II Weierstrass models of elliptic curves

- We start with elliptic curves. For this special case, there is a theory of models which is more elementary than both minimal regular models and Néron models

1) Weierstrass equations

def 1 Let R be a ring. A <u>Weierstrass equation</u> with coefficients in R is an homogeneous equation of the form

$$y^2 z + a_1 x y z + a_3 y z^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3.$$

- Our goal is to derive the main properties of such equations with as little computations as possible, and then forget about the equations !

(not recommended if you actually want to compute things)

Lemma 2 Let k be a field, $W \subseteq \mathbb{P}_k$ given by a Weierstrass equation. Then W is geometrically integral, smooth at $[0:1:0]$.

Proof: – Writing the equation as $F(x,y,z) = 0$, we have

$$\frac{\partial F}{\partial z} = y^2 + a_1 x y + 2 a_3 y z - a_2 x^2 - 2 a_4 x z - 3 a_6 z^2$$

Hence $\frac{\partial F}{\partial z}([0:1:0]) = 1 \neq 0 \Rightarrow W$ is smooth at $[0:1:0]$.

- $W \cap V(z) = \{[0:1:0]\}$. Since $V(z)$ intersects all irreducible components of W (2 curves in $\mathbb{P}^2_k$ intersect !), and $[0:1:0]$ is a smooth point, we get W irreducible.
- W irreducible cubic curve. If W non-reduced, then necessarily $F = L^3$, and W everywhere non-reduced $\lightning$
- This argument applies over any field extension

$\Rightarrow W$ geometrically integral. $\square$

Prop 3 Let (E, e) be an elliptic curve over a field k.
There exists $x, y \in K(E)$ such that the birational map $\phi : E \longrightarrow \mathbb{P}^2_k, [x : y : 1]$
is a closed immersion whose image is cut out by a Weierstrass equation, and $\phi(e) = [0:1:0]$ unique pt in the line at ∞

Proof:

- RR $\Rightarrow \forall n \geq 1$, $\dim H^0(E, \mathcal{O}(ne)) = n$.
- Choose meromorphic functions x, y such that

$\{1, x\}$ basis of $H^0(E, \mathcal{O}(2e))$,

$\{1, x, y\}$ basis of $H^0(E, \mathcal{O}(3e))$.

- Then $\{1, x, y, x^2, xy, y^2, x^3\} \subseteq H^0(E, \mathcal{O}(6e))$

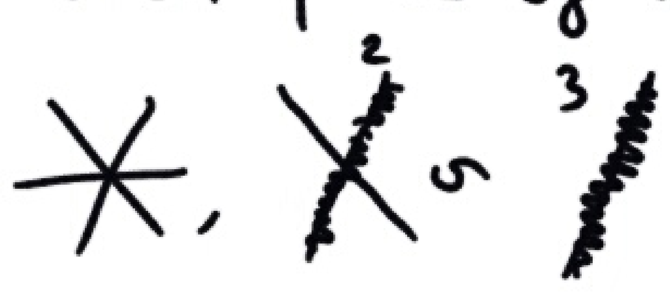
(Recall that if $k = \mathbb{C}$, $E^{an} = \mathbb{C}/\Lambda$ then can choose $x = \wp$, $y = \wp'$ with $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$ Weierstrass elliptic function.)

$\Longrightarrow \exists A_1, \dots, A_7 \in k$, not all 0,

$$A_1 + A_2 x + A_3 y + \dots + A_7 x^3 = 0$$

- By considering pole order at e and using $\dim H^0(E, \mathcal{O}(5e)) = 5$ see that $A_6 \neq 0$ and $A_7 \neq 0$.
- Substitute $x \rightsquigarrow -A_6 A_7 x$, $y \rightsquigarrow A_6 A_7^2 y$ $\Longrightarrow$ (Weierstrass equat°) $\cdot A_6^3 A_7^4$
- Get $\phi : E \dashrightarrow \mathbb{P}^2$. Since E is smooth of $\dim \leq 1$ and $\mathbb{P}^2$ is proper, ϕ is automatically a morphism. Then $\mathrm{Im}(\phi)$ is given by Weierstrass equation.
- By composing with proj to $\mathbb{P}^1$ via $[x:1], [y:1]$, we see that $\deg(\phi)$ divides 2 and 3 $\Rightarrow \deg(\phi) = 1$.
- To conclude, it remains to show that $\mathrm{Im}(\phi)$ is smooth. By Lemma 2, $\mathrm{Im}(\phi)$ is an integral cubic curve in $\mathbb{P}^2$.

