Model Reference Adaptive Control

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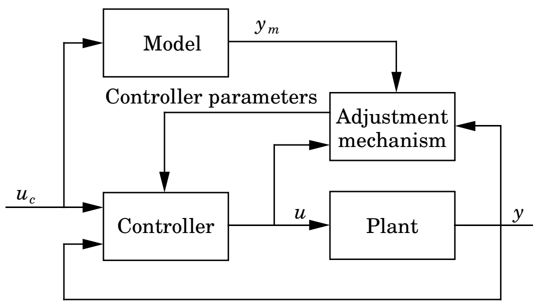
Model Reference Adaptive Control

Introduction

- Driven by flight control - a servo problem
- MRAS and the MIT rule Whitaker 1959
- Empirical evidence of instability - modified adaptation laws
- Lyapunov theory - Time domain
  - Butchart and Shackloth, Synthesis of model reference systems by Lyapunov's second method. 1965
- Passivity theory - Frequency domain
  - A .I. Lur'e problem - Linear system with one nonlinerity 1944
  - V. M. Popov, Absolute stability of nonlinear systems of automatic control, Automat. Remote Control, 22, 837-875, 1961
  - Landau 1969
  - Kalman Yakubuvich Popov Lemma
- The augmented error Monopoli 1974
- Stability of MRAS
  - Counterexamples Feuer and Morse 1978
  - Egardt 1979
  - Goodwin Ramage Caines 1980
  - Narendra 1980
  - Morse 1980

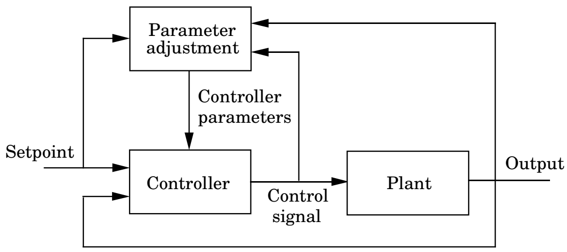
Direct Adaptive Control

Controller parameters are adjusted directly



Indirect Adaptive Control

Controller parameters are adjusted indirectly by first estimating parameters of a process model and then designing a controller



Model Reference Adaptive Control

Flight Control – Servo Problem

P. C. Gregory March 1959. Proceedings of the Self Adaptive Flight Control Systems Symposium. Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

Most of you know that with the advent a few years ago of hypersonic and supersonic aircraft, the Air Force was faced with a control problem. This problem was two-fold; one, it was taking a great deal of time to develop a flight control system; and two, the system in existence were not capable of fulfilling future Air Force requirements. These systems lacked the ability to control the aircraft satisfactorily under all operating conditions.

Test flights start summer 1961

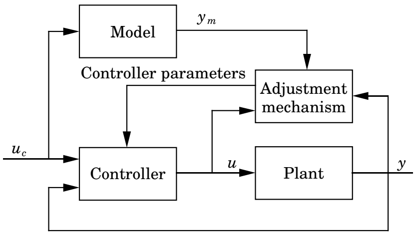
- Honeywell self oscillating adaptive system X-15
- MIT model reference adaptive system on F-101A

Mishkin, E. and Brown, L Adaptive Control Systems. Mc-Graw-Hill New York, 1961

Model Reference Adaptive Control – P. Whitaker MIT 1959

We have further suggested the name model-reference adaptive system for the type of system under consideration. A model-reference system is characterized by the fact that the dynamic specification for a desired system output are embodied in a unit which is called the model-reference for the system, and which forms part of the equipment installation. The commandsignal input to the control system is also fed to the model. The difference between the output signal of the model and the corresponding output quantity of the system is then the response error. The design objectie of the adaptive portion of this type of system is to minimze this response error under all operational conditions of the system. Specifically the adjustment is done by the MIT Rule.

Model Reference Adaptive Control – MRAS



- Servo problem
- Desired response y_m to command signal u_c is specified by the model
- How to find the parameter adjustment algorithm?

Gradient Algorithms

Tracking error

$$e = y - y_m$$

Introduce

$$J(\theta) = \frac{1}{2} e^2$$

Change parameters such that

$$\frac{d\theta}{dt} = -\gamma \frac{\partial J}{\partial \theta} = -\gamma e \frac{\partial e}{\partial \theta}$$

where $\partial e / \partial \theta$ is the sensitivity derivative

$$\frac{dJ}{dt} = e \frac{de}{dt} = e \frac{\partial e}{\partial \theta} \frac{d\theta}{dt} = -\gamma e^2 \left(\frac{\partial e}{\partial \theta} \right)^2$$

Many other alternatives

$$J(e) = |e|$$

gives

$$\frac{d\theta}{dt} = -\gamma \frac{\partial J}{\partial \theta} = -\gamma \frac{\partial e}{\partial \theta} \operatorname{sign}(e)$$

Feedforward Gain

Process

$$y = kG(s)$$

Desired response

$$y_m = k_0 G(s) u_c$$

Controller

$$u = \theta u_c$$

$$e = y - y_m = kG(p)\theta u_c - k_0 G(p) u_c$$

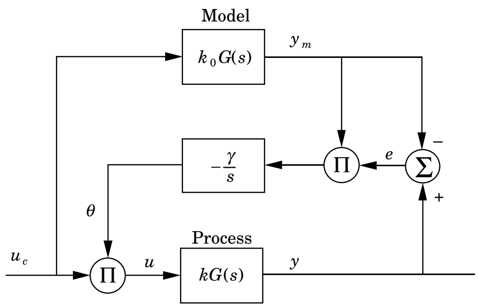
Sensitivity derivative

$$\frac{\partial e}{\partial \theta} = kG(p) u_c = \frac{k}{k_0} y_m$$

MIT rule

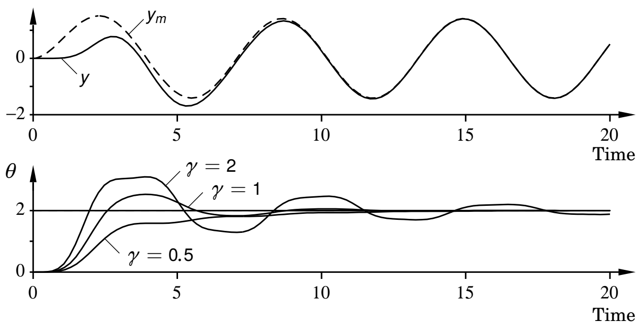
$$\frac{d\theta}{dt} = -\gamma' \frac{k}{k_0} y_m e = -\gamma y_m e$$

