Variational Method, Finite Element Implementation and Their Applications to Liquid Crystal Elastomers

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Preface

Basically, this essay grows up from the reading notes by the writer. Nonlinear elasticity becomes more and more important as it can capture behaviours of Liquid Crystal Elastomers. Simulating LCEs requires PDEs in hand. However, PDEs are often derived from certain functionals representing the "energy", which is physically more intuitive via the variational method. This addresses the importance to be proficient in variational method to adapt many new behaviour of advanced materials that are observed at laboratory. Readers educated as continuum mechanician may not really understand the functionals and the notions related such as convexity, differentiability of functionals, etc. The mission of the present essay is to fill the gap. Researchers can quickly acquire basic definitions and usages of variational method by reading the short essay.

At last, some features may have your attention:

- Theorems, examples and propositions, etc. with (♠) can be omitted for the first reading.
- All theorems and propositions are proved rigorously, yet some computations are completed in a rush. It will be a great practice to work them out again independently.

- Length of the present essay is limited to the minimum, so that it can be lectured in a fortnight seminar.
- If you want to report mistakes from the book or want to raise up your suggestions, please E-mail 12231147@mail.sustech.edu.cn

Contents

Chapter 1

Introduction

1.1 Facts about LCEs

LCEs are supposed now to be the most hopeful next generation materials that may be applied to biological substitutes, aerospace engineering and compliant robots.

If designed properly, the thin LCE plates could be completely simulated

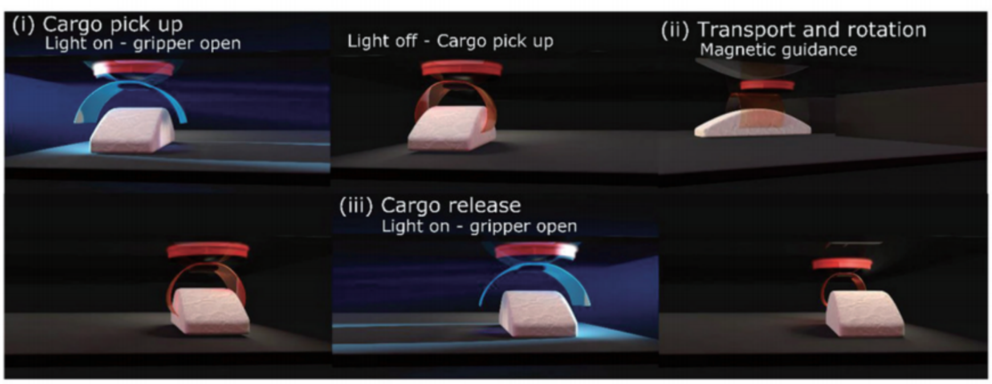
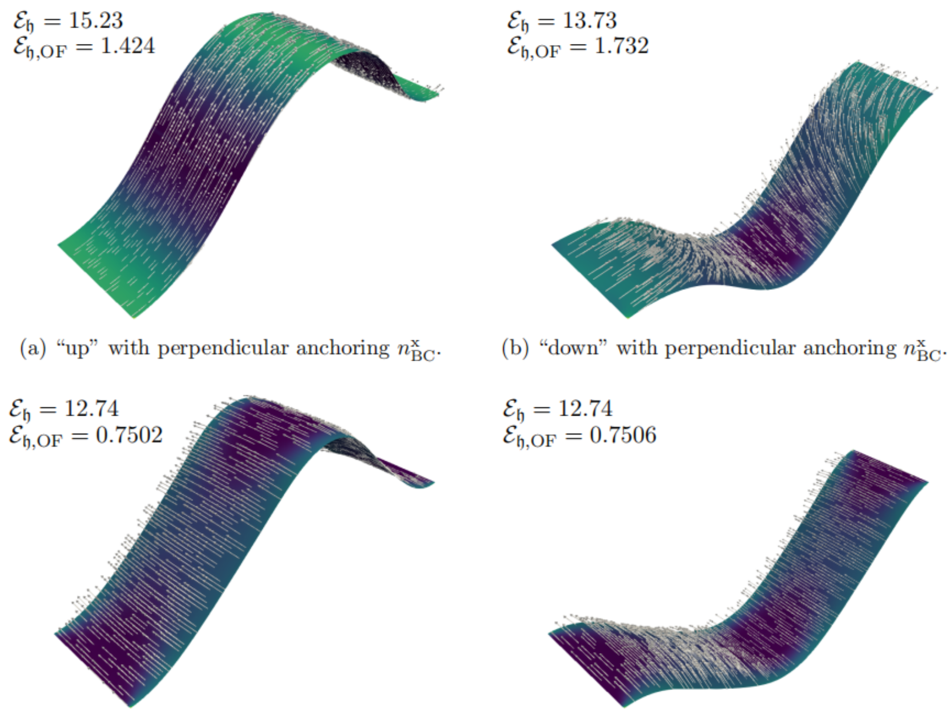


Figure 1.1: The tiny robot is grabbing an object



(a) "up" with perpendicular anchoring $n^{\mathrm{x}}_{\mathrm{BC}}$.

(b) "down" with perpendicular anchoring $n^{\mathrm{x}}_{\mathrm{BC}}$.

(c) "up" with tangential anchoring $n^{\mathrm{y}}_{\mathrm{BC}}$.

(d) "down" with tangential anchoring $n^{\mathrm{y}}_{\mathrm{BC}}$.

Figure 1.2: Computer simulations of LCE plate. Reproduced from [4]

by your computers.

Part I

Variational Methods

Chapter 2

Convexity

The goal of this chapter is the problem:

$$\begin{cases} \text{Minimize } \mathcal{F}[u] \triangleq \displaystyle\int_\Omega f(x, u(x), \nabla u(x))\, dx \\ \text{over all } u \in W^{1,p}(\Omega; \mathbb{R}^m) \text{ with } u|_{\partial\Omega} = g \end{cases} \tag{2.1}$$

We shall minimize a functional over a so-called admissible set (here the $W^{1,p}(\Omega; \mathbb{R}^m)$ space). Problem (2.1) is constantly encountered in the study of nonlinear elasticity since many stored energy functions take the form

$$f(x, \nabla u(x)) \tag{2.2}$$

x is the spatial point and $u(x)$ is the deformation mapping.

2.1 Direct method

We call the first method direct method because we prove the existence of solutions to minimization problem without the detour through a differential equation. Let X be a ***complete metric space***; objective functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ satisfies

(H1) **Coercivity:** For all $\Lambda \in \mathbb{R}$, sublevel set

$$\{u \in X : \mathcal{F}[u] \leq \Lambda\} \text{ is sequentially precompact}$$

that is, if $\mathcal{F}[u_j] \leq \Lambda$ for a sequence $\{u_j\} \subset X$ and some $\Lambda \in \mathbb{R}$, then $\{u_j\}$ has a converging subsequence in X.

(H2) **Lower semicontinuity(l.s.c.):** For all sequences $\{u_j\} \subset X$ with $u_j \to u$ in X, it holds that

$$\mathcal{F}[u] \leq \liminf_{j\to\infty} \mathcal{F}[u_j]$$

We tacitly assume *sequential* notions of compactness and l.s.c. from now on.

Direct method for problem

$$\text{Minimize } \mathcal{F}[u] \text{ over all } u \in X \tag{2.3}$$

is given in the following theorem

Theorem 2.1. *Suppose $\mathcal{F}$ satisfies (H1) and (H2), then abstract minimization problem (2.3) has at least one solution, i.e., $\exists u_* \in X$ with $\mathcal{F}[u_*] = \min\{\mathcal{F}[u] : u \in X\}$.*

Proof. Without any loss of generality, assume $\exists u \in X$ such that $\mathcal{F}[u] < +\infty$; otherwise the problem degenerates and any $u \in X$ is a solution.

To construct a minimizer, take a minimizing sequence $\{u_j\} \subset X$ such that

$$\lim_{j\to\infty} \mathcal{F}[u_j] \to \alpha := \inf \mathcal{F}[u] : u \in X < +\infty \tag{2.4}$$

This sequence exists by the definition of infimum. Then, for a number sequence, convergence implies boundedness, so $\mathcal{F}[u_j] \leq \Lambda$, $\forall j \in \mathbb{N}$ for some $\Lambda \in \mathbb{R}$. By (H1), select a subsequence such that

$$u_j \to u_* \in X.$$

From (H2), we conclude

$$\alpha \leq \mathcal{F}[u_*] \leq \liminf_{j \to \infty} \mathcal{F}[u_j] = \alpha.$$

The first inequality is by the definition of α and that $u_* \in X$. The second inequality is an applicaiton of l.s.c. The last inequality holds because limit equals lower limit for a convergent sequence.

Thus, $\mathcal{F}[u_*] = \alpha$ and u_* is the sought minimizer. $\square$

Example 2.1. Using the direct method, one can easily see that the l.s.c. function

$$h(t) := \begin{cases} 1 - t & \text{if } t < 0, \\ t & \text{if } t \geq 0. \end{cases} \tag{2.5}$$

has the minimizer $t = 0$.

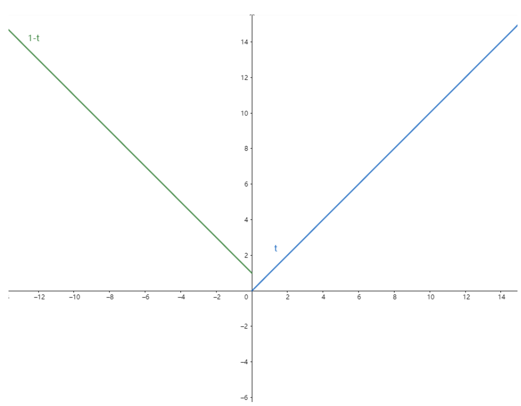


Figure 2.1:

The direct method reduce the existence problem into establishing coercivity and lower semicontinuity. It is useful to gain a perceptual understanding that: if topology on X is stronger, then there're fewer converging sequences, it is easier for $\mathcal{F}$ to be l.s.c., but harder to be coercive. The opposite holds if we choose a weaker topology. Therefore, in a practical view, it's crucial to choose an appropriate topology in which both coercivity and l.s.c can be established. It's interesting to see that the *mathematically* convenient topology turns out to be also *physically* relevant.

In this work, X will always be an infinite dimensional Banach space or its subset. We have to choose weak topology since strong compactness of infinite space is quite restrictive. However, lower semicontinuity with respect to weakly converging sequences is a delicate work that will cost considerable time of us to deal with.