- Assume that $\mathrm{Im}(\phi)$ is singular and let p be a singular point. Then p is of multiplicity exactly 2 (otherwise, $\mathrm{Im}(\phi)$ would be either



, hence not integral). This implies that the projection map $\mathbb{P}^2 \setminus \{p\} \longrightarrow \mathbb{P}^1$ restricts to a degree 1 rational map

$\mathrm{Im}(\phi) \dashrightarrow \mathbb{P}^1$, which composed with $E \to \mathrm{Im}(\phi)$ gives $E \dashrightarrow \mathbb{P}^1$ of degree 1 $\lightning$ $\square$

<u>rmk:</u> If $\mathrm{char}(k) \neq 2, 3$, there are simplified forms of Weierstrass equations; since we try to develop a purely geometric theory this is not so important for us.

<u>prop 4:</u> Let $W \subseteq \mathbb{P}^2_k$ be defined by a Weierstrass equation. Then W smooth $\iff$ $(W, [0:1:0])$ elliptic curve.

<u>proof:</u>

$\Leftarrow$ follows from prop 3.

$\Rightarrow$ follows from Hurwitz's genus formula and $[0:1:0] \in W(k) \neq \emptyset$.

Alternative argument for $\Rightarrow$: from Weierstrass equation, can write down

$$\omega = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3x^2 + 2a_2 x + a_4} \in \Omega^1_{K(W)/k}$$

and show (assuming W smooth) that $\omega \in H^0(W, \Omega^1_{W/k})$ and that ω is everywhere non-vanishing.

A curve with such a 1-form has genus 1 $\square$

Here are some further geometric properties of Weierstrass equations which are not strictly necessary for what follows and which we state without proofs.

def 5
- The discriminant Δ of a Weierstrass equation is
$$\Delta = 2^{-4} \operatorname{disc}_X\left(4\left(X^3 + a_2 X^2 + a_4 X + a_6\right) + \left(a_1 X + a_3\right)^2 \right) \in k.$$
- We also define
$$c_4 = (a_1^2 + 4a_4)^2 - 24(2a_4 + a_1 a_3) \in k$$

prop 6
Any two Weierstrass equation for the same curve are related by a change of variables of the form
$$\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases} \quad \text{with } u \in k^\times,\ r, s, t \in k.$$
- Such a change has the effect that $\begin{cases} c_4 \rightsquigarrow u^{-4} c_4 \\ \Delta \rightsquigarrow u^{-12} \Delta \end{cases}$.

prop 7
If $\operatorname{char}(k) \neq 2,3$, using such a change of variable, we can put the equation in the form: $y^2 = x^3 + Ax + B$
and then $\begin{cases} \Delta = -16(4A^3 + 27B^2) \\ c_4 = AB. \end{cases}$

prop 8
Let k be a field and W given by a Weierstrass equation.
(i) W non-singular $\iff \Delta \neq 0$.
(ii) If $\Delta = 0$, $c_4 \neq 0$, then W has a unique geometric singularity which is a node defined over k.
(note that the branches at the node may not be defined over k.)
(iii) If $\Delta = 0$, $c_4 = 0$, then W has a unique geometric singularity which is a cusp. It is defined over k unless $\operatorname{char}(k) \in \{2,3\}$ and k is imperfect.

- All this can be found in [Silverman, chap III] except the discussion of non-rational cusps on non-perfect fields which is nicely explained in [Conrad_models, p.15].

2) Weierstrass models

We now turn to models. For simplicity, we start with $S = \mathrm{Spec}(R)$, R discrete valuation ring,

In this case, it turns out that Weierstrass equations provide rather good models of elliptic curves.

<u>lemma 1</u>: Let $W/_K$ be defined by a Weierstrass equation. Then W is isomorphic to a curve defined by a Weierstrass equation with coeffs in R.

<u>proof</u>: Let $u \in K$. One checks that the substitution

$$\begin{cases} x = u^{-2}x' \\ y = u^{-3}y' \end{cases}$$

acts on coefficients by $a_i' = u^i a$

Hence by taking $v_K(u) >> 0$, we can ensure that $a_i \in R$.