Block Diagram



$$\frac{d\theta}{dt} = -\gamma' \frac{k}{k_0} y_m e = -\gamma y_m e$$

Simulation $\gamma = 1$



A First Order System

Process

$$\frac{dy}{dt} = -ay + bu$$

Model

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Controller

$$u(t) = \theta_1 u_c(t) - \theta_2 y(t)$$

Ideal controller parameters

$$\theta_1 = \theta_1^0 = \frac{b_m}{b}$$

$$\theta_2 = \theta_2^0 = \frac{a_m - a}{b}$$

Find a feedback that changes the controller parameters so that the closed loop response is equal to the desired model

MIT Rule - First Order System

The error

$$e = y - y_m$$

$$y = \frac{b\theta_1}{p + a + b\theta_2} u_c \qquad p = \frac{d}{dt}$$

$$\frac{\partial e}{\partial \theta_1} = \frac{b}{p + a + b\theta_2} u_c$$

$$\frac{\partial e}{\partial \theta_2} = -\frac{b^2 \theta_1}{(p + a + b\theta_2)^2} u_c = -\frac{b}{p + a + b\theta_2} y$$

Approximate

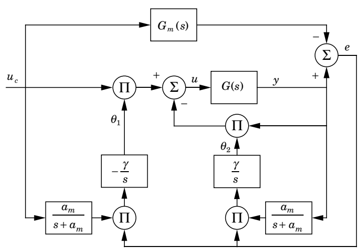
$$p + a + b\theta_2 \approx p + a_m$$

Hence

$$\frac{d\theta_1}{dt} = -\gamma \left(\frac{a_m}{p + a_m} u_c \right) e$$

$$\frac{d\theta_2}{dt} = \gamma \left(\frac{a_m}{p + a_m} y \right) e$$

Block Diagram



$$\frac{d\theta_1}{dt} = -\gamma \left(\frac{a_m}{p + a_m} u_c \right) e$$

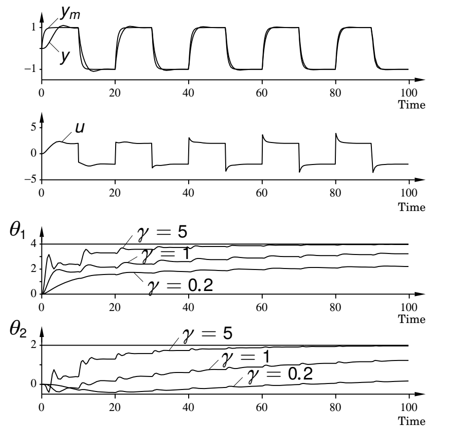
$$\frac{d\theta_2}{dt} = \gamma \left(\frac{a_m}{p + a_m} y \right) e$$

Example $a = 1$, $b = 0.5$, $a_m = b_m = 2$.

Simnon Code

```
CONTINUOUS SYSTEM mras
"MRAS for first-order system with Gm=bm/(s+am)
INPUT y uc
OUTPUT u
STATE ym th1 th2 x1 x2
DER dym dth1 dth2 dx1 dx2
u=th1*uc-th2*y
dym=-am*ym+bm*uc
dx1=-am*x1+am*uc
dx2=-am*x2-am*y
e=y-ym
dth1=-gamma*e*x1
edth2=-gamma*e*x2 am:4 "model parameter
bm:2 "model parameter
gamma:2 "adaptation gain
END
```

Simulation $a = 1$, $b = 0.5$, $a_m = b_m = 2$

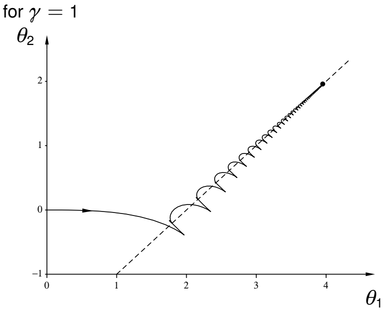


Good Control but Bad Parameters?

The closed loop transfer function is

$$G_{cl}(s) = \frac{\theta_1 G(s)}{1 + \theta_2 G(s)} = \frac{\theta_1 b}{s + a + \theta_2 b}$$

Parameters for $\gamma = 1$



Error and Parameter Convergence

Consider adaptation of feedforward gain

$$e = (k\theta - k_0)u_c = k(\theta - \theta^0)u_c$$

with $\theta^0 = k_0/k$

$$\frac{d\theta}{dt} = -\gamma k^2 u_c^2 (\theta - \theta^0)$$

Solution

$$\theta(t) = \theta^0 + (\theta(0) - \theta^0)e^{-\gamma k^2 I_t}$$

where

$$I_t = \int_0^t u_c^2(\tau)\, d\tau$$

Convergence rate depends on the input!

MIT Rule Does Not Guarantee Stability with Unmodeled Dynamics

Adaptation of Feedforward Gain: $G(s) = \dfrac{1}{s^2 + a_1 s + a_2}$

$$y = kG(p)u, \quad y_m = k_0 G(p) u_c$$

$$u = \theta u_c, \quad e = y - y_m$$

$$\frac{d\theta}{dt} = -\gamma y_m e$$

Parameter equation

$$\frac{d\theta}{dt} + \gamma y_m \left(kG(p)\theta u_c\right) = \gamma y_m^2$$

Approximate!

$$\frac{d\theta}{dt} + \gamma y_m^o u_c^o \left(kG(p)\theta\right) = \gamma (y_m^o)^2$$

Characteristic equation

$$s^3 + a_1 s^2 + s_2 s + \gamma y_m^o u_c^o k = 0$$

Stable if $\mu = \gamma y_m^o u_c^o k < a_1 a_2$, $\gamma < 1$ for $k = a_1 = a_2 = u_c^0 = y_m^0 = 1$

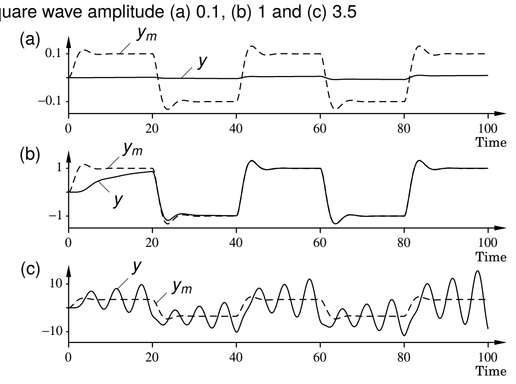
MIT Rule Does Not Guarantee Stability 2

Process: $G(s) = \frac{1}{s^2 + a_1 s + a_2}$.

