Prop 15 Let $(\mathcal{C}_\alpha)_{\alpha \in J}$ be a family of ∞-categories. Then the canonical map

$$h\left(\prod_{\alpha \in J} \mathcal{C}_\alpha\right) \longrightarrow \prod_{\alpha \in J} h(\mathcal{C}_\alpha)$$

is an isomorphism.

("h commutes with products")

proof: • First, we know from Prop 7 that $\prod \mathcal{C}_\alpha$ is an ∞-category so this makes sense.

• Recall that in Cat, products are formed by taking products of all objects and all morphisms.

• The map of the statement is thus a bijection on objects and surjective on morphisms, so it remains to see that it is injective on morphisms. If $([f_\alpha])_{\alpha \in J} = ([g_\alpha])_{\alpha \in J}$ in $\prod_\alpha h\mathcal{C}_\alpha$, we get a collection of $t_\alpha \in (\mathcal{C}_\alpha)_2$ giving homotopies, but then $(t_\alpha) \in (\prod \mathcal{C}_\alpha)_2$ implies that

$$[(f_\alpha)] = [(g_\alpha)] \text{ in } h\left(\prod_\alpha \mathcal{C}_\alpha\right). \quad \square$$

Cor 16: Let $F: \mathcal{C} \to \mathcal{C}'$ be an equivalence of ∞-categories. Then $hF: h\mathcal{C} \to h\mathcal{C}'$ is an equivalence of categories.

proof: Step 1) We show that any natural transformation $\alpha: F \Rightarrow G$ of functors $F, G: \mathcal{C} \to \mathcal{D}$ induces a natural transformation $h\alpha: hF \Rightarrow hG$.

- We have

$$h(\mathcal{C} \times \Delta^1) \simeq h\mathcal{C} \times [1]$$

Prop 15

α is by definition a morphism

$$\mathcal{C} \times \Delta^1 \longrightarrow \mathcal{D}$$

$$\Rightarrow h\alpha: h\mathcal{C} \times [1] \simeq h(\mathcal{C} \times \Delta^1) \longrightarrow h\mathcal{D}$$

is a natural transformation $hF \Rightarrow hG$.

- Step 2) Assume now that α is a natural isomorphism, that is, we have $\beta: G \Rightarrow F$

and

$$t, t' : C \times \Delta^2 \longrightarrow D \quad \text{with}$$

$$\begin{tikzcd} & G \arrow[dr, "\beta"] & \\ F \arrow[ur, "\alpha"] \arrow[rr, "id_F"'] \arrow[rr, phantom, shift left=3, "t"] & & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & F \arrow[dr, "\alpha"] & \\ G \arrow[ur, "\beta"] \arrow[rr, "id_G"'] \arrow[rr, phantom, shift left=3, "t'"] & & G \end{tikzcd}$$

Then by using $R(C \times \Delta^2) \underset{\text{Prop 15}}{\simeq} RC \times [2]$

we get that $\begin{cases} R\beta \circ R\alpha \overset{Rt}{=} R\,id_F = id_{RF} \\ R\alpha \circ R\beta \overset{Rt'}{=} R\,id_G = id_{RG} \end{cases}$

so $R\alpha$ is also a natural isomorphism.

<u>Step 3)</u> Assume F is an equivalence of ∞-categories with "inverse" $G : D \to C$. Then we have natural isomorphisms $\begin{cases} F \circ G \overset{\alpha}{\simeq} id_D \\ G \circ F \overset{\beta}{\simeq} id_C \end{cases}$

and by Step 2) : $\begin{cases} RF \circ RG = R(F \circ G) \overset{R\alpha}{\simeq} id_{RD} \\ RG \circ RF = R(G \circ F) \overset{R\beta}{\simeq} id_{RC} \end{cases}$

So hF is an equivalence of 1-categories. $\blacksquare$

⚠ The converse is not at all true!

def 17 Let C be an ∞-category.
A morphism $f \in C_1$ is an *isomorphism* if $[f] \in \mathrm{Mor}(hC)$ is an isomorphism, i.e if $\exists\, g \in C_1$ with $[f] \circ [g]$ and $[g] \circ [f]$ identities in hC.

Ex: · $\forall\, x \in C_0$, $id_x = s_0(x)$ is an iso.