We given a weak version of theorem 2.1 based on above discussion:

Theorem 2.2. *X is a* ***reflexive*** *Banach space or a closed affine subset of a reflexive Banach space; $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$. If the following two conditions hold,*

(WH1) ***Weak coercivity:*** *$\forall \Lambda \in \mathbb{R}$ the sublevel set*

$$\{u \in X : \mathcal{F}[u] \leq \Lambda\} \text{ is sequentially weakly precompact,}$$

that is, given a sequence $\{u_j\} \subset X$, if $\mathcal{F}[u_j] \leq \Lambda$, $\forall u_j$, then $\{u_j\}$ has a weakly converging subsequence.

(WH2) ***Weak lower semicontinuity:*** *$\forall$ sequences $\{u_j\} \subset X$ with $j \rightharpoonup u$ in X, it holds that*

$$\mathcal{F}[u] \leq \liminf_{j \to \infty} \mathcal{F}[u_j].$$

then problem

$$\text{Minimize } \mathcal{F}[u] \text{ over all } u \in X$$

has at least one solution.

Proof. Analogous to proof of theorem 2.1, also take into account the fact that all strongly closed affine subsets of a Banach space are weakly closed. □

2.2 Functionals with Convex Integrands

In 2.1, the importance of l.s.c. is shown. To apply 2.2, one have to establish l.s.c. of the functional.In many practical uses, say, elasticity, the functionals (stored energy functions) are given as integral functionals. In this section we shall find l.s.c. of the integral funcitional is intimately related to *convexity* properties of the integrand.

A simple yet fundamental integral functional is

$$\mathcal{F}[u] := \int_\Omega f(x, \nabla u(x))\, dx \tag{2.6}$$

We want to minimize $\mathcal{F}[u]$ over all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, recall we have tacitly assumed $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $p \in (1, \infty)$ is chosen depending on growth properties of f. For kwonledge of Sobolev spaces, the reader is referred to [5].

(2.6) is well-defined only if the integrand f is Lebesgue measurable, only if f is a Carathéodory integrand, which from now on we assume. Recall that

Definition 2.1.

(I) The smallest σ-ring that contains all open sets in $\mathbb{R}^N$ is called Borel σ-ring. Elements of Borel σ-ring is called **Borel sets**;

(II) Similar to the definition of Lebesgue measurable set, suppose $B \subset \mathbb{R}^N$ is a Borel set, on which $f(x)$ is defined. If $\forall \alpha \in \mathbb{R}$, $x \in B | f(x) > \alpha$ is a Borel set, then we say $f(x)$ is **Borel measurable** on B.

Lemma 2.1. Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory integrand, that is,

(i) $x \mapsto f(x, A)$ is Lebesgue measurable for every fixed $A \in \mathbb{R}^N$

(ii) $A \mapsto f(x, A)$ is continuous for a.e. $x \in \Omega$.

Then, for any Borel measurable map $V : \Omega \to \mathbb{R}^N$, the composition $x \mapsto f(x, V(x))$ is Lebesgue measurable.

Proof. First assume that V is a simple function, and adopt the unique canonical form (coefficients are distinct and nonzero, sets are disjoint)

$$V = \sum_{k=1}^{m} \nu_k \chi_{E_k}$$

where $E_k \subset \Omega$ are Borel measurable, $\bigcup_{k=1}^{m} E_k = \Omega$ and $\nu_k \in \mathbb{R}$. For $t \in \mathbb{R}$ we have

$$\{x \in \Omega : f(x, V(x)) > t\} = \bigcup_{k=1}^{m} \{x \in E_k : f(x, \nu_k) > t\}$$

which is a Lebesgue measurable set by assumption (i). Hence, $x \mapsto f(x, V(x))$ is Lebesgue measurable.

Now consider the general case. A fact is that every Borel measurable function (thus Lebesgue measurable) V can be approximated by simple functions ([6], Chapter 1, Theorem 4.2; [2], Lemma A.5) V_k with

$$f(x, V_k(x)) \to f(x, V(x)), \quad \forall x \in \Omega, \quad \text{as } k \to \infty$$

Therefore the RHS is Lebesgue measurable as the pointwise limit of Lebesgue measurable functions ([6], Chapter 1, Section 4, property 4). □

Now that we can make compound integrand in $\mathcal{F}$ measurable, it's still possible that the integral is not well-defined. To avoid these pathological cases, we require
either

$$f \geq 0 \tag{2.7}$$

or f is imposed with the **p-growth bound**

$$|f(x, A)| \leq M(1 + |A|^p), \quad (x, A) \in \Omega \times \mathbb{R}^{m \times d} \tag{2.8}$$

for some $M > 0$.

The name "p-growth bound" is justified because it implies the finiteness of $\mathcal{F}[u]$ for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. For example, let m=n=2, then

$$\nabla u = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} \tag{2.9}$$

then

$$|A| = |\nabla u| = \sqrt{\sum_{i,j}^{2} \left(\frac{\partial u_i}{\partial x_j} \right)^2} \tag{2.10}$$

$$\begin{aligned} |\mathcal{F}[u]| &= |\int_\Omega f(x, \nabla u(x))\, dx| \\ &\leq \int_\Omega |f(x, \nabla u)|\, dx \\ &\leq M|\Omega| + M \int_\Omega \left(\sqrt{\sum_{i,j}^{2} \left(\frac{\partial u_i}{\partial x_j} \right)^2} \right)^p dx \end{aligned} \tag{2.11}$$

The last term in (2.11) is controlled by

$$\begin{aligned}\int_\Omega \left(\sqrt{\sum_{i,j}^{2}\left(\frac{\partial u_i}{\partial x_j}\right)^2}\right)^p dx &\le \int_\Omega \left(\sum_{i,j}^{2}\left|\frac{\partial u_i}{\partial x_j}\right|\right)^p dx \\ &\le \int_\Omega 4^{p-1}\sum_{i,j}^{2}\left|\frac{\partial u_i}{\partial x_j}\right|^p dx \\ &< +\infty\end{aligned} \tag{2.12}$$

The last inequality is from the definition of Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$. Substitute (2.12) into (2.11), finally $\mathcal{F}[u]$ is finite since Ω is bounded thus has finite measure $|\Omega| < +\infty$.

So far so good, we've checked the measurability and finiteness of the integral functional. Next we shall check the coercivity. The most basic assumption is **p-coercivity bound**

$$\mu|A|^p \le f(x,A) \quad \forall(x,A)\in\Omega\times\mathbb{R}^{m\times d} \tag{2.13}$$

for some $\mu > 0$. (2.13) also determine the Sobolev exponential p where we look for solutions.

Note $\mu|A|^p - C \le f(x,A)$, $\mu, C > 0$ doesn't increase the generality since we may define a modified integrand $\tilde{f}(x,A) := f(x,A) + C$, which satisties (2.13), without changing the minimization problem.

Proposition 2.1. If Carathéodory integrand $f : \Omega\times\mathbb{R}^{m\times d} \to [0,\infty)$ satisfies p-coercivity bound (2.13),
then $\mathcal{F}$ is weakly coercive on the space

$$W_g^{1,p}(\Omega;\mathbb{R}^m) = \{u \in W^{1,p}(\Omega;\mathbb{R}^m) : u|_{\partial\Omega} = g\} \tag{2.14}$$

where $g \in W^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{R}^m)$.

Proof. By 1, given any sequence $\{u_j\} \subset W_g^{1,p}(\Omega;\mathbb{R}^m)$ with

$$\sup_{j\in\mathbb{N}} \mathcal{F}[u_j] < +\infty,$$

it suffices to show that $\{u_j\}$ is sequentially weakly precompact. (2.13) implies

$$\mu \cdot \sup_{j\in\mathbb{N}} \int_\Omega |\nabla u_j|^p \, dx \le \sup_{j\in\mathbb{N}} \mathcal{F}[u_k] < \infty,$$

whereby

$$\sup_{j\in\mathbb{N}} ||\nabla u_j|| < \infty.$$

Fix $u_0 \in W_g^{1,p}(\Omega;\mathbb{R}^m)$. Then $u_j - u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\sup_{j\in\mathbb{N}} ||\nabla(u_j - u_0)|| < \infty$. Apply Poincaré's inequality and triangle inequality to get

$$\sup_{j\in\mathbb{N}} ||u_j||_{W^{1,p}} \le \sup_{j\in\mathbb{N}} ||u_j - u_0||_{W^{1,p}} + ||u_0||_{W^{1,p}} < \infty.$$

$W^{1,p}(\Omega;\mathbb{R}^m)$ is reflexive ([7], Proposition 9.1), then the convergent subsequence exists guaranteed by ([7], Corollary 3.18) $\square$

Now that weak coercivity is settled, we shall investigate the weak lower semicontinuity.

Theorem 2.3 (Tonelli 1920 & Serrin 1961)**.** *Let $f : \Omega \times \mathbb{R}^{m\times d} \to [0,\infty)$ be a Carathéodory integrand such that*

$$f(x,\cdot) \text{ is convex for a.e. } x \in \Omega. \tag{2.15}$$

Then, $\mathcal{F}$ is weakly l.s.c. on $W^{1,p}(\Omega;\mathbb{R}^m)$ for any $p \in (1,\infty)$.

Proof. Step 1. First establish that $\mathcal{F}$ is strongly l.s.c. Let $u_i \to u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, then after passing to a subsequence, we have $\nabla u_j \to \nabla u$

a.e. ([7], Theorem 4.9). Assume $f(x, \nabla u_j(x)) \geq 0$ as indicated in (2.7). Applying Fatou's Lemma, we draw that

$$\begin{aligned}\mathcal{F}[u] &= \int_\Omega f(x, \nabla u(x))\, dx \\ &= \int_\Omega \liminf_{j\to\infty} f(x, \nabla u_j(x))\, dx \\ &\leq \liminf_{j\to\infty} \int_\Omega f(x, \nabla u_j(x))\, dx = \liminf_{j\to\infty} \mathcal{F}[u_j].\end{aligned}$$

where the second "=" is due to the continuity hypothesis of Carathéodory integrand.

Since the above inequality holds for all subsequences of $\{u_i\}$ (subseuqence of convergent sequence converges to the same limit), lemma 2.2 shows that this hold for the original sequence $\{u_i\}$, i.e.

$$\mathcal{F}[u] \leq \liminf_{i\to\infty} \mathcal{F}[u_i] \tag{2.16}$$

Step 2. For the weak lower semicontinuity, take $\{u_i\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $u_i \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. Need to show that

$$\mathcal{F}[u] \leq \liminf_{i\to\infty} \mathcal{F}[u_i] =: \alpha. \tag{2.17}$$

Taking a subsequence $\{u_j\}$ such that $\mathcal{F}[u_j] \to \alpha$ as $j \to \infty$.