Approximate characteristic equation: $s^3 + a_1 s^2 + a_2 s + \gamma y_m^o u_c^o k = 0$.

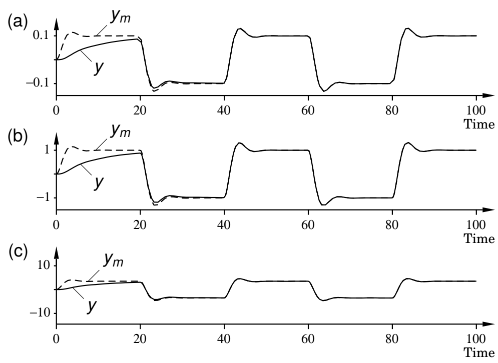
Stability condition: $\gamma y_m^o u_c^o k < a_1 a_2$.

Square wave amplitude (a) 0.1, (b) 1 and (c) 3.5



Normalized Adaptation Law

Replace MIT rule: $\frac{d\theta}{dt} = \gamma \varphi e$ by normalized rule $\frac{d\theta}{dt} = \frac{\gamma \varphi e}{\alpha + \varphi^T \varphi}$



Summary

- Servoproblem
  - Model following
  - The MIT rule
  - Good excitation through reference signal
- The error equation

$$e(t) = (G(p, \theta) - G_m(p))u_c(t)$$

- Gradient procedure

$$\frac{d\theta}{dt} = \gamma \varphi e, \qquad \varphi = \frac{\partial G(p, \theta)}{\partial \theta} u_c$$

- Normalized adaptation laws

$$\frac{d\theta}{dt} = \gamma \frac{\varphi e}{\alpha + \varphi^T \varphi}$$

Laws based on recursive system identification give normalization automatically

Model Reference Adaptive Control

1. Introduction
2. The MIT Rule
3. Lyapunov Theory
    - Ordinary differential equations
    - Stability
    - Lyapunov's idea
    - Finding Lyapunov functions
4. Adaptation Laws based on Lyapunov Theory
5. Passivity
6. Adaptation Laws based on Passivity
7. Nonlinear Systems
8. Summary

Alexandr Lyapunov 1857-1918

- MS Physico-Math Dept St Petersburg University 1876
- Chebyshev $\Rightarrow$ Markov
- PhD Moscow University The general problem of the stability of motion 1892
- Chair of Mechanics Kharkov University 1885
- Professor Applied Math St Petersburg 1902

Ordinary Differential Equations

Consider the solution $x(t) = 0$ to

$$\frac{dx}{dt} = f(x) \qquad f(0) = 0$$

Existence and uniqueness

$$\|f(x) - f(y)\| \le L\|x - y\| \qquad L > 0$$

Many solutions

$$\frac{dx}{dt} = \sqrt{x}$$
$$x(0) = 0$$
$$x(t) = \begin{cases} 0 & \text{if } t \le t_0 \\ t^2 & \text{if } t > t_0 \end{cases}$$

Finite escape time

$$\frac{dx}{dt} = x^2$$
$$x(0) = 1$$
$$x(t) = \frac{1}{1-t}$$

Lyapunov Stability

Definition

The solution $x(t) = 0$ is *stable* if for given $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that all solutions with initial conditions

$$\|x(0)\| < \delta$$

have the property

$$\|x(t)\| < \varepsilon \quad \text{for} \quad 0 \le t < \infty$$

The solution is *unstable* if it is not stable. The solution is *asymptotically stable* if it is stable and δ can be found such that all solutions with $\|x(0)\| < \delta$ have the property that $\|x(t)\| \to 0$ as $t \to \infty$.

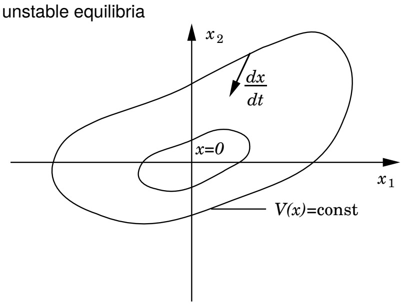
Notice

- Stability of a particular solution
- Local concept
- How to make it global?

Lyapunov's Idea

Inspiration from mechanics - energy function

Stable and unstable equilibria



Condition

$$\frac{\partial V^T}{\partial x} f(x) < 0$$

Formalities

Definition

A continuously differentiable function $V : R^n \to R$ is called *positive definite* in a region $U \subset R^n$ containing the origin if

1. $V(0) = 0$
2. $V(x) > 0, \quad x \in U$ and $x \ne 0$

A function is called *positive semidefinite* if Condition 2 is replaced by $V(x) \ge 0$.

Theorem

If there exists a function $V : R^n \to R$ that is positive definite such that

$$\frac{dV}{dt} = \frac{\partial V^T}{\partial x}\frac{dx}{dt} = \frac{\partial V^T}{\partial x} f(x) = -W(x)$$

is negative semidefinite, then the solution $x(t) = 0$ is stable. If dV/dt is negative definite, then the solution is also asymptotically stable.

Time-Varying Systems

$$\frac{dx}{dt} = f(x, t)$$

Definition

The solution $x(t) = 0$ is *uniformly stable* if for $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \qquad \forall t \ge t_0 \ge 0$$

The solution is *uniformly asymptotically stable* if it is uniformly stable and there is $c > 0$, independent of t_0, such that $x(t) \to 0$ as $t \to \infty$, uniformly in t_0, for all $\|x(t_0)\| < c$.

Definition

A continuous function α: $[0, a) \to [0, \infty)$ is said to belong to *class K* if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Lyapunov Functions for Linear Systems

Let the linear system

$$\frac{dx}{dt} = Ax$$

be stable. Pick Q positive definite. The *Lyapunov equation*

$$A^T P + PA = -Q$$

has always a unique solution with P positive definite and the funtion

$$V(x) = x^T Px$$

is a Lyapunov function

Lyapunov Theorem

Theorem
Let $x = 0$ be an equilibrium point and $D = \{x \in R^n \mid \|x\| < r\}$. Let V be a continuously differentiable function such that

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|)$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \le -\alpha_3(\|x\|)$$

for $\forall t \ge 0$, where α_1, α_2, and α_3 are class K functions. Then $x = 0$ is uniformly asymptotically stable.

