$$S[\varphi(x,t)] = \int dt\,dx \;\; \mathcal{L}(\varphi, \partial_x\varphi, \partial_t\varphi)$$

$$\frac{\delta S}{\delta\varphi(x,t)} = \frac{\partial\mathcal{L}}{\partial\varphi} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial(\partial_x\varphi)} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}$$

$$\int \frac{\delta S}{\delta\varphi}\,\delta\varphi \;dx\,dt = \delta S = \int dt\,dx \quad \mathcal{L}_x\,\delta\varphi + \mathcal{L}_{\varphi_x}\,\delta\varphi_x + \mathcal{L}_{\varphi_t}\,\delta\varphi_t$$

$$\int dx \;\; \mathcal{L}_{\varphi_x}\underbrace{\delta\varphi_x}_{\partial_x\delta\varphi} = \Big[\mathcal{L}_{\varphi_x}\delta\varphi\Big]_{x_-}^{x_+} - \int dx\;\partial_x(\mathcal{L}_{\varphi_x})\,\delta\varphi$$

$$\int dt \;\; \mathcal{L}_{\varphi_t}\underbrace{\delta\varphi_t}_{\partial_t\delta\varphi} = \Big[\partial\varphi_t\,\delta\varphi\Big]_{t_-}^{t_+} - \int dt\;\partial_t(\mathcal{L}_{\varphi_t})\,\delta\varphi$$

$$= \int dt\,dx \;\; \overbrace{\mathcal{L}_x - \partial_t(\mathcal{L}_{\varphi_t}) - \partial_x(\mathcal{L}_{\varphi_x})}^{=\frac{\delta S}{\delta\varphi(x,t)}}$$

c.f. functional derivative. n.b. F maps a function to a number, for our kind of function this map involves an integral over the domain of the argument (g).

$$\frac{\delta F[g]}{\delta g(x)} = \lim_{\epsilon \to 0} \frac{F[g(y) + \epsilon\,\delta(y-x)] - F[g]}{\epsilon}.$$

a functional of g and a function of x

(even if we just pick g values at point we can do with: $\int dx\; g(x)\left[\delta(x-x_1) + \delta(x-x_2)\ \text{etc}\right]$.

has property that

$$\delta F = F[g(x) + \delta g(x)] - F[g(x)] = \frac{\delta F[g]}{\delta g(x)}\,\delta g(x) \quad \text{to first order in } \delta g$$

$$H[f] = \int_a^b dy \quad g(f(y))$$

$$\frac{\delta H[f]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \int_a^b dy. \quad \frac{g\left(f(y) + \epsilon\, \delta(y - x_0)\right) - g(f(y))}{\epsilon}.$$

$$= \lim_{\epsilon \to 0} \int_a^b dy \quad g'(f(y)) \frac{\epsilon\, \delta(y - x_0)}{\epsilon}$$

$$= \begin{cases} g'(f(x_0)) & a \leq x_0 \leq b \\ 0 & \text{otherwise.} \end{cases}$$

OR (OLD WAY)

$$\delta H[f] = \int_a^b dy \quad g(f(y) + \delta f(y)) - g(f(y)) = \int_a^b dy \quad g'(f(y))\, \delta f(y)$$

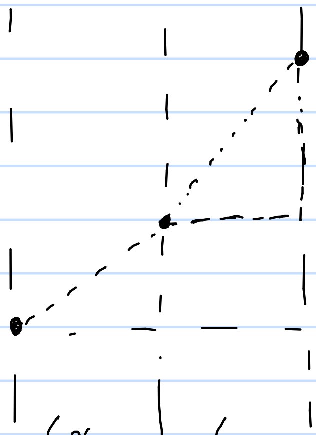
to first order in δf

$F = -kx$

$F = -\nabla V$

$V = \frac{k}{2} x^2$

$-\frac{\partial}{\partial x} V = -kx$



potential energy of string to right of point x_i

$$V_i = \frac{k}{2} \left\{ \left(\psi_{x_{i+1}} - \psi_{x_i} \right)^2 + \cancel{\delta x^2} - \cancel{\delta x^2} \right\}$$

$$= \frac{k}{2} \left(\frac{\Delta \psi}{\delta x} \delta x \right)^2$$

$$V_{tot} = \sum_i V_i \qquad \lim_{N \to \infty} \sum_i^n \frac{V_i}{\delta x} \delta x$$

$$V = \frac{k}{2} \int_0^l \left(\frac{\partial \psi}{\partial x} \right)^2 dx$$

$$\sim \sum_{i=0}^{n} \frac{k}{2} \left[\frac{\left(\psi_{x_{i+1}} - \psi_i \right)}{\delta x_i} \right]^2 \delta x_i$$

$$Z_0[J] = \exp\left(-\frac{1}{2}\overbrace{\int dx\,dy \quad J(x)\,\Delta(x-y)\,J(y)}^{J\cdot\Delta\cdot J}\right)$$

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\exp\left(-\frac{1}{2}\int dx\,dy \quad \big(J(x)+\epsilon\,\delta(x-z_1)\big)\,\Delta(x-y)\,\big(J(y)+\epsilon\,\delta(y-z_1)\big)\right) - Z_0[J]\right.$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\quad\frac{Z_0[J]}{\epsilon}\left\{\exp^{-\frac{1}{2}\int dx\,dy\ \epsilon\left[\delta(x-z_1)J(y)+\delta(y-z_1)J(x)\right]A(x-y)} - 1\right\}$$

$$= \quad Z_0[J]\left\{-\frac{1}{2}\int dy \ J(y)\,A(z_1-y) - \frac{1}{2}\int dx\,J(x)\,A(x-z_1)\right\}$$

now,

$$A[x] = A[-x]$$

$$= -\int dx\,J(x)\,A(z_1-x) \qquad Z_0[J]$$

$$T_{AB}(\theta_1) = \frac{l_{AO}}{V_1} + \frac{l_{OB}}{V_2}$$

$$\frac{dT_{AB}(\theta_1)}{d\theta_1} = 0$$

$$\frac{V_1}{V_2} = -\frac{dl_{AO}/d\theta_1}{dl_{OB}/d\theta_1} = -\frac{l_{AO}'}{l_{OB}'} = -\frac{A'}{B'}$$

$$l_{AO}' \sin\theta_1 + l_{AO}\cos\theta_1 + l_{OB}' \sin\theta_2 + l_{OB}\,\theta_2' \cos\theta_2 = 0$$

$$l_{AO}' \cos\theta_1 - l_{AO}\sin\theta_1 + l_{OB}' \cos\theta_2 - l_{OB}\,\theta_2' \sin\theta_2 = 0$$

