

Prop 15 Let $(C_\alpha)_{\alpha \in J}$ be a family of ∞ -categories

Then the canonical map

$$R\left(\prod_{\alpha \in J} C_\alpha\right) \longrightarrow \prod_{\alpha \in J} R(C_\alpha)$$

is an isomorphism in Cat .

(" R commutes with products")

Proof: • First, we know from Prop 7 that

$\prod C_\alpha$ is an ∞ -category so this makes sense.

• Recall that in Cat , products are formed by taking products of all objects and all morphisms.

• The map of the statement is thus a bijection on objects and surjective on morphisms, so it remains to

see that it is injective on morphisms. If

$$\left([f_\alpha]\right)_{\alpha \in J} = \left([g_\alpha]\right)_{\alpha \in J} \text{ in } \prod_{\alpha} R(C_\alpha),$$

we get a collection of $t_\alpha \in (C_\alpha)_2$ giving homotopies, but then $(t_\alpha) \in (\prod C_\alpha)_2$ implies that

$$[(f_\alpha)] = [(g_\alpha)] \text{ in } R\left(\prod_{\alpha} C_\alpha\right). \quad \square$$

Cor 16: Let $F: C \rightarrow C'$ be a

an equivalence of ∞ -categories. Then

$R F: RC \rightarrow RC'$ is an equivalence of categories.

proof: Step 1) We show that any natural transformation $\alpha: F \Rightarrow G$ of functors $F, G: C \rightarrow D$ induces a natural transformation $R\alpha: RF \Rightarrow RG$.

- We have

$$R(C \times \Delta^1) \cong RC \times [1]$$

Prop 15

α is by definition a morphism

$$C \times \Delta^1 \rightarrow D$$

$$\Rightarrow R\alpha: RC \times [1] \cong R(C \times \Delta^1) \rightarrow RD$$

is a natural transformation $RF \Rightarrow RG$.

- Step 2) Assume now that α is a natural isomorphism, that is, we have $B: G \Rightarrow f$

and

$$t, t' : C \times \Delta^2 \rightarrow D \quad \text{with}$$

$$\begin{array}{ccc} & \alpha \nearrow G & \\ F & \xrightarrow{id_F} & F \\ & \beta \searrow F & \end{array} \quad \text{and} \quad \begin{array}{ccc} & F \nearrow \alpha & \\ G & \xrightarrow{id_G} & G \\ & \beta \searrow G & \end{array}$$

$$\text{Then by using } R(C \times \Delta^2) \cong RC \times [2]$$

$$\left\{ \begin{array}{l} R\beta \circ R\alpha = R\text{id}_F = \text{id}_{RF} \\ R\alpha \circ R\beta = R\text{id}_G = \text{id}_{RG} \end{array} \right.$$

Prop 15

we get that $R\alpha$ is also a natural isomorphism.

Step 3) Assume F is an equivalence of ∞ -categories with "inverse" $G : D \rightarrow C$. Then

$$\text{we have natural isomorphisms } \left\{ \begin{array}{l} F \circ G \cong \text{id}_D \\ G \circ F \cong \text{id}_C \end{array} \right.$$

$$\text{and by Step 2) : } \left\{ \begin{array}{l} RF \circ RG = R(F \circ G) \cong \text{id}_{RD} \\ RG \circ RF = R(G \circ F) \cong \text{id}_{RC} \end{array} \right.$$

$R\alpha$ $R\beta$

So RF is an equivalence of 1-categories.



⚠ The converse is not at all true!

Ex: K Kan complex.

$\begin{cases} K \rightarrow \Delta^0 \text{ is an eq. of } \infty\text{-categories} \Rightarrow K \text{ contractible} \\ RK \simeq \pi_1 |K|; RK \rightarrow [0] \text{ is an eq} \Rightarrow |K| \text{ is connected} \text{ and simply-connected.} \end{cases}$

def 17 Let C be an ∞ -category.

