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Msc Thesis - Dynamics of adaptive neuronal networks.
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1 Writing out the whole system

When we assemble the whole expression for the Ott-Antonsen manifold as found in [1] we obtain the following:

$$
\begin{aligned}
\frac{\partial z(\boldsymbol{k},t)}{\partial t} &= -\mathrm{i}\frac{(z(\boldsymbol{k},t)-1)^2}{2} + \frac{(z(\boldsymbol{k},t)+1)^2}{2}\cdot I(\boldsymbol{k}) \\
I(\boldsymbol{k}) &= -\Delta(\boldsymbol{k}) + \mathrm{i}\eta_0(\boldsymbol{k}) + id_n\kappa\cdot H_n(\boldsymbol{k},t) \\
H_n(\boldsymbol{k},t) &= \frac{a_n}{\langle k\rangle}\sum_{\boldsymbol{k'}} P(\boldsymbol{k'})\, a(\boldsymbol{k'}\to\boldsymbol{k})\cdot\left[A_0 + \sum_{p=1}^{n} A_p\left(z(\boldsymbol{k'},t)^p + \overline{z}(\boldsymbol{k'},t)^p\right)\right]
\end{aligned}
\tag{1}
$$

Here $z(\boldsymbol{k},t)\in\mathbb{C}^{M_{\boldsymbol{k}}}$. Following [2], $H_2(\boldsymbol{k},t)$ is computed as:

$$
H_2(\boldsymbol{k},t) = \frac{1}{\langle k\rangle}\sum_{\boldsymbol{k'}} P(\boldsymbol{k'})\, a(\boldsymbol{k'}\to\boldsymbol{k})\cdot\left(1 + \frac{z(\boldsymbol{k'},t)^2+\overline{z}(\boldsymbol{k'},t)^2}{6} - \frac{4}{3}\mathrm{Re}(z(\boldsymbol{k'},t))\right) \tag{2}
$$

2 Fixpoint iteration

In [1] a fixpoint iteration is suggested to find attractive fixpoints of the system (1). If we set $\frac{\partial z(\boldsymbol{k},t)}{\partial t}=0$ we can solve the following system:

$$
\begin{aligned}
\mathrm{i}\frac{(z(\boldsymbol{k},t)-1)^2}{2} &= \frac{(z(\boldsymbol{k},t)+1)^2}{2}\cdot I(\boldsymbol{k}) \\
\mathrm{i}\left(\frac{z(\boldsymbol{k},t)-1}{z(\boldsymbol{k},t)+1}\right)^2 &= I(\boldsymbol{k}) \\
\frac{z(\boldsymbol{k},t)-1}{z(\boldsymbol{k},t)+1} &\equiv b(\boldsymbol{k},t) \\
z(\boldsymbol{k},t)-1 &= b(\boldsymbol{k},t)z(\boldsymbol{k},t)+b(\boldsymbol{k},t) \\
z(\boldsymbol{k},t)\cdot(1-b(\boldsymbol{k},t)) &= b(\boldsymbol{k},t)+1
\end{aligned}
$$

We can then obtain the stable equilibria from:

$$
\mathrm{i}b(\boldsymbol{k},t)^2 = I(\boldsymbol{k}) \qquad z(\boldsymbol{k},t)_{\pm} = \frac{1\pm b(\boldsymbol{k},t)}{1\mp b(\boldsymbol{k},t)} \tag{3}
$$

where the signs are chosen so that $|z(\boldsymbol{k},t)|\leq 1$.

3 A Newton-Raphson iteration for all fixpoints

3.1 Theory behind the method

The fixpoint iteration only gives us the stable equilibria of the system (1). We can obtain all equilibria and the Jacobian from a Newton-Raphson iteration. We define the equilibria $\boldsymbol{x}^*\in\mathbb{R}^n$ of a multivariate function $\boldsymbol{f}(\boldsymbol{x}):\mathbb{R}^n\to\mathbb{R}^n$ with $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{0}$. Expanding $\boldsymbol{f}$ as a Taylor series, we obtain:

$$
f_i(\boldsymbol{x}+\delta\boldsymbol{x}) = f_i(\boldsymbol{x}) + \sum_{j=1}^{n}\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\delta x_j + O\left(\delta\boldsymbol{x}^2\right) \approx f_i(\boldsymbol{x}) + \sum_{j=1}^{n}\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\delta x_j, \quad (i=1,\cdots,n) \tag{4}
$$

We can also write this in vector notation, by setting $\boldsymbol{J}(\boldsymbol{x}) = \nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{d}{d\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$

$$\boldsymbol{f}(\boldsymbol{x}+\delta\boldsymbol{x}) \approx \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_N(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_N \end{bmatrix} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{J}(\boldsymbol{x})\delta\boldsymbol{x} \tag{5}$$

By assuming $\boldsymbol{f}(\boldsymbol{x}+\delta\boldsymbol{x}) = 0$ we can find that $\delta\boldsymbol{x} = -\boldsymbol{J}^{-1}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$ so that $\boldsymbol{x}+\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{J}^{-1}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$. This expression converges to $\boldsymbol{x}^*$. When the equations are nonlinear, the equations converge to the real root as $\boldsymbol{x}_k = \boldsymbol{x}_k - \boldsymbol{J}^{-1}(\boldsymbol{x}_k)\boldsymbol{f}(\boldsymbol{x}_k)$.