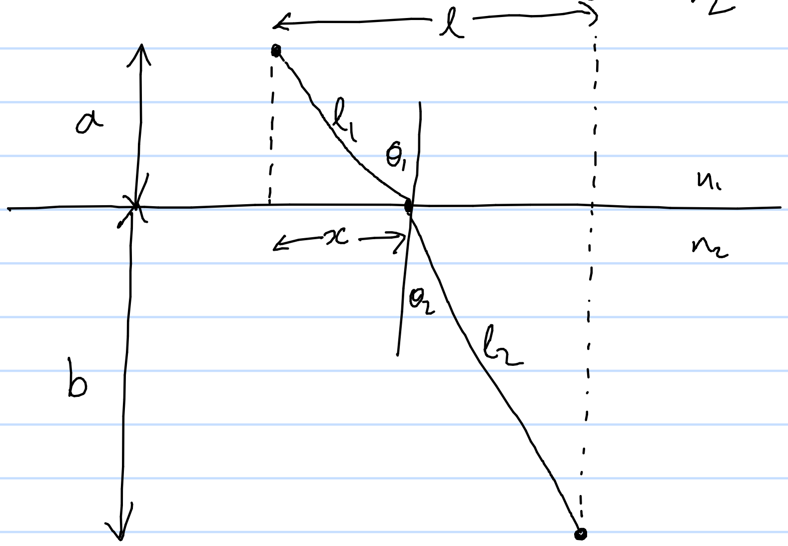
$$l_{AO}\sin\theta_1 + l_{OB}\sin\theta_2 = x \qquad AS_1 + BS_2 = X$$

$$l_{AO}\cos\theta_1 + l_{OB}\cos\theta_2 = y \qquad AC_1 + BC_2 = Y$$

$$A'S_1 + AC_1 + B'S_2 + B\theta' C_2$$

$$= A'C_1 - AS_1 + B'C_2 - B\theta' S_2$$

Snell's Law $\quad \dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$



$$T(x) = \frac{(a^2+x^2)^{1/2}}{n_1} + \frac{(b^2+(l-x)^2)^{1/2}}{n_2}$$

$$\text{''} + \frac{(b^2+x^2+l^2-2lx)^{1/2}}{n_2}$$

$T' = 0$

$$= \frac{l_1}{n_1} + \frac{l_2}{n_2}$$

$$\frac{n_1}{n_2} = -\frac{l_1'}{l_2'}$$

$\sin\theta_1 = x/l_1$

$\sin\theta_2 = (l-x)/l_2$

$$l_1' = \frac{1}{2l_1} 2x = \sin\theta_1$$

$$l_2' = \frac{1}{2l_2}(2x-2l) = -\sin\theta_2$$

$$\frac{n_1}{n_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

(a function of x).

$$H[f] = \int G(x,y) f(y)\, dy$$

$$I[f] = \int_{-1}^{1} f(x)\, dx$$

$$J[f] = \int \left(\frac{df}{dy}\right)^2 dy$$

$$\frac{\delta H[f]}{\delta f(z)} = \int dy \; G(x,y)\, \delta(y-z) = G(x,z)$$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\, \delta f(x_1)}$$

$$\frac{\delta I[f^3]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \int_{-1}^{1} \frac{[f + \epsilon\, \delta(x-x_0)]^3 - f^3}{\epsilon}$$

$$= \int 3\, \delta(x-x_0) f^2$$

$$= 3 f(x_0)^2$$

$$\frac{\delta^2 I[f^3]}{\delta f(x_0)\, \delta f(x_1)} = 3 \lim_{\epsilon \to 0} \left\{ \frac{\left[f(x_0) + \epsilon\, \delta(x_0 - x_1)\right]^2 - f(x_0)^2}{\epsilon} \right\}$$

$$= 3 \cdot 2\, f(x_0)\, \delta(x_0 - x_1)$$

$$= 6\, f(x_0)\, \delta(x_0 - x_1)$$

$$J[f] = \int \left(\frac{df}{dy}\right)^2 dy$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int \left[\frac{d}{dy}\left(f + \epsilon \delta(y-x)\right)\right]^2 - \left(\frac{df}{dy}\right)^2 dy.$$

$$= \quad \left(f' + \epsilon \delta'(y-x)\right)^2 - f'^2$$

$$= \int 2f' \delta'(y-x) \, dy$$

$$= \left[2f' \delta(y-x)\right] - \int 2f'' \delta(y-x) \, dy.$$

iff x in interval

$$= -2f''(x)$$

1.3 $G[f] = \int g(y,f)\ dy$

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\ g(y, f + \epsilon\delta(y-x)) - g(y,f)$$

$$= \int dy\ \delta(y-x) \frac{\partial g}{\partial f}$$

$$= \frac{\partial g(x,f)}{\partial f}$$

$$H[f] = \int g(y, f, f')\ dy$$

$$\frac{\delta H[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int g(y, f + \epsilon\delta(y-x), [f + \epsilon\delta(y-x)]') - g(y,f,f')$$

