Referential Solutions (HW2)

Section 3

Problem 1

(a)

$$
\begin{aligned}
G(A+U) &= [tr(A+U)]\,(A+U)\\
&= (trA + trU)\,(A+U)\\
&= (trA)A + (trA)U + (trU)A + o(U)\\
D\ G(A) &= (trA)U + (trU)A
\end{aligned}
$$

(b)

$$
\begin{aligned}
G(A+U) &= (A+U)B(A+U)\\
&= (A+U)\,(BA+BU)\\
&= ABA + ABU + UBA + o(U)\\
D\ G(A) &= ABU + UBA
\end{aligned}
$$

(c)

$$
\begin{aligned}
G(A+U) &= (A+U)^T(A+U)\\
&= A^TA + A^TU + U^TA + o(U)\\
D\ G(A) &= A^TU + U^TA
\end{aligned}
$$

(d)

$$
\begin{aligned}
G(A+U) &= [u\cdot(A+U)u]\,(A+U)\\
&= (u\cdot Au)A + (u\cdot Au)U + (u\cdot Uu)A + o(U)\\
D\ G(A) &= (u\cdot Au)U + (u\cdot Uu)A
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
D[\det(A^2)] &= \det(A^2)tr[D(A^2)A^{-2}] \\
&= \det(A^2)tr[(AU+UA)A^{-2}] \\
&= \det(A^2)tr(AUA^{-2}+UAA^{-2}) \\
&= 2\det(A^2)tr(UA^{-1})
\end{aligned}
$$

Problem 4

$$D[e^{\boldsymbol{v}^2}] = e^{\boldsymbol{v}^2}D[\boldsymbol{v}^2] = 2e^{\boldsymbol{v}^2}(\boldsymbol{v}\cdot\boldsymbol{u})$$

Problem 7

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$Q = e^{Wt} = I + Wt + \frac{1}{2}(Wt)^2 + \cdots$$

$$Q^T = e^{(Wt)^T} = I - Wt + \frac{1}{2}(Wt)^2 + \cdots$$

$$QQ^T = Q^TQ = [I + tW + o(Wt)][I - tW + o(Wt)] = I + o(Wt)$$

$$\lim_{t\to 0} QQ^T = \lim_{t\to 0}[I + o(Wt)] = I$$

$$e^{Wt}G(A)(e^{Wt})^T = G(e^{Wt}A)$$

Take the partial derivative with respect to t on both sides, we have

$$We^{Wt}G(A)(e^{Wt})^T + e^{Wt}G(A)W^T(e^{Wt})^T = [D\ G(e^{Wt}A)](We^{Wt}A) \qquad (*)$$

Take the limit $\lim_{t\to 0} e^{Wt} = I$, equation (*) is reduced to

$$WG(A) + G(A)W^T = [D\ G(A)](WA)$$

Problem 9

$$
\begin{aligned}
D(I_1) &= D(trA) \\
&= trU \\
D(I_2) &= D\left\{\frac{1}{2}[(trA)^2 - tr(A^2)]\right\} \\
&= \frac{1}{2}[(trA)(trU) + (trU)(trA) - tr(AU + UA)] \\
&= (trA)(trU) - tr(AU) \\
D(I_3) &= D(\det A) \\
&= (\det A)tr(UA^{-1})
\end{aligned}
$$

Section 4

Problem 1

(a)

$$
\nabla(\alpha\varphi) = (\nabla\alpha)\varphi + \alpha(\nabla\varphi)
$$

(b)

$$
\begin{aligned}
\nabla[(\vec{u}\cdot\vec{v})\vec{w}] &= \vec{w}\nabla(\vec{u}\cdot\vec{v}) + (\vec{u}\cdot\vec{v})\nabla\vec{w} \\
&= \vec{w}[(\nabla\vec{u})'\cdot\vec{v} + (\nabla\vec{v})'\cdot\vec{u}] + (\vec{u}\cdot\vec{v})\nabla\vec{w}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\nabla\cdot(\varphi \mathrm{S}\vec{v}) &= (\nabla\varphi)\cdot(\mathrm{S}\vec{v}) + \varphi\nabla\cdot(\mathrm{S}\vec{v}) \\
&= (\nabla\varphi)\cdot(\mathrm{S}\vec{v}) + \varphi(\nabla\cdot\mathrm{S})\cdot\vec{v} + \varphi\mathrm{S}':(\nabla\vec{v})
\end{aligned}
$$

(d)

$$\begin{aligned}\Delta(\vec{v}\cdot\vec{w}) &= \nabla\cdot\nabla(\vec{v}\cdot\vec{w})\\ &= \nabla\cdot[(\nabla\vec{v})'\cdot\vec{w}+(\nabla\vec{w})'\cdot\vec{v}]\\ &= \nabla\vec{v}:\nabla\vec{w}+(\Delta\vec{w})\cdot\vec{v}+\nabla\vec{w}:\nabla\vec{v}+(\Delta\vec{v})\cdot\vec{w}\\ &= (\Delta\vec{w})\cdot\vec{v}+(\Delta\vec{v})\cdot\vec{w}+2(\nabla\vec{w}:\nabla\vec{v})\end{aligned}$$

Problem 4

From $(7)_4$, we have

$$[\mathrm{div}(v\otimes w)]_i = \frac{\partial(v_i w_j)}{\partial x_j} = \frac{\partial v_i}{\partial x_j}w_j + v_i\frac{\partial w_j}{\partial x_j}$$