Use Mazur's theorem to find convex combinations ([7], Corollary 3.8)

$$\nu_j = \sum_{\text{finite sum}} \theta_{n(j)} u_{n(j)}, \tag{2.18}$$

where

$$\theta_{n(j)} \in [0,1] \text{ and } \sum_{\text{finite sum}} \theta_{n(j)} = 1$$

such that $\nu_j \to u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Now use our pivotal convexity assumption (2.15),

$$\begin{aligned}\mathcal{F}[\nu_j] &= \int_\Omega f\left(x, \sum_{\text{finite sum}} \theta_{n(j)} \nabla u_{n(j)}(x)\right) dx \\ &\leq \sum_{\text{finite sum}} \theta_{n(j)} \mathcal{F}[u_{n(j)}].\end{aligned} \tag{2.19}$$

Beware that we can let any $n(j+1)$ in the representation of ν_{j+1} greater than any $n(j)$ in that of ν_j, which means that $\mathcal{F}[u_{n(j)}] \to \alpha$ as $j \to \infty$. The proof is a little subtle: in the j-th turn, cut the first $\max_n\{n(j)\}$ terms in sequence $\{u_j\}_j$, which doesn't affect that $u_j \rightharpoonup u$. Hence u still belongs to the weak closure of $C = conv(\bigcup_{\max_n\{n(j)\}}^\infty u_j)$, then $u \in \bar{C}$, the strong closure of C ([7], Theorem 3.7).

Since $\mathcal{F}[u_{n(j)}] \to \alpha$ as $j \to \infty$ as we've argued just now, and $\sum_{\text{finite sum}} \theta_{n(j)} = 1$, we arrive at

$$\liminf_{j\to\infty} \mathcal{F}[\nu_j] \leq \alpha. \tag{2.20}$$

On the other hand, $\nu_j \to u$ strongly, step 1 implies that

$$\mathcal{F} \leq \liminf_{j\to\infty} \mathcal{F}[\nu_j] = \alpha.$$

which is precisely (2.17). □

Lemma 2.2. $\mathcal{F}: X \to \mathbb{R}$, X is a complete metric space. If **every** subsequence $\{u_j\}$ of the sequence $\{u_i\} \subset X$ with $u_i \to u$ in X has a further subsequence $\{u_{j(k)}\}_k$ such that

$$\mathcal{F}[u] \leq \liminf_{k\to\infty} \mathcal{F}[u_{j(k)}] \tag{2.21}$$

then for the whole sequence, we have

$$\mathcal{F}[u] \leq \liminf_{k\to\infty} \mathcal{F}[u_i] \tag{2.22}$$

Proof. The proof is just a simple exercise of calculus. Assume without loss of generality that $\liminf_{k\to\infty} \mathcal{F}[u_i] < +\infty$, otherwise (2.22) will always hold. Now, if (2.22) doesn't hold, we can select a subsequence $\{u_j\}$ such that

$$\mathcal{F}[u] > \lim_{j\to\infty} \mathcal{F}[u_j]$$

Since any subsequence of convergent sequence always converges to the same limit, we conclude that

$$\mathcal{F}[u] > \liminf_{k\to\infty} \mathcal{F}[u_{j(k)}]$$

which contradicts to our hypothesis (2.21). □

We can summarize findings so far in the following existence theorem.

Theorem 2.4. *Let $f : \Omega \times \mathbb{R}^{m\times d} \to [0,\infty)$ be a nonnegative Carathéodory integrand such that*

(i) f satisfies the p-coercivity bound (2.13) with $p \in (1,\infty)$;

(ii) $f(x,\cdot)$ is convex for a.e. $x \in \Omega$.

Then, the associated functional $\mathcal{F}$ has a minimizer over $W_g^{1,p}(\Omega;\mathbb{R}^m)$, where $g \in W^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{R}^m)$.

Proof. It is just 2.2 with $X := W_g^{1,p}(\Omega;\mathbb{R}^m)$. The two conditions are verified by proposition 2.1 and theorem 2.3 respectively. □

Example 2.2 (Dirichlet functional)*.* The **Dirichlet functional** (or **Dirichlet integral**) is defined as

$$\mathcal{F}[u] := \int_\Omega \frac{1}{2}|\nabla u(x)|^2\,dx, \quad u \in W^{1,2}(\Omega;\mathbb{R}^m). \tag{2.23}$$

Here, $f(x, \nabla u) = \frac{1}{2}|\nabla u(x)|^2$ satisfies the 2-coercivity bound $\frac{1}{2}|\nabla u|^2 \leq f(x, \nabla u(x))$ and is convex in $\nabla u(x)$ for a.e. x. By 2.4, there exists a minimizer for any prescribed boundary values $g \in W^{1/2,2}(\partial\Omega; \mathbb{R}^m)$. The physical interpretation is that: when u represent the electrical potential, the Dirichlet functional is the electrostatic energy. Indeed, the total electric energy of charge distribution ρ is

$$U_E := \frac{1}{2}\int_{\mathbb{R}^3} \rho u \, dx = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} (\nabla \cdot E)\phi \, dx,$$

where we have used Gauss law for electricity

$$\nabla \cdot E = \operatorname{div} E = \frac{\rho}{\epsilon_0}.$$

Using identity

$$(\nabla \cdot E)u = \nabla \cdot (Eu) - E \cdot (\nabla u)$$

the Gauss-Green theorem, and the reasonable assumption that ϕ vanishes at infinity, we obtain

$$U_E = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} \nabla \cdot (Eu) - E \cdot (\nabla u) \, dx = -\frac{1}{2}\int_{\mathbb{R}^3} E \cdot (\nabla u) \, dx = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

Above discussion provide that there exists a solution to minimize the total electric energy, indeed it is the solution of electric pde, which once again verified the "least action principle".

There's a a converse to the Tonelli-Serrin theorem 2.3:

Proposition 2.2. Let $\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$, $p \in [1, \infty)$ be an integral functional. The integrand $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is continuous and dosen't dependent explicitly on x.

$$\mathcal{F}[u(x)] = \int_\Omega f(\nabla u(x)) \, dx$$

If both of

(i) $\mathcal{F}$ is weakly l.s.c. on $W^{1,p}(\Omega;\mathbb{R}^m)$

(ii) either $m=1$ or $d=1$ (scalar case and 1-D case, respectively)

hold, then f is convex.

Proof. Only consider the case where $m=1$ and d arbitrary; the other case is proved in a similar manner and left as an exercise.

Set $\nu := \theta a + (1-\theta)$, $n := b-a$, where $a,b \in \mathbb{R}^d$ with $a \neq b$, and $\theta \in (0,1)$ is arbitrary but fixed once chosen. Set

$$u_j(x) := \nu \cdot x + \frac{1}{j}\varphi_0(jx\cdot n - \lfloor jx \cdot n \rfloor), \quad x \in \Omega,$$

and

$$\varphi_0(t) := \begin{cases} -(1-\theta)t & \text{if } t \in [0,\theta), \\ \theta t - \theta & \text{if } t \in [\theta) \end{cases}$$

see figure 2.2. By direct computation, we have

$$\nabla u_j(x) = \begin{cases} \theta a + (1-\theta)b - (1-\theta)(b-a) = a & \text{if } jx\cdot n - \lfloor jx \cdot n \rfloor \in [0,\theta), \\ \theta a + (1-\theta)b + \theta(b-a) = b & \text{if } jx\cdot n - \lfloor jx \cdot n \rfloor \in [\theta,1). \end{cases}$$

Hence, $\{u_j\} \subset W^{1,\infty}$ because Ω is always bounded in out assumption which controls term $\nu \cdot x$. Note that zero-measure set is unimportant because functions differ at a zeore-measure set belong to the same equivalent class in Sobolev space $W^{1,\infty}(\Omega)$. Furthermore, $u_j \rightharpoonup (x \mapsto \nu \cdot x)$ in $W^{1,p}$ since the second term in the definition of u_j converges to zero uniformly and the dual space of L^p is L^{p^*} $(\frac{1}{p}+\frac{1}{p^*}=1)$. Now by l.s.c. assumption, we conclude that

$$|\Omega| f(\nu) = \mathcal{F}[\nu \cdot x] \leq \liminf_{j\to\infty} \mathcal{F}[u_j].$$

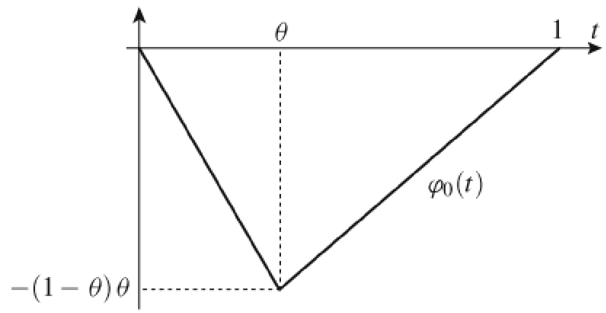


Figure 2.2: Sketch of $\varphi_0(t)$, reproduced from [2]

The proposition is proved once we can show

$$\liminf_{j\to\infty} \mathcal{F}[u_j] = |\Omega| \cdot (\theta f(a) + (1-\theta) f(b))$$

$$\begin{aligned} \mathcal{F}[u_j] &= \int_\Omega f(\nabla u_j)\, dx \\ &= \int_{portion1} f(a) + \int_{portion2} f(b) \end{aligned}$$

by portion 1 we mean that the portion of x over whole $\partial\Omega$ that $jx\cdot n - \lfloor jx \cdot n \rfloor \in [0,\theta)$; similar for portion 2. Geometrically, condiser the ray line from origin to a boundary point. Discretize this line by unit-length segment. Then portion 1 is approximately equal to θ if the discretization is finer and finer uniformly for all ray line from origin point. This is what j do for us.

□

In the simple case, convexity of integrand function is necessary condition for weak lower semicontinuity. But in the vectorial case, things become much complex. We postpone the discussion to Chapters later on.

Someone may ask: if there is one or are many minimizers to a specific variational problem?