$$= \int dy\ \delta(y-x) \frac{\partial g}{\partial f} + \delta'(y-x) \frac{\partial g}{\partial f'}$$

$$= \frac{\partial g}{\partial f} + \left[\delta(y-x) \frac{\partial g}{\partial f'} \right] - \int \delta(y-x) \frac{d}{dy} \frac{\partial g}{\partial f'}$$

$$= \frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'}$$

$$J[f] = \int g(y, f, f', f'')\, dy$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy \quad \overset{\text{as prior}}{\cdots\cdots} + \epsilon\, \delta''(y-x) \frac{\partial g}{\partial f''} \qquad - g(y, f, f', f'')$$

$$= \cdots + \int dy \quad \delta''(y-x) \frac{\partial g}{\partial f''}$$

$$\left[\delta'(y-x) \frac{\partial g}{\partial f''} \right] - \int dy \quad \delta'(y-x) \left(\frac{\partial g}{\partial f''} \right)'$$

$\delta'(y-x) = 0$ on boundary.

$$= -\left[\delta(y-x) \left(\frac{\partial g}{\partial f''} \right)' \right] + \int dy \quad \delta(y-x) \left(\frac{\partial g}{\partial f''} \right)''$$

$$\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{d}{dx} \frac{\partial g}{\partial f'} + \frac{d^2}{dx^2} \frac{\partial g}{\partial f''}$$

$$x_j = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k \, e^{ikja} \quad \ldots$$ show this uniquely defines $\tilde{x}_k$

can do this provided e^{ikja} are linearly independent.

inner product.

i.e. $(A_j, B_j) = \frac{1}{N} \sum_{j=1}^{N} A^* B$

$$\left(e^{ikja}, e^{ilja}\right) = \frac{1}{N} \sum_{j=1}^{N} e^{i(l-k)ja} = \delta_{l,k}$$

n.b. $S = \sum_{j=1}^{N} \omega^j$

$\omega = e^{i(l-k)a}$

$\omega^{N+1} = e^{i(l-k)(N+1)a} = \omega$ by periodicity.

$$\omega S - S = \omega^{N+1} - \omega$$

$$S = \frac{\omega^{N+1} - \omega}{\omega - 1}$$

This means that we can use our inner product to recover $\{\tilde{x}_k\}$ from $\{x_k\}$

$$\left(\sqrt{N} \, e^{ilja}, x_j\right) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} x_j \, e^{-ilja}$$

$$= \frac{1}{N} \sum_{k,j=1}^{N} \tilde{x}_k \, e^{i(k-l)ja}$$

$$= \sum_k \tilde{x}_k \, \delta_{k,l} = \tilde{x}_l$$

Example 3.6.

$$\langle p'q'|qp\rangle = \langle 0|\hat{a}_{p'}\hat{a}_{q'}\hat{a}^\dagger_q\hat{a}^\dagger_p|0\rangle$$

recall $\hat{a}_p\hat{a}^\dagger_{p'} = \delta(p-p') \pm \hat{a}^\dagger_{p'}\hat{a}_p$

$=$

fermions	bosons
$\{\hat{c}^\dagger_i, \hat{c}^\dagger_j\} = 0 \Rightarrow \hat{c}^\dagger_i\hat{c}^\dagger_i = 0$	$[\hat{a}^\dagger_i, \hat{a}^\dagger_j] = 0$
$\{\hat{c}_i, \hat{c}_j\} = 0$	$[\hat{a}_i, \hat{a}_j] = 0$
$\{\hat{c}_i, \hat{c}^\dagger_j\} = \delta_{ij}$	$[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij}$

Bosons.

$$\langle 0|\hat{a}_{p'}\hat{a}_{q'}\hat{a}^\dagger_q\hat{a}^\dagger_p|0\rangle$$

$$\langle 0|\hat{a}_{p'}\left(\delta(q'-q) \pm \hat{a}^\dagger_q\hat{a}_{q'}\right)\hat{a}^\dagger_p|0\rangle$$

$$= \langle 0|\left[\delta(q'-q)\left[\delta(p'-p) \pm \hat{a}^\dagger_p\hat{a}'_p\right]\right]|0\rangle \text{ ①} \quad = \delta(q'-q)\,\delta(p'-p)$$

$$\pm \langle 0|\hat{a}_{p'}\hat{a}^\dagger_q\left(\delta(q'-p) \pm \hat{a}^\dagger_p\hat{a}_{q'}\right)|0\rangle$$

②

$$\pm \delta(q'-p)\ \langle 0|\left(\delta(p'-q) \pm \hat{a}^\dagger_q\hat{a}_{p'}\right)|0\rangle \quad \pm \delta(q'-p)\,\delta(p'-q)$$

Q.E.D

Example 3.6.

check $\langle q|p\rangle = \delta(q-p)$ is correct by

given: $\langle x|p\rangle = \phi_p(x)$

and $|x\rangle = \int dq\, \underbrace{|q\rangle\langle q|}^{\text{resolution of identity.}} x\rangle = \int dq\, \phi_q^*(x)\,|q\rangle$

we calculate
$$\langle p|x\rangle = \int dq\, \phi_q^*(x)\,\langle p|q\rangle$$
$$= \int dq\, \phi_q^*(x)\,\delta(p-q)$$

which is what we are checking.

$$= \phi_p^*(x) \quad \text{as required.}$$

Similarly we want to check:

$$\langle p'q'|pq\rangle = \delta(p'-p)\delta(q'-q) \pm \delta(p'-q)\delta(q'-p)$$

$$|xy\rangle = \frac{1}{\sqrt{2!}}\int dp'\,dq'\;\phi_{p'}^*(x)\,\phi_{q'}^*(y)\,|p'q'\rangle$$

$$\langle xy|pq\rangle = \frac{1}{\sqrt{2!}}\int dp'\,dq'\;\phi_{p'}(x)\,\phi_{q'}(y)\,\langle p'q'|pq\rangle$$

$$= \frac{1}{\sqrt{2}}\underbrace{\left(\phi_p(x)\,\phi_q(y) \pm \phi_q(x)\,\phi_p(y)\right)}_{\text{which is the known result.}}$$

