# Continuum Mechanics - A (2022 Fall)

**(Reference Solutions Manual)**

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评分总则：满分 200，共 7 大题，12 小问。每小问 16 分, 总计 $12 \times 16 = 192$ 分，剩余 8 分为保底分。按照预期每次作业占总评 100 分的 2 分，也就是作业的每 100 分相当于总评的 1 分。接受误判及分数错误求和这两种情况的 argument。

## Section 6 pp.52

**Exercise 1.** Show that:

1. given a point $x_0$, a tensor $F \in \mathcal{L}in^+$, $\exists!$ homogeneous deformation $f$ with $\nabla f = F$ and $f(x_0) = x_0$
2. $f$, $g$ are homogeneous deformations, then so is $f \circ g$ and

$$\nabla(f \circ g) = (\nabla f)(\nabla g) \tag{1}$$

   Moreover, if $f, g$ fix $x_0$, then so does $f \circ g$.

*Solution.*

For 1, uniqueness: let $\nabla f_1 = \nabla f_2 = F$, then

$$f_1(x) = f_1(x_0) + F(x - x_0) = x_0 + F(x - x_0)$$
$$f_2(x) = f_2(x_0) + F(x - x_0) = x_0 + F(x - x_0)$$

then $f_1(x) = f_2(x)$; existence: it suffices to define $f(x) := x_0 + F(x - x_0)$.

For 2,

$$\begin{aligned} f \circ g(x) = f\left(g(x_1) + G(x - x_1)\right) &= f(x_2) + F\left(g(x_1) + G(x - x_1) - x_2\right) \\ &= \left[f(x_2) - F(x_2) + F\left(g(x_1)\right) - FG(x_1)\right] + FG(x) \end{aligned}$$

since the first four terms are constants, taking derivatives w.r.t. x will lead to

$$\nabla(f \circ g) = FG = (\nabla f)(\nabla g)$$

Furthermore, if $f, g$ fix $x_0$, just choose $x_1 = x_2 = x_0$, then

$$f \circ g(x_0) = \left[f(x_0) - F(x_0) + F\left(g(x_0)\right) - FG(x_0)\right] + FG(x_0) = f(x_0) = x_0.$$

评分细则：第一问中存在性和唯一性缺一不可，少一个扣 3 分；第二问未说明复合映射是均匀映射扣 3 分、漏证不动点扣 5 分，扣分不叠加。

**Exercise 2.** Pure shear is a homogeneous deformation of the form:

$$x_1 = p_1 + \gamma p_2,$$
$$x_2 = p_2,$$
$$x_3 = p_3$$

Compute:

1. the matrices $F$, $C$ and $B$;
2. the list of principal invariants of $C$(or $B$);
3. the principal stretches.

*Solution.*

For 1,

$$F = \nabla f = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$C = F^T F = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$B = F F^T = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For 2,

$$p(t) = \det(tI - C) = \begin{vmatrix} t-1 & -\gamma & 0 \\ -\gamma & t-1-\gamma^2 & 0 \\ 0 & 0 & t-1 \end{vmatrix} = t^3 - (3+\gamma^2)t^2 + (3+\gamma^2)t - 1$$
$$(I_1, I_2, I_3) = \big(\pm(3+\gamma^2), \pm(3+\gamma^2), \pm 1\big)$$

For 3,

$$(\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}) = \left(1, (\frac{2+\gamma^2+|\gamma|\sqrt{\gamma^2+4}}{2})^{1/2}, (\frac{2+\gamma^2-|\gamma|\sqrt{\gamma^2+4}}{2})^{1/2}\right)$$

评分细则：极分解过程中 $U = \sqrt{F^T F} = \sqrt{C}$，因而三个方向上的拉伸应为 C 对应特征值的算术平方根，错算扣 10 分，因为这属于本质错误；如果公式全对也开根了，纯计算错误则扣 5 分。矩阵计算可以不写过程，但是不变量和特征值计算必须有依据和过程，否则扣 5 分。

**Exercise 4.** Show that a deformation is isochoric iff $\det C = 1$.

*Solution.* $\det F = 1 \Leftrightarrow \det C = 1$, since we require $\det F > 0$. Then it suffices to show that the deformation is isochoric iff $\det F = 1$, which is obvious. Indeed, let $\Omega$, $\Omega_f$ denote the reference configuration and the deformed configuration by $f$, respectively. By restriction of regularity of the deformation mapping $f(X)$ and the boundary $\partial\Omega$, change of variable formula is valid, then

$$meas(\Omega) = \int_\Omega 1 \cdot dX,$$
$$meas(\Omega_f) = \int_{\Omega_f} 1 \cdot dx = \int_\Omega |\det F| \, dX$$

Equating these two volumes leads to

$$\int_\Omega |\det F(X)| - 1 \, dX = 0,$$

since the reference domain is chosen arbitrarily provided enough regularity, then it follows from the continuity of $|\det F(X)| - 1$ that $\det F \equiv 1$, $\forall x \in \Omega$.

