# Continuum Mechanics - A (2022 Fall)

(Reference Solutions Manual)

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评分总则：满分 200，共 8 大题，每大题 25 分。保底分为 15 分，换言之即便一大题全写错也能得 15 分。按照预期每次作业占总评 100 分的 2 分，也就是作业的每 100 分相当于总评的 1 分。即便作业的**每道大题都乱写也至多会扣总评的 *0.8* 分**。评分规则不接受 argue；只有误判或者分数求和出错这两种情况接受 argue。

## Section 1.1 pp.9

**Exercise 1.** Choose $\mathbf{a} \in \mathscr{V}$ and let $\psi : \mathscr{V} \to \mathbb{R}$ be defined by $\psi(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v}$. Show that $\mathbf{a} = \sum_i \psi(\mathbf{e}_i)\mathbf{e_i}$.

*Solution.* The theorem is precisely saying the famous Riesz representation thm: any functionals in dual space of $\mathscr{V}$ can be writen as an inner product with a unique fixed element (depending on the functional) in the linear space. This result can be generalized to infinite dimension spaces, e.g., $L^p$ space, Hilbert spaces, etc.

For finite dimensional space as considered here, just let $e_1, e_2, e_3$ be an orthonormal basis of $\mathscr{V}$.

$$\begin{aligned}\psi(v) &= \psi\left(\langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 + \langle v, e_3\rangle e_3\right)\\ &= \langle v, e_1\rangle \psi(e_1) + \langle v, e_1\rangle \psi(e_1) + \langle v, e_1\rangle \psi(e_1)\\ &= \langle v, \psi(e_1)e_1 + \psi(e_2)e_2 + \psi(e_3)e_3\rangle\end{aligned}$$

Then let $a = \psi(e_1)e_1 + \psi(e_2)e_2 + \psi(e_3)e_3$.

Indeed, $a$ defined above is unique. Otherwise suppose $\psi(v) = \langle v, a_1\rangle = \langle v, a_2\rangle$, then by linearity,

$$0 = \langle v, a_1 - a_2\rangle, \quad \forall v \in \mathscr{V}$$

Substitute $v = a_1 - a_2$ shows that $a_1 = a_2$, and we've done.

评分细则：三个向量依次点积无结合律；没有 $e = (e_1, e_2, e_3)$。错用，乱用记号扣 2 分；错用，乱用数学法则扣 5 分。满分 25。

**Exercise 5.** Show that the tensor product $\mathbf{a} \otimes \mathbf{b}$ is a tensor.

*Solution.* Verify linearity in addition and scalar multiplication.

评分细则：很多同学这道题被扣分。解答中点出线性映射，但是未加以验证加法、数乘的线性性者，扣 5 分；既没验证线性性，又没点出线性映射，扣 10 分。满分 25。

**Exercise 6.** Prove that

- (a) $S(a \otimes b) = (Sa) \otimes b$

- (b) $(a \otimes b)S = a \otimes (S^T b)$
- (c) $\sum_i (Se_i) \otimes e_i = S$

*Solution.* Just apply both sides of each equation an arbitrary vector and verify that you are returned with the same thing.

评分细则：本题给了几乎所有同学满分，但是部分同学使用指标记法显得过于繁琐，这里推荐使用作用于任意向量再验证相等的做法。满分 25。

**Exercise 13.** Show that $Q$ is orthogonal if and only if $H = Q - I$ satisfies

$$H + H^T + HH^T = 0, \quad HH^T = H^T H$$

*Solution.* Direct computation shows the results. By the way, the geometry interpretation of $H$ is the rigid motion. As already stated in class notes, $H$ is an isometry. For the other direction, since $H = Q - I$, computing $H + H^T + HH^T = 0$ implies $QQ^T = I$, then $HH^T = H^T H$ implies that $QQ^T = Q^T Q = I$. By definition, $Q^{-1} = Q^T$ shows that $Q$ is orthogonal.

