Homogenization of Elliptic Operators — in Modern Perspective

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Preface

The present book is divided into three parts. Part I is devoted to introduce the basic ideas and techniques related to qualitative homogenization. It's the writer' s belief that reading the classic books is indispensable for one who wish to be familiar with some branches of modern theoretic mechanics or mathematics. The reason is straightforward, classics have already become the solid basis, upon which further researches continue to develop. When compared with these classics, although new textbooks have the advantages that they're more freshman-friendly and sometimes contain recent progresses on the subject, researchers prefer to consult and refer to those classics in their own research papers. One cannot avoid reading them before reading research papers. However, it turns out that classic books and monographs are hard to be thoroughly understood due to the old date. This is the point where I grow up the idea that, instead of writing a brand-new book on the homogenization theory, it' s more beneficial for readers to read classic books rewritten in a modern way with clear motivations of applications along with detailed proofs of theorems. So basically, part I is designed to improve readability of the classic book written by J.-L. Lions et al. Part II concerns with the homogenization theory with emphasis on quantitative aspect. It draws heavily from the monograph by S. Zhongwei published in 2018. During this part, plentiful analyses are implemented to exhibit beautiful results regarding, say, regularity of solutions and rates of convergence. One who have no analytical bases, especially advanced PDE and harmonic analysis tools may find the massive techniques used here difficult to understand. Those who can't wait to examine the applications of homogenization theory can skip part II entirely for the moment, and come back there when indicated in the book in subsequent chapters. Some miscellaneous topics on applications of homogenization are chosen and presented in part III. We collect some original works by the author and colleagues. These topics are strongly biased by the author' s personal tastes and preferences, including Saint-Venant problem for composite cylinders, the connection with perturbation theory and some numerical procedures.

2022 April, in Shenzhen, China
Wu Simeng

Dedicated to my parents W. Huaijun
& L. Li
and my piggy ZRL qwq

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Part I

Periodic Homogenization

Chapter 1

A Brief Introduction to Homogenization

We are constantly confronted with composite materials in Mechanics, Physics, Chemistry and Engineering. To determine the macroscopic properties of periodic microstructures, one is led to the following boundary value problem: Given a family of partial differential operators A^ϵ, where ϵ is the ratio of micro-to-macro, a small quantity, the coefficients of A^ϵ are periodic functions in all or some variables.

$$\begin{cases} A^\epsilon u_\epsilon = f \ in \ \Omega \\ u_\epsilon \ \text{ subject to appropriate boundary conditions} \end{cases} \tag{1.1}$$

The point is, the coefficient function of operator oscillate very fast, making the question in hand difficult.

Throughout the 20th century, specialists in mechanics and mathematicians have developed a complete theory to handle ((1.1)), named “Homogenization”. Why chooses this name? Indeed, the general idea towards 1.1 consists of two steps:

1. One seek for an expansion of u_ϵ

$$u_\epsilon = u_0 + \epsilon u_1 + ... \tag{1.2}$$

2. find a convergence theorem as $\epsilon \to 0$

In many cases, with appropriate notion of convergence, $u_\epsilon \to u_0$ as $\epsilon \to 0$. u_0 is the solution of the “homogenized” boundary value problem.

$$\begin{cases} \mathcal{A}u_0 = f \text{ in } \Omega, \\ u_\epsilon \ \text{ subject to appropriate boundary conditions} \end{cases} \tag{1.3}$$

In general, $\mathcal{A}$ is a new partial differential operator with simple, even constant coefficient functions; it is called the homogenized operator of the family A^ϵ. Look at what we've done, we have converted the difficult B.V.P (1.1) with strongly oscillating coefficients to a much simpler one (1.3). We approximate u_ϵ by u_0 with the aid of the two steps mentioned above, then the problem is reduced to solve the homogenized problem. This justifies the name of this theory. The coefficients of $\mathcal{A}$ are called, by definition, the **effective coefficients** describing macroscopic properties of the underlying medium.

The explicit analytical construction of $\mathcal{A}$ is probably the most difficult part. It requires, typically, the solution of a B.V.P. within a single cell, which we refer to as the cell problem.