For (1), we can compute the Jacobian for the diagonal and off-diagonal elements separately. But as $z(\boldsymbol{k},t)$ is a complex function, first we need to understand what the derivative of a complex function is.

3.2 Derivatives of complex functions

For $z = x + \mathrm{i}y \in \mathbb{C}$ and $x, y \in R$ the conjugate is defined as $\overline{z} = x - \mathrm{i}y$. That means that we can write the real and imaginary parts as:

$$x = \frac{z+\overline{z}}{2} \text{ and } y = -\mathrm{i}\frac{z-\overline{z}}{2}$$

Using the chain rule, we can write the partial derivative with respect to z in function of x and y as x and y are functionally independent and find the first Wirtinger operator:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial z}\frac{\partial}{\partial \overline{y}} \longrightarrow \frac{\partial x}{\partial z} = \frac{1}{2} \text{ and } \frac{\partial y}{\partial z} = -\frac{\mathrm{i}}{2} \longrightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)$$

We note the following properties:

$$\frac{\partial}{\partial z} z = 1 \qquad \frac{\partial}{\partial z}\overline{z} = \frac{1}{2}\left(1 - \mathrm{i}^2\right) = 0$$

Interesting for is the result of the following:

$$\begin{aligned} \overline{z}^2 &= (x - \mathrm{i}y)^2 = x^2 - y^2 - \mathrm{i}2xy \\ \frac{\partial}{\partial z}\overline{z}^2 &= \frac{1}{2}\cdot\left(2x - \mathrm{i}2y - \mathrm{i}\cdot(-2y - \mathrm{i}2x)\right) = x - \mathrm{i}y + \mathrm{i}y - \mathrm{i}x = 0 \end{aligned}$$

3.3 Derivatives of the complex mean field equations

We can now compute derivatives of the complex functions $z(\boldsymbol{k},t)$. When setting $z(\boldsymbol{k},t) = z_k$ the diagonal elements are found as

$$\begin{aligned} \frac{\partial}{\partial z_k}\left(\frac{\partial z_k}{\partial t}\right) &= -\mathrm{i}(z_k - 1) + (z_k+1)\cdot I(z_k) + \frac{(z_k+1)^2}{2}\cdot\frac{\partial I(z_k)}{\partial z_k} \\ \frac{\partial I(z_k)}{\partial z_k} &= \mathrm{i}\kappa\cdot\frac{\partial H_{2,k}}{\partial z_k} \\ \frac{\partial H_{2,k}}{\partial z_k} &= \frac{1}{\langle k\rangle}P_k a_{kk}\cdot\left(\frac{2z_k}{6} - \frac{4}{3}\cdot\frac{1}{2}\right) = \frac{1}{\langle k\rangle}P_k a_{kk}\cdot\frac{z_k-2}{3} \end{aligned} \tag{6}$$

When setting $z(\boldsymbol{k'},t) = z_{k'}$ the off-diagonal elements are found as

$$\begin{aligned} \frac{\partial}{\partial z_{k'}}\left(\frac{\partial z_k}{\partial t}\right) &= \frac{(z_k+1)^2}{2}\cdot\frac{\partial I(z_k)}{\partial z_{k'}} \\ \frac{\partial I(z_k)}{\partial z_{k'}} &= \mathrm{i}\kappa\cdot\frac{\partial H_{2,k}}{\partial z_{k'}} \\ \frac{\partial H_{2,k}}{\partial z_{k'}} &= \frac{1}{\langle k\rangle}P_{k'} a_{k'k}\cdot\frac{z_{k'}-2}{3} \end{aligned} \tag{7}$$

References

[1] S. Chandra, D. Hathcock, K. Crain, T. Antonsen, M. Girvan, and E. Ott, *Modeling the Network Dynamics of Pulse-Coupled Neurons*. *Chaos (Woodbury, N.Y.)* **27** (03, 2017) 10.

[2] C. Bick, M. Goodfellow, C. Laing, and E. Martens, *Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: a review*. *Journal of Mathematical Neuroscience* **10** no. 1, (Dec., 2020) .