评分细则：注意本题既要证明充分性又要证明必要性，只证明一个方向者，扣 5 分。注意到两个矩阵 A，B 的乘积 AB=0 并不能推出 A=0 或 B=0，一个简单的反例如下：

$$A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

都是非零矩阵，但其乘积却为零。利用这点证明的扣 5 分。一小题证错扣 5 分。满分 25。

**Exercise 15.** Let $Q$ be an orthogonal tensor, and let $e$ be a vector with

$$Qe = e$$

- (a) Show that $Q^T e = e$
- (b) Let $w$ be the axial vector corresponding to the skew part of $Q$. Show that $w$ is parallel to $e$.

*Solution.* (a) For orthogonal tensor $Q$, $Q^T = Q^{-1}$, then multiply $Q^{-1}$ from left on both sides of $Qe = e$, $e = Q^{-1}e = Q^T e$.

(b)

$$w \wedge e = \frac{Q - Q^T}{2} e = \frac{1}{2}\left(Qe - Q^T e\right) = 0$$

Hence by definition of outer product, $w$ is parallel to $e$.

评分细则：只有方阵才能定义行列式，向量和一般矩阵是没有行列式的定义的，错用行列式定义者扣 5 分。满分 25。

Section 1.2 pp.16

**Exercise 1.** Determine the spectrum, the characteristic spaces, and a spectral decomposition for each of the following tensors:

$$A = \alpha I + \beta m \otimes m$$

$$B = m \otimes n + n \otimes m$$

*Solution.* First noting that both $A$ and $B$ are symmetric so there're enough eigenvectors to generate a basis. For $A$, obviously, $m$ is an eigenvector with corresponding eigenvalue $\alpha+\beta$, the characteristic space is $span\{m\}$. The hyperplane orthogonal to $m$, i.e., $m^\perp = \{v|\langle v, m\rangle = 0\}$ is another characteristic space, every nonzero vectors in this hyperplane is an eigenvector with corresponding eigenvalue $\alpha$; For $B$, we assume $m, n$ is linearly independent, or equivalently, $m$ is not a multiple of $n \neq 0$. Observe that vector $m \wedge n$ that is orthogonal to the plane spanned by $m, n$ is an eigenvector with corresponding eigenvalue 0. The characteristic space consists of all multiples of $m \wedge n$. Now suppose other eigenvectors is a linear combination of $m, n$. Write $v = \alpha m + \beta n$. Then finally it requires that $\beta^2\langle n, n\rangle = \alpha^2\langle m, m\rangle$. So let $\beta = 1/|n|, \alpha = \pm 1/|m|$, the eigenvector is $v = \pm\frac{1}{|m|}m + \frac{1}{|n|}n$, with correspongding eigenvalue $\langle m, n\rangle \pm |m||n|$

评分细则：特征值，特征向量，特征空间和谱分解四者缺一不可，按照漏答个数酌情扣分。注意若 $m, n$ 单位化且互相正交，则答案变得很简单，解答中考虑的是一般的情况，即 $m, n$ 仅满足线性无关。满分 25。

**Exercise 3.** Let $D \in Sym$, $Q \in Orth$. Show that the spectrum of $D$ equals the spectrum of $QDQ^T$. Show further that if $e$ is an eigenvector of $D$, then $Qe$ is an eigenvector of $QDQ^T$ corresponding to the same eigenvalue.