A morphism $g \in C_1$ is an **isomorphism**

if $[g] \in \text{Mor}(RC)$ is an isomorphism,

i.e. if $\exists g \in C_1$ with $[g] \circ [g]$ and

$[g] \circ [g]$ identities in RC , i.e. $\exists t, t' \in C_2$

$$\begin{array}{ccc} \overset{g}{\nearrow} \overset{\bullet}{t} \overset{g}{\searrow} & \text{and} & \overset{g}{\nearrow} \overset{\bullet}{t'} \overset{g}{\searrow} \\ \bullet = \bullet & & \bullet = \bullet \end{array}$$

Ex: • $\forall x \in C_0$, $\text{id}_x = s_0(x)$ is an iso.

• If C is a 1-category, then isomorphisms

in NC are exactly theisos. in C .

$$(RNC \xrightarrow{\sim} C)$$

def 18: An ∞ -groupoid is an ∞ -category
in which every morphism is an iso.

lemma 19: Kan complexes are ∞ -groupoids.

Proof: same argument as part II of Prop 4.
($\subset 1\text{-cat}, \subset \text{groupoid} \Leftrightarrow NC \text{ Kan}$) \square

The converse is true but much more difficult
we will see (at least parts of) the proof later.

thm 20: ∞ -groupoids are Kan complexes.

Rmk: • Try to prove this by hand!

Even proving the lifting property for
 $\Delta_0^3 \hookrightarrow \Delta^3$ is challenging.

(Kan $\stackrel{\text{def}}{\Leftrightarrow}$ lifting for all $\Delta_h^n \hookrightarrow \Delta^n$, $0 \leq h \leq n$
 ∞ -groupoid " \Leftrightarrow " $\begin{array}{c} 0 < h < n \\ + \end{array}$ $\Delta_0^2 \hookrightarrow \Delta^2, \Delta_2^2 \hookrightarrow \Delta^2$)

Rmk: Thm 20 implies
that the (not yet defined) ∞ -category
of ∞ -groupoids is equivalent to the
 ∞ -category of Kan complexes,
which by simplicial Homotopy theory
is equivalent to the ∞ -cat. of topological
spaces. \Rightarrow Grothendieck's Homotopy
Hypothesis holds in the quasicategory
model of $(\infty, 1)$ -categories.

- Another important theorem, which
will be our main goal in the next section
of the course, is

thm 21: Let $K \in \text{sSet}$, $C \in \text{Cat}_{\infty}^1$.

Then the simplicial set $\text{Fun}(K, C)$,

with $\text{Fun}(K, C)_n = \text{sSet}(K \times \Delta^n, C)$,

is an ∞ -category.

□

- $C, D \in \infty$ -categories. Then

$$\text{Fun}(C, D)_0 = \{ \text{functors } C \rightarrow D \}$$

$$\text{Fun}(C, D)_1 = \{ \text{natural transformations } C \times \Delta^1 \rightarrow D \}$$

- In fact, the definition of natural isomorphisms and categorical equivalences can be reformulated as:

$$\bullet \left(\begin{matrix} \alpha : F \Rightarrow G \\ (\text{natural iso}) \end{matrix} \right) \Leftrightarrow \begin{matrix} \alpha : F \rightarrow G \text{ isomorphism} \\ \text{in } \text{Fun}(C, D) \end{matrix}$$

$$\bullet \left(\begin{matrix} C \xrightarrow{F} D \\ (\text{categorical equivalence}) \end{matrix} \right) \Leftrightarrow \left(\begin{matrix} \exists G : D \rightarrow C, \\ F \circ G \text{ iso. to } id_D \text{ in } \text{Fun}(D, D) \\ G \circ F \xrightarrow{\quad} id_C \text{ in } \text{Fun}(C, C) \end{matrix} \right)$$

. Another application of the Homotopy category
is to the notion of subcategory:

def 22: A **subcategory** C' of an ∞ -category

C is a simplicial subset which satisfies

moreover: $\forall n \geq 2, \forall \gamma \in C_n,$

$\gamma \in C'_n \iff \gamma|_{I^n}$ has edges in C'_1 .

Hence C' is determined by C'_0 and C'_1 .

C' is a **full subcategory** if

$\forall n \geq 1, \forall \gamma \in C_n,$

$\gamma \in C'_n \iff$ the vertices of γ lie in C'_0 .

C' is then determined entirely by C'_0 .

