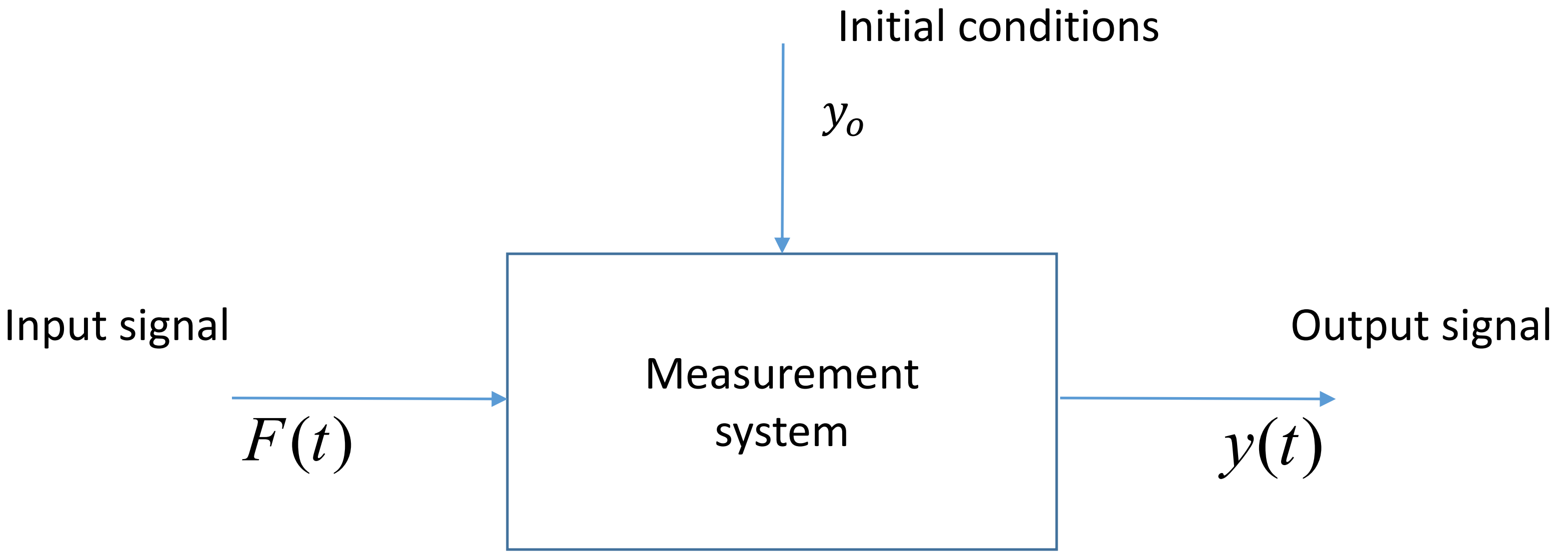
Measurement system response

Covers material from Chapter 3

General measurement system response



$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_o y = F(t)$$

Zero order system response

Almost nothing is zero order unless you wait for all time dependence to die out.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots a_1 \frac{dy}{dt} + a_o y = F(t)$$

