

1 Hebbian Learning

When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic change takes places in one or both cells such that A's efficiency, as one of the cells firing B, is increased. [1]

In this section, we will construct a framework in which we can formulate rules to make networks of neurons adaptive. We model the strength of the synapse between θ_i and θ_j as K_{ij} and in that sense we will mostly speak of θ_i as the *presynaptic* neuron and of θ_j as the *postsynaptic* neuron.

1.1 Modelling spike-based Hebbian learning

1.1.1 Locality

Changes to K_{ij} only depend on the membrane potential u of θ_i and θ_j . Therefore we can write:

$$\frac{dK_{ij}(t)}{dt} = F(K_{ij}(t), u_i^{\text{pre}}(t'), u_j^{\text{post}}(t'')) \quad (1)$$

F is thus a functional of the time course of the neuron membrane potentials at the synapse. The timescales t' and t'' signify that changes in the synaptic strength depend on the current membrane potentials and their history ($t', t'' < t$). F could in fact also depend on other intrinsic properties of the neuron, though we assume those already have an effect on the membrane potential and are therefore left out [2].

1.1.2 Cooperativity

Activity from both the pre- and postsynaptic neuron is required to induce changes of K_{ij} and response to the activity is causal. This requires the activity to be almost simultaneous, and we can implement this into (1) by making a Volterra expansion of F about the rest state $u_i^{\text{pre}} = u_j^{\text{post}} = u_{\text{rest}}$ where u_{rest} is set to 0 for simplicity.

$$\frac{dK_{ij}}{dt} = c_0(K_{ij}) + \int_0^\infty I_1^{\text{pre}}(K_{ij}, s) \cdot u_i^{\text{pre}}(t-s)ds + \int_0^\infty I_1^{\text{post}}(K_{ij}, s) \cdot u_j^{\text{post}}(t-s)ds \quad (2)$$

$$+ \int_0^\infty \int_0^\infty I_2^{\text{corr}}(K_{ij}, s, s') \cdot u_i^{\text{pre}}(t-s) \cdot u_j^{\text{post}}(t-s')ds'ds + \dots \quad (3)$$

Here I

1.1.3 Synaptic depression

1.1.4 Boundedness

1.1.5 Competition

1.1.6 Long term stability

2 Spike Timing Dependant Plasticity

Write about STDP, see the 'Postulates.pdf'.

With spike timing dependant plasticity we want to study how two neurons change their synaptic strength based on the time delay between spikes. When $\theta(t)_j > \pi$ we say that the neuron θ_j spikes at time t . Let us say that θ_i spikes at time t_i and θ_j spikes at t_j . Taking the time difference Δt_{ij} as $t_j - t_i$, we can say that when $\Delta t_{ij} > 0$ the spikes are correlated (there exists a temporally causal relation), and we can model an increase in synaptic strength of the connection K_{ij} from θ_i to θ_j . In the same fashion we can decrease K_{ji} when $\Delta t_{ij} < 0$ as there is no causal relation.

The functions $W(t)$ that relate Δt_{ij} to ΔK_{ij} are called *learning windows*, as they define a range in which K_{ij} is able to adapt. When signals between neurons show a very large time difference

(negative or positive) we do not expect them to be correlated. Because the learning windows are generally not symmetrical we can also expect an asymmetrical adjacency matrix. Another characteristic is the integral over the learning window. A window with a negative integral directs synaptic strengths mostly towards inhibitory behaviour, and vice versa with a positive integral. An integral of zero would mean that both inhibitory and excitatory synapses are stimulated equally.

2.1 Method

Now that we have a feeling of how *STDP* works, we need to formulate the behaviour exactly. One by one, we will denote our ideas into mathematics.