Model Reference Adaptive Control

Adaptation Laws based on Lyapunov Theory

- Replace ad hoc with desings that give guaranteed stability
- Lyapunov function $V(x) > 0$ positive definite

$$\frac{dx}{dt} = f(x),$$
$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \frac{DV}{dx} f(x) < 0$$

- Determine a controller structure
- Derive the Error Equation
- Find a Lyapunov function
- $\frac{dV}{dt} \le 0$ Barbalat's lemma
- Determine an adaptation law

Adaptation of Feedforward Gain

Process model: $\frac{dy}{dt} = -ay + ku$

Desired response: $\frac{dy_m}{dt} = -ay_m + k_o u_c$

Controller: $u = \theta u_c$

Introduce the error $e = y - y_m$ and the error equation becomes

$$\frac{de}{dt} = -ae + (k\theta - k_o)u_c$$

Candidate Lyapunov function

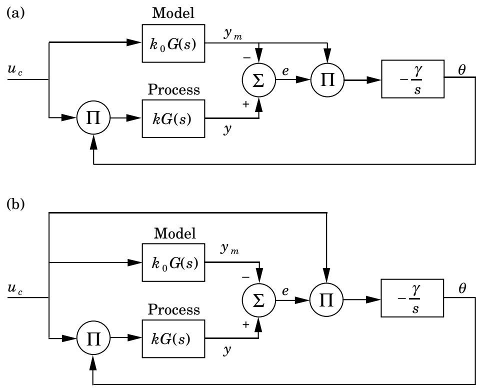
$$V(e,\theta) = \frac{\gamma}{2}e^2 + \frac{k}{2}(\theta - \frac{k_0}{k})^2$$

Time derivative

$$\frac{dV}{dt} = \gamma e\Big(-ae + (k\theta - k_0)u_c\Big) + k\Big(\theta - \frac{k_0}{k}\Big)\frac{d\theta}{dt}$$
$$= -\gamma a e^2 + (k\theta - k_0)\Big(\frac{d\theta}{dt} + \gamma u_c e\Big)$$

Adaptation of Feedforward Gain

MIT Rule a: $\frac{d\theta}{dt} = -\gamma\, ye$ Lyapunov rule b: $\frac{d\theta}{dt} = -\gamma\, u_c e$

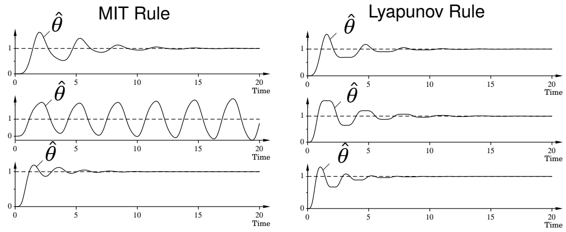


A minor change of architecture (moving one wire) has dramatic effect!

Adaptation of Feedforward Gain: MIT Rule and Lyapunov Rules

$$G(s) = \frac{1}{s+1} \qquad \text{MIT Rule: } \frac{d\theta}{dt} = -\gamma\, ye$$

Sinusoidal input of varying frequency



First Order System

Process model and desired behavior

$$\frac{dy}{dt} = -ay + bu, \qquad \frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Controller and error

$$u = \theta_1 u_c - \theta_2 y, \qquad e = y - y_m$$

Ideal parameters

$$\theta_1 = \frac{b}{b_m}, \qquad \theta_2 = \frac{a_m - a}{b}$$

The derivative of the error

$$\frac{de}{dt} = -a_m e - (b\theta_2 + a - a_m)y + (b\theta_1 - b_m)\, u_c$$

Candidate for Lyapunov function

$$V(e, \theta_1, \theta_2) = \frac{1}{2}\left(e^2 + \frac{1}{b\gamma}(b\theta_2 + a - a_m)^2 + \frac{1}{b\gamma}(b\theta_1 - b_m)^2\right)$$

Derivative of Lyapunov Function

$$V(e, \theta_1, \theta_2) = \frac{1}{2}\left(e^2 + \frac{1}{b\gamma}(b\theta_2 + a - a_m)^2 + \frac{1}{b\gamma}(b\theta_1 - b_m)^2\right)$$

Derivative of error and Lyapunov function

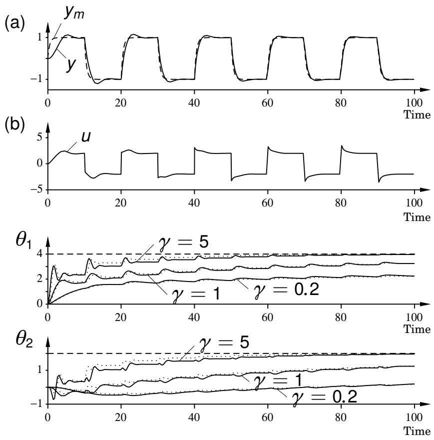
$$\frac{de}{dt} = -a_m e - (b\theta_2 + a - a_m)y + (b\theta_1 - b_m)\, u_c$$
$$\frac{dV}{dt} = e\frac{de}{dt} + \frac{1}{\gamma}(b\theta_2 + a - a_m)\frac{d\theta_2}{dt} + \frac{1}{\gamma}(b\theta_1 - b_m)\frac{d\theta_1}{dt}$$
$$= -a_m e^2 + \frac{1}{\gamma}(b\theta_2 + a - a_m)\left(\frac{d\theta_2}{dt} - \gamma ye\right)$$
$$+ \frac{1}{\gamma}(b\theta_1 - b_m)\left(\frac{d\theta_1}{dt} + \gamma u_c e\right)$$

Adaptation law

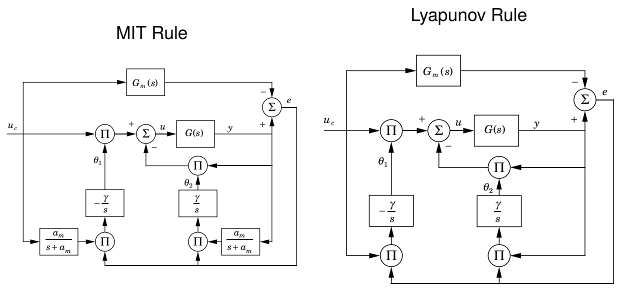
$$\frac{d\theta_1}{dt} = -\gamma u_c e, \qquad \frac{d\theta_2}{dt} = \gamma ye \Rightarrow \frac{de}{dt} = -e^2$$

Error will always go to zero, what about parameters?