From $(7)_2$, we have

$$\frac{\partial v_i}{\partial x_j}w_j + v_i\frac{\partial w_j}{\partial x_j} = (\nabla\vec{v})\cdot\vec{w} + (\mathrm{div}\vec{w})\vec{v}$$

From $(7)_3$, we have

$$\mathrm{div}(\mathrm{S}\vec{v}) = \frac{\partial(S_{ij}v_j)}{\partial x_i} = \frac{\partial S_{ij}}{\partial x_i}v_j + S_{ij}\frac{\partial v_j}{\partial x_j}$$

From $(7)_2$ and $(7)_4$, we have

$$\frac{\partial S_{ij}}{\partial x_i}v_j + S_{ij}\frac{\partial v_j}{\partial x_j} = \vec{v}\cdot\mathrm{divS} + \mathrm{S}\cdot(\nabla\cdot\vec{v})$$

Problem 6

(a)

$$\begin{cases}\boldsymbol{r}(\boldsymbol{x}+\boldsymbol{u}) = \boldsymbol{x}+\boldsymbol{u}-\boldsymbol{o}\\ \boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{x}-\boldsymbol{o}\end{cases} \Rightarrow \boldsymbol{r}(\boldsymbol{x}+\boldsymbol{u}) - \boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{u} = \mathrm{I}\boldsymbol{u} \Rightarrow \nabla\boldsymbol{r} = \mathrm{I}$$

(b)

$$\begin{aligned}(\nabla\boldsymbol{e})\boldsymbol{e} &= \left(\nabla\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\right)\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\\ &= \left[\frac{1}{|\boldsymbol{r}|}\nabla\boldsymbol{r} + \boldsymbol{r}\left(\nabla\frac{1}{|\boldsymbol{r}|}\right)\right]\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\\ &= \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2} + \left(\nabla\frac{1}{|\boldsymbol{r}|}\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\right)\boldsymbol{r}\\ &= \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2} + \left(-\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|}\right)\boldsymbol{r}\\ &= \left(\frac{1}{|\boldsymbol{r}|^2} - \frac{1}{|\boldsymbol{r}|^2}\right)\boldsymbol{r} = 0\end{aligned}$$

Problem 8

$$\boldsymbol{u}=\frac{\boldsymbol{r}}{|\boldsymbol{r}|^{3}}=\frac{(x_1,x_2,x_3)}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

For the sake of simplicity, denote $\Delta\boldsymbol{u}$ by $(\Delta u_1,\Delta u_2,\Delta u_3)$

$$\begin{aligned}\Delta u_1&=\frac{15x_1x_1^2}{|\boldsymbol{r}|^{7}}-\frac{9x_1}{|\boldsymbol{r}|^{5}}+\frac{15x_1x_2^2}{|\boldsymbol{r}|^{7}}-\frac{3x_1}{|\boldsymbol{r}|^{5}}+\frac{15x_1x_3^2}{|\boldsymbol{r}|^{7}}-\frac{9x_1}{|\boldsymbol{r}|^{5}}\\&=\frac{15x_1(x_1^2+x_2^2+x_3^2)}{|\boldsymbol{r}|^{7}}-\frac{15x_1}{|\boldsymbol{r}|^{5}}\\&=\frac{15x_1}{|\boldsymbol{r}|^{5}}-\frac{15x_1}{|\boldsymbol{r}|^{5}}\\&=0\end{aligned}$$

Therefore, $\boldsymbol{u}$ is harmonic.

$$\nabla\left(\lambda\frac{1}{|\boldsymbol{r}|^{t}}\right)=\frac{-\lambda t}{|\boldsymbol{r}|^{2+t}}\boldsymbol{r}\Rightarrow\begin{cases}2+t=3\\-\lambda t=1\end{cases}\Rightarrow\begin{cases}t=1\\\lambda=-1\end{cases}\Rightarrow\nabla\left(-\frac{1}{|\boldsymbol{r}|}\right)=\boldsymbol{u}$$

Problem 10

$$\begin{aligned}\operatorname{div}(\vec{u}\times\vec{v})&=\frac{\partial(\varepsilon_{ijk}u_jv_k)}{\partial x_i}=v_k\varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}+u_j\varepsilon_{ijk}\frac{\partial v_k}{\partial x_i}\\&=v_k\varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}-u_j\varepsilon_{jik}\frac{\partial v_k}{\partial x_i}=\vec{v}\cdot\operatorname{curl}\vec{u}-\vec{u}\cdot\operatorname{curl}\vec{v}\end{aligned}$$

Section 5

Problem 2

$(\Rightarrow)$

$$\int_{\partial P}(\boldsymbol{v}\cdot\boldsymbol{n})dA=\int_P(\nabla\cdot\boldsymbol{v})dV=V_P(\nabla\cdot\boldsymbol{v})|_{P_0}$$

$$\therefore\nabla\cdot\boldsymbol{v}=\lim_{d(P)\to0}\frac{\int_{\partial P}(\boldsymbol{v}\cdot\boldsymbol{n})dA}{V_P}=0$$

$(\Leftarrow)$

$$\int_{\partial P}(\boldsymbol{v}\cdot\boldsymbol{n})dA=\int_P(\nabla\cdot\boldsymbol{v})dV=\int_P0dV=0$$