Continuum Mechanics - A (2022 Fall)

(Reference Solutions Manual)

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评分总则：满分 200，共 8 大题。每大题 25 分。按照预期每次作业占总评 100 分的 2 分，也就是作业的每 100 分相当于总评的 1 分。接受误判及分数错误求和这两种情况的 argument。

Section 7 pp.57

Exercise 1. Given one-parameter family of deformations with $|\nabla u_\epsilon| = \epsilon$ small, show that:

1. $E_\epsilon = U_\epsilon - I + o(\epsilon) = V_\epsilon - I + o(\epsilon)$
2. $\det F_\epsilon - 1 = div(u_\epsilon) + o(\epsilon)$

Give a physical interpretation of $\det F_\epsilon - 1$.

Solution.

For the sake of briefness, we omit the subscript ϵ.

For 1, recall that $U = \sqrt{C} = \sqrt{F^T F}$, $C = I + \nabla u + \nabla u^T + \nabla u^T \nabla u$, thus

$$C = I + 2E + \nabla u^T \nabla u$$

Let $H := \nabla u$, by definition of Frechet derivative, along with the smiling inequality $|AB| \leq C(n)|A||B|$, we have

$$\begin{aligned} C(H) &= C(0) + DC(0)[H] + o(H) \\ &= C(0) + 2DU(0)[H] + o(H) \\ &= C(0) + 2E(H) + o(H) \end{aligned}$$

Therefore,

$$\begin{aligned} U(H) &= U(0) + DU(0)[H] + o(\epsilon) \\ &= U(0) + E + o(\epsilon) \\ &= I + E + o(\epsilon) \end{aligned}$$

Thus $E = U - I + o(\epsilon)$. $E = V - I + o(\epsilon)$ is obtain in an analogous way.

For 2,

$$F = \begin{pmatrix} u_{1,1}+1 & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2}+1 & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3}+1 \end{pmatrix}$$

Then,

$$\begin{aligned}\det F &= (u_{1,1}+1)(u_{2,2}+1)(u_{3,3}+1)+o(\epsilon)\\ &= u_{1,1}+u_{2,2}+u_{3,3}+o(\epsilon)\\ &= div(u)+o(\epsilon)\end{aligned}$$

Physical interpretation is given in solution to exercise 4 in hw3: that is, the volume change per unit volume under the deformation f_ϵ.

评分细则:

Exercise 4. Let $W=\frac{1}{2}(\nabla u-\nabla u^T)$. Show that

1. $|E|^2+|W|^2=|\nabla u|^2$

2. $|E|^2-|W|^2=\nabla u\cdot\nabla u^T$

Solution.

$$\begin{aligned}|E|^2 &= \frac{1}{2}(\nabla u+\nabla u^T)\cdot\frac{1}{2}(\nabla u+\nabla u^T)\\ &= \frac{1}{4}\left(\nabla u\cdot\nabla u+\nabla u\cdot\nabla u^T+\nabla u^T\cdot\nabla u+\nabla u^T\cdot\nabla u^T\right)\\ |W|^2 &= \frac{1}{2}(\nabla u-\nabla u^T)\cdot\frac{1}{2}(\nabla u-\nabla u^T)\\ &= \frac{1}{4}\left(\nabla u\cdot\nabla u-\nabla u\cdot\nabla u^T-\nabla u^T\cdot\nabla u+\nabla u^T\cdot\nabla u^T\right)\end{aligned}$$

Therefore,

$$\begin{aligned}|E|^2+|W|^2 &= \frac{1}{2}\nabla u\cdot\nabla u+\frac{1}{2}\nabla u^T\cdot\nabla u^T\\ &= \frac{1}{2}|\nabla u|^2+\frac{1}{2}|\nabla u|^2\\ &= |\nabla u|^2\\ |E|^2-|W|^2 &= \frac{1}{2}\nabla u\cdot\nabla u^T+\frac{1}{2}\nabla u^T\cdot\nabla u\\ &= \nabla u\cdot\nabla u^T\end{aligned}$$

评分细则:

Exercise 5. (Korn's inequality) Let $u\in C^2(\bar{\mathscr{R}})$ and suppose that $u=0$ on $\partial\mathscr{R}$. Show that

$$\int_{\mathscr{R}}|\nabla u|^2\,dx\le 2\int_{\mathscr{R}}|E|^2\,dx.$$

Solution. By exercise 4,

$$\begin{aligned}2\int_{\mathscr{R}}|E|^2\,dx &= \int_{\mathscr{R}}(|E|^2+|W|^2)+(|E|^2-|W|^2)\,dx\\ &= \int_{\mathscr{R}}|\nabla u|^2\,dx+\int_{\mathscr{R}}\nabla u\cdot\nabla u^T\,dx\end{aligned}$$