· If C is a 1-category, then isomorphisms in NC are exactly the isos. in C.

def 18: An ∞-groupoid is an ∞-category in which every morphism is an iso.

lemma 19: Kan complexes are ∞-groupoids.

proof: Same argument as part !! of Prop 4. $\square$

The converse is true but much more difficult
we will see (at least parts of) the proof later.

thm 20: ∞-groupoids are Kan complexes.

Rmk: Try to prove this by hand!
Even proving the lifting property for $\Lambda^3_0 \hookrightarrow \Delta^3$ is challenging.

Rmk: Thm 20 implies that the (not yet defined) ∞-category of ∞-groupoids is equivalent to the ∞-category of Kan complexes, which by simplicial homotopy theory is equivalent to the ∞-cat. of topological spaces. $\Rightarrow$ Grothendieck's homotopy hypothesis holds in the quasicategory model of $(\infty,1)$-categories.

- Another important theorem, which will be our main goal in the next section of the course, is

thm 21: Let $K \in sSet$, $C \in Cat_\infty^1$.

Then the simplicial set $Fun(K,C)$,

with $Fun(K,C)_n = sSet(K \times \Delta^n, C)$,

is an ∞-category. $\square$

In fact, the definition of natural isomorphisms and categorical equivalences can be reformulated as:

- $\begin{pmatrix} \alpha : F \Rightarrow G \\ \text{natural iso} \end{pmatrix} \iff \begin{matrix} \alpha : F \to G \text{ isomorphism} \\ \text{in } Fun(C,D) \end{matrix}$

- $\begin{pmatrix} C \xrightarrow{F} D \\ \text{categorical} \\ \text{equivalence} \end{pmatrix} \iff \begin{pmatrix} \exists\, G : D \to C, \\ F \circ G \text{ iso. to } id_D \text{ in } Fun(D,D) \\ G \circ F \text{ — } id_C \text{ in } Fun(C,C) \end{pmatrix}$

. Another application of the homotopy category is to the notion of subcategory:

def 22: A subcategory C' of an ∞-category C is a simplicial subset which satisfies moreover: $\forall\, n \geq 2$, $\forall\, \gamma \in C_n$,

$$\gamma \in C'_n \iff \gamma_{|I^n} \text{ has edges in } C'_1.$$

Hence C' is determined by C'_0 and C'_1.

C' is a full subcategory if

$\forall\, n \geq 1$, $\forall\, \gamma \in C_n$,

$\gamma \in C'_n \iff$ the vertices of γ lie in C'_0.

C' is then determined entirely by C'_0.

lemma 23: A subcategory C' of an ∞-category C an ∞-category.

proof: This follows immediately from $I^n \subseteq \Lambda^n_k$.

$\square$

Prop 25: Let C be an ∞-category and $\eta : C \longrightarrow N\,h\,C$ be the unit map.

Then a) If D is a subcategory of the 1-category hC, then the pull back $\eta^{-1}(ND) := ND \times_{NhC} C$ is a subcategory of C.

$$\begin{tikzcd}
\eta^{-1}(ND) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
NC' \arrow[r, hook] & NhC
\end{tikzcd}$$

b) This construction gives a bijection

$$\{\text{Subcategories of } hC\} \simeq \{\text{Subcategories of } C\}$$

which restricts to a bijection of full subcategories.

proof: We only do the case of subcategories, the full case is easier.

a)

Claim | If C is a 1-category and C' is a (full) subcategory of C, then NC' is a (full) subcategory of NC.

pf: obvious from the unique lifting property of nerves along $I^n \hookrightarrow \Delta^n$.

Claim: | the pullback of a subcategory is a subcategory.

pf: Let $C' \subseteq C$ be a subcategory of an ∞-cat. and $E \xrightarrow{f} C$ be any morphism. Then $E' := E \times_C C'$ is a simplicial subset of E (pullback of mono is mono in any category)

By definition, we have for $n \in \mathbb{N}$: $E'_n = E_n \times_{C_n} C'_n$, so for $z \in E_n$, we have:

$$z \in E'_n \overset{\text{pullback}}{\Longleftrightarrow} f(z) \in C'_n$$

$$\overset{\text{subcategory}}{\Longleftrightarrow} f(z)_{|I^n} \text{ has edges in } C'_1$$

$$\Longleftrightarrow f(z_{|I^n}) \text{ ——— } C'_1$$

pullback
$\iff \exists\, I^n \longrightarrow E_1'$;

so that E' is a subcategory.