Simulation $a = 1$, $b = 0.5$, $a_m = b_m = 2$, $\gamma = 1$



Comparison with MIT rule



A minor change of architecture (removing two filters) has dramatic effect!

State Feedback

Process model

$$\frac{dx}{dt} = Ax + Bu$$

Desired response to command signals

$$\frac{dx_m}{dt} = A_m x_m + B_m u_c$$

Control law

$$u = Mu_c - Lx$$

The closed-loop system

$$\frac{dx}{dt} = (A - BL)x + BMu_c = A_c(\theta)x + B_c(\theta)u_c$$

Parametrization

$$A_c(\theta^0) = A_m$$
$$B_c(\theta^0) = B_m$$

Compatibility conditions

$$A - A_m = BL$$

The Error Equation

Process

$$\frac{dx}{dt} = Ax + Bu$$

Desired response

$$\frac{dx_m}{dt} = A_m x_m + B_m u_c$$

Control law

$$u = Mu_c - Lx$$

Error

$$e = x - x_m$$

$$\frac{de}{dt} = \frac{dx}{dt} - \frac{dx_m}{dt} = Ax + Bu - A_m x_m - B_m u_c$$

Hence

$$\begin{aligned}\frac{de}{dt} &= A_m e + (A - A_m - BL)\,x + (BM - B_m)\,u_c \\ &= A_m e + (A_c(\theta) - A_m)\,x + (B_c(\theta) - B_m)\,u_c \\ &= A_m e + \Psi\,(\theta - \theta^0)\end{aligned}$$

The Lyapunov Function

The error equation

$$\frac{de}{dt} = A_m e + \Psi\,(\theta - \theta^0)$$

Try

$$V(e,\theta) = \frac{1}{2}\left(\gamma e^T P e + (\theta - \theta^0)^T(\theta - \theta^0)\right)$$

Hence

$$\begin{aligned}\frac{dV}{dt} &= -\frac{\gamma}{2} e^T Q e + \gamma(\theta - \theta^0)\Psi^T P e + (\theta - \theta^0)^T \frac{d\theta}{dt} \\ &= -\frac{\gamma}{2} e^T Q e + (\theta - \theta^0)^T\left(\frac{d\theta}{dt} + \gamma \Psi^T P e\right)\end{aligned}$$

where Q positive definite and

$$A_m^T P + P A_m = -Q$$

Adaptation law: $\frac{d\theta}{dt} = -\gamma \Psi^T P e \quad \Rightarrow \quad \frac{dV}{dt} = -\frac{\gamma}{2} e^T Q e$

Output Feedback

Process

$$\frac{dx}{dt} = Ax + B(\theta - \theta^0)u_c$$
$$e = Cx$$

Adaptation law

$$\frac{d\theta}{dt} = -\gamma u_c B^T P x$$

Can we find P such that

$$B^T P = C$$

The adaptation law then becomes

$$\frac{d\theta}{dt} = -\gamma\, u_c e$$

Kalman-Yakubovich Lemma

Definition

A rational transfer function G with real coefficients is *positive real* (PR) if

$$\mathrm{Re}\, G(s) \ge 0 \quad \text{for} \quad \mathrm{Re}\, s \ge 0$$

A transfer function G is *strictly positive real* (SPR) if $G(s - \varepsilon)$ is positive real for some real $\varepsilon > 0$.

Lemma

The transfer function

$$G(s) = C(sI - A)^{-1}B$$

is strictly positive real if and only if there exist positive definite matrices P and Q such that

$$A^T P + PA = -Q$$

and

$$B^T P = C$$

Summary

- Lyapunov Stability Theory
  - Stability concept
  - Lyapunovs theorem
  - How to use it?
- Adaptive laws with Guaranteed Stability
- Simple design procedure
  - Find control law
  - Derive Error Equation
  - Find Lyapunov Function
  - Choose adjustment law so that $dV/dt \le 0$
- Remark
  - Strong similarities with MIT rule
  - Often simpler
  - No normalization
  - Connection to passivity!

Model Reference Adaptive Control

1. Introduction
2. The MIT Rule
3. Lyapunov Theory
4. Adaptation Laws based on Lyapunov Theory
5. Passivity
   - Input-Output view of systems
   - The notions of gain and phase
   - The small gain theorem
   - The passivity theorem
6. Adaptation Laws based on Passivity
7. Nonlinear Systems
8. Summary

The Input-Output View of Systems

- Introduction
  - Conceptually - the table
  - White Boxes and Black Boxes
  - Input-output descriptions
  - How to generalize from linear to nonlinear?
- The Small Gain Theorem (SGT)
  - The notion of gain
  - Examples
  - The main result
- The Passivity Theorem (PT)
  - Passivity and phase
  - Examples
  - The Passivity Theorem
  - Relations between SGT and PT
- Applications to Adaptive Control
  - The augmented error
  - MRAS and STR
- Conclusions

The Notion of Gain

Signal spaces

L_2: $\|u\| = \left(\int_{-\infty}^{\infty} u^2(t)\,dt\right)^{\frac{1}{2}}$

L_∞: $\|u\| = \sup_{0 \le t < \infty} |u(t)|$

Extended spaces

$$x_T(t) = \begin{cases} x(t) & 0 \le t \le T \\ 0 & t > T \end{cases}$$

$u \in X_e$ if $x_T \in X$

The notion of gain (=operator norm)

$$\gamma(S) = \sup_{u \in X_e} \frac{\|Su\|}{\|u\|}$$

The gain of S is the smallest value γ such that

$$\|Su\| \le \gamma(S)\|u\| \quad \text{for all } u \in X_e$$

Examples

Linear systems with signals in L_{2e}

$$\|y\| \le \max_\omega |G(i\omega)| \cdot \|u\|$$

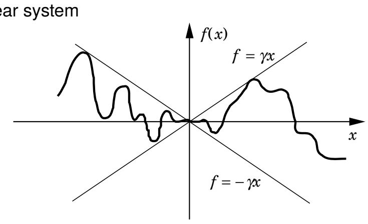
$$u_0 = \sin \omega t$$

Linear Systems with signals in L_∞

$$\gamma(G) = \int_0^\infty |h(\tau)|\,d\tau$$

$$u_0(s) = u_0\,\mathrm{sign}(h(t-s))$$

Static nonlinear system



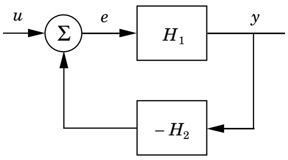
The Small Gain Theorem

Definition

A system is called *bounded-input, bounded-output (BIBO) stable* if the system has bounded gain.

