MFRs December 1, 2020
Msc Thesis - Dynamics of adaptive neuronal networks.
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1 Writing out the whole system

The network dynamics are described as follows:

$$\begin{aligned} \frac{d\theta_i}{dt} &= (1-\cos\theta_i) + (1+\cos\theta_i)\cdot(\eta_i + I_i) \qquad \theta_i \in \mathbb{T}^N \\ I_i &= \frac{\kappa}{\langle k\rangle}\sum_{j=1}^{N} A_{ij} P_n\left(\theta_j\right) \\ P_n\left(\theta_j\right) &= a_n(1-\cos\theta_j) \end{aligned} \tag{1}$$

We observe synchronisation through the order parameter

$$Z(t) = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j} \qquad Z(t)\in\mathbb{C} \tag{2}$$

For a fixed degree network it has been proven that the order parameter follows:

$$\dot{Z}(t) = -\mathrm{i}\frac{(Z-1)^2}{2} + \frac{(Z+1)^2}{2}\cdot\left(-\Delta + \mathrm{i}\eta_0 + \mathrm{i}\kappa\cdot\left(1 + \frac{Z^2+\overline{Z}^2}{6} - \frac{4}{3}\mathrm{Re}(Z)\right)\right) \tag{3}$$

For an arbitrary network the order parameter follows a trajectory per degree. When we assemble the whole expression for the Ott-Antonsen manifold as found in [1] with $H_2(\boldsymbol{k},t)$ as in [2], we obtain the following:

$$\begin{aligned} \frac{\partial z(\boldsymbol{k},t)}{\partial t} &= -\mathrm{i}\frac{(z(\boldsymbol{k},t)-1)^2}{2} + \frac{(z(\boldsymbol{k},t)+1)^2}{2}\cdot I(\boldsymbol{k},t) \qquad z(\boldsymbol{k},t)\in\mathbb{C}^{M_{\boldsymbol{k}}} \\ I(\boldsymbol{k},t) &= -\Delta(\boldsymbol{k}) + \mathrm{i}\eta_0(\boldsymbol{k}) + i\kappa\cdot H_2(\boldsymbol{k},t) \\ H_2(\boldsymbol{k},t) &= \frac{1}{\langle k\rangle}\sum_{\boldsymbol{k'}} P\left(\boldsymbol{k'}\right) a\left(\boldsymbol{k'}\to\boldsymbol{k}\right)\cdot\left(1 + \frac{z(\boldsymbol{k'},t)^2 + \overline{z}(\boldsymbol{k'},t)^2}{6} - \frac{4}{3}\mathrm{Re}(z(\boldsymbol{k'},t))\right) \end{aligned} \tag{4}$$

$\boldsymbol{k}$ represents a two-dimensional vector of the in- an out degree as $\boldsymbol{k} = (k^{\text{in}}, k^{\text{out}})$ and has unique entries as it forms the support of the vector of degrees of (1) as $\deg(\theta_i)$. So $z(\boldsymbol{k},t)$ is really a vector in the complex plane that represents the mean-field dynamics on any node with degree $\boldsymbol{k}$, so we could also index as $z(t)_{\boldsymbol{k}}$. $\boldsymbol{k'}$ represents $\boldsymbol{k}$ when $\boldsymbol{k}$ is already in use.

We can then find the mean field dynamics through

$$\overline{Z}(t) = \frac{1}{N}\sum_{\boldsymbol{k}} P(\boldsymbol{k}) z(\boldsymbol{k},t) \qquad \overline{Z}(t)\in\mathbb{C} \tag{5}$$

It is important to notice that in (4) and (5) we actually compute an inner vector product, which is non-commutative for complex numbers:

$$a\cdot b = \overline{b\cdot a} \qquad a,b\in\mathbb{C}^r \tag{6}$$

This is the result of the *Conjugate* or *Hermitian* symmetry of the inner product. This is especially important in the MATLAB implementation.