Together the two claims imply a).

b) <u>Injectivity</u>:

The morphism $C \longrightarrow N\mathfrak{h}C$ induces

- a bijection $C_0 \xrightarrow{\sim} (N\mathfrak{h}C)_0$
- a surjection $C_1 \twoheadrightarrow (N\mathfrak{h}C)_1$.

So if D is a subcategory of $\mathfrak{h}C$, then

$\eta^{-1}\mathfrak{h}D$ determines $\begin{cases} D_0 = \eta\big((\eta^{-1}ND)_0\big) \\ D_1 = \eta\big((\eta^{-1}ND)_1\big) \end{cases}$

$\mathfrak{h}$ence determines D.

This proves the injectivity.

<u>Surjectivity</u>: Let $C' \subseteq C$ be a subcategory.

The natural candidate for D is hC', so let's try that! We first check that $hC' \to hC$ is a subcategory. We have $\mathrm{Ob}(hC') = C'_0 \hookrightarrow C_0 = \mathrm{Ob}(hC)$.

<u>Q</u>: $\mathrm{Mor}(hC') = C'_1/_{\text{homotopy}} \overset{?}{\hookrightarrow} C_1/_{\text{homotopy}} = \mathrm{Mor}(hC)$

If $f, g \in C'_1$ are homotopic in C,

it means there exists
$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "t"] & & y \end{tikzcd} \in C_2$$

but $t \in C'_2 \subset C_2$ because f, id_y lie in C'_1.
and C' is a subcategory.

$\Rightarrow$ hC' is a subcategory of hC.

- It remains to show that
$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ NhC' \arrow[r] & NhC \end{tikzcd}$$

is a pullback square. On n-simplices, this says that

$$z \in C'_n \overset{?}{\iff} \text{“the edges in } z|_{I^n} \text{ are homotopic in } C \text{ to morphisms in } C'\text{.”}$$

But as we saw, this last condition is $\iff$ "the edges of $z|_{I^n}$ are morphisms of C'_1"

which by definition of a subcategory is precisely $\iff z \in C'_n$. $\square$

Ex: • If $C \in \mathrm{Cat}$, then

$$\{\text{(full) subcategories of } C\} \xrightarrow{\simeq} \{\text{(full) subcategories of } NC\}$$

$$D \longmapsto ND$$

- To finish this introductory section, we need to see at least one example which does not come from nerves or Kan complexes.

The idea is that, in the same way that the nerve N gives

$$N : \{ 1\text{-categories} \} \subset \{ \infty\text{-categories} \}$$

there should exist a "fully faithful" functor:

$$N_2 : \{ (2,1)\text{-categories} \} \subset \{ \infty\text{-categories} \}$$

This does exist! However:

Pb: The definition of general (weak) (2,1)-categories is complicated!

Even strict (2,1)-categories require

ideas that I don't want to discuss (yet).

Sol: We know one example of a strict $(2,1)$-category Cat^2:

$$\begin{cases} \mathrm{Ob}\ \mathrm{Cat}^2 : \text{small categories} \\ \mathrm{Mor}\ \mathrm{Cat}^2 : \text{functors} \\ 2\text{-}\mathrm{Mor}\ \mathrm{Cat}^2 : \text{natural isomorphisms.} \end{cases}$$

So the plan is to construct an ∞-category $N_2(\mathrm{Cat}^2)$ from this; the recipe then generalizes to any weak $(2,1)$-category (N_2 is called the Duskin nerve).

