Simon Blaue Sheet 3

P1

$$m\dot{v} = -\gamma v + \xi(t)$$

a) $$v(t) = v_0 e^{-\gamma t/m} + \frac{1}{m}\int_0^t dt' \, e^{-\gamma(t-t')/m} \xi(t')$$

$$\Rightarrow \dot{v}(t) = -\frac{v_0}{m}\gamma e^{-\gamma t/m} + \frac{1}{m}\frac{d}{dt}\int_0^t dt' \, e^{-\gamma(t-t')/m}\xi(t')$$

$$= -\frac{v_0}{m}\gamma e^{-\gamma t/m} - \frac{\gamma}{m^2}\int_0^t dt' \, e^{-\gamma(t-t')/m}\xi(t') + e^{-\gamma(t-t)/m}\xi(t)$$

$$\Rightarrow m\dot{v} = -\gamma\left[v_0 e^{-\gamma t/m} + \frac{1}{m}\int_0^t dt' \, e^{-\gamma(t-t')/m}\xi(t')\right] + \xi(t)$$

$$= -\gamma v + \xi(t)$$

b) $$\langle [\Delta v(t)]^2\rangle = \langle [v(t) - \langle v(t)\rangle]^2\rangle = \langle v^2(t)\rangle - \langle v(t)\rangle^2$$

$$\langle v(t)\rangle = \left\langle v_0 e^{-\gamma t/m} + \frac{1}{m}\int_0^t dt' \, e^{-\gamma(t-t')/m}\xi(t')\right\rangle$$

$$= \underbrace{\langle v_0 e^{-\gamma t/m}\rangle}_{\text{independent of } x} + \frac{1}{m}\int_0^t dt' \, e^{-\gamma(t-t')/m}\underbrace{\langle\xi(t')\rangle}_{=0}$$

$$= v_0 e^{-\gamma t/m}$$

$$\langle v^2(t)\rangle = v_0^2 e^{-2\gamma t/m} + \frac{2}{m} v_0 e^{-\gamma\frac{t}{m}}\int_0^t dt' \, e^{-\gamma(t-t')/m}\langle\xi(t')\rangle + \frac{1}{m^2}\int_0^t\int_0^t dt' dt'' \, e^{-\gamma(2t-t'-t'')/m}\langle\xi(t')\xi(t'')\rangle$$

$$= v_0^2 e^{-2\gamma t/m} + \frac{1}{m^2}\int_0^t\int_0^t dt' dt'' \, e^{-\gamma(2t-t'-t'')/m}\, 2D\gamma^2\delta(t'-t'')$$

$$= v_0^2 e^{-2\gamma t/m} + \frac{2D\gamma^2}{m^2}\int_0^t dt' \, e^{-2\gamma(t-t')/m}$$

$$= v_0^2 e^{-2\gamma t/m} + \frac{\cancel{2}D\gamma^{\cancel{2}}}{m^{\cancel{2}}} e^{-2\gamma t/m}\left[\frac{\cancel{m}}{\cancel{2\gamma}} e^{2\gamma t'/m}\right]_0^t$$

$$= v_0^2 e^{-2\gamma t/m} + \frac{D\gamma}{m}\left(1 - e^{-2\gamma t/m}\right)$$

$$\Rightarrow \langle \Delta v(t)^2\rangle = \frac{D\gamma}{m}\left(1 - e^{-2\gamma\frac{t}{m}}\right)$$

$$\langle \Delta v(t)^2 \rangle \xrightarrow{t\to\infty} \frac{D\gamma}{m}$$

$$\langle \Delta v(t)^2 \rangle \xrightarrow{t\ll 1} \lim_{t\to 0} \frac{D\gamma}{m}\left(1 - e^{-2\gamma t/m}\right) \simeq \frac{D\gamma}{m}\left(1 - \left(1 - 2\gamma t/m\right)\right) = \frac{2D\gamma^2 t}{m^2}$$

Autocorrelation:

$$\langle \Delta v(t)\, \Delta v'(t) \rangle = \left\langle \left(\frac{1}{m}\int_0^t dt_1\, e^{-\gamma(t-t_1)/m} \xi(t_1)\right)\left(\frac{1}{m}\int_0^{t'} dt_2\, e^{-\gamma(t'-t_2)} \xi(t_2)\right)\right\rangle$$

$$= \frac{1}{m^2}\int_0^t\int_0^{t'} dt_1\, dt_2\; e^{-\gamma(2t-t_1-t_2)} \underbrace{\langle \xi(t_1)\xi(t_2)\rangle}_{2D\gamma^2\delta(t_1-t_2)}$$

$$= \frac{2D\gamma^2}{m^2}\int_0^{t'} dt_2\, e^{-\gamma(t+t'-2t_2)/m}$$

$$= \frac{\cancel{2}D\gamma^{\cancel{2}}}{m^{\cancel{2}}}\, e^{-\gamma(t+t')/m}\int_0^{t'} dt_2\, e^{2\gamma t_2/m}$$