It suffices to show that the second term is nonnegative. Use exercise 9 (b) in section 4, along with divergence theorem,

$$
\begin{aligned}
\int_{\mathscr{R}} \nabla u \cdot \nabla u^T \, dx &= \int_{\mathscr{R}} \operatorname{div}\left((\nabla u)u - (\operatorname{div} u)u\right) + (\operatorname{div} u)^2 \, dx \\
&= \int_{\mathscr{R}} (\operatorname{div} u)^2 \, dx + \int_{\partial\mathscr{R}} \left((\nabla u)u - (\operatorname{div} u)u\right) \cdot \vec{\nu} \, ds \\
&\xlongequal{u|_{\partial\mathscr{R}}=0} \int_{\mathscr{R}} (\operatorname{div} u)^2 \, dx \\
&\geq 0.
\end{aligned}
$$

评分细则：

注：广义 Korn's 不等式的意义在于定义索伯列夫空间 $H_0^1(\mathscr{R})$ 上 u 的等价范数，进而方便处理弹性力学中的 PDE 边界值问题。这里是具有二阶正则性时的 strong version。

Exercise 6. Pure torsion is given as

$$
\begin{aligned}
x_1 &= r\cos(\theta), \quad & p_1 &= R\cos(\Theta), \\
x_2 &= r\sin(\theta), \quad & p_2 &= R\sin(\Theta), \\
x_3 &= z, & p_3 &= Z,
\end{aligned}
$$

where

$$
r = R, \ \theta = \Theta + \alpha Z, \ z = Z
$$

Find the displacement components in terms of $p_i, \ i = 1, 2, 3$ and $\beta = \alpha p_3$. Furthermore, show that

1. $$\nabla u \to 0 \text{ and } u \to 0 \text{ as } \alpha \to 0;$$

2. $$
\begin{aligned}
u_1(p, \alpha) &= -\alpha p_2 p_3 + o(\epsilon), \\
u_2(p, \alpha) &= \alpha p_1 p_3 + o(\epsilon).
\end{aligned}
$$

Solution.

$$
\begin{aligned}
u_1(p) &= R\cos(\Theta + \alpha Z) - R\cos(\Theta) \\
&= R\left[\cos(\Theta)\cos(\beta) - \sin(\Theta)\sin(\beta)\right] - R\cos(\Theta) \\
&= p_1\cos(\beta) - p_2\sin(\beta) - p_1 \\
&= p_1(\cos(\beta) - 1) - p_2\sin(\beta)
\end{aligned}
$$

In the same fashion, it can be easily concluded that

$$u_2(p) = p_2(\cos(\beta) - 1) + p_1 \sin(\beta)$$

Finally,

$$u_3(p) = z - Z = 0$$

Since $\sin(\gamma) \sim \gamma,\ \cos(\gamma) \sim 1$ for small γ, the asymptotic expressions for u_1, u_2 as $\alpha \to 0$ are

$$\begin{aligned} u_1(p,\alpha) &= p_1(\cos(\beta) - 1) - p_2 \sin(\beta) \\ &= -p_2\beta + o(\alpha) \\ &= -\alpha p_2 p_3 + o(\alpha), \quad \text{as } \alpha \to 0 \end{aligned}$$

$$\begin{aligned} u_2(p,\alpha) &= p_2(\cos(\beta) - 1) + p_1 \sin(\beta) \\ &= p_1\beta + o(\alpha) \\ &= -\alpha p_1 p_3 + o(\alpha), \quad \text{as } \alpha \to 0 \end{aligned}$$

Compute the values at $\alpha = 0$:

$$u(p,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\nabla_p u(p,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By the sequential continuity of u and ∇u, we obtain

$$u \xrightarrow{\alpha\to 0} 0, \quad \nabla u \xrightarrow{\alpha\to 0} 0$$

评分细则：没有过程扣 5 分。

Section 8 pp.66

Exercise 2. Show that

$$\dot{C} = 2F^T D_m F$$

Solution.

$$\begin{aligned} \dot{C}(p,t) &= F^T\dot{F} + (\dot{F^T})F \\ &= F^T\dot{F} + (\dot{F})^T F \end{aligned}$$

$$\begin{aligned} 2F^T D_m F &= F^T(\operatorname{grad} v)_m F + F^T\left[(\operatorname{grad} v)^T\right]_m F \\ &= F^T(\operatorname{grad} v)_m F + F^T\left[(\operatorname{grad} v)_m\right]^T F \end{aligned} \tag{1}$$

To prove $\dot{C} = 2F^T D_m F$, it suffices to show that

$$\dot{F}(p,t) = (\operatorname{grad} v)_m F(p,t)$$

because $\dot{F}^T = F^T \left[(\operatorname{grad} v)_m\right]^T$. Equation (1) is already given in eqn. $(8)_1$ in section 8.