The classical procedure is drawn in the following:

To solve (1.1)

Becomes $\downdownarrows$

Construction of $\mathcal{A}$ by solving the cell problem

Becomes $\Downarrow$

To solve (1.3)

The last stage is classical and there're vast literatures available on market regarding to the much simpler B.V.P. To list a few, we recommend those who are totally not familiar with PDE to Wang & Tang’s lecture notes on undergraduate level PDE, for more mature researchers, graduate students and ambitious undergraduate students, we recommend them to Evans’ book which aims at graduate students in PDE fields.

This booklet mainly deals with the second procedure, i.e., to obtain $\mathcal{A}$ from A^ϵ, the most non-trivial procedure and novel idea in homogenization theory. More specifically, we deal with elliptic operators only.

We use **multiple scales** asymptotic expansions and **energy estimates** mostly.

The use of multiple scales is well known in many contexts including modern perturbation theory but may not be clearly and rigorously articulated. We assume there're at least two natural spatial length scales with one measuring variations within one period cell (the **fast** scale) and the other measuring variations within the whole region of interest (the **slow** scale). It's worth noting that the theory is equivalent to averaging method in ODE and is recently exploited in transport theory.

Energy estimates overcome the arising difficulty of ϵ-independent estimates of derivatives of coefficients of A^ϵ caused by its rapidly oscillating nature. To this end, one must pass to the limit in weak sense. One uses integration by parts and suitable test functions to achieve the goal.

Chapter 2

Setting of the Model Problem

2.1 Description of the operators A^{ϵ}

Boundary setting

The domain considered here is a bounded, open and connected set $O \subset \mathbb{R}^n$ with C^1-boundary (very important), or alternatively speaking, bounded domain O is of C^1-class.

Definition 2.1 (Boundary Regularity)**.**

(i) We say that $\partial\Omega$ is C^k-smooth, $k \geq 1$, if $\forall x \in \partial\Omega, \exists r > 0$ such that after moderate rotation and translation,

$$\Omega^+ \triangleq B_r(x_0) \cap \Omega = \{x \in B_r(x_0)\,|x_n > \varphi(x_1, ..., x_{n-1})\,\}$$

$$B_r(x_0) \cap \partial\Omega = \{x \in B_r(x_0)|x_n = \varphi(x_1, ..., x_{n-1})\}$$

for some $\varphi \in C^k(\mathbb{R}^{N-1})$

(ii) Similarly, we say $\partial\Omega$ is Lipschitz continuous if φ is a Lipschitz continuous function, i.e.,

$$|\varphi(x') - \varphi(y')| \leq C\,|x' - y'|\,, \quad \forall x', y'$$

C is a positive constant.

In other words, near x, the boundary is the local graph of a C^k (resp. Lipschitz continuous class) functions, and the domain lies above the graph in x_n-direction.

Remark 2.1.

(i) Boundaries of open balls in $\mathbb{R}^n$ are of C^∞-class; Polyhedrons, cubes in $\mathbb{R}^n$ are of Lipschitz continuous class.

(ii) Let's mention that if boundary $\partial\Omega$ is of C^1-class, then unit outer normal vector field exists everywhere on the boundary.

(iii) If the boundary is of Lipschitz continuous class, then one can define surface measure on $\partial\Omega$. In particular, $L(\partial\Omega)^2$ is well defined almost everywhere with respect to the surface measure.

As we will encounter the global regularity estimation, curved boundary is complicated to perform analysis on. For a later use, we now state how to "stretch" the boundary into "straight" subspace.

Theorem 2.1 (Straightening the Boundary)**.**

The boundary Γ can be straightened by a C^k-class function.

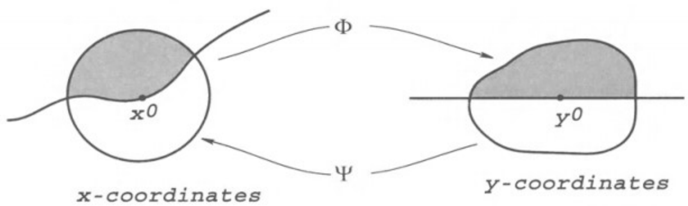

Figure 2.1: Straighten the boundary

Function setting

The homogenization theory relies on periodicity, to illustrate the idea, we shall define the periodic cell.