*Solution.* Note that $D$ and $QDQ^T$ are the different matrix representation of the same linear mapping under different bases. Thus they have the same spectrum since eigenvalues are invariants. An elementary proof goes as below: suppose $\lambda$ is an eigenvalue of D, i.e., $De = \lambda e$, then

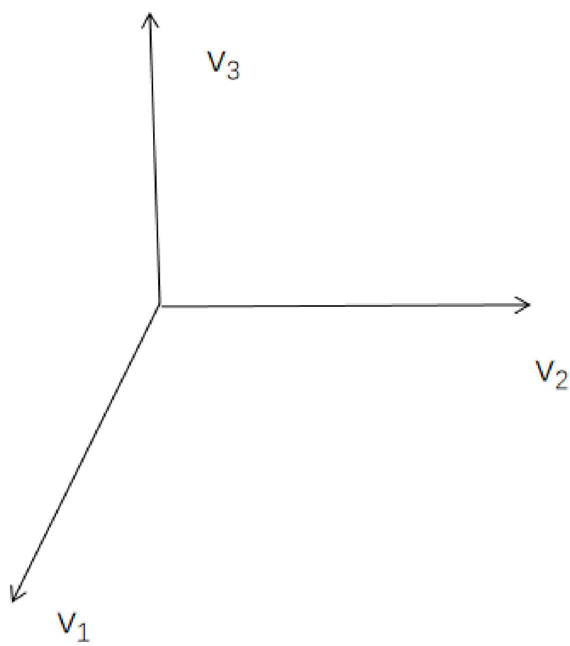
$$QDQ^T(Qe) = QDIe = Q(De) = Q(\lambda e) = \lambda(Qe)$$

which means that any eigenvalue of D is also an eigenvalue of $QDQ^T$ with corresponding eigenvector $Qe$. Since $D$ is symmetric, notice that $QDQ^T$ is also symmetric. By linear algebra, there is a basis of $\mathscr{V}$ consists of eigenvectors of $D$, since $Q$ is nonsingular, $\ker Q = \{0\}$, which means that if $\{e_1, e_2, e_3\}$ is such a eigen-basis, then $\{Qe_1, Qe_2, Qe_3\}$ is also a basis since entries of this are linearly independent and the number is the dimension of the present vector space. The latter basis consists of eigenvectors of $QDQ^T$ corresponding to the same eigenvalue with $D$, and we've done.

评分细则：只说明 $D$ 的特征值同时也是其共轭的特征值是不充分的，扣 2 分。这是因为同一个特征值可以对应多个特征向量，即还需验证重数相同。满分 25。

**Exercise 5.** Show that $S = \omega I$ iff $S$ commutes with each skew-symmetric matrix $W$.

*Solution.* One direction is obviously verified. For the other direction, since what we are discussing belong to 3-d linear space, the exercise can be finished by a brute force procedure. Suppose $S \neq 0$, otherwise $\omega = 0$ and we've done. Then there exists a unit vector $v_1$ such that $Sv_1 \neq 0$, set up an orthonormal basis $\{v_1, v_2, v_3\}$, see the Figure below. The main idea




is take different $W$ to verify $Sv_i = \lambda v_i$. Take $w = -v_2$, then

$$SWv_1 = S(w \wedge v_1) = WSv_1 = w \wedge Sv_1$$

since $-v_2 \wedge v_1 = v_3$, we have

$$S(v_3) = -v_2 \wedge Sv_1$$

Next by letting $w = -v_1$, we obtain

$$SWv_1 = S(-v_1 \wedge v_1) = S(0) = 0 = WSv_1 = -v_1 \wedge Sv_1$$

Thus, $Sv_1 // v_1$, from what we can let $Sv_1 = \lambda v_1$. Hence we can compute