2 Initial conditions

As the systems in eqs. (1) to (5) describe the same dynamics for fully connected networks, it is important to be able to transform initial conditions between systems. When transforming from $\theta_i(t) \to z(\boldsymbol{k}, t) \to Z(t)$ we go from $\mathbb{T}^N \to \mathbb{C}^{M_{\boldsymbol{k}}}$ to $\mathbb{C}^{M_{\boldsymbol{k}}} \to \mathbb{C}$. If we have the same initial conditions, then all systems will predict the same behaviour. We will only map everything to $\mathbb{C}$.
The following maps can be used to transform the initial conditions, but as they do not give any qualitative information on the dynamics or distributions of the variables, they are not valid for transforming between dynamics. We discard $t = 0$ for clarity.
Mapping operations onto the order parameter in the complex plane is straightforward:

$$\theta_i \longrightarrow Z = \frac{1}{N} \sum_{j=1}^{N} e^{\mathrm{i}\theta_j}$$
$$z(\boldsymbol{k}) \longrightarrow Z = \frac{1}{N} \sum_{\boldsymbol{k}} P(\boldsymbol{k}) z(\boldsymbol{k}, t)$$

Taking the inverse maps, we can make use of the fact that the average of a set of identical values is the value itself. For $z(\boldsymbol{k})$ we have a weighed average which we need to undo, making sure that the whole sums up to N.

$$Z \longrightarrow \theta_i = -\mathrm{i} \cdot \log\left(Z\right)$$
$$Z \longrightarrow z(\boldsymbol{k}) = \overline{\frac{Z \cdot n(\boldsymbol{k})}{P(\boldsymbol{k})}}$$

Then, transferring between θ_i and $z(\boldsymbol{k})$, we need to filter θ_i per degree:

$$z(\boldsymbol{k}) \longrightarrow \theta_i = -\mathrm{i} \cdot \log\left(\frac{z(\boldsymbol{k}) \cdot P(\boldsymbol{k})}{n(\boldsymbol{k})}\right) \qquad \text{if } \deg(\theta_i) = \boldsymbol{k}$$
$$\theta_i \longrightarrow z(\boldsymbol{k}) = \sum_{\boldsymbol{k}} e^{\mathrm{i}\vartheta_{\boldsymbol{k}}} \qquad \vartheta_{\boldsymbol{k}} = \sum_{i=1}^{n(\boldsymbol{k})} \theta_i \in \{\theta_i | \deg(\theta_i) = \boldsymbol{k}, \forall i \leq N\}$$

We can see how $\lim_{N\to+\infty} n(\boldsymbol{k}) = P(\boldsymbol{k})$, which makes these maps exact for any network size.

3 Fixpoint iteration

In [1] a fixpoint iteration is suggested to find attractive fixpoints of the system (4). If we set $\frac{\partial z(\boldsymbol{k},t)}{\partial t} = 0$ we can solve the following system:

$$\mathrm{i}\frac{(z(\boldsymbol{k}, t) - 1)^2}{2} = \frac{(z(\boldsymbol{k}, t) + 1)^2}{2} \cdot I(\boldsymbol{k})$$
$$\mathrm{i}\left(\frac{z(\boldsymbol{k}, t) - 1}{z(\boldsymbol{k}, t) + 1}\right)^2 = I(\boldsymbol{k})$$
$$\frac{z(\boldsymbol{k}, t) - 1}{z(\boldsymbol{k}, t) + 1} \equiv b(\boldsymbol{k}, t)$$
$$z(\boldsymbol{k}, t) - 1 = b(\boldsymbol{k}, t) z(\boldsymbol{k}, t) + b(\boldsymbol{k}, t)$$
$$z(\boldsymbol{k}, t) \cdot (1 - b(\boldsymbol{k}, t)) = b(\boldsymbol{k}, t) + 1$$

We can then obtain the stable equilibria from:

$$\mathrm{i}b(\boldsymbol{k}, t)^2 = I(\boldsymbol{k}) \qquad z(\boldsymbol{k}, t)_{\pm} = \frac{1 \pm b(\boldsymbol{k}, t)}{1 \mp b(\boldsymbol{k}, t)} \tag{7}$$

where the signs are chosen so that $|z(\boldsymbol{k}, t)| \leq 1$. This works.