Example: We define a simplicial set Cat^2 as follows: an n-dimensional element in $(\mathrm{Cat}^2)_n$ is the datum of:

- $\forall\ 0 \leq i \leq n$, a small category C_i
- $\forall\ 0 \leq i \leq j \leq n$, a functor $F_{i,j}: C_i \longrightarrow C_j$

- $\forall\ 0 \le i \le j \le k \le n$, a natural iso. $\alpha_{i,j,k} : F_{i,k} \Rightarrow F_{j,k} F_{i,j}$

such that :
- $F_{i,i} = id_{C_i}$
- $\alpha_{i,i,j}$ and $\alpha_{i,j,j}$ are identities.
- $\forall\ 0 \le i \le j \le k \le l \le n$,

the diagram
$$\begin{tikzcd} F_{i,l} \arrow[r, Rightarrow, "\alpha_{i,j,l}"] \arrow[d, Rightarrow, "\alpha_{i,k,l}"'] \arrow[dr, phantom, "\circledast"] & F_{j,l} F_{i,j} \arrow[d, Rightarrow, "\alpha_{j,k,l}"] \\ F_{k,l} F_{i,k} \arrow[r, Rightarrow, "\alpha_{i,j,k}"] & F_{k,l} F_{j,k} F_{i,j} \end{tikzcd}$$
commutes.

- For $\delta : [m] \longrightarrow [n]$, we define

$$\delta^*(C_i, F_{i,j}, \alpha_{i,j,k}) = (C_{\delta(i)}, F_{\delta(i),\delta(j)}, \alpha_{\delta(i),\delta(j),\delta(k)})$$

Rmk: $N\,Cat$ is isomorphic to the simplicial subcomplex of Cat^2 on the elements where

$$F_{i,k} = F_{j,k} F_{i,j} \quad \text{and} \quad \alpha_{i,j,k} = id .$$

So we have added some new higher morphisms to $N\,Cat$, corresponding to non-id. natural isomorphisms.

prop 25: Cat^2 is an ∞-category.

proof: • The proof is similar to the proof that the nerve of a category is an ∞-category; we just need to go "one dimension higher".

• By definition of Cat^2, an n-simplex is determined by its restriction to $Sk_3(\Delta^n)$ ("Cat^2 is 3-coskeletal"). Since we have

$$Sk_3(\Lambda^n_k) = Sk_3(\Delta^n) \quad \text{for all } n \geq 4,$$

we only need to check the inner horn liftings for $n \leq 3$.

__n=2__ A map $\Lambda^2_1 \longrightarrow Cat^2$ is just a datum

$$\begin{tikzcd} & c_1 \arrow[dr, "F_{12}"] & \\ c_0 \arrow[ur, "F_{01}"] & & c_2 \end{tikzcd} \quad \text{which we fill up with} \quad \begin{tikzcd} & c_1 \arrow[dr, "F_{12}"] \arrow[d, phantom, "\parallel"] & \\ c_0 \arrow[ur, "F_{01}"] \arrow[rr, "F_{12}F_{01}"'] & {} & c_2 \end{tikzcd}$$

__Note:__ this is not unique in general, we can take any natural iso $G \simeq F_{12}F_{01}$.

$n=3$ A map $\Lambda^3_1 \longrightarrow \mathrm{Cat}^2$ is a diagram:

$$\begin{tikzcd}
C_0 \arrow[dd, "F_{01}"'] \arrow[drr, "F_{02}"] \arrow[drrr, bend left, "F_{03}"] & & & {} \\
{} \arrow[r, phantom, "\overset{\alpha_{012}}{\Longrightarrow}"] & {} & C_2 \arrow[r, "F_{23}"] \arrow[ur, phantom, "\Downarrow \alpha_{023}"] \arrow[dr, phantom, "\Uparrow \alpha_{123}"] & C_3 \\
C_1 \arrow[urr, "F_{12}"] \arrow[urrr, bend right, "F_{13}"'] & & & {}
\end{tikzcd}$$

By the condition ⊛ in the definition, we must fill this in with

$$\begin{tikzcd}
C_0 \arrow[dd, "F_{01}"'] \arrow[dr, bend left, "F_{03}"] & {} \\
{} \arrow[r, phantom, "\Downarrow \alpha_{013}"] & C_3 \\
C_1 \arrow[ur, bend right, "F_{13}"'] & {}
\end{tikzcd}$$

with $\alpha_{013} = \alpha_{123}^{-1} \circ \alpha_{012} \circ \alpha_{023}$

and it works. $\square$

III Lifting calculus

- The definition of ∞-categories is very combinatorial and the proofs so far have been by explicit manipulations of simplices.
- To go further and prove results like Thm 20-21, we need a more systematic way to keep the combinatorics manageable.