Theorem

Consider the system.



Let γ_1 and γ_2 be the gains of the systems H_1 and H_2. The closed-loop system is BIBO stable if

$$\gamma_1\gamma_2 < 1$$

and its gain is less than

$$\gamma = \frac{\gamma_1}{1 - \gamma_1\gamma_2}$$

Passivity

- The idea
  - Energy dissipation
  - Capacitors, induktors, resistances
  - Mass, spring, dashpot
  - Circuit Theory
  - Mechatronics
- Mathematical Formalization
  - The Notion of Phase
  - Examples
  - Postive real linear system
  - The passivity theorem
  - Using passivity in system design

A Formal Statement

Definition

A system with input u and output y is *passive* if

$$\langle y \,|\, u \rangle \ge 0$$

The system is *input strictly passive* (ISP) if there exists $\varepsilon > 0$ such that

$$\langle y \,|\, u \rangle \ge \varepsilon\|u\|^2$$

and *output strictly passive* (OSP) if there exists $\varepsilon > 0$ such that

$$\langle y \,|\, u \rangle \ge \varepsilon\|y\|^2$$

Intuitively

- Think about u and v as voltage and current or force and velocity
- Causality?

The Notion of Phase

Let the signal space have an inner product

The phase for a given input u can then be defined as

$$\cos\varphi = \frac{\langle y \,|\, u \rangle}{\|u\|\,\|y\|} = \frac{\langle Hu \,|\, u \rangle}{\|u\|\,\|Hu\|}$$

Passivity implies that the phase is in the range

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

Linear Time-invariant Systems

$$\langle y \mid u\rangle = \int_0^\infty y(t)u(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(i\omega)U(-i\omega)\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(i\omega)U(i\omega)U(-i\omega)\,d\omega$$
$$= \frac{1}{\pi}\int_0^{\infty} \operatorname{Re}\{G(i\omega)\}\,U(i\omega)U(-i\omega)\,d\omega$$

Definition

A rational transfer function G with real coefficients is *positive real* (PR) if

$$\operatorname{Re} G(s) \ge 0 \quad \text{for} \quad \operatorname{Re} s \ge 0$$

A transfer function G is *strictly positive real* (SPR) if $G(s-\varepsilon)$ is positive real for some real $\varepsilon > 0$.

Characterizing Positive Real Transfer Functions

Theorem

A rational transfer function $G(s)$ with real coefficients is PR if and only if the following conditions hold.

(i) The function has no poles in the right half-plane.

(ii) If the function has poles on the imaginary axis or at infinity, they are simple poles with positive residues.

(iii) The real part of G is nonnegative along the $i\omega$ axis, that is,

$$\operatorname{Re}(G(i\omega)) \ge 0$$

A transfer function is SPR if conditions (i) and (iii) hold and if condition (ii) is replaced by the condition that $G(s)$ has no poles or zeros on the imaginary axis.

Examples

Recall

$$\langle y \mid u\rangle = \frac{1}{\pi}\int_0^{\infty} \operatorname{Re}\{G(i\omega)\}\,U(i\omega)U(-i\omega)\,d\omega$$

- Positive real PR

$$\operatorname{Re} G(i\omega) \ge 0$$

- Input strictly passive ISP

$$\operatorname{Re} G(i\omega) \ge \varepsilon > 0$$

- Output stricly passive OSP

$$\operatorname{Re} G(i\omega) \ge \varepsilon |G(i\omega)|^2$$

$G(s) = s+1$ SPR and ISP not OSP
$G(s) = \frac{1}{s+1}$ SPR and OSP not ISP
$G(s) = \frac{s^2+1}{(s+1)^2}$ OSP and ISP not OSP
$G(s) = \frac{1}{s}$ PR not SPR, OPS or ISP

Nonlinear Static Systems $y = f(u)$

$$\langle y \mid u\rangle = \int_0^\infty f(u(t))u(t)\,dt$$

- Passive if $xf(x) \ge 0$
- Input strictly passive (ISP) if $xf(x) \ge \delta |x|^2$
- Output strictly passive if

$$xf(x) \ge \delta f^2(x)$$

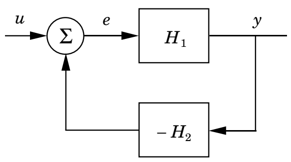
Geometric Interpretation
Example

- $f(x) = x + x^3$ input strictly passive
- $f(x) = x/(1+|x|)$ output strictly passive.

The Passivity Theorem

Theorem

Consider a system obtained by connecting two systems H_1 and H_2 in a feedback loop

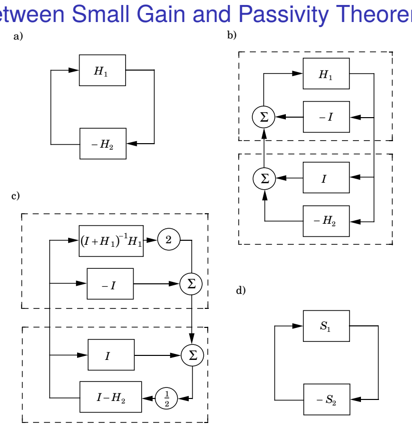


Let H_1 be strictly output passive and H_2 be passive. The closed-loop system is then BIBO stable.

Intuitive: Think about phase
Use of passivity in system design

- Force control in robotics
- Remote manipulator - Mark Spong

Relations Between Small Gain and Passivity Theorems



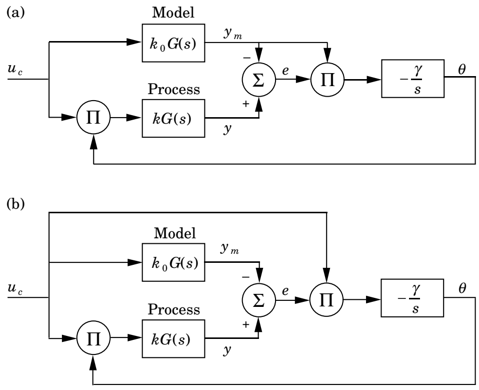
$S_1 = (H_1+I)^{-1}(H_1-I) \quad H_1 = -(S_1+I)^{-1}(S_1-I)$ *LHP to unit circle*

Summary

- Passivity is a very powerful idea
- Cascade of two passive systems is passive
- Related to
  - Energy
  - Phase shift
- Can be used in many different ways
  - Stable control laws
  - Remote control
  - Adaptive control

Model Reference Adaptive Control

Adaptation of Feedforward Gain



Redraw b)

Analysis

Lemma

Let r be a bounded square integrable function, and let $G(s)$ be a transfer function that is positive real. The system whose input-output relation is given by

$$y = r\,(G(p)ru)$$

is then passive.

