We can also apply the free cocompletion property to prove that $PSh(C)$ (and in part. sSet) is cartesian closed.

def 14 A category C with finite products is cartesian closed is for all $a \in C$, the functor

$$a \times - : C \longrightarrow C$$

has a right adjoint, which is then denoted $\underline{\mathrm{Hom}}(a,-)$ and called the internal Hom or exponentiation. $\square$

Prop 15 $PSh(C)$ is cartesian closed and

$$\boxed{\underline{\mathrm{Hom}}(F,G)(X) = PSh(C)(F \times y(X), G)}$$

Proof: Can be done by hand, but also as applicat° of Thm 9.b):

- $F \times -$ is colimit-preserving (because products / colimits are computed objectwise) $\rightsquigarrow$ it has a right adjoint, given by this formula. $\square$

3) Structure of Δ and applications

- We now go into the structure of Δ and what it means for sSet.

Notation:

Following [Rezk], we write

$$f = \langle f_0 \cdots f_n \rangle : [n] \longrightarrow [m]$$
$$k \longmapsto f_k$$

def 16 There are distinguished morphisms for every $0 \leq i \leq n$:

$$\delta^i := \langle 0 \cdots \hat{i} \cdots n \rangle : [n-1] \hookrightarrow [n]$$

(face morphisms)

$$\sigma^i := \langle 0 \cdots i\, i \cdots n \rangle : [n+1] \twoheadrightarrow [n]$$

(degeneracy morphisms)

If $X \in sSet$, we write

$$\begin{cases} d_i = (\delta^i)^* : X_n \longrightarrow X_{n-1} & \text{(face maps)} \\ s_i = (\sigma^i)^* : X_n \longrightarrow X_{n+1} & \text{(degeneracy maps)} \end{cases}$$

Lemma 17:

a) We have the simplicial identities:

$$d_i d_j = d_{j-1} d_i \,, \quad i < j$$

$$s_i s_j = s_{j+1} s_i \,, \quad i \leq j$$

$$d_i s_j = \begin{cases} 1 & , \ i = j, j+1 \\ s_{j-1} d_i & , \ i < j \\ s_j d_{i-1} & , \ i > j+1 \end{cases}$$

b) Every morphism $\delta : [n] \to [m]$ can be written as

$$[n] \overset{S}{\twoheadrightarrow} [r] \overset{D}{\hookrightarrow} [m]$$

with $\begin{cases} S & \text{composite of deg. morphisms} \\ D & \text{——— face morphisms} \end{cases}$

c) The datum of a simplicial object is equivalent to a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right=40, "d_0"'] \arrow[l, bend left=40, "d_1"] \arrow[r, bend left=20, "s_0"] \arrow[r, bend right=20, "s_1"'] & X_2 \arrow[l, bend right=50, "d_0"'] \arrow[l, "d_1"] \arrow[l, bend left=50, "d_2"] & \cdots
\end{tikzcd} \qquad (*)$$

satisfying the simplicial identities.

$\square$

Proof: a) is an exercise.

b) f factors uniquely into a surjective map follows by an injective map:

$$\begin{tikzcd}
{[n]} \arrow[r, two heads] \arrow[dr, two heads, "S"'] & \mathrm{Im}(f) \arrow[r, hook] \arrow[d, "\exists! \cong"] & {[m]} \\
 & {[r]} \arrow[ur, hook, "D"'] &
\end{tikzcd}$$

so it is enough to show that

- S is composition of deg. morphisms
- D ———————— faces ————

Let's do the case of S (D is similar)

This is an induction on $m - r \geq 0$.