Lemma 23: A subcategory C' of an ∞ -category C is

an ∞ -category.

proof: This follows immediately from $I^n \subseteq \Delta^n$.

$$I^n \hookrightarrow \Delta^n \rightarrow C' \hookrightarrow C$$

□

prop 25: Let C be an ∞ -category and

$\eta: C \rightarrow NHC$ be the unit map.

Then a) If D is a subcategory of the
1-category HC , then the pull back

$\eta^*(ND) := ND \times_{NHC} C$ is a subcategory of C .

$$\begin{array}{ccc} \eta^*(ND) & \hookrightarrow & C \\ \downarrow & \lrcorner & \downarrow \eta \\ ND & \hookrightarrow & NHC \end{array}$$

b) This construction gives a bijection

$$\{ \text{Subcategories of } HC \} \cong \{ \text{Subcategories of } C \}$$

which restricts to a bijection of full
subcategories.

proof: We only do the case of subcategories,
the full case is easier.

a)

Claim | If C is a γ -category and C' is a (full) subcategory
of C , then NC' is a (full) subcategory of NC .

Pf: obvious from the unique lifting property of
nerves along $I^n \hookrightarrow \Delta^n$.

Claim: the pullback of a subcategory is a subcategory.

Pf: Let $C' \subseteq C$ be a subcategory of an ∞ -cat.

and $E \xrightarrow{g} C$ be any morphism. Then $E' := E \times_C^{C'}$
is a simplicial subset of E (pullback of mono is
mono in any category)

By definition, we have for $n \in \mathbb{N}$: $E'_n = E_n \times_{C_n}^{C'_n}$,
so for $z \in E_n$, we have:

$$z \in E'_n \stackrel{\text{pullback}}{\iff} g(z) \in C'_n$$

$$\stackrel{\text{subcategory}}{\iff} g(z)|_{I^n} \text{ has edges in } C'_n$$

$$\iff g(z|_{I^n}) \subset C'_n$$

$\xrightarrow{\text{pullback}}$ $\mathcal{D} \wr \mathcal{I}^h \longrightarrow E'$

so that E' is a subcategory.

Together the two claims imply a).

b) Injectivity:

The morphism $C \longrightarrow NRC$ induces

- a bijection $C_0 \xrightarrow{\sim} (NRC)_0$
- a surjection $C_1 \twoheadrightarrow (NRC)_1$.

So if D is a subcategory of RC , then

$$\gamma^{-1} h D \text{ determines } \begin{cases} \text{Ob}(D) = \gamma((\gamma^{-1} N D)_0) \\ \text{Mor}(D) = \gamma((\gamma^{-1} N D)_1) \end{cases}$$

Hence determines D .

This proves the injectivity.

Surjectivity: Let $C' \subseteq C$ be a subcategory.

The natural candidate for D is hC' , so let's

try that! We first check that $hC' - hC$

is a subcategory. We have $\text{Ob}(hC') = C'_0 \cap C_0 = \text{Ob}(hC)$.

$$\text{Q: } \text{Mor}(hC') = C'_1 /_{\text{homotopy}} \xrightarrow{?} C_1 /_{\text{homotopy}} = \text{Mor}(hC)$$

If $g, g' \in C'_1$ are homotopic in C ,

it means there exists

$$g \xrightarrow{\quad t \quad} \begin{matrix} y \\ \parallel \\ x \end{matrix} \xrightarrow{\quad g' \quad} y \in C_2$$

but $t \in C'_2 \subseteq C_2$ because g, id_y lie in C'_1 .
and C' is a subcategory.

$\Rightarrow hC'$ is a subcategory of hC .

• It remains to show that

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \gamma \downarrow & & \downarrow \gamma \\ N hC' & \longrightarrow & N hC \end{array}$$

is a pullback square. On n -simplices,
this says that

$\beta \in C'_n \stackrel{?}{\iff} \text{"the edges in } \beta|_{I^n} \text{ are homotopic in } C \text{ to morphisms in } C'.$

But as we saw, this last condition is \iff "the edges of $\beta|_{I^n}$ are morphisms of C'_1 "

which by definition of a subcategory
is precisely $\iff \beta \in C'_n$. □

Ex: If $C \in \text{Cat}$, then

$$\left\{ (\text{full}) \text{ subcategories of } C \right\} \hookrightarrow \left\{ (\text{full}) \text{ subcategories} \right. \\ \left. \text{of } NC \right\}$$

$$D \longrightarrow ND$$

Rmk: The notion of subcategory in Cat is
not invariant under equivalences.