Definition 2.2 (Cell)**.**

$$Y = \prod_{j=0}^{n}(0, y_j^\circ) \ \subset \mathbb{R}^n$$

an open rectangular in $\mathbb{R}^n$, is said to be a cell.

Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Y-periodic if is admits period y_j° in y_j-component, $j = 1, ..., n$. To be more specific, f is Y-period if and only if

$$f(x + ky_j^\circ e_j) = f(x), \quad \forall k \in \mathbb{Z},$$

e_j is the standard basis of $\mathbb{R}^n$.

Now we give the precise description of the operators of our interest.

Consider functions $a_{ij}(y): \ O \to \mathbb{R},\ i,j = 1,...,n$ such that:

$$\begin{cases} (periodic) & a_{ij}\ is\ Y-periodic \\ (essentially\ bounded) & a_{ij} \in L^\infty(\mathbb{R}^n) \\ (elliptic)\ \ \displaystyle\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \sum_{i,j=1}^{n} \alpha\xi_i^2 \quad a.e.\ in\ y,\ \forall \boldsymbol{\xi} \in \mathbb{R}^n,\ \alpha > 0 \end{cases} \tag{2.1}$$

We shall make some explanations.

The first condition says that what we' re going to consider below are restricted to cases where the coefficient matrices consist of the highest-order derivatives are periodic in all n variables $y_j,\ j = 1,...,n$ for different period. This is actually a very strong condition to be guaranteed for technical usage. Naturally one may ask whether we can drop the periodicity condition, the answer is definite. But cases there are more complicated to contend with, and advanced perturbation techniques are required. Therefore, we postponed the discussion to the last chapter. To the best knowledge of the author, quasiperiodic operator is the most general case that can be addressed in existing literatures.

The second condition says that once i and j are fixed, function a_{ij} is bounded except on a set with zero Lebesgue measure. This is constantly referred to as "essentially bounded" because we have $|f(x)| \leq \|f\|_\infty \ a.e.\ on\ \mathbb{R}^n$. It occurs naturally based on the fact: whenever we' re dealing with the weak solution of some PDEs, we shall always confine the coefficients of the operator to be essentially bounded.

Note that the last condition in (2.1) is precisely the ellipticity condition for a 2^{nd}-order equation. LHS (left hand side) = quadratic form associated with the coefficient matrix $(a_{ij})_{n\times n}$ and any vector ξ; RHS (right hand side) = $\alpha|\xi|^2$. It says that the coefficient matrix $(a_{ij})_{n\times n}$ is positive definite (possibly not symmetric, for now) and all of its eigenvalues have uniform, positive lower bound $\alpha > 0$.

From another aspect, the last inequality in (2.1) implies the invertibility of coefficient matrix $A(x)$ a.e. on $\mathbb{R}^n$. Indeed, theorem 7.C. of Shelton Axler proved that A is positive operator, i.e., $(Av, v) \geq 0$ if and only if A is self-adjoint, then by spectral theorem,

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

with all the eigenvalues $\lambda_i \geq 0$. Now substitute $\xi_i = (0, ..., 1, 0, ..., 0)$ with 1 appears on the i-th slot, the coercivity then shows that $\lambda_i > 0$ strictly.

Remark 2.2. Some people define ellipticity as

$$\sum a_{ij}\xi_i\xi_j > 0, \quad \forall \text{ nonzero vector } \xi \tag{2.1'}$$

This is actually equivalent to the third item of our definition (2.1). One direction is obvious, it suffices to consider how (2.1') implies the 3rd item of (2.1). Indeed, $A = (a_{ij})_{n\times n}$ is a real symmetric matrix, by linear algebra, $A = N^T BN$, where

$$B = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

is a diagonal matrix with positive eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ appearing on the diagonal, and N is an orthogonal matrix representing the change of basis. Probably the most important feature of orthogonal matrix is that it preserves inner products, in particular, preserves norms. Therefore, $|N\xi|^2 = |\xi|^2$. We shall find a constant $\lambda_0 > 0$ such that the following inequality holds:

$$\xi^T A\xi > \lambda_0|\xi|^2$$

Let $\eta \triangleq N\xi$, it becomes $\lambda_1\eta_1^2 + \cdots + \lambda_n\eta_n^2 > \lambda_0\eta_1^2 + \cdots + \lambda_0\eta_n^2$. Taking $\lambda_0 = min\{\lambda_1, ..., \lambda_n\}$ gives what we want.