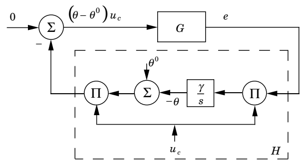
Example: PI adjustments

$$\theta(t) = -\gamma_1 u_c(t)e(t) - \gamma_2 \int^{t} u_c(\tau)e(\tau)\,d\tau$$

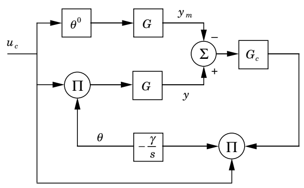
Explore the advantages of PI adjustments analytically and by simulation!

A modified algorithm

Change



To



Make G_cG SPR. Still a problem with pole excess > 1.

The Augmented Error

Consider the error

$$\begin{aligned} e &= G(\theta-\theta^0)u_c \\ &= G(\theta-\theta^0)u_c + (\theta-\theta^0)Gu_c - (\theta-\theta^0)Gu_c \end{aligned}$$

Introduce the *augmented error*

$$\epsilon = e + \eta$$

where

$$\eta = G(\theta-\theta^0)u_c - (\theta-\theta^0)Gu_c = G\theta u_c - \theta G u_c$$

Notice that η is zero under stationary conditions

Use the adaptation law

$$\frac{d\theta}{dt} = -\gamma\epsilon G_2 u_c$$

Stability now follows from the passivity theorem

The idea can be extended to the general case, details are messy.

A Minor Extension

Factor

$$G = G_1G_2$$

where the transfer function G_1 is SPR. The error $e = y - y_m$ can then be written as

$$\begin{aligned} e &= G(\theta-\theta^0)u_c = (G_1G_2)(\theta-\theta^0)u_c \\ &= G_1\big(G_2(\theta-\theta^0)u_c + (\theta-\theta^0)G_2u_c - (\theta-\theta^0)G_2u_c\big) \end{aligned}$$

Introduce

$$\varepsilon = e + \eta$$

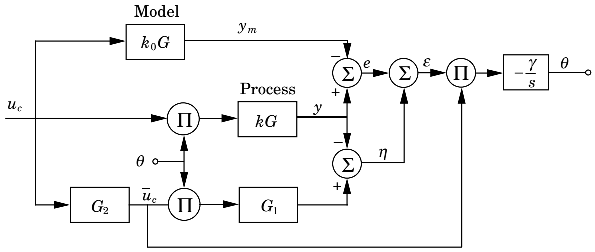
where η is the *error augmentation* defined by

$$\begin{aligned} \eta &= G_1(\theta-\theta^0)G_2u_c - G(\theta-\theta^0)u_c \\ &= G_1(\theta G_2 u_c) - G\theta u_c \end{aligned}$$

Use adaptation law

$$\frac{d\theta}{dt} = -\gamma\epsilon G_2 u_c$$

MRAS with Augmented Error - Monopoli



Summary

- The concepts
  - Notions of gain, phase and passivity
  - Positive real PR and strictly positive real SPR
- The key results
  - The small gain theorem
  - The passivity theorem
  - Equivalence - complex variable LHP interior of unit circle

Model Reference Adaptive Control

1. Introduction
2. The MIT Rule
3. Lyapunov Theory
4. Adaptation Laws based on Lyapunov Theory
5. Passivity
6. Adaptation Laws based on Passivity
7. Nonlinear Systems
   - Feedback linerization
   - Adaptive feedback linearization
8. Summary

Feedback Linearization - Example

Consider the system

$$\frac{dx_1}{dt} = x_2 + f(x_1) \quad \frac{dx_2}{dt} = u$$

f is a differentiable function, introduce new coordinates

$$\xi_1 = x_1 \quad \xi_2 = x_2 + f(x_1)$$

Then

$$\frac{d\xi_1}{dt} = \xi_2 \quad \frac{d\xi_2}{dt} = \xi_2 f'(\xi_1) + u$$

Introduce the control law

$$u = -a_2\xi_1 - a_1\xi_2 - \xi_2 f'(\xi_1) + v$$

gives the closed loop system

$$\frac{d\xi}{dt} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

This system is linear with the characteristic equation

$$s^2 + a_1 s + a_2 = 0$$

Feedback Linearization - Example ...

$$\frac{d\xi_1}{dt} = \xi_2, \quad \frac{d\xi_2}{dt} = \xi_2 f'(\xi_1) + u$$

$$\xi_1 = x_1, \xi_2 = x_2 + f(x_1)$$

Transforming back to the original coordinates the control law becomes

$$u = -a_2 x_1 - \big(a_1 + f'(x_1)\big)\big(x_2 + f(x_1)\big) + v$$

The design method used in the example can be interpreted as the analog equivalent of pole placement design for linear system. The closed-loop system obtained will behave like a linear system. This is the reason why the method is called *feedback linearization*

The system in Exampleis very special but the method can be applied in several other systems for example.

$$\frac{dx}{dt} = f(x) + ug(x)$$

Feedback Linearization

Consider

$$\frac{dx}{dt} = f(x) + ug(x)$$

first pick

$$\xi_1 = h(x)$$

where $h(x)$ is chosen so that $h'(x)g(x) = 0$ as a new state variable. The time derivative of ξ_1 is

$$\frac{d\xi_1}{dt} = h'(x)\big(f(x) + ug(x)\big)$$

Since $h'(x)g(x) = 0$, we introduce the new state variable $\xi_2 = h'(x)f(x)$ We proceed as long as the control variable u does not appear explicitly on the right-hand side. In this way we obtain the state variables $\xi_1 \ldots \xi_r$, which are combined to the vector $\xi \in R^r$, where $r \le n$. We also introduce the new state variable $\eta_1 \ldots \eta_{n-r}$, which are combined into the vector $\eta \in R^{n-r}$. This can be done in many different ways.