Periodic boundary conditions

$$p_m = \frac{2\pi m}{L}$$

$$\delta_N(x-y) = \frac{1}{L}\sum_{m=-N}^{N} e^{-ip_m(x-y)}$$

def^n

Strictly, def^n of $\delta(x-y)$ is

$$\lim_{N\to\infty}\int dx\, \delta_N(x-y)\, f(x) = f(y)$$

Scaling.

$$\int dx\, \delta(\alpha(x-y)) = \frac{1}{\alpha}\int d(\alpha x)\, \delta(\alpha x-\alpha y)\, f\left(\frac{\alpha x}{\alpha}\right)$$

$$= \frac{1}{\alpha} f\left(\frac{\alpha y}{\alpha}\right) = \frac{1}{\alpha} f(y) \Rightarrow$$

* $\delta(\alpha x) = \frac{1}{\alpha}\delta(x)$

$$\theta = \frac{2\pi x}{L}$$

*

$$\delta_N(\theta) = \frac{L}{2\pi}\delta_N(x) = \frac{1}{2\pi}\sum_{m=-N}^{N} e^{-im\theta}$$

$$S_N = e^{+iN\theta}\sum_{m=0}^{2N} e^{-im\theta}$$

$$= e^{iN\theta}\,\frac{1-e^{-i(2N+1)\theta}}{1-e^{-i\theta}}$$

$$= \frac{e^{i(N+\frac{1}{2})\theta}}{e^{i\theta/2}}\cdot\frac{1-e^{-i(2N+1)\theta}}{1-e^{-i\theta}}$$

$$= \frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\theta/2)}$$

The "Dirichlet Kernel"

recall

$$S = \sum_{n=0}^{N}\omega^n$$

$$S - \omega S = 1-\omega^{N+1}$$

$$S = \frac{1-\omega^{N+1}}{1-\omega}$$

So

$$\delta_N(\theta) = \frac{1}{2\pi}\,\frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\theta/2)}$$

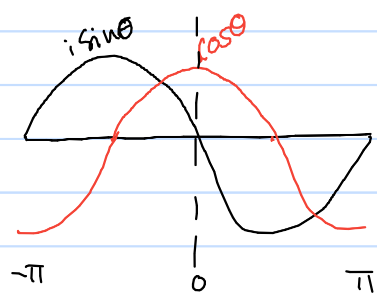
$$\tilde{\psi}(p) = \frac{1}{\sqrt{L}}\int_0^L dx\, e^{-ipx}\psi(x)$$

$$\psi(y) = \frac{1}{\sqrt{L}}\sum_p e^{ipy}\tilde{\psi}(p)$$

$$\Rightarrow \psi(y) = \int_0^L dx\, \underbrace{\frac{1}{L}\sum_p e^{-ip(x-y)}}_{=\delta(x-y)}\,\psi(x)$$

We require

$$1 = \int_{-\pi}^{\pi} d\theta\, \frac{1}{2\pi}\sum_{m=-N}^{N} e^{-im\theta}$$



$$\frac{1}{2\pi}\sum_{-N}^{N}\underbrace{\int_{-\pi}^{\pi} d\theta\,(\cos m\theta - i\sin m\theta)} =$$

$$\left[m(\sin m\theta + i\cos m\theta)\right]_{-\pi}^{\pi} = 0$$

$\sin(m\pi) = 0$

unless $m = 0$

$\cos(m\pi) = \cos(-m\pi)$

which gives 1.

p52 Writing down Lagrangians.

We're going to do the relativistic theory of a charged particle, so we need something Lorentz invariant for the action. A good choice is

$$S = \varepsilon \int_{\tau_1}^{\tau_2} d\tau \qquad \text{for some constant } \varepsilon$$

This integral is the proper time interval, the time interval measured along the path of the integral (which is the path the particle takes through space time).

now $\quad c\bar{t} = \gamma(ct - \beta x)$ is the Lorentz transform

$\quad \bar{x} = \gamma(x - \beta ct)$ t, x & v of the particle are measured in the lab frame

$\gamma^{-2} = 1 - \beta^2 \qquad \beta = \frac{v}{c}$

$\bar{t} = \tau$
$\bar{x}$ are in the particle frame

So if I in the lab frame ascribe coords (t, x) to an event, an observer in the particle frame will ascribe $(\bar{t}, \bar{x})$.
for $\beta \ll 1$ we get $\quad \bar{t} = t \quad \bar{x} = x - vt$

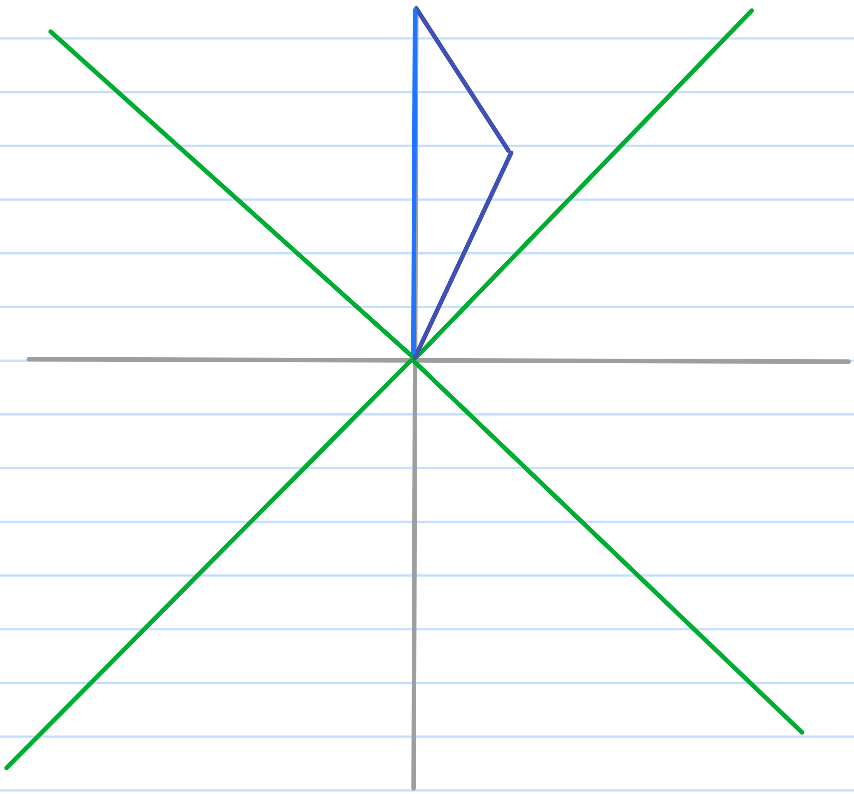
$$\begin{pmatrix} c\bar{t} \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