In an analogous way, let another function a_0 satisfy

$$\begin{cases} a_0 \ is \ Y - periodic \\ a_0 \ \in L^\infty(\mathbb{R}^n) \\ a_0(y) \geq a_o > 0 \quad a.e. \ y, \ a_o \in \mathbb{R} \end{cases} \tag{2.2}$$

Define a family of divergence form operators associated with functions a_{ij}, a_0:

$$A^\epsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}(\frac{x}{\epsilon})\frac{\partial}{\partial x_j}\right) + a_0(\frac{x}{\epsilon}) \tag{2.3}$$

Einstein summation convention is always assumed without specification. Don't ignore the "-" in front of the first term on the RHS. Here $\epsilon > 0$ is a parameter ranging from $[0, \epsilon_0)$.

Remark 2.3.

(i) When applying A^ϵ to $u \in$ some function spaces, we first take derivative w.r.t (abbr. with respect to) x_j, multiply to a_{ij}, then take derivative w.r.t x_i. Again, the process is summed up both in indices i and j as required by Einstein summation.

(ii) Why we call these operators divergence form? It's readily checked that the first term in RHS of (**??**) is $-div\Big(A(\nabla)\Big)$, i.e., coefficient matrix as a linear map transforms the gradient vector $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_j}, ..., \frac{\partial}{\partial x_n})$ into a new vector before taking divergence div of the transformed vector.

(iii) By constructions of a_{ij}, a_0, the family $\{A^\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ consists of 2nd-order elliptic operators that are uniformly elliptic in parameter ϵ.

(iv) Though in (**??**) we restrict $a_0 > 0$, sometimes $a_0 = 0$ is allowed.

The core task we deal with can now be formulated in a clearer way: For each ϵ, suppose u_ϵ is the solution (in classical sense or weak sense) of the following boundary value problem

$$(B.V.P.)\begin{cases} A^\epsilon u_\epsilon = f \ in \ O \\ u_\epsilon \ subject \ to \ boundary \ conditions \ on \ \Gamma = \partial O \end{cases} \tag{2.4}$$

we study the behavior of u_ϵ as $\epsilon \to 0$.

2.2 Variational formulation of the problem

The variational formulation (in weak sense) is necessary for, even if a_{ij} are smooth, which is hardly satisfied in technical use. Derivatives of $a_{ij}(\frac{x}{\epsilon})$ will still be of order $\frac{1}{\epsilon}$. Therefore ϵ-independent priori estimates on u_ϵ can be obtained only by the use of variational formulation.

We begin with some usual function spaces. Those who are unfamiliar with these concepts are referred to Chapter 9 of H. Brezis’ s book and the references therein.

Let $H^1(O) = \left\{ v \,|v, v_{x_1}, ..., v_{x_n} \in L^2(O) \right\}$ be the Sobolev space $W^{1,2}(O)$, with all-order weak derivatives belonging to $L^2(O)$ space. Letter “H” stands for a Hilbert space –a complete normed vector space. Here H^1 is such a vector space accompanied with the following inner product

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right) = \int_O uv + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \tag{2.5}$$

The associated norm is

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{1/2} \tag{2.6}$$

$H_0^1(O)$ is defined to be the closure of $C_0^\infty(O)$ in $H^1(O)$ under the usual topology.

Proposition 2.1. Under the assumption that O is of C^1-class, we have

$$H_0^1(O) \cap C(\bar{O}) = \{v | v \in H^1(O) \cap C(\bar{O}), \quad v = 0 \ on \ \Gamma\} \tag{2.7}$$

Proof. See H. Brezis's book for a similar statement. □

Moral of the story: The intersection with the set of all continuous functions on $\bar{O}$ is vital here. Roughly speaking, $H_0^1(O)$ consists of functions that “are 0 on Γ”. We know that changing values of function on a zero-measure set has no impact on Lebesgue integral, this directly implies that Sobolev spaces, which are defined by Lebesgue integral, are rather made up of equivalent classes of functions, not usual functions. These functions are equivalent if and only if they only differ on a zero-measure set. Thus, it is subtle to say that $v = 0 \ on \ \Gamma$ because *Gamma* is a zero-measure set, and ν could be arbitrary value on Γ! However, if set O is smooth enough and the functions are made to be continuous, we see that there does exist functions in $H_0^1(O)$ that vanish on Γ.