1) Fibrations and anodyne maps

Let us formalize a concept we have already seen a lot:

def 1: . Let C be a category. A lifting problem in C is a commutative square in C:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "p"] \\ B \arrow[r, "v"'] & Y \end{tikzcd}$$

- A solution / lift of the lifting problem is a diagonal map:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "p"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

- If S, T are collections of morphisms in $\mathcal{C}$, we say that S has the left lifting property with respect to T

  or equivalently that

  T has the right lifting property with respect to S.

  and write $S \boxslash T$ if every lifting problem as above with $f \in S$ and $p \in T$ has a solution.

  In particular, if $S = \{f\}$ (resp. $T = \{p\}$) is a singleton, we say that f has the left lifting property w.r.t to T (resp. p has the right lifting property w.r.t. to S) and write $f \boxslash T$ (resp. $S \boxslash p$).

- We define the left complement of S, resp. the right complement of T, by

$$S^{\boxslash} := \{ p \in \mathrm{Mor}(C) \mid g \boxslash p \text{ for all } g \in S \}$$

$$\left(\text{resp. } {}^{\boxslash}T := \{ g \in \mathrm{Mor}(C) \mid g \boxslash p \text{ for all } p \in T \}\right)$$

[alternative notations: $RLP(S)$ / $LLP(S)$]
$S \perp T$

Ex: in $sSet$, we have already several examples of this pattern:

- $\{\text{Kan fibrations}\} := \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k \leq n \right)^{\boxslash}$
- $\{\text{trivial Kan fibrations}\} := \left(\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 1 \right)^{\boxslash}$
- $C \in sSet$ is an ∞-category iff the map $C \to *$ lies in $\left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n \right)^{\boxslash}$.
- On the other hand, the Grothendieck-Segal condition for $X \simeq N(C)$ with $C \in Cat$ cannot be expressed this way because it requires imposing the <u>uniqueness</u> of solutions.

Ex: Some other examples you may be familiar with:

- $\underline{\mathcal{C} = \mathrm{Set}}$: $(\text{injections}) \boxslash (\text{surjections})$

  and in fact:

$$\begin{cases} (\text{inj.}) = {}^{\boxslash}(\text{surj.}) = {}^{\boxslash}(* \amalg * \to *) \\ (\text{surj.}) = (\text{inj.})^{\boxslash} = (\emptyset \to *)^{\boxslash} \end{cases}$$

- $\underline{\mathcal{C} = \mathrm{Ab} \text{ (or any abelian category)}}$:

  - $P \in \mathrm{Ab}$ is called projective if the unique map $P \to 0$ lies in $(\text{mono.})^{\boxslash}$.

  - $I \in \mathrm{Ab}$ is called injective if the unique map $0 \to I$ lies in ${}^{\boxslash}(\text{epi})$.

- $\underline{\mathcal{C} = \mathrm{Top}}$:

  A morphism $p : X \to Y$ is called a Serre fibration if $p \in (D^n \hookrightarrow D^n \times I)^{\boxslash}$

a Hurewicz Fibration if $p \in (X \hookrightarrow X \times I \mid X \in \mathrm{Top})^{\boxslash}$.

Lemma 2: With the notations of def 1:

a) $R \subseteq S \Rightarrow \begin{cases} S^{\boxslash} \subseteq R^{\boxslash}, \text{ and} \\ {}^{\boxslash}S \subseteq {}^{\boxslash}R. \end{cases}$

b) $S \subseteq {}^{\boxslash}(S^{\boxslash})$ and

$$S \subseteq ({}^{\boxslash}S)^{\boxslash}.$$

c) $S^{\boxslash} = ({}^{\boxslash}(S^{\boxslash}))^{\boxslash}$ and

$${}^{\boxslash}S = {}^{\boxslash}(({}^{\boxslash}S)^{\boxslash}).$$

proof: · a) is clear: we have more lifting problems to solve with S than with R.

b) By duality, we only need to prove the first.

