

# Goldbach's Conjecture via Complements and Factor-Size Filtering

Simon Berry Byrne

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## Abstract

We present a proof of Goldbach's Conjecture using a novel method based on complements and factor-size filtering. We prove that every even integer  $n \geq 4$  can be expressed as the sum of two prime numbers by demonstrating that the complement set must always intersect the set of primes.

## 1 Introduction

Goldbach's Conjecture, first proposed in a 1742 letter from Christian Goldbach to Leonhard Euler, asserts that every even integer  $n \geq 4$  can be expressed as the sum of two prime numbers. A related problem, known as the ternary Goldbach Conjecture, states that every odd integer  $n \geq 7$  can be expressed as the sum of three primes. The ternary version has been fully resolved, notably by Helfgott in 2013 [1], but the binary Goldbach Conjecture remains unproven despite extensive mathematical effort.

Significant analytic progress on the binary problem includes Chen's Theorem, which shows that every sufficiently large even integer can be written as the sum of a prime and a number that is either a prime or a semiprime [2]. Extensive computational verification has further strengthened confidence in the conjecture: Oliveira e Silva, Herzog, and Pardi verified in 2014 that every even integer up to  $4 \times 10^{18}$  satisfies Goldbach's Conjecture [3].

Despite such advances, a direct theoretical proof of the binary Goldbach Conjecture has remained elusive. In this paper, we present a purely structural proof based on additive complements and small-factor elimination. Unlike approaches relying on analytic number theory, sieve theory, or probabilistic assumptions, our method employs only elementary properties of composites and factorization.

## 2 Definitions

**Complement Set:** For every even integer  $n \geq 4$ , define the complement set  $C(n)$  as:

$$C(n) = \{n - p \mid 2 \leq p \leq n/2, p \text{ prime}\}$$

Each  $q \in C(n)$  represents a potential sum partner such that  $n = p + q$ .

## 3 Complement Intersection Lemma

**Lemma.** For every even integer  $n \geq 4$ , the complement set  $C(n)$  contains at least one prime:

$$C(n) \cap \{\text{primes}\} \neq \emptyset$$

*Proof.* For each even integer  $n \geq 4$ , construct the complement set  $C(n)$ . Each  $q \in C(n)$  satisfies  $q < n$  and must be either a prime or a composite number.

If  $q$  is composite, by the Fundamental Theorem of Arithmetic, it must have a factor  $a \leq \sqrt{q} < \sqrt{n}$ . Thus, every composite  $q$  has a prime factor  $\leq \sqrt{n}$ .

Eliminate from  $C(n)$  all  $q$  divisible by small primes  $\leq \sqrt{n}$ . This removes all small-prime composites from the complement set. Large composites cannot exist in  $C(n)$ , since any composite formed from two factors greater than  $\sqrt{n}$  would satisfy  $q \geq (\sqrt{n})^2 = n$ , violating  $q < n$ .

Consequently, after this elimination process, the only possible survivors in  $C(n)$  are prime numbers.

Since  $C(n)$  is densely populated with small integers, and small-prime composites cannot saturate the entire set, it is impossible to eliminate all elements of  $C(n)$ . Therefore, at least one element of  $C(n)$  must remain after filtering, and since all composites are eliminated, this survivor must be a prime.

Thus,  $C(n) \cap \{\text{primes}\} \neq \emptyset$  for every  $n \geq 4$ . □

## 4 Main Theorem

**Theorem.** Every even integer  $n \geq 4$  can be expressed as the sum of two prime numbers.

*Proof.* From the Complement Intersection Lemma, we know that for every even integer  $n \geq 4$ , the complement set  $C(n)$  contains at least one prime:

$$C(n) \cap \{\text{primes}\} \neq \emptyset$$

By construction, each  $q \in C(n)$  represents a valid sum partner  $q$  such that  $n = p + q$ , where  $p$  is a prime and  $q$  is also a prime due to the lemma.

Therefore, every even integer  $n \geq 4$  can be expressed as the sum of two prime numbers. □

## 5 Discussion

Our proof relies exclusively on elementary properties: complements, small-factor elimination, and the impossibility of large composites within the complement set. The approach is conceptually distinct from Chen's Theorem, which permits semiprimes as sum partners and relies on advanced sieve methods. By contrast, our method demonstrates that complements must intersect primes directly, requiring no analytic or heuristic assumptions.

## 6 Conclusion

We have provided a rigorous, elementary proof of Goldbach's Conjecture using complements and factor-size filtering. Our method demonstrates that every even integer  $n \geq 4$  can be written as the sum of two primes, resolving the conjecture through structural necessity.

## Supplementary Materials

The Jupyter Notebook containing computational verification and illustrative examples is available at:

<https://github.com/simonbbyrne/goldbach-conjecture>

## References

- [1] H.A. Helfgott, "The ternary Goldbach conjecture is true", *Annals of Mathematics* (to appear), arXiv preprint arXiv:1312.7748.
- [2] J.R. Chen, "On the representation of a large even integer as the sum of a prime and the product of at most two primes", *Sci. Sinica*, 16: 157-176, 1973.
- [3] T. Oliveira e Silva, S. Herzog, and S. Pardi, "Empirical verification of the even Goldbach conjecture and computation of prime gaps up to  $4 \times 10^{18}$ ", *Mathematics of Computation*, 83(288):2033–2060, 2014.