4 A Newton-Raphson iteration for all fixpoints

4.1 Theory behind the method

The fixpoint iteration only gives us the stable equilibria of the system (4). We can obtain all equilibria and the Jacobian from a Newton-Raphson iteration. We define the equilibria $\boldsymbol{x}^* \in \mathbb{R}^n$ of a multivariate function $\boldsymbol{f}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$ with $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$. Expanding $\boldsymbol{f}$ as a Taylor series, we obtain:

$$f_i(\boldsymbol{x} + \delta\boldsymbol{x}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{n} \frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\delta x_j + O\left(\delta\boldsymbol{x}^2\right) \approx f_i(\boldsymbol{x}) + \sum_{j=1}^{n} \frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\delta x_j, \qquad (i = 1, \cdots, n) \tag{8}$$

We can also write this in vector notation, by setting $\boldsymbol{J}(\boldsymbol{x}) = \nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{d}{d\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$

$$\boldsymbol{f}(\boldsymbol{x} + \delta\boldsymbol{x}) \approx \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_N(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_N \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{J}(\boldsymbol{x})\delta\boldsymbol{x} \tag{9}$$

By assuming $\boldsymbol{f}(\boldsymbol{x} + \delta\boldsymbol{x}) = 0$ we can find that $\delta\boldsymbol{x} = -\boldsymbol{J}^{-1}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$ so that $\boldsymbol{x} + \delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{J}^{-1}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$. This expression converges to $\boldsymbol{x}^*$. When the equations are nonlinear, the equations converge to the real root as $\boldsymbol{x}_k = \boldsymbol{x}_k - \boldsymbol{J}^{-1}(\boldsymbol{x}_k)\boldsymbol{f}(\boldsymbol{x}_k)$.

For (4), we can compute the Jacobian for the diagonal and off-diagonal elements separately. But as $z(\boldsymbol{k}, t)$ is a complex function, first we need to understand what the derivative of a complex function is.

4.2 Derivatives of complex functions

For $z = x + \mathrm{i}y \in \mathbb{C}$ and $x, y \in R$ the conjugate is defined as $\overline{z} = x - \mathrm{i}y$. That means that we can write the real and imaginary parts as:

$$x = \frac{z + \overline{z}}{2} \text{ and } y = -\mathrm{i}\frac{z - \overline{z}}{2}$$

Using the chain rule, we can write the partial derivative with respect to z in function of x and y as x and y are functionally independent and find the first Wirtinger operator:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial z}\frac{\partial}{\partial \overline{y}} \longrightarrow \frac{\partial x}{\partial z} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial z} = -\frac{\mathrm{i}}{2} \longrightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)$$

We note the following properties:

$$\frac{\partial}{\partial z}z = 1 \qquad \frac{\partial}{\partial z}\overline{z} = \frac{1}{2}\left(1 - \mathrm{i}^2\right) = 0$$

Interesting for is the result of the following:

$$\begin{aligned} \overline{z}^2 &= (x - \mathrm{i}y)^2 = x^2 - y^2 - \mathrm{i}2xy \\ \frac{\partial}{\partial z}\overline{z}^2 &= \frac{1}{2} \cdot (2x - \mathrm{i}2y - \mathrm{i} \cdot (-2y - \mathrm{i}2x)) = x - \mathrm{i}y + \mathrm{i}y - \mathrm{i}x = 0 \end{aligned}$$

4.3 Derivatives of the complex mean field equations

We can now compute derivatives of the complex functions $z(\boldsymbol{k}, t)$. We will set $z(\boldsymbol{k}, t) = z_{\boldsymbol{k}}$ and rewrite system (4) to help the reader:

$$
\begin{aligned}
\frac{\partial z_{\boldsymbol{k}}}{\partial t} &= -\mathrm{i}\frac{(z_{\boldsymbol{k}}-1)^2}{2} + \frac{(z_{\boldsymbol{k}}+1)^2}{2} \cdot I_{\boldsymbol{k}} \qquad z_{\boldsymbol{k}} \in \mathbb{C}^{M_{\boldsymbol{k}}} \\
I_{\boldsymbol{k}} &= -\Delta_{\boldsymbol{k}} + \mathrm{i}\eta_{0_{\boldsymbol{k}}} + i\kappa \cdot H_{2_{\boldsymbol{k}}} \\
H_{2_{\boldsymbol{k}}} &= \frac{1}{\langle k \rangle} \sum_{\boldsymbol{k}'} P_{\boldsymbol{k}} a_{\boldsymbol{k}'\boldsymbol{k}} \cdot \left(1 + \frac{z_{\boldsymbol{k}'}^2 + \overline{z}_{\boldsymbol{k}'}^2}{6} - \frac{4}{3}\mathrm{Re}(z_{\boldsymbol{k}'})\right)
\end{aligned}
\tag{10}
$$

The diagonal elements are found as:

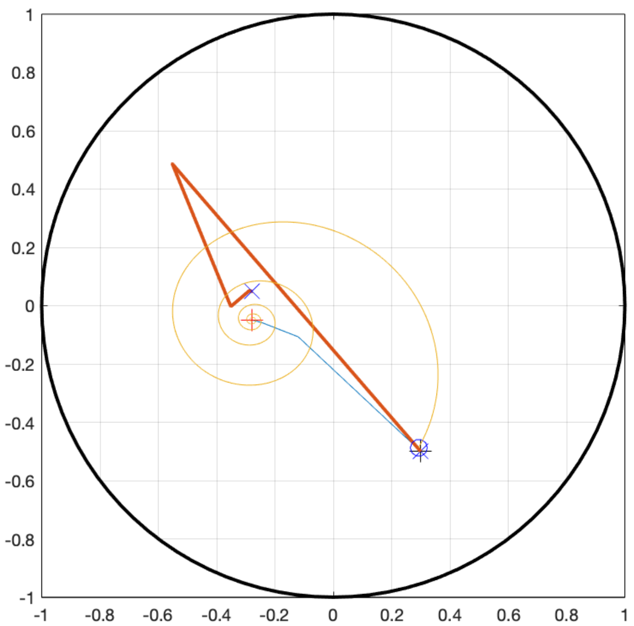
$$
\begin{aligned}
\frac{\partial}{\partial z_{\boldsymbol{k}}}\left(\frac{\partial z_{\boldsymbol{k}}}{\partial t}\right) &= -\mathrm{i}(z_{\boldsymbol{k}}-1) + (z_{\boldsymbol{k}}+1) \cdot I_{\boldsymbol{k}} + \frac{(z_{\boldsymbol{k}}+1)^2}{2} \cdot \frac{\partial I(z_{\boldsymbol{k}})}{\partial z_{\boldsymbol{k}}} \\
\frac{\partial I_{\boldsymbol{k}}}{\partial z_{\boldsymbol{k}}} &= \mathrm{i}\kappa \cdot \frac{\partial H_{2_{\boldsymbol{k}}}}{\partial z_{\boldsymbol{k}}} \\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial z_{\boldsymbol{k}}} &= \frac{1}{\langle k \rangle} P_{\boldsymbol{k}} a_{\boldsymbol{k}\boldsymbol{k}} \cdot \left(\frac{2z_{\boldsymbol{k}}}{6} - \frac{4}{3} \cdot \frac{1}{2}\right) = \frac{1}{\langle k \rangle} P_{\boldsymbol{k}} a_{\boldsymbol{k}\boldsymbol{k}} \cdot \frac{z_{\boldsymbol{k}}-2}{3}
\end{aligned}
\tag{11}
$$

The off-diagonal elements are found as

$$
\begin{aligned}
\frac{\partial}{\partial z_{\boldsymbol{k}'}}\left(\frac{\partial z_{\boldsymbol{k}}}{\partial t}\right) &= \frac{(z_{\boldsymbol{k}}+1)^2}{2} \cdot \frac{\partial I_{\boldsymbol{k}}}{\partial z_{\boldsymbol{k}'}} \\
\frac{\partial I_{\boldsymbol{k}}}{\partial z_{\boldsymbol{k}'}} &= \mathrm{i}\kappa \cdot \frac{\partial H_{2_{\boldsymbol{k}}}}{\partial z_{\boldsymbol{k}'}} \\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial z_{\boldsymbol{k}'}} &= \frac{1}{\langle k \rangle} P_{\boldsymbol{k}'} a_{\boldsymbol{k}'\boldsymbol{k}} \cdot \frac{z_{\boldsymbol{k}'}-2}{3}
\end{aligned}
\tag{12}
$$

4.4 First results

I cannot seem to converge close enough.