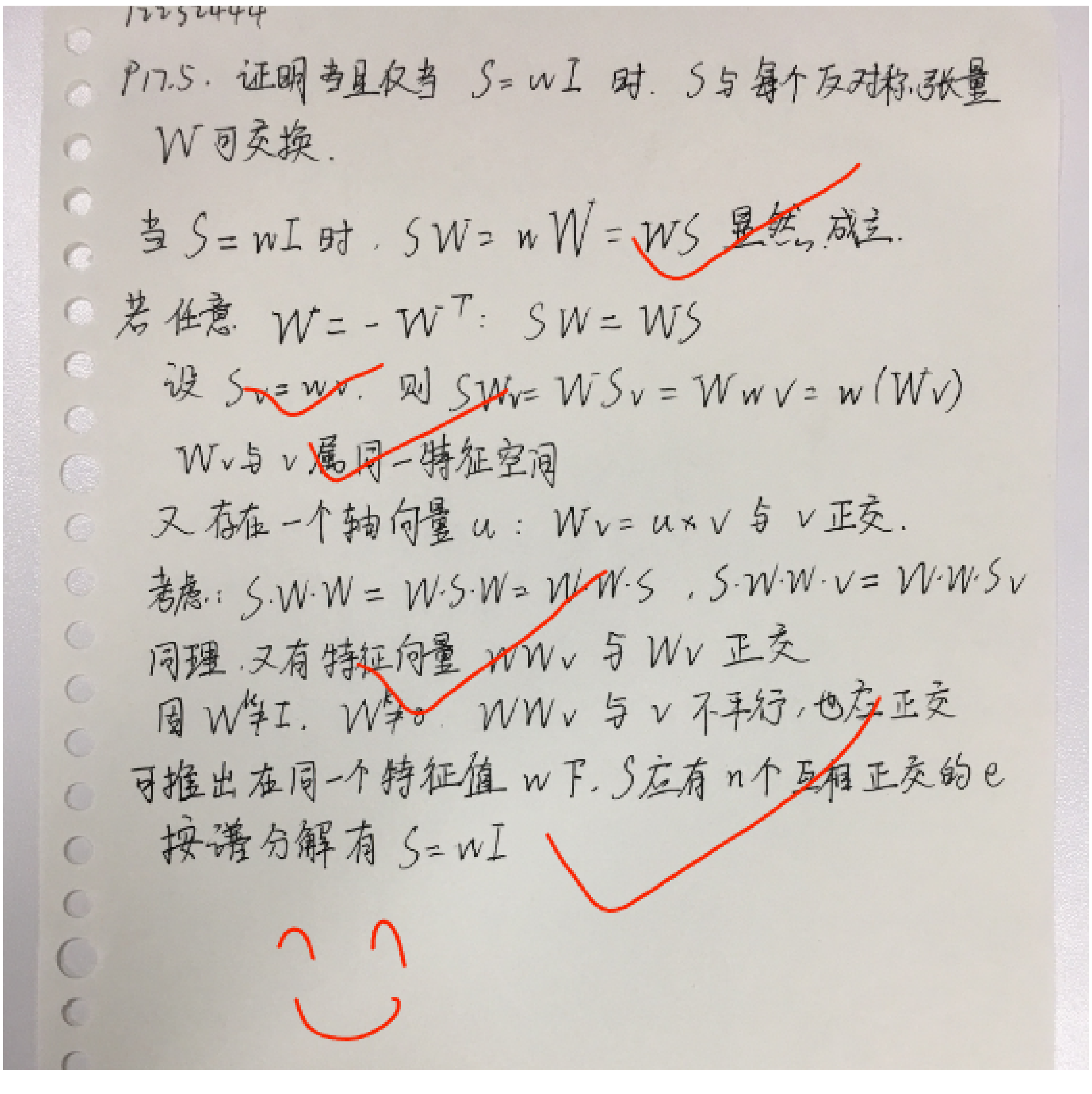
$$Sv_3 = -v_2 \wedge Sv_1 = -v_2 \wedge \lambda v_1 = -\lambda v_2 \wedge v_1 = \lambda v_3$$

Taking $w = -v_3$ and going through the same procedure will give $Sv_2 = \lambda v_2$. Finally, for any $u \in V$, $u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$$Su = \alpha_1 Sv_1 + \alpha_2 Sv_2 + \alpha_3 Sv_3 = \lambda u$$

which means that $S = \lambda I$.

黄宣铭同学 gives an elegant proof without considering basis:



P17.5. 证明当且仅当 $S = wI$ 时. $S$ 与每个反对称张量 $W$ 可交换.

当 $S = wI$ 时, $SW = wW = WS$ 显然成立.

