EVERYTHING YOU WANTED TO KNOW ABOUT THE MATHEMATICAL PENDULUM BUT WERE AFRAID TO ASK

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Abstract. We discuss basic facts regarding the motion of mathematical pendulum both in the linearised and fully nonlinear setting. In particular, we focus on the calculation of the period of oscillations both via the elliptic integral and a perturbation expansion.

Contents

1. Introduction

We investigate the motion of the mathematical pendulum. This dynamical system is arguably the most important dynamical system since it constitutes a prominent example of periodic motion. Concerning the mathematical point of view, the objective of this tutorial is to

- deepen knowledge about the dynamics governed by a second order ordinary differential equation,
- show that a certain quantitative piece of information about the system dynamics can be obtained without explicitly solving the governing equations.

2. Physical background

We consider a pendulum as shown in Figure 1. The pendulum length is l and the pendulum is moving in a homogeneous gravitational field with gravitational acceleration g.

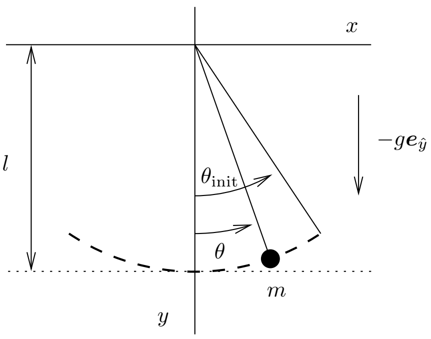

Figure 1. Mathematical pendulum.

There are several techniques how to obtain the governing equations for the motion of pendulum, in any case the governing equation reads

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0. \tag{2.1}$$

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(Check your notes from an elementary lecture on classical mechanics. It would be worthwhile to recall the derivation based on the use of Lagrangian function. We will use this concept heavily in the following lectures.) The problem with equation (2.1) is that it is a *nonlinear* equation, which means that it is difficult to find an explicit solution in terms of elementary functions.

In the case of small amplitude oscillations, that is if θ is small for all time, one can think of using the approximation $\sin\theta \approx \theta$ and replace the nonlinear equation (2.1) by the linearised equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0. \tag{2.2}$$

The question is whether the *dynamics* of the linearised system is similar to the dynamics of the nonlinear system. It turns out that some insight into the dynamics governed by (2.1) can be obtained by the study of approximation (2.2) or for that matter by better approximations of the same type.

3. Period of oscillations

Since we do not have an explicit formula for the solution, it seems that it is impossible to find a formula for the period of oscillations T. It however turns out that the period of oscillations can be found without solving the governing equations. The period of oscillations as predicted by the linearised governing equation (2.2) is

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3.1}$$

Regarding the pendulum governed by the nonlinear equation (2.1), we can conjecture that the period of oscillations of the pendulum that is released with zero velocity from the initial position given by the angle θ_{init} is

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + a\theta_{\mathrm{init}}^2 + b\theta_{\mathrm{init}}^4 + \cdots\right), \tag{3.2}$$

where a, $b \in R$ are real constants. (In virtue of the symmetry of the problem we know that there are no odd terms in the expansion.) We now derive an exact formula for the period of oscillations, and we find the value of the coefficient a in the expansion (3.2).

3.1. **Direct computation via approximation of elliptic integral.**

The multiplication of (2.1) by $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - \frac{g}{l}\cos\theta\right) = 0. \tag{3.3}$$

(This equation is the conservation of energy.) Equation (3.3) implies that

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \frac{2g}{l}\cos\theta + C, \tag{3.4}$$

where C is an integration constant. The value of the constant is fixed by the initial condition. If $\theta = \theta_{\mathrm{init}}$ we want the pendulum to have the zero angular velocity $\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{\theta=\theta_{\mathrm{init}}} = 0$, hence we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_{\mathrm{init}}}. \tag{3.5}$$

(Let us assume that we investigate the motion of the pendulum in the time interval where $\frac{\mathrm{d}\theta}{\mathrm{d}t} \geq 0$.) Note that (3.5) or for that matter even (3.4) imply that the velocity $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ remains bounded for all times! This is an important qualitative feature and we will exploit it later.

Now we use the theorem on the derivative of inverse functions, and we arrive at

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_{\mathrm{init}}}}. \tag{3.6}$$

This is a differential equation for the time expressed as a function of the angle. We solve this equation by separation of variables,

$$\mathrm{d}t = \frac{\mathrm{d}\theta}{\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_{\mathrm{init}}}}. \tag{3.7}$$

If we integrate the equation from $\theta = 0$ to $\theta = \theta_{\mathrm{init}}$, then on the left-hand side we get $\frac{T}{4}$. (One quarter of period is necessary to move from the vertical position $\theta = 0$ to the extreme position $\theta = \theta_{\mathrm{init}}$.) Consequently, the formula for the period of oscillation reads

$$\frac{T}{4} = \int_{\theta=0}^{\theta_{\mathrm{init}}} \frac{\mathrm{d}\theta}{\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_{\mathrm{init}}}}. \tag{3.8}$$

The integral on the right-hand side is difficult to evaluate in the sense that we have no simple explicit formula for the integral. However, if our objective is to derive an asymptotic expansion for the period of oscillations, we can try to approximate the integral as well.

Before we manipulate the integral we rewrite it in a more convenient form. One of the objectives is to eliminate the annoying singularity in the integrand. We recall the trigonometric identity $\sin^2 \frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$, and we use it in the integrand

$$T = 2\sqrt{\frac{l}{g}} \int_{\theta=0}^{\theta_{\mathrm{init}}} \frac{\mathrm{d}\theta}{\sqrt{\sin^2 \frac{\theta_{\mathrm{init}}}{2} - \sin^2 \frac{\theta}{2}}}. \tag{3.9}$$

The integrand has now the structure $\frac{1}{\sqrt{1-y^2}}$, and we know that if we want to calculate $\int \frac{\mathrm{d}y}{\sqrt{1-y^2}}$, then the good substitution is $y = \sin\varphi$. Motivated by this observation we manipulate the integral as follows

$$T = 2\sqrt{\frac{l}{g}} \int_{\theta=0}^{\theta_{\mathrm{init}}} \frac{\mathrm{d}\theta}{\sqrt{\sin^2 \frac{\theta_{\mathrm{init}}}{2} - \sin^2 \frac{\theta}{2}}} = \left| \begin{gathered} \sin\varphi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_{\mathrm{init}}}{2}} \\ \cos\varphi \,\mathrm{d}\varphi = \frac{1}{2}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta_{\mathrm{init}}}{2}}\,\mathrm{d}\theta \\ \sqrt{1 - \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta_{\mathrm{init}}}{2}}}\mathrm{d}\varphi = \frac{1}{2}\frac{\sqrt{1-\sin^2\frac{\theta}{2}}}{\sin\frac{\theta_{\mathrm{init}}}{2}}\mathrm{d}\theta \\ \sqrt{\sin^2\frac{\theta_{\mathrm{init}}}{2} - \sin^2\frac{\theta}{2}}\mathrm{d}\varphi = \frac{1}{2}\sqrt{1 - \sin^2\frac{\theta_{\mathrm{init}}}{2}\sin^2\frac{\varphi}{2}}\mathrm{d}\theta \end{gathered} \right| = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \tag{3.10}$$

where we have denoted $k =_{\mathrm{def}} \sin\frac{\theta_{\mathrm{init}}}{2}$. The integral on the right-hand side looks much better than the original integral. In particular the integrand is now well defined for all $\varphi \in \left[0, \frac{\pi}{2}\right]$

Let us summarise our findings so far. The formula for the period of oscillations is

$$T = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \tag{3.11a}$$

where

$$k =_{\mathrm{def}} \sin\frac{\theta_{\mathrm{init}}}{2}. \tag{3.11b}$$

The integral in (3.11a) is referred to as the *elliptic integral*. The elliptic integral can be completely evaluated for all k using a series expansion, see, for example Abramowitz and Stegun (1964). We however need something less complicated. Using the Taylor expansion for small θ_{init}, and consequently for small k, we see that

$$\frac{1}{\sqrt{1-k^2\sin^2\varphi}} \approx 1 + \frac{1}{2}k^2\sin^2\varphi \approx 1 + \frac{1}{8}\theta_{\mathrm{init}}^2\sin^2\varphi, \tag{3.12}$$

where we have used the fact that $\sin\alpha \approx \alpha$ and $\frac{1}{\sqrt{1-y}} \approx 1 + \frac{1}{2}y$. Making use of the expansion (3.12) in (3.11a) we arrive at

$$T = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} \approx 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\frac{\pi}{2}} \left(1 + \frac{1}{8}\theta_{\mathrm{init}}^2\sin^2\varphi\right)\mathrm{d}\varphi = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_{\mathrm{init}}^2}{16}\right). \tag{3.13}$$

3.2. **Perturbation method for the nonlinear equation.** We can attack the problem from a different perspective. We can try to directly expand the nonlinear term in (2.1) using the Taylor expansion

$$\sin\theta = \sum_{n=1}^{+\infty}(-1)^n\frac{\theta^{2n+1}}{(2k+1)!} \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots, \tag{3.14}$$

and then solve the corresponding nonlinear ordinary differential equation. Thus far the only simplification is that the nonlinear function $\sin\theta$ has been replaced by a polynomial function. This is not a big deal. The only benefit—so far—is that the polynomials are a little easier to work with.

Next we need to articulate the idea that the nonlinear terms in the expansion (3.14) are not as important as the lower order terms. The crucial question is how to choose the parameter that should be "small". In our case the small parameter should be the *amplitude* of oscillation. The motivation is the following. If we consider the linear equation (2.2), with initial conditions

$$\left.\theta\right|_{t=0} = \theta_{\mathrm{init}}, \tag{3.15a}$$

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{t=0} = 0, \tag{3.15b}$$

that is if the pendulum is released from the initial position/angle θ_{init} with zero (angular) velocity, then the solution takes the form

$$\theta(t) = \theta_{\mathrm{init}}\cos\left(\sqrt{\frac{g}{l}}t\right). \tag{3.16}$$

We note that the initial position/angle θ_{init} is equal to the amplitude of the oscillation, which is a good message. If the initial position/angle is small, so is the solution to the linear equation.

Based on the analysis of the linear system, we can make the following wild guess. We assume that the angle θ is given as

$$\theta(t) = \varepsilon\theta_A(t) + \varepsilon^2\theta_B(t) + \varepsilon^3\theta_C(t) + \cdots, \tag{3.17}$$

where $\{\theta_i(t)\}_{i=A,B,C,\dots}$ are the functions that must be found. (We are looking for the solution in the form of *power series expansion.*) In fact the function that we look for must be an *odd* function of the initial position/angle θ_{init}, hence we in fact have

$$\theta(t) = \varepsilon\theta_A(t) + \varepsilon^3\theta_B(t) + \cdots, \tag{3.18}$$

where we have relabeled the functions. (The even terms in ε vanish since the solution that starts with the initial position/angle θ_{init} and the solution that starts with the initial position/angle $-\theta_{\mathrm{init}}$ should be identical up to the sign.) Furthermore, we will also need to adjust the angular frequency ω of oscillations. In the linearised setting the frequency is given by the simple formula

$$\omega_0 =_{\mathrm{def}} \sqrt{\frac{g}{l}}, \tag{3.19}$$

in the nonlinear setting the angular frequency and consequently also the period of oscillations is to be expected to depend on the initial position/angle θ_{init}. This motivates us to search for the angular frequency in the form

$$\omega^2 = \omega_0^2 + \varepsilon^2\alpha + \varepsilon^4\beta + \cdots. \tag{3.20}$$

(The angular frequency must depend only on the magnitude of the initial position/angle θ_{init}, hence the expansion (3.20) contains only *even* terms in ε.) To conclude we assume that

$$\theta(t) = \varepsilon\theta_A(t) + \varepsilon^3\theta_B(t) + \cdots, \tag{3.21a}$$

$$\omega^2 = \omega_0^2 + \varepsilon^2\alpha + \varepsilon^4\beta + \cdots, \tag{3.21b}$$

and we want to find a systematic procedure that will allow us to find unknown functions $\{\theta_i(t)\}_{i=A,B,C,\dots}$ and numbers $\{i\}_{i=\alpha,\beta,\gamma,\dots}$.

Recalling that ε is linked to the initial position/angle θ_{init} which is a small quantity, we can substitute (3.21) into (2.1), that is into

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2\sin\theta = 0, \tag{3.22}$$

and make use of (3.14). We get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\varepsilon\theta_A + \varepsilon^3\theta_B + \dots\right] + \left[\omega^2 - \varepsilon^2\alpha - \varepsilon^4\beta + \cdots\right]\left[\left(\varepsilon\theta_A + \varepsilon^3\theta_B + \dots\right) - \frac{\left(\varepsilon\theta_A + \varepsilon^3\theta_B + \dots\right)^3}{3!} + \cdots\right] = 0. \tag{3.23}$$

Regarding the initial conditions (3.15) we need to rewrite them in terms of $\{\theta_i(t)\}_{i=A,B,C,\dots}$. We set

$$\left.\theta_A\right|_{t=0} = \widetilde{\theta_{\mathrm{init}}}, \tag{3.24a}$$

$$\left.\frac{\mathrm{d}\theta_A}{\mathrm{d}t}\right|_{t=0} = 0, \tag{3.24b}$$

$$\left.\theta_B\right|_{t=0} = 0, \tag{3.24c}$$

$$\left.\frac{\mathrm{d}\theta_B}{\mathrm{d}t}\right|_{t=0} = 0, \tag{3.24d}$$

and so forth, where we have denoted

$$\widetilde{\theta_{\mathrm{init}}} =_{\mathrm{def}} \frac{\theta_{\mathrm{init}}}{\varepsilon}. \tag{3.25}$$

(Recall that we need to satisfy the condition $\left.\theta\right|_{t=0} = \left.\varepsilon\theta_A + \varepsilon^3\theta_B + \cdots\right|_{t=0} = \theta_{\mathrm{init}}$.) Now we go back to the nonlinear equation (3.23), and we collect terms with the same order of ε. The first order (linear) terms in ε and the third order (cubic) terms in ε are

$$\frac{\mathrm{d}^2\theta_A}{\mathrm{d}t^2} + \omega^2\theta_A = 0, \tag{3.26a}$$

$$\frac{\mathrm{d}^2\theta_B}{\mathrm{d}t^2} + \omega^2\theta_B = \alpha\theta_A + \frac{\omega^2\theta_A^3}{3!}. \tag{3.26b}$$

Note that we have obtained a hierarchy of *linear* ordinary differential equations. We solve the first equation, that is (3.26a), with initial conditions (3.24a) and (3.24b), and we get

$$\theta_A = \widetilde{\theta_{\mathrm{init}}}\cos(\omega t). \tag{3.27}$$

Having obtained the solution to (3.26a), we see that we have in fact identified the right-hand side of (3.26b), hence we can proceed with the solution of this equation. Substituting (3.27) into (3.26b) yields

$$\frac{\mathrm{d}^2\theta_B}{\mathrm{d}t^2} + \omega^2\theta_B = \alpha\widetilde{\theta_{\mathrm{init}}}\cos(\omega t) + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3\cos^3(\omega t)}{3!}, \tag{3.28}$$

which upon using the trigonometric identity $\cos^3\varphi = \frac{1}{4}(3\cos\varphi + \cos 3\varphi)$ further reduces to

$$\frac{\mathrm{d}^2\theta_B}{\mathrm{d}t^2} + \omega^2\theta_B = \alpha\widetilde{\theta_{\mathrm{init}}}\cos(\omega t) + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{3!}\frac{1}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right). \tag{3.29}$$

We rearrange the terms on the right-hand side, and we finally get

$$\frac{\mathrm{d}^2\theta_B}{\mathrm{d}t^2} + \omega^2\theta_B = \left(\alpha\widetilde{\theta_{\mathrm{init}}} + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{8}\right)\cos(\omega t) + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{24}\cos(3\omega t). \tag{3.30}$$

We are now in a position to exploit basic fact regarding the solution of linear ordinary differential equations with constant coefficients and with non-zero right-hand side. (See especially the section on the method of undetermined coefficients for the special right-hand side.) The solution to such an equation is constructed as a sum of a *particular solution* $\theta_{B,\mathrm{p}}$ and the *homogeneous* solution $\theta_{B,\mathrm{h}}$. The homogeneous solution $\theta_{B,\mathrm{h}}$ is a solution to the equation "without the right-hand side", that is $\theta_{B,\mathrm{h}}$ solves

$$\frac{\mathrm{d}^2\theta_{B,\mathrm{h}}}{\mathrm{d}t^2} + \omega^2\theta_{B,\mathrm{h}} = 0, \tag{3.31}$$

which leads to

$$\theta_{B,\mathrm{h}}(t) = c_1\cos(\omega t) + c_2\sin(\omega t), \tag{3.32}$$

where c_1 and c_2 are integration constants. The particular solution $\theta_{B,\mathrm{p}}$ is obtained as a solution to the equation with the right-hand side, while we can first treat the first term on the right-hand side

$$\frac{\mathrm{d}^2\theta_{B,\mathrm{p}_1}}{\mathrm{d}t^2} + \omega^2\theta_{B,\mathrm{p}_1} = \left(\alpha\widetilde{\theta_{\mathrm{init}}} + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{8}\right)\cos(\omega t) \tag{3.33}$$

and then the second term on the right-hand side,

$$\frac{\mathrm{d}^2\theta_{B,\mathrm{p}_2}}{\mathrm{d}t^2} + \omega^2\theta_{B,\mathrm{p}_2} = \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{24}\cos(3\omega t). \tag{3.34}$$

Solution to (3.33) is

$$\theta_{B,\mathrm{p}_1}(t) = \frac{\alpha\widetilde{\theta_{\mathrm{init}}} + \frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^3}{8}}{4\omega^2}\left[\cos(\omega t) + 2t\omega\sin(\omega t)\right], \tag{3.35}$$

while the solution to (3.34) reads

$$\theta_{B,\mathrm{p}_2}(t) = \frac{\widetilde{\theta_{\mathrm{init}}}^3}{192}\cos(3\omega t). \tag{3.36}$$

We note that the solution (3.35) is not bounded as $t \to +\infty$ due to the presence of the term $t\sin(\omega t)$. (This is not a coincidence, recall the method of undetermined coefficients and the rules for the construction of the *ansatz* for the particular solution. The frequency of the term $\cos(\omega t)$ on the right-hand side of (3.33) coincides with the natural frequency ω.) Note also that (3.33) can be in an abstract way read as follows

$$\mathcal{L}\boldsymbol{x} = \boldsymbol{f}, \tag{3.37}$$

where $\mathcal{L}$ is a linear operator. This equation has a bounded solution provided that $\boldsymbol{f} \in (\mathcal{N}(\mathcal{L}))^{\perp}$, where $\mathcal{N}$ denotes the kernel of the corresponding operator. (You can recall this construction in advanced lectures during the discussion of the so-called *Fredholm alternative*.) The unbounded growth is a serious problem, since we have shown that this can not happen. (We are anyway interested in periodic solutions.) Consequently, we must avoid it at all costs. Fortunately, the expansion (3.20) gives us a chance to save the day. Indeed, if we set

$$\alpha = -\frac{\omega^2\widetilde{\theta_{\mathrm{init}}}^2}{8}, \tag{3.38}$$

then the coefficient on the right-hand side of (3.33) is equal to zero, and the danger of unbounded growth is averted. The solution to the original problem (3.26b) then reads

$$\theta_B(t) = \theta_{B,\mathrm{h}} + \theta_{B,\mathrm{p}_2}(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{\widetilde{\theta_{\mathrm{init}}}^3}{192}\cos(3\omega t), \tag{3.39}$$

which after determining the integration constants via the application of initial conditions reduces to

$$\theta_B(t) = \frac{\widetilde{\theta_{\mathrm{init}}}^3}{192}\left[\cos(3\omega t) - \cos(\omega t)\right], \tag{3.40}$$

Substituting (3.27) and (3.40) into the expansion (3.21a) then yields the *approximation* of the solution to the nonlinear equation (3.22)

$$\theta(t) \approx \theta_{\mathrm{init}}\cos(\omega t) + \frac{\theta_{\mathrm{init}}^3}{192}\left[\cos(3\omega t) - \cos(\omega t)\right] + \cdots, \tag{3.41}$$

with respect to the initial position/angle θ_{init}. The corresponding angular frequency ω is read from the expansion (3.21b) with the help of (3.38), which yields

$$\omega^2 \approx \omega_0^2 - \frac{\omega^2\theta_{\mathrm{init}}^2}{8} + \cdots, \tag{3.42}$$

which can be also rewritten as $\omega^2 \approx \frac{\omega_0^2}{1+\frac{\theta_{\mathrm{init}}^2}{8}}$. In terms of the period of oscillation $T =_{\mathrm{def}} \frac{2\pi}{\omega}$ it means that

$$T \approx \frac{2\pi}{\omega_0}\left(1 + \frac{\theta_{\mathrm{init}}^2}{8} + \cdots\right)^{\frac{1}{2}} = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_{\mathrm{init}}^2}{16} + \cdots\right), \tag{3.43}$$

where we have used the Taylor expansion $\sqrt{1+y} \approx 1 + \frac{y}{2}$. This is the same result as (3.13). The advantage of the current approach is that we have not only an approximation of the period of oscillation, but we also have an approximation of the solution itself. The method we have presented is referred to as the Poincaré–Lindstedt method, and it is named after Henry Poincaré and Anders Lindstedt.

4. Huygens cycloidal pendulum – isochronous oscillations

Let us now try to identify the curve that will lead to *isochronous* oscillations. The problem is the following. If we deal with the nonlinear equation (2.1), then the period of oscillation T depends on the initial position/angle θ_{init}. This is annoying from the perspective of measuring the time. We would prefer a "pendulum" with the period of oscillation independent on the initial position/angle. Such oscillations are called *isochronous*.

In the case of mathematical pendulum, the motion of the point particle is restricted to a *circle*, and the particle is allowed to freely fall—as long as it stays on the circle—under the action of gravity. If we want to get isochronous osciallations, the idea is to replace the *circle* with a different curve, and restrict the motion of the point particle to this curve, and then let it freely fall under the action of gravity. If the new curve is well designed, maybe we can get period of oscillation that does not depend on the initial position.

4.1. **Divine inspiration – cycloid.**

We now check that the curve that leads to isochronous oscillations is the *cycloid*. Let us consider the cycloid given by the parametric equation

$$x = R\left(\varphi + \sin\varphi\right), \tag{4.1a}$$

$$y = -R\left(1 + \cos\varphi\right). \tag{4.1b}$$

(The cycloid corresponds to the curve traced by a point on a circle of radius R as it rolls along the x-axis, see Figure 2.) The formula for the period of oscillation is now

$$\frac{T}{4} = \int_{\varphi=0}^{\varphi_{\mathrm{init}}} \frac{\sqrt{2R^2\left(1+\cos\varphi\right)}}{\sqrt{2g\left(\cos\varphi - \cos\varphi_{\mathrm{init}}\right)}}\,\mathrm{d}\varphi \tag{4.2}$$

This follows by the same argument as before, namely

$$\frac{T}{4} = \int_{t=0}^{\frac{T}{4}} \mathrm{d}t = -\int_{\varphi=\varphi_{\mathrm{init}}}^{0} \frac{\mathrm{d}s}{v} = -\int_{\varphi=\varphi_{\mathrm{init}}}^{0} \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2}\mathrm{d}\varphi}{v} = -\int_{\varphi=\varphi_{\mathrm{init}}}^{0} \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\varphi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2}}{\sqrt{2g\left(y_{\mathrm{init}} - y\right)}}\mathrm{d}\varphi, \tag{4.3}$$

where we have exploited the conservation of energy,

$$\frac{1}{2}mv^2 + mgy = mgy_{\mathrm{init}}. \tag{4.4}$$

(We recall that the point particle starts at position $[x_{\mathrm{init}}, y_{\mathrm{init}}]$ and that it has initially zero velocity.) Equation (4.2) is the special case of (4.3) for the curve parametrised by equations (4.1).

Having obtained (4.2) we now want to show that the final formula for the period of oscillation does not depend on φ_{init}. In fact we are more ambitious than that. We explicitly evaluate the integral. It holds

$$\frac{T}{4} = \int_{\varphi=0}^{\varphi_{\mathrm{init}}} \frac{\sqrt{2R^2\left(1+\cos\varphi\right)}}{\sqrt{2g\left(\cos\varphi - \cos\varphi_{\mathrm{init}}\right)}}\,\mathrm{d}\varphi = \sqrt{\frac{R}{g}}\int_{\varphi=0}^{\varphi_{\mathrm{init}}} \frac{\cos\frac{\varphi}{2}}{\sqrt{\cos^2\frac{\varphi}{2} - \cos^2\frac{\varphi_{\mathrm{init}}}{2}}}\,\mathrm{d}\varphi = \left|\begin{aligned} u &= \arctan\left(\frac{\sin\frac{\varphi}{2}}{\sqrt{\cos^2\frac{\varphi}{2} - \cos^2\frac{\varphi_{\mathrm{init}}}{2}}}\right) \\ \mathrm{d}u &= \frac{1}{2}\frac{\cos\frac{\varphi}{2}}{\sqrt{\cos^2\frac{\varphi}{2} - \cos^2\frac{\varphi_{\mathrm{init}}}{2}}}\mathrm{d}\varphi \end{aligned}\right|$$

$$= 2\sqrt{\frac{R}{g}}\int_{u=0}^{\frac{\pi}{2}} \mathrm{d}u = \pi\sqrt{\frac{R}{g}}, \tag{4.5}$$

hence we finally get

$$T = 4\pi\sqrt{\frac{R}{g}}, \tag{4.6}$$

and the period of oscillation indeed does not depend on the initial position/angle φ_{init}.

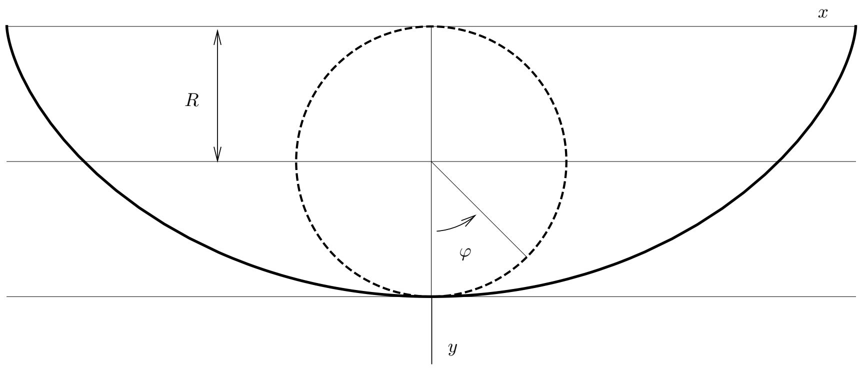

Figure 2. Cycloid.

4.2. **Systematic brute force approach.** Now we can ask how to figure out that the right curve that guarantees the isochoronous property is the cycloid. The solution reported below is due to Niels Henrik Abel. We again start with the energy conservation, and we assume that the curve is parametrised via the vertical coordinate y. That is we look for a function $x(y)$ such that the curve

$$x = x(y), \tag{4.7a}$$

$$y = y, \tag{4.7b}$$

has the isochronous property. The conservation of energy leads to

$$t = \int_{t=0}^{t} \mathrm{d}t = -\int_{y=y_{\mathrm{init}}}^{y_{\min}} \frac{\mathrm{d}s}{v} = -\int_{y=y_{\mathrm{init}}}^{y_{\min}} \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\,\mathrm{d}y}{v} = \int_{y=y_{\min}}^{y_{\mathrm{init}}} \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}}{\sqrt{2g\left(y_{\mathrm{init}} - y\right)}}\,\mathrm{d}y, \tag{4.8}$$

where t is the travel time for the descent from y_{init} to $y_{\min}$. (Position $y_{\min}$ is the lowest position on the curve.) Now we introduce the notation

$$\psi(\xi) =_{\mathrm{def}} \frac{1}{\sqrt{2g\xi}}, \tag{4.9a}$$

$$\phi(\xi) =_{\mathrm{def}} \sqrt{\left(\left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_{y=\xi}\right)^2 + 1}. \tag{4.9b}$$

If we use this notation, then (4.8) reduces to

$$t = \int_{y=0}^{y_{\mathrm{init}}} \phi(y)\psi\left(y_{\mathrm{init}} - y\right)\,\mathrm{d}y, \tag{4.10}$$

where t is a constant. (We want the curve to generate isochronous oscillations.) Equation (4.10) is an *integral equation* for an unknown function $\phi(y)$, and the unknown function $\phi(y)$ enters the equation via *convolution* with a known function $\psi\left(y_{\mathrm{init}} - y\right)$. A good tool for solving such an integral equation is the *Laplace transform.*

Taking the Laplace transform of (4.10) with respect to y_{init} yields

$$\frac{t}{s} = \mathcal{L}\left[\phi(y_{\mathrm{init}})\right](s)\,\mathcal{L}\left[\psi(y_{\mathrm{init}})\right](s), \tag{4.11}$$

where we have used the fact that the Laplace transform of the convolution of two functions is the product of the corresponding Laplace transforms. (The symbol $\mathcal{L}\left[f(y_{\mathrm{init}})\right](s)$ denotes the Laplace transform of function f with respect to the variable y_{init} evaluated at point s.) The Laplace transform of ψ is known,

$$\mathcal{L}\left[\psi(y_{\mathrm{init}})\right](s) = \mathcal{L}\left[\frac{1}{\sqrt{2g}\sqrt{y_{\mathrm{init}}}}\right](s) = \frac{1}{\sqrt{2g}}\frac{\sqrt{\pi}}{\sqrt{s}}, \tag{4.12}$$

hence for the unknown function $\phi(y_{\mathrm{init}})$ we have the following equation

$$\mathcal{L}\left[\phi(y_{\mathrm{init}})\right](s) = t\frac{\sqrt{2g}}{\sqrt{\pi s}}. \tag{4.13}$$

Taking the inverse Laplace transform, we get

$$\phi(y_{\mathrm{init}}) = t\frac{\sqrt{2g}}{\pi}\frac{1}{\sqrt{y_{\mathrm{init}}}}. \tag{4.14}$$

Having identified the function $\phi(\xi)$, we can go back to (4.9b) and solve the differential equation for the parametrisation $x = x(y)$,

$$t\frac{\sqrt{2g}}{\pi}\frac{1}{\sqrt{y}} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}. \tag{4.15}$$

The equation can be manipulated into the form

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = t^2 \frac{2g}{\pi^2}\frac{1}{y} - 1, \tag{4.16}$$

which is the differential equation of type

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = \frac{2R}{y} - 1, \tag{4.17}$$

which is the differential equation for the cycloid generated by the circle of radius R. Consequently, if we want the curve with isochronous property and with the period of oscillation T identical to that of linearised pendulum $T = 2\pi\sqrt{\frac{l}{g}}$, then we need to set $t = \frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{l}{g}}$, which means that the radius of the generating circle R must be set as

$$R = \frac{t^2 g}{\pi^2} = \frac{l}{4}, \tag{4.18}$$

see also (4.6) for the period of oscillation on the cycloid generated by the circle of radius R. (We recall that the descent from y_{init} to y_{min} is just one quarter of the period of oscillation.) The fact that the cycloid is the curve that generates isochronous oscillations was discovered by Christian Huygens. (Huygens of course used a different proof – he did not know anything about Laplace transform.) Based on his analysis, he also proposed a modification of pendulum clocks, see Figure 3, and note the cycloidal shaped metal sheets close to the pendulum pivot.

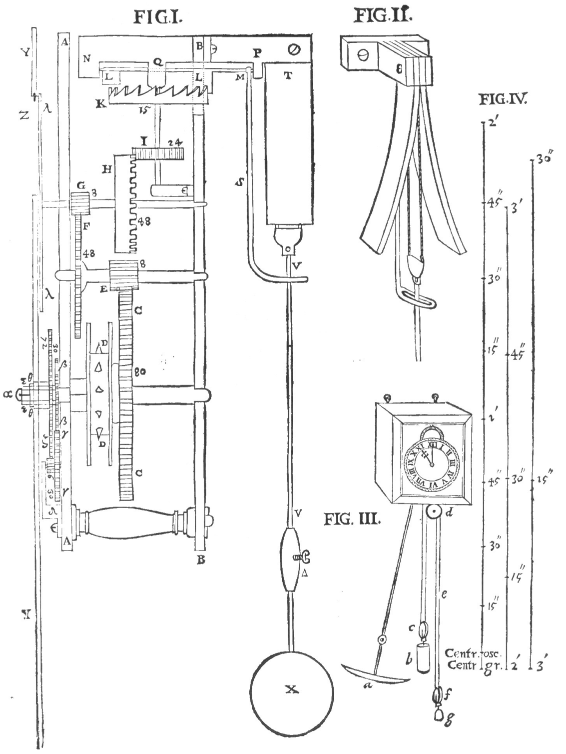

FIGURE 3. Huygens pendulum.

The design of the pendulum clock is based on the following observation. If we want to restrict the motion of a point particle to a cycloid, and if we still want the particle to be hanging on a (flexible and massless) rod, then all we need to do is to restrict the motion of the rod close to the pendulum pivot by suitably designed guides. The shape of the guides restricting the motion of the rod must be such that if we take a flexible rod of length l, which as it is swinging is allowed to freely attach/detach to/from the restricting guides, then the freely moving tip of the rod draws the desired curve. In technical terms we are looking for *involute* of the cycloid, which turns out to be another cycloid. Let us show that the last statement is true.

4.3. **Involute of a cycloid.** We consider the cycloid $\boldsymbol{\gamma}$ parametrised by equations

$$x = R(\varphi - \sin\varphi), \tag{4.19a}$$

$$y = -R(1 - \cos\varphi), \tag{4.19b}$$

and a flexible rod of length L with one end fixed at origin, see Figure 4.

The involute $\boldsymbol{\Gamma}$ with the rod length of L to the curve $\boldsymbol{\gamma}$ is given by the parametrisation

$$\boldsymbol{\Gamma}(\varphi) = \boldsymbol{\gamma}(\varphi) + \frac{\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\varphi}}{\left|\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\varphi}\right|}\left(L - \int_{\xi=0}^{\varphi}\left|\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\xi}\right|\mathrm{d}\xi\right). \tag{4.20}$$

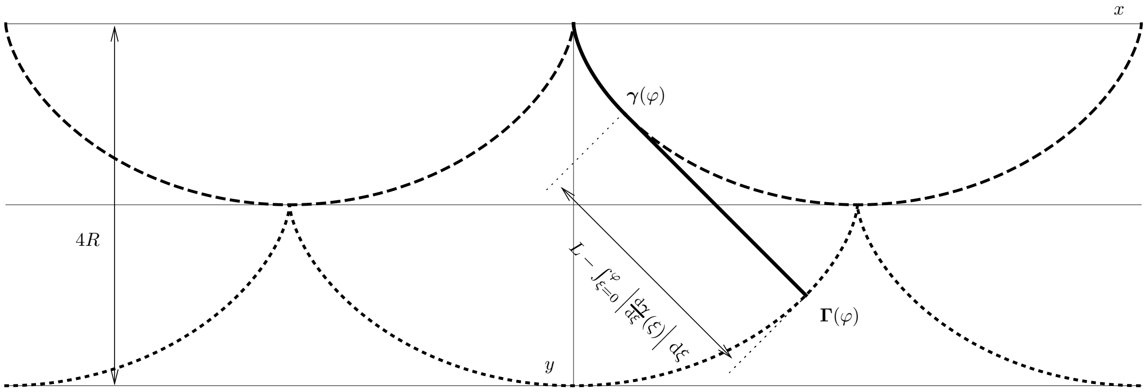

FIGURE 4. Cycloidal pendulum – construction of the involute.

(We are moving from a point $\boldsymbol{\gamma}(\varphi)$ along the (normalised) tangent vector $\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\varphi}$ and length of the straight line is given by the formula $L - \int_{\xi=0}^{\varphi} \left|\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\xi}\right| \mathrm{d}\xi$, which represents the total length of the rod minus the length of the rod that is already attached to the curve $\boldsymbol{\gamma}$.) If we evaluate the formula (4.20) in our special case, then we get $\left|\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\xi}\right| = 2R \sin\frac{\varphi}{2}$ and consequently

$$\boldsymbol{\Gamma}(\varphi) = \begin{bmatrix} R(\varphi - \sin\varphi) \\ -R(1-\cos\varphi) \end{bmatrix} + \frac{1}{2R\sin\frac{\varphi}{2}} \begin{bmatrix} R(1-\cos\varphi) \\ -R\sin\varphi \end{bmatrix} \left(L - \int_{\xi=0}^{\varphi} 2R \sin\frac{\xi}{2}\,\mathrm{d}\xi \right). \tag{4.21}$$

If we consider $L = 4R$, then we get after some algebraic manipulation based on our favourite formulas $\sin^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{2}$, $\cos^2\frac{\varphi}{2} = \frac{1+\cos\varphi}{2}$ and $\sin\varphi = 2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$ we find that

$$\boldsymbol{\Gamma}(\varphi) = \begin{bmatrix} 0 \\ -R \end{bmatrix} + \begin{bmatrix} R(\varphi - \sin\varphi) \\ -R(1+\cos\varphi) \end{bmatrix}, \tag{4.22}$$

which is the parametrisation of a cycloid centered at the origin and shifted along the y axis by R. We note that the cycloid is generated by the same circle as the original cycloid.

5. Further reading

The grand old masters regarding the theory of nonlinear oscillations are Minorsky (1947) and Krylov and Bogoliubov (1943) and Andronov et al. (1966), frequently used books in the engineering community are Nayfeh and Mook (1979); Nayfeh (2000). A recent treatise on nonlinear oscillations is Mickens (2010). A nice (popular) book on the use of clocks in maritime navigation is Sobel (2007).

6. Problems

6.1. **Tasks.** If you want to practice, you can try to solve the following problems:

(1) Write a code that numerically solves the nonlinear governing equation (2.1) with the corresponding initial conditions. Experiment with the settings for the numerical solver (accuracy goal, choice of numerical method). Does the solution satisfy the conservation of energy?
(2) Use your code and find the period of oscillation in the nonlinear setting.
(3) Carry out the perturbation technique one step further, that is obtain a formula for the next order correction of the period of oscillation and the approximate analytical solution. Show that the theoretical prediction matches with your numerical solution.
(4) What if the pendulum moves in an inhomogeneous gravitational field? Would it be possible to apply some of the methods discussed above?
(5) Consider the linearised equation for the mathematical pendulum and assume that the pendulum is released with zero velocity from position/angle θ_{init}. Find the probability that the pendulum position θ is, during one cycle, in the interval $[\theta, \theta + \mathrm{d}\theta]$. You should obtain the formula $p_{[\theta,\theta+\mathrm{d}\theta]} = \frac{\mathrm{d}\theta}{\pi\sqrt{1-\left(\frac{\theta}{\theta_{\mathrm{init}}}\right)^2}}$. This example motivates the formula

$$\lim_{n\to+\infty} \int_{x=0}^{2\pi} f(\sin(nx))\,\mathrm{d}x = \int_{x=0}^{2\pi} \left(\int_{y=-1}^{1} f(\lambda) \frac{\mathrm{d}\lambda}{\sqrt{1-\lambda^2}} \right) \mathrm{d}x,$$

see the Young measure entry in the Encyclopedia of Mathematics. Try to grasp what is going on!

6.2. **Implementation hints.** You might find the following commands (Wolfram Mathematica) useful: `DSolve` (symbolic solution of a differential equation), `NDSolve` (numerical solution of a differential equation, note that during the numerical computation you can exploit `EventLocator` and `EventAction` constructs), `Series` (series expansion), `SeriesCoefficient` (series expansion, coefficients), `TrigReduce` (symbolic manipulation with trigonometric functions), `LaplaceTransform` and `InverseLaplaceTransform` (Laplace transform), `ParametricPlot` (plot of a curve described by parametric equations), `VectorPlot` (plot of a vector field, useful in construction of phase field plots), `StreamPlot` (stream plot of the vector field).

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