HW Q6: Linear Transformations

nnd4zb

February 2026

Let $X = \{x_1, \dots, x_N\}$ be a sample. Define the sample mean

$$m(X) = \frac{1}{N} \sum_{i=1}^{N} x_i,$$

and for two samples $X = \{x_i\}$ and $Y = \{y_i\}$ define the (course) sample covariance

$$\operatorname{cov}(X, Y) = \frac{1}{N} \sum_{i=1}^{N} (x_i - m(X))(y_i - m(Y)).$$

Also recall the (course) sample variance of X:

$$s^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - m(X))^2.$$

We study linear transformations $Y = a + bX$, meaning $y_i = a + bx_i$ for each i.

1. Show that $m(a + bX) = a + b\, m(X)$.

Compute the mean of the transformed values:

$$m(a + bX) = \frac{1}{N} \sum_{i=1}^{N} (a + bx_i) = \frac{1}{N} \left(\sum_{i=1}^{N} a + b \sum_{i=1}^{N} x_i \right).$$

Since $\sum_{i=1}^{N} a = Na$,

$$m(a + bX) = \frac{1}{N}(Na) + \frac{b}{N} \sum_{i=1}^{N} x_i = a + b\, m(X).$$

2. Show that $\operatorname{cov}(X, X) = s^2$.

By definition,

$$\operatorname{cov}(X, X) = \frac{1}{N} \sum_{i=1}^{N} (x_i - m(X))(x_i - m(X)) = \frac{1}{N} \sum_{i=1}^{N} (x_i - m(X))^2.$$

But this is exactly the definition of s^2, so $\operatorname{cov}(X, X) = s^2$.

3. Show that $\mathrm{cov}(X, a + bY) = b\,\mathrm{cov}(X, Y)$.

First note the mean:

$$m(a + bY) = a + b\,m(Y)$$

(from part 1, applied to Y). Then

$$(a + by_i) - m(a + bY) = (a + by_i) - (a + bm(Y)) = b(y_i - m(Y)).$$

So,

$$\mathrm{cov}(X, a + bY) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))\big((a + by_i) - m(a + bY)\big)$$

$$= \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X)) \cdot b(y_i - m(Y)) = b\left(\frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))\right) = b\,\mathrm{cov}(X, Y).$$

4. Show that $\mathrm{cov}(a + bX, a + bY) = b^2\,\mathrm{cov}(X, Y)$.

Using part 3 with X replaced by $a + bX$:

$$\mathrm{cov}(a + bX, a + bY) = b\,\mathrm{cov}(a + bX, Y).$$

Now apply part 3 again to $\mathrm{cov}(a + bX, Y)$ (scaling the first argument):

$$\mathrm{cov}(a + bX, Y) = b\,\mathrm{cov}(X, Y).$$

Therefore,

$$\mathrm{cov}(a + bX, a + bY) = b \cdot \big(b\,\mathrm{cov}(X, Y)\big) = b^2\,\mathrm{cov}(X, Y).$$

As a special case, if $Y = X$, then

$$\mathrm{cov}(bX, bX) = b^2\,\mathrm{cov}(X, X) = b^2 s^2.$$

5. Suppose $b > 0$ and let the median of X be $\mathrm{med}(X)$.

Is it true that $\mathrm{med}(a + bX) = a + b\,\mathrm{med}(X)$? Is $\mathrm{IQR}(a + bX) = b\,\mathrm{IQR}(X)$?

When $b > 0$, the transformation $x \mapsto a + bx$ is strictly increasing, so it preserves order. That means the sample median transforms the same way:

$$\mathrm{med}(a + bX) = a + b\,\mathrm{med}(X).$$

For quartiles, the same order-preservation implies

$$Q_1(a + bX) = a + bQ_1(X), \quad Q_3(a + bX) = a + bQ_3(X).$$

Thus the interquartile range (IQR) satisfies

$$\mathrm{IQR}(a+bX) = Q_3(a+bX) - Q_1(a+bX) = (a+bQ_3(X)) - (a+bQ_1(X)) = b(Q_3(X) - Q_1(X)) = b\,\mathrm{IQR}(X).$$

6. Show by example that $m(X^2) \neq (m(X))^2$ and $m(\sqrt{X}) \neq \sqrt{m(X)}$ in general.

Example: let $X = \{0, 4\}$ (so $N = 2$). Then

$$m(X) = \frac{0+4}{2} = 2 \quad \Rightarrow \quad (m(X))^2 = 4.$$

But

$$X^2 = \{0^2, 4^2\} = \{0, 16\} \quad \Rightarrow \quad m(X^2) = \frac{0+16}{2} = 8.$$

So $m(X^2) = 8 \neq 4 = (m(X))^2$.

For the square root:

$$\sqrt{X} = \{\sqrt{0}, \sqrt{4}\} = \{0, 2\} \quad \Rightarrow \quad m(\sqrt{X}) = \frac{0+2}{2} = 1.$$

Meanwhile,

$$\sqrt{m(X)} = \sqrt{2}.$$

So $m(\sqrt{X}) = 1 \neq \sqrt{2} = \sqrt{m(X)}$.