The reason that this method does not work is because the field $H_2(\boldsymbol{k},t)$ is non-holomorphic. We can verify this with the Cauchy-Riemann equations:

$$\begin{aligned}
z(\boldsymbol{k},t) &= x(\boldsymbol{k},t) + \mathrm{i}\cdot y(\boldsymbol{k},t)\\
f\left(z(\boldsymbol{k},t)\right) &= u\left(x(\boldsymbol{k},t), y(\boldsymbol{k},t)\right) + \mathrm{i}v\left(x(\boldsymbol{k},t), y(\boldsymbol{k},t)\right)\\
&= \frac{1}{\langle k\rangle}\sum_{\boldsymbol{k'}} P\left(\boldsymbol{k'}\right) a\left(\boldsymbol{k'}\to\boldsymbol{k}\right)\cdot\left(1+\frac{z(\boldsymbol{k'},t)^2+\overline{z}(\boldsymbol{k'},t)^2}{6}-\frac{4}{3}\mathrm{Re}(z(\boldsymbol{k'},t))\right)\\
&= \frac{1}{\langle k\rangle}\sum_{\boldsymbol{k'}} P\left(\boldsymbol{k'}\right) a\left(\boldsymbol{k'}\to\boldsymbol{k}\right)\cdot\left(1+\frac{x(\boldsymbol{k'},t)^2}{3}-\frac{4}{3}x(\boldsymbol{k'},t)\right)
\end{aligned}$$

This leaves us with only u defined as a real-valued function, so that the Cauchy-Riemann equations do not hold: the function is non-holomorphic.

4.5 Mapping onto $\mathbb{R}^2$

$\bar{z}_{\boldsymbol{k}}$ and $\mathrm{Re}(z_{\boldsymbol{k}})$ are non-holomorphic. Try again by separating the real and imaginary part: $z(\boldsymbol{k},t) = x(\boldsymbol{k},t) + \mathrm{i}y(\boldsymbol{k},t)$.

$$\begin{aligned}
\dot{x_{\boldsymbol{k}}} &= f_{\boldsymbol{k}}\left(x_{\boldsymbol{k}}, y_{\boldsymbol{k}}, \eta_0, \Delta, \kappa\right)\\
&= (x_{\boldsymbol{k}}-1)y_{\boldsymbol{k}} - \frac{(x_{\boldsymbol{k}}+1)^2-y_{\boldsymbol{k}}^2}{2}\Delta - (x_{\boldsymbol{k}}+1)\cdot y_{\boldsymbol{k}}\cdot\left[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}\right]\\
\dot{y_{\boldsymbol{k}}} &= g_{\boldsymbol{k}}\left(x_{\boldsymbol{k}}, y_{\boldsymbol{k}}, \eta_0, \Delta, \kappa\right)\\
&= -\frac{(x_{\boldsymbol{k}}-1)^2-y_{\boldsymbol{k}}^2}{2} - (x_{\boldsymbol{k}}+1)y_{\boldsymbol{k}}\cdot\Delta + \frac{(x_{\boldsymbol{k}}+1)^2-y_{\boldsymbol{k}}^2}{2}\cdot\left[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}\right]
\end{aligned}$$

The Jacobian can the be found from taking the derivatives?

