

Some complements on Kan complexes
 (based on Ex. sheet 2).

Lemma 32: (Exercise 2.4)

X. Kan complex. The relation on X_0 .

given by : $x \sim y \iff \exists t \in X_1, \begin{cases} d_1(t) = x \\ d_0(t) = y \end{cases}$

is an equivalence relation.

$$\Rightarrow \pi_0(X) = X_0 / \sim .$$

Proof: Reflexivity $d_0 s_0(x) = d_1 s_0(x) = x$ (does not require X Kan)

Symmetry Assume $x \sim y$ witnessed by $p \in X_1$.

We have $\Delta_0^2 = \Delta_0^1 \coprod_{\Delta_0^1} \Delta_0^1$, i.e.

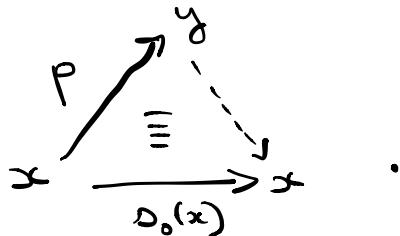
$$\begin{array}{ccc} \Delta_0^1 & \xrightarrow{\quad \sim_0 \quad} & \Delta_0^1 \\ \downarrow & & \downarrow \\ \Delta_0^1 & \xrightarrow{\quad \tau \quad} & \Delta_0^2 \end{array}$$

$$\text{so } \text{sSet}(\Delta_0^2, X) = \left\{ (p_1, p_2) \in X_1^2 \mid d_1(p_1) = d_1(p_2) \right\}$$

Consider $\Delta_0^2 \rightarrow X$ given by $(p, d_0(x))$.

Apply the Kan lifting property $\Rightarrow \Delta^2 \xrightarrow{t} X$.

Then $d_0 t \in X_n$ witnesses $y \sim x$:



Transitivity Same construction with Λ^2_1 : \square

def 33: A simplicial set X_\cdot is discrete if for all $g \in \Delta([m], [n])$, $g^*: X_n \rightarrow X_m$ is a bijection.

Lemma 33: (Exercises 1.6, 2.5)

a) The functors

$$\text{Set} \longrightarrow \text{sSet}^{\text{discr}}$$

$S \longmapsto \underline{S}$:= constant presheaf with value S :

$$[n] \mapsto S, g \mapsto \text{id}_S.$$

and $\text{sSet}^{\text{discr}} \xrightarrow{\text{ev}_{[0]}} \text{Set}$ yield an equivalence of categories between discrete simplicial sets and sets.

b) discrete simplicial sets are Kan complexes.

proof: a) We have $\underline{\text{ev}_0}(\underline{\Delta}) \cong S$ naturally in S .

It remains to show $\underline{\text{ev}_0}(X_\cdot) \cong X_\cdot$ for X_\cdot discrete.

There is actually a natural transformation $\underline{\text{ev}_0}(-) \xrightarrow{\epsilon} \text{id}$ of functors $s\text{Set}^G$, given on X_\cdot by:

$$(\varepsilon_{X_\cdot})_n : \underline{\text{ev}_0(X_\cdot)}_n = X_0 \xrightarrow{\langle \circ \dots \circ \rangle^*} X_n. \quad (\text{This is a map of } s\text{-sets because } [\circ] \text{ is the final obj. of } \Delta)$$

If X_\cdot is discrete, then by definition ε_{X_\cdot} is an isomorphism.
and we are done.

b) By part a) we can assume $X_\cdot = \underline{\Delta}$ (we could also argue directly)

We must show: $\forall n \geq 1, \forall 0 \leq i \leq n, s\text{Set}(\Delta^n, \underline{\Delta}) \cong S$

$$\downarrow \\ s\text{Set}(\Lambda^n_i, \underline{\Delta})$$

I will explain in general how to compute $s\text{Set}(X_\cdot, \underline{\Delta})$.

Claim 1: A map $X_\cdot \xrightarrow{f} \underline{\Delta}$ is determined by $X_0 \xrightarrow{f_0} S$.

Pf: $\langle \circ \dots \circ \rangle^* f_n(x) = f_0(\langle \circ \dots \circ \rangle^* x)$; but $\langle \circ \dots \circ \rangle^*$ is a bijection in $\underline{\Delta}$.

Claim 2: $f_0 : X_0 \rightarrow S$ factors through $X_0 \xrightarrow{\pi_0} \pi_0(X) \xrightarrow{\sim} S$

Pf: Let $x, y \in X_0$ and $p \in X_1$, $d_1(p) = x$ and $d_0(p) = y$.

Then $\begin{cases} d_1 f(p) = f(d_1 p) = f(x) & \text{but in } S, d_0 = d_1 = \text{id}_S \\ d_0 f(p) = f(d_0 p) = f(y) \end{cases}$

This is enough to finish the exercise; indeed $\pi_0(\Lambda_R^n)$ has one element (because $I^n \subset \Lambda_R^n$, hence $0 \mapsto 1 \mapsto \dots \mapsto n$).

Claim 3: $sSet(X_*, \underline{\leq}) \simeq \text{Set}(\pi_0(X_*), S)$.

Pf: We have constructed an injective map \hookrightarrow .

It remains to see that any map $\pi_0(X_*) \xrightarrow{g} S$ can be realized this way. There is a natural candidate:

$$x \in X_n \longmapsto g([<0\dots 0]^* x) \in S$$

I leave it to you to check that this is well-defined and gives an inverse. □

Lemma 34: (Exercise 2.6)

Trivial Kan fibrations are Kan fibrations.

Proof: Let $f: X \rightarrow Y$ be a trivial Kan fibration, and $0 \leq R \leq n$. We are given a diagram:

$$\begin{array}{ccc} \Lambda_R^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

and look for a diagonal map filling it.

We have a pushout square

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \Lambda_B^n \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow{\quad} & \partial\Delta^n \end{array} \quad \left(\begin{array}{c} \triangle \xrightarrow{\quad} \square \\ \downarrow \quad \downarrow \\ \triangle \xrightarrow{\quad} \square \end{array} \right)$$

To prove it is a pushout, can use the lemma from the proof of Prop 24.

We thus have

$$\begin{array}{ccccc} \partial\Delta^{n-1} & \longrightarrow & \Lambda_B^n & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow g \\ \Delta^{n-1} & \xrightarrow{\quad} & \partial\Delta^n & \xrightarrow{\quad} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & & & Y \end{array}$$

with $\text{---} \rightarrow$ exists because g is a trivial Kan fibration

$\text{---} \rightarrow$ exists because of the universal property of the pushout square.

$\text{---} \rightarrow$ exists because f is a trivial Kan fibration.



II) Infinity-categories

1) Nerves of categories

Our model of ∞ -categories

will be a particular type of

simplicial sets. We also

want to be able to consider

ordinary categories as

∞ -categories.

