the bottom path classifies $P_{12}^* \Lambda(\mathcal{L}) \otimes P_{13}^* \Lambda(\mathcal{L})$ (with appropriate rigidification)

Hence the result follows from the triviality of $\Lambda(\mathcal{L})$ and the remark just after the theorem on cubical structure. \square

def: $|K(\mathcal{L})| := \text{Ker } \phi_L$ subgroup scheme of A .

3) Ample line bundles

The construction $\mathcal{L} \mapsto \phi_{\mathcal{L}}$ allows one to study general line bundles through the one in $\text{Pic}_{A/R}^\circ$. The clearest case of this is for ample line bundles, which cannot lie in $\text{Pic}_{A/R}^\circ$. One small issue is that $\phi_{(\mathcal{L}^{-1})}(x) = \phi_{\mathcal{L}}(-x)$ so it is difficult to distinguish between ample and anti-ample!

thm (Characterization of ample line bundles)

Let \mathcal{L} be a line bundle. The following conditions are equivalent:

(i) \mathcal{L} is ample.

(ii) Some power of \mathcal{L} has a global section + $K(\mathcal{L})$ is finite over R .

Proof: (i) \Rightarrow (ii): The first condition is clear. For the second, we can assume $R = \bar{R}$.

Put $B = K(\mathcal{L})_{\text{red}}^\circ \subset A$. This is a group scheme ("G grp scheme $\Rightarrow G_{\text{red}}$ " for R perfect), reduced \Rightarrow smooth and connected. So B is an abelian variety and we must show that $B = 0$. We have: $\forall b \in B(R)$, $t_b^* \mathcal{L} \cong \mathcal{L}$. This implies that $\phi_{\mathcal{L}|_B}: B \rightarrow \text{Pic}_{B/R}^\circ$ is 0. Since $\mathcal{L}|_B$ is the restriction of an ample line bundle, it is still ample. So it remains to prove:

Lemma: If $A \neq 0$ is an abelian var. and \mathcal{L} is ample, then $\phi_{\mathcal{L}} \neq 0$.

Proof: By construction, $\phi_{\mathcal{L}} = 0 \iff \Lambda(\mathcal{L}) = \mathcal{O}_{A \times A}$. Let us restrict

this iso to the anti-diagonal $\Delta^- = \{(x, -x) \in A \times A\} \cong A$. We get

$$\mathcal{O}_A \cong \mathcal{L}^{-1} \otimes [-1]^* \mathcal{L}^{-1} \text{ hence by inversion } \mathcal{O}_A \cong \mathcal{L} \otimes [-1]^* \mathcal{L}.$$

But \mathcal{L} is ample and $[-1]^* \mathcal{L}$ is ample (as $[-1]: A \xrightarrow{\sim} A$). So \mathcal{O}_A is ample. This contradicts the fact that A is proper and of dimension > 0 . \square

(ii) \Rightarrow (i): We can replace \mathcal{L} by a power and assume $\mathcal{L} \cong \mathcal{O}_A(D)$ with D effective Cartier divisor. From $K(L)$ finite, we deduce that $\{x \in A(\mathbb{R}) \mid t_x^* D = D\}$ is finite. Ampleness can be checked after base extension [Conrad-Ab, Prop 3.4.2] \Rightarrow WLOG $R = \mathbb{R}$. We can also clearly assume $\dim(A) > 0$.

Claim: $\mathcal{L}^{\otimes 2}$ is base-point free.

We have to show that, for all $a \in A(\mathbb{R})$, there exists an effective divisor $D' \sim 2D$ such that $a \notin D'$. By the theorem of the square, we have:

$\forall x \in A(\mathbb{R})$, $t_x^* D + t_{-x}^* D \sim 2D$. So we just need to find x such that $a \notin t_x^* D$ and $a \notin t_{-x}^* D$. This is equivalent to $\pm x \notin -a + D$. Since $-a + D$ is irreducible of codimension 1, and $\mathbb{R} = \overline{\mathbb{R}}$, such an x exists.

Claim: The induced morphism $A \xrightarrow{f} \mathbb{P}(\Gamma(A, \mathcal{L}^{\otimes 2}))$ is finite.

Since f is proper, it is enough to show that it is quasi-finite.

We have the following cool result about morphisms from abelian varieties:

Prop: [Moonen-VanderGeer, prop 2.20] A abelian variety $/_{\mathbb{R}} = \overline{\mathbb{R}}$. Let $f: A \rightarrow X$ be a morphism with X separated. Then, for all $a \in A(\mathbb{R})$ we have

$$f^{-1}(f(a))_{\text{red}}^\circ = f^{-1}(f(0))_{\text{red}}^\circ + a, \text{ and } f^{-1}(f(0))_{\text{red}}^\circ \text{ is an abelian subvar.}$$

The proof is yet another application of the rigidity theorem.

This shows that, to prove f quasi-finite, it is enough to prove that $F_0 := f^{-1}(f(0))_{\text{red}}^\circ$ is trivial. We are going to show $F_0 \subseteq K(L)_{\text{red}}^\circ = \{0\}$.