Furthermore, one can handle with (2.7) under trace meaning. Trace theorem gives a unique way to understand what it means that vanishing on boundary.

Theorem 2.2 (Trace)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. Then, there exists a unique linear continuous (= bounded) map*

$$T: \quad H^1(\Omega) \to L^2(\partial\Omega)$$

such that if $u \in H^1(\Omega) \cap C(\bar{\Omega})$, then $T[u](x) = u|_{\partial\Omega}(x)$ for all $x \in \partial\Omega$.

Proof. See L.C. Evans, chapter5, section5, Theorem1 and Cioranescu Theorem 3.28. □

Corollary 2.1 (Characterization of $H_0^1(\Omega)$)**.**

$$H_0^1(\Omega) = \left\{ u \,\middle|\, u \in H^1(\Omega), \quad T[u] = 0 \right\} \tag{2.8}$$

Theorem 2.3 (Sobolev Imbedding)**.** *Here we list 3 function spaces imbedding relations:*

(i)

$$1 \tag{2.9}$$

(ii)

$$1 \tag{2.10}$$

(iii)

$$1 \tag{2.11}$$

Proof. You find the proof in any advanced PDE textbooks. □

On $H_0^1(O)$, as a corollary of above theorem, we have the famous and useful Poincaré’ s inequality:

Corollary 2.2 (Poincaré)**.** *It shows that, there’ s a constant C, independent of u, such that*

$$\forall u \in H_0^1(O), \quad \|u\|_p \leq C\|\nabla u\|_p, \tag{2.12}$$

where $C = C(\Omega)$ is a function of the geometry of the domain, more precisely, the diameter of domain.

The above inequality shows that, L^p-norm of any u can be totally controlled by L^p-norm of its gradient. By Poincaré' s inequality, one can show that $\|\nabla v\|_{L^2}$ is an equivalent norm of $\|v\|_{H_0^1}$. See exercise 1.

Next, we introduce tools from functional analysis. Then explain how they' re used in solving variational equation.

Every A^ϵ associates with a bilinear form. An appropriate choice of bilinear benefits our study on existence theorem a lot, thanks to the famous Lax-Milgram Theorem that we' re going to show.

Definition 2.4. Suppose V is a real Banach space. Map $a : V \times V \to \mathbb{R}$ is called a bilinear form if and only if

$$a(u, \cdot) : V \ni v \mapsto a(u, v) \in \mathbb{R}$$
$$a(\cdot, v) : V \ni u \mapsto a(u, v) \in \mathbb{R}$$

are both linear maps.

Definition 2.5. The bilinear form $a(u, v)$ on $V \times V$ is said to be

- bounded, if $|a(u, v)| \leq C||u||_V||v||_V, \quad \forall u, v \in V$
- coercive, if $a(u, u) \geq \lambda_0||u||_V^2, \quad \forall u \in V$
- symmetric, if $a(u, v) = a(v, u), \quad \forall u, v \in V$
- positive, if $a(u, u) \geq 0, \quad \forall u \in V$

Proposition 2.2 (for bilinear form, bounded = continuous)**.** Let $a(u, v)$ be a bilinear form, then the following statements are equivalent: (i) a is bounded, i.e., $|a(u, v)| \leq C||u||_V||v||_V, \ \forall u, v \in V$, for some constant $C > 0$. (ii) a is continuous on $V \times V$, i.e., $|a(u_1, v_1) - a(u_2, v_2)| \leq \epsilon$, provided u_1 is close to u_2, and v_1 is close to v_2.

Proof. □

Now let H be a Hilbert space, necessarily H is a real Banach space. Suppose $F \in H^*$, $*$ denotes the dual space, the variational problem is to find an $u \in H$ such that the variational equation

$$a(u, v) = \langle F, v\rangle_{H^*,H}, \quad \forall v \in H \tag{2.13}$$

holds

The following theorem is well-known, it has been constantly taught in courses *Linear Algebra* in finite dimension version and *Functional Analysis* in infinite dimension version.

Theorem 2.4 (Riesz Representation)**.**

Proof. We show the general Poincaré' s inequality based on the perspective of eigenvalue. □

When talking about variational method, one recalls the direct method at a glance. Direct method requires some functionals to be minimized.