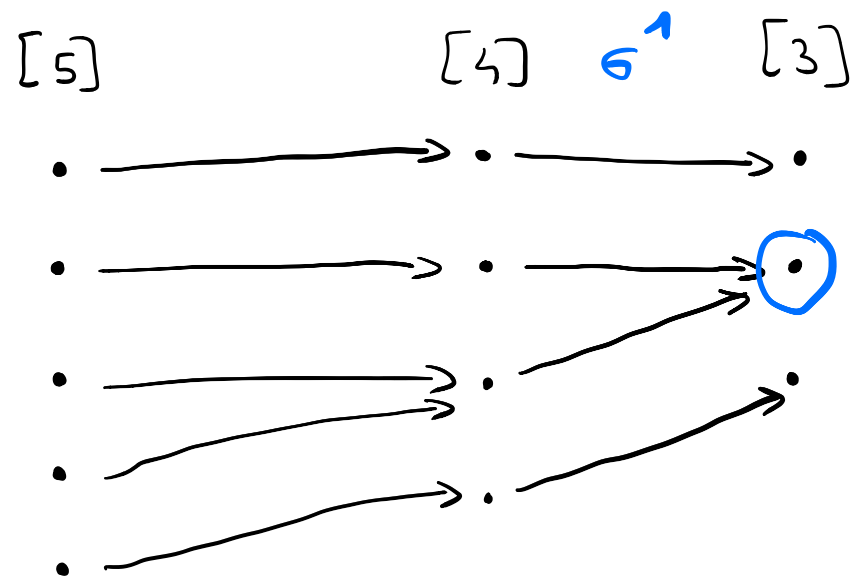
Assume $m > r$;

then $\exists\, i \in [r]$ such that $|D^{-1}(i)| > 1$.

Split $D^{-1}(i)$ into two non-empty sets to get a factorisation

$$D : [n] \xrightarrow{D'} [n+1] \xrightarrow{\sigma^i} [n].$$

<u>ex</u>

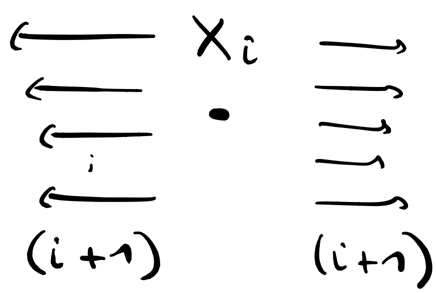


c) Almost follows from a) + b); it remains to check that the simplicial identities imply that different choices of factorisations of

D, S yield the same map in a diagram $(*)$. $\square$

Rmk

a) Mnemotechnics



b) A cosimplicial object is similar:

$$
\begin{tikzcd}
X^0 \arrow[r, shift left=2] \arrow[r, shift right=2] & X^1 \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & X^2 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] & \cdots
\end{tikzcd}
$$

but imbalanced # of arrows...

def 18

- For $X_\bullet \in sSet$, a simplicial subset $Y_\bullet \subset X_\bullet$ is the datum of $\{Y_n \subset X_n\}_n$, stable under f^* for all $f : [m] \to [n]$ in Δ. $\square$

Examples "Shape repertoire"!

* Let $S \subseteq [n]$. We write

$$\Delta^S \subseteq \Delta^n$$ for the S-face of Δ^n:

$$(\Delta^S)_k = \{ f \in (\Delta^n)_k \mid \mathrm{Im}(f) \subseteq S \}.$$

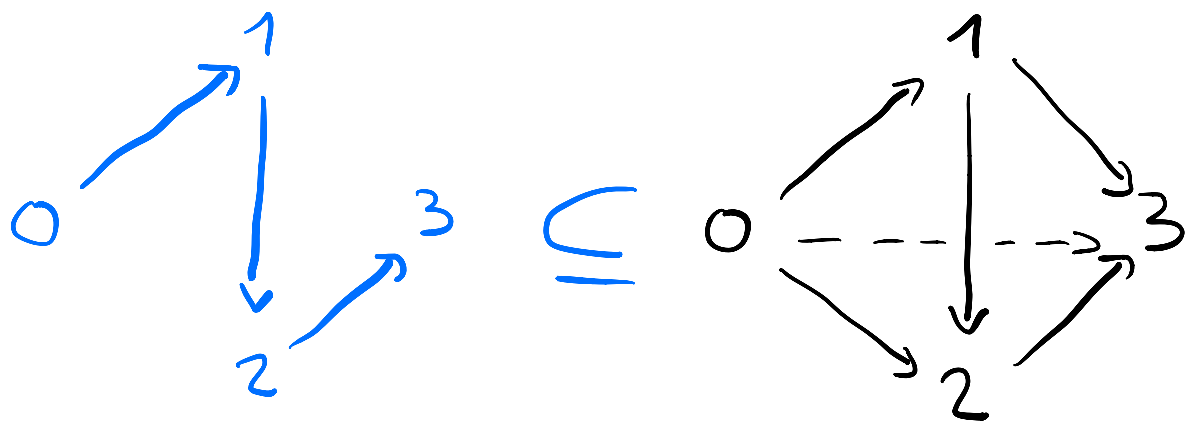
$\exists !$ isomorphism $\Delta^S \simeq \Delta^{|S|}$.