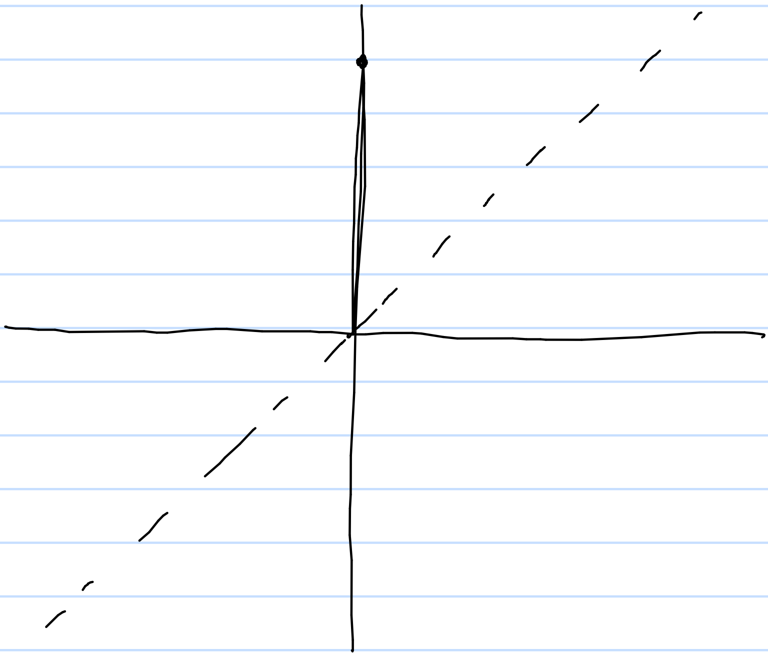
particle hasn't moved in own frame.   particle has moved in lab frame.

Minkowski Metric $\quad c^2 d\tau^2 = c^2 dt^2 - dx^2$

from Metric $\quad c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \right)$

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}}$$
$$= \frac{dt}{\gamma}$$





10.2 Noether's Theorem

$$\varphi \to \varphi + \delta\varphi$$

Change in Lagrangian density is.

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta(\partial_\mu \varphi)$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\mu \delta\varphi$$

$$= \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right]}\delta\varphi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi\right]$$

n.b. If $\bullet = 0$ then φ satisfies the equations of motion and

$$\delta \mathcal{L} = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi\right] = \partial_\mu\left[\Pi^\mu \delta\varphi\right] \quad (1)$$

now suppose we have a continuous symmetry. transformation. which we know does not change parameterised by λ

$$\delta\varphi = D\varphi\, \delta\lambda$$

the action $0 = \delta S = \int d^4x\, \delta\mathcal{L}$

then we know $\delta\mathcal{L} = \delta\lambda\, \partial_\mu\left[K^\mu\right]$ (2) for some $K^\mu(x)$

(since only a divergence annihilate the integral).

Now lets apply the symmetry transform to a φ which is a solution of the equations of motion.

by (1) $\quad \delta \mathcal{L} = \delta\lambda\, \partial_\mu\left[\Pi^\mu D\varphi\right]$

by (2) $\quad \delta \mathcal{L} = \delta\lambda\, \partial_\mu\left[K^\mu\right]$

which satisfies. $\int d^3A\, \hat{n}_\mu K^\mu = 0$

so $\quad \partial_\mu\left[\Pi^\mu D\varphi - K^\mu\right] \stackrel{\text{def}}{=} \partial_\mu\left[J^\mu_N\right] = 0$

and $J^\mu_N(x)$ is a locally conserved Noether current.

$$\int d^3x \, [\pi(x)\dot{\phi}(x) - \mathcal{L}(x)] = \int d^3x \, \mathcal{H} = E$$

$\pi(x) = \dfrac{\delta \mathcal{L}(\phi, \dot{\phi}, \partial_x \phi)}{\delta \dot{\phi}(x)}$ the conjugate momentum.

in analogy to

$p_i = \dfrac{\partial L(q, \dot{q}, \partial_x q)}{\partial \dot{q}_i}$ $\qquad \dot{p}_i = -\partial$

very!

naively, one might expect the conserved momentum to

p^k be $\sim \int d^3x \, \pi(x)$ $\qquad$ but it is $\int d^3x \, \pi(x) \, \partial^k \phi(x)$

↑ doesn't even have a direction!

It's Noether's theorm which helps us identify it as such.