$$= \dots \left[\frac{\cancel{m}}{\cancel{2}\cancel{\gamma}} e^{2\gamma t_2/m}\right]_0^{t'} = \frac{D\gamma}{m} e^{-\gamma(t+t')/m}\left(e^{2\gamma t'/m} - 1\right)$$

$$\overset{t,t' \text{ large}}{=} \frac{D\gamma}{m} e^{\gamma(t'-t')/m} e^{-\gamma(t-t')/m}$$

$$= \frac{D\gamma}{m} e^{-\gamma\Delta t/m} = \frac{D\gamma}{m} e^{-\Delta t/\tau_v} \Rightarrow \underline{\underline{\tau_v = \frac{m}{\gamma}}}$$

c) $$x(t) = \int_0^t dt'\, v(t') = \int_0^t dt'\, v_0 e^{-\gamma t'/m} + \int_0^t dt' \int_0^{t'} dt''\, \frac{1}{m} e^{-\gamma(t'-t'')/m} \xi(t'')$$

$$= \frac{m v_0}{\gamma}\left(1 - e^{-\gamma t/m}\right) + \frac{1}{m}\int_0^t\int_0^{t'} dt'\, dt''\, e^{-\gamma(t'-t'')/m} \xi(t'')$$

$$\Rightarrow \langle x(t) \rangle = \frac{m v_0}{\gamma}\left(1 - e^{-\gamma t/m}\right) + \frac{1}{m}\int_0^t\int_0^{t'} dt'\, dt''\, e^{-\gamma(t'-t'')/m} \underbrace{\langle \xi(t'')\rangle}_{=0}$$

$$= \frac{m v_0}{\gamma}\left(1 - e^{-\gamma t/m}\right)$$

d) $\langle [x(t) - x_0]^2 \rangle = \langle x^2(t) \rangle - 2x_0\, x(t) + x_0^2$

$$\langle x^2(t)\rangle = \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + \frac{2v_0}{\gamma}\left(1 - e^{-\gamma t/m}\right)\int_0^t\int_0^{t'} dt' dt'' e^{-\gamma(t'-t'')/m} \langle \xi(t'')\rangle$$

$$+ \frac{1}{m^2}\int_0^t dt_1' \int_0^{t'} dt_1'' \int_0^{t_2''} dt_2'' e^{-\gamma(t_1'+t_2'-t_1''-t_2'')/m} \underbrace{\langle \xi(t_1'')\xi(t_2'')\rangle}_{=2D\gamma^2\delta(t_1''-t_2'')}$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + \frac{2D\gamma^2}{m^2}\int_0^t dt_1' \int_0^{t_1'} dt_1'' \int_0^t dt_2' e^{-\gamma(t_1'+t_2'-2t_1'')/m}$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + \frac{2D\gamma^2}{m^2}\int_0^t dt_1' \int_0^{t_1'} dt_1'' \underbrace{\left[-\frac{m}{\gamma} e^{-\gamma(t_1'+t_2'-2t_1'')/m}\right]_0^t}_{=\frac{m}{\gamma}e^{-\gamma(t'-2t_1'')}\left(1-e^{-\gamma t/m}\right)}$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + \frac{2D\gamma}{m}\left(1 - e^{-\gamma t/m}\right)\int_0^t dt' \underbrace{\int_0^{t'} dt'' e^{-\gamma(t'-2t'')/m}}_{\frac{m}{2\gamma}e^{-\gamma t'/m}\left(e^{2\gamma t'/m}-1\right)}$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + D\left(1 - e^{-\gamma t/m}\right)\int_0^t dt' \underbrace{e^{\gamma t'/m} - e^{-\gamma t'/m}}_{2\sinh(\gamma t'/m)}$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + D\left(1 - e^{-\gamma t/m}\right)\left[\frac{2m}{\gamma}\cosh(\gamma t'/m)\right]_0^t$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - e^{-\gamma t/m}\right)^2 + \frac{2mD}{\gamma}\left(1 - e^{-\gamma t/m}\right)\left(\cosh(\gamma t/m) - 1\right)$$

$$\Rightarrow \langle [x(t) - x_0]^2\rangle = \langle x^2(t)\rangle - 2x_0\langle x(t)\rangle + x_0^2$$

$$= \frac{m^2 v_0^2}{\gamma^2}\left(1 - \cancel{e^{-\gamma t/m}}\right)^2 + \frac{2mD}{\gamma}\left(1 - \cancel{e^{-\gamma t/m}}\right)\left(\cosh\left(\frac{\gamma}{m}t\right) - 1\right)$$

$$- \frac{2m v_0 x_0}{\gamma}\left(1 - \cancel{e^{-\gamma t/m}}\right) + x_0^2$$

$\underset{\sim\sim}{t \to \infty}$ $\Rightarrow$ $$\ldots = \frac{m^2 v_0^2}{\gamma^2} + \frac{2mD}{\gamma}\left(\underbrace{\cosh\left(\frac{\gamma}{m}t\right)}_{\text{exp. dominant}} - 1\right) - \frac{2m v_0 x_0}{\gamma} + x_0^2$$

$$= \frac{2mD}{\gamma}\cosh\left(\frac{\gamma}{m}t\right) \propto \frac{mD}{\gamma}e^{\gamma t/m}$$ $\longrightarrow$ displacement grows exponential

↳ Diffusion

For $t \ll 1$, Taylor exp-Terms to linear order:

$$\Rightarrow \ldots = \frac{m^2 v_0^2}{\gamma^2}\left(\frac{\gamma}{m}t\right)^2 + \frac{2mD}{\gamma}\left(\frac{\gamma}{m}t\right)(0) - \frac{2m v_0 x_0}{\gamma}\left(\frac{\gamma}{m}t\right) + x_0^2$$

$$= v_0^2 t^2 - 2 v_0 x_0 t + x_0^2 = (v_0 t - x_0)^2$$

$\rightarrow$ Displacement grows constant $\rightarrow$ ballistic.