$$\begin{cases} \Delta^{\{i\}} \subseteq \Delta^n \text{ are vertices} \\ \Delta^{\{i<j\}} \subseteq \Delta^n \text{ are edges} \end{cases}$$

$$\begin{tikzcd}
 & & \Delta^{\{1\}} \arrow[d, equal] & & \\
 & & 1 \arrow[dr, "\Delta^{\{1,2\}}"] & & \\
\Delta^{\{0\}} \arrow[r, equal] & 0 \arrow[ur, "\Delta^{\{0,1\}}"] \arrow[rr, "\Delta^{\{0,2\}}"'] & & 2 & \Delta^{\{2\}} \arrow[l, equal]
\end{tikzcd}$$

* The spine $I^n \subseteq \Delta^n$ is

$$(I^n)_i = \left\{ \langle a_0 \cdots a \rangle \in (\Delta^n)_i \;\middle|\; a_i \leq a_0 + 1 \right\}$$



$$I^3 \subseteq \Delta^3$$

* The boundary, or

simplicial n-sphere $\partial\Delta^n \subseteq \Delta^n$

is

$$(\partial\Delta^n)_i = \{ f \in (\Delta^n)_i \mid \mathrm{Im}(f) \neq [n] \}$$

We have:

$$\partial\Delta^n = \bigcup_{j=0}^{n} \Delta^{[n]\setminus j}$$

The name comes from:

$$|\partial\Delta^n| \simeq S^n$$

* For $0 \leq k \leq n$, the k-th horn $\Lambda^n_k \subseteq \Delta^n$ is

$$(\Lambda^n_k)_i = \{ f \in (\Delta^n)_k \mid i \notin \mathrm{Im}(f) \}$$

So:

$$\Lambda^n_k = \bigcup_{j \neq k} \Delta^{[n] \setminus k}$$

$$|\Lambda^n_k| \simeq D^n \quad \text{n-disk}$$

Λ^2_0	$\Lambda^2_1 \, (= I^2)$	Λ^2_2
$0 \to 1,\ 0 \to 2$	$0 \to 1,\ 1 \to 2$	$1 \to 2,\ 0 \to 2$

* Horns Λ^n_k with

- $0 < k < n$ are inner horns.
- $k = 0, n$ are outer horns.

Lemma 20

a). Inner horns contain spines:

$$0 < k < n \implies I^n \subseteq \Lambda^n_k$$

b) $\forall\, n \geq 3$, $\begin{cases} I^n \subseteq \Lambda^n_0 \\ I^n \subseteq \Lambda^n_n \end{cases}$

c) $I^1 \nsubseteq \Lambda^1_0, \Lambda^1_1$

$I^2 \nsubseteq \Lambda^2_0, \Lambda^2_2$ $\square$

4) Skeletal filtration

Presheaves are colimits of representables, but for simplicial sets we can be more precise.

Idea: Add first all the 0-simplices
then all the 1-simplices
...

def 21 Let $X_\bullet \in sSet$ and $x \in X_n$. We say that x is degenerate if $n > 0$ and the following equivalent conditions hold:

* $x \in \mathrm{Im}(s_i : X_{n-1} \longrightarrow X_n)$ for some i.
* x factors through Δ^m for some $m<n$:

$$x : \Delta^n \longrightarrow \Delta^m \longrightarrow X_\bullet$$

Otherwise we say that x is non-degenerate. $\square$

Notation

· $X_n = X_n^{nd} \amalg X_n^{deg}$

⚠ $X_\bullet^{nd}, X_\bullet^{deg}$ are not simplicial subsets of $X_\bullet$.

· **Example**

$$(\Delta^n)^{nd}_k = \{ [k] \hookrightarrow [n] \}$$

correspond to the "true" k-dimensional faces of $|\Delta^n|$.

Prop 22: (Eilenberg-Zilber lemma)

Let $X_\bullet \in sSet$, $n \geq 0$ and $x \in X_n$. Then $x : \Delta^n \to X$ can be factored uniquely as

$$\boxed{x : \Delta^n \xrightarrow{p^*} \Delta^m \xrightarrow{\tau} X_\bullet}$$

with :

- $p : [m] \twoheadrightarrow [n]$ surjective
- τ non-degenerate m-simplex .

proof:

Existence

Let $m \geq 0$ be minimal for the existence of a factorisation

$$x : \Delta^n \xrightarrow{\delta^*} \Delta^m \xrightarrow{\tau} X .$$

Then :

- δ^* is surjective (otherwise we would have a factorisation through $\mathrm{Im}(\delta^*)\big)\big) \Rightarrow \delta$ is surjective.

- τ is non-degenerate (otherwise we would have a factorisation

$$x : \Delta^n \to \Delta^m \to \Delta^{m'} \to X.$$

with $m' < m$).

<u>Uniqueness:</u>

Let $x : \Delta^n \xrightarrow{(\delta')^*} \Delta^{m'} \xrightarrow{\tau'} X.$

be another such factorisation.

Write $\alpha = \delta^*$, $\alpha' = (\delta')^*$.

* α, α' surjective

$$\Updownarrow$$

γ, γ' surjective

$$\Downarrow$$

γ, γ' admit sections

$$\Downarrow$$

α, α' admit sections β, β'.

* We get

$$\begin{aligned} \tau &= \tau \circ \alpha \circ \beta \\ &= x \circ \beta \\ &= \tau' \circ \alpha' \circ \beta . \end{aligned}$$

* Because τ is non-deg,

$\alpha' \circ \beta$ inj $\Rightarrow m \leq m'$.

By symmetry, $m = m'$.

$\Rightarrow \alpha' \circ \beta = id_{[m]}$

$$\Rightarrow \begin{cases} \tau = \tau' \circ \alpha' \circ \beta = \tau' \\ \alpha = \alpha' \circ \beta \circ \alpha = \alpha' \end{cases}$$ $\blacksquare$

def 23 (Skeleta)

$X_\bullet \in sSet$, $k \geq -1$.

$$Sk_k(X_n) := \left\{ x \in X_n, \exists \text{ fact}^\circ \ \Delta^n \to \Delta^m \to X_\bullet \text{ with } m \leq k \right\}$$

$Sk_k(X_\bullet) \subseteq X_\bullet$ is a simplicial subset of $X_\bullet$, the **k-th skeleton** of $X_\bullet$. $\square$

By construction:

$$\begin{cases} \emptyset = Sk_{-1}(X_\bullet) \subseteq Sk_0(X_\bullet) \subseteq \cdots \\ \bigcup_{k \geq -1} Sk_k(X_\bullet) = X_\bullet \end{cases}$$

and $X_n^{nd} \cap Sk_k(X_\bullet) = \begin{cases} \emptyset, & k < n \\ X_n^{nd}, & k \geq n \end{cases}$

Rmk Sk_k induces a functor

$$Sk_k(-) : sSet \longrightarrow sSet$$

with interesting properties (see Exercise Sheet 2).

Prop 24: Let $X_\cdot \in sSet$, $k \geq 0$. There is a pushout square

$$\begin{tikzcd} \coprod_{X_k^{nd}} \partial \Delta^k \arrow[r] \arrow[d] & \coprod_{X_k^{nd}} \Delta^k \arrow[d] \\ Sk_{k-1}(X_\cdot) \arrow[r] & Sk_k(X_\cdot) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

More generally, for any $A_\bullet \subseteq X_\bullet$ subcomplex, there is a pushout square

$$\begin{tikzcd} \coprod_{X_k^{nd} \setminus A_k^{nd}} \partial\Delta^k \arrow[r] \arrow[d, hook] & \coprod_{X_k^{nd} \setminus A_k^{nd}} \Delta^k \arrow[d, hook] \\ A_\bullet \cup Sk_{k-1}(X_\bullet) \arrow[r] & A_\bullet \cup Sk_k(X_\bullet) \end{tikzcd}$$

proof: Let's do the particular case $A_\bullet = \emptyset$.

We have $x \in X_k^{nd} \Rightarrow x \in Sk_k(X_\bullet)_k$.

and the faces of $x \in X_k^{nd}$ are in $Sk_{k-1}(X) \Rightarrow$ we have the commutative square of the statement.

- We observe that we have

$$\left(\coprod_{X_k^{nd}} \Delta^k\right)_n \setminus \left(\coprod_{X_k^{nd}} \partial\Delta^k\right)_n$$

$$\text{IS}$$

$$\left\{ \delta^* x \mid x \in X_k^{nd}, \delta : [n] \twoheadrightarrow [k] \right\}$$

$$\text{IS (Eilenberg-Zilber)}$$

$$Sk_k(X_\cdot)_n \setminus Sk_{k-1}(X_\cdot)_n \, .$$

- Moreover, the square is clearly a pullback.
- It remains to show

Lemma If
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r, hook] & D \end{tikzcd}$$
in $s\mathrm{Set}$

satisfies: $\forall n \geq 0$, $B_n \setminus A_n \xrightarrow{\sim} D_n \setminus C_n$

then it is a pushout.

which reduces to the same statement in Set (since (co)limits are computed objectwise) and is then an exercise. $\square$

Cor 25 Let $X_\bullet \in sSet$. Then the geometric realisation is a CW-complex, whose k-cells are in bijection with X_k^{nd}.
If $A_\bullet \subseteq X_\bullet$ is a simplicial subset, then $|A_\bullet| \subseteq |X_\bullet|$ is a CW-subcomplex.

proof: $|-|$ is a left adjoint $\Rightarrow$ commutes with colimits.
Hence: $$|X| = \bigcup_{k \geq -1} |Sk_k(X_\bullet)|$$
and we have a pushout

$$\begin{tikzcd}
\coprod\limits_{X_k^{nd}} S^k \arrow[r] \arrow[d] & \coprod\limits_{X_k^{nd}} D^k \arrow[d] \\
|Sk_{k-1}(X_\bullet)| \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & |Sk_k(X_\bullet)|
\end{tikzcd}$$

$\blacksquare$

Rmk This shows that, to model homotopy types, one could forget about degeneracies and work with $PSh(\Delta^{inj})$.

Not so for our purpose!

Examples

$$\begin{cases} Sk_0(I^n) = \coprod \Delta^{\{i\}} \\ Sk_i(I^n) = I^n \quad \text{for all } i \geq 1 \end{cases}$$

and $(I^n)_1^{nd} = \left\{ \Delta^{\{i,i+1\}} \,\middle|\, 0 \leq i \leq n-1 \right\}$

We deduce :

$$\boxed{I^n = \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod \cdots \Delta^{\{n-1,n\}}}$$

· Similar arguments show :

$$\partial \Delta^n = \coprod_{\Delta^{[n] \setminus \{i,j\}}} \Delta^{[n] \setminus i}$$

$$\Lambda^n_k = \coprod_{\substack{\Delta^{[n] \setminus \{i,j\}} \\ i \neq k}} \Delta^{[n] \setminus i}$$

5) Kan complexes

Simplicial sets model homotopy types via geometric realisation. But if one tries to develop homotopy theory directly in sSet with $[0,1] \rightsquigarrow \Delta^1$, things do not work very well:

Rmk: In general, the relation on X_0 defined by

$$x \sim y \iff \exists\, \Delta^1 \xrightarrow{h} X, \quad d^0(h) = x,\ d^1(h) = y$$

is neither symmetric ($X = \Delta^1$)

nor transitive ($X = I^2$)

def 26 Let $X_\bullet \in sSet$.

$$\pi_0(X_\bullet) := X_0 / \simeq$$

with $\simeq$ the equivalence relation generated by $\sim$. $\square$

Rmk $\pi_0(X_\bullet) \simeq \operatorname{Colim}(X_\bullet : \Delta^{op} \to Set)$

This is already unsatisfactory and things get worse for π_n for $n \geq 1$.

