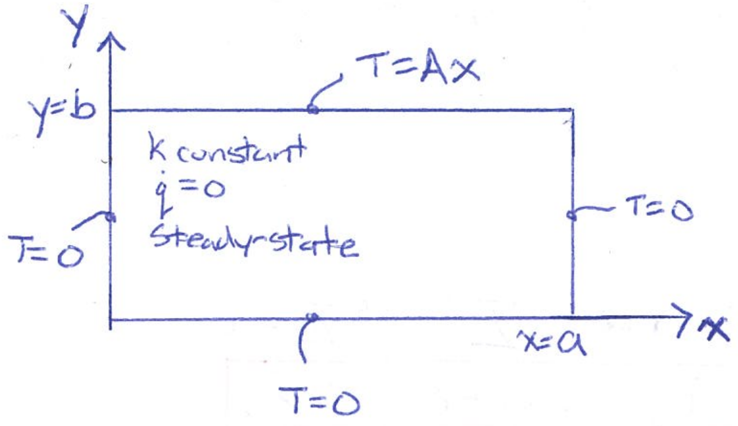
Problem 4.4



Heat Diffusion Eqn. Reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (I)$$

subjected to BCs

1. $T=0$, at $x=0$
2. $T=0$, at $x=a$
3. $T=0$, at $y=0$
4. $T=Ax$, at $y=b$

For this particular example there is no need to use the "excess temperature, θ concept as there is only one non-homogeneous boundary conditions

Separation of Variables

Assume that $T(x,y) = X(x)\cdot Y(y)$ and substitute into the heat diffusion eqn, (I).

$$\frac{\partial^2[X(x)Y(y)]}{\partial x^2} + \frac{\partial^2[X(x)\cdot Y(y)]}{\partial y^2} = 0$$

Note: X is only a function of x.
Y " y.

$$\Rightarrow Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow -\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

The only way for an equation in x to be equal to an equation in y is for both to be equal to a constant, commonly chosen as λ^2.

so

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda^2$$

what we have just done is reduced a partial differential eqn, PDE, into two ordinary differential eqns, ODEs, above. Therefore we are now looking for solutions to

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \lambda^2 \quad (IIa)$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda^2 \quad (IIb)$$

Where $T(x,y) = X(x)Y(y)$ subjected to the BCs above.

We will now solve $\text{II}_{a,b}$ considering 3 cases for λ^2.

$\lambda^2 > 0$ [positive]

$\lambda^2 = 0$ [zero]

$\lambda^2 < 0$ [negative]

If $\underline{\underline{\lambda^2 > 0}}$: then $X'' = -\lambda^2 X \Rightarrow X'' + \lambda^2 X = 0$

a solution to this O.D.E is $\boxed{X = B\sin(\lambda x) + C\cos(\lambda x)}$

we can check that this is a solution,

$$X' = +B\lambda\cos(\lambda x) - C\lambda\sin(\lambda x)$$
$$X'' = -B\lambda^2\sin(\lambda x) - C\lambda^2\cos(\lambda x)$$
$$X'' = -\lambda^2[B\sin(\lambda x) + C\cos(\lambda x)]$$
$$X'' = -\lambda^2 X \quad \checkmark$$

additionally $Y'' = \lambda^2 Y \Rightarrow Y'' - \lambda^2 Y = 0$

a solution to this ODE is $\boxed{Y = De^{-\lambda y} + Ee^{\lambda y}}$

again we can check that this is a solution,

$$Y' = -\lambda De^{-\lambda y} + \lambda Ee^{\lambda y}$$
$$Y'' = \lambda^2 De^{-\lambda y} + \lambda^2 Ee^{\lambda y}$$
$$Y'' = \lambda^2[De^{-\lambda y} + Ee^{\lambda y}]$$
$$Y'' = \lambda^2 Y \quad \checkmark$$

another common form of a solution to this ODE is

$$\boxed{Y = D\sinh(\lambda y) + E\cosh(\lambda y)}$$

Aside:

$\frac{d}{dx}\sinh x = \cosh x$

$\frac{d}{dx}\cosh x = \sinh x$

again checking that this is a solution,

$$Y' = \lambda D\cosh(\lambda y) + \lambda E\sinh(\lambda y)$$
$$Y'' = \lambda^2 D\sinh(\lambda y) + \lambda^2 E\ \ sh(\lambda y)$$
$$Y'' = \lambda^2[D\sinh(\lambda y) + E\cosh(\lambda y)]$$
$$Y'' = \lambda^2 Y \quad \checkmark$$

We will proceed using $Y = De^{-\lambda y} + Ee^{\lambda y}$

So if $\lambda^2 > 0$: $T(x,y) = XY = [B\sin(\lambda x) + C\cos(\lambda x)][De^{-\lambda y} + Ee^{+\lambda y}]$

APPLY BCs to determine unknown coefficients

BC 1: $T=0$, at $x=0$:

$$T(0,y) = [B\underbrace{\sin(0)}_{0} + C\underbrace{\cos(0)}_{=1}][De^{-\lambda y} + Ee^{+\lambda y}]$$

$$T(0,y) = 0 = C[De^{-\lambda y} + Ee^{-\lambda y}]$$

This means that

$$\boxed{C=0} \text{ -or- } De^{-\lambda y} + Ee^{+\lambda y} = 0$$

however if $De^{-\lambda y} + Ee^{+\lambda y} = 0$ then $T(x,y) = 0$, which does NOT satisfy BC 4, $T = Ax$, at $y=b$, therefore we conclude $De^{-\lambda y} + Ee^{+\lambda y} \neq 0$

so rewriting our temperature field knowing $C=0$, results in

$$T(x,y) = B\sin(\lambda x)[De^{-\lambda y} + Ee^{+\lambda y}]$$

BC 3: $T=0$, at $y=0$:

$$T(x,0) = B\sin(\lambda x)[D\underbrace{e^{0}}_{1} + E\underbrace{e^{0}}_{1}]$$

$$0 = B\sin(\lambda x)[D+E]$$

This means that

$$B=0 \text{ -or- } \sin(\lambda x) = 0 \text{ -or } D+E=0$$

however if either B or $\sin(\lambda x)$ equals zero, then $T(x,y)=0$, which again does NOT satisfy BC 4, $T=Ax$, at $y=b$, therefore we conclude

$$B \neq 0, \ \sin(\lambda x) \neq 0, \text{ and } \boxed{D=-E}$$

so rewriting our temperature field knowing $D=-E$, results in

$$T(x,y) = B\sin(\lambda x)[De^{-\lambda y} - De^{+\lambda y}]$$

$$T(x,y) = BD\sin(\lambda x)[e^{-\lambda y} - e^{+\lambda y}]$$

$$T(x,y) = 2BD\sin(\lambda x)\sinh(\lambda y)$$

Aside:

$$\sinh x = \frac{e^{-x} - e^{x}}{2}$$

$$\cosh x = \frac{e^{-x} + e^{x}}{2}$$

let $F = 2BD$, $T(x,y) = F\sin(\lambda x)\sinh(\lambda y)$

BC ②: $T=0$, at $x=a$:

$$T(a,y) = F\sin(\lambda a)\sinh(\lambda y)$$
$$0 = F\sin(\lambda a)\sinh(\lambda y)$$

this means that

$F=0$ -or- $\sin(\lambda a)=0$ -or- $\sinh(\lambda y)=0$

however if either F or $\sinh(\lambda y)$ equal zero, then $T(x,y)=0$, which again does NOT satisfy BC ④, $T=Ax$ at $y=b$, therefore we conclude $F\neq 0$, $\sinh(\lambda y)\neq 0$

so this means $\sin(\lambda a)=0$,

since sine is a periodic function, infinitely many solutions satisfy the above equation.

sine is equal to zero for arguments that are multiples of π

$$\lambda a = n\pi \Rightarrow \lambda = \frac{n\pi}{a}, \text{ for } n = 1,2,\dots,\infty$$