$\square$

def 2 Let (E, e) be an elliptic curve over K.
A <u>planar Weierstrass model</u> (PWM) of (E, e) is a pair (W, i) with $W \subseteq \mathbb{P}^2_R$ defined by a Weierstrass equation and

$$i : W_K \xrightarrow{\sim} E$$
$$[0:1:0] \mapsto e .$$

prop 3 Let $W/_S$ be a planar Weierstrass model.
(i) W is proper flat over S.
(ii) W_σ has geometrically integral fibers, and is smooth at $[0:1:0]$. Equivalently, W_σ is smooth at $\varepsilon(\sigma)$ for $\varepsilon \in W(R)$ the unique section extending $i^{-1}(e)$.
(iii) W is normal.

proof: <u>(i)</u> - properness $\Leftarrow$ projectivity.

- flatness $/_S$ $\iff$ $\mathcal{O}_W$ torsion-free over R ; can be checked on affine patches ; have to check π unif of R does not divide F in the factorial ring $R[x,y]$; follows from monicity of Weierstrass eq.

<u>(ii)</u> follows from previous prop.

<u>(iii)</u> Want to apply Serre's criterion.

<u>Recall</u> (R, m) local ring. A <u>regular sequence</u> $r_1, \dots, r_k \in m$ satisfies r_i non zero-divisor in $R/(r_1, \dots, r_{i-1})$.
The <u>depth</u> of R is the maximal length of a regular sequence.

<u>Serre's criteria:</u>

[St 0310] X locally noetherian scheme. Then X $\overset{\text{reduced}}{\text{normal}}$ iff

$\overset{R0}{(R1)}$ $\forall x \in X$, $\dim(\mathcal{O}_{X,x}) \leq \overset{0}{1} \Rightarrow \mathcal{O}_{X,x}$ regular

$\overset{S1}{(S2)}$ $\forall x \in X$, $\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min\{\overset{1}{2}, \dim(\mathcal{O}_{X,x})\}$.

- W/S is flat, W_η/η is smooth, W_σ/σ is gen. smooth ($\Leftarrow W_\sigma/\sigma$ is geom. reduced)

$\Longrightarrow W \smallsetminus \underbrace{W^{sm}}_{S\text{-smooth locus}}$ is a finite set of points in W_σ

$\Longrightarrow W$ is (R1).

- W is reduced, hence S1, so it remains to show that for all x codimension 2 pt, we have $\operatorname{depth}(\mathcal{O}_{W,x}) = 2$. Such an x lies in W_σ.

- Lemma [St 090R] | (R,m) noeth. local ring, $\pi \in m$. π non-zero divisor $\Rightarrow \operatorname{depth}(R/(\pi)) = \operatorname{depth}(R) - 1$.

This reduces us to show $\operatorname{depth}(\mathcal{O}_{W_\sigma,x}) = 1$.
W_σ is reduced, hence (S1), and we are done.

[Alternative argument for (S2): W is an hypersurface in $\mathbb{P}^2_S$, hence local complete intersection, hence Cohen-Macaulay, hence (S_k) $\forall k$.]

$\square$

def 4 | Let $(E,e)/\eta$ be an elliptic curve.
An abstract Weierstrass model (AWM) of (E,e) is a pair (W,i) with
- W/S proper flat with W normal and geometrically integral fibers.
- $i: W_K \xrightarrow{\sim} E$
- W_σ smooth at $\varepsilon(\sigma)$ for $\varepsilon \in W(R)$ unique section extending $i^{-1}(e)$.

- By the above, any planar Weierstrass model is an abstract Weierstrass model. Conversely:

thm 5 | Every abstract Weierstrass model is isomorphic to a planar one.

proof Follows the same strategy as over a field with RR replaced by Serre duality for W_σ.

• Serre duality:

def 6 Let k be a field, X/k projective equidimensional of dim. $n \geq 0$.
A <u>dualizing sheaf</u> for X is a pair (ω_X, t)
with
- ω_X coherent sheaf on X
- $t : H^n(X, \omega_X) \xrightarrow{\sim} k$ (<u>trace map</u>)

such that for any coherent sheaf $\mathcal{F}$ on X and $i \in \mathbb{N}$, the natural pairing

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \to H^n(X, \omega_X) \xrightarrow[\sim]{t} k$$

is perfect. Ext cup-product

rmk ω_X is unique up to a unique iso if it exists.

thm 7 ("concrete cases")

1) Let X/k be smooth projective of dimension n.
Then $\Omega^n_{X/k} := \bigwedge^n \Omega^1_{X/k}$ is a dualizing sheaf

2) Let $X \overset{i}{\hookrightarrow} Y$ be a closed embedding of proj. equidimensional k-schemes, of dim. n and N.
Assume Y has a dualizing sheaf ω_Y. Then X has a dualizing sheaf given by

$$\omega_X \simeq i^{-1} \mathcal{E}xt^{N-n}(i_* \mathcal{O}_X, \omega_Y).$$

3) $j : U \hookrightarrow X$ open embedding of proj. equidimensional k-schemes of dim n.
Assume X has a dualizing sheaf ω_X.
Then U has a dualizing sheaf given by

$$\omega_U \simeq j^{-1} \omega_X.$$

- If we already knew that W was a planar Weierstrass model, we would see that $W_6 \overset{i}{\hookrightarrow} \mathbb{P}^2_6$ has a dualizing sheaf $\omega_6 := \omega_{W_6/6}$ (given by $\omega_6 \simeq i^{-1}\mathcal{E}xt^1(i_*\mathcal{O}_{W_6}, \mathcal{O}_{\mathbb{P}^2}(-3))$)

<u>thm 8</u> | X/k projective equidimensional of dim n. If X is Cohen-Macaulay, then X has a dualizing sheaf.

<u>cor 9</u> | Any reduced curve $/k$ has a dualizing sheaf.

- So in particular W_6 has a dualizing sheaf ω_6.

<u>lemma 10</u> | i) ω_6 is torsion-free.

ii) ω_6 is generically invertible.

<u>proof</u>: i) If ω_6 has torsion, it would contain a torsion subsheaf $\mathcal{C}$. By duality:

$$\mathrm{Hom}(\mathcal{C}, \omega_6) \simeq H^1(W_6, \mathcal{C})^\vee \underset{\underset{\mathcal{C}\text{ torsion}}{\uparrow}}{\simeq} 0 \quad \lightning$$

ii) By part 3) of the previous thm:

$$(\omega_6)|_{W_6^{sm}} \simeq \omega_{W_6^{sm}} \simeq \Omega^1_{W_6^{sm}}$$

is invertible.

$\square$

- We can now start the proof in earnest.

By hypothesis, $\mathcal{E}$ lies in $W^{om}(R)$, hence $\mathcal{J}_{\mathcal{E}}$ is invertible.

Write $\mathcal{O}(\mathcal{E}) = \mathcal{J}_{\mathcal{E}}^{-1}$, $\mathcal{O}(n\mathcal{E}) = \mathcal{O}(\mathcal{E})^{\otimes n}$.

Write $\bar{\mathcal{E}} \in W_6^{om}(k)$ for the reduction, and $\mathcal{O}(\bar{\mathcal{E}})$, $\mathcal{O}(n\bar{\mathcal{E}})$ ...

<u>Lemma 11</u> | $\forall n \geq 1$, $H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}})) = 0$

<u>proof</u>: • By duality:

$$H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}}))^\vee \simeq \mathrm{Hom}(\mathcal{O}(n\bar{\mathcal{E}}), \omega_6)$$
$$\simeq \mathrm{Hom}(\mathcal{O}, \omega_6(-n\bar{\mathcal{E}})) \quad (\mathcal{O}(n\bar{\mathcal{E}}) \text{ invertible})$$

- Assume there is a non-zero map $\mathcal{O} \xrightarrow{f} \omega_6(-n\bar{\mathcal{E}})$. Since $\omega_6(-n\bar{\mathcal{E}})$ is torsion-free and generically invertible, f is injective and its cokernel is torsion.