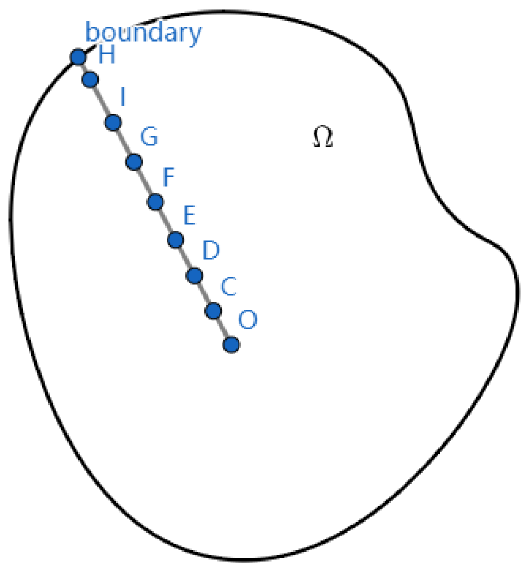


Figure 2.3: Discretization of an arbitrary line by unit-length segment

Proposition 2.3 (Uniqueness of minimizer)**.** Let $\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$, $p \in [1, \infty)$, be an integral functional with Carathéodory integrand $f : \Omega \times \mathbb{R}^{m\times d} \to \mathbb{R}$. If f is **strictly convex**, i.e.,

$$f(x, \theta A + (1-\theta)B) < \theta f(x, A) + (1-\theta) f(x, B) \tag{2.24}$$

for all $x \in \Omega$, $A, B \in \mathbb{R}^{m\times d}$ with $A \neq B$ and $\theta \in (0, 1)$,
then $g \in W^{1-\frac{1}{p},p}(\partial\Omega; \mathbb{R}^m)$, the minimizer $u_* \in W^{1,p}_g(\Omega; \mathbb{R}^m)$ of $\mathcal{F}$, if it exists, is **unique**.

Proof. Argue by contradiction. Assume there're 2 different minimizers $u, v \in W^{1,p}_g(\Omega; \mathbb{R}^m)$ of $\mathcal{F}$. Then set

$$w := \frac{1}{2}u + \frac{1}{2}v \in W^{1,p}_g(\Omega; \mathbb{R}^m)$$

and observe that

$$\mathcal{F}[w] = \int_\Omega f\left(x, \frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\right) < \frac{1}{2}\mathcal{F} + \frac{1}{2}\mathcal{F}[v] = \min_{W_g^{1,p}(\Omega;\mathbb{R}^m)} \mathcal{F}$$

which makes a contradiction to definition of minimizer. □

2.3 Integrals with u-Dependence

Remember our goal in this chapter, we try to extend results to more general functionals

$$\mathcal{F}[u] := \int_\Omega f(x, u(x), \nabla u(x))\, dx.$$

However, Mazur's lemma cannot be directly used here because it would be crazy to image "pulling out" convex combination from 2 arguments

$$\int_\Omega f\left(x, \sum \theta_{n(j)} u_n(j), \sum \theta_{n(j)} \nabla u_{n(j)}\right)$$

so we devise a substitution to determine the *weak lower semicontinuity.*

Definition 2.2. $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ is Carathéodory integrand if

(i) $x \mapsto f(x, v, A)$ is Lebesgue measurable for every fixed $(v, A) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$;

(ii) $(v, A) \mapsto f(x, v, A)$ is continuous for a.e. fixed $x \in \Omega$.

Theorem 2.5. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to [0, \infty)$ be a Carathéodory integrand.*
Assume also that f is convex in the third position

$$f(x, v, \cdot) \text{ is convex for every } (x, v) \in \Omega \times \mathbb{R}^m \tag{2.25}$$

Then, for $p \in (1, \infty)$, the functional

$$\mathcal{F}[u] := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^m) \tag{2.26}$$

is weakly l.s.c.

Proof. See [8], Section 3.2.6. □

We end the section with an interesting example from elasticity.

Example 2.3. Consider a solid occupies $\Omega \subset \mathbb{R}^3$, Ω is connected and $\partial\Omega$ is Lipshitz continuous. We call Ω the reference configuration because it deforms by $y : \Omega \to y(\Omega)$ — a differentiable bijection, to a deformed configuration $y(\Omega)$. We require y to be orientation-preserving, i.e.,

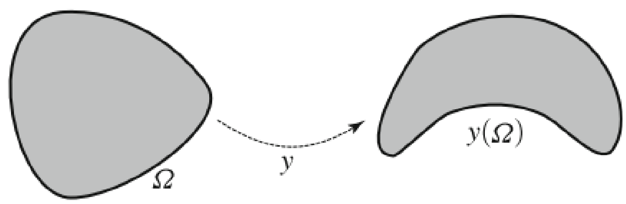


Figure 2.4: Deform the solid, reproduced of [2]

$$\det \nabla y(x) > 0. \tag{2.27}$$

The geometric meaning is that: for any two different regular curves $\alpha(t)$ and $\beta(t)$, $t \in (-\epsilon, \epsilon)$ restricted in Ω such that $\alpha(0) = \beta(0)$, the corresponding curves in deformed configuration are $y(\alpha(t))$ and $y(\beta(t))$, respectivily. By chain rule, the tangent vectors satisfy

$$\frac{d}{dt}y(\alpha(t))|_{t=0} = [\nabla y(\alpha(0))]\alpha'(0), \tag{2.28}$$

$$\frac{d}{dt}y(\beta(t))|_{t=0} = [\nabla y(\beta(0))]\beta'(0) \tag{2.29}$$

For an arbitrary invertible operator $T \in \mathcal{L}(V)$, by polar decomposition, there exists an isometry $S \in \mathcal{L}(V)$

$$T = S\sqrt{T * T} \tag{2.30}$$

Now, let $T = \nabla y$ and $V = \mathbb{R}^3$

$$\det(\nabla y) = (\det S)(\det \sqrt{T * T}) \tag{2.31}$$

Since $\sqrt{T * T}$ is a positive operator, the sign of $\det \nabla y(x)$ depends on the sign of the isometry S, which further depends on whether there are odd or even number of one-dimensional subspaces are reversed by S.

Besides, the volume element satisfy:

$$dx^y = \det(\nabla y(x))dx \tag{2.32}$$

Introduce the displacement

$$u(x) := y(x) - x. \tag{2.33}$$

We also need a measure of local "strechting", called a strain tensor. Intuitively, rigid body motions

$$u(x) = Rx + u_0, \quad R \in SO(3) \tag{2.34}$$

should not cause strain. On the contrary, strain measures the deviation of the deformation from a rigid body motion. A common choice is the Green-St. Venant strain tensor

$$G := \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u). \tag{2.35}$$

When consider finite strain, it becomes nonlinear elasticity. Assume existence of stored-energy density function $W : \mathbb{R}^{3\times 3} \to [0, \infty]$ and an external body force field $b : \Omega \to \mathbb{R}^3$ (e.g. gravity) such that

$$\mathcal{F}[y] := \int_\Omega W(\nabla y(x)) - b(x) \cdot y(x)\, dx \tag{2.36}$$

represents the total energy stored in the elasticity system. If ignore body force,

$$\mathcal{F}[y] := \int_{\Omega} W(\nabla y(x))\,dx \tag{2.37}$$

then the material is called **hyperelastic**. There're some physically justified restrictions imposed on W:

(i) Norming: $W(Id) = 0$. The undeformed state has no energy.

(ii) Frame-Indifference: $W(QA) = W(A)$ for all $Q \in SO(3)$, $A \in \mathbb{R}^{3\times 3}$.

(iii) Infinite compression costs infinite energy: $W(A) \to +\infty$ as $(\det A) \downarrow 0$.

(iv) Infinite strechting costs infinite energy: $W(A) \to +\infty$ as $|A| \to \infty$.

But at this stage, let's restricted us to the linearized elasticity, after making the "small strain" assumption. ∇u is so small that we can omit the quadratic term in (2.35) to work with the linearized strain tensor

$$\mathscr{E} u := \frac{1}{2}(\nabla u + \nabla u^T). \tag{2.38}$$

The motion that doesn't cause strain is

$$u(x) = Wx + u_0 \tag{2.39}$$

with skew symmetric matrix $W^T = -W$. This is precise the rigid motion studied in classical mechanics course. The even more fascinating aspect is algebra: The Lie group $SO(3)$ of rotations has as its Lie algebra $Lie(SO(3)) = \mathfrak{so}(3)$, the spaces of all skew-symmetric matrices, which then can be seen as "infinitesimal rotations".

The energy associated to linearized elasticity is the special quadratic form

$$\mathscr{W}[u] := \int_\Omega \frac{1}{2}\mathscr{E}u(x) : \mathbf{C}(x) : \mathscr{E}u(x)\,dx \tag{2.40}$$

$\mathbf{C}(x) = \mathbf{C}^{ik}_{jl}(x)$ $x \in \Omega$ is a symmetric, positive definite ($A : \mathbf{C}A \geq c|A|^2$ for some $c > 0$) 4^{th}-order tensor — the elasticity tensor. Note that by the usual major and minor symmetry of elasticity tensor $\mathbf{C}$, equation (2.40) can be also writen as

$$\mathscr{W}[u] := \int_\Omega \frac{1}{2}\nabla u(x) : \mathbf{C}(x) : \nabla u(x)\,dx$$

For homogeneous, isotropic material, $\mathbf{C}$ doesn't depend on spatial variable x or the direction of strain

$$(AQ) : \mathbf{C}(AQ) = A : \mathbf{C}A, \quad \text{for all } A \in \mathbb{R}^{3\times 3}, Q \in SO(3) \tag{2.41}$$

It can be shown that $\mathscr{W}$ simplifies to

$$\mathscr{W}[u] = \int_\Omega \mu|\mathscr{E}u(x)|^2 + \frac{1}{2}(\kappa - \frac{2}{3}\mu)|tr\mathscr{E}u(x)|^2\,dx \tag{2.42}$$

$\mu > 0$ is shear modulus and $\kappa > 0$ is bulk modulus. Taking body force into consideration leads to the nonlinear setting. We shall minimize the total energy

$$\mathscr{W}[u] = \int_\Omega \mu|\mathscr{E}u(x)|^2 + \frac{1}{2}(\kappa - \frac{2}{3}\mu)|tr\mathscr{E}u(x)|^2 - b(x)\cdot u(x)\,dx \tag{2.43}$$

$b : \Omega \to \mathbb{R}^3$ is external body force.