Note $n=0$ is not included b/c $n=0 \Rightarrow \lambda=0$ which would give $T(x,y)=0$, which does not satisfy B.C ④

rewriting our temperature field,

$$T(x,y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

for each n, we have to find a new coefficient that will satisfy B.C. ④, hence the subscript 'n' denotes F is dependent on 'n'

To find F_n, use BC ④

BC ④: $T=Ax$, at $y=b$:

$$T(x,b) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

$$Ax = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

by invoking orthonality by multiplying both sides by $\sin\left(\frac{m\pi}{a}x\right)$ and integrate from 0 to a

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = \int_0^a \sin\left(\frac{m\pi}{a}x\right)\sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)dx$$

since $\sin\left(\frac{m\pi}{a}x\right)$ is not dependent on n, we can move it inside the Σ

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = \int_0^a \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)dx$$

swap the $\int_0^a$ and $\sum_{n=1}^{\infty}$,

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = \sum_{n=1}^{\infty} \int_0^a F_n \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)dx$$

not functions of x so can pull it out of the integral

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = \sum_{n=1}^{\infty} F_n\sinh\left(\frac{n\pi}{a}b\right)\underbrace{\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx}_{=\begin{cases} a/2, & n=m \\ 0 & \text{otherwise}\end{cases}}$$

b/c the $\int_0^a dx$ is only non-zero for when $n=m$ the only term for the summation occurs when $n=m$, thus the RHS is

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = F_m\sinh\left(\frac{m\pi}{a}b\right)\cdot\frac{a}{2}$$

Evaluating the LHS: $\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx$

$$\int_0^a Ax\sin\left(\frac{m\pi}{a}x\right)dx = A\int_0^a x\sin\left(\frac{m\pi}{a}x\right)dx$$

use integration by parts

$u = x \qquad v = -\frac{a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)$

$du = dx \qquad dv = \sin\left(\frac{m\pi}{a}x\right)dx$

$$A\left[\frac{-a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)\cdot x\Big|_{x=0}^{a} - \int_0^a \frac{-a}{m\pi}\cos\left(\frac{m\pi}{a}x\right)dx\right]$$

$$A\left[\frac{-a}{m\pi}\left(\cos\left(\frac{m\pi}{a}\cdot a\right)\cdot a - \cos\left(\frac{m\pi}{a}\cdot 0\right)\cdot 0\right) + \frac{a}{m\pi}\int_0^a \cos\left(\frac{m\pi}{a}x\right)dx\right]$$

$$A\left[\frac{-a}{m\pi}\left(\cos(m\pi)\cdot a - \underbrace{\cos(0)}_{=1}\cdot 0\right) + \frac{a}{m\pi}\frac{a}{m\pi}\sin\left(\frac{m\pi x}{a}\right)\Big|_{x=0}^{a}\right]$$

$$A\left[\frac{-a}{m\pi}\left(a\cos(m\pi)\right)+\left(\frac{a}{m\pi}\right)^2\left(\sin\left(\frac{m\pi}{a}a\right)-\sin\left(\frac{m\pi}{a}\cdot 0\right)\right)\right]$$

$$A\left[\frac{-a^2}{m\pi}\cos(m\pi)+\left(\frac{a}{m\pi}\right)^2\left(\underbrace{\sin(m\pi)}_{0}-\underbrace{\sin(0)}_{0}\right)\right]$$

$$-\frac{Aa^2}{m\pi}\underbrace{\cos(m\pi)}_{=(-1)^m} = LHS \qquad LHS = -\frac{Aa^2}{m\pi}(-1)^m \qquad \begin{aligned}&-(-1)^m\\ &=(-1)^1(-1)^m\\ &=(-1)^{m+1}\end{aligned}$$

$$\boxed{LHS=\frac{Aa^2}{m\pi}(-1)^{m+1}}$$

when $m=1$ $\cos(\pi)=-1$
$=2$ $\cos(2\pi)=1$
$=3$ $\cos(3\pi)=-1$
$=4$ $\cos(4\pi)=1$
$=5$ $\cos(5\pi)=-1$
$\vdots$

so,

$$\frac{Aa^2}{m\pi}(-1)^{m+1}=F_m\sinh\left(\frac{m\pi}{a}b\right)\frac{a}{2}$$

$$\Rightarrow\quad F_m=\frac{2Aa(-1)^{m+1}}{m\pi\sinh\left(\frac{m\pi}{a}b\right)}$$

m is just a variable representing an integer, so we'll call it n

$$\boxed{F_n=\frac{2Aa(-1)^{n+1}}{n\pi\sinh\left(\frac{n\pi}{a}b\right)}}$$

Rewriting our temperature field results in,

$$T(x,y)=\sum_{n=1}^{\infty}\frac{2Aa(-1)^{n+1}}{n\pi\sinh\left(\frac{n\pi}{a}b\right)}\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$\boxed{T(x,y)=\frac{2Aa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{a}x\right)\frac{\sinh\left(\frac{n\pi}{a}y\right)}{\sinh\left(\frac{n\pi}{a}b\right)}}\qquad(\lambda^{2\,2}>0)$$

This is the temperature field when $\lambda^2>0$; we now need to check if there are any solutions when $\lambda^2<0$ and $\lambda^2=0$. If there are then we can use superposition to add all possible solutions

If $\lambda^2 < 0$: then $X'' = +|\lambda^2| X \rightarrow X'' - |\lambda^2| X = 0$,

a solution to this ODE is $\boxed{X = Fe^{-\lambda x} + Ge^{+\lambda x}}$

additionally, $Y'' = -|\lambda^2| Y \rightarrow Y'' + |\lambda^2| Y = 0$

$$\boxed{Y = H\sin(\lambda y) + J\cos(\lambda y)}$$

so if $\lambda^2 < 0$: $T(x,y) = XY = [Fe^{-\lambda x} + Ge^{+\lambda x}][H\sin(\lambda y) + J\cos(\lambda y)]$

APPLY BCs to determine unknown coefficients

BC ①: $T = 0$, at $x = 0$:

$$T(0,y) = [F\underbrace{e^0}_{1} + G\underbrace{e^0}_{1}][H\sin(\lambda y) + J\cos(\lambda y)]$$

$$0 = [F + G][H\sin(\lambda y) + J\cos(\lambda y)]$$

this means that

$$F + G = 0 \quad \text{-or-} \quad H\sin(\lambda y) + J\cos(\lambda y) = 0$$

however if $H\sin(\lambda y) + J\cos(\lambda y) = 0$ then $T(x,y) = 0$, which does not satisfy BC ④, $T = Ax$, at $y = b$, therefore we conclude

$H\sin(\lambda y) + J\cos(\lambda y) \neq 0$ and $\boxed{F = -G}$

so rewriting our temperature field knowing $F = -G$, results in

$$T(x,y) = [Fe^{-\lambda x} - Fe^{+\lambda x}][H\sin(\lambda y) + J\cos(\lambda y)]$$

$$T(x,y) = F(e^{-\lambda x} - e^{+\lambda x})[H\sin(\lambda y) + J\cos(\lambda y)]$$

$$T(x,y) = 2F\sinh(\lambda x)[H\sin(\lambda y) + J\cos(\lambda y)]$$

BC ③: $T = 0$, at $y = 0$:

$$T(x,0) = 2F\sinh(\lambda x)[H\underbrace{\sin(\lambda \cdot 0)}_{0} + J\underbrace{\cos(0)}_{1}]$$

$$0 = 2F\sinh(\lambda x)J$$

$$0 = FJ\sinh(\lambda x)$$

this means that

$$F = 0 \quad \text{-or-} \quad J = 0 \quad \text{-or-} \quad \sinh(\lambda x) = 0$$

however if F or $\sinh(\lambda x)$ ~~equal~~ equal zero then $T(x,y) = 0$, which does NOT satisfy B.C. ④, $T = Ax$ at $y = b$, therefore we conclude

$F \neq 0$, $\sinh(\lambda x) \neq 0$, and $\boxed{J=0}$

so rewriting our temperature field knowing $J=0$, results in

$$T(x,y) = 2F\sinh(\lambda x)\, H\sin(\lambda y)$$

$$T(x,y) = 2FH\sinh(\lambda x)\sin(\lambda y)$$

let $K = 2FH$, $T(x,y) = K\sinh(\lambda x)\sin(\lambda y)$

BC ②: $T=0$, at $x=a$:

$$T(a,y) = K\sinh(\lambda a)\sin(\lambda y)$$

$$0 = K\sinh(\lambda a)\sin(\lambda y)$$

this means that

$$K=0 \text{ -or- } \sinh(\lambda a) = 0 \text{ -or- } \sin(\lambda y) = 0$$

however if either K or $\sin(\lambda y)$ ~~=~~ equal zero, then $T(x,y)=0$, which again does NOT satisfy BC ④, $T=Ax$ at $y=b$, thus

$$\sinh(\lambda a) = 0$$

hyperbolic sine is NOT periodic and only equals zero when its argument equals 0, i.e.

$$\lambda a = 0 \Rightarrow \boxed{\lambda = 0}$$

rewriting our temperature field knowing $\lambda = 0$ results in

$$T(x,y) = K\sinh(0\cdot x)\sin(0\cdot y)$$

$$= K\underbrace{\sinh(0)}_{0}\underbrace{\sin(0)}_{0}$$

$$T(x,y) = 0$$

← this solution will NOT satisfy the fourth B.C. ④, $T=Ax$, at $y=b$, therefore $\lambda^2 < 0$ does NOT ~~have~~ result in any solutions for this problem

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Lastly need to check the case if $\lambda^2 = 0$,

then $X'' = 0 \Rightarrow \boxed{X = Lx + M}$

Similarly $Y'' = 0 \Rightarrow \boxed{Y = Ny + P}$

so if $\lambda^2 = 0$, $T(x,y) = XY = (Lx+M)(Ny+P)$

APPLY BCs to determine unknown coefficients

BC ①: $T=0$, at $x=0$:

$$T(0,y) = (L\cdot 0 + M)(Ny+P)$$

$$0 = M(Ny+P)$$

this means that

$$M=0 \quad \text{-or-} \quad Ny+P=0$$

however if $Ny+P=0$, then $T(x,y)=0$, which does NOT satisfy BC ④, $T=Ax$ at $y=b$ thus we conclude

$$Ny+P \neq 0 \quad \text{and} \quad \boxed{M=0}$$

rewriting our temperature field knowing $M=0$, results in

$$T(x,y) = Lx(Ny+P)$$

BC ③: $T=0$, at $y=0$:

$$T(x,0) = Lx(N\cdot 0+P)$$

$$0 = LxP$$

this means that

$$L=0 \quad \text{-or-} \quad x=0 \quad \text{-or-} \quad P=0$$

however if $L=0$, $T(x,y)=0$, which again does not satisfy BC ④, $T=Ax$ at $y=b$, $x=0$ does NOT make sense b/c we are dealing w/ a 2D problem and it would also make, $T(x,y)=0$.
So we conclude that $\boxed{P=0}$

re-writing our temperature field knowing $P=0$, results in

$$T(x,y) = LxNy$$

$$T(x,y) = LNxy$$

let $Q=LN$, $T(x,y) = Qxy$

BC ②: $T=0$, at $x=a$:

$$T(a,y) = Qay$$

$$0 = Qay \Rightarrow 0 = Qy$$

this means that

$Q=0$ -or- $y=0$ which again would result in $T(x,y)=0$ and would not satisfy BC④, $T=Ax$ at $y=b$.

Therefore $\lambda^2 = 0$ does <u>NOT</u> result in any solutions for this problem.

so our final answer (which only has a contribution from $\lambda^2 > 0$) is

$$\boxed{T(x,y) = \frac{2Aa}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{a}x\right) \frac{\sinh\left(\frac{n\pi}{a}y\right)}{\sinh\left(\frac{n\pi}{a}b\right)}}$$