$$n = 0$$

$$a_o y(t) = F(t)$$

$$y(t) = KF(t)$$

$$K = 1 / a_o$$

$$K = static\ calibration\ constant$$

First order system response to a step function

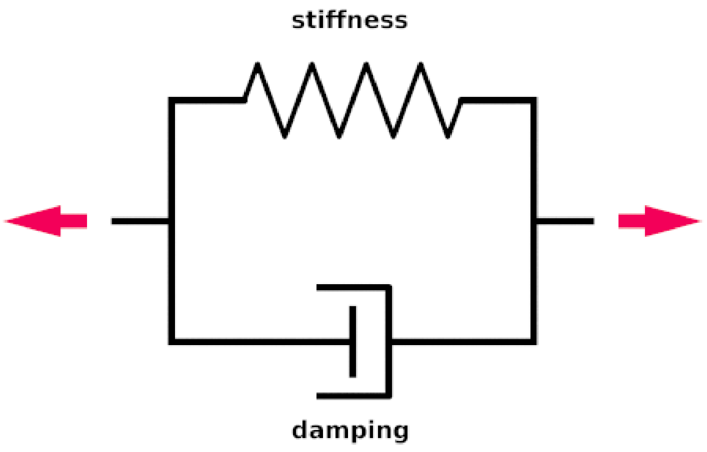
$a_1\dot{}$ $F(t)$

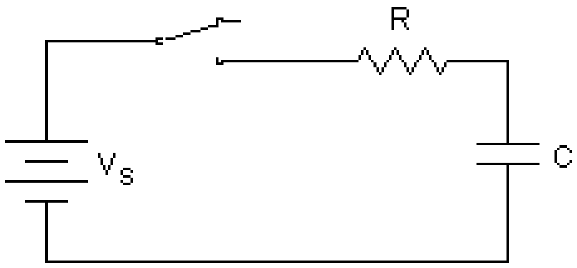
$$F(t) = AU(t)$$

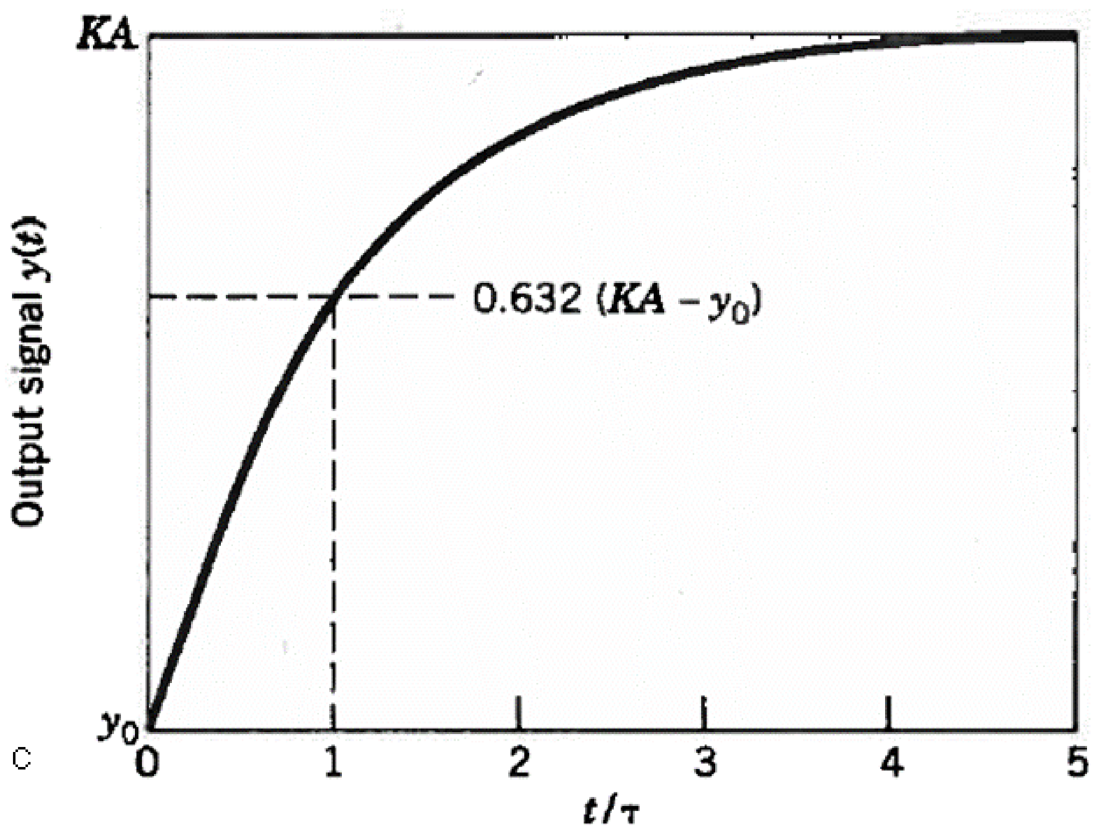
$$U(t) = 0;\ \ t \le 0-$$

$$U(t) = 1;\ \ t \ge 0+$$

$$y(t) = KA + \left(y_o - KA\right)e^{-t/\tau};\ \frac{a_1}{a_o} = \tau$$



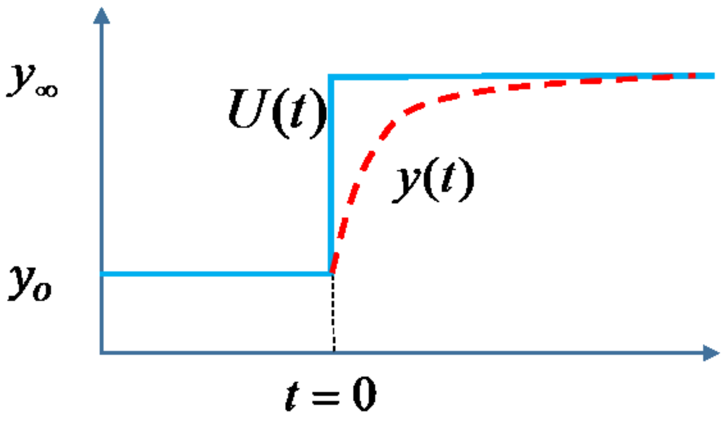




http://megapov.inetart.net/manual-1.2.1/img/con_vk.png

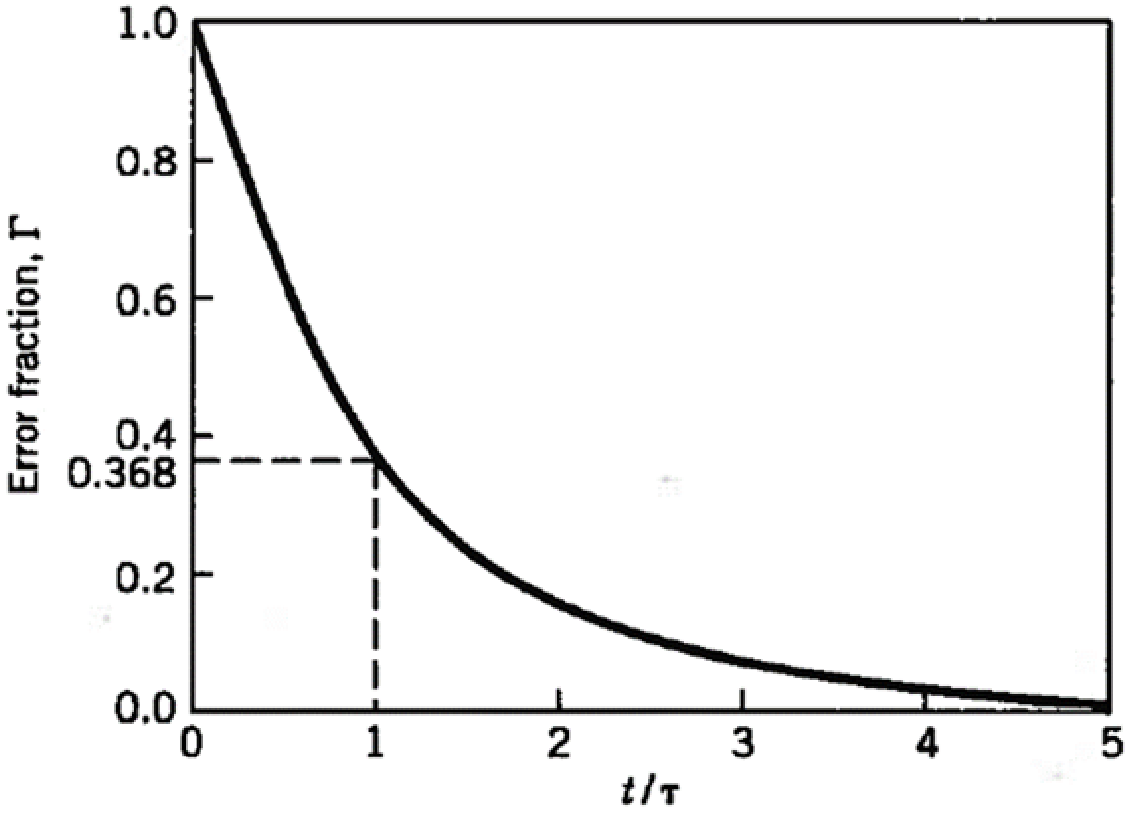
http://www.electronicspoint.com/attachments/seriesrc-gif.12542/

Use of error fraction to determine time constant, τ.



$$\Gamma = \frac{y_\infty - y(t)}{y_\infty - y_o} = e^{-t/\tau}$$

This is different than the book expression. I think it makes more intuitive sense.



Three ways to determine the time constant from $\Gamma(t)$ data

- Find the time corresponding to $\Gamma = 0.368$ (i.e. $1/e$)
  - If there is noise on the data, there is time uncertainty about when $\Gamma = 0.368$.
  - Uncertainty in $t = 0$.
- Find the initial slope of Γ vs. t around $t = 0$. The slope is $-1/\tau$.
  - You must decide how far in time you should take the slope.
  - Uncertainty in $t = 0$.
- Fit Γ vs. t to an exponential.
  - Must truncate the fit when $\Gamma \rightarrow 0$ say at $\Gamma \sim 0.1$.
  - Must use correct fit i.e. an exponential is $y = Ae^{bx}$. The $A = 1$ for Γ vs. t
  - Uncertainty in $t = 0$.