This is the topic of simplicial homotopy theory and leads to:

def 27 $X_\bullet \in sSet$ is a Kan complex or Kan simplicial set if it has the Kan lifting property:

$\forall\, n \geq 1,\ \forall\, 0 \leq k \leq n,$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$X_\bullet$ is a contractible Kan complex if $\forall n \geq 1$,

$$\begin{tikzcd} \partial \Delta^n \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

More generally, $X_\bullet \to Y_\bullet$ is a Kan fibration if

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y_\bullet \end{tikzcd}$$

and a trivial Kan fibration

if
$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"'] & Y_\bullet \end{tikzcd}$$

Rmk It is not easy to really motivate these defs without going into simplicial homotopy theory.

The basic idea is:

"For a Kan complex $X_\bullet$, the

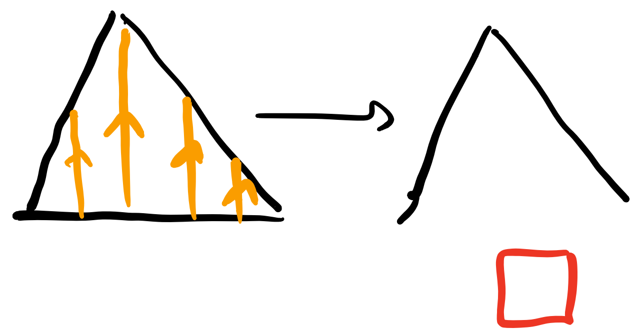
Homotopical properties of $|X_\cdot|$ can be expressed purely simplicially. »

Prop 28 Let $A \in \mathrm{Top}$. Then the singular simplicial set $\mathcal{S}(A)$ is a Kan complex.

Proof: By adjunction, we have to show:

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & A \\ {|\Delta^n|} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

But $|\Lambda^n_k| \hookrightarrow |\Delta^n|$ admits a continuous retraction.

In pictures: 

def 29 The homotopy category of Kan complexes $h\mathrm{Kan}$

has: - objects = Kan complexes

- morphisms = Δ^1-homotopy classes of morphisms

(need to show composition is well-defined)

· A morphism $f: X_\bullet \to Y_\bullet$ in sSet is a weak homotopy eq.

if $|f|: |X_\bullet| \to |Y_\bullet|$ is an homotopy equivalence of CW-complexes. $\square$

The main result of s. homotopy theory is:

Thm We have a diagram

$$\begin{tikzcd} h\mathrm{Kan} \arrow[r, "|\cdot|"] \arrow[d, "S"] & h\mathrm{CW} \arrow[d, "S"] \\ s\mathrm{Set}[\mathrm{w.h.eq}^{-1}] \arrow[r, "|\cdot|", "\sim"'] & \mathrm{Top}[\mathrm{w.h.eq}^{-1}] \end{tikzcd}$$

Remark This still does not explain why horns appear!

Lemma 30: Monomorphisms of simplicial sets are "generated" by the inclusions

$$\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$$

under
- pushouts
- transfinite composition

proof Follows from the existence of the skeletal filtration and Proposition 24. $\square$

The fundamental reason that horns appear in the definition of Kan complexes is the analoguous result:

Prop 31: Monomorphisms of simplicial sets which are also weak homotopy equivalences are "generated by" the horn inclusions

$$\left\{ \Lambda^n_k \hookrightarrow \Delta^n \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ 0 \leq k \leq n \end{array} \right\}$$

under
- pushouts
- retracts
- transfinite composition.

$\square$

4) Nerves of categories

Our model of ∞-categories will be a particular type of simplicial sets. We also want to be able to consider ordinary categories as ∞-categories.