$p_i \sim \dfrac{tL}{q}$ $\qquad \pi \sim \dfrac{tL}{V\phi}$ so need to multiply by $\dfrac{V\phi}{x}$

q "=" x

to get a momentum.

$$0 = \delta S = \int dt\, \delta L$$

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta(\dot{q}_i)$$

integral over space is Einstein sum over indexes wrt.

$$= \left[\frac{\partial L}{\partial q_i} - \partial_t \frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i + \partial_t\left[\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right]$$

$$p_i \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{q}_i} \qquad \dot{p}_i = \frac{\partial L}{\partial q_i}$$

so p_i is conserved if L is indep of q_i

$$L = T - V = \frac{1}{2} m \dot{q}_i^2 \qquad p_i = m\dot{q}_i$$

Example 10.3

$$X^{\lambda\mu\nu} \overset{\text{def}}{=} - X^{\mu\lambda\nu}$$

$$\partial_\mu \partial_\lambda X^{\lambda\mu\nu} = \partial_\lambda \partial_\mu X^{\lambda\mu\nu} = \underset{\text{relabel}}{\partial_\mu \partial_\lambda X^{\mu\lambda\nu}} = -\partial_\mu \partial_\lambda X^{\lambda\mu\nu}$$

$$= 0.$$

i.e. symmetric operation on antisymmetric object gives 0.

$$D \mathcal{L}\, \delta\lambda = \delta x^\rho \partial_\rho \mathcal{L}$$

$$\delta x^\mu = \omega^{\mu\nu} x_\nu \, \delta\lambda$$

$$D\mathcal{L} = \omega^{\mu\nu} x_\nu \partial_\mu \mathcal{L}$$

$$= \omega_{\mu\nu} x^\nu \partial^\mu \mathcal{L}$$

now

$$\delta[f(x)]$$

$$\int dx \ \delta[\alpha(x-y)] f(x)$$

$$= \frac{1}{\alpha} \int d(\alpha x) \ \delta(\alpha x - \alpha y) \ f\left(\frac{\alpha x}{\alpha}\right)$$

$$= \frac{1}{\alpha} \int dx \ \delta(x - \alpha y) \ f\left(\frac{x}{\alpha}\right)$$

$$= \frac{1}{\alpha} f\left(\frac{\alpha y}{\alpha}\right)$$

$$= \frac{1}{\alpha} f(y) \qquad = \qquad \delta(\alpha(x-y)) = \frac{1}{\alpha} \delta(x-y)$$

$$\int dx \ \delta(g(x)) \ f(x)$$

$$= \sum_{x_0 : g(x_0)=0} \int_{x_0-\epsilon}^{x_0+\epsilon} dx \ \delta((x-x_0) g'(x_0)) \ f(x)$$

$$= \sum \int_{x_0-\epsilon}^{x_0+\epsilon} dx \ \frac{\delta(x) f(x)}{g'(x_0)}$$

$$S = \sum_{n=1}^{N} e^{-n\pi a/x} = e^{-\pi a/x} \sum_{n=0}^{N-1} e^{-n\pi a/x}$$

$$= e^{-\pi a/x} \frac{1 - e^{-N\pi a/x}}{1 - e^{-\pi a/x}}$$

$$= \frac{1}{e^{\pi a/x} - 1}$$

$$\frac{1}{2} \frac{\partial}{\partial a} \left(e^{\pi a/x} - 1 \right)^{-1}$$

$$= \frac{1}{2} \left(\frac{-\pi/x \; e^{\pi a/x}}{(e^{\pi a/x} - 1)^2} \right)$$

$$-\frac{\pi}{2x} \frac{e^{\pi a/x}}{(1 - e^{\pi a/x})^2}$$

$\xi = -1$ fermions.

$\xi = 1$ bosons

$$[\hat{a}_i^\dagger, a_j^\dagger]_\xi = 0$$

$$[\hat{a}_i, \hat{a}_j]_\xi = 0$$

$$[\hat{a}_i, \hat{a}_j^\dagger]_\xi = \delta_{i,j}$$

$$\langle 0|N[c^\dagger_{p_1-q}\, c^\dagger_{p_2+q}\, \hat{c}_{p_2}\, \hat{c}_{p_1}]|0\rangle$$

Given the VEV of fermionic operators, we can apply Wick's theorem as follows

$$\begin{aligned}
\langle 0|\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}|0\rangle &= \langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}\right]|0\rangle \\
&= \langle 0|N\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1} + \hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1} + \hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}\right]|0\rangle
\end{aligned}$$

$$\boxed{\langle 0|\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}|0\rangle = -\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}_{\mathbf{p}_2}\right]|0\rangle\,\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_1}\right]|0\rangle + \langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}_{\mathbf{p}_1}\right]|0\rangle\,\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\right]|0\rangle}$$

Consider $\langle 0|N[c^\dagger_{p_1-q}\, c^\dagger_{p_2+q}\, \hat{c}_{p_2}\, \hat{c}_{p_1}]|0\rangle$

Notes on Wick's Theorem

Tim Evans
(23rd November 2018)

From Wick contraction notes

eq. (14) is wrong.

$$\Delta_{ij} = \phi_i \phi_j$$

$$\Theta(t_i - t_j)\,[\phi_i^+, \phi_j^-] + \Theta(t_j - t_i)\left\{[\phi_j^+, \phi_i^+] + [\phi_j^+, \phi_i^-] + [\phi_j^-, \phi_i^-]\right\} \qquad (13)$$

$$\Delta_{ij} - \Delta_{ji} =$$

$$\Theta(t_i - t_j)\,\cancel{[\phi_i^+, \phi_j^-]} + \Theta(t_j - t_i)\left\{[\phi_j^+, \phi_i^+] + \cancel{[\phi_j^+, \phi_i^-]} + [\phi_j^-, \phi_i^-]\right\}$$

$$- \Theta(t_j - t_i)\,\cancel{[\phi_j^+, \phi_i^-]} - \Theta(t_i - t_j)\left\{[\phi_i^+, \phi_j^+] + \cancel{[\phi_i^+, \phi_j^-]} + [\phi_i^-, \phi_j^-]\right\}$$

$$= \left\{\Theta(t_j - t_i) + \Theta(t_i - t_j)\right\}\left\{[\phi_j^+, \phi_i^+] + [\phi_j^-, \phi_j^-]\right\}$$

$$= [\phi_j^+, \phi_i^+] + [\phi_j^-, \phi_j^-]$$

n.b. $\Theta(0) = \frac{1}{2}$ under half maximum convention.