→ We need a fully

faithful functor

$$N: \text{Cat} \longrightarrow s\text{Set} .$$

- Posets give rise to categories:

$$\text{Poset} \xrightarrow{\text{Fully faithful}} \text{Cat}$$

$$P \longrightarrow \text{Ob} : x \in P$$

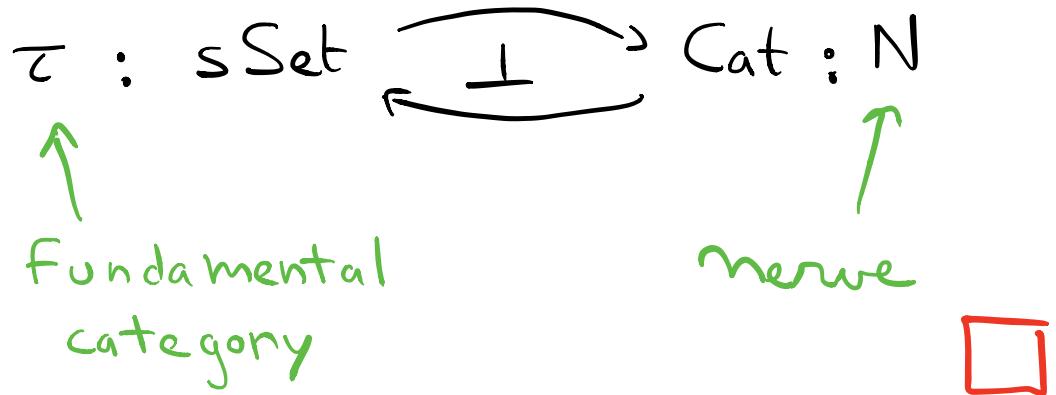
$$P(x, y) = \begin{cases} *, & x \leq y \\ \emptyset, & \text{otherwise} \end{cases}$$

In particular we have a
fully faithful cosimplicial category:

$$\Delta \xrightarrow{Q^\bullet} \text{Cat}$$

def 1 By the free cocompletion

property, Q^\bullet induces an adjunction:



By construct°:

$$\left\{ \begin{array}{l} \tau(\Delta^n) = [n] \\ \tau \text{ commutes with colimits.} \end{array} \right.$$

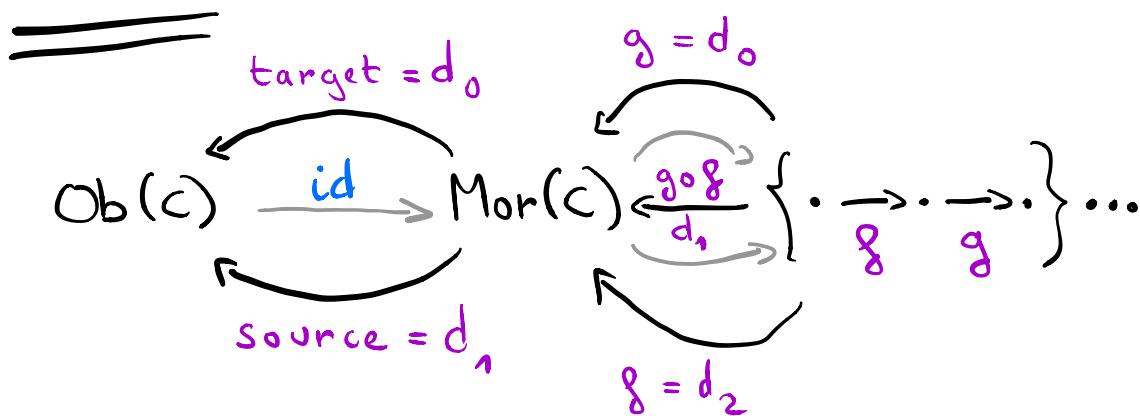
and

$$N(C)_n = \text{Cat}([n], C)$$

$$= \left\{ \begin{array}{l} n\text{-tuples of} \\ \text{composable morphisms} \end{array} \right\}$$

Rmk We have $N([n]) \simeq \Delta^n$
 $(\Leftarrow Q \text{ fully faithful})$.

$N(C)$:



This makes it clear that

We can reconstruct C from $N(C)$.

Prop 2 N is fully faithful

Proof: For C, D in Cat , we must show

$$\text{Cat}(C, D) \xrightarrow{\sim} \text{sSet}(N_C, N_D).$$

Injectivity: A functor is determined by its effect on objects and morphisms. Since $(Nc)_0 = \text{Ob}(c)$ and $(Nc)_1 = \text{Mor}(c)$, we are done.

Surjectivity Let $\alpha: Nc \rightarrow Nd$.

We define a candidate F for the preimage functor using again

$$\begin{cases} (Nc)_0 = \text{Ob}(c) \\ (Nc)_1 = \text{Mor}(c) \end{cases}$$

To check that F is a functor, we need to see:

• F compatible with source/target:

Let $g : c \rightarrow c'$ in C

$$\left\{ \begin{array}{l} s F(g) \stackrel{\text{def}}{=} d_1 \alpha(g) = \alpha d_1(g) \stackrel{\text{sSet}}{=} F(c) \\ t F(g) \stackrel{\text{def}}{=} d_0 \alpha(g) = \alpha d_0(g) \stackrel{\text{def}}{=} F(c') \end{array} \right.$$

• F compatible with identities

$$F(id_c) \stackrel{\text{def}}{=} g s_0(c) \stackrel{\text{sSet}}{=} s_0 g(c) \stackrel{\text{def}}{=} id_{F(c)}$$

• F compatible with compositions

$$F(g \circ f) = \alpha(g \circ f) = \alpha(d_1(g, f))$$

$$= d_1 \alpha(g, f) = F(g) \circ F(f)$$

$\rightsquigarrow F$ is a functor.

By construction, $N(F)$ and α coincide on 0 and 1-simplices, and it is easy to see that it forces them to be equal

(because simplices in $(NC)_i$ for $i > 2$ are uniquely determined by their 1-simplices).



We also need a precise description of fundamental categories.