Response of first order system to cyclic input

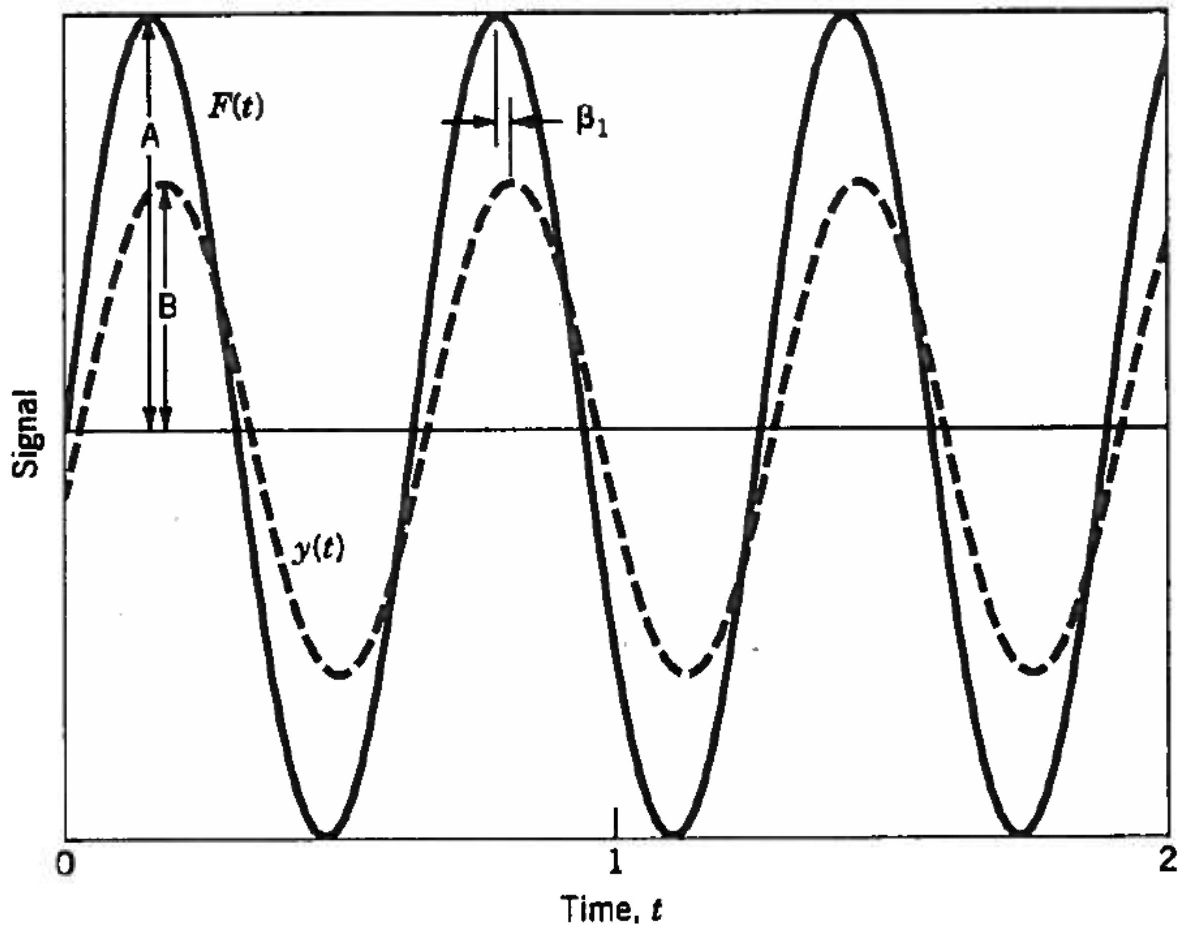
τ $\sin \omega t$

$$y = Ce^{-t/\tau} + B(\omega)\sin\left(\omega t + \Phi(\omega)\right)$$

$$B(\omega) = \frac{KA}{\sqrt{1+\left(\omega\tau\right)^2}};\ M\left(\omega\right) = \frac{B(\omega)}{KA}$$

$$\Phi(\omega) = -tan^{-1}\left(\omega\tau\right);\ \beta_1 = \frac{\Phi(\omega)}{\omega}$$

$$y = Ce^{-t/\tau} + B(\omega)\sin\left(\omega\left(t+\beta_1\right)\right)$$



$$y = Ce^{-t/\tau} + B(\omega)\sin\left(\omega t + \Phi(\omega)\right)$$

$$B(\omega) = \frac{KA}{\sqrt{1+(\omega\tau)^2}};\ M(\omega) = \frac{B(\omega)}{KA}$$

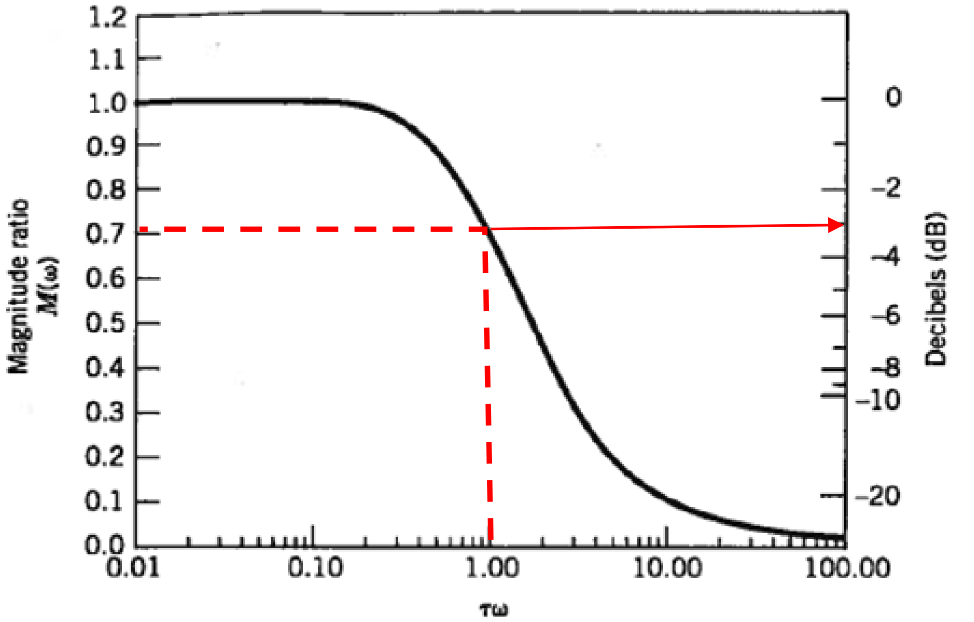
$$\Phi(\omega) = -tan^{-1}(\omega\tau);\ \beta_1 = \frac{\Phi(\omega)}{\omega}$$