$$\begin{aligned}
\frac{\partial f_{\boldsymbol{k}}}{\partial x_{\boldsymbol{k}}} &= y_{\boldsymbol{k}} - (x_{\boldsymbol{k}}+1)\cdot\Delta - y_{\boldsymbol{k}}\cdot[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}] - (x_{\boldsymbol{k}}+1)\cdot y_{\boldsymbol{k}}\cdot\left(\kappa\cdot\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}}}\right)\\
\frac{\partial f_{\boldsymbol{k}}}{\partial y_{\boldsymbol{k}}} &= (x_{\boldsymbol{k}}-1) + y_{\boldsymbol{k}}\cdot\Delta - (x_{\boldsymbol{k}}+1)\cdot[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}]\\
\frac{\partial g_{\boldsymbol{k}}}{\partial x_{\boldsymbol{k}}} &= -(x_{\boldsymbol{k}}-1) - y_{\boldsymbol{k}}\cdot\Delta + (x_{\boldsymbol{k}}+1)\cdot[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}] + \left(\frac{(x_{\boldsymbol{k}}+1)^2-y_{\boldsymbol{k}}^2}{2}\right)\cdot\left(\kappa\cdot\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}}}\right)\\
\frac{\partial g_{\boldsymbol{k}}}{\partial y_{\boldsymbol{k}}} &= y_{\boldsymbol{k}} - (x_{\boldsymbol{k}}+1)\cdot\Delta - y_{\boldsymbol{k}}\cdot[\eta_0+\kappa\cdot H_{2_{\boldsymbol{k}}}]\\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}}} &= \frac{1}{\langle k\rangle}P_{\boldsymbol{k}}a_{\boldsymbol{k}\boldsymbol{k}}\cdot(x_{\boldsymbol{k}}-2)\cdot\frac{2}{3}\\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial y_{\boldsymbol{k}}} &= 0
\end{aligned}$$

And the off-diagonal elements:

$$\begin{aligned}
\frac{\partial f_{\boldsymbol{k}}}{\partial x_{\boldsymbol{k}'}} &= -(x_{\boldsymbol{k}}+1)\cdot y_{\boldsymbol{k}}\cdot\left(\kappa\cdot\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}'}}\right)\\
\frac{\partial f_{\boldsymbol{k}}}{\partial y_{\boldsymbol{k}'}} &= 0\\
\frac{\partial g_{\boldsymbol{k}}}{\partial x_{\boldsymbol{k}'}} &= \left(\frac{(x_{\boldsymbol{k}}+1)^2-y_{\boldsymbol{k}}^2}{2}\right)\cdot\left(\kappa\cdot\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}'}}\right)\\
\frac{\partial g_{\boldsymbol{k}}}{\partial y_{\boldsymbol{k}'}} &= 0\\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial x_{\boldsymbol{k}'}} &= \frac{1}{\langle k\rangle}P_{\boldsymbol{k}'}a_{\boldsymbol{k}'\boldsymbol{k}}\cdot(x_{\boldsymbol{k}'}-2)\cdot\frac{2}{3}\\
\frac{\partial H_{2_{\boldsymbol{k}}}}{\partial y_{\boldsymbol{k}'}} &= 0
\end{aligned}$$

What dimensions does the Jacobian have now? $2n\times 2n$.

4.6 From the order parameter

What if we take the Jacobian straight from the OA order parameter? If $z(\boldsymbol{k},t)=x(\boldsymbol{k},t)+\mathrm{i}y(\boldsymbol{k},t)$:

$$\overline{Z}(t)=\frac{1}{N}\sum_{\boldsymbol{k}}P(\boldsymbol{k})\cdot(x(\boldsymbol{k},t)+\mathrm{i}y(\boldsymbol{k},t))=\frac{1}{N}\sum_{\boldsymbol{k}}P(\boldsymbol{k})x(\boldsymbol{k},t)+\mathrm{i}\frac{1}{N}\sum_{\boldsymbol{k}}P(\boldsymbol{k})y(\boldsymbol{k},t) \tag{13}$$

References

[1] S. Chandra, D. Hathcock, K. Crain, T. Antonsen, M. Girvan, and E. Ott, *Modeling the Network Dynamics of Pulse-Coupled Neurons.* *Chaos (Woodbury, N.Y.)* **27** (03, 2017) 10.

[2] C. Bick, M. Goodfellow, C. Laing, and E. Martens, *Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: a review.* *Journal of Mathematical Neuroscience* **10** no. 1, (Dec., 2020) .