Let $x \in F_0(\mathbb{R})$. We have $f \circ t_x = f$. Hence, if $s \in H^0(A, \mathcal{L}^{\otimes 2})$, then s and $t_x^* s$ have the same divisor of zeroes. We apply this to $s = v^{\otimes 2}$ with $v \in H^0(A, \mathcal{L})$ with $\text{div}(v) = D$. This implies $t_x^* D = D$, hence

$t_x^* \mathcal{L} = \mathcal{L}$, hence $x \in K(L)$. So $F_0 \subseteq K(L)$ hence by integrality of F_0 , we have $F_0 \subseteq K(L)_{\text{red}}^\circ \Rightarrow F_0 = \{0\}$.

We have found a finite morphism f and an ample line bundle

$\mathcal{O}(1)$ such that $f^*(\mathcal{O}(1)) \cong \mathcal{L}^{\otimes 2}$. By Serre's cohomological criterion, $\mathcal{L}^{\otimes 2}$ is ample. \square

con: Abelian varieties are projective.

proof: We need to find an effective divisor D such that $K(G(D))$ is finite. Can assume $\mathbb{R} = \bar{\mathbb{R}}$.
Let $U \subset A$ be any affine open neighbourhood. Put $D := (A \setminus U)_{\text{red}}$. Then
 D is of pure codimension 1 (because U is affine in a normal var.) Hence defines
a reduced Cartier divisor. From the proof of the previous prop, we see that it
is enough to show that $H(D)$ is finite, with $H(D)$ the reduced closed subscheme
defined by $H(D)(\mathbb{R}) = \{x \in A(\mathbb{R}) \mid t_x^* D = D\}$ (note that $H(D) \subseteq K(G(D))$, and
that we have shown $H(D)$ finite
 $\Rightarrow G(D)$ ample)

If $x \in H(D)(\mathbb{R})$ then $x + D = D \Rightarrow x + U = U$.

Since $0 \in U$ we get $H(D) \subseteq U$. So $H(D)$ is proper (closed in A) and
contained in U affine. So $H(D)$ is finite \square

rmk: Any smooth projective variety of dimension g over an infinite field
can be embedded into \mathbb{P}^{2g+1} . It turns out that, for abelian varieties,
this is sharp most of the time, for instance if $g \geq 3$. See [Moonen-Vandenberg]
and mathoverflow/questions/14177. This means that it is difficult
to write explicit equations for abelian varieties.

rmk: We have only scratched the surface. The theory of ample line bundles
and their spaces of sections (often called theta functions) is a very
rich topic, developed by Mumford in his papers "On the equations
defining abelian varieties I-II".

4) Smoothness

thm: $\text{Pic}_{A/\mathbb{R}}^\circ$ is an abelian variety, called the dual abelian variety of A .

proof: Let $g = \dim A$. Choose \mathcal{L} ample line bundle. We have $\text{Ker } \Phi_{\mathcal{L}}$ finite,
hence $\Phi_{\mathcal{L}}$ is finite by properness. Hence by quasi-finiteness, $\dim \text{Pic}_{A/\mathbb{R}}^\circ \geq \dim A$.

So it is enough to prove $\dim T_0 \text{Pic}_{A/\mathbb{R}}^\circ = \dim H^*(A, \mathcal{O}_A) \leq g$. (holds for any
group scheme, as the proof shows)

Put $R := \bigoplus_i H^*(A, \mathcal{O}_A)$. This is a graded ring, and $R^i = 0$ for $i > g$.

R is anticommutative. We have $R^\circ = R$. By Serre duality, using that the
cotangent sheaf of any group scheme is trivial, we get that $R^g \cong \mathbb{R}$ and
 $R^i \times R^{g-i} \rightarrow R^g \cong \mathbb{R}$ is a perfect pairing. Using the multiplication and the
Künneth formula for coherent cohomology, we get a comultiplication

$$\mu : R \rightarrow R \otimes R.$$

We check that this makes R into a graded Hopf algebra over \mathbb{R} .

- We can now use a structure theorem of Borel:

thm: Let R be a finite dimensional \mathbb{N} -graded commutative Hopf algebra over \mathbb{k} .
 Assume $\cdot R^0 \cong R^g \cong \mathbb{k}$
 $\cdot R^i \times R^{g-i} \rightarrow R^g \cong \mathbb{k}$ perfect for all i .

Then $\dim R^i \leq g$ with equality iff the natural map $\bigwedge^i R^i \rightarrow R^i$ is an isomorphism.

- This concludes the proof (and shows that $H^i(A, \mathcal{O}_A) \cong \bigwedge^i H^1(A, \mathcal{O}_A)$ for all i) \square

5) Final remarks

- To conclude, let us discuss what happens over a general base.
- thm: (Raynaud) Let S be an integral normal scheme. Then any abelian scheme over S is projective. Also, there exist non-projective abelian schemes over noetherian 1-dimensional bases.
- See [Faltings-Chai, Chap 1] and [Raynaud-ample].
- thm: (Raynaud) A smooth proper group algebraic space with geometrically connected fibers is a scheme.

Again see [Faltings-Chai, Chap 1].

- Combining this with the general theory of Artin and the results of this chapter, one can then prove:

Cor: Let A/S be an abelian scheme (over any base S). Then $\text{Pic}_{A/S}^0$ is representable by an abelian scheme, the dual abelian scheme \hat{A} of A .

- The next topic in logical order would be Polarizations, i.e. morphisms $\phi: A \rightarrow \hat{A}$ which after base change to $\bar{\mathbb{K}}$ are of the form ϕ_L with L ample. We change gear and go to Néron models.