评分细则：

Exercise 4. Given a motion $\mathscr{E}$

$$\begin{aligned} x_1 &= p_1 e^t, \\ x_2 &= p_2 + t, \\ x_3 &= p_3. \end{aligned}$$

in some Cartesian frame. Compute the spatial velocity field v and determine the streamlines.

Solution. The reference mapping is

$$\begin{cases} p_1 = x_1 e^{-t} \\ p_2 = x_2 - t \\ p_3 = x_3 \end{cases}$$

The velocity field is by definition,

$$\dot{x}(p,t) = \begin{pmatrix} p_1 e^t \\ 1 \\ 0 \end{pmatrix}$$

Applying the reference mapping p to obtain the spatial velocity field:

$$v(x,t) = \begin{pmatrix} x_1 \\ 1 \\ 0 \end{pmatrix}$$

The steamlines ODEs are

$$\begin{cases} \dot{s}_1 = s_1 \\ \dot{s}_2 = 1 \\ \dot{s}_3 = 0 \end{cases}$$

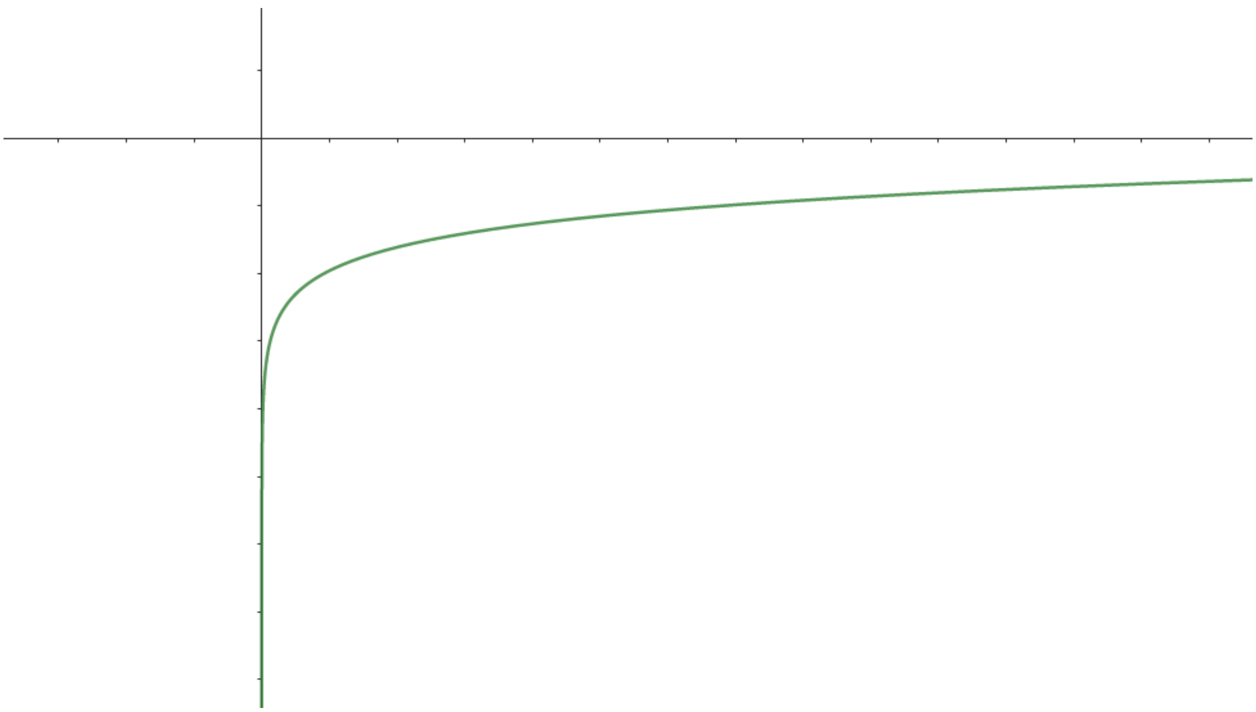
Integration shows that

$$\begin{aligned} \int \frac{ds_1}{s_1} &= \int d\lambda \\ \ln^{|s_1|} &= \lambda + \tilde{C} \\ s_1(\lambda) &= Ce^{\lambda} \end{aligned}$$