Replete subcategories in $C = C'$ subcategory such
that $x \in C, x \cong y \in C'$
 $\Rightarrow x \in C'$.

• To finish this introductory section, we need to see at least one example which does not come from nerves or Kan complexes.

The idea is that, in the same way that the nerve N gives

$$N : \{1\text{-categories}\} \subset \{\infty\text{-categories}\}$$

there should exist a “fully faithful” functor:

$$N_2 : \{(2,1)\text{-categories}\} \subset \{\infty\text{-categories}\}$$

This does exist! However:

Pb: The definition of general (weak) $(2,1)$ -categories is complicated!

Even strict $(2,1)$ -categories require

ideas that I don't want to discuss yet).

Sol: We know one example of
a strict $(2,1)$ -category Cat^2 :

$\left\{ \begin{array}{l} \text{Ob } \text{Cat}^2 : \text{ small categories} \\ \text{Mor } \text{Cat}^2 : \text{ functors} \\ 2\text{-Mor } \text{Cat}^2 : \text{ natural isomorphisms.} \end{array} \right.$

So the plan is to construct an ∞ -category $N_2(\text{Cat}^2)$ from this; the recipe then generalizes to any weak $(2,1)$ -category (N_2 is called the **Duskin nerve**).

Example: We define a simplicial set $N_2(\text{Cat}^2)$ follows: an n -dimensional element in $(N_2(\text{Cat}^2))_n$ is the datum of:

- $\forall 0 \leq i \leq n$, a small category C_i
- $\forall 0 \leq i \leq j \leq n$, a functor $F_{i,j}: C_i \rightarrow C_j$

• $\forall 0 \leq i \leq j \leq k \leq n$, a natural iso. $\alpha_{i,j,k} : F_{i,k} \Rightarrow F_{j,k} F_{i,j}$

such that : • $F_{i,i} = id_{C_i}$

• $\alpha_{i,i,j}$ and $\alpha_{i,i,j}$ are identities.

• $\forall 0 \leq i \leq j \leq k \leq l \leq n$,

the diagram $F_{i,l} \xrightarrow{\alpha_{i,j,l}} F_{j,l} F_{i,j}$ commutes.

$$\begin{array}{ccc} & \alpha_{i,k,l} \downarrow & \circledast \downarrow \\ F_{i,l} & \xrightarrow{\alpha_{i,j,l}} & F_{j,l} F_{i,j} \\ \downarrow \alpha_{i,k,l} & & \downarrow \alpha_{j,k,l} \\ F_{k,l} F_{i,k} & \xrightarrow{\alpha_{i,j,k}} & F_{k,l} F_{j,k} F_{i,j} \end{array}$$

• For $\delta : [m] \rightarrow [n]$, we define

$$\delta^*(C_i, F_{i,j}, \alpha_{i,j,k}) = (C_{\delta(i)}, F_{\delta(i), \delta(j)}, \alpha_{\delta(i), \delta(j), \delta(k)})$$

Rmk: $N\text{Cat}$ is isomorphic to the simplicial

subcomplex of $N_2(\text{Cat}^2)$ on the elements where

$$F_{i,k} = F_{j,k} F_{i,j} \quad \text{and} \quad \alpha_{i,j,k} = id.$$

So we have added some new higher morphisms

to $N\text{Cat}$, corresponding to non-id. natural isomorphisms.

prop 25: $N_2(\text{Cat}^2)$ is an ∞ -category.

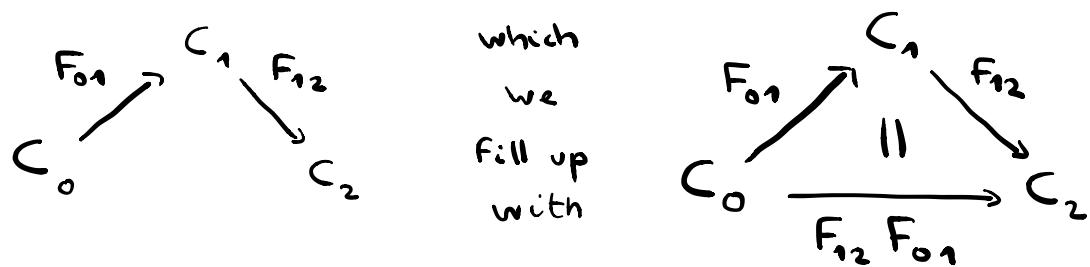
proof: The proof is similar to the proof that the nerve of a category is an ∞ -category; we just need to go "one dimension higher".

