

Systems of linear equations

Mathematics – RRMATA

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Basic concepts

Definition (System of linear equations)

A **system of m linear equations in n unknowns** is a collection of equations

$$(*) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Variables x_1, x_2, \dots, x_n are called **unknowns**. Numbers a_{ij} are called **coefficients of the left-hand sides** and numbers b_i are called **coefficients of the right-hand sides**.

A **solution** of the system is an ordered n -tuple of real numbers t_1, t_2, \dots, t_n that make each equation true statement when the values t_1, t_2, \dots, t_n are substituted for x_1, x_2, \dots, x_n , respectively.

If $b_1 = b_2 = \cdots = b_m = 0$, the system is called **homogenous**.

Definition (Coefficient matrix, augmented matrix)

- The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called the **coefficient matrix** of system (*).

- The matrix

$$\tilde{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

is called the **augmented matrix** of system (*).

Matrix notation of (*)

Denote

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the vector of the right-hand sides and unknowns, respectively. System (*) can be written as the **matrix equation**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

i.e.,

$$A\vec{x} = \vec{b}.$$

Theorem (Frobenius)

System () has a solution if and only if the rank of the coefficient matrix of (*) is equal to the rank of the augmented matrix of this system, i.e.,*

$$\text{rank}A = \text{rank}\tilde{A}.$$

Remark

System (*) may have no solution, exactly one solution, or infinitely many solutions.

- If $\text{rank}A < \text{rank}\tilde{A}$, then (*) has **no solution**.
- If $\text{rank}A = \text{rank}\tilde{A} = n$, then (*) has **exactly one solution**.
- If $\text{rank}A = \text{rank}\tilde{A} < n$, then (*) has **infinitely many solutions**. In this case the unknowns can be computed in terms of $n - \text{rank}A$ **parameters (free variables)**.

Homogeneous linear systems have either exactly one solution (namely, $x_1 = 0, x_2 = 0, \dots, x_n = 0$, called the **trivial solution**) or an infinite number of solutions (including the trivial solution).

Gauss method

- ① We convert the augmented matrix \tilde{A} into its row echelon form (using row operations). We find $\text{rank } \tilde{A}$ and $\text{rank } A$ to determine the solvability or nonsolvability of $(*)$ (Frobenius theorem).
- ② If $\text{rank } A = \text{rank } \tilde{A}$, we rewrite back the row echelon form of \tilde{A} into a system of linear equations (in the original unknowns). This system has the same set of solutions as the original system $(*)$.
- ③ We solve this new system from below:
 - If $\text{rank } A = \text{rank } \tilde{A} = n$, there is exactly one “new” unknown in each equation of the system. (Other unknowns have been computed from the equations below.)
 \Rightarrow exactly one solution
 - If $\text{rank } A = \text{rank } \tilde{A} < n$, then there exists at least one equation with $k > 1$ “new” unknowns. In this case, we solve one arbitrary of these unknowns through the other $k - 1$ unknowns. These $k - 1$ unknowns are called free variables and can be considered as parameters, i.e., they can take any real values \Rightarrow infinitely many solutions. The choice of the free unknowns is not unique, hence the set of solutions can be written in different forms.

Example (One solution)

Solve the system:

$$\begin{array}{l} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + 4x_2 + 7x_3 = 8 \\ 3x_1 + 5x_2 + 10x_3 = 10 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 4 & 7 & 8 \\ 3 & 5 & 10 & 10 \end{array} \right) \xrightarrow{\substack{-2 \\ + \\ +}} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 2 & 4 & 10 \end{array} \right) \xrightarrow{\substack{-1 \\ +}} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Rank of the coefficient matrix
(denote A) and of the augmented
matrix (denote \tilde{A}):

$$\text{rank}(A) = \text{rank}(\tilde{A}) = 3$$

number of variables: $n = 3$

\Rightarrow 1 solution

From the last matrix (solved from below):

$$x_3 = 2$$

$$2x_2 + 3 \cdot 2 = 8 \Rightarrow x_2 = 1$$

$$x_1 + 1 + 2 \cdot 2 = 0 \Rightarrow x_1 = -5$$

Example (Infinitely many solution, 1 parameter)

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 4$$

$$x_2 - x_3 + x_4 = -3$$

$$x_1 + 3x_2 - 3x_4 = 1$$

$$-7x_2 + 3x_3 + x_4 = -3$$

Solve the system:

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 1 & 3 & 0 & -3 & 1 \\ 0 & -7 & 3 & 1 & -3 \end{array} \right) \xrightarrow[-1]{+} \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & \boxed{1} & -1 & 1 & -3 \\ 0 & 5 & -3 & 1 & -3 \\ 0 & -7 & 3 & 1 & -3 \end{array} \right) \xrightarrow[-5]{+} \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & -3 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 & 12 \\ 0 & 0 & -4 & 8 & -24 \end{array} \right) \mid :2 \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{rank}(A) = \text{rank}(\tilde{A}) = 3$$

$$\text{number of variables: } n = 4$$

$\Rightarrow \infty$ solutions, 1

parameter

$$x_3 - 2x_4 = 6 : \boxed{x_4 = t, t \in \mathbb{R}} \Rightarrow \boxed{x_3 = 6 + 2t}$$

$$x_2 - (6 + 2t) + t = -3 \Rightarrow \boxed{x_2 = 3 + t}$$

$$x_1 - 2(3 + t) + 3(6 + 2t) - 4t = 4 \Rightarrow \boxed{x_1 = -8}$$

Example (Infinitely many solutions, 2 parameters)

Solve the system:

$$\begin{array}{rcl} x_1 + 2x_2 + 4x_3 - 3x_4 = 0 \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 = 0 \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 = 0 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & 0 \\ 4 & 5 & -2 & 3 & 0 \\ 3 & 8 & 24 & -19 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} -3 \\ + \\ + \\ + \end{array}} \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -3 & -18 & 15 & 0 \\ 0 & 2 & 12 & -10 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{rank}(A) = \text{rank}(\tilde{A}) = 2 \quad -x_2 - 6x_3 + 5x_4 = 0 : \boxed{x_4 = t, x_3 = s, t, s \in \mathbb{R}}$$

number of variables: $n = 4$

$\Rightarrow \infty$ solutions, 2

parameters

$$\Rightarrow \boxed{x_2 = -6s + 5t}$$

$$x_1 + 2(-6s + 5t) + 4s - 3t = 0 \Rightarrow \boxed{x_1 = 8s - 7t}$$

Example (No solution)

$$x_1 + 2x_2 + 3x_3 = 1$$

Solve the system:

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 1 \\ 4x_1 + 5x_2 + 8x_3 &= 2 \end{aligned}$$

$$\begin{array}{l} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 5 & 8 & 2 \end{array} \right) \xrightarrow[-2]{\square+} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 1 & 0 & 0 \end{array} \right]^{-4} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -1 \\ 0 & -3 & -4 & -2 \end{array} \right) \xrightarrow[-1]{\square+} \\ \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 0 & -1 \end{array} \right) \end{array}$$

$$\text{rank}(A) = 2, \quad \text{rank}(\tilde{A}) = 3$$

$\text{rank}(A) \neq \text{rank}(\tilde{A}) \implies$ the system has no solution.

Systems with regular coefficient matrices

Theorem (Properties of regular matrices)

Let A be an $n \times n$ square matrix. Then the following statements are equivalent:

- ① A is invertible, i.e., A^{-1} exists.
- ② $\det A \neq 0$
- ③ $\text{rank } A = n$.
- ④ The rows (columns) of A are linearly independent.
- ⑤ System of linear equations $A\vec{x} = \vec{b}$ has a unique solution for any vector \vec{b} .

Method of matrix inversion

Next we present a method which can be used for solving the system $A\vec{x} = \vec{b}$ in case when A is regular.

Theorem (Method of matrix inversion)

Let A be an $n \times n$ matrix and suppose that A is invertible. Then system of equations $A\vec{x} = \vec{b}$ has a unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

Example

$$\begin{array}{l} x_1 + x_2 + 2x_3 = 1 \\ \text{Solve the system: } \quad 2x_1 + x_2 + 3x_3 = 2 \\ \qquad \qquad \qquad x_1 + x_2 + x_3 = 3 \end{array}$$

The coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

The vector of the right-hand sides:

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The inverse matrix of A :

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The vector of solutions: $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$$\implies \boxed{x_1 = 3, x_2 = 2, x_3 = -2}.$$

Using the computer algebra systems

Solve the system using Wolfram Alpha (<http://www.wolframalpha.com/>):

$$\begin{aligned}x_1 + x_2 + 2x_3 &= 1 \\2x_1 + x_2 + 3x_3 &= 2 \\x_1 + x_2 + x_3 &= 3\end{aligned}$$

Solution:

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solve x1+x2+2*x3=1,2x1+x2+3x3=2,x1+x2+x3=3
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