2.1.1 From Kempter1999

Following the notation in [3], we will denote the spike train coming from each neuron θ_i as $S_i^{\text{out}}(t) = \sum_n \delta(t - t_i^n)$, where t_i^n is the time that θ_i has fired. Similarly, we will denote the spike train coming into each neuron θ_i as $S_i^{\text{in}}(t) = \sum_f \delta(t - t_i^f)$. Now we can say that the synaptic strengths are adjusted as:

$$\Delta K_{ij} = \int_t^{t+\mathcal{T}} w^{\text{out}} S_i^{\text{out}}(\tau) + w^{\text{in}} S_j^{\text{in}}(\tau) d\tau + \iint_t^{t+\mathcal{T}} W(\tau' - \tau) S_i^{\text{out}}(\tau) S_j^{\text{in}}(\tau') d\tau d\tau' \quad (4)$$

$$= \sum_{t_i^n \in \mathcal{T}} w^{\text{out}} + \sum_{t_j^f \in \mathcal{T}} w^{\text{in}} + \sum_{t_j^f, t_i^n \in \mathcal{T}} W(t_j^f - t_i^n) \quad (5)$$

(4) is the most general formulation of *STDP* and includes a small change in weights when a signal is merely received. Whether this is a process that influences *STDP* is still under investigation, but we will neglect the contribution.

2.2 Biphasic Learning windows

2.2.1 Kempter 1999

$$W(t)_K = \eta \begin{cases} \left[A_p \left(1 - \frac{t}{\tilde{\tau}_p} \right) + A_n \left(1 - \frac{t}{\tilde{\tau}_n} \right) \right] \cdot \exp\left(\frac{t}{\tau_{\text{syn}}}\right) & \text{for } t \leq 0 \\ A_p \cdot \exp\left(-\frac{t}{\tilde{\tau}_p}\right) + A_n \cdot \exp\left(-\frac{t}{\tilde{\tau}_n}\right) & \text{for } t > 0 \end{cases} \quad (6)$$

Here t is the delay between presynaptic spike arrival and postsynaptic firing, η is a small learning parameter and all τ are time constants. Numerical values are usually $\eta = 0.05$, $\tau_{\text{syn}} = 5$ ms, $\tau_p = 1$ ms, $\tau_n = 20$ ms and $A_p = 1$ and $A_n = -1$. $\tilde{\tau}_p \equiv \tau_{\text{syn}} \tau_p / (\tau_{\text{syn}} + \tau_p)$ and $\tilde{\tau}_n \equiv \tau_{\text{syn}} \tau_n / (\tau_{\text{syn}} + \tau_n)$. $\int W(s)_K ds = 2.56 \times 10^{-4}$.

2.2.2 Song 2000

The first formulation of *STDP* as a mathematical model was in [4].

$$W(t)_S = \begin{cases} A_p \cdot \exp\left(\frac{-t}{\tau_p}\right) & \text{for } s > 0 \\ A_n \cdot \exp\left(\frac{t}{\tau_n}\right) & \text{for } s \leq 0 \end{cases} \quad (7)$$

where we will use the values as proposed in [5]: $A_p = 0.1$, $A_n = -0.12$ and $\tau_p = \tau_n = 20$. $\int W(s)_S ds = -3.70 \times 10^{-4}$.