- By definition of $N_2(\text{Cat}^2)$, an n -simplex is determined by its restriction to $\text{Sh}_3(\Delta^n)$ (" Cat^2 is 3-coskeletal"). Since we have

$$\text{Sh}_3(\Lambda_n^k) = \text{Sh}_3(\Delta^n) \text{ for all } n \geq 4,$$

we only need to check the inner horn liftings for $n \leq 3$.

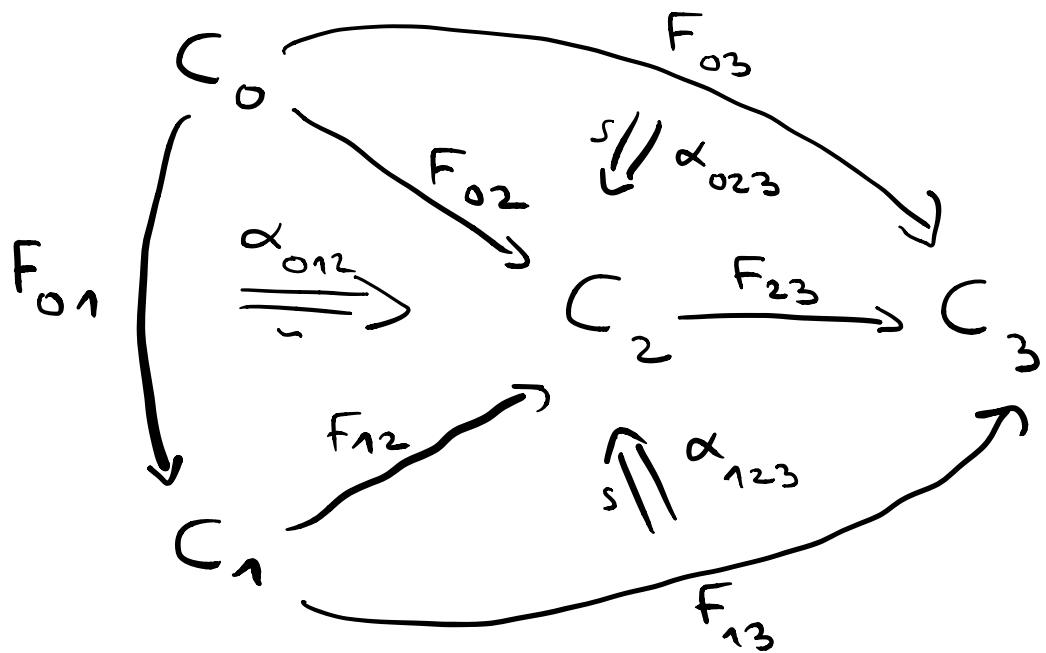
$n=2$ A map $\Lambda_1^2 \rightarrow N_2(\text{Cat}^2)$ is just a datum



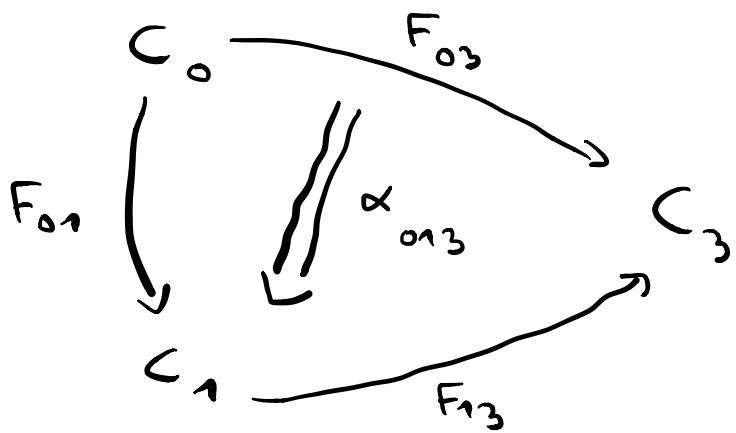
Note: this is not unique in general, we can take any natural iso $G \cong F_{12}F_{01}$.