We can neglet m if γv dominates over $m\dot{v}$, hence in the overdamped case.

P2

a) $$\frac{\partial A_1}{\partial t} \overset{!}{=} L_{11} A_1(t) + L_{12} \int_0^t dt' \, e^{L_{22}(t-t')} L_{21} A_1(t') + L_{12} \, e^{L_{22} t} A_2(0)$$

$$\Rightarrow A_2(t) = \int_0^t dt' \, e^{L_{22}(t-t')} L_{21} A_1(t') + L_{12} \, e^{L_{22} t} A_2(0)$$

$$\rightarrow \frac{\partial A_2}{\partial t} = L_{22} \int_0^t dt' \, e^{L_{22}(t-t')} + e^{\cancel{L_{22}(t-t)}} L_{21} A_1(t) + L_{12} L_{22} \, e^{L_{22} t} A_2(0)$$

$$= L_{22} A_1(t) + L_{22} A_2(t) \checkmark$$

The equation for $A_1(t)$ is not Markovian, as it is integrated over $A_1(t)$ over time, which means it depends on its history.

b) $$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = \mathbb{1} - P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$PLP = \begin{pmatrix} L_{11} & 0 \\ 0 & 0 \end{pmatrix} \qquad PLQ = \begin{pmatrix} 0 & L_{12} \\ 0 & 0 \end{pmatrix}$$

$$QLP = \begin{pmatrix} 0 & 0 \\ L_{21} & 0 \end{pmatrix} \qquad QLQ = \begin{pmatrix} 0 & 0 \\ 0 & L_{22} \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial t}\vec{A} = PLP\vec{A}(t) + \int_0^t dt' \, PLQ \, e^{QLQ(t-t')} QLP\vec{A}(t) + PLQ \, e^{QLQt} Q\vec{A}(0)$$

c) $PB = (B, A^T)(A, A^T)^{-1} A \quad , \quad e^{\mathcal{L}t} = e^{Q\mathcal{L}t} + \int_0^t dt' \, e^{\mathcal{L}(t-t')} P\mathcal{L} e^{Q\mathcal{L}t'}$

$\Omega A = (\mathcal{L}A, A^T)(A, A^T)^{-1} A = P(\mathcal{L}A)$

$K(t) A = (\mathcal{L}F(t), A^T)(A, A^T)^{-1} A = P(\mathcal{L}F(t))$

$\frac{\partial}{\partial t} A = \mathcal{L} A \Rightarrow A(t) = e^{\mathcal{L}t} A$

$$\Rightarrow \frac{\partial A}{\partial t} = e^{Q\mathcal{L}t} Q\mathcal{L}A + \int_0^t dt' \, e^{\mathcal{L}(t-t')} \mathcal{L}P\mathcal{L} e^{Q\mathcal{L}t'} A + e^{\mathcal{L}(t-t)} P\mathcal{L} e^{Q\mathcal{L}t} A$$

$$= F(t) - \left[e^{\mathcal{L}(t-t')} P\mathcal{L} e^{Q\mathcal{L}t'} A \right]_0^t + \int_0^t dt' \, e^{\mathcal{L}(t-t')} P\mathcal{L} e^{Q\mathcal{L}t'} Q\mathcal{L}A + P\mathcal{L} e^{Q\mathcal{L}t} A$$

$$= F(t) - P\mathcal{L} e^{Q\mathcal{L}t} A + e^{\mathcal{L}t} P\mathcal{L}A + \int_0^t dt' \, e^{\mathcal{L}(t-t')} P\mathcal{L} F(t) + P\mathcal{L} e^{Q\mathcal{L}t} A$$

$$= e^{\mathcal{L}t} \Omega A + \int_0^t dt' \, e^{\mathcal{L}(t-t')} K(t) A + F(t)$$

ΩA and $K(t) A$ are observables, thus $e^{\mathcal{L}t} \Omega A \, e^{\mathcal{L}t} = \Omega A$ and $e^{\mathcal{L}t} K(t) A \, e^{\mathcal{L}t} = K(t) A$

$$\Rightarrow \ldots = e^{\mathcal{L}t} \Omega A \, e^{-\mathcal{L}^+ t} + \int_0^t dt' \, e^{\mathcal{L}(t-t')} K(t) A(t-t') e^{-\mathcal{L}(t-t')} + F(t)$$

$$= \Omega A + \int_0^t dt' \, K(t) A(t-t') + F(t)$$