To be specific, consider

$$\begin{cases} \text{Minimize} \quad \mathscr{F}[u] = \displaystyle\int_\Omega \mu|\mathscr{E}u|^2 + \frac{1}{2}(\kappa - \frac{2}{3}\mu)|tr\mathscr{E}u|^2 - b\cdot u\,dx \\ \text{over all} \quad u \in W^{1,2}_0(\Omega;\mathbb{R}^3) \end{cases} \tag{2.44}$$

$\mu, \kappa > 0$ are constants, $b \in L^2(\Omega; \mathbb{R}^3)$ and $0 \equiv g \in W^{\frac{1}{2},2}(\partial\Omega; \mathbb{R}^3)$. $\mathscr{F}$ has quadratic growth. Assume $\kappa - \frac{2}{3}\mu > 0$ and $g \equiv 0$ for simplicity.
Claim

$$||\nabla u||_{L^2} \le \sqrt{2}||\mathscr{E}u||_{L^2} \quad \forall u \in W_0^{1,2}(\Omega; \mathbb{R}^3) \tag{2.45}$$

Indeed, for test function $\varphi \in C_c^\infty(\Omega; \mathbb{R}^3)$, direct calculation shows

$$2(\mathscr{E}\varphi : \mathscr{E}\varphi) - \nabla\varphi : \nabla\varphi = \operatorname{div}[(\nabla\varphi)\varphi - (\operatorname{div}\varphi)\varphi] + (\operatorname{div}\varphi)^2. \tag{2.46}$$

Thus, by divergence theorem,

$$\begin{aligned} 2||\mathscr{E}\varphi||_{L^2}^2 - ||\nabla\varphi||_{L^2}^2 &= \int_\Omega \operatorname{div}[(\nabla\varphi)\varphi - (\operatorname{div}\varphi)\varphi]\,dx + \int_\Omega (\operatorname{div}\varphi)^2\,dx \\ &= \int_\Omega (\operatorname{div}\varphi)^2\,dx \\ &\ge 0 \end{aligned} \tag{2.47}$$

(2.45) is thus proved for test function φ. Validation of (2.45) for $u \in W_0^{1,2}(\Omega; \mathbb{R}^3)$ follows from the fact that $C_0^\infty(\Omega; \mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega; \mathbb{R}^3)$. Then just take a sequence of test functions that satisfy (2.45) converging to u in $W^{1,p}$ norm, thus in L^2 norm.
Exploit Hölder's inequality (check this componentwise)

$$\int_\Omega |b \cdot u|\,dx \le ||b||_{L^2}||u||_{L^2}, \tag{2.48}$$

variant Young's inequality for $p = p* = 2$

$$(\sqrt{\delta}a)(\frac{1}{\sqrt{\delta}}b) \le \frac{1}{2}(\sqrt{\delta}a)^2 + \frac{1}{2}(\frac{1}{\sqrt{\delta}}b)^2 = \frac{1}{2}\delta a^2 + \frac{1}{2\delta}b^2, \tag{2.49}$$

and vectororial Poincaré inequality

$$\begin{aligned}
\frac{1}{\sqrt{m}}||\nabla u||_{L^2} &= (\int_\Omega \frac{1}{m}\sum u_{i,j}^2\,dx)^{\frac{1}{2}} \\
&= (\int_\Omega \frac{1}{m}|\nabla u_1|^2 + \cdots + \frac{1}{m}|\nabla u_m|^2\,dx)^{\frac{1}{2}} \\
&\geq \frac{1}{m}(\int_\Omega |\nabla u_1|^2\,dx)^{\frac{1}{2}} + \cdots + \frac{1}{m}(\int_\Omega |\nabla u_m|^2\,dx)^{\frac{1}{2}} \\
&\geq \frac{C}{m}||u_1||_{W_0^{1,2}} + \cdots + \frac{C}{m}||u_m||_{W_0^{1,2}} \\
&\geq \frac{C_p}{\sqrt{m}}||u||_{W_0^{1,2}}
\end{aligned} \tag{2.50}$$

We have used the concavity of $x \mapsto x^{\frac{1}{2}}$ in the third line, used scalar Poincaré's inequality in the fourth line, and used scalar triangle inequality inthe fifth line in above inequalities.
Then for any $\delta > 0$,

$$\begin{aligned}
\mathscr{F}[u] &\geq \mu||\mathscr{E}u||_{L^2}^2 - ||b||_{L^2}||u||_{L^2} \\
&\geq \mu||\mathscr{E}u||_{L^2}^2 - \frac{1}{2\delta}||b||_{L^2}^2 - \frac{\delta}{2}||u||_{L^2}^2 \\
&\geq \frac{\mu}{2}||\nabla u||_{L^2}^2 - \frac{1}{2\delta}||b||_{L^2}^2 - \frac{C_p^2\delta}{2}||\nabla u||_{L^2}^2
\end{aligned} \tag{2.51}$$

Choosing $\delta = \mu/(2C_p^2)$, we obtain the coercivity estimate

$$\mathscr{F}[u] \geq \frac{\mu}{4}||\nabla u||_{L^2}^2 - \frac{C_p^2}{\mu}||b||_{L^2}^2 \tag{2.52}$$

Applying the Poincaré inequality one more time, we see that $\mathscr{F}[u]$ controls $||u||_{W_0^{1,p}}$. Therefore, $\mathscr{F}$ is weakly coercive because any bounded sequence in $W_0^{1,2}$ — a reflexive Banach space, has a weakly convergent subsequence ([7], Theorem 3.18). Moreover, the integrand is convex in the $\mathscr{E}u$ argument, thus theorem 2.5 asserts that $\mathscr{F}$ is weakly l.s.c. Finally theorem 2.2 guarantees the existence of a solution $u_* \in W_0^{1,2}(\Omega;\mathbb{R}^3)$ to our problem (2.44).

2.4 The Lavrentiev Gap Phenomenon

Can we expand the class of candidate functions? If so, do we need additional conditions? This section we answer this interesting and practical problem. Besides, examples in this section show that direct numerical simulation (such as standard conforming Finite Element Method) of the abstract minimization problem is impossible due to the Lavrentiev gap phenomenon. This phenomenon is also of great concern in nonlinear elasticity theory. This section could serve as a demand for the next chapter.

2.5 Integral Side Constraints

In real minimization problems, the class of candidate functions is often restricted with constraints expressed by an integral equation. It is urgent to extend the Direct Method to this scenario.

Theorem 2.6. *Let X be a Banach space or a closed affine hyperplane of a Banach space. Let $\mathscr{F}, \mathscr{H} : X \to \mathbb{R} \cup \{+\infty\}$. Assume the following:*

(WH1) ***Weak coercivity of $\mathscr{F}$:*** *$\forall \Lambda \in \mathbb{R}$ the sublevel set*

$$\{u \in X : \mathcal{F}[u] \leq \Lambda\} \text{ is sequentially weakly precompact,}$$

that is, given a sequence $\{u_j\} \subset X$, if $\mathcal{F}[u_j] \leq \Lambda$, $\forall u_j$, then $\{u_j\}$ has a weakly converging subsequence.

(WH2) ***Weak lower semicontinuity of $\mathscr{F}$:*** *$\forall$ sequences $\{u_j\} \subset X$ with $u_j \rightharpoonup u$ in X, it holds that*

$$\mathcal{F}[u] \leq \liminf_{j\to\infty} \mathcal{F}[u_j].$$

(WH3) ***Weak continuity of $\mathscr{H}$:*** *$\forall$ sequences $\{u_j\} \subset X$ with $u_j \rightharpoonup u$ in X, it holds that*

$$\mathscr{H}[u_j] \to \mathscr{H}[u] \quad as\ j \to \infty$$

Assume also that the zero set of $\mathscr{H}$ is nonempty.
Then the minimization problem

$$\text{Minimize } \mathscr{F}[u] \text{ over all } u \in X \text{ with } \mathscr{H}[u] = 0 \tag{2.53}$$

has a solution.

Proof. The proof is nearly the same with theorem 2.2, except that we should choose u_j such that $\mathscr{H}[u_j] = 0$ for the minimizing sequence. Then by (WH3), this property also holds for any weak limit u_* of a subsequence of the u_j's, which then is the sought minimizer. Be aware that, though the function space is shrinked to $\{u \in X | \mathscr{H}[u] = 0\}$, the weak limit is always exist, since one can take $\{u_j\}_{j=1}^{\infty} \subset \{u \in X | \mathscr{H}[u] = 0\}$

$$\lim_{j \to \infty} \mathscr{F}[u_j] = \alpha := \inf_{u \in X, \mathscr{H}[u]=0} \mathscr{F}[u]$$

then (WH1) shows the existence of the weak limit. □

To use theorem 2.6, we need to verify (WH1) - (WH3), especially (WH3), the next lemma is very useful to this end.

Lemma 2.3. Let Ω be bounded and of C^1 class. As usual, suppose $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory integrand. Let $p \in [1, \infty)$ such that $\exists M > 0$ with

$$\begin{cases} |h(x,v) \le M(1 + |v|^q)|, \quad (x,v) \in \Omega \times \mathbb{R}^m, q \in [1, \dfrac{dp}{(d-p)}) & \text{if } p \le d \\ \text{no growth condition,} & \text{if } p > d \end{cases} \tag{2.54}$$

Here, d is the dimension of the ambient Euclidean space of domain Ω while m is the dimension of the target space of function u. If we have function $u : \Omega \to \mathbb{R}^m$. Then, the functional $\mathscr{H} : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ defined by

$$\mathscr{H}[u] := \int_\Omega h(x, u(x))\, dx, \quad u \in W^{1,p}(\Omega;\mathbb{R}^m) \tag{2.55}$$

is *weakly continuous*.

Proof. Let's say, for example we're in the case $p \leq d$.

Let $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. By proposition 3.5 of [7], $\{||u_j||\}_j$ is bounded sequence. Thus by Rellich-Kondrachov Compactness Theorem (theorem 1 of section 5.7 in [9]), we know $\{||u_j||\}$ is precompact in $L^q(\Omega)$, where $q \in [1, \frac{dp}{(d-p)})$ if $p < d$ and $q \in [p, \infty)$ if $p = d$. More precisely, exists some subsequence such that $u_{k_j} \to u$ in $L^q(\Omega)$. Finally, L^q convergence implies pointwise convergence a.e. (theorem 4.9 in [7]). By growth condition (2.54),

$$\pm h(x, v) + M(1 + |v|^q) \geq 0$$

Applying Fatou's lemma (cf. Lemma 4.1 in [7]) separately to these two integrands, we have

$$\begin{aligned}
&\liminf_{j\to\infty} \left(\pm\mathscr{H}[u_{k_j}] + \int_\Omega M(1 + |u_{k_j}|^q)\, dx \right) \\
&\geq \int_\Omega \liminf_{j\to\infty} \pm h(x, u_{k_j}(x))\, dx + \int_\Omega \liminf_{j\to\infty} M(1 + |u_{k_j}(x)|^q)\, dx \\
&= \pm\mathscr{H}[u] + \int_\Omega M(1 + |u(x)|^q)\, dx
\end{aligned}$$

The equality in the third line comes from the definition of Carathéodory integrand: for vector $u \in \mathbb{R}^m$, $u \mapsto h(x, u)$ is continuous for a.e. $x \in \Omega$. Indeed, we've showed the result that $u_j(x) \to u(x)$ a.e. x as $j \to \infty$, thus $\liminf_{j\to\infty} \pm h(x, u_j(x)) = \lim_{j\to\infty} \pm h(x, u_j(x)) = \pm h(x, u(x))$ except on a zero measure set. Now, since $||u_{k_j}||_{L^q} \to ||u||_{L^q}$,

$$\int_\Omega |u_{k_j}|^q \, dx \to \int_\Omega |u|^q \, dx$$

which means

$$\liminf_{j\to\infty} \left(\int_\Omega M(1 + |u_{k_j}|^q) \, dx \right) = \int_\Omega M(1 + |u(x)|^q) \, dx$$

Thus,

$$\liminf_{j\to\infty} \mathscr{H}[u_{k_j}] \geq \mathscr{H}[u]$$

and

$$\liminf_{j\to\infty} -\mathscr{H}[u_{k_j}] \geq -\mathscr{H}[u]$$

$$\limsup_{j\to\infty} \mathscr{H}[u_{k_j}] \leq \mathscr{H}[u]$$

Hence

$$\limsup_{j\to\infty} \mathscr{H}[u_{k_j}] \leq \mathscr{H}[u] \leq \liminf_{j\to\infty} \mathscr{H}[u_{k_j}]$$

by sandwiched theorem, $\mathscr{H}[u_{k_j}] \to \mathscr{H}[u]$. At last, since this holds for a subsequence of any subsequence of $\{u_j\}$, arguing in the same spirit of lemma 2.2, it turns out to be true for our original sequence. □

Here comes our main theorem in this section, which help taking a large class of side constraints into account for minimization problems.

Theorem 2.7. *Let $f : \Omega \times \mathbb{R}^{m\times d} \to [0, \infty)$ and $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory integrands such that*

(i) f satisfies the p-coercivity bound (2.13), where $p \in (1, \infty)$;

(ii) $f(x, \cdot)$ is convex for all $x \in \Omega$;

(iii) h satisfies the q-growth condition (2.54) for some $q \in [1, \frac{dp}{d-p})$ if $p \leq d$, or no growth condition if $p > d$.

Then, there exists a minimizer $u_* \in W_g^{1,p}(\Omega;\mathbb{R}^m)$, *where* $g \in W^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{R}^m)$ *of the functional*

$$\mathscr{F}[u] := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \quad u \in W_g^{1,p}(\Omega;\mathbb{R}^m) \tag{2.56}$$

under the side constraint

$$\mathscr{H}[u] := \int_\Omega h(x, u(x))\, dx = 0$$

Proof. Combing lemma 2.3 with theorem 2.4, theorem 2.6 and the Rellich-Kondrachov theorem. □

For further development on the constrained minimization subject, see section 3.3.

Chapter 3

Variations

In chapter 2, we focus on proving the existence results of minimization problem. However, it is a more important thing to find them out. This task is undertaken by calculations with variations, which is our topic in the present chapter.

Let $\mathcal{F} : W_g^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a functional with minimizer $u_* \in W_g^{1,p}(\Omega;\mathbb{R}^m)$. Take a path $t \mapsto u_t \in W_g^{1,p}(\Omega;\mathbb{R}^m)$ with $u_0 = u_*$, if the map $t \mapsto \mathcal{F}[u_t]$ is differentiable around $t = 0$, then its derivative at $t = 0$ vanishes.

Definition 3.1 (first variation)**.** The **first variation** $\delta\mathcal{F}[u]$ of $\mathcal{F}$ at $u \in W_g^{1,p}(\Omega;\mathbb{R}^m)$ is the linear map

$$\delta\mathcal{F}[u] : C_c^\infty(\Omega;\mathbb{R}^m) \to \mathbb{R}$$

defined as

$$\delta\mathcal{F}[u][\Psi] := \lim_{h\downarrow 0} \frac{\mathcal{F}[u+h\Psi] - \mathcal{F}[u]}{h}, \quad \Psi \in C_c^\infty(\Omega;\mathbb{R}^m) \tag{3.1}$$

Thus, for every minimizer u_*, $\delta\mathcal{F}[u_*] = 0$ provided (3.1) exists. This induce a pde, called *Euler-Lagrange equation*, which minimizers necessarily satisfy in the weak sense. It turns out to be sufficient for a map to be a minimizer under additional convexity assumption on $\mathcal{F}$. This gives us a way to find a minimizer — solving the Euler-Lagrange equation.

Besides, we also consider side constraint for the path $t \mapsto u_t$ in this chapter, which leads to the generalized Euler-Lagrange equation, involving *Lagrange multiplier*.

Finally, we exploit these techiniques to unveil the "hidden" consevation laws — the invariance of the integral functional, i.e., a nontrivial path $t \mapsto u_t$ along which $\mathcal{F}[u_t]$ is constant.

3.1 The Euler-Lagrange Equation

We remark that this section serves as a bridge connecting calculus of variations and PDE theory.

Definition 3.2 (directional derivative at A in direction B)**.** The **directional derivative** $D_A f(x, v, A) \in \mathbb{R}^{m\times d}$ of $f(x, v, \cdot)$ at A in direction B is defined as

$$D_A f(x, v, A) := (\partial_{A_k^j} f(x, v, A))_k^j \tag{3.2}$$

such that

$$D_A f(x, v, A) : B = \lim_{h\downarrow 0} \frac{f(x, v, A + hB) - f(x, v, A)}{h}, \quad A, B \in \mathbb{R}^{m\times d} \tag{3.3}$$

Remark 3.1. Recall the usual directional derivative vector $D_v f(x, v, A)$ of $f(x, \cdot, A)$ is given in component as

$$D_v f(x, v, A) := (\partial_{v^j} f(x, v, A))^j \tag{3.4}$$

such that

$$D_v f(x, v, A) \cdot w = \lim_{h\downarrow 0} \frac{f(x, v + hw, A) - f(x, v, A)}{h}, \quad v, w \in \mathbb{R}^d$$

In fact, the matrix $D_A f(x, v, A)$ is defined in an analogous way.

The following theorem furnishes the connection pointed out in the beginning of this section.

Theorem 3.1. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ be a Carathéodory integrand that is continuously differentiable in the second and third arguments (i.e., can be treated as multivariable functions in differentiation sense) and that satisfies the growth bounds: $\forall (x, v, A) \in D_A f(x, v, A)$ and $p \in [1, \infty)$,*

$$|D_v f(x, v, A)|,\ |D_A f(x, v, A)| \le C(1 + |v|^p + |A|^p), \tag{3.5}$$

If $u_* \in W^{1,p}_g(\Omega; \mathbb{R}^m)$, where $g \in W^{1-1/p,p}(\partial\Omega; \mathbb{R}^m)$, minimizes the functional

$$\mathcal{F}[u] := \int_\Omega f(x, u(x), \nabla u(x))\, dx, \quad u \in W^{1,p}_g(\Omega; \mathbb{R}^m) \tag{3.6}$$

*then u_* is a **weak solution** of the **Euler-Lagrange** equation (B.V.P.)*

$$\begin{cases} -\operatorname{div}[D_A f(x, u, \nabla u)] + D_v f(x, u, \nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

Here, $u_* \in W^{1,p}(\Omega; \mathbb{R}^m)$ is called a weak solution of (3.7) if

$$\int_\Omega D_A f(x, u_*, \nabla u_*) : \nabla\psi + D_v f(x, u_*, \nabla u_*) \cdot \psi\, dx = 0, \quad \forall \psi \in C^\infty_c(\Omega; \mathbb{R}^m) \tag{3.8}$$

Proof. For any $\psi \in C^\infty_c(\Omega; \mathbb{R}^m)$ and all $0 < h < 1$, $u_* + h\psi \in W^{1,p}_g(\Omega; \mathbb{R}^m)$ is admissible in the minimization. We have

$$\mathcal{F}[u_*] \le \mathcal{F}[u_* + h\psi]$$

Therefore,

$$
\begin{aligned}
0 &\leq \int_\Omega \frac{f(x,u_* + h\psi, \nabla u_* + h\nabla\psi) - f(x,u_*,\nabla u_*)}{h}\,dx \\
&= \int_\Omega \int_0^1 \frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}\,t}[f(x,u_* + th\psi, \nabla u_* + th\nabla\psi)]\,dt\,dx \\
&= \int_\Omega \int_0^1 D_A f(x,u_* + th\psi, \nabla u_* + th\nabla\psi) : \nabla\psi \\
&\qquad + D_v f(x,u_* + th\psi, \nabla u_* + th\nabla\psi)\cdot\psi\,dt\,dx
\end{aligned}
$$

where we have used chain rule of differentiation in the last inequality. By growth bounds (3.5) on the derivative and convexity of function $x \mapsto (x)^p,\ x > 0$, the integrand can be seen to have an h-uniform majorant, namely $C(1 + |u_*|^p + |\nabla\psi|^p + |\nabla\psi|^p)$. The majorant is obviously integrable.

$$
\begin{aligned}
0 &\leq \lim_{h\downarrow 0}\int_\Omega \frac{f(x,u_* + h\psi, \nabla u_* + h\nabla\psi) - f(x,u_*,\nabla u_*)}{h}\,dx \\
&= \int_\Omega \lim_{h\downarrow 0}\frac{f(x,u_* + h\psi, \nabla u_* + h\nabla\psi) - f(x,u_*,\nabla u_*)}{h}\,dx
\end{aligned}
$$

The existence of this limit is guaranteed by the differentiability conditions we imposed on the second and third arguments. Applying the Lebesgue dominated convergence theorem to pass $h \downarrow 0$ into the double integral shows

$$
0 \leq \int_\Omega D_A f(x,u_*,\nabla u_*) : \nabla\psi + D_v f(x,u_*,\nabla u_*)\cdot\psi\,dx
$$

The proof is finished after replacing test function ψ with $-\psi$. □

Remark 3.2.

- Note that the Euler-Lagrange equation is actually a boundary value problem of a system of PDEs;
- Boundary condition $u = g$ on $\partial\Omega$ is to be understood in trace sense (see, for example, [9]);

- If we want to allow the wider test function space $\psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ in weak formulation (3.8), we have to assume the stronger growth condition

$$|D_v f(x,v,A)|,\ |D_A f(x,v,A)| \le C(1+|v|^{p-1}+|A|^{p-1}) \tag{3.9}$$

in order for (3.8) to be well-defined and finite. This is simply because $\nabla\psi \in W_0^{1,p-1}(\Omega;\mathbb{R}^{m\times d})$, the integral in (3.8) is controlled by

$$\begin{aligned}
&\left|\int_\Omega D_A f(x,u_*,\nabla u_*) : \nabla\psi + D_v f(x,u_*,\nabla u_*)\cdot\psi\, dx = 0, \quad \forall\psi\in C_c^\infty(\Omega;\mathbb{R}^m)\right| \\
&\le C\int_\Omega \left(1+|u_*|^{p-1}+|\nabla u_*|^{p-1}\right)\left(|\psi|+|\nabla\psi|\right)\, dx \\
&\le C'\left(|\Omega|+||u_*||_{L^p}^{p-1}+||\nabla u_*||_{L^p}^{p-1}\right)||\psi||_{W^{1,p}}
\end{aligned}$$

The first inequality is due to Cauchy-Schwartz inequality, and the second one is due to Hölder’s inequality with $p'=\frac{p}{p-1}$. Then argue by density theorem, $\exists\varphi_k \in C_c^\infty(\Omega;\mathbb{R}^m)$ such that $\varphi_k \to \psi$ in $W^{1,p}$-norm for any $\psi\in W_0^{1,p}(\Omega;\mathbb{R}^m)$

$$\int_\Omega D_A f(x,u_*,\nabla u_*) : \nabla\varphi_k + D_v f(x,u_*,\nabla u_*)\cdot\varphi_k\, dx = 0,\ \forall\varphi_k\in C_c^\infty(\Omega;\mathbb{R}^m) \tag{3.10}$$

then by triangle inequality for integral and Cauchy-Schwartz inequality, we have the estimate for the first term in (3.10)

$$\begin{aligned}
&\left|\int_\Omega D_A f(x,u_*,\nabla u_*) : \nabla\varphi_k - \int_\Omega D_A f(x,u_*,\nabla u_*) : \nabla\psi\right| \\
&\le \int_\Omega |D_A f(x,u_*,\nabla u_*) : (\nabla\varphi_k - \nabla\psi)| \\
&\le \int_\Omega |D_A f(x,u_*,\nabla u_*)|\cdot|(\nabla\varphi_k-\nabla\psi)|
\end{aligned}$$

Letting $k\to\infty$, $\int_\Omega |(\nabla\varphi_k - \nabla\psi)| \to 0$ the original difference thus tends to 0. The second term in (3.10) is estimated in an analogous way. Finally we broaden the test function space to $W_0^{1,p}(\Omega;\mathbb{R}^m)$.

Alternatively, the claim of this item in the language of linear operator can be found in pp.50 in [2].

Using the notion of first variation (3.1), obviously we have the following necessary condition for u to be a minimizer

Corollary 3.1. $\delta\mathscr{F}[u_*] = 0$ *if* u_* *minimizes* $\mathscr{F}$ *over* $W^{1,p}_g(\Omega;\mathbb{R}^m)$.

In fact, any solution of the Euler-Lagrange equation is called a **critical point** of $\mathscr{F}$, which is further classified into a minimizer, a maximizer, or a saddle point.

However, under a convexity assumption, the solution to the Euler-Lagrange equation is always a minimizer:

Proposition 3.1. In the situation of Theorem 3.1, assume furthermore that

1. the stronger growth conditions (3.9) hold;
2. $(v, A) \mapsto f(x, v, A)$ is jointly convex $\forall x \in \Omega$, i.e.,

$$f(x, \lambda v+(1-\lambda)w, \lambda A+(1-\lambda)B) \leq \lambda f(x, v, A)+(1-\lambda)f(x, w, B) \tag{3.11}$$

for any $\lambda \in [0, 1]$. (3.11) implies that $\mathscr{F}$ is convex:

$$\begin{aligned}\mathscr{F}[tu + (1-t)v] &= \int_\Omega f(x, tu + (1-t)v, t\nabla u + (1-t)\nabla v)\,dx \\ &\leq t\int_\Omega f(x, u, \nabla u)\,dx + (1-t)\int_\Omega f(x, v, \nabla v)\,dx \\ &\leq t\mathscr{F}[u] + (1-t)\mathscr{F}[v]\end{aligned} \tag{3.12}$$

If $u_* \in W^{1,p}_g(\Omega;\mathbb{R}^m)$ solves Euler-Lagrange equation (3.7), then u_* is a minimizer of $\mathscr{F}$.

Proof. Let $v \in W^{1,p}_g(\Omega;\mathbb{R}^m)$, set $\psi := v - u_* \in W^{1,p}_0(\Omega;\mathbb{R}^m)$. Construct the function

$$g(t) := \mathscr{F}[u_* + t\psi], \quad t \in \mathbb{R}$$

Clearly, g inherits convexity from $\mathscr{F}$. In the same way to the proof of theorem 3.1 (as a fact of calculus, multivalued function is (continuously) differentiable if and only if every partial derivative exists and is continuous) g is differntiable. Since g solves the Euler-Lagrange equation, we can apply Lebesgue dominated convergence theorem to equations below (3.8), then by (3.8), we get

$$\left.\frac{d}{dt}g(t)\right|_{t=0} = 0$$

By convexity, the secant is always above the tangent line at $t = 0$ (see fig 3.1), so

$$g(t) \geq g(0) + tg'(0) = g(0) = \mathscr{F}[u_*], \quad t \geq 0$$

Take $t = 1$, we have $\mathscr{F}[v] \geq \mathscr{F}[u_*]$. As $v \in W^{1,p}_g(\Omega;\mathbb{R}^m)$ was arbitrary, u_* must be a minimizer. □

The proposition we just proved clarifies what we said in the very beginning of this section — Euler-Lagrange equation gives a way to find a minimizer. Now look at some examples.

Example 3.1. Revisit the Dirichlet functional (2.23), suppose $m = d = 3$, the associated Euler-Lagrange equation is the Laplace equation

$$-\Delta u = 0 \ \text{ in } \Omega$$

where

$$\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$$

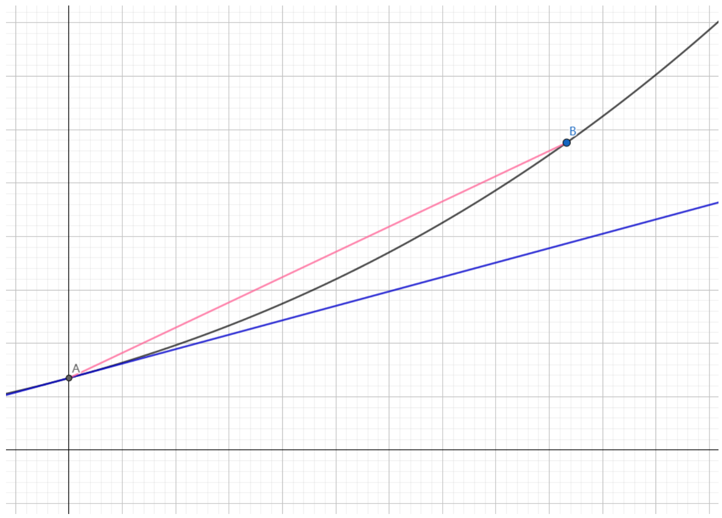


Figure 3.1: Secants is above tangent lines for convex functions

is the 3-D Laplace operator. Since the Dirichlet functional is convex, applying 3.1 wo see that all harmonic functions (solutions to the Laplace equation) are in fact minimizers of the Dirichlet functional. Proposition **??** shows that solutions of the Laplace equation are unique for given boundary values, which is compatible with our PDE knowledge. The same assertions also apply to

$$\mathscr{F}[u] := \int_\Omega \frac{1}{2}|\nabla u(x)|^2 - h(x) \cdot u(x)\, dx, \quad u \in W^{1,2}(\Omega; \mathbb{R}^m) \tag{3.13}$$

with $h \in L^2(\Omega; \mathbb{R}^m)$. This time the Euler-Lagrange equation is the Poisson equation

$$-\Delta u = h \quad \text{in } \Omega.$$

Example 3.2. In example 2.3, the prototypical problem of linearized elasticity is (2.44). The Euler-Lagrange equation is

$$\begin{cases} -\operatorname{div}\left[2\mu\mathscr{E}u + \left(\kappa - \frac{2}{3}\mu\right)(tr\mathscr{E}u)Id\right] = b & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

Theorem 3.1 says we can study minimizers by using PDE methods. Immediately, one may ask what is the type of Euler-Lagrange PDE. We have a nice result in this respect.

Definition 3.3. Define

$$A : TB := \sum_{i,j} \sum_{k,l} \mathbf{T}_{jl}^{ik} A_j^i B_l^k$$

We say that $\mathbf{T}$ is symmetric (major symmetry) if $A : \mathbf{T}B = B : \mathbf{T}A$ for all matricecs A, B, or equivalently, $\mathbf{T}_{jl}^{ik} = \mathbf{T}_{lj}^{ki}$; we say that $\mathbf{T}$ is positive definite if $A : \mathbf{T}A > 0$ for all A.

Proposition 3.2. In the situation of theorem 3.1, assume furthermore that f does **NOT** depend on v and is quadratic in A, i.e.,

$$\begin{aligned} f(x, v, A) &= \frac{1}{2} A : \mathbf{S}(x) A, \\ &= \frac{1}{2} \sum_{i,j} \sum_{k,l} \mathbf{S}_{jl}^{ik} A_l^k A_j^i, \quad (x, v, A) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \end{aligned} \tag{3.15}$$

for a 4^{th}-order symmetric tensor $\mathbf{S}(x) = \mathbf{S}_{jl}^{ik}(x),\ x \in \Omega$.