$n=3$ A map $\Lambda_1^3 \rightarrow N_2(\text{Cat}^\ell)$ is

a diagram:



By the condition \circledast in the definition,
we must fill this in with



$$\text{with } \alpha_{013} = \alpha_{123}^{\circ -1} \circ \alpha_{012} \circ \alpha_{023}$$

and it works.



III Lifting calculus

- The definition of ∞ -categories is very combinatorial and the proofs so far have been by explicit manipulations of simplices.
- To go further and prove results like Thm 20-21, we need a more systematic way to keep the combinatorics manageable.

1) Fibrations and anodyne maps

Let us formalize a concept we have already seen a lot:

def 1: Let C be a category. A **lifting problem**

in C is a commutative square in C :

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ g \downarrow & \equiv & \downarrow p \\ B & \xrightarrow{v} & Y \end{array}$$

- A lift of the lifting problem is a diagonal map:

$$\begin{array}{ccc}
 A & \xrightarrow{u} & X \\
 g \downarrow & \nearrow \text{---} \quad \text{---} \nearrow & \downarrow p \\
 B & \xrightarrow{v} & Y
 \end{array}$$

If S, T are collections of morphisms in C , we say

that S has the left lifting property with respect to T
or equivalently that

T has the right lifting property with respect to S .

and write $S \boxtimes T$ if every lifting problem as

above with $g \in S$ and $p \in T$ has a solution.

In particular, if $S = \{g\}$ (resp. $T = \{p\}$) is a singleton,
we say that g has the left lifting property w.r.t to T
(resp. p has the right lifting property w.r.t. to S)

and write $g \boxtimes T$ (resp. $S \boxtimes p$).

- We define the right complement of S , resp.
the left complement of T , by

$$S^{\square} := \{ p \in \text{Mor}(C) \mid g \square p \text{ for all } g \in S \}$$

$$(\text{resp. } T^{\square} := \{ g \in \text{Mor}(C) \mid g \square p \text{ for all } p \in T \})$$

[alternative notations: RLP(S) / LLP(T)]

$S \perp T \ (\Rightarrow S \square T)$

Ex: in $s\text{Set}$, we have already several examples of this pattern:

- $\{\text{Kan fibrations}\} := \left(\Delta_{k_2}^n \hookrightarrow \Delta^n \mid 0 \leq k \leq n \right)^{\square}$
- $\{\text{trivial Kan fibrations}\} := \left(\partial \Delta^n \hookrightarrow \Delta^n \mid n \geq 1 \right)^{\square}$
- $C \in s\text{Set}$ is an ∞ -category iff the map $C \rightarrow \Delta^0$ lies in $(\Delta_{k_2}^n \hookrightarrow \Delta^n \mid 0 < k < n)^{\square}$. | 
- On the other hand, the Grothendieck-Segal condition for $X \simeq N(C)$ with $C \in \text{Cat}$ cannot be expressed this way because it requires imposing the uniqueness of solutions.

Ex: Some other examples you may be familiar with:

- $C = \text{Set}$: (injections) \square (surjections)

and in fact:

$$\begin{cases} (\text{inj.}) = \square (\text{surj.}) = \square (* \xrightarrow{\perp} *) \\ (\text{surj.}) = (\text{inj.})^\square = (\emptyset \rightarrow *) \square \end{cases}$$

- $C = \text{Ab}$ (or any abelian category):

- $I \in \text{Ab}$ is called injective if
the unique map $I \rightarrow 0$ lies in $(\text{mono.})^\square$.

- $P \in \text{Ab}$ is called projective if
the unique map $0 \rightarrow P$ lies in $\square(\text{epi})$.