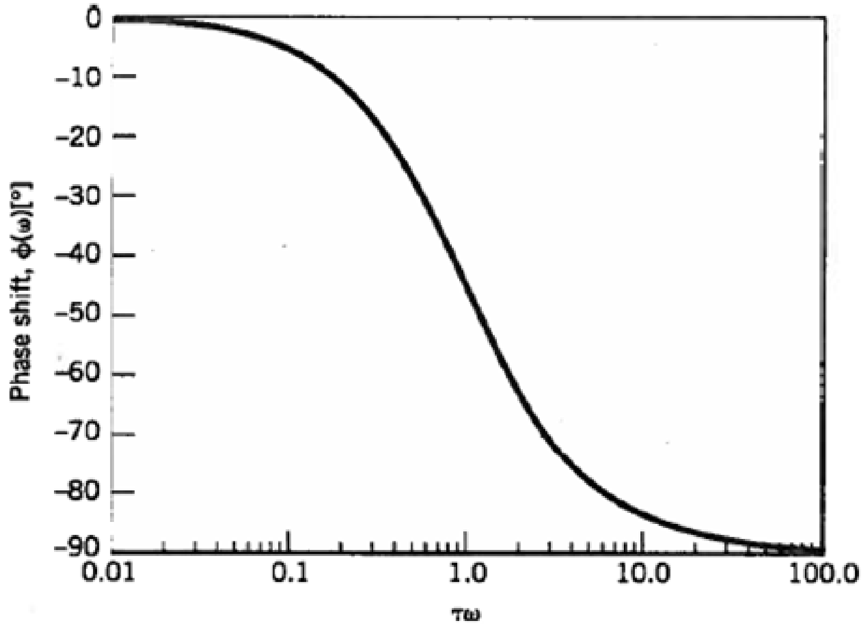
$$y = Ce^{-t/\tau} + B(\omega)\sin\left(\omega(t+\beta_1)\right)$$

M is the magnitude ratio. It compares the input, KA, to the actual system response.

$$dB = 20\log M(\omega)$$

The value of M = KA/√2 = 0.7071KA at $\tau\omega$=1.
The dB is above -3 dB when $\tau\omega \leq 1$





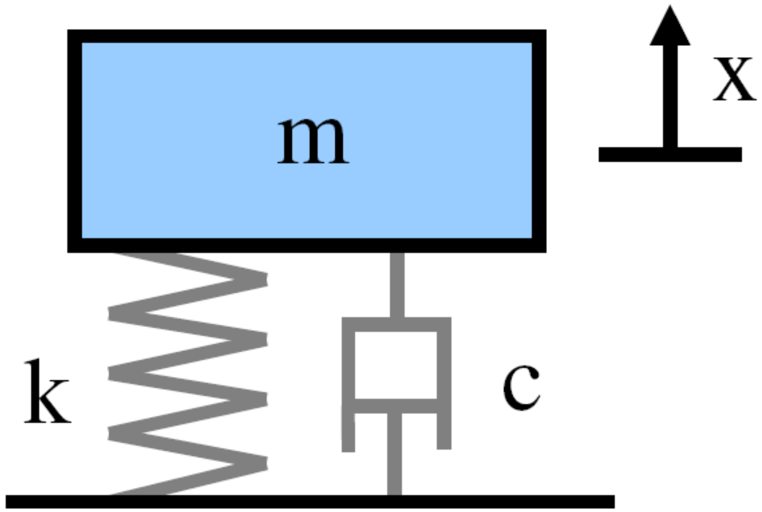
Response of second order systems to step and cyclic inputs

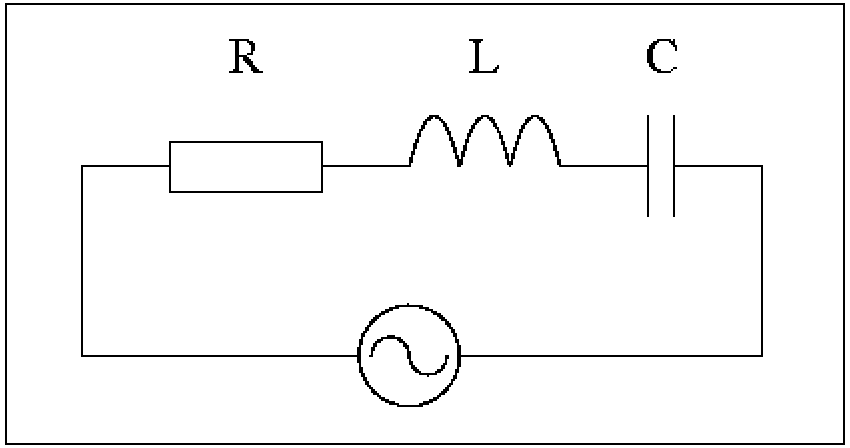
$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_o y = F(t)$$

$$m\ddot{\ } \qquad \dot{\ } \qquad F(t) \quad \text{mass–spring–damper}$$

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\,d\tau = V(t) \quad \text{RLC circuit}$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dV(t)}{dt}$$





Solutions to second order equation.

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_o y = F(t)$$

$$\frac{1}{\omega_n^2} \ddot{\ } \qquad \frac{\ }{\omega_n} \dot{\ } \qquad F(t)$$

$$\omega_n = natural\ \ frequency;\ \zeta = damping\ \ ratio$$

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$\mathrm{y}_n(t) = Ce^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t - \Theta\right) \ \ 0 \le \zeta < 1\ underdamped$$

$$\mathrm{y}_n(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \zeta = 1\ \ critically\ \ damped$$

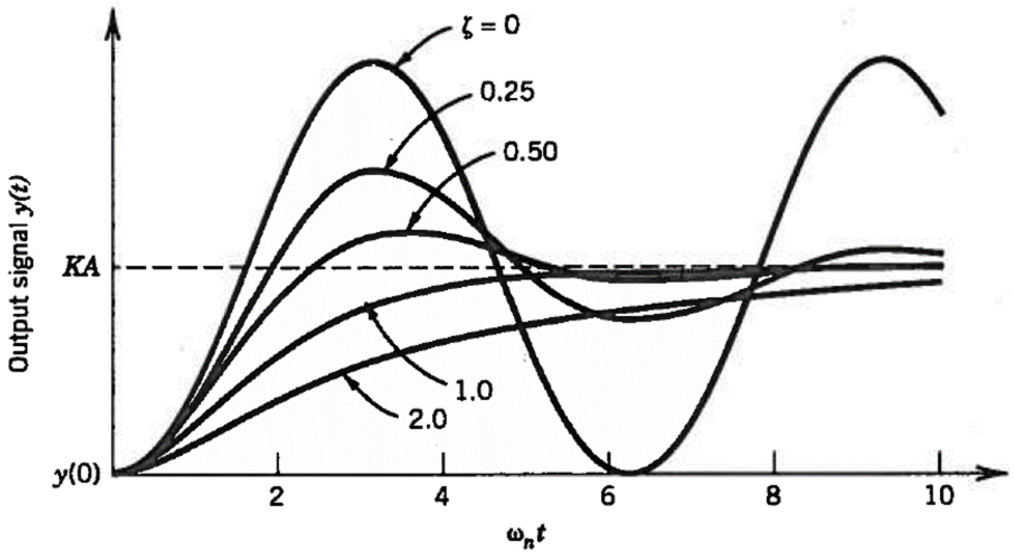
$$\mathrm{y}_n(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \zeta > 1\ overdamped$$