$$\cdot \quad S \subseteq {}^{\boxslash}(S^{\boxslash}) : \quad \begin{tikzcd} A \arrow[r] \arrow[d, "S \ni"'] & X \arrow[d, "\in S^{\boxslash}"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

c) Again by duality we prove only the first:

$$\begin{cases} S^{\boxslash} \overset{b)}{\subseteq} \left({}^{\boxslash}(S^{\boxslash})\right)^{\boxslash} \\ \left({}^{\boxslash}(S^{\boxslash})\right)^{\boxslash} \overset{a)+b)}{\subseteq} S^{\boxslash} \end{cases}$$

$\square$

---

def 3: We define collections of morphisms in sSet:

$\cdot$ (inner fibrations) $:= \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n\right)^{\boxslash}$

$\cdot$ (left fibrations) $:= \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k < n\right)^{\boxslash}$

$\cdot$ (right fibrations) $:= \left(\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k \leq n\right)^{\boxslash}$

$$\Big[\cdot\ (\text{Kan fibrations}) = \left(\Lambda^n_k \hookrightarrow \Delta^n \;\middle|\; 0 \le k \le n\right)^{\boxslash}\Big]$$

$$\cdot \begin{pmatrix} \text{inner anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{inner fibrations})$$

$$\cdot \begin{pmatrix} \text{left anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{left fibrations})$$

$$\cdot \begin{pmatrix} \text{right anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{right fibrations})$$

$$\cdot \begin{pmatrix} \text{anodyne} \\ \text{morphisms} \end{pmatrix} := {}^{\boxslash}(\text{Kan fibrations})$$

Rmk: • From the definition and Lemma 2:

$$\begin{tikzcd}
& \text{Kan fibration} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{left fibration} \arrow[dr, Rightarrow] & & \text{right fibration} \arrow[dl, Rightarrow] \\
& \text{inner fibration} &
\end{tikzcd}$$

We concentrate → inner fibration

on these in this chapter.

$$\begin{tikzcd}[column sep=small]
& \text{inner anodyne} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{left anodyne} \arrow[dr, Rightarrow] & & \text{right anodyne} \arrow[dl, Rightarrow] \\
& \text{anodyne} &
\end{tikzcd}$$

- Moreover, it is easy to see that

f Kan fibration $\Leftrightarrow$ f left and right fibration.

⚠ f inner anodyne $\not\Leftrightarrow$ f left and right anodyne.

- $X \in sSet$,

X ∞-category $\Leftrightarrow$ $X \to *$ inner fibration.

- We will see later that any functor $C \xrightarrow{F} D$ of ∞-categories is an inner fibration "up to equivalence":

$$\begin{tikzcd}
C \arrow[rr, "F"] \arrow[dr, "\sim", "eq." '] & & D \\
& C' \arrow[ur, "\text{inner fibration.}" '] &
\end{tikzcd}$$

- "anodyne" is due to Gabriel and Zisman (1967) and means $\sim$ "without pain" in Greek. Central notion in simplicial homotopy theory, the left/right/inner versions are natural extensions in the context of quasicategories.

Lemma 4: Let $X \in sSet$ and $C \in Cat$. Then $X \longrightarrow NC$ is an inner fibration iff X is an ∞-category.

In particular, if $F: C \to D$ is a functor of 1-categories, then $NF: NC \to ND$ is an inner fibration.

proof: Exercise. Hint: use the $\tau \dashv N$ adjunction and consider the maps $\tau(\Lambda^n_k) \longrightarrow \tau(\Delta^n)$ for $0<k<n$

$\square$

Collections of morphisms of the form ${}^{\boxslash}S$ (or $S^{\boxslash}$) have some remarkable "closure" properties.

def 5: Let $S \subseteq \mathrm{Mor}(\mathcal{C})$. We say that:

* S is closed under pushouts if for every pushout diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\gamma"'] & A' \arrow[d] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$
, $\gamma \in S \Rightarrow \gamma' \in S$.

* S is closed under retracts if for every retract diagram in C:

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "id"] \arrow[d, "g"'] & A' \arrow[r] \arrow[d, "g'"'] & A \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "id"'] & B' \arrow[r] & B \end{tikzcd}$$

(expressing that g' is a retract of g)

we have $g \in S \Rightarrow g' \in S$.