Prop 3: Let $X_0 \in \text{Set}$. The fundamental category τX_0 admits a presentation by generators and relations :

- $\text{Ob } \tau X_0 = X_0$.
- $\text{Mor } \tau X_0$ is generated by X_n ; for all $n \geq 0$ and $g_1, \dots, g_n \in X_n$ such that $d_1 g_i = d_0 g_{i-1}$, we have $\overline{g_n} \circ \dots \circ \overline{g_1} \in \text{Mor } \tau X_0(d_0 g_n, d_0 g_1)$.
- We have relations:
 $\forall x \in X_0, \overline{d_0(x)} = \text{id}_x \in \tau X_0(x, x);$
 $\forall t \in X_2, \text{ we have } \overline{d_0 t} \circ \overline{d_2 t} \simeq \overline{d_1 t}.$

In other words, for any other category D , we have $\text{Cat}(\tau X_0, D)$ is in natural bijection with the set of pairs

$(F_0: X_0 \longrightarrow \text{Ob } D,$ such that:
 $F_1: X_1 \longrightarrow \text{Mor } D)$

- $\forall g \in X_1, \begin{cases} \text{source } (F_1(g)) = F_0(d_1 g) \\ \text{target } (F_1(g)) = F_0(d_0 g) \end{cases}$

$$\forall x \in X_0, F_1(d_0 x) = \text{id}_{F_0(x)}.$$

- $\forall t \in X_2^{(\text{ind})}, F_1(d_1 t) = F_1(d_0 t) \circ F_1(d_2 t)$

Proof: We use the skeletal filtration together

with the facts that $\begin{cases} \tau \text{ commutes with colimits.} \\ \tau(\Delta^n) = [n]. \end{cases}$

$$\cdot \tau(X_\cdot) = \underset{n \geq 0}{\text{colim}} \tau(\text{Sk}_n X_\cdot)$$

category with
one morphism.
↓

$$\cdot \tau(\text{Sk}_0 X) = \tau\left(\coprod_{x \in X_0} \Delta^0\right) = \coprod_{x \in X_0} [0] = \coprod_{x \in X_0} *$$

$$\cdot \partial \Delta^\sim = \Delta^\circ \amalg \Delta^\circ \Rightarrow \tau \Delta^\sim = * \amalg *$$

$$\begin{array}{ccc} \coprod_{X_1^{\text{nd}}} \partial \Delta^\sim & \longrightarrow & \coprod_{X_1^{\text{nd}}} \Delta^\sim & \coprod_{X_1^{\text{nd}}} (* \amalg *) \longrightarrow \coprod_{X_1^{\text{nd}}} [1] \\ \downarrow & & \downarrow & \downarrow \\ \text{Sk}_0(X) & \longrightarrow & \text{Sk}_n(X) & \coprod_{X_0} * \longrightarrow \tau(\text{Sk}_n(X)) \end{array}$$

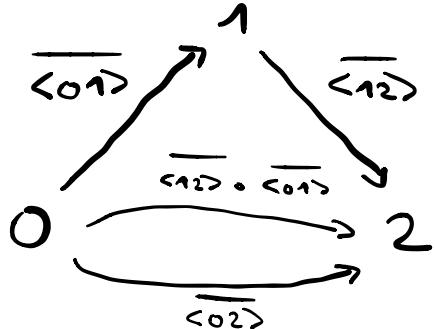
$\Rightarrow \tau(\text{Sk}_n(X_\cdot))$ is the free category generated by X_1^{nd} .

Since $X_1 = X_1^{\text{nd}} \amalg s_0(X_0)$, this is equivalent to generating by X_1 and imposing $\overline{s_0(x)} = \text{id}_x$.

- $\partial\Delta^2 = \text{Sh}_1(\partial\Delta^2)$

\amalg

$\tau(\partial\Delta^2)$ is given by:



- So the map $\tau(\partial\Delta^2) \rightarrow \tau(\Delta^2) = [2]$ identifies the two morphisms between 0 and 2.

$$\begin{array}{ccc} \coprod_{X_2^{\text{nd}}} h(\partial\Delta^2) & \longrightarrow & \coprod_{X_2^{\text{nd}}} [2] \\ \downarrow & & \downarrow \\ \tau(\text{Sh}_1 X) & \longrightarrow & \tau(\text{Sh}_2 X) \end{array}$$

thus imposes exactly
the relations of the
statement for $t \in X_2^{\text{nd}}$.

- Any $t \in X_2^{\text{deg}}$ imposes a relation with identities which is already there.

- $\tau(\partial\Delta^3) = \tau(\text{Sh}_2 \partial\Delta^3)$ ($\partial\Delta^3$ is 2-skeletal)

$$= \tau \left(\begin{array}{c} \nearrow \downarrow \searrow \\ \cdot \end{array} \right)$$

$$= [3]$$

(use previous
step)

$$\Rightarrow \tau(\partial\Delta^3) \xrightarrow{\sim} \tau(\Delta^3)$$

$$\Rightarrow \forall X \in \text{sSet}, \tau(Sh_2 X) \xrightarrow{\sim} \tau(Sh_3 X)$$

*skeletal
filtration*

$$\Rightarrow \tau(\partial\Delta^4) \simeq \tau(Sh_2 \partial\Delta^4) \simeq \tau(\Delta^4)$$

$$\Rightarrow \forall X \in \text{sSet}, \tau(Sh_3 X) \xrightarrow{\sim} \tau(Sh_4 X)$$

⋮

$$\Rightarrow \forall X \in \text{sSet}, \tau(X) \xrightarrow{\sim} \tau(Sh_2 X)$$



Rmk The terminology "fundamental category" comes from topology:

- For $A \in \text{Top}$, $\tau(\text{Sing } A) \simeq \pi_{\leq 1} A$ fundamental groupoid of A .

- For $X \in \text{Kan}$, $\tau(X) \simeq \pi_{\leq 1} |X| \longrightarrow |X|$

Notation $X \in \text{sSet}$, $x \in X_n$, $\langle g_0 \dots g_m \rangle : [m] \rightarrow [n]$.

We write $x_{g_0 g_1 \dots g_m} := \langle g_0 \dots g_m \rangle^*(x)$.

In particular, x_0, \dots, x_n are the vertices of x

$x_{i,j}$ are the edges, and so on.

Thm 3 : (characterisation of nerves)

Let $X_\cdot \in \text{sSet}$. TFAE:

- (1) $X_\cdot \simeq NC$ for some $C \in \text{Cat}$.
- (2) The unit $X_\cdot \xrightarrow{\eta} N h X_\cdot$ is an isomorphism.
- (3) X_\cdot satisfies the unique

spine extension property :

For all $n \geq 2$,

$$I^n \longrightarrow X_\cdot$$
$$\begin{matrix} & I^n & \longrightarrow & X_\cdot \\ \downarrow & \nearrow & \nearrow & \\ \Delta^n & & \nearrow & \end{matrix}$$

(4) $X.$ satisfies the inner
horn unique extension property:

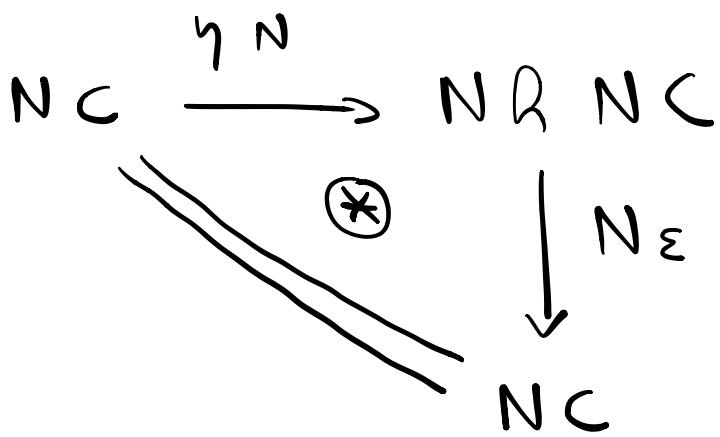
For all $n \geq 2$ and $0 < k < n$:

$$\begin{array}{ccc} \Delta^n_k & \longrightarrow & X. \\ \downarrow & \nearrow \text{exists!} & \end{array}$$

Proof: $(2) \Rightarrow (1)$ ✓

$(1) \Rightarrow (2)$: Assume $X \simeq NC$.