Feedback Linearization

We obtain the equations

$$\frac{d\xi_1}{dt} = \xi_2$$

$$\frac{d\xi_2}{dt} = \xi_3$$

$$\vdots$$

$$\frac{d\xi_r}{dt} = \alpha(\xi, \eta) + u\beta(\xi, \eta)$$

$$\frac{d\eta}{dt} = \gamma(\xi, \eta)$$

The state variables ξ represents a chain of r integrators, where the integer r is the nonlinear equivalent of pole excess. The variables η will not appear if $r = n$. This case corresponds to a system without zeros.

Feedback Linerization

A design procedure, which is the nonlinear analog of pole placement, can be constructed if $\beta(\xi, \eta) \ne 0$. If this is the case, we can introduce the feedback law

$$u = \frac{1}{\beta(\xi, \eta)}\big(-a_r\xi_1 - a_{r-1}\xi_2 - \ldots - a_1\xi_r - \alpha(\xi, \eta) + b_0 v\big)$$

and the closed-loop system becomes

$$\frac{d\xi}{dt} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \\ \vdots & & & & \\ -a_r & -a_{r-1} & -a_{r-2} & \ldots & -a_1 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} v$$

$$\frac{d\eta}{dt} = \gamma(\xi, \eta)$$

The relation between v and ξ_1 is given by a *linear* dynamical system with the transfer function

$$G(s) = \frac{b_0}{s^r + a_1 s^{r-1} + \ldots a_r}$$

Feedback Linerization

The differential equation

$$\frac{d\xi}{dt} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \\ \vdots & & & & \\ -a_r & -a_{r-1} & -a_{r-2} & \ldots & -a_1 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} v$$

$$\frac{d\eta}{dt} = \gamma(\xi, \eta)$$

has a triangular structure. The vector ξ is a governed by a linear system that is decoupled from the variable η. If $\xi = 0$, η is governed by

$$\frac{d\eta}{dt} = \gamma(0, \eta)$$

This equation represents the *zero dynamics*. It is necessary for this system to be stable if the proposed control design is going to work. For linear systems the zero dynamics are also the dynamics associated with the zeros of the transfer function. Feedback linearization is the nonlinear analog of pole placement for linear systems.

Adaptive Feedback Linerization

Consider the system

$$\frac{dx_1}{dt} = x_2 + \theta f(x_1) \quad \frac{dx_2}{dt} = u$$

where θ is an unknown parameter and f is a known differentiable function. Applying the certainty equivalence principle gives the following control law:

$$u = -a_2 x_1 - \big(a_1 + \hat\theta f'(x_1)\big)\big(x_2 + \hat\theta f(x_1)\big) + v$$

Introducing this into the system equations gives an error equation that is nonlinear in the parameter error. This makes it very difficult to find a parameter adjustment law that gives a stable system. Therefore it is necessary to use another approach.

Adaptive Feedback Linerization ..

$$\frac{dx_1}{dt} = x_2 + \theta f(x_1) \qquad \frac{dx_2}{dt} = u$$

Introduce the new coordinates

$$\xi_1 = x_1 \qquad \xi_2 = x_2 + \hat\theta f(x_1)$$

where $\hat\theta$ is an estimate of θ, we have

$$\frac{d\xi_1}{dt} = \frac{dx_1}{dt} = x_2 + \theta f(x_1) = \xi_2 + (\theta - \hat\theta) f(\xi_1)$$

$$\frac{d\xi_2}{dt} = \frac{d\hat\theta}{dt} f(x_1) + \hat\theta\big(x_2 + \theta f(x_1)\big) f'(x_1) + u$$

The control law

$$u = -a_2\xi_1 - a_1\xi_2 - \hat\theta\big(x_2 + \hat\theta f(x_1)\big) f'(x_1) - f(x_1)\frac{d\hat\theta}{dt} + v$$

gives the closed loop system

$$\frac{d\xi}{dt} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \xi + \begin{bmatrix} f(\xi_1) \\ \hat\theta f(\xi_1) f'(\xi_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

Adaptive Feedback Linerization ..

Introduce the reference model

$$\frac{dx_m}{dt} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ a_2 \end{bmatrix} u_m$$

Let $e = \xi - x_m$ and $v = a_2 u_m$ the error equation becomes

$$\frac{de}{dt} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} e + \begin{bmatrix} f(\xi_1) \\ \hat{\theta} f(\xi_1) f'(\xi_1) \end{bmatrix} \tilde{\theta} = Ae + B\tilde{\theta}$$

The matrix A has all eigenvalues in the left half-plane if $a_1, a_2 > 0$ we can then find a matrix P such that

$$A^T P + PA = -I$$

Choosing the Lyapunov function

$$V = e^T Pe + \frac{1}{\gamma} \tilde{\theta}^2$$

gives

$$\frac{dV}{dt} = e^T (A^T P + PA)e + 2\tilde{\theta} B^T Pe + \frac{2}{\gamma} \tilde{\theta} \frac{d\tilde{\theta}}{dt}$$

Adaptive Feedback Linerization ..

$$\frac{dV}{dt} = e^T (A^T P + PA)e + 2\tilde{\theta} B^T Pe + \frac{2}{\gamma} \tilde{\theta} \frac{d\tilde{\theta}}{dt}$$

The adaptation law

$$\frac{d\hat{\theta}}{dt} = \gamma B^T P$$

gives

$$\frac{d\tilde{\theta}}{dt} = \frac{d}{dt} (\theta - \hat{\theta}) = -\frac{d\hat{\theta}}{dt} = -\gamma B^T Pe$$

and the derivative of the Lyapunov function becomes

$$\frac{dV}{dt} = -e^T e$$

This function is negative as long as any component of the error vector is different from zero and the tracking error will thus always go to zero.

Summary

- Passivity is a powerful concept
- Admits design of stable adaptive systems
- Strongly intuitive
- Straight forward for linear systems
- Nonlinear systems difficult
- Feedback linearization
- There are other methods like backstepping

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Summary

- Model reference systems are useful
- Servoproblem
- MIT rule
    - Simple and straight forward
    - Does not guarantee stability
    - Passivity gives insight
    - MRAS has been used in aerospace
- Lyapunov theory
    - Guaranteed stability
- Passivity
    - Guaranteed stability
    - Augmented error
- Some results for nonlinear systems