Write $0 \to \mathcal{O}_{W_6} \xrightarrow{f} \omega_6(-n\bar{\mathcal{E}}) \to \mathcal{C} \to 0$

- We have $\chi(\omega_6(-n\bar{\mathcal{E}})) = \chi(\mathcal{O}_{W_6}) + \chi(\mathcal{C}) = \chi(\mathcal{C}) \geq 0$

$$\chi(\mathcal{O}_{W_6}) \overset{\text{flatness}}{=} \chi(\mathcal{O}_{W_\eta}) \underset{\text{ell. curve}}{=} 0$$

- By duality, $\chi(\omega_6(-n\bar{\mathcal{E}})) = -\chi(\mathcal{O}(n\bar{\mathcal{E}}))$

$$= -\deg(n\bar{\mathcal{E}}) - \underbrace{\chi(\mathcal{O}_{W_6})}_{=0}$$

$$< 0 \quad ↯$$

□

- Consider the R-module $H^0(W, \mathcal{O}(n\mathcal{E}))$.

$\begin{cases} \text{We have seen that } H^1(W_6, \mathcal{O}(n\bar{\mathcal{E}})) = 0 \\ \text{We also have } H^1(W_\eta, \mathcal{O}(n\mathcal{E}_\eta)) = 0 \end{cases} \Rightarrow R^1(W/R)_* \mathcal{O}(n\mathcal{E}) = 0$

By general theory of cohomology and base change, this implies

$H^0(W, \mathcal{O}(n\varepsilon))$ is a finite free module and its formation commutes with arbitrary base change on S.
By the elliptic curve case, it is of rank n.

- By the same argument as in the field case, we construct

$$W \longrightarrow \mathrm{Proj}_S\left(\mathrm{Sym}\left(H^0(W, \mathcal{O}(3\varepsilon))\right)\right) \simeq \mathbb{P}^2_S$$

with image a planar Weierstrass model W' of E.
The following lemma then finishes the proof.

<u>Lemma 12</u> Let $\phi : W \to W'$ be a morphism of abstract Weierstrass models. Then ϕ is an isomorphism.

<u>proof:</u>

- The map $\phi : W \to W'$ is proper birational. (since ϕ_η is an iso. by hypothesis)
The map ϕ_σ is a non-constant map between <u>integral</u> curves, hence it is finite.
So ϕ is finite birational. Since W' is normal, we deduce that ϕ is an isomorphism. □

<u>Lemma 13</u> Let W, W' be planar Weierstrass model. Then (iso)morphisms between W and W' as models are given by linear changes of coordinates

$$\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases} \quad \text{with } u \in R^\times,\ r, s, t \in R.$$

There is in fact at most one such isomorphism.

<u>proof</u> Follows from the intrinsic construction of the embedding $W \hookrightarrow \mathbb{P}^2_R$ described in previous proof + explicit manipulation of equations as in field case. See [Conrad-models, Cor 2.9] □

Over a more general Dedekind scheme:

- Let S be a Dedekind scheme and $E/_\eta$ an elliptic curve. We can define AWMs for E in the same way, and, when S is affine, PWMs. (one should really allow coefficients in a line bundle on S, which would allow "PWM" beyond S affine, but I could not find a good reference).

Then any PWM is an AWM (same proof) and one can ask when the converse holds. This is not always true (ex. later!)

- Let $S = \mathrm{Spec}(R)$ affine Dedekind. The criterion for when an AWM W is a PWM involves the <u>relative dualizing sheaf</u> $\omega_{W/S}$. We do not want to introduce too much machinery this early in the course, so let us just say that one can prove that $W/_S$ is a local complete intersection and that this implies that there exists $\omega_{W/S}$ invertible sheaf on W with fibers the dualizing sheaves $\omega_{W_s/s}$ for all $s \in S$.

thm 14 Let $\pi : W \longrightarrow S$ be an AWM. Then

(i) $\pi_* \omega_{W/S} \simeq (R^1 \pi_* \mathcal{O}_W)^\vee$ (instance of <u>relative</u> duality)

(ii) $R^1 \pi_* \mathcal{O}_W$ and $\pi_* \omega_{W/S}$ are locally free.

(iii) W is a PWM <u>iff</u> $R^1 \pi_* \mathcal{O}_W$ is free
<u>iff</u> $\pi_* \omega_{W/S}$ is free.

cor 15: S affine + $\mathrm{Pic}(S) = 0 \Rightarrow$ Every AWM is a PWM.

rmk: The criterion in terms of $\pi_* \omega_{W/S}$ looks more complicated than the one for $R^1 \pi_* \mathcal{O}_W$ but it is more directly related to computations with diff. forms.