By a general property of
adjunctions, we have



• Because N is fully faithful,

$\varepsilon : RNC \xrightarrow{\sim} C$ is an iso

$\Rightarrow N\varepsilon \text{ iso} \xrightarrow{\textcircled{*}} \gamma N \text{ iso}$

$\Rightarrow X \xrightarrow[\gamma]{\sim} NRX.$

(1) \Rightarrow (3):

$sSet(\Delta^n, N(C))$

$\left\{ c_0 \xrightarrow{\delta_1} c_1 \rightarrow \dots \xrightarrow{\delta_n} c_n \text{ in } C \right\}$

is

$$\{ g_1, \dots, g_n \in N(C)_1 \mid d_0(g_i) = d_1(g_{i+1}) \}$$

$$IS \leftarrow I^n = \Delta^{\{0,1\}} \amalg \cdots \amalg \Delta^{\{n-1,n\}}$$

$$SSet(I^n, N(C))$$

(3) \Rightarrow (1):

We have to construct a category C and an isomorphism $X \xrightarrow{\sim} NC$.

We put $Ob C = X_0$, $Mor C = X_1$.

with $source(g) = d_1(g)$, $target(g) = d_0(g)$.

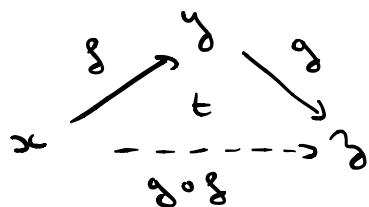
We put $id_x := s_0(x)$.

Let $g, h \in X_1$, $g: x \rightarrow y$ and $h: y \rightarrow z$

we get $(g, h): I^2 \rightarrow X$.

By assumption there is a unique extension t

to Δ^2 , and we define $g \circ h$ to be $d_1(t)$:



The unitality then follows from the triangles

$$s_0(g) : \begin{array}{ccc} & \nearrow^x & \downarrow^g \\ x & \xrightarrow[g]{} & y \end{array} \quad \text{and} \quad s_1(g) : \begin{array}{ccc} & \nearrow^y & \parallel \\ x & \xrightarrow[g]{} & y \end{array}$$

Associativity follows from the uniqueness of liftings along $I^3 \hookrightarrow \Delta^3$.

\Rightarrow we have constructed a category C .

- We construct a morphism $X \xrightarrow{\psi} NC$.

Let $x \in X_n = s\text{Set}(\Delta^n, X)$. Then

$x|_{I^n}$ determines a sequence of composable morphisms in $C \rightarrow$ a simplex $\psi(x) \in (NC)_n$.

It is easy to see that $X \xrightarrow{\psi} NC$ is

indeed a morphism of simplicial sets,

with $\begin{cases} X_0 \xrightarrow{\sim} (NC)_0 \\ X_n \xrightarrow{\sim} (NC)_n. \end{cases}$

We now contemplate the diagram:

$$\begin{array}{ccc} X_n & \longrightarrow & (NC)_n \\ S \downarrow (3) & & \downarrow S((1) \Rightarrow (3)) \end{array}$$

$$X_n \times_{X_0} X_1 \times \dots \times X_n \xrightarrow{\sim} (NC) \times \dots \times (NC)$$

$\uparrow (NC)_1$

and see that $X_n \xrightarrow{\sim} (NC)_n$

$(1) \Rightarrow (4)$:

Let $n \geq 1$ and $0 < k < n$.

We compute:

$$SSet(\Lambda_k^n, NC)$$

$$\simeq \text{Cat}(\tau(\Lambda_k^n), C)$$

$$\simeq \text{Cat}(\tau(\text{Sk}_2(\Lambda_h^n)), \circ)$$

We now observe that

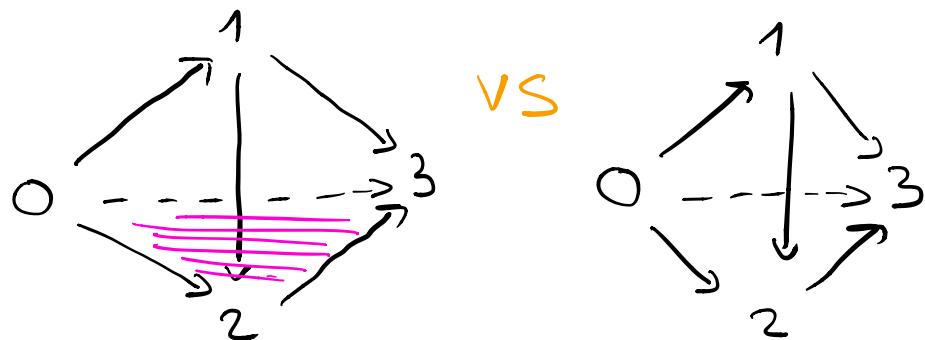
$$\left\{ \begin{array}{l} \forall n \geq 4, \text{ Sk}_2(\Lambda_h^n) \simeq \text{Sk}_2(\Delta^n) \\ \Lambda_1^2 = I^2 \end{array} \right.$$

So it remains to show

$$\left\{ \begin{array}{l} \tau(\Lambda_1^3) \simeq \tau(\Delta^3) \\ \tau(\Lambda_2^3) \simeq \tau(\Delta^3) \end{array} \right.$$

The two cases are similar, we

do Λ_1^3 . First a picture:



Λ^3_1 and Δ^3 have the same objects and edges. The only additional relation in $\tau(\Delta^3)$ is

$$\overline{\langle 03 \rangle} = \overline{\langle 23 \rangle} \circ \overline{\langle 02 \rangle}$$

but in $\tau(\Lambda^3_1)$ we have

$$\overline{\langle 03 \rangle} = \overline{\langle 13 \rangle} \circ \overline{\langle 01 \rangle}$$

back

face

$$\downarrow = (\overline{\langle 23 \rangle} \circ \overline{\langle 12 \rangle}) \circ \overline{\langle 01 \rangle}$$

left

front

face

$$= \overline{\langle 23 \rangle} \circ (\overline{\langle 12 \rangle} \circ \overline{\langle 01 \rangle})$$

$$= \overline{\langle 23 \rangle} \circ \overline{\langle 02 \rangle}$$

$$\triangle \left\{ \begin{array}{l} \tau(\Delta^3_0) \neq \tau(\Delta^3) \\ \tau(\Delta^3_3) \neq \tau(\Delta^3) \end{array} \right.$$

$\Rightarrow N(c)$ does not have a lifting property for outer horns.