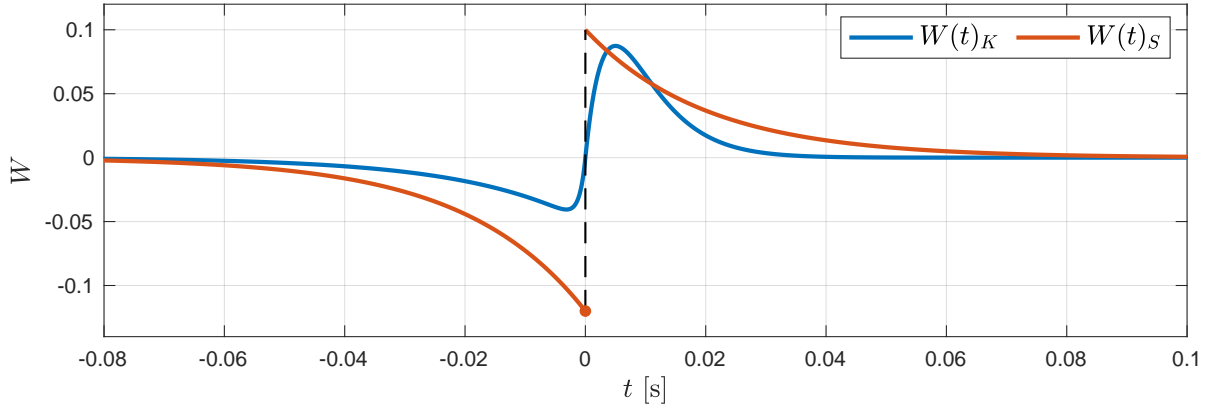


Figure 1: Two different biphasic learning windows. We can see how in $W(t)_S$ a larger weight is put on the anti-Hebbian learning.

2.3 Triphasic Learning windows

2.3.1 Chrol-Cannon 2012

$$W(t)_C = A_p \cdot \exp\left(\frac{-(t-15)^2}{\tau_p}\right) - A_n \cdot \exp\left(\frac{-(t-20)^2}{\tau_n}\right) \quad (8)$$

where $A_p = 0.23$, $A_n = 0.15$, $\tau_p = 200$ and $\tau_n = 2000$. $\int W(s)_C ds = -60.0 \times 10^{-4}$.

2.3.2 Waddington 2014

$$W(t)_W = A \left[1 - \frac{(t-\alpha)^2}{\alpha^2} \right] \cdot \exp\left(\frac{-|t-\alpha|}{\alpha}\right) \quad (9)$$

We will use $A = 0.1$ and $\alpha = 4.0$ ms. $\int W(s)_W ds = -8.0 \times 10^{-4}$.

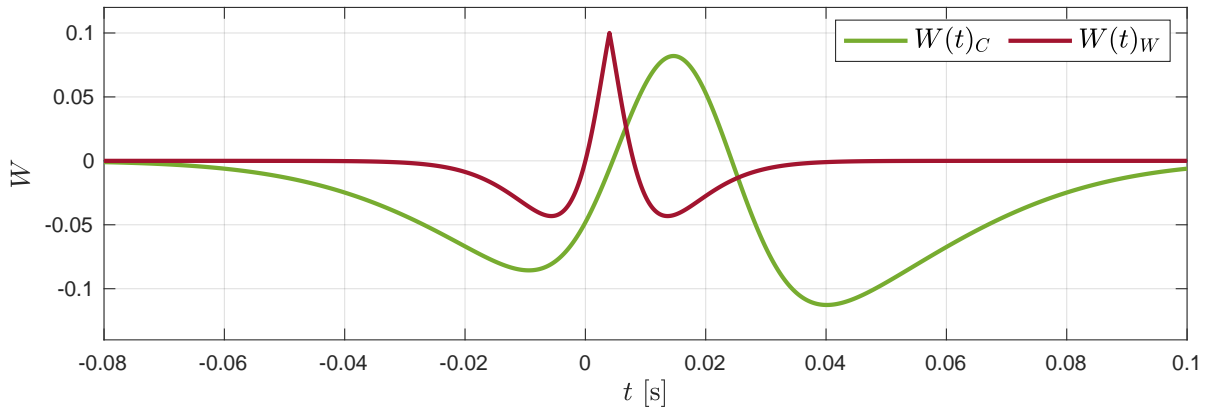


Figure 2: Two different triphasic learning windows. We can see how in $W(t)_C$ a much higher penalty is given to signals that arrive too early or too late. The moments at which signals should peak also differ by about 10 ms.

2.4 Interpretation

The learning windows generally have $W(t^*) = 0$ for $t^* > 0$. This means that no learning will take place when the delay between neuron spikes is exactly $t^* > 0$. The triphasic windows show two of those points.