where $\tilde{C}$ is integration constant. $s_1(0) = x_1(p, 0) = y_1$, thus, $C = y_1$. After performing the same process for s_2, s_3, we finally obtain the streamlines equations at time t passing through (y_1, y_2, y_3) at $\lambda = 0$:

$$\begin{cases} s_1(\lambda) = y_1 e^{\lambda} \\ s_2(\lambda) = \lambda + y_2 \\ s_3(\lambda) = y_3 \end{cases}$$

The sketch is



Exercise 5. Consider the motion ξ defined by

$$x(p, t) = p_0 + U(t)[p - p_0],$$

where

$$U(t) = \sum_{i=1}^{3} \alpha_i(t) e_i \otimes e_i$$

$\alpha_i > 0$ are smooth and $\{e_i\}$ is an orthonormal basis. Compute p, v, L and determine the streamlines.

Solution.

We make the convention that superscript (i) always stands for the $i - th$ Cartesian component. Since

$$\begin{cases} x^{(1)} = p_0^{(1)} + \alpha_1(t)[p^{(1)} - p_0^{(1)}] \\ x^{(2)} = p_0^{(2)} + \alpha_2(t)[p^{(2)} - p_0^{(2)}] \\ x^{(3)} = p_0^{(3)} + \alpha_3(t)[p^{(3)} - p_0^{(3)}] \end{cases}$$

We have the reference mapping p:

$$\begin{cases} p^{(1)} = p_0^{(1)} + \dfrac{x^{(1)} - p_0^{(1)}}{\alpha_1(t)} \\ p^{(2)} = p_0^{(2)} + \dfrac{x^{(2)} - p_0^{(2)}}{\alpha_2(t)} \\ p^{(3)} = p_0^{(3)} + \dfrac{x^{(3)} - p_0^{(3)}}{\alpha_3(t)} \end{cases}$$

The velocity field:

$$\dot{x}(p,t) = \begin{pmatrix} \dot{\alpha}_1(t)[p^{(1)} - p_0^{(1)}] \\ \dot{\alpha}_2(t)[p^{(2)} - p_0^{(2)}] \\ \dot{\alpha}_3(t)[p^{(3)} - p_0^{(3)}] \end{pmatrix}$$

Apply reference mapping p to obtain the spatial velocity field:

$$v(x,t) = \begin{pmatrix} \dot{\alpha}_1(t)\frac{x^{(1)} - p_0^{(1)}}{\alpha_1(t)} \\ \dot{\alpha}_2(t)\frac{x^{(2)} - p_0^{(2)}}{\alpha_2(t)} \\ \dot{\alpha}_3(t)\frac{x^{(3)} - p_0^{(3)}}{\alpha_3(t)} \end{pmatrix}$$

The velocity gradient L:

$$L = \operatorname{grad} v = \begin{pmatrix} \frac{\dot{\alpha}_1(t)}{\alpha_1(t)} & 0 & 0 \\ 0 & \frac{\dot{\alpha}_2(t)}{\alpha_2(t)} & 0 \\ 0 & 0 & \frac{\dot{\alpha}_3(t)}{\alpha_3(t)} \end{pmatrix}$$

Streamlines equations are given by

$$\dot{s}(\lambda) = \begin{pmatrix} \dot{\alpha}_1(\tau)\frac{s^{(1)}(\lambda) - p_0^{(1)}}{\alpha_1(t)} \\ \dot{\alpha}_2(\tau)\frac{s^{(2)}(\lambda) - p_0^{(2)}}{\alpha_2(t)} \\ \dot{\alpha}_3(\tau)\frac{s^{(3)}(\lambda) - p_0^{(3)}}{\alpha_3(t)} \end{pmatrix}$$

By change of variables $w^{(i)} = s^{(i)} - p_0^{(i)}$, it becomes ODEs of $w^{(i)}$'s. It follows that

$$w^{(i)}(\lambda) = Ce^{\frac{\dot{\alpha}^{(i)}(\tau)}{\alpha^{(i)}(\tau)}\lambda}$$

Integration constant C is determined by the initial conditions. In conclusion, the streamlines are:

$$s^{(i)}(\lambda) = p_0^{(i)} + (y^{(i)} - p_0^{(i)})e^{\frac{\dot{\alpha}^{(i)}(\tau)}{\alpha^{(i)}(\tau)}\lambda}$$