corrected *

$$\phi_i \phi_j - \phi_j \phi_i = [\phi_j^+, \phi_i^+] + [\phi_j^-, \phi_j^-] \qquad (14)$$

So contraction is symetic if

was OK.

$$[\phi_j^+, \phi_i^+] + [\phi_j^-, \phi_j^-] = 0 \qquad (16)$$

n.b. this is for an arbitrary split, if we need creation & annihilation ops then. (for equal time operators)

$$[\hat{a}_p, \hat{a}_q]_\xi = 0$$

$$[\hat{a}_p^+, \hat{a}_q^+]_\xi = 0$$

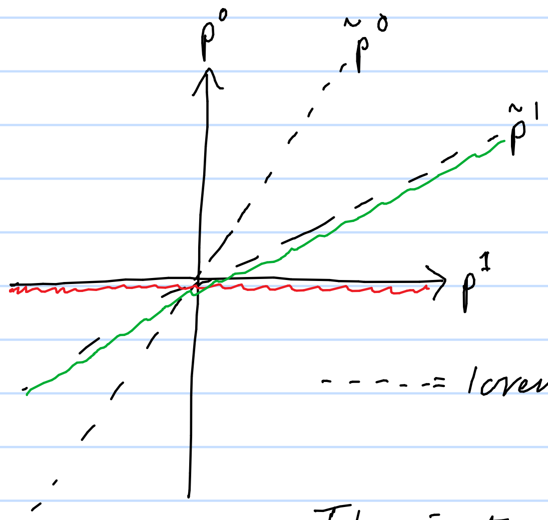
$$[\hat{a}_p, \hat{a}_q^+]_\xi = \delta_{p,q}$$

So contraction symetric for bosons.

can I use $[\,,\,]_\xi$ in argument for fermions?

Show that $\int d^3p$ is not Lorentz invariant

but $\int d^4p$ is.



$$\int_{-\infty}^{\infty} dp^1 \, f(p^\mu p_\mu) \neq \int_{-\infty}^{\infty} d\tilde{p}^1 \, f(\tilde{p}^\mu \tilde{p}_\mu)$$

$$p^2 \qquad = \qquad p^2$$

- - - - = lorentz boosted frame.

The integration of Lorentz scalars is different because the domain of integration is different.

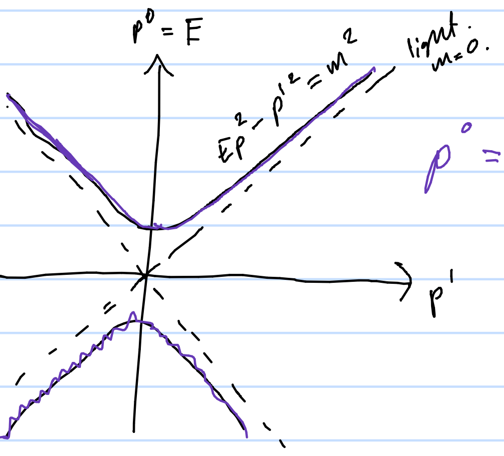
$\int d^4p$ is the whole 4volume, unchanged by boost/rotation/translation (Poincare group transformation)

$$\int d^4p \, \delta(p^2 - m^2)\, \Theta(p_0)$$

$p^2 = p^{0\,2} - \vec{p}^{\,2} = m^2 \overset{\text{def of } E_p}{=} E_p^2 - \vec{p}^{\,2}$

mass shell condition.

$E_p = \pm\sqrt{\vec{p}^{\,2} + m^2}$



$p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2} = \pm E_p$

note that this surface does not change with Boost.

or The δ function is of lorentz scalars. p^2 and m^2

$$\delta(p^2 - m^2) = \delta(p_0^2 - E_p^2) = \frac{\delta(p_0 - E_p) + \delta(p_0 + E_p)}{2E_p}$$

Scaling of a δ function

consider $\int dx \, \delta(\alpha x) f(x) = \frac{1}{\alpha}\int d(\alpha x)\, \delta(\alpha x)\, f\left(\frac{\alpha x}{\alpha}\right)$

$= \frac{1}{\alpha} f\left(\frac{0}{\alpha}\right) = \frac{1}{\alpha} f(0)$.

and since $\delta(x) = \delta(-x)$ we have $\frac{1}{|\alpha|} f(0)$

So, if you "go through" a δ function "more quickly" you get "less impulse" if you go through more slowly you get more ...

δ function of a function

consider $\int dx \quad \delta(g(x))\, f(x)$

$$= \sum_{x_i : g(x_i) = 0} \int_{x_i - \epsilon}^{x_i + \epsilon} dx \; \delta\big(\underbrace{\cancel{g(x_i)}}_{=0} + (x - x_i)\, g'(x_i)\big)\, f(x)$$

$$\int_{x_i - \epsilon}^{x_i + \epsilon} dx \quad \frac{\delta(x - x_i)}{|g'(x_i)|}\, f(x)$$

So $\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$

roots

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \qquad (13.17)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is just the Lagrangian for the electro-

$$\Pi^{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial^\nu A^\mu - \partial^\mu A^\nu.$$

1 =

$$-\frac{1}{4} g^{\mu\sigma} g^{\nu\tau} (\underset{1}{\partial_\mu A_\nu} - \underset{2}{\partial_\nu A_\mu})(\underset{3}{\partial_\sigma A_\tau} - \underset{4}{\partial_\tau A_\sigma})$$

n.b.