3) Minimal Weierstrass models (S trait again)

- Given an elliptic curve E, there are many non-isomorphic Weierstrass models, obtained by suitable transformations of x, y.

It turns out there is a "best" one, the minimal Weierstrass model $W^{min}(E)$.

There are several ways to pin it down:

a) From a Weierstrass equation, one can extract the discriminant

$$\Delta := 2^{-4} \operatorname{disc}\left(4(X^3 + a_2 X^2 + a_4 X + a_6) + (a_1 X + a_3)^2 \right)$$
$$\in R$$

Then $v(\Delta) \in \mathbb{N}$ is an invariant of the model, and $W^{min}(E)$ is the unique model with minimal $v(\Delta)$.

$\left(v(\Delta_{min}) = 0 \iff E \text{ has good reduct}^{\circ} \iff W^{min}(E) \text{ smooth} \right)$

- Using the explicit form of the changes of variables, we see that $\begin{cases} \Delta' = u^{-12} \Delta \\ c_4' = u^{-4} c_4 \end{cases}$

Lemma 1 | $v(\Delta) < 12$ or $v(c_4) < 4 \implies W$ minimal.
$\Longleftarrow$ $(\mathrm{char}(k) \neq 2,3)$ [Silverman, ex. VII 7.1]

def 2 | Let E be an elliptic curve over K with minimal Weierstrass model W. We say that E

- has <u>good reduction</u> if W_k is smooth $(\iff v(\Delta) = 0)$
- has <u>multiplicative reduction</u> if W_k has a node $\left(\iff \begin{matrix} v(\Delta) > 0 \\ v(c_4) = 0 \end{matrix}\right)$
- has <u>additive reduction</u> if W_k has a cusp $\left(\iff \begin{matrix} v(\Delta) > 0 \\ v(c_4) > 0 \end{matrix}\right)$

. For this approach see [Silverman, chap 7].

b) For any Weierstrass model $W/_S$, one can prove that $H^0(W^{sm}, \Omega^1_{W^{sm}/R})$ is a free rk 1 R-submodule of $H^0(E, \Omega^1)$. [Conrad_models, thm 2.6] Then $W^{min}(E)$ is the unique WM with maximal $H^0(W^{sm}, \Omega^1)$.
[Conrad_models, cor 2.10]

c) $W^{min}(E)$ is the only Weierstrass model of E with rational singularities. [Conrad_models, Cor 8.4]

- Pt of view a) is classical and suitable for computations (incl. with a computer).

- Pt of view b) is useful to relate Weierstrass models with the main objects of the course. (see later)

- Pt of view c) is nice because of the distinguished role of rational singularities in resolution of singularities of surfaces (see later).

- The relation between a) and b) is based on the direct computation:

__Lemma 3__ | Let $E/_K$ be an elliptic curve with a fixed Weierstrass equation. We have $\begin{cases} \Delta \in k^\times \\ \omega = \dfrac{dx}{2y + a_1 x + a_3)} \in H^0(E, \omega_{E/K}) \end{cases}$.
Then $\Delta\, \omega^{\otimes 12} \in (H^0(E, \omega_{E/K}))^{\otimes 12}$ is independent of the choice of the equation.

i.e. "Δ is a weight 12 modular form" !

Semistable reduction for elliptic curves

- We look at the first instance where the properties of a model can be improved by passing to an extension of K.

<u>thm 4:</u> Let $E/_K$ be an elliptic curve.
There exists a finite separable extension $L/_K$ such that E_L has good or multiplicative reduction.

- Unfortunately I could not find an (elementary) equation-free proof of this in the litterature, so I refer you to [Silverman, Prop 5.4.(c)].

Situation over a Dedekind scheme:

- Let S be a general Dedekind scheme, and $(E,e)/_\eta$ be an elliptic curve. Then E admits a <u>minimal AWM</u> $W^{min}(E)/_S$ i.e., an AWM such that for every $s \in S^{(0)}$, $W \times_S \mathrm{Spec}(\mathcal{O}_{S,s})/_{\mathrm{Spec}(\mathcal{O}_{S,s})}$ is minimal.
- Assume S affine. Then $W^{min}(E)$ is planar
  <u>iff</u> $\pi_* \omega_{W^{min}(E)/_S}$ is free
  <u>iff</u> the "<u>Weierstrass ideal</u>" $\mathfrak{a}_{E/K} \subseteq R$ is principal.