* S is closed under transfinite compositions if for all ordinals α and a functor $F : [\alpha] \longrightarrow C$ with $\forall\ 0 < \lambda \leq \alpha$ with λ limit ordinal,

$$F(\lambda) \simeq \operatorname*{colim}_{\gamma < \lambda} F(\gamma)$$

$\left(\text{for all } \beta < \alpha ,\ F(\beta < \beta+1) \in S \right) \Rightarrow F(0 < \alpha) \in S$

[ex For $\alpha = \omega = (\mathbb{N}, \leq)$, then $[\omega] = \omega + 1 = \mathbb{N} \cup \{\infty\}$

and the condition says that if

$$F(\infty) \simeq \operatorname*{colim}_{n \in \mathbb{N}} \left(F(0) \underset{\in S}{\to} F(1) \underset{\in S}{\to} F(2) \underset{\in S}{\to} \cdots \right)$$

then $F(0) \longrightarrow F(\infty)$ is in S.]

* S is (weakly) saturated if it is stable under pushouts, retracts and transfinite compositions.

Rmk: • There is a related notion of "strongly saturated" class which plays a role in the theory of localizations of (∞-) categories, hence the terminology.

• The transfinite composition for $\alpha = 0$ (resp. $\alpha = 2$) means that S contains all isos (resp. is stable by compositions).

Lemma 6: Let C admit arbitrary coproducts. Then any saturated collection in C is stable under coproducts.

proof: Let S be a saturated collection, and $(\delta_\alpha : A_\alpha \to B_\alpha)$ be a set of morphisms such that the coproduct

$$\coprod \delta_\alpha : \coprod A_\alpha \longrightarrow \coprod B_\alpha \text{ exists.}$$

We can construct $\coprod \delta_\alpha$ as a transfinite composition of pushouts. Ex:

$$\begin{tikzcd}
A_0 \arrow[rr] \arrow[d] & & B_0 \arrow[r, "id"] \arrow[d] & B_0 \arrow[d] \\
A_0 \amalg A_1 \arrow[rr] & & A_1 \amalg B_0 \arrow[r] \arrow[ul, phantom, "\ulcorner"] & B_0 \amalg B_1 \\
\vdots & & & \vdots \\
\coprod_{n \in \mathbb{N}} A_n \arrow[rrr, "\coprod f_\alpha"] & & & \coprod_{n \in \mathbb{N}} B_n
\end{tikzcd}$$

$\square$

This notion seems very artificial at first, but as we will see, it is in fact quite natural in view of Quillen's "<u>small object argument</u>".

First, we have:

prop 7: Let $S \subseteq \mathrm{Mor}(\mathcal{C})$ be any collection of morphisms. Then ${}^{\boxslash}S$ is saturated.

proof:

<u>Stable under pushouts:</u>

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "{}^{\boxslash}S \ni", swap] \arrow[d, "f"] & A' \arrow[r] \arrow[d] & X \arrow[d, "\in S"] \\
B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & Y
\end{tikzcd}$$

where $\longrightarrow$ exists because $f \in {}^{\boxslash}S$

$\longrightarrow$ exists because of universal property of the pushout.

Stable under retracts:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "{}^{\boxslash}S \ni f"'] & A' \arrow[r] \arrow[d, "f'"'] \arrow[rr, bend left] & A \arrow[d, "f"] & X \arrow[d, "\in S"] \\
B \arrow[r] \arrow[urrr, dashed] & B' \arrow[r] \arrow[urr, dashed] \arrow[rr, bend right] & B \arrow[ur, dashed] & Y
\end{tikzcd}$$

where the two (equal) $\longrightarrow$ exist because of $f \in {}^{\boxslash}S$, the $\longrightarrow$ is just defined as composition $\longrightarrow \nearrow \nearrow$, and everything commutes because of the retract property $\xrightarrow{\mathrm{id}}$ $\xrightarrow{\mathrm{id}}$.

