Proof Convergence Theorem Part 2

First, use the Chapman-Kolmogorov equation:

$$P^t(x,y) = \sum_{k\in\Omega} P(x,k)\, P^{(t-1)}(k,y)$$

Fix k_* and decompose with $\delta > 0$: $P(x,k_*) = (P(x,k_*) - \delta\,) + \delta$

$$= \delta\, P^{(t-1)}(k_*,y) \;+ \sum_{k\setminus\{k_*\}} P(x,k)P^{(t-1)}(k,y) + (P(x,k_*) - \delta\,)\, P^{(t-1)}(k_*,y)$$

$$\geq \delta\, P^{(t-1)}(k_*,y) \;+\; (1-\delta\,)\; \min_{l\in\Omega}\, P^{(t-1)}(l,y)$$

Let $m_t = \min_{x\in\Omega} P^t(x,y)$ and $M_t = \max_{x\in\Omega} P^t(x,y)$. Then:

$$m_t \geq \delta\, P^{(t-1)}(k_*,y) + (1-\delta)\, m_{t-1} \quad \text{and}$$

$$M_t \leq \delta\, P^{(t-1)}(k_*,y) + (1-\delta)\, M_{t-1}$$

Now, conclude:

$$\begin{aligned}
|P^t(x,y) - \pi(y)| &= |P^t(x,y) - \sum_k \pi(y)P^t(k,y)| \qquad\qquad \textit{here we use (1)} \\
&= |\sum_k \pi(y)(P^t(x,y) - P^t(k,y))| \\
&\leq |\sum_k \pi(y)(M_t - m_t)| \\
&= (M_t - m_t) =: \Delta_t
\end{aligned}$$

From above we know:

$$\begin{aligned}
\Delta_t &\leq (1-\delta)\,\Delta_{(t-1)} \\
&\leq \ldots \\
&\leq (1-\delta)^t\,\Delta_0 \\
&\leq (1-\delta)^t \qquad\qquad \textit{here we need aperiodicity}
\end{aligned}$$