(4) \Rightarrow (1):

We follow the same strategy as for (3) \Rightarrow (1). We construct a category C in exactly the same way, except that associativity is proved using

$$\Delta^3_n \hookrightarrow \Delta^3 \text{ rather than } I^3 \hookrightarrow \Delta^3.$$

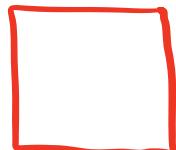
Then we have $X \xrightarrow{\psi} Nc$ defined as before with $\begin{cases} X_0 \cong (Nc)_0 \\ X_1 \cong (Nc)_1. \end{cases}$

We prove that ψ_n is a bijection

by induction on $n \geq 2$.

$$\begin{array}{ccc}
 X_n & \xrightarrow[\sim]{(\zeta)} & \text{sSet}(\Lambda_1^n, C) \\
 \downarrow & \equiv & \downarrow \text{○☆} \\
 (N C)_n & \xrightarrow[\sim]{(1) \Rightarrow (\zeta)} & \text{sSet}(\Lambda_1^n, N C)
 \end{array}$$

But $\Lambda_1^n = \text{Sh}_{n-1}(\Lambda_1^n)$ is
 colimit of standard simplices
 of dimension $< n$, so ○☆
 is a bijection by induction.



Rmks

- We did not use the full strength of (\hookrightarrow) , only $\overset{\text{h}}{\wedge} \hookrightarrow \overset{\text{h}}{\Delta}$.
For later purposes it is (\hookrightarrow) which is relevant.
- Moreover, we saw in the course of the proof that NC has the lifting property for $\overset{\text{h}}{\wedge}_k \hookrightarrow \overset{\text{h}}{\Delta}$, $0 \leq k \leq n$, as long as $n \geq h$.

$$\begin{array}{ccc}
 \text{Prop} \hookrightarrow & \text{NC} & \text{Kan complex} \\
 & \uparrow \downarrow & \\
 & \text{C groupoid} &
 \end{array}$$

proof: \Downarrow : Let $g: X \rightarrow Y$ be a morphism in C . We look at

$$\begin{array}{ccc}
 \Delta^2_0 & \longrightarrow & \text{NC} \quad g \nearrow Y \\
 \text{given by:} & & X = X
 \end{array}$$

By the Kan lifting property, there exists $\sigma \in (\text{NC})_2$

$$\begin{array}{ccc}
 & g \nearrow Y & \\
 \text{with} & \sigma \nearrow \downarrow g = \delta^\circ(\sigma) & \\
 & \sigma \searrow & \\
 & X = X &
 \end{array}$$

but then $\alpha \circ \varphi = \text{id}$.

--- o o --- x

Similarly, using $\Delta_2^2 \hookrightarrow \Delta^2$,

$\exists g': Y \rightarrow X$, $g \circ g' = \text{id}_Y$.

But then $g = g'$ and g is
an isomorphism.

$$(g = g \circ (g \circ g') = (g \circ g) \circ g' = g')$$

↑↑: By Thm 34 and the
remark above,
it is enough to prove the
lifting property for Δ_0^2 , Δ_2^2 ,
 Δ_0^3 (and Δ_3^3).

$$\underline{\Delta^2_0} : \begin{array}{ccc} & \bullet & \\ g \nearrow & & \rightsquigarrow \\ \bullet & \xrightarrow{g} & \bullet \\ & g & \end{array} \rightsquigarrow \begin{array}{ccc} & \bullet & \\ \overline{g} \nearrow & \downarrow g \circ g^{-1} & \\ \bullet & \xrightarrow{g} & \bullet \\ & g & \end{array}$$

$$\underline{\Delta^3_0} : \begin{array}{ccc} & \bullet & ? \\ g \nearrow & & \downarrow \\ \bullet & \xrightarrow{g} & \bullet \\ & g & \end{array} \quad \text{has an}$$

extension to $\Delta^3 \rightarrow NC$

$$\underline{\text{iff}} \quad ? = R g.$$

By considering the other faces, we see that

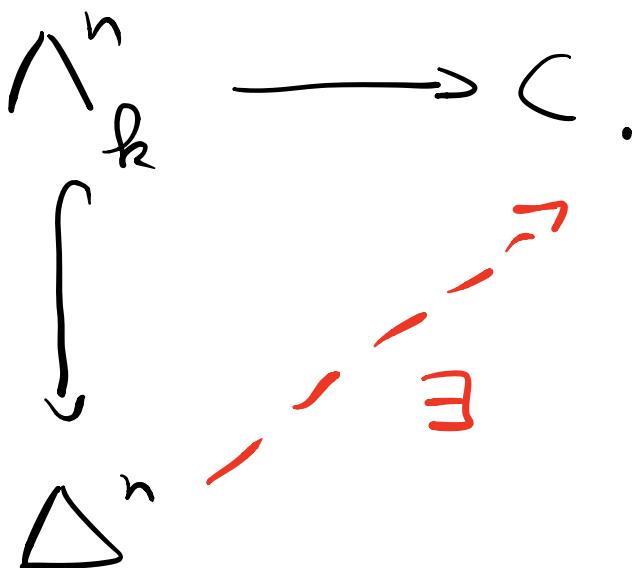
$$? \circ g = R g g$$

which is enough since g is iso. \square

5) ∞ - categories

def 5 An ∞ -category (or quasicategory) is a simplicial set $C_\cdot \in s\text{Set}$ satisfying the inner Horn extension property:

$\forall n \geq 2, \forall 0 < k < n,$



- A **functor** $F: \mathcal{C}_\cdot \rightarrow \mathcal{D}_\cdot$ between ∞ -categories is simply a morphism of simplicial sets. This defines a (1-)category Cat_{∞}^1 of ∞ -categories: $\text{Cat}_{\infty}^1 \hookrightarrow \text{sSet}$.
- A **natural transformation** $\mathcal{C}_\cdot \xrightarrow{\alpha \amalg} \mathcal{D}_\cdot$ is a morphism $\alpha: \mathcal{C}_\cdot \times \Delta^1 \rightarrow \mathcal{D}_\cdot$ with $\alpha|_{\mathcal{C}_\cdot \times \{0\}} = F$ and $\alpha|_{\mathcal{C}_\cdot \times \{1\}} = G$. □

Basic examples:

- By $\begin{cases} \text{the def. of Kan complexes} \\ \text{Prop 33 and Thm 34} \end{cases}$, we have fully faithful functors:

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{N} & \mathbf{Cat}_{\infty}^1 \hookrightarrow \mathbf{sSet} \\ \text{Kan} \curvearrowleft & & \end{array}$$

- We will see later that there are many examples which are not of these forms.

History

- This definition is due to Boardman-Vogt

(1973) in the context of homotopy theory (Homotopy coherent algebraic structures, infinite loop spaces). They proved some basic results which we will review next.