评分细则：注意这是一个充要条件，这就意味着少一个方向将会扣 10 分；按照过程给分，言之有理即可。推荐使用重积分换元。从积分相等推导被积函数相等的过程推荐使用反证法或者积分均值定理，使用前者需要强调被积函数的连续性和积分区域的任意性；使用后者则要刻画出极限过程。考虑到书上有现成结论，这次只证明了 $F, B$ 行列式等于一相互等价也不扣分。

注：这里为了简洁，对变形和边界都做了正则性的假设，但是实际生活中我们发现很多固体有边、角等局部非光滑的区域，因此发展一套非正则的全局估计理论似乎是很必要的事情。事实上这些疑惑已经在上世纪下半叶被解决了并且已然成为 PDE 和变分分析等学科的基石并成功运用于力学中。弱化条件的过程就像抽丝剥茧，虽更接近问题的本质，但却常常牵涉出盘根错节的现代分析学理论。为了内容的连贯性和集中性这里就不再赘述，只是粗暴地假设我们需要的各阶偏导数都存在且在光滑的区域内部连续。

**Exercise 5.** Show that

$$C = I + \nabla u + (\nabla u)^T + (\nabla u)^T(\nabla u)$$

*Solution.* Almost trivial.

$F = \nabla f = \nabla u + I$, then $C = F^T F = (\nabla u + I)^T(\nabla u + I) =$ desired result.

评分细则：没有过程扣 5 分。

**Exercise 8.** Plane strain is a deformation of the following form

$$x_1 = f_1(p_1, p_2),$$
$$x_2 = f_2(p_1, p_2),$$
$$x_3 = p_3$$

Show that the principal stretch in the $p_3$ direction satisfies $\lambda_3 = 1$. Furthermore, the deformation is isochoric iff the other two principal stretches $\lambda_\alpha$ and $\lambda_\beta$ satisfy

$$\lambda_\alpha = \frac{1}{\lambda_\beta}.$$

*Solution.* Clearly,

$$C = F^T F = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\lambda_3 = \sqrt{1} = 1$.

$\det C = (\det F)^2 = (\lambda_\alpha^2 \cdot \lambda_\beta^2 \cdot \lambda_3^2)^2 = (\lambda_\alpha \cdot \lambda_\beta)^4$. Use the result from exercise 5: isochoric $\Leftrightarrow$ $\det C = 1 \Leftrightarrow \lambda_\alpha \cdot \lambda_\beta = 1$

评分细则：特征值算错扣 10 分，证明没有过程扣 10 分。

**Exercise 11.** Under the context of p. 47, define

$$\frac{dA_x}{dA_p} := \lim_{\alpha \to 0} \frac{|\mathbf{d}_\alpha \times \mathbf{e}_\alpha|}{|\alpha \mathbf{d} \times \alpha \mathbf{e}|}$$

Show that

$$m(x) \frac{dA_x}{dA_p} = cof(F_p) n(p)$$

with the help of identity $(Sa) \times (Sb) = cof(S)(a \times b)$, where

$$\begin{cases} m(x) := \lim_{\alpha \to 0} \dfrac{\mathbf{d}_\alpha \times \mathbf{e}_\alpha}{|\mathbf{d}_\alpha \times \mathbf{e}_\alpha|} \\ n(x) := \dfrac{\mathbf{d} \times \mathbf{e}}{|\mathbf{d} \times \mathbf{e}|} \end{cases}$$

*Solution.* Direct calculations give the desired result.

**Exercise 13.**

$$\mathscr{R} := \{\mathbf{p} | 0 \le p_1 < \infty\}$$

Construct mapping $f : \mathscr{R} \to f(\mathscr{R})$ by $(p_1, p_2, p_3) \mapsto (-\frac{1}{p_1+1}, p_2, p_3)$

1. Verify $f$ is one-to-one and $\det \nabla f > 0$.

2. Compute $f(\mathscr{R})$ and use this to demonstrate that $f$ is not a deformation.

3. Show that

$$f(\partial \mathscr{R}) = \partial f(\mathscr{R})$$

is not satisfied.

*Solution.*

For 1, suffices to verify injectivity of $f$, then verify $\det \nabla f > 0$.

For 2,

$$f(\mathscr{R}) = [-1, 0) \times \mathbb{R} \times \mathbb{R}$$

Note that $[0, \infty) \times \mathbb{R} \times \mathbb{R}$ is closed in $\mathbb{R}^3$, while $[-1, 0) \times \mathbb{R} \times \mathbb{R}$ is not. Recall the topology of metric space, this is equivalent to say that $f^{-1}$ is not continuous, which makes $f$ not being an homeomorphism.

For 3,

$$\begin{aligned} f(\partial\mathscr{R}) &= \{-1\} \times \mathbb{R} \times \mathbb{R} \\ \partial f(\mathscr{R}) &= \{-1\} \times \mathbb{R} \times \mathbb{R} \cup \{0\} \times \mathbb{R} \times \mathbb{R} \end{aligned}$$

评分细则：必须证明单射，否则扣 5 分。用函数单调性或者反证法都行；证明 $f$ 不是变形需要考虑定义域以范数诱导的标准拓扑，需要展开说明。但是这次只要正确点出集合的开闭性都不扣分；要有 $f(\partial\mathscr{R}), \partial f(\mathscr{R})$ 的计算结果，或者有文字说明边界映射后少了哪一部分。

警告：制定第一次关于张量代数作业的评分标准时，因为考虑到同学们大学的代数课程设置不同，所以不会写的题都有高额的保底分。这是一个帮助适应课程的缓冲带，不能成为少部分人投机钻营的“过墙梯”。


图 1: 这样写作业很酷，改这样的作业也很酷

如今第三次作业的内容已经和课本完全自洽连贯，如果不会，可以请教同学，请教Google，至少可以写出哪里不会而不是徒然留白。

因此，一个对那些认真完成作业的同学公平的方案是：**空多少分的题就扣多少分**。