Then, the Euler-Lagrange equation is the linear PDE

$$\begin{cases} -\operatorname{div}[\mathbf{S}\nabla u] = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.16}$$

Moreover,

$$B : \mathbf{S}B = D_A^2 f(x, v, A)[B, B] := \frac{d^2}{dt^2} f(x, v, A + tB)\Big|_{t=0} \tag{3.17}$$

for all $x \in \Omega$, $v \in \mathbb{R}^{m \times d}$. Consequently, $\mathbf{S}$ is positively semidefinite if and only if $f(x, v, \cdot)$ is convex. In this case, (3.16) is (possibly degenerate) **elliptic**.

Proof. With patient computation, (3.16) follows immediately from the general Euler-Lagrange equation. For (3.17),

$$\begin{aligned}\frac{d^2}{dt^2}f(x,v,A+tB) &= \frac{d}{dt}\left[D_{A+tB}f(x,v,A+tB):B\right]\\ &= \left[D_{A+tB}(D_{A+tB}f(x,v,A+tB)):B\right]:B\end{aligned}$$

Note that in the second line we applied "D_{A+tB}" twice, so it became a 4^{th}-order tensor. Then after taking $t=0$, we attain (3.17).

Finally, first suppose that $f(A)$, which is a shorthand of $f(x,v,\cdot)$ is convex. We claim that $f(A+tB)$ is convex in t. Indeed,

$$\begin{aligned}f(A+(\mu p+(1-\mu)q)B) &= f(\mu A+(1-\mu)A+(\mu p+(1-\mu)q)B)\\ &= f(\mu(A+pB)+(1-\mu)(A+qB))\\ &\leq \mu f(A+pB)+(1-\mu)f(A+qB)\end{aligned}$$

Thus the second order derivative $\frac{d^2}{dt^2}f(x,v,A+tB)\Big|_{t=0}\geq 0$ and $\mathbf{S}$ is positively semidefinite by (3.17). Suppose, conversely, that $\mathbf{S}$ is positively semidefinite. Then also by (3.17), we know that $f(A+tB)$ is convex, similarly,

$$\begin{aligned}\mu f(A+pB)+(1-\mu)f(A+qB) &\geq f(A+(\mu p+(1-\mu)q)B)\\ &= f(\mu A+(1-\mu)A+(\mu p+(1-\mu)q)B)\end{aligned}$$

Taking $p=0$ and B,q such that for arbitrary $C\in\mathbb{R}^{m\times d}$, $C=A+qB$, then we have

$$\begin{aligned}\mu f(A)+(1-\mu)f(C) &= \mu f(A+pB)+(1-\mu)f(A+qB)\\ &\geq f(A+(1-\mu)qB)\\ &= f(\mu A+(1-\mu)(A+qB))\\ &= f(\mu A+(1-\mu)C)\end{aligned}$$

Hence, $f(A)$ is convex. □

Let's see some examples that use the Euler-Lagrange equation to find concrete solutions of variational problems.

Example 3.3 (Linearized Elasticity System). Recall equation (2.40) and its variant following (2.40), the Euler-Lagrange equation associated with linear elasticity is generally given as

$$\begin{cases} -\operatorname{div}(\mathbf{C}\nabla u) = f & \text{in } \Omega \\ \mathbf{C}\nabla u \cdot n = g & \text{on } \Gamma_1 \\ u = u_b & \text{on } \Gamma_2 \end{cases}$$

where n is the outer normal director, $f = (f_1, f_2, f_3)$ denotes the volume density of applied body forces, $g = (g_1, g_2, g_3)$ denotes the density of surface forces on force boundary portion and u_b denotes displacement on the displacement boundary portion. Keep in mind that

$$\begin{aligned} \mathbf{C}\nabla u &= \mathbf{C}^{ik}_{jl}\frac{1}{2}\left(\frac{\partial u_k}{\partial u_l} + \frac{\partial u_l}{\partial u_k}\right) \\ &= \mathbf{C}\mathscr{E}u \\ &= \sigma^i_j \end{aligned}$$

The last equality is the famous Hooke's Law. Hence $\mathbf{C}\nabla u$ means the "stress" physically.

Example 3.4 (♠). This example shows application to financial science. The optimal saving model is constructed in section 1.5 of [2]. Here we omit the modeling process.

$$\begin{cases} \text{Minimize} & \mathscr{F}[S] := \displaystyle\int_0^T -\ln(1 + w + \rho S(t) - \dot{S}(t))\,dt \\ \text{subject to} & S(0) = 0, S(T) = S_T \geq 0, C(t) := w + \rho S(t) - \dot{S}(t) \geq 0 \end{cases} \tag{3.18}$$

Can you tell the best savings strategy for a worker such that it can be the happiest?

At last, it's natural to ask whether the weak solution of Euler-Lagrange equation (3.7) is also a strong solution or a classical solution. Recall that u is called a strong solution if $u \in W^{2,2}(\Omega;\mathbb{R}^m)$ satisfies

$$\begin{cases} -\operatorname{div}[D_A f(x,u(x),\nabla u(x))] + D_v f(x,u(x),\nabla u(x)) = 0 & \text{for } \textbf{a.e. } x \in \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.19}$$

u is called a classical solution if $u \in C^2\Omega;\mathbb{R}^m \cap C^0(\bar{\Omega};\mathbb{R}^m$ satisfies

$$\begin{cases} -\operatorname{div}[D_A f(x,u(x),\nabla u(x))] + D_v f(x,u(x),\nabla u(x)) = 0 & \text{for } \textbf{every } x \in \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.20}$$

The strong solution is always a weak solution (try to show this claim. Hint: Integration by parts and Gauss-Green theorem), the converse is not always true, but we have the following proposition

Proposition 3.3. Let u be a weak solution of the Euler-Lagrange equation (3.7), assume that $u \in W^{2,2}(\Omega;\mathbb{R}^m)$ and the integrand f is twice continuously differentiable with respect to v and A. Then u solves (3.7) in the strong sense.

Proof. Since $u \in W^{2,2}$ is a weak solution, then

$$\int_\Omega D_A f(x,u,\nabla u) : \nabla\psi + D_v f(x,u,\nabla u)\cdot\psi \, dx = 0, \quad \forall \psi \in C_c^\infty(\Omega;\mathbb{R}^m)$$

since f is twice continuously differentiable, the second Gauss-Green theorem (integration by parts) gives

$$\int_\Omega \Big(-\operatorname{div}[D_A f(x,u,\nabla u)] + D_v f(x,u,\nabla u)\Big)\cdot\psi \, dx = 0, \quad \forall \psi \in C_c^\infty(\Omega;\mathbb{R}^m)$$

Since the term in the outer parenthesis is L^1-integrable and Ω is an open set, then corollary 4.24 of [7] shows that

$$-\operatorname{div}[D_A f(x,u,\nabla u)] + D_v f(x,u,\nabla u) = 0 \quad \text{a.e. in } \Omega \tag{3.21}$$

□

3.2 Regularity of Minimizers

From proposition 3.3, we find that once we have higher regularity i.e., the weak differentiability of the solution u of Euler-Lagrange equation and the differentiability of integrand f, solution u is improved to be strong solution. More generally, one would like to know *how much* regularity we can expect from solutions of a variational problem. This is the content of famous Hilbert's 19th problem.

We remark at the outset that many regularity results are sensitive to the dimensions of the domain and the target space. In particular, the behaviour of the scalar case ($m=1$) and the vector case ($m>1$) is fundamentally different.

Definition 3.4. In the spirit of Hilbert's 19th problem, call

$$\mathscr{F}[u] := \int_\Omega f(\nabla u(x))\,dx, \quad u \in W^{1,2}(\Omega;\mathbb{R}^m) \tag{3.22}$$

a **regular variational integral** if $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is twice continuously differentiable and $\exists$ constants $\mu, M > 0$ with

$$\mu|B|^2 \le D_A^2 f(A)[B,B] \le M|B|^2, \quad A,B \in \mathbb{R}^{m\times d} \tag{3.23}$$

where, as in (3.17),

$$D_A^2 f(A)[B,B] := \frac{d^2}{dt^2} f(A+tB)\Big|_{t=0} \tag{3.24}$$

By the lower bound of (3.23), (3.24) and the proof of proposition 3.2, the regular variational problems are **strongly convex**.

It can be easily verified that the Dirichlet functional 2.2 is a regular variational integral.

Since the regularity result needs plenty of estimates using difference quotient and bootstrapping method, we omit details under the consideration of concision.

Theorem 3.2. *Let $\mathscr{F}$ be a regular variational integral. Then, for any minimizer $u_* \in W^{1,2}(\Omega;\mathbb{R}^m)$, it holds that*

$$u_* \in W^{2,2}_{loc}(\Omega;\mathbb{R}^m) \tag{3.25}$$

Moreover, the Caccioppoli inequality holds, as a consequence of which, the Euler-Lagrange equation is satisfied strongly,

$$-\operatorname{div} Df(\nabla u_*) = 0 \quad a.e. \ \ in\ \Omega \tag{3.26}$$

This theorem is generalized by De Giorgi (1957), Nash (1958) and Moser (1960) to

Theorem 3.3 (**De Giorgi 1957 & Nash 1958 & Moser 1960**)**.** *Let $\mathscr{F}$ be a regular variational integral with n-times continuously differentiable integrand $f : \mathbb{R}^{d\times d} \to \mathbb{R}$, where $n \in \{2, 3, ...\}$.*

If $u_* \in W^{1,2}(\Omega;\mathbb{R}^d)$ minimizes $\mathscr{F}$, then $u_* \in C^{n-1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. In particular, if f is smooth, then $u_* \in C^\infty(\Omega)$.

3.3 Lagrange Multipliers

3.4 Invariances and Noether's Theorem

In mechanics and physics, symmetries are frequently referred to. Symmetry is interpreted as invariant property under variation. Actually the Noether theorem that will be introduced in this section roughly says that "differentiable invariances of a functional give rise to conservation laws".

Part II

Finite Element Methods

Chapter 4

Theory

Chapter 5

Implementation

There're some commercial "black-boxes" such as Abaqus, Comsol and Matlab. They are quite enough for the industrial use. But we prefer open sources softwares due to their modifiable creativity. *Netgen/NGSolve* and *FreeFem++* use C++ to solve pde, while our protagonist — SfePy uses Python libraries. SfePy is quite open and clear. What's more, the community is much better for these days!

5.1 SfePy for Elasticity

Part III

Applications to Liquid Crystal Elastomers

Chapter 6

Models

The field of modeling, analysis and numerical computation of LCEs is best fields combining both homogenization theory ([5]) with variational methods.

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