Let H be a Hilbert space, define functional on H as

$$J(u) = \frac{1}{2}a(u, u) - \langle F, u\rangle_{H^*,H} \quad \forall u \in H \tag{2.14}$$

Our goal is to look for a $u \in H$ such that

$$J(u) = \inf_{w\in H} J(w) \tag{2.15}$$

Theorem 2.5 (Minimization of functionals)**.** *Suppose H is a Hilbert space, $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ is a bounded, symmetric and positive bilinear form,*

Then, $u \in H$ solves the variational equation (2.13) if and only if u solves (2.15)

Proof. □

Troubles occur as the symmetry of the bilinear form is broken. Naturally this provoke the famous Lax-Milgram theorem.

Theorem 2.6 (Lax-Milgram)**.**

L-M theorem greatly generalizes the applicablility of Riesz's theorem.

For our operators defined in previous chapter, define the family of bilinear form depending on parameter ϵ:

$$a^{\epsilon}(u,v) = \int_O a^{\epsilon}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx \; + \int_O a_0^{\epsilon} uv\,dx \tag{2.16}$$

Superscript stands for $a_{ij}^{\epsilon}(x) = a_{ij}(x/\epsilon)$.

To apply L-M theorem, one has to check boundedness and coercivity of (2.16). Let's prove it in detail.

Boundedness of $a^\epsilon(\cdot,\cdot)$: By imposition on a_{ij} (2.1) and a_0 (2.2),

$$
\begin{aligned}
|a^\epsilon(u,v)| &= \left| \int_O a^\epsilon(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx \ + \int_O a_0^\epsilon uv\,dx \right|, \text{(Einstein summation)}\\
&\le \left| \int_O \sum_{i,j} a_{ij}^\epsilon(x) \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx \right| \\
&\qquad + \left| \int_O a_0^\epsilon uv\,dx \right|, \text{(triangle inequality for real numbers)}\\
&= \left| \int_O \sum_i w_i v_{x_i}\,dx \right| + \left| \int_O a_0^\epsilon uv\,dx \right|, \quad w_i \triangleq \sum_j a_{ij}^\epsilon u_{x_j}\\
&\le \int_O \left| \sum_i w_i v_{x_i} \right| \,dx \\
&\qquad + \int_O |a_0^\epsilon uv| \,dx, \text{(triangle inequality for Lebesgue integrals)}\\
&\le \int_O |w||v_{x_i}| \\
&\qquad + \|a_0^\epsilon\|_\infty |u||v|\,dx, \text{(Cauchy-Schwartz inequality for vectors)}\\
&\le \int_O M_1 |\nabla u||\nabla v| + \|a_0^\epsilon\|_\infty |u||v|\,dx, \quad M_1 \triangleq \left(\sum_{i,j} \|a_{ij}^\epsilon\|_\infty^2 \right)^{1/2}\\
&\le M_1 \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|a_0^\epsilon\|_\infty \|u\|_{L^2}\|v\|_{L^2}, \text{(Hölder's inequality)}\\
&\le M \|u\|_{H^1}\|v\|_{H^1},\\
&\qquad M = max\{M_1, \|a_0^\epsilon\|_\infty\} > 0, \text{(mean-value inequality)}.
\end{aligned}
\tag{2.17}
$$

Then the bilinear form $a^\epsilon(u,v)$ is controlled by norms of u and v with a strict positive coefficient M. Note that we have used the condition $a_{ij}^\epsilon, a_0 \in L^\infty(O)$. Boundedness fails if this condition is dropped.

Coercivity of $a^\epsilon(\cdot,\cdot)$: Since we seek for a lower bounded of $a^\epsilon(\cdot,\cdot)$, the first term of RHS of (2.16) must be controlled by some assumptions. Recall the ellipticity condition (2.1) for a^{ϵ}_{ij}.

$$\begin{aligned} a^{\epsilon}(u,u) &\geq \alpha \int_{O} |\nabla u|^2 + a^{\epsilon}_0 u^2 \, dx, \quad \alpha > 0 \\ &\geq min\{\alpha, a_o\} \|u\|^2_{H^1} \\ &\geq 0 \end{aligned} \tag{2.18}$$

The last inequality is guaranteed by (**??**), $a_0 \geq a_o \geq 0 \quad a.e.$

Next, we introduce a closed subspace V of $H^1(O)$ such that $H^1_0(O) \subseteq V \subseteq H^1(O)$.

Let $\Gamma_0 \subseteq \Gamma$ be a smooth subset of Γ with positive surface measure, we define V to be

$$\begin{aligned} V = \Big\{ v \in H^1(O) \Big| &\text{if } v \in H^1(O) \cap C(\overline{O}), \\ &\text{then } v = 0 \text{ on } \Gamma_0, \frac{\partial v}{\partial \vec{\nu}} = 0 \text{ on } \Gamma \backslash \Gamma_0 \Big\} \end{aligned} \tag{2.19}$$

If $a^{\epsilon}_0 = 0$, then

$$a^{\epsilon}(u,v) = \int_{O} a^{\epsilon}_{ij}(x) u_{x_j} v_{x_i} \, dx \tag{2.20}$$

is the degenerate bilinear form. We claim that (2.20) is still bounded and coercive. Indeed, the boundedness follows in analogous way to (2.17), so it suffices to prove that

$$a^{\epsilon}(u,u) \geq c\|u\|^2_{H^1}, c > 0, \forall u \in V \tag{2.21}$$

The trick is that, if we are handling with H^1_0 space, we can control the missing term $\int |u|^2 = \|u\|^2_{L^2}$ by norm of its gradient with the help of Poincaré' s inequality, but in general, $V \supset H^1_0$. So, we reasonably hope that Poincaré' s inequality still holds on the larger space V. It turns out that our idea is true, provided $V \subset H^1$ strictly!

Proposition 2.3. Suppose $V \subset H^1$ strictly, then $a^{\epsilon}(u,u) \geq c\|u\|^2_{H^1}, \forall u \in V$. $c > 0$ is a constant independent of ϵ.

Proof. We show the general Poincaré’ s inequality based on the perspective of eigenvalue. □

2.3 The 1-dimensional case

Let $\Omega = (d_1, d_2)$ be an interval in $\mathbb{R}$ and consider the problem

$$\begin{cases} -\dfrac{d}{dx}\left(a^\epsilon \dfrac{du^\epsilon}{dx}\right) = f, & \text{in } (d_1, d_2) \\ u^\epsilon(d_1) = u^\epsilon(d_2) = 0 \end{cases} \tag{2.22}$$

where, as will continue to be used in the rest of the book, $a^\epsilon(x) = a\left(\frac{x}{\epsilon}\right)$.

We assume $a(x) > 0$ is a positive function in $L^\infty(0, l_1)$ such that

$$\begin{cases} a \text{ is } l_1\text{-periodic} \\ 1 < \alpha \leq a(x) \leq \beta < +\infty \end{cases} \tag{2.23}$$

where α, β are constants. Then we have the following result,

Theorem 2.7. *Let $f \in L^2(d_1, d_2)$ and a^ϵ satisfies (2.23). Let $u^\epsilon \in H_0^1(d_1, d_2)$ be the solution of problem (2.22). Then,*

$$u^\epsilon \rightharpoonup u^\circ \quad \textit{weakly in } H_0^1(d_1, d_2)$$

where u° is the unique solution in $H_0^1(d_1, d_2)$ to problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_{(0,l_1)}(\frac{1}{a})} \dfrac{du^\circ}{dx}\right) = f, & \textit{in } (d_1, d_2) \\ u^\circ(d_1) = u^\circ(d_2) = 0 \end{cases} \tag{2.24}$$

here $\mathcal{M}_\Omega(f) := \frac{1}{|\Omega|}\int_\Omega f(y)\,dy$ is the integral mean value of function f over region Ω.

Proof. □

Before going through next section, let's remark on the homogenized problem (2.24).

Chapter 3

Exercises

(1) Try to show that $\|\nabla v\|_{L^2}$ is an equivalent norm of $\|v\|_{H_0^1}$

Part II

Stochastic Homogenization

Chapter 4

Digest of Probability Theory

4.1 Probability space and properties

4.2 Random variables

Chapter 5

Digest of Stochastic Process and Ergodicity

Part III

Computational Homogenization

Chapter 6

Review on HMM

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