2.5 Results

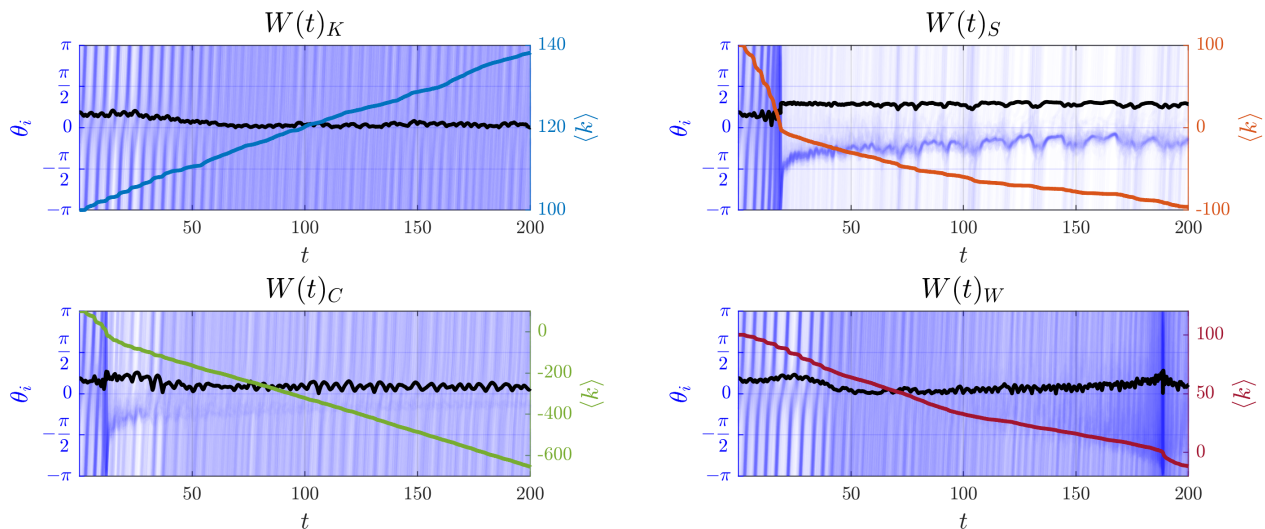


Figure 3: Two different triphasic learning windows. We can see how in $W(t)_C$ a much higher penalty is given to signals that arrive too early or too late. The moments at which signals should peak also differ by about 10 ms.

2.6 Synaptic scaling

There is no upper or lower bound on the synapse strength. Also generally, connection strengths are nonzero. This means that the notion of a *network* is lost.

One technique we can apply to keep the strengths within a definitive range is to scale homeostatically - a method where increases in synaptic strength will balance out any decreases by scaling:

$$K_{ij}^s = K_{ij} \frac{\frac{1}{N} \sum_{i,j} K_{ij}}{\sum_i K_{ij}} \quad (10)$$

2.7 Custom learning windows

Some of the behaviour we want to observe is currently not accounted for: synaptic strengths are unbounded and are generally nonzero. How can we model the behaviour where synaptic strengths can also settle on being zero? Perhaps a better idea than updating the synaptic strength by adding a new value, we can scale it.

- When Δt_{ij} is very large (both positive and negative) we expect no change in the synaptic strength: $W(-\infty) = W(\infty) = 0$.
- We expect a specific positive time delay to yield the most amount of synaptic strengthening: $\Delta t_{\text{best}} = \arg \max_x W(t)$
- We expect small connections that have been reducing in size to quickly become zero,
- A better scaling system would be that not the out- or in-degree is constant, but

References

- [1] D. O. Hebb, *The organization of behaviour: a neurophysical theory*. John Wiley and sons, 1949.
- [2] W. Gerstner and W. Kistler, *Mathematical Formulations of Hebbian Learning*. *Biological cybernetics* **87** (01, 2003) 404–15.

- [3] R. Kempter, W. Gerstner, and L. van Hemmen, *Hebbian learning and spiking neurons*. *Phys. Rev. E* **59** (04, 1999) .
- [4] S. Song, K. Miller, and L. Abbott, *Competitive Hebbian learning through spike timing-dependent plasticity* *Nature neuroscience* **3** (10, 2000) 919–26.
- [5] J. Chrol-Cannon, A. Grüning, and Y. Jin, *The emergence of polychronous groups under varying input patterns, plasticity rules and network connectivities* in *The 2012 International Joint Conference on Neural Networks (IJCNN)*, pp. 1–6, IEEE. 2012.