评分细则：

Exercise 7. Consider a surface $\mathscr{P} = \{p \in \mathscr{D} \subset \mathscr{R} | \varphi(p) = 0\}$, where $\mathscr{D}$ is an open subset of $\mathscr{R}$. φ is a smooth function defined on $\mathscr{D}$, whose gradient doesn't vanish on $\mathscr{P}$. Consider the trajectory $\mathscr{P}_t$ associated with motion x:

$$\mathscr{P}_t = \{x \in \mathscr{D}_t | \psi(x,t) = 0\},$$

where $\mathscr{D}_t = x(\mathscr{D}, t)$, $\psi(x,t) = \varphi(p(x,t))$.

Show that

(a) $\nabla\varphi(\mathrm{p})$ $(p \in \mathscr{P})$ is normal to $\mathscr{P}$;

(b) $\operatorname{grad}\psi(x,t)$ $(x \in \mathscr{P}_t)$ is normal to $\mathscr{P}_t$;

(c) $\nabla\varphi = F^T(\operatorname{grad}\psi)_m$, and hence $\operatorname{grad}\psi(x,t)$ never vanishes on $\mathscr{P}_t$;

(d) $|\nabla\varphi|^2 = (\operatorname{grad}\psi)_m \cdot B(\operatorname{grad}\psi)_m$, $B = FF^T$;

(e) $\psi' = -v \cdot \operatorname{grad}\psi$.

Solution.

Firstly, note that 0 is a regular value of φ, the surface is thus a regular surface, for which tangent plane and the normal is well defined.

By observation, (b) follows immediately from (a). Indeed, it suffices to formally replace counterparts for (b) in (a); (d) follows immediately from (c). Indeed, by asscociative law,

$$|\nabla\varphi|^2 = \nabla\varphi \cdot \nabla\varphi = (F^T(\operatorname{grad}\psi)_m)^T(F^T(\operatorname{grad}\psi)_m) = (\operatorname{grad}\psi)_m \cdot FF^T(\operatorname{grad}\psi)_m.$$

Besides, (e) follows immediately from eqn. (4) in section 8, since $\dot{\psi} = (\dot{\varphi_\sigma}) = (\dot{\varphi})_\sigma = 0$.

Therefore, it suffices to validate (a) and (b).

For (a): Given any but fixed $p \in \mathscr{P}$, we arbitrarily take a regular curve restricted on surface $\mathscr{P}$ $\alpha : (-\epsilon, \epsilon) \mapsto \mathbb{R}^3$ such that $\alpha(0) = p$.

$$\varphi \circ \alpha(t) = \varphi\left(\alpha_1(t), \alpha_2(t), \alpha_3(t)\right) = 0$$

Taking derivative w.r.t. t at $t = 0$ leads to

$$\varphi_{p_1} \cdot \dot{\alpha}_1 + \varphi_{p_2} \cdot \dot{\alpha}_2 + \varphi_{p_3} \cdot \dot{\alpha}_3 = \nabla\varphi(p) \cdot \dot{\alpha}(t) = 0$$

Thus, $\nabla\varphi(p)$ is normal to any tangents at p.

For (c):

$$\psi_m\left(x(p,t),t\right) = \psi(\cdot,t)\circ x(p,t)$$

$$\nabla\psi_m = \nabla_p\psi_m = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_2}{\partial p_1} & \frac{\partial x_3}{\partial p_1} \\ \frac{\partial x_1}{\partial p_2} & \frac{\partial x_2}{\partial p_2} & \frac{\partial x_3}{\partial p_2} \\ \frac{\partial x_1}{\partial p_3} & \frac{\partial x_2}{\partial p_3} & \frac{\partial x_3}{\partial p_3} \end{pmatrix} \begin{pmatrix} \frac{\partial\psi}{\partial x_1}(x(p,t),t) \\ \frac{\partial\psi}{\partial x_2}(x(p,t),t) \\ \frac{\partial\psi}{\partial x_3}(x(p,t),t) \end{pmatrix} = F^T(\operatorname{grad}\psi)_m$$

One the other hand, recall that $\varphi(p(x,t)) = \psi(x,t)$, hence

$$\begin{aligned} \psi_m(p,t) &= [\varphi(p(x,t))]_m \\ &= \varphi\left(p\left(x\left(p,t\right),t\right)\right) \\ &= \varphi(p) \end{aligned}$$

and $\nabla\psi_m = \nabla\varphi$. We've done.