- The idea of taking quasicategories as a model for $(\infty, 1)$ -categories is due to Joyal (late 90's) and he developed most of the results from in the first

half of this course. Then Lurie came and pushed the theory even further!

Terminology For $X \in \text{sSet}$ (and in particular for ∞ -categories), we call

- objects of X , the elements of X_0
- (1-)morphisms of X , $\underline{\quad}$ X_1

For $g \in X_1$, we say that the source (resp. the target) of f is $d_1(g)$ (resp. $d_0(g)$) and we write $g: d_1(g) \rightarrow d_0(g)$.

- For $x \in X_0$, we write $\text{id}_x = d_0(x)$ and call it the identity morphism of x .

Slogan: An ∞ -category is like the nerve of a category, except that composition of chains of composable morphisms is only well-defined up to homotopy, and these homotopies are compatible in a precise way.

In particular, it should be possible to get a 1-category by identifying all those homotopies. This amounts to giving a simple description of the fundamental category πX , when X is an ∞ -category. This was achieved by Boardman-Vogt.

The first tool we want to have in any “category theory” is duality.

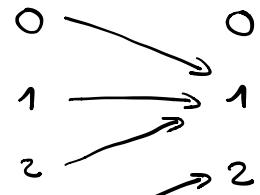
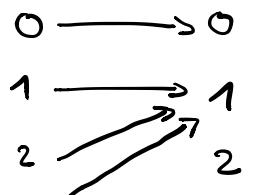
Def 6 The order-reversing functor

$\rho : \Delta \rightarrow \Delta$ is defined as

the identity on objects and, for $g : [m] \rightarrow [n]$,

$$\rho(g)(i) := n - g(m - i).$$

Ex:



$$\rho(g)$$

$\rho^* : \text{sSet} \rightarrow \text{sSet}$ is the functor “precomposition by ρ ”. For $X \in \text{sSet}$, the opposite simplicial set is $X^\text{op} := \rho^*(X)$. □

Examples • $(\Delta^n)^\text{op} \simeq \Delta^n$, $(I^n)^\text{op} \simeq I^n$, $(\partial \Delta^n)^\text{op} \simeq \partial \Delta^n$.

$$\cdot (\Lambda_i^n)^\text{op} \simeq \Lambda_{n-i}^n$$

• X -infinity-category (resp Kan)

$\Leftrightarrow X^\text{op}$ -infinity-category (resp Kan).

- $N(C^{\text{op}}) \simeq N(C)$ for $C \in \text{Cat}$.

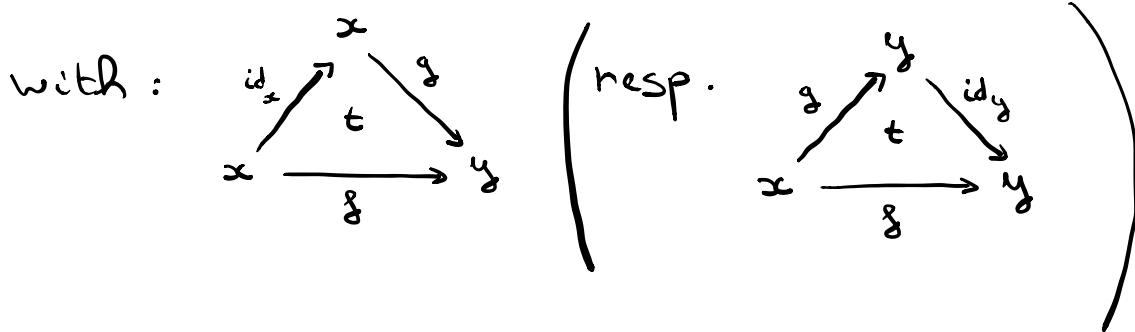
We now turn to the Boardman-Vogt result.

Def 7: Let $X_+ \in \text{sSet}$, $x, y \in X_+$.

We say that $f, g: x \rightarrow y$ are **left**

homotopic (resp. **right homotopic**), written

$f \sim_l g$ (resp. $f \sim_r g$) if $\exists t \in X_+$



Left and right homotopy are not necc.

equivalence relations; however we have

Lemma 8 Let C be an ∞ -category. Then

left and right homotopy coincide and
is an equivalence relation.

Proof: Let $f, g, h: x \rightarrow y$ in $C(x, y)$.

We prove:

a) $\tilde{g} \underset{\ell}{\sim} g$.

b) $\tilde{g} \underset{\ell}{\sim} g$ and $g \underset{\ell}{\sim} h$ imply $\tilde{g} \underset{\ell}{\sim} h$.

c) $\tilde{g} \underset{\ell}{\sim} g$ implies $\tilde{g} \underset{n}{\sim} g$.

d) $\tilde{g} \underset{n}{\sim} g$ implies $g \underset{\ell}{\sim} \tilde{g}$.

a): $t := g_{001}$ works:

$$\begin{array}{ccc} & \xrightarrow{id_x} & x \\ & \searrow \delta_{001} & \downarrow g \\ x & \xrightarrow{g} & y \end{array} .$$

b), c), d): They are proven in the same way:

- construct from the given 2-simplices a map

$$\Lambda_i^3 \rightarrow C, \text{ with } i = \begin{cases} 1, & \text{for b) \& c)} \\ 2, & \text{for d)} \end{cases}$$

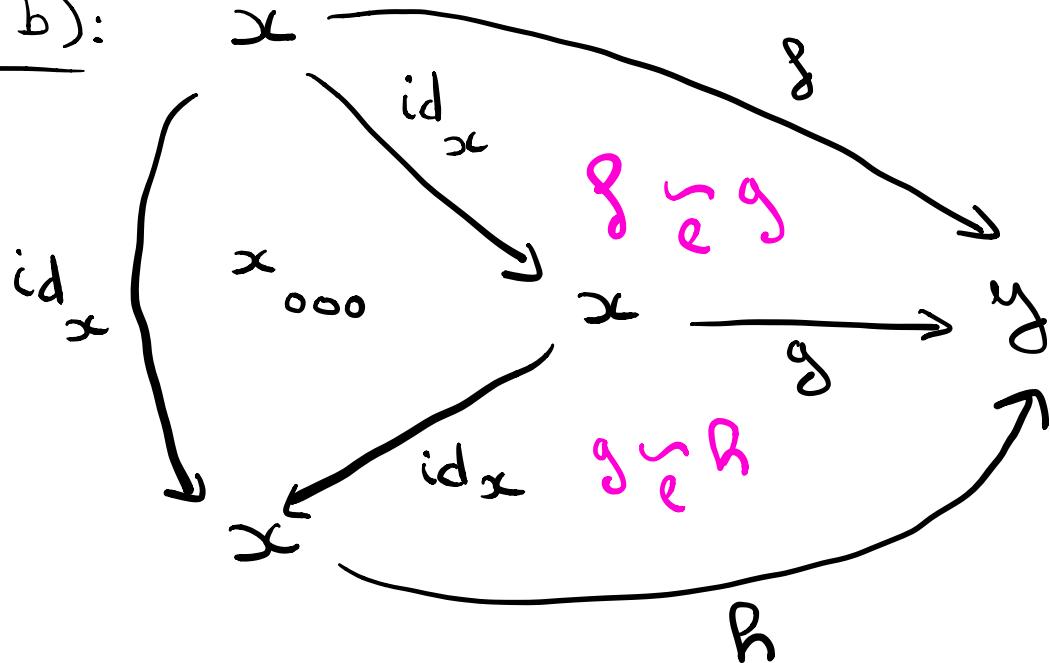
- appeal to the inner horn extension property
to get $\Delta^3 \rightarrow C$.

- restrict to the "new" 2-simplex.

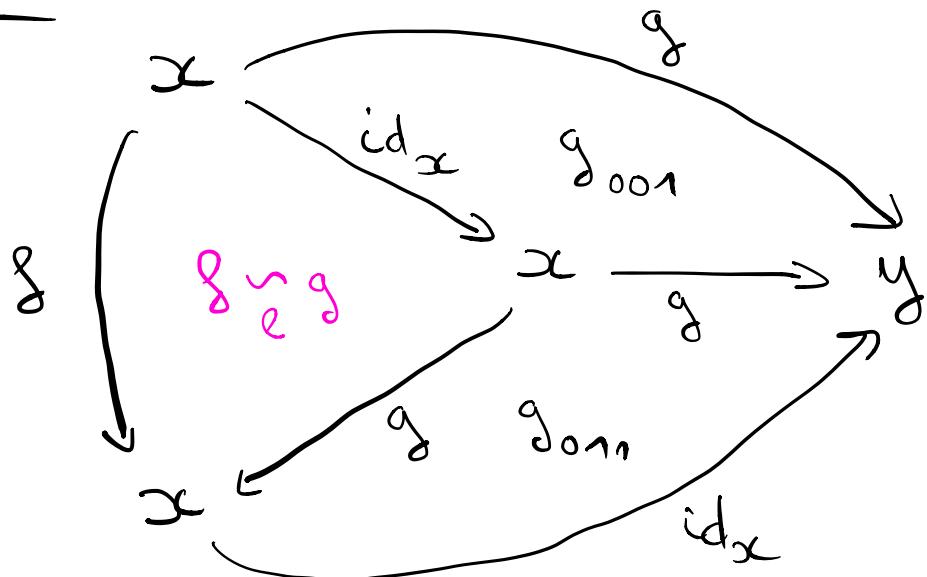
It is easier in this case to display
simplices so that the missing face is the back

of the picture:

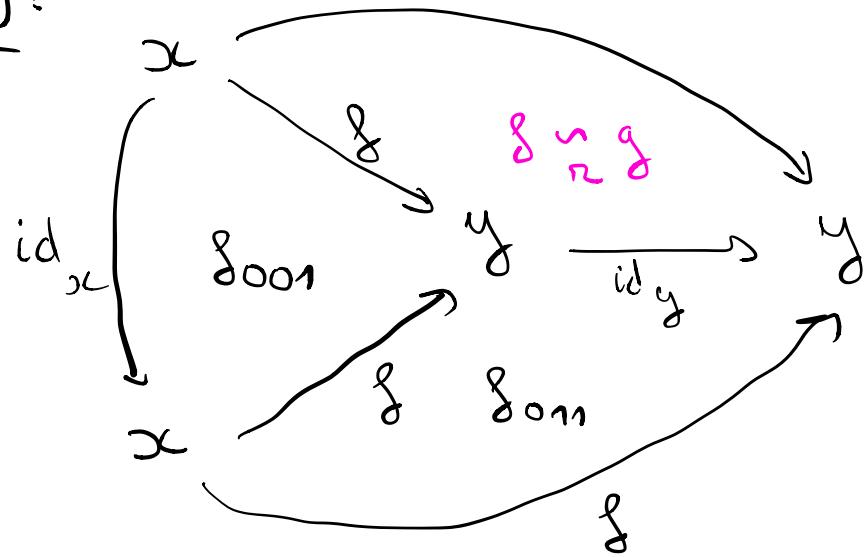
For b):



For c):



For d):



Finally:

- c), d) $\Rightarrow \sim_e$ is symmetric.

+ a), b) $\Rightarrow \sim_e$ equivalence relation

+ c), d) $\Rightarrow \sim_e = \sim_n$

□

In this case we write $\sim \approx$ for $\sim_e = \sim_n$.

and we write $[g]$ for the homotopy class of g in $C(x, y)$.

def g: Let $C \in \text{Cat}_\infty^1$, $g: x \rightarrow y, g: y \rightarrow z$

and $R: x \rightarrow z$. We say that R is a composition

of g and g if there is $t \in C_2$ with

$$\begin{array}{ccc} & y & \\ g \nearrow & t & \searrow g \\ x & \xrightarrow{R} & z \end{array}$$

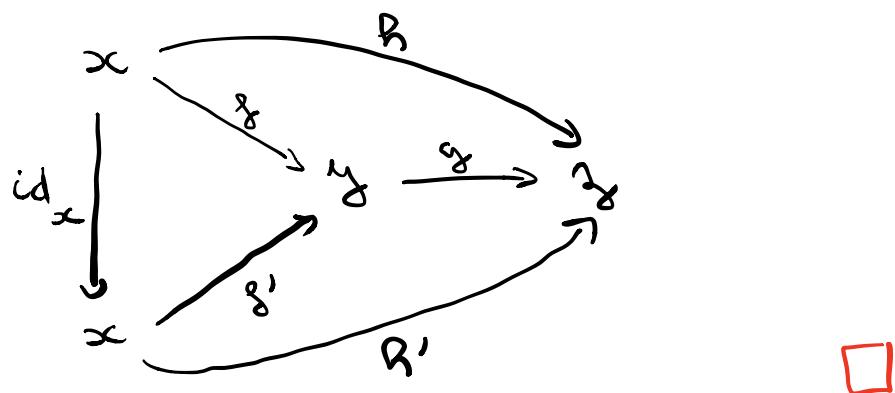
□

Prop 10: In an ∞ -category, compositions exist ; their homotopy class is well-defined and depends only on the homotopy classes of the morphisms being composed.

proof: • The existence is simply the extension property for $\Lambda_1^2 \hookrightarrow \Delta^2$.

- Let $\begin{cases} g \approx g': x \rightarrow y \\ g \approx g': x' \rightarrow y \end{cases}$ and let $\begin{cases} h \text{ be a composition of } g \text{ and } g' \\ h' \text{ _____ } g' \text{ and } g' \end{cases}$

We must prove that $R \approx R'$. It is enough to treat separately the cases $g = g'$ and $g = g'$. By working in C^{op} , we reduce to $g = g'$. As before we construct a horn Λ_2^3 , extends and restrict;



Prop 11: The resulting composition on homotopy classes of morphisms is associative and unital.

proof: same method as Prop 39: left as exercise □

def 12 Let C be an ∞ -category. Its **homotopy category** RC has:

$$\begin{cases} \text{Ob}(RC) = C_0 \\ RC(x, y) = C(x, y)/\sim \end{cases}$$

By Prop 39-40, this is indeed a 1-category.