- $C = \text{Top}$:

A morphism $p : X \rightarrow Y$ is called
a Serre fibration if $p \in (D^n \hookrightarrow D^n \times I)^\square$

a Hurewicz fibration if $p \in (X \hookrightarrow X \times I |_{X \in \text{Top}})^\square$. \square

Lemma 2: With the notations of def 1 :

$$a) R \subseteq S \Rightarrow \left\{ \begin{array}{l} S^\square \subseteq R^\square, \text{ and} \\ S^\square \subseteq R^\square. \end{array} \right.$$

$$b) S \subseteq {}^\square(S^\square) \text{ and}$$

$$S \subseteq ({}^\square S)^\square.$$

$$c) S^\square = ({}^\square(S^\square))^\square \text{ and}$$

$${}^\square S = {}^\square(({}^\square S)^\square).$$

proof: . a) is clear : we have more lifting

problems to solve with S than with R .

b) By duality, we only need to prove the first.

- $S \subseteq \square(S^\square)$: $\begin{array}{ccc} A & \longrightarrow & X \\ s \Rightarrow \downarrow & \nearrow & \downarrow \in S^\square \\ B & \longrightarrow & Y \end{array}$

c) Again by duality we prove only the first:

$$\left\{ \begin{array}{l} S^\square \stackrel{b)}{\subseteq} (\square(S^\square))^\square \\ (\square(S^\square))^\square \stackrel{a)+b)}{\subseteq} S^\square \end{array} \right. \quad \square$$

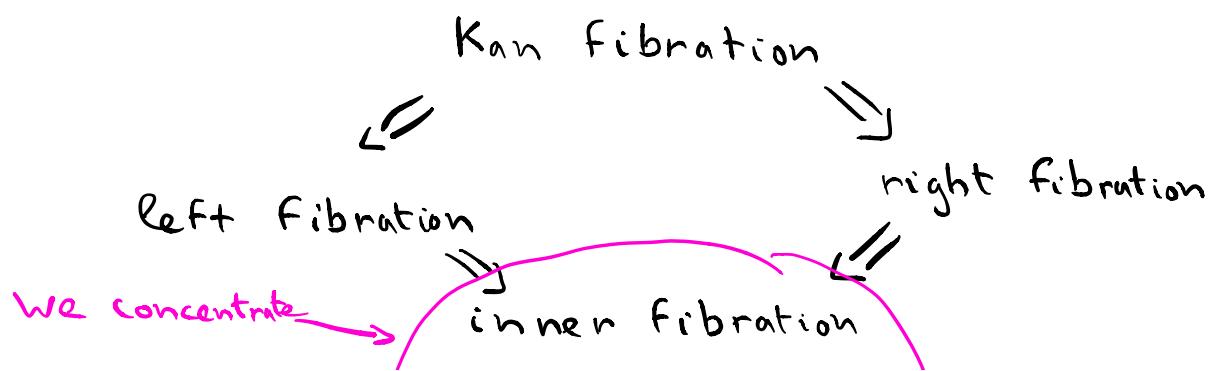
def 3: We define collections of morphisms in $sSet$:

- (inner fibrations) := $(\Lambda_k^n \hookrightarrow \Delta^n \mid 0 < k < n)^\square$
- (left fibrations) := $(\Lambda_k^n \hookrightarrow \Delta^n \mid 0 \leq k < n)^\square$
- (right fibrations) := $(\Lambda_k^n \hookrightarrow \Delta^n \mid 0 < k \leq n)^\square$

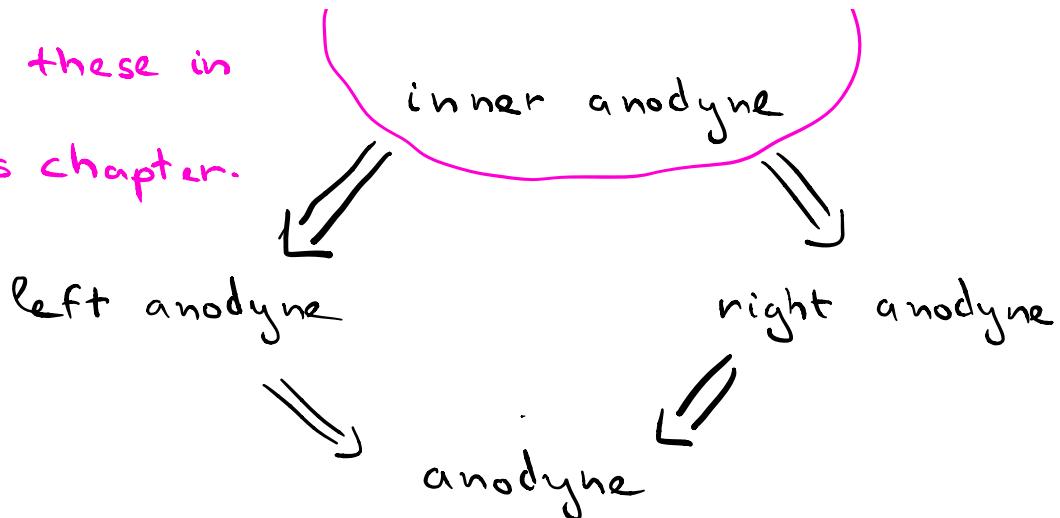
$$[\cdot \text{ (Kan fibrations)} = (\Delta_h^n \hookrightarrow \Delta^n \mid 0 \leq h \leq n)]^{\square}$$