$\rightsquigarrow$ We need a fully faithful functor

$$N : \mathrm{Cat} \hookrightarrow s\mathrm{Set}.$$

- Posets give rise to categories:

$$\mathrm{Poset} \xhookrightarrow{\text{fully faithful}} \mathrm{Cat}$$

$$P \longmapsto \mathrm{Ob}: x \in P$$

$$P(x,y) = \begin{cases} *, & x \leq y \\ \emptyset, & \text{otherwise} \end{cases}$$

In particular:

$$\Delta \xhookrightarrow{\iota} \mathrm{Cat}$$

def 32 By the free cocompletion property, ι induces an adjunction:

$$\begin{tikzcd} h : sSet \arrow[r, bend left] \arrow[r, phantom, "\perp"] & Cat : N \arrow[l, bend left] \end{tikzcd}$$

Fundamental category

nerve

$\square$

By construct°:

$$\begin{cases} h(\Delta^n) = [n] \\ h \text{ commutes with colimits} \end{cases}$$

and

$$N(C)_n = Cat([n], C)$$

$$= \left\{ \begin{array}{l} n\text{-tuples of} \\ \text{composable morphisms} \end{array} \right\}$$

Rmk We have $N([n]) \cong \Delta^n$

($\Leftarrow \iota$ fully faithful).

$N(C)$:

$$\begin{tikzcd}
\mathrm{Ob}(C) \arrow[r, "id"] & \mathrm{Mor}(C) \arrow[l, bend right, "source"'] \arrow[l, bend left, "target"] \arrow[r, bend left] \arrow[r, bend right] & \{\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot\} \arrow[l, bend right=60, "f"'] \arrow[l, "g \circ f"'] \arrow[l, bend left=60, "g"] & \cdots
\end{tikzcd}$$

This makes it clear that

we can reconstruct C from $N(C)$.

Prop 33 N is fully faithful

proof: For C, D in Cat, we must show

$$\mathrm{Cat}(C, D) \xrightarrow{\sim} \mathrm{sSet}(NC, ND).$$

Injectivity: A functor is determined by its effect on objects and morphisms. Since $(NC)_0 = Ob(C)$ and $(NC)_1 = Mor(C)$, we are done.

Surjectivity Let $\alpha : NC \longrightarrow ND$. We define a candidate F for the preimage functor using again

$$\begin{cases} (NC)_0 = Ob(C) \\ (NC)_1 = Mor(C) \end{cases}$$

To check that F is a functor, we need to see:

- **F compatible with source/target:**

Let $f : c \longrightarrow c'$ in C

$$\begin{cases} sF(f) \overset{\text{def}}{=} d^0\alpha(f) \overset{\text{sSet}}{=} \alpha d^0(f) \overset{\text{def}}{=} F(c) \\ tF(f) \overset{\text{def}}{=} d^1\alpha(f) \overset{\text{sSet}}{=} \alpha d^1(f) \overset{\text{def}}{=} F(c') \end{cases}$$

- **F compatible with identities**

$$F(id_c) \overset{\text{def}}{=} g\, s^0(c) \overset{\text{sSet}}{=} s^0 g(c) \overset{\text{def}}{=} id_{F(c)}$$

- F compatible with compositions

$$F(gf) = \alpha(gf) = \alpha(d^1(g,f))$$

$$= d^1\alpha(g,f) = F(g) \circ F(f)$$

$\rightsquigarrow$ F is a functor.

By construction, $N(F)$ and α coincide on 0 and 1-simplices, and it is easy to see that it forces them to be equal (because simplices in $(NC)_i$ for $i > 2$ are uniquely determined by their 1-simplices). $\square$

Thm 34 : (characterisation of nerves)

Let $X_\bullet \in sSet$. TFAE:

(1) $X_\bullet \simeq NC$ for some $C \in Cat$.

(2) $X_\bullet$ satisfies the unique spine extension property:

For all $n \geq 2$,

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

(3) $X_\bullet$ satisfies the inner horn unique extension property:

For all $n \geq 2$ and $0 < k < n$:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>proof:</u>