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial_\nu A_\mu H^{\nu\mu}\right) = H^{\mu\nu}$$

n.b. in the sum indexes are dummy.

$$\frac{\partial}{\partial(A_{\mu\nu})}\left(A_{\alpha\beta}B^{\alpha\beta}\right) \equiv \frac{\partial}{\partial(A_{\alpha\beta})}\left(A_{\alpha\beta}B^{\alpha\beta}\right)$$

then $\alpha \to \mu$, $\beta \to \nu$

from 1 $-\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$

2 $+\frac{1}{4} g^{\mu\sigma} g^{\nu\tau}(\partial_\sigma A_\tau - \partial_\tau A_\sigma) = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$

$$\frac{\partial \; 3}{\partial(\partial_\sigma A_\tau)} = -\frac{1}{4}\left(\partial^\sigma A^\tau - \partial^\tau A^\sigma\right) \quad \begin{matrix}\sigma \to \mu \\ \tau \to \nu\end{matrix} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\frac{\partial \; 4}{\partial(\partial_\tau A_\sigma)} = \frac{1}{4}\left(\partial^\sigma A^\tau - \partial^\tau A^\sigma\right) \quad \begin{matrix}\tau \to \mu \\ \sigma \to \nu\end{matrix} = -\frac{1}{4}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

Q: If the mode expansion is just a F.T. why does it contain annihilators as well as creators?

It is because:

e.g.

$$a(k) = \sqrt{\frac{\omega}{2}} \left(\phi(k) + \frac{i}{\omega} \pi(k) \right)$$

so $\quad a(k) + a(k)^{\dagger} = \sqrt{2\omega}\, \phi(k)$

I think the best way to think of it is:

Motivated by analogy with S.H.M we define a_k, $a_k^{\dagger}$ with:

$$\phi(x) = \int \frac{d\vec{p}^3}{(2\pi)^{3/2}} \frac{1}{(2E_p)^{1/2}} \left\{ a(\vec{p})\, e^{i p^{\mu} x_{\mu}} + a(\vec{p})^{\dagger} e^{-i p^{\mu} x_{\mu}} \right\}$$

where integral is $\left\{\begin{array}{l}\text{over space like } \vec{p} \\ \text{on surface}\end{array}\right.$ $p_{\mu} p^{\mu} = m^2$ ("mass ||")

$E^2 = p^{0\,2} - \vec{p}^{\,2} = m^2$ i.e. $p^0 = +\sqrt{\vec{p}^{\,2} + m^2} = E_p$

When we put this form of $\phi(x)$ in the Lagrangian density

and calculate $\pi^{\mu}(x) = \dfrac{\partial \mathcal{L}}{\partial(\partial_{\mu}\ (x))}$ then

get a Hamiltonian density from

$$\mathcal{H} = \pi^{\mu}(x)\, \partial_{\mu} \phi(x) - \mathcal{L}(x)$$

we find that $a(\vec{p})$, $a(\vec{p})^{\dagger}$ behave as ladder operators on the eigenvectors of this Hamiltonian.

Example 14.8

Step I: The Lagrangian is still

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu), \tag{14.26}$$

although we'll need to remember that our choice of gauge dictates $A^0 = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. Actually, this will be implemented at step III.

Step II: We find that the $\Pi^{\mu\nu}$ tensor has components

$$\Pi^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -(\partial^\mu A^\nu - \partial^\nu A^\mu), \tag{14.27}$$

just as in the case of massive vector field theory. Again, the momentum density of the zeroth component $\Pi^{00} = 0$, but we don't care since we have decided to eliminate $A^0 = 0$. The momentum component conjugate to the ith component of the field is $\Pi^{0i} = E^i$, that is, the electric vector field $\boldsymbol{E}(x)$. The Hamiltonian[9] is then

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2). \tag{14.28}$$

Step III: Next we need commutation relations. Our first guess might be that $\left[\hat{A}^\mu(\boldsymbol{x}), \hat{\Pi}^{0\nu}(\boldsymbol{y})\right] = \mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})g^{\mu\nu}$, similar to the massive vector field. This won't work here though. Rewriting yields $\left[\hat{A}^i(\boldsymbol{x}), \hat{E}^j(\boldsymbol{y})\right] = -\mathrm{i}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})g^{ij}$, which looks fine until you take the divergence (with respect to $\boldsymbol{x}$) of this equation and get

[9] You should find

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \boldsymbol{E}\cdot\boldsymbol{\nabla}A^0.$$

The last term is removed using the same method as employed in the previous chapter, noting that $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$ and that all fields should vanish at infinity.

$$F^{\mu\nu} = \begin{array}{c} \begin{array}{cccc} \nu=0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \end{array} \begin{array}{c} \mu=0 \\ 1 \\ 2 \\ 3 \end{array} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

* reminder sign as this is the displacement current enters with same sign as current.

$$A = \begin{pmatrix} \phi \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \qquad \vec{E} = -\nabla\phi$$

$$\nabla\cdot E = \rho$$
$$\nabla\cdot B = 0$$
$$\nabla\times B = J \overset{*}{+} \partial_t E$$
$$\nabla\times E = -\partial_t B$$

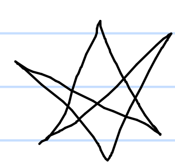
$$\vec{B} = \nabla\times\vec{A}$$

$$E_i = cF_{0i} \qquad B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}$$

$$F_{\mu\nu} = g_{\mu\lambda}g_{\nu\tau}F^{\lambda\tau} \qquad \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}F^{T\nu\mu}$$

$$= \mathrm{Tr}\begin{pmatrix} E^2 & & & \\ & E_x^2 + B_z^2 + B_y^2 & \cdot & \\ & & E_y^2 + B_z^2 + B_x^2 & \cdot \\ \cdot & \cdot & & E_z^2 + B_y^2 + B_x^2 \end{pmatrix} \qquad c \to 1$$



$$= 2(E^2 + B^2)$$