Prop 13: The construction of the homotopy category

defines a functor $R: \text{Cat}_{\infty}^1 \rightarrow \text{Cat}$ which

is a left adjoint to $N: \text{Cat} \rightarrow \text{Cat}_{\infty}^1$.

($\Rightarrow \tau C \simeq RC$ naturally in C)

proof: The functoriality follows from the

fact that for $F: C \rightarrow D$, $g \simeq g'$ in $C \Rightarrow F(g) \simeq F(g')$

which is clear.

Let C be an ∞ -category. We construct

a natural equivalence (in fact isomorphism) of

categories $\tau C \xrightarrow{\cong} RC$.

Because of Prop 3, to define φ it is

enough to define $\begin{cases} C_0 \rightarrow \text{Ob } \mathcal{R}\mathcal{C} \\ C_1 \rightarrow \text{Mor } \mathcal{R}\mathcal{C} \end{cases}$

Satisfying some relations given by identities and C_2 .

We put $C_0 = \text{Ob } \mathcal{R}\mathcal{C}$

$C_1 \rightarrow \text{Mor } \mathcal{R}\mathcal{C}$, $g \mapsto [g]$

and the relations are satisfied by def 12.

• By construction, q is $\begin{cases} \text{bijective on objects} \\ \text{surjective on morphisms.} \end{cases}$

It remains to show q is faithful.

Because of the lifting property for Δ^2 in Δ^2 ,

any morphism in $\tau\mathcal{C}$ can be written as \bar{g}

for $g \in C_1$. Now suppose that $g, g' : x \rightarrow y$

satisfy $[g] = [g']$. By definition, there is

an homotopy $\begin{array}{ccc} & \nearrow^{g'} & \\ g & \searrow & \\ x & \xrightarrow{g} & y \end{array}$, but this also

implies $\text{id}_x \circ \bar{g}' = \bar{g}$ in $\tau\mathcal{C}$ and we are done. \square

Let us see some basic ways to construct new ∞ -categories.

Prop 15: 1) Arbitrary products and coproducts

of ∞ -categories (in $sSet$) are ∞ -categories.

2) Filtered colimits of ∞ -categories are ∞ -categories.

proof: 1)

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of ∞ -categories.

• Let $0 \leq h \leq n$. We have

$$sSet(\Delta^n, \prod_\alpha X_\alpha) \longrightarrow sSet(\Lambda_{\alpha}^n, \prod_\alpha X_\alpha)$$

IS

IS

$$\prod_\alpha sSet(\Delta^n, X_\alpha) \xrightarrow{\text{product of surjections}} \prod_\alpha sSet(\Lambda_{\alpha}^n, X_\alpha)$$

↑
 product of
 surjections is
 a surjection

$\Rightarrow \prod_\alpha X_\alpha$ is an ∞ -category.

- For the coproduct, we need to compute
 $sSet(\Lambda_R^n, \coprod X_\alpha)$ and $sSet(\Delta^n, \coprod X_\alpha)$

By Yoneda, $sSet(\Delta^n, \coprod X_\alpha) = (\coprod_\alpha X_\alpha)^{[n]}$

colimits
 are objectwise
 $= \coprod_\alpha X_\alpha^{[n]}$

$$= \coprod_\alpha sSet(\Delta^n, X_\alpha)$$

So it suffices to show that the natural map

$$\coprod_\alpha sSet(\Lambda_R^n, X_\alpha) \longrightarrow sSet(\Lambda_R^n, \coprod_\alpha X_\alpha)$$

is a bijection. For this one can use

$$\Lambda_R^n = \coprod_{\substack{\Delta^{[n]-\{i,j\}} \\ i \neq R}} \Delta^{[n]-i} \quad \text{and the Yoneda}$$

trick above; the key point is that the various $\Delta^{[n]-i}$ must be sent to the same X_α because they are connected via

the $(n-2)$ -faces $\Delta^{\{n\} - \{i,j\}}$. Or in other words,

one can show that

$\pi_0(\coprod X_\alpha) \simeq \coprod_\alpha X_\alpha$ while $\pi_0(\Lambda_R^n)$ has one element (Λ_R^n is connected).

2) Let J be a filtered category and

$X: J \rightarrow \text{sSet}$ be a diagram so that

each $X(\alpha)$ is an ∞ -category. Once again we

have $(\operatorname{colim}_J X)_n = \operatorname{colim}_J X(n)$ because

colimits are computed objectwise.

We want to show that the canonical map

$$\operatorname{colim}_{\alpha \in J} \text{sSet}(\Lambda_R^n, X(\alpha)) \rightarrow \text{sSet}(\Lambda_R^n, \operatorname{colim}_J X)$$

is a bijection. We will show this holds for

Λ_R^n replaced by any $Y \in \text{sSet}$ with finitely many non-degenerate simplices.

Let $\mathcal{C} = \left\{ Y \in \text{sSet} \mid \begin{array}{l} \text{sSet}(Y, -) \text{ commutes} \\ \text{with filtered colimits} \end{array} \right\}$.

As remarked above, $\Delta^n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Let's show that \mathcal{C} is closed under finite colimits. Let K be a finite category and $Y : K \rightarrow \mathcal{C}$ be a diagram.

We have

$$\begin{aligned} \operatorname{Colim}_{\alpha \in J} \operatorname{sSet}\left(\operatorname{Colim}_K Y, X(\alpha)\right) &\longrightarrow \operatorname{sSet}\left(\operatorname{Colim}_K Y, \operatorname{Colim}_J X\right) \\ \text{IS } \operatorname{Colim} \text{ prop.} & \quad \text{IS } \operatorname{Colim} \text{ prop.} \\ \operatorname{Colim}_{\alpha \in J} \lim_{\beta \in K} \operatorname{sSet}(Y(\beta), X(\alpha)) & \quad \lim_{\beta \in K} \operatorname{sSet}(Y(\beta), \operatorname{Colim}_J X) \\ \text{Filtered colimits} \swarrow \text{commutes} \quad \text{with finite limits} \quad \text{in Set.} & \quad \text{IS } X(\alpha) \in \mathcal{C} \\ \lim_{\beta \in K} \operatorname{Colim}_{\alpha \in J} \operatorname{sSet}(Y(\beta), X(\alpha)) \end{aligned}$$

$\Rightarrow \operatorname{Colim}_K Y \in \mathcal{C}$.

- By the skeletal filtration,

$$\left\{ \text{s.sets with } < \infty \text{ non-deg. simplices} \right\} = \left\{ \begin{array}{l} \text{finite colimits of} \\ \text{standard simplices} \end{array} \right\}$$

and we are done.