($\mathfrak{a}_{E/K}$ is defined as a product of local terms obtained from turning a fixed PWM into a minimal one at that point, see [Silverman, Chap $\overline{\underline{\mathrm{VIII}}}$]; $\mathfrak{a}_{E/K} \sim \frac{1}{12}(\Delta$ of any PWM of E))

Relationship with general theory:

- $W^{min}(E)$ is neither the minimal regular model (it's not regular in general) nor the Néron model (it's not smooth in general.

- Let us sketch the relationship with the introduction. Write
  - $\mathcal{E}^{reg}$ for the minimal regular model of E over S.
  - $\mathcal{N}(E)$ for the Néron model of E over S.

Then:

- There is a canonical morphism $\mathcal{E}^{reg} \longrightarrow W^{min}(E)$ of models which contracts all the irreducible components of the special fibers which do not meet the unique section $\varepsilon \in \mathcal{E}^{reg}(S)$ which extends $e \in E(\eta)$.
- There is a canonical isomorphism $(\mathcal{E}^{reg})^{sm} \simeq \mathcal{N}(E)^{0}$ with $(\mathcal{E}^{reg})^{sm}$ the S-smooth locus and $\mathcal{N}(E)^{0}$ is the union of the identity components of the fibers of the group scheme $\mathcal{N}(E)$.

- These results and others will show up in the next instances of:

Elliptic curves, the running example

- The other moral is that equations can fail us, already in the case of elliptic curves over global fields. Needless to say, for higher genus curves / higher dimensional abelian varieties, the situation will not improve and "abstract" algebraic geometry will be essential.

Examples:

- $S = \operatorname{Spec}\mathbb{Z}$: the situation is especially simple: $\mathbb{Z}$ PID $\Rightarrow$ minimal PWM exists.

+ For any $E/_{\mathbb{Q}}$, there exists a unique MPW equation with $a_1, a_3 \in \{0,1\}$, $a_2 \in \{-1,0,1\}$. ("reduced min equat°")

- Here is a Weierstrass equation $/\mathbb{Z}$: $W : y^2 = x^3 - 432$

We have $\Delta_W = -2^{12} \cdot 3^9$, and $W_{\mathbb{F}_2} : y^2 = x^3$ is singular.

On the other hand, if we put $x = 4x'$, $y = 8y' - 4$, we get

$W' : y'^2 - y' = x'^3 - 7$ with $\Delta_{W'} = -3^9$.

- So $\begin{cases} W \text{ is not minimal at } 2, \\ W' \text{ is minimal at } 2 \ (\Leftarrow \text{good red at } 2). \end{cases}$

- What about at 3? By looking at the possible changes of variables of Weierstrass equation, one sees that $v_p(\Delta)$ at a prime p changes by multiples of 12. So $v_3(\Delta_W) = v_3(\Delta_{W'}) = 9 < 12$ implies that both W and W' are minimal at 3; $c_4(W) = 0 \Rightarrow$ additive reduction.

- In conclusion W' is the reduced MPWE of its generic fiber E.

- E is a very cool elliptic curve:
  - it is isomorphic to the Fermat cubic $X^3 + Y^3 + Z^3 = 0$
  - it has complex multiplication by $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$
  - it is the modular curve $X_0(27)$.

- E acquires semistable reduction only after an extension of degree 12: $\mathbb{Q}(2^{1/3}, (-3)^{1/4}, (-1)^{1/3})$.

- $\underline{\operatorname{Pic}(S) \neq 0}$: Let us describe an example of elliptic curve with no MPWM. We need $\operatorname{Pic}(S) \neq 0$.

Given any number field K with class number > 1, a result of Silverman states that there exists an elliptic curve with no MPWM. For instance, for $K = \mathbb{Q}(\sqrt{-10})$:

$\operatorname{Pic}(\mathcal{O}_K) = \langle (5, \sqrt{-10}) \rangle \simeq \mathbb{Z}/_{2\mathbb{Z}}$.

$E : y^2 = x^3 + 125$

has Weierstrass ideal equal to $(5, \sqrt{-10})$, so

E has no MPWM.