- $\left(\begin{array}{l} \text{inner anodyne} \\ \text{morphisms} \end{array} \right) := \boxed{\square} \left(\text{inner fibrations} \right)$
 - $\left(\begin{array}{l} \text{left anodyne} \\ \text{morphisms} \end{array} \right) := \boxed{\square} \left(\text{left fibrations} \right)$
 - $\left(\begin{array}{l} \text{right anodyne} \\ \text{morphisms} \end{array} \right) := \boxed{\square} \left(\text{right fibrations} \right)$
 - $\left(\begin{array}{l} \text{anodyne} \\ \text{morphisms} \end{array} \right) := \boxed{\square} \left(\text{Kan fibrations} \right)$

Rmk: From the definition and Lemma 2:



on these in
this chapter.



- Moreover, it is easy to see that

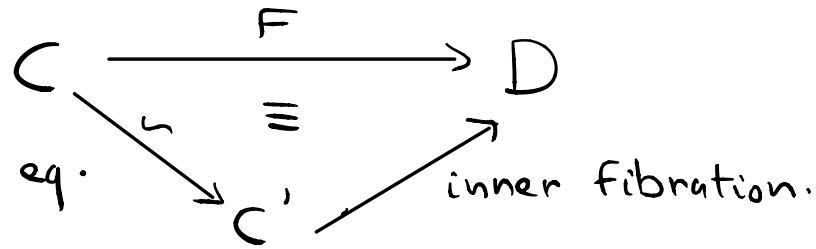
\mathcal{G} Kan fibration $\Leftrightarrow \mathcal{G}$ Left and right fibration.

$\left(\begin{array}{l} \Delta^! \\ \mathcal{G} \text{ inner anodyne } \Leftrightarrow \mathcal{G} \text{ left and right} \\ \text{anodyne.} \end{array} \right)$

- $X \in \mathbf{sSet}$,

$X \infty\text{-category} \Leftrightarrow X \rightarrow \Delta^\circ$ inner fibration.

- We will see later that any
Functor $C \xrightarrow{F} D$ of ∞ -categories is an
inner fibration "up to equivalence":



- “anodyne” is due to Gabriel and Zisman (1967) and means “without pain” in Greek. Central notion in simplicial homotopy theory, the left/right/inner versions are natural extensions in the context of quasicategories.

Lemma 4: Let $X \in \text{sSet}$ and $C \in \text{Cat}$.

Then $X \rightarrow NC$ is an inner fibration iff X is an ∞ -category.

In particular, if $F: C \rightarrow D$ is a functor of 1-categories, then $NF: NC \rightarrow ND$ is an inner fibration.

proof: Exercise. Hint: use the $\mathcal{I} \dashv \mathcal{N}$

adjunction and consider the maps

$$\mathcal{I}(\Lambda_k^n) \rightarrow \mathcal{I}(\Delta^n) \text{ for } 0 < k < n$$



Collections of morphisms of the form S^\square (or $S^{\square \square}$) have some remarkable "closure" properties.

def 5: Let $S \subseteq \text{Mor}(C)$. We say that:

* S is closed under pushouts if for every pushout diagram $A \xrightarrow{f} A'$, $g \in S \Rightarrow g' \in S$.

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ g \downarrow & & \downarrow \\ B & \xrightarrow{g'} & B' \end{array}$$