Project IV: Detection theory

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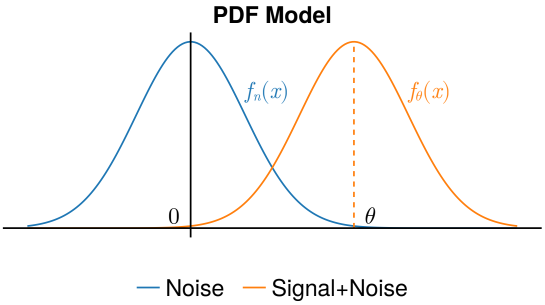
PDF of measurement

We first want to define the PDF of $f_\theta(x)$ which is the PDF of the *signal* + *noise*

We have a SNR-PDF model as shown on the right. We notice that:

- Noise PDF is i.i.d. and sampled around 0
- Signal PDF f_θ average at θ
- The noise is additive to the signal



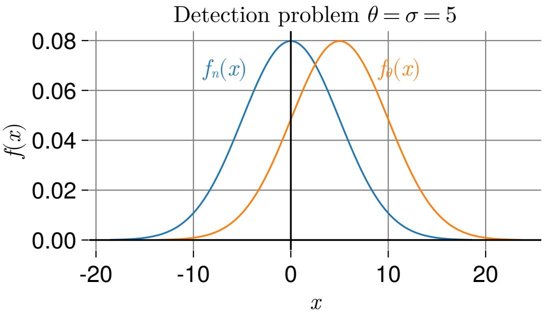
With the noise characterized and distributed $n \sim \mathcal{N}(0, \sigma)$, we have that the signal is distributed $s \sim \mathcal{N}(\theta, \sigma)$ which gives the PDF

$$f_\theta(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

Specific detection problem

Now we assume that $\theta = 5$ and $\sigma = 5$. Plotting this yields the following plot

```
θ = σ = 5
fn = Normal(0,σ); fs = Normal(θ,σ)
lines!(fn); lines!(fs)
```



I would say that this detection problem is quite tricky, because the two distributions are quite close together. The overlap of the false alarm and the detected signal is large, and so actually classifying what is noise and what is a true signal is difficult.

CFAR detector with $P_{FA} = 0.1$

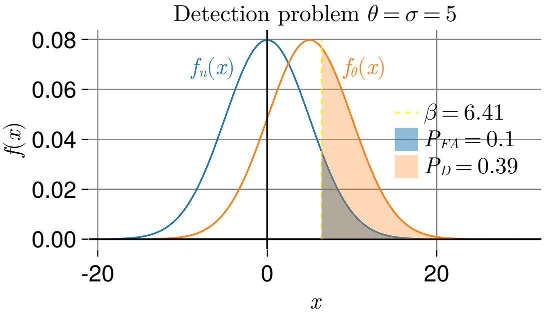
We want a threshold β such that $P_{FA} = 0.1$, which is given by

$$\int_{\beta}^{\infty} f_n(x)dx = P_n(\beta \leq x) = 1 - F_n(\beta)$$

where $F_n(x)$ is the CDF of the noise PDF. We know that the quantile function $Q : [0, 1] \to \mathbb{R}$ is defined as the inverse CDF, s.t. $Q_n(q) = F_n^{-1}(q)$. This gives the threshold $\beta = Q_n\,(1 - P_{FA})$. With β given, we can easly find P_D by $1 - F_s\,(\beta)$.

```
β = cquantile(fn, 0.1)
P_fa = ccdf(fn, β)
P_d  = ccdf(fs, β)
```

As we can see in the plot and evaluations, we we get a threshold at $\beta = 6.41$ with a $P_D = 0.39$. That is, there is a 39% chance for a detection.



ROC of detector

We now want to find the ROC of the detector. We do this by evaluating the detector at every possible threshold value.

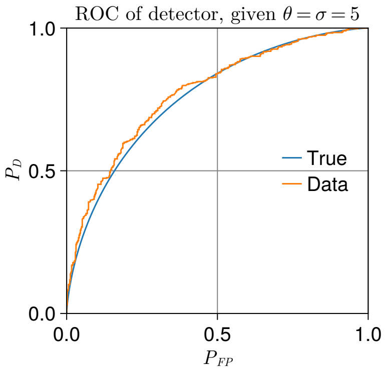
Create the data as done in assignmnet

```
θ = σ = 5
nData = 1000
s = θ * (randn(nData,1) .>= 0); # True data
x = s + σ * randn(size(s))
```

Then evaluate the true ROC and using the data

```
# Theoretical ROC
βs = 20:-0.01:-20
TPR_true,TFP_true = ccdf.(fs,βs), ccdf.(fn,βs)
# Emprirical ROC
detector = map(β -> β .< x, βs)
TPR_data = map(result -> mean(result[s .> 0.50]),
        detector)
TFP_data = map(result -> mean(result[s .< 0.50]),
        detector);
```

By plotting TFP against TPR, we get the following ROC plot

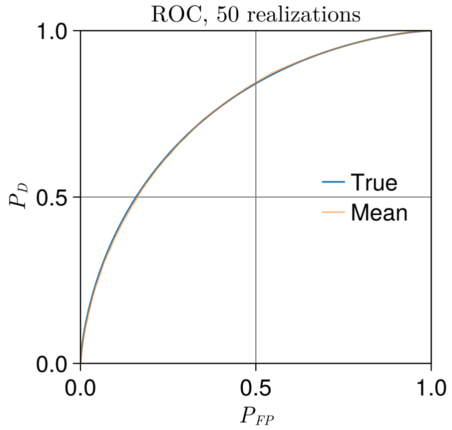


Several realizations

We rap the code above into a function `roc_realization()` and evaluate the ROC for a few realization.

```
many_runs = [ roc_realization() for _ in 1:50 ]
# Each row is an roc of threshold and columns are each run
# → β × run
TPR_many = mapreduce(run -> run[1], hcat, many_runs)
TFP_many = mapreduce(run -> run[2], hcat, many_runs)
# Takse the mean along runs
TPR_mean = mean(TPR_many, dims=2)
TFP_mean = mean(TFP_many, dims=2);
```

* We can see that increasing the number of realizations will make the ROC of the data approach the theoretical ROC.



Multi-measure over transmit

We now want to measure multiple times per transmit, such that

$$
\begin{aligned}
H_0' &: x_n = n_n \qquad \text{for } n = 1, \ldots, N \\
H_1' &: x_n = \theta + n_n \ \text{for } n = 1, \ldots, N
\end{aligned}
$$

* Both distribution are sampled N times and we take the mean.

$$Z = \frac{1}{N}\sum_{i=1}^{N} X_i$$

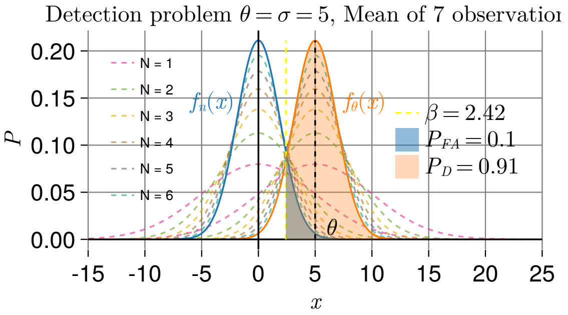
* We use the central limit theorem to find the distribution of each variable

$$
\begin{aligned}
H_0' &: z_0 \sim \mathcal{N}\left(0, \sigma/\sqrt{n}\right) \\
H_1' &: z_\theta \sim \mathcal{N}\left(\theta, \sigma/\sqrt{n}\right)
\end{aligned}
$$

* PDFs are Normal distributions with $\sigma_N = \sigma/\sqrt{(N)}$

```
multi_fn = Normal(0, σ/sqrt(N))
multi_fs = Normal(θ, σ/sqrt(N))
β = cquantile(multi_fn, 0.1)
P_fa, P_d = ccdf(multi_fn, β), ccdf(multi_fs, β)
```

This gives $P_{FA} = 0.1$ to be $\beta = 2.42$ which gives a $P_D = 0.91$.



ROC with multiple measurements

We want to find the ROC for different number of measurements N per observation.

Define some functions, and loop over each value of N

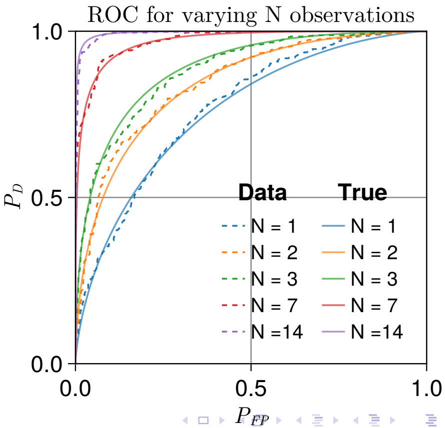
```
function make_observations(signal, noise_pdf, N)
    S = repeat(signal, 1, N)
    X = S .+ rand(noise_pdf, size(S))
    return X
end

mean_detector(observations, threshold) = mean(
        observations, dims=2) .>= threshold
```

Then iterate over each value of N

```
for (i,n) in enumerate([1,2,3,7,14])
    noise_pdf, signal_pdf = mean_cfar(fn, fs, n)
    X = make_observations(signal, fn, n)
    detector = map(β -> mean_detector(X, β), βs)
    # ...
end
```

Plotting the ROC for each detector, yields the following plot

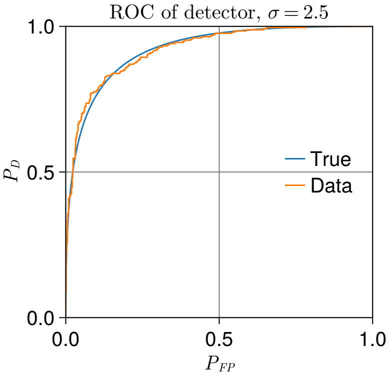
* Increasing N gives a better detector
* Data follows true roc as expected



Detector problem when $P_0 \neq P_\theta$

We now want to look at a detection problem where the ratio between 0 and θ differ from 1.

* We find the ROC for this detector problem



* There is no difference between the ROC plot when assuming $P_0 = P_\theta$ and with $P_0 \neq P_\theta$ * When assuming equal probability $P_0 = P_1$, ignoring cost gives the maximum likelihood (ML) criterion

$$\frac{f_\theta(\boldsymbol{y})}{f_0(\boldsymbol{y})} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

* With *a priori* probability $P_0 \neq P_1$, ignoring cost gives the maximum *a posteriori* (MAP) criterion

$$\frac{f_\theta(\boldsymbol{y})}{f_0(\boldsymbol{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1}$$

* Only difference is threshold value given observations