Step function response of a second order system

$$y(t) = KA - KAe^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2}\right) + \cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right] \quad 0 \le \zeta < 1$$

$$y(\mathrm{t}) = KA - KA\left(1+\omega_n t\right)e^{-\zeta\omega_n t} \quad \zeta = 1$$

$$y(t) = KA - K\left[\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + \frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}\right] \zeta > 1$$



Characterization of underdamped response

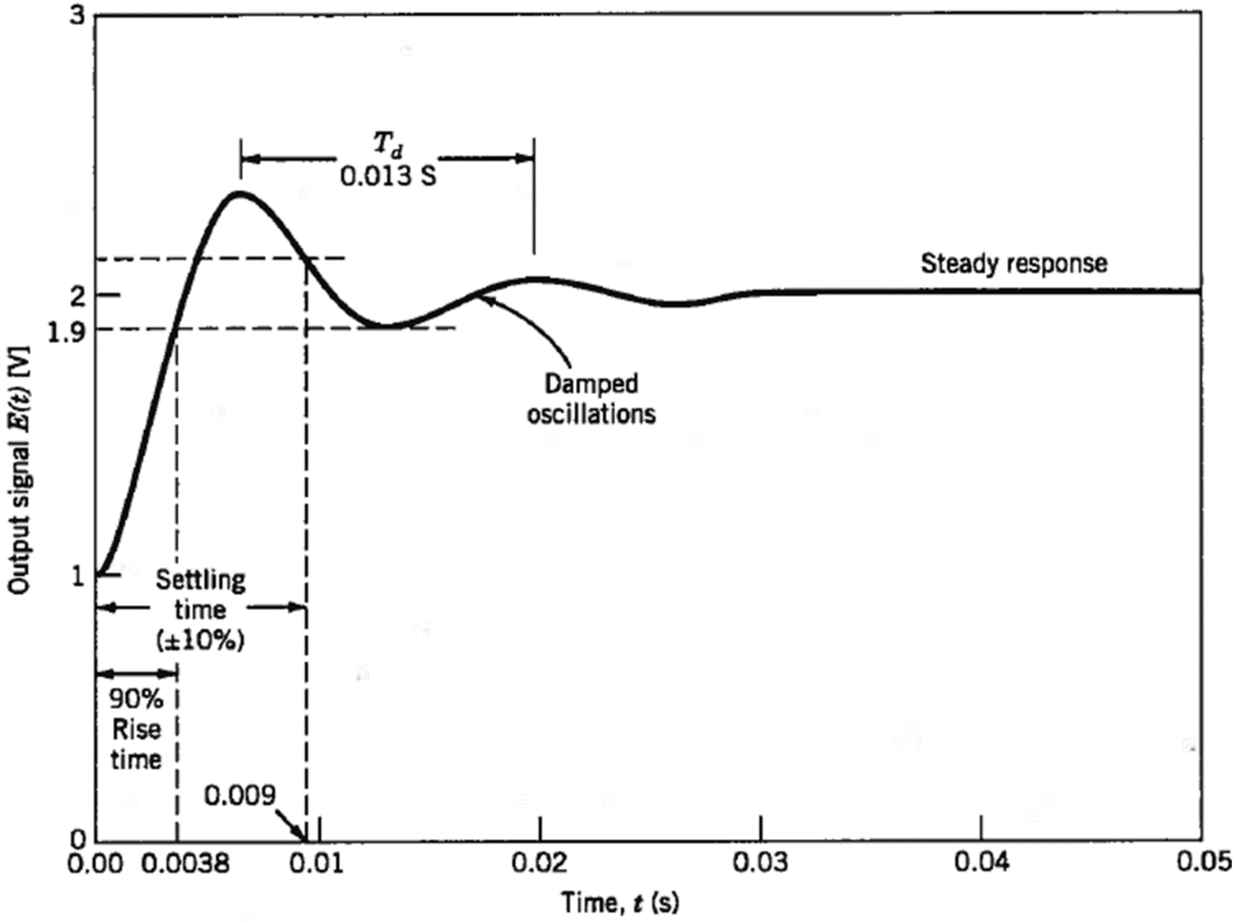
Damping period and damping frequency
Second order time constant

$$T_D = \frac{2\pi}{\omega_d}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\tau = \frac{1}{\zeta \omega_n}$$

The 90% rise time is self-explanatory. The settling time defined as signal being within some specified percentage of the final value. Some people define it as $\pm$10%, some define it as $\pm$2%,



How to experimentally determine the damping ratio?

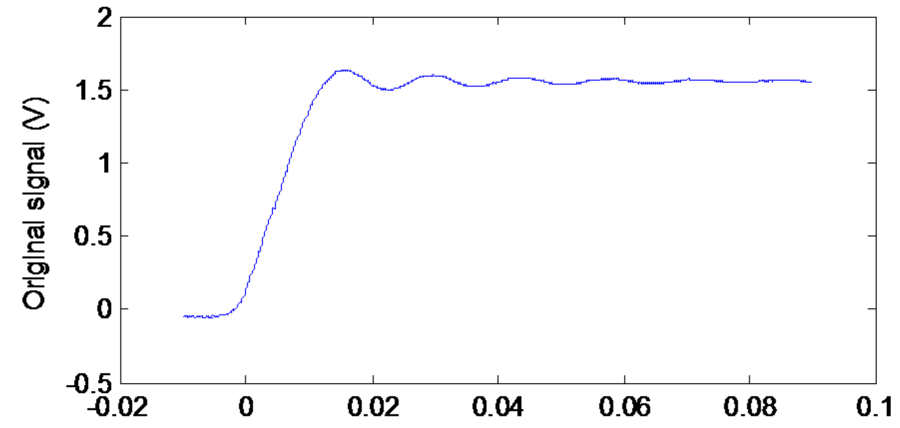
Amplitudes defined with respect to fully settled value.

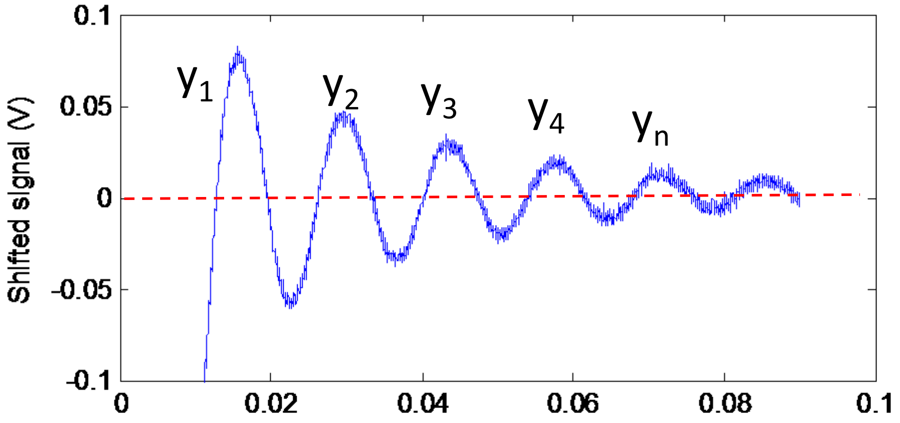
$$y_n = y(t) - KA$$

Damping ratio definition uses all of the peaks.

$$\zeta = \frac{\frac{1}{n-1}\ln\left(\frac{y_1}{y_n}\right)}{\sqrt{4\pi^2 + \left[\frac{1}{n-1}\ln\left(\frac{y_1}{y_n}\right)\right]^2}}$$

Ogata, page 395





Use of Matlab step function to simulate first and second order system response.
Ogata, Chapter 4 describes Transfer Function Approach to Modeling Dynamic Systems
See Ogata, Example 4-15, p. 144

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_o y = F(t)$$

Apply Laplace Transform

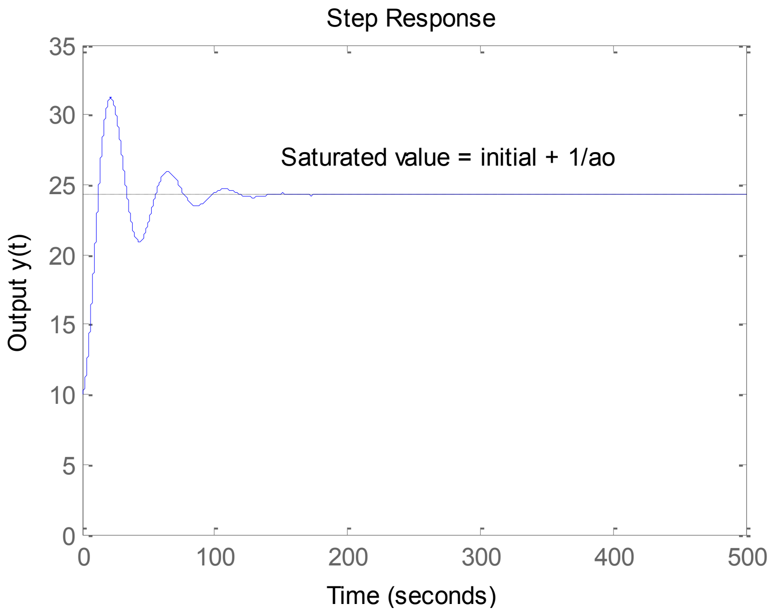
$$\left(a_2 s^2 + a_1 s + a_0\right) Y(s) = \mathrm{F(s)}$$

Find transfer function

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1/a_o}{\frac{a_2}{a_0} s^2 + \frac{a_1}{a_0} s + 1}$$

```
close all; clear all;
ao=.07;
a2=ao*45;
a1=ao*3;
num=1/ao;
den=[a2/ao a1/ao ao/ao];
t=0:0.01:500;
initialsig=10;
sys=tf(num,den);
step(sys+initialsig,t)
title ('Step Response')
xlabel('Time')
ylabel('Output y(t)')
axis([0 500 0 35])
text(150,27,'Saturated value = initial + 1/ao')
```



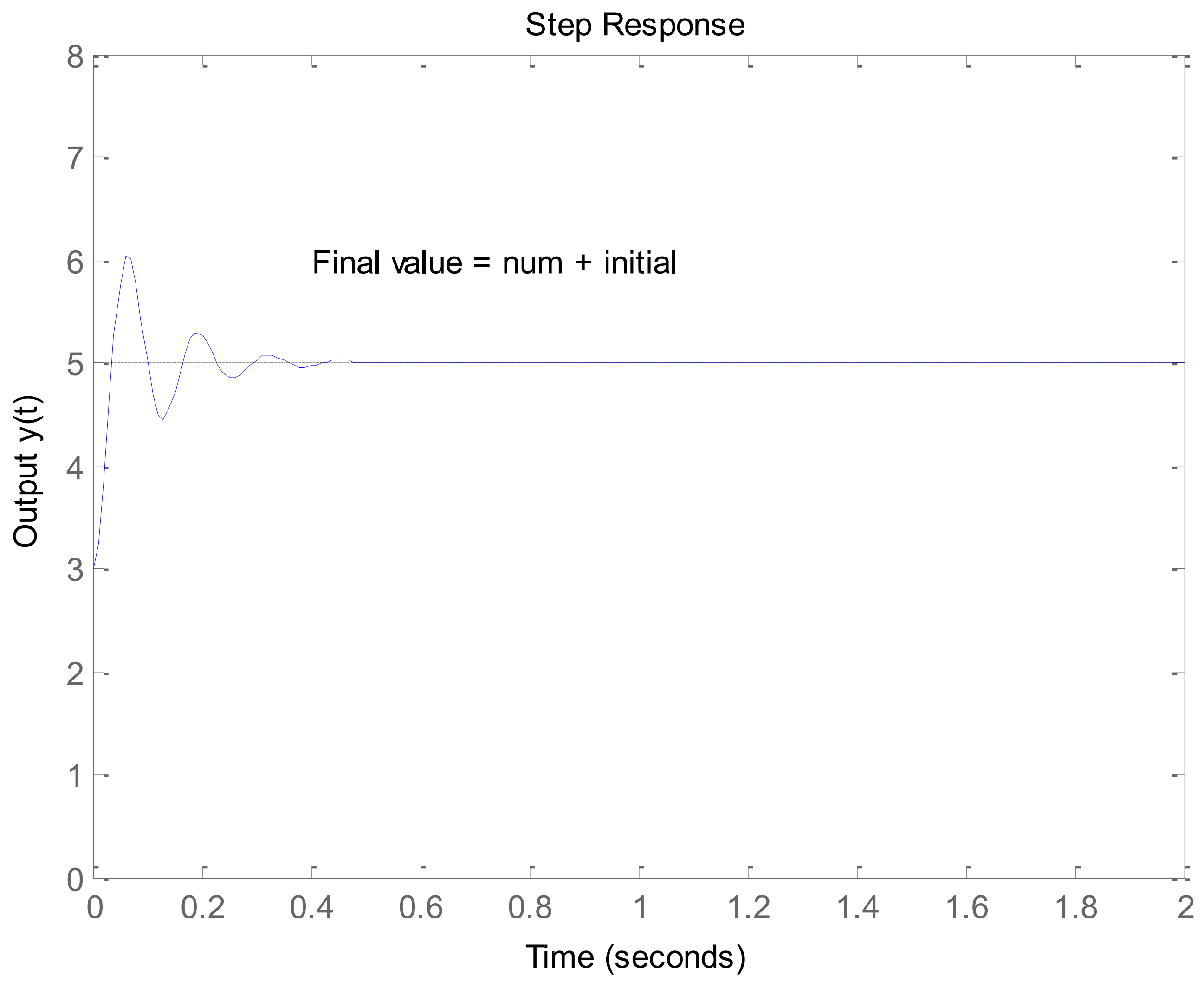
Calculating the transfer function for a generalized second order system

$$\frac{1}{\omega_n^2}\ddot{\ } \qquad \frac{2\zeta}{\omega_n}\dot{\ } \qquad\qquad F(t)$$

$$\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1\right)X(s)=KF(\mathrm{s})$$

$$\mathrm{G(s)}=\frac{X(s)}{F(s)}=\frac{K}{\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}s+1\right)}$$

```
close all; clear all;
wn=50;
zeta=0.2;
num=2; %K value
den = [1/wn^2 2*zeta/wn 1];
t=0:0.01:2;
initialsig=3;
sys=tf(num,den);
step(sys+initialsig,t)
title ('Step Response')
xlabel('Time')
ylabel('Output y(t)')
axis([0 2 0 8])
text(0.4,6,'Final value = num + initial')
```


Step Response
Final value = num + initial
Output y(t)
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5
6
7
8

Useful for Lab 3 and 6

```
close all; clear all;
wn=50; %Natural frequency
zeta=0.08; %Damping ratio
num=2; %Final value - initial value
den = [1/wn^2 2*zeta/wn 1]; %denominator of transfer function
t=0:0.01:2;% time vector, could be from your data
initialsig=3;% initial signal
sys=tf(num,den);%creates transfer function
step(sys+initialsig,t) %creates step response plot for transfer function, sys
% Could create vector containing system response
% y=step(sys,t) will create a y vector you could plot and process
% This would be useful to determine the difference between the experimental
% and predicted response e.g. diff=exp-y; would create a vector of residuals
title ('Step Response')
xlabel('Time')
ylabel('Output y(t)')
axis([0 2 0 8])
text(0.4,6,'Final value = num + initial')
```

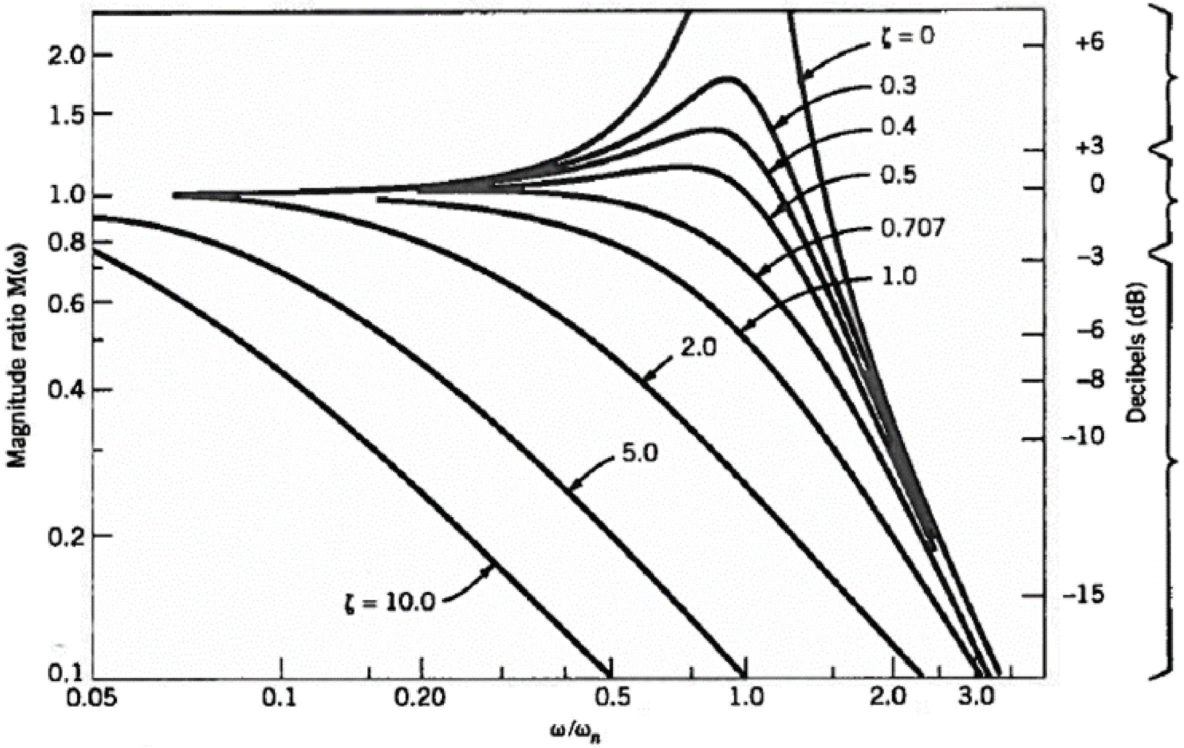
Solutions for underdamped case with sinusoidal forcing

$$y(t) = y_h + KAM(\omega)\sin\left(\omega t + \Phi(\omega)\right)$$

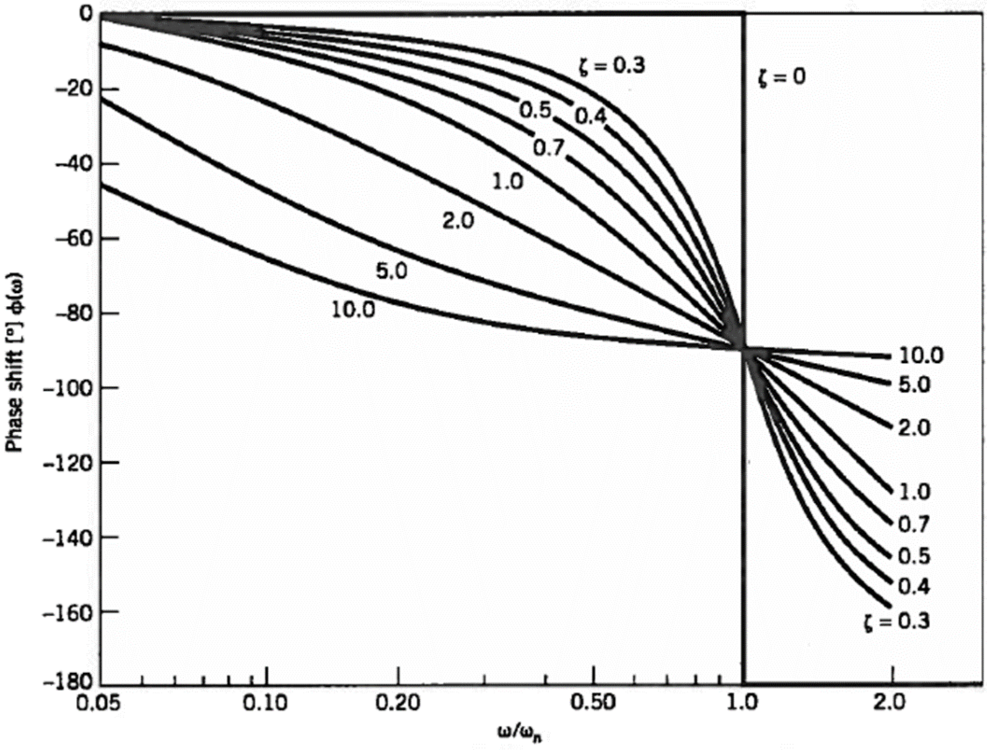
$$y_{ss}(t) = KAM(\omega)\sin\left(\omega t + \Phi(\omega)\right)$$

$$M(\omega) = \left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\zeta\omega / \omega_n\right]^2\right\}^{-1/2}$$

$$\beta_1 = \Phi / \omega = \frac{tan^{-1}\left(\frac{-2\zeta\omega / \omega_n}{1 - \left(\omega / \omega_n\right)^2}\right)}{\omega}$$



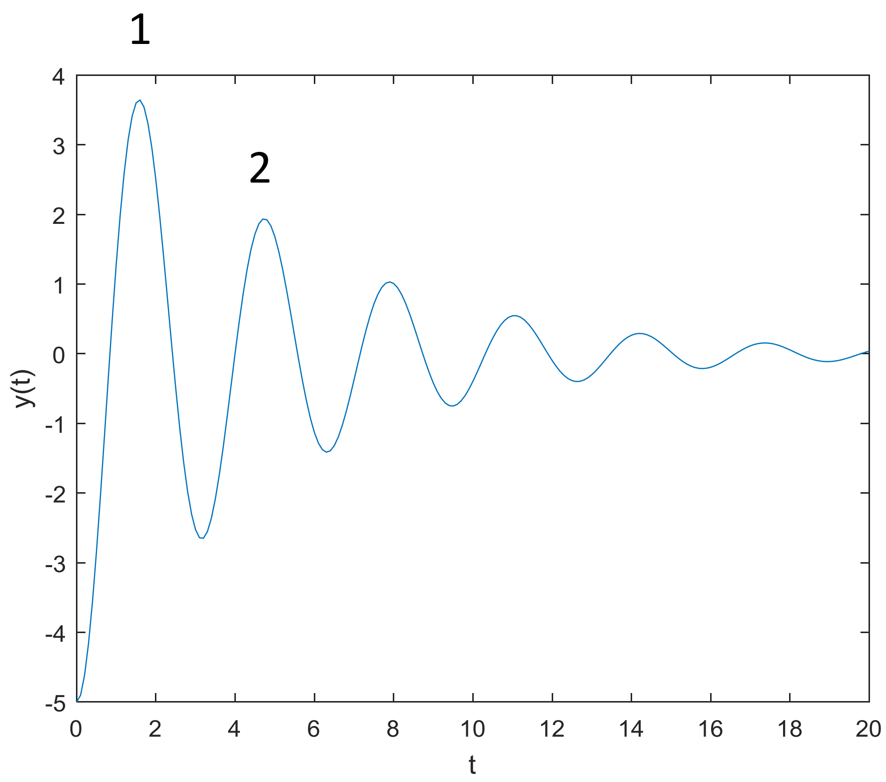
Magnitude ratio M(ω)
Decibels (dB)
ω/ωₙ
ζ = 0
0.3
0.4
0.5
0.707
1.0
2.0
5.0
ζ = 10.0
2.0
1.5
1.0
0.8
0.6
0.4
0.2
0.1
+6
+3
0
−3
−6
−8
−10
−15
0.05
0.1
0.20
0.5
1.0
2.0
3.0



Phase shift [°] φ(ω)
ω/ωₙ
ζ = 0.3
ζ = 0
0.5
0.4
0.7
1.0
2.0
5.0
10.0
10.0
5.0
2.0
1.0
0.7
0.5
0.4
ζ = 0.3
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
0.05
0.10
0.20
0.50
1.0
2.0

Problem 3.43 – A sensor mounted to a cantilever beam indicates beam motion with time. When the beam is deflected and released (step test), the sensor signal indicates that the beam oscillates as an underdamped second-order system with a ringing frequency of ω_d =10 rad/sec. The maximum displacement amplitudes are measured at three different times corresponding to the 1st, 16th, and 32nd and found to be 17, 9, and 5 mV. Estimate the damping ratio and natural frequency of the beam based on this measured signal assuming K = 1 mm/mV.

This is an initial condition problem. The equations are slightly different than that for a step input.



$$y(t) = KAe^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2}\right)+\cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right]$$

$$= KAe^{-\zeta\omega_n t}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)+\cos(\omega_d t)\right] \quad (\textit{note use of } \omega_d \textit{ and } \omega_n)$$

$$\text{Repeats when } t = \frac{2n\pi}{\omega_d} = \frac{2n\pi}{\omega_n\sqrt{1-\zeta^2}} \quad n = 1, 2 \ldots n$$

$$\mathrm{y}(n=1)=y_1=KAe^{-\zeta\omega_n\left(\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t\right)+\cos\left(\omega_d t\right)\right]$$

$$\mathrm{y}(n)=y_n=KAe^{-\zeta\omega_n\left(\frac{2n\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t\right)+\cos\left(\omega_d t\right)\right]$$

$$\frac{y_1}{y_n}=e^{-\zeta\omega_n\left(\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}-\frac{2n\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}=e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}(n-1)}=e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)(n-1)}$$

$$\ln\left(\frac{y_1}{y_n}\right)=\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)(n-1)$$

$$\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)=\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)=\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\left(\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)\right)^2=\frac{4\pi^2\zeta^2}{1-\zeta^2}$$

$$\left(1-\zeta^2\right)\left(\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)\right)^2=\zeta^2 4\pi^2$$

$$\left(\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)\right)^2=\zeta^2\left(4\pi^2+\left(\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)\right)^2\right)$$

$$\zeta=\frac{\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)}{\sqrt{4\pi^2+\left(\frac{1}{(n-1)}\ln\left(\frac{y_1}{y_n}\right)\right)^2}}$$

$$\ln\left(\frac{y_1}{y_n}\right)=\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)(n-1)$$

n	y	y_1/y_n
1	17	1
16	9	9/17
32	5	5/17

$$Y=\alpha(n-1)$$

$$\alpha=\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\alpha^2\left(1-\zeta^2\right)=4\pi^2\zeta^2$$

$$\alpha^2=\left(4\pi^2+\alpha^2\right)\zeta^2$$

$$\zeta=\frac{\alpha}{\sqrt{4\pi^2+\alpha^2}}$$