Stable under transfinite compositions:

For notational simplicity I explain the case $\omega = (\mathbb{N}, \leq)$, the general argument is the same:

$$\begin{tikzcd}
F(0) \arrow[r] \arrow[d, "{}^{\boxslash}S \ni"'] & X \arrow[ddd, "\in S"] \\
F(1) \arrow[ur, dashed] \arrow[d, "{}^{\boxslash}S \ni"'] & \\
F(2) \arrow[uur, dashed] \arrow[d, dotted, no head] & \\
F(\infty) \arrow[r] \arrow[uuur, dashed] & Y
\end{tikzcd}$$

where:
- $\dashrightarrow$'s are constructed by induction using that $F(i < i+1) \in {}^{\boxslash}S$.
- $\dashrightarrow$ is constructed from the universal property of colimits. $\square$

An intersection of saturated collections is saturated. This implies the following is well-defined:

def 8: Let C be a category and S_0 be a collection of morphisms in C. There is a smallest saturated collection $\overline{S_0}$ which contains S_0. We call it the saturated collection generated by S_0.

As a first example, we have:

prop 9: The collection of all monomorphisms in sSet is the saturated collection generated by $(\partial\Delta^n \hookrightarrow \Delta^n)_{n \geq 0}$.

proof: The fact that (monos) is saturated is easy and left as an exercise.

$$\Rightarrow \overline{(\partial\Delta^n \hookrightarrow \Delta^n)} \subseteq (\text{monos}).$$

- The fact that $(\text{monos}) \subseteq \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}$ follows from the existence of the skeletal filtration (Prop I.24) for any monomorphism. $\square$

Cor 10: 1) Let $f : X \to Y$ in sSet. TFAE

a) f is a trivial fibration.

b) f has the right lifting property wrt monos.

2) Let $f : X \to Y$ be a trivial Kan fibration. Then f admits a section $s : Y \to X$. Moreover, the composition $s \circ f : X \to X$ is homotopic to id_X over Y: there exists

$$\begin{tikzcd} X \times \Delta^1 \arrow[rr, "h"] \arrow[dr] & \equiv & X \arrow[dl] \\ & Y & \end{tikzcd} \quad \text{with } \begin{cases} h_{|X \times 0} = s \circ f \\ h_{|X \times 1} = \mathrm{id}_X \end{cases}$$

proof: __1):__ b) $\Rightarrow$ a) is clear.

__a) $\Rightarrow$ b):__ Let $S = {}^{\boxslash} g$. We know that $(\partial\Delta^n \hookrightarrow \Delta^n) \subseteq S$ by a) and we want to prove that $(\text{monos}) \subseteq S$.

We know by $\begin{cases} \text{Prop 5 that } S \text{ is saturated.} \\ \text{Prop that } (\text{monos}) = \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}. \end{cases}$

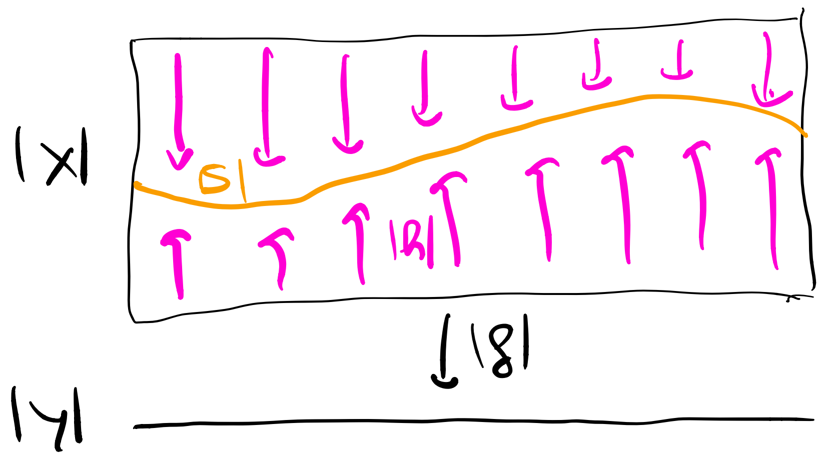
and we are done.

2) s (resp. h) is solution of:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, "\underline{\text{mono}}"'] & X \arrow[d, "g"] \\ Y \arrow[r, equal, "id"] \arrow[ur, dashed, "s"] & Y \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} \partial\Delta^n \times X \arrow[r, "{(s\circ g, id)}"] \arrow[d] & X \arrow[d, "g"] \\ \Delta^n \times X \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

which exist by part 1). $\square$

Rmk: Part 2) means that, not only are the fibers of a trivial fibration contractible Kan complexes, but they can be contracted "simultaneously": after geometric realisation:



Our next objective is to get similar description of inner/left/right anodyne maps as saturated classes.