若任意 $W = -W^T$: $SW = WS$

设 $Sv = wv$. 则 $SWv = WSv = Wwv = w(Wv)$

$Wv$ 与 $v$ 属同一特征空间

又存在一个轴向量 $u$: $Wv = u \times v$ 与 $v$ 正交.

考虑: $S \cdot W \cdot W = W \cdot S \cdot W = W \cdot W \cdot S$, $S \cdot W \cdot W \cdot v = W \cdot W \cdot Sv$

同理. 又有特征向量 $WWv$ 与 $Wv$ 正交

因 $W \neq I$. $W \neq 0$. $WWv$ 与 $v$ 不平行, 也应正交

可推出在同一个特征值 $w$ 下. $S$ 应有 $n$ 个互相正交的 $e$

按谱分解有 $S = wI$

黄同学这里默认 $S$ 存在一个特征值 $w$，这是正确的，源自线性代数:

**5.21 Operators on complex vector spaces have an eigenvalue**

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

另一种做法来源于王开元同学，与暴力做法本质上相同：

5. Suppose that $S$ commutes with $\forall W \in Skw$.

Let $v$ be the axial vector of $W$

$S W v = W S v = 0$

$S v \in \mathrm{span}\{v\}$

Since $v$ is unique for every skew $W$

$S u \in \mathrm{span}\{u\}$ holds for all $u \in V$

第一步证明是数乘

Let $u_1, u_2$ be linearly independent vectors, $S u_1 = \lambda_1 u_1$, $S u_2 = \lambda_2 u_2$

$S(u_1+u_2) = \lambda_1 u_1 + \lambda_2 u_2 = \omega(u_1+u_2)$

$\lambda_1 = \lambda_2 = \omega$

第二步证明各向量对应的数乘是同一个数

$\therefore S = \omega I$

Suppose that $S = \omega I$

$\forall W \in Skw$, $u \in V$

$S W v = \omega I W v = \omega W I v = W S v$

$S$ commutes with $W$

For more general cases in higher dimensional linear spaces ($n > 3$), we refer to the the class notes in Wechat group.

评分细则：一个方向验证出来获得 5 分。另一个方向漏做、本质上做错者均扣 5 分。部分同学对 $S$ 分类讨论，分为对称和反对称两种情况，那如果是更一般的情况呢？这种情况也是扣 5 分。满分 25。

**16** Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $T$ is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

16. Let $v_1, \dots, v_n$ be basis of $V$. First, we prove that for any $1 \leq i \leq n$, there exists $C_i \in \mathbf{F}$ so $Tv_i = C_i v_i$. Indeed, we can choose linear map $S$ so $Sv_i = v_i$ and $Sv_j = 2v_j$ for $1 \leq j \leq n, j \neq i$. This follows $STv_i = T(Sv_i) = Tv_i$. Hence, if $Tv_i = \sum_{k=1}^{n} \alpha_k v_k$ then

$$\sum_{k=1}^{n} \alpha_k v_k = Tv_i = STv_i = S\left(\sum_{k=1}^{n} \alpha_k v_k\right) = \alpha_i v_i + 2\sum_{k \neq i} \alpha_k v_k.$$

This follows $\sum_{k \neq i} \alpha_k v_k = 0$, which happens when $\alpha_k = 0$ for $k \neq i$. Thus, $Tv_i = \alpha_i v_i = C_i v_i$.

Next, we prove that for any two $1 \leq j, i \leq n$ so $j \neq i$ then $C_i = C_j = C$. Indeed, pick an invertible operator $S$ so $Sv_i = v_j$ then $TSv_i = Tv_j = C_j v_j$ but $STv_i = S(C_i v_i) = C_i v_j$. Hence, $C_i v_j = C_j v_j$ so $C_i = C_j = C$. Thus, $Tv_i = Cv_i$ for all $1 \leq i \leq n$, or $T$ is scalar multiple of $I$.

图 1: