Sheet 2 ASP

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1) $m = N_+ - N_-$

$$\binom{N}{N_+} = \frac{N!}{N_+!\,(\underbrace{N-N_+}_{N_-})!} = \frac{N!}{\left(\frac{\overbrace{N_+ + N_-}^{N} + \overbrace{N_+ - N_-}^{m}}{2}\right)!\left(\frac{\overbrace{N_+ + N_-}^{N} \overbrace{- N_+ + N_-}^{m}}{2}\right)!} = \frac{N!}{\left(\frac{N+m}{2}\right)!\,\left(\frac{N-m}{2}\right)!}$$

$$\Rightarrow P(m) = \binom{N}{N_+}\frac{1}{Z}e^{-\beta H} = \binom{N}{N_+}\frac{1}{Z}e^{\beta\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j}$$

$$\sum_{i,j=1}^{N}\sigma_i\sigma_j = \sum_i^N \sigma_i \sum_j^N \sigma_j = m\cdot m = m^2$$

$$\Rightarrow P(m) = \frac{N!}{\left(\frac{N+m}{2}\right)!\,\left(\frac{N-m}{2}\right)!}\frac{1}{Z}e^{\beta\frac{J}{2N}m^2}$$

b)

$$I(m) = -\lim_{N\to\infty}\frac{1}{N}\ln(P(m))$$

Stirling: $\ln(N!) \approx N\ln(N) - N$

$$\ln(P(m)) = -\ln(Z) + \ln(N!) - \ln\left(\left(\frac{N+m}{2}\right)!\right) - \ln\left(\left(\frac{N-m}{2}\right)!\right) + \beta J\, m^2/2N$$

$$= \beta J m^2 N/2 + N\ln(N) - N - \frac{N+m}{2}\ln\left(\frac{N+m}{2}\right) + \frac{N+m}{2} - \frac{N-m}{2}\ln\left(\frac{N-m}{2}\right) + \frac{N-m}{2} - \ln(Z)$$

$$= \beta J m^2\frac{N}{2} + N(\ln(N)-1) - N\left(\frac{1+m}{2}\right)\left(\ln(N) + \ln\left(\frac{1+m}{2}\right)\right) + N\left(\frac{1+m}{2}\right)$$

$$- N\left(\frac{1-m}{2}\right)\left(\ln(N) + \ln\left(\frac{1-m}{2}\right)\right) + N\left(\frac{1-m}{2}\right) - \ln(Z)$$

$$\Leftrightarrow \frac{\ln(P(m))}{N} = \beta J m^2/2 + \ln(N) \cancel{-1} - \frac{1+m}{2}\left(\ln(N) + \ln\left(\frac{1+m}{2}\right)\right) + \frac{1+m}{2} \cancel{+1}$$

$$- \frac{1-m}{2}\left(\ln(N) + \ln\left(\frac{1-m}{2}\right)\right) + \frac{1-m}{2} - \frac{1}{N}\ln(Z)$$

$$= \beta J m^2/2 + \ln(N) - \frac{1}{2}\left(2\ln(N) + \ln\left(\frac{1+m}{2}\right) + \ln\left(\frac{1-m}{2}\right)\right) - \frac{m}{2}\left(0 + \ln\left(\frac{1+m}{2}\right) - \ln\left(\frac{1-m}{2}\right)\right) - \ln(Z)\frac{1}{N}$$

$$= \beta J\frac{m^2}{2} - \frac{1}{2}\left(\ln\left(\frac{1+m}{2}\right) + \ln\left(\frac{1-m}{2}\right)\right) - \frac{m}{2}\left(\ln\left(\frac{1+m}{2}\right) - \ln\left(\frac{1-m}{2}\right)\right) - \ln(Z)\frac{1}{N}$$

$$= \beta J\, m^2/2 - \frac{1}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1}{2}\ln\left(\frac{1-m}{2}\right) - \frac{m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{m}{2}\ln\left(\frac{1-m}{2}\right) - \ln(Z)\frac{1}{N}$$

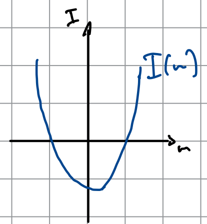
$$= \beta J m^2/2 - \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) - \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right) - \frac{1}{N}\ln(Z)$$

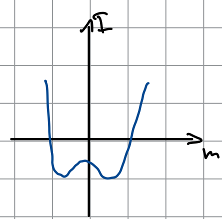
$$\Rightarrow I(m) = -\beta J \frac{m^2}{2} + \frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\left(\frac{1-m}{2}\right) + \underbrace{\lim_{N\to\infty}\frac{1}{N}\ln\left(\sum_{conf} e^{-\beta H}\right)}$$

$= \underline{\text{const.}}$ because $\sum_{conf}$ growth with 2^N

$Z \propto 2^N \rightsquigarrow \lim_{N\to\infty} \frac{1}{N} N\ln(2)$

$\beta J < 1$ $\beta J > 1$





$\hookrightarrow$ Need to take convex hull, so we get not differentiable points $\rightarrow$ indicate phase transition

c)

$$\frac{1}{\sqrt{2\pi\beta J/N}} \int dx\, e^{-Nx^2/(2\beta J) + Mx} = \ldots \int dx\, e^{-\frac{N}{2\beta J}\left(x^2 + \frac{2\beta J M}{N}x\right)} = \ldots \int dx\, e^{-\frac{N}{2\beta J}\left((x+a)^2 - \left(\frac{\beta J M}{N}\right)^2\right)}$$

with $a = \frac{\beta J M}{N}$

$$\int_{-\infty}^{\infty} e^{-\alpha(x+\beta)^2} dx = \sqrt{\frac{\pi}{\alpha}} \quad \text{Gaussian Integral}$$

$$\ldots = \frac{1}{\sqrt{2\pi\beta\frac{J}{N}}} e^{\frac{\beta J M^2}{2N}} \int dx\, e^{-\frac{N}{2\beta J}(x+a)^2} = \frac{1}{\sqrt{2\pi\beta\frac{J}{N}}} e^{\frac{\beta J M^2}{2N}} \sqrt{\frac{\pi\, 2\beta J}{N}}$$

$$= \sqrt{\frac{\pi\, 2\beta J N}{2\pi\beta N J}}\, e^{\beta J \frac{M^2}{2N}} = e^{\beta J \frac{M^2}{2N}}$$

$$Z = \mathrm{Tr}\, e^{-\beta(H - hM)} = \sum_{conf} e^{-\beta\left(-\frac{1}{2N}JM^2 - hM\right)}$$

$$= \sum_{conf} e^{\beta J M^2/2N}\, e^{hM} = \sum_{conf} e\, e^{hM}$$

$$= \sum_{conf} \int \frac{dx}{\sqrt{2\pi\beta J/N}}\, e^{-Nx^2/2\beta J + Mx}\, e^{\beta h M}$$

$$= \sum_{conf} \int \frac{dx}{\sqrt{2\pi\beta J/N}}\, e^{-\frac{N}{2\beta J}x^2}\, e^{M(x+\beta h)}$$

$$= \sum_{\mathrm{conf}} \int \frac{dx}{\sqrt{2\pi\beta J/N}} e^{-\frac{N}{2\beta J}x^2} e^{M(x+\beta h)}$$

$$= \int \frac{dx}{\sqrt{2\pi\beta J/N}} e^{-\frac{N}{2\beta J}x^2} \sum_{\mathrm{conf}} e^{M(x+\beta h)}$$

$$= \int \dots \sum_{\sigma_i=\pm 1} e^{\sum^N \sigma_i (x+\beta h)} = \int \dots \sum_{\sigma_i=\pm 1} \prod_{i=1}^N e^{\sigma_i (x+\beta h)} = \int \dots \prod_{i=1}^N \sum_{\sigma_i=\pm 1} e^{\sigma_i (x+\beta h)}$$

$$= \int \dots \prod^N 2\cosh(x+\beta h) = \int \dots (2\cosh(x+\beta h))^N$$

$$= \frac{2^N}{\sqrt{2\pi\beta J/N}} \int dx\, e^{-\frac{N}{2\beta J}x^2} \cosh(\beta h + x)^N$$

$F = -k_B T \ln(Z)$

$\rightarrow$ Saddle point $N \to \infty$ and $\int \Rightarrow \min_x$

$$f(h) = \min_x \underbrace{\left[\frac{x^2}{2\beta^2 J} - \frac{1}{\beta}\ln(2\cosh(\beta h + x))\right]}_{g(x)}$$

$$g'(x) = \frac{2x}{2\beta^2 J} - \frac{1}{\beta} 2\sinh(\beta h + x)\frac{1}{2\cosh(\beta h + x)}$$

$$= \frac{x}{\beta^2 J} - \frac{1}{\beta}\tanh(\beta h + x) \doteq 0$$

$$\Rightarrow \frac{1}{\beta J} x^* = \tanh(\beta h + x^*)$$

$$\Rightarrow x^* = \beta J \tanh(\beta h + x^*)$$

d) $m = -f'(h)$

ignoring $\min_x$ at first

$$= - \frac{\cancel{1}}{\cancel{\beta}} \cancel{\beta}\, \cancel{2} \sinh(\beta h + x) \frac{1}{\cancel{2}\cosh(\beta h + x)}$$

$$m = \tanh(\beta h + x) \longrightarrow \text{Plug in } x^* \text{ (we show after that this way around is legit)}$$

$$= \tanh(\beta h + \beta J \tanh(\beta h + x^*))$$

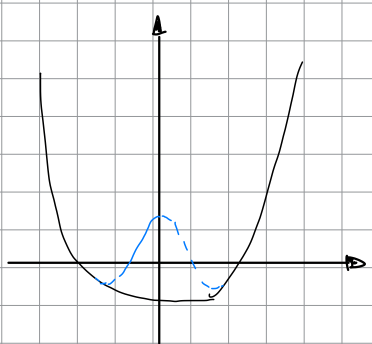
$$= \tanh(\beta(h + Jm))$$

Show that this way was fine:

$$f'(h, x^*(h)) = \underbrace{\frac{\partial x^*}{\partial h} \frac{\partial f}{\partial x^*}}_{=0 \text{ after minimum condition above}} + \frac{\partial f}{\partial h}$$

$$= \underline{\underline{\frac{\partial f}{\partial h}}}$$

e) $$I^*(m) = \sup_\beta \left(hm - \min_x \frac{x^2}{2\beta^2 J} - \frac{1}{\beta} \ln(2\cosh(\beta h + x)) - \min_x \frac{x^2}{2\beta^2 J} - \frac{1}{\beta} \ln(2\cosh(x)) \right)$$



As we seen in lecture the GE-Theorem yields the convex-Hull, indicating phase transitions.

A2

a)

$$C_{AB}(t,t') = \mathrm{Tr}\; A(t)\, B(t')\, \rho(0)$$

$$C^*_{BA}(t',t) = \left(\mathrm{Tr}\; B(t')\, A(t)\, \rho(0)\right)^* = \mathrm{Tr}\left(B(t')\, A(t)\, \rho(0)\right)^\dagger$$

$$= \mathrm{Tr}\left(\rho(0)^\dagger\, A(t)^\dagger\, B(t')^\dagger\right) = \mathrm{Tr}\left(A(t)\, B(t')\, \rho(0)\right)$$

$$= C_{AB}(t,t')$$

b) $C_{AB}(t,t') = (A(t), B(t')) = \mathrm{Tr}\; AB\rho^{eq.}$

$\mathcal{L}A = \frac{i}{\hbar}[H,A]$

$\mathcal{L}A = \{H,A\}$

$$\left.\begin{array}{l} ① \; \mathrm{Tr}\; A(\mathcal{L}B)\rho^{eq} = \mathrm{Tr}\,(\mathcal{L}^\dagger A)\, B\rho^{eq.} \\ ② \; \mathcal{L}A = -\mathcal{L}^\dagger A \iff -\mathcal{L}A = \mathcal{L}^\dagger A \end{array}\right\} \text{known from lecture}$$

$$(A, \mathcal{L}B) = -(\mathcal{L}A, B)$$

$$\iff \mathrm{Tr}\; A(\mathcal{L}B)\rho^{eq} = -\mathrm{Tr}\,(\mathcal{L}A)\, B\rho^{eq}$$

$$\overset{①}{=} \mathrm{Tr}\,(\mathcal{L}^\dagger A)\, B\rho^{eq}$$

$$\iff \overset{②}{=} \mathrm{Tr}\, -(\mathcal{L}A)\, B\rho^{eq} = -\mathrm{Tr}\,(\mathcal{L}A)\, B\rho^{eq} \quad ✓$$

$$(A, e^{\mathcal{L}t'}B) = \mathrm{Tr}\; A(e^{\mathcal{L}t'}B) = \mathrm{Tr}\; A\left(\sum_{n=0}^{\infty} \frac{(\mathcal{L}t')^n}{n!}\, B\right)$$

$$= \mathrm{Tr}\; \frac{A}{n!}\left(\sum_n^{\infty} \mathcal{L}^n t'^n\, B\right) = \sum_n^{\infty} \frac{1}{n!} t'^n\; \mathrm{Tr}\; A(\mathcal{L}^n B) = \sum_n^{\infty} \frac{1}{n!} t'^n\, (A, \mathcal{L}^n B)$$

$$= \dots\; (-\mathcal{L}^n A, B) = \mathrm{Tr}\left(\sum_n^{\infty} \frac{-\mathcal{L}^n t'^n}{n!} A\right) B = \mathrm{Tr}\left(e^{-\mathcal{L}t'} A\right) B = \underline{\underline{(e^{-\mathcal{L}t'} A, B)}}$$

c) $\langle [A(\Delta t) - A(0)]^2 \rangle = \langle [A(\Delta t)A(\Delta t) - 2A(\Delta t)A(0) + A(0)A(0)] \rangle$

$$= \mathrm{Tr}\; A(\Delta t)\, A(\Delta t)\, \rho^{eq} - 2\,\mathrm{Tr}\left(A(\Delta t)A(0)\,\rho^{eq}\right) + \mathrm{Tr}\left(A(0)A(0)\,\rho^{eq}\right)$$

$\mathrm{Tr}\; A(t)\, B(t+\Delta t) = C_{AB}(\Delta t) \qquad -\Delta t \equiv \Delta t$

$$= C_{AA}(0) \quad - 2C_{AA}(-\Delta t) \quad + C_{AA}(0) \;\geq 0 \iff C_{AA}(0) \geq C_{AA}(\Delta t)$$

A3

a) $\mathcal{L}(\cdot) = \{\cdot, H\} = \sum_{i\alpha} \frac{\partial H}{\partial p_{i\alpha}} \frac{\partial}{\partial r_{i\alpha}}(\cdot) - \frac{\partial H}{\partial r_{i\alpha}} \frac{\partial}{\partial p_{i\alpha}}(\cdot) = \frac{p}{m}\frac{\partial}{\partial r} - m\omega^2 x \frac{\partial}{\partial p}$

b)
$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial p}{\partial \phi} \\ \frac{\partial x}{\partial J} & \frac{\partial p}{\partial J} \end{pmatrix}}_{B}{}^{-1} \begin{pmatrix} \partial_\phi \\ \partial_J \end{pmatrix}$$

$$\frac{\partial x}{\partial \phi} = -\sqrt{\frac{2J}{m\omega}} \sin(\phi)$$

$$\frac{\partial x}{\partial J} = \frac{2}{m\omega}\frac{1}{2}\frac{1}{\sqrt{\frac{2J}{m\omega}}} \cos(\phi) = \frac{1}{m\omega}\sqrt{\frac{m\omega}{2J}} \cos(\phi)$$

$$= \frac{1}{\sqrt{2Jm\omega}} \cos\phi$$

$$\Lambda = \frac{1}{\det B} \begin{pmatrix} \frac{\partial p}{\partial J} & -\frac{\partial p}{\partial \phi} \\ -\frac{\partial x}{\partial J} & \frac{\partial x}{\partial \phi} \end{pmatrix}$$

$$\frac{\partial p}{\partial \phi} = \sqrt{2Jm\omega} \cos(\phi)$$

$$\frac{\partial p}{\partial J} = 2m\omega \frac{1}{2} \frac{1}{\sqrt{2Jm\omega}} \sin\phi = \sqrt{\frac{m\omega}{2J}} \sin(\phi)$$

$$\det B = \frac{\partial x}{\partial \phi}\frac{\partial p}{\partial J} - \frac{\partial p}{\partial \phi}\frac{\partial x}{\partial J} = -\sqrt{\frac{2J}{m\omega}} \sin(\phi) \sqrt{\frac{m\omega}{2J}} \sin\phi - \sqrt{2Jm\omega} \cos(\phi) \frac{1}{\sqrt{2Jm\omega}} \cos\phi$$

$$= -1$$

$$\Rightarrow \Lambda = \begin{pmatrix} -\sqrt{\frac{m\omega}{2J}} \sin\phi & \sqrt{2Jm\omega} \cos\phi \\ \frac{1}{\sqrt{2Jm\omega}} \cos\phi & \sqrt{\frac{2J}{m\omega}} \sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \partial_p \end{pmatrix} = \Lambda \begin{pmatrix} \partial_\phi \\ \partial_J \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{m\omega}{2J}} \sin\phi \frac{\partial}{\partial \phi} + \sqrt{2Jm\omega} \cos\phi \frac{\partial}{\partial J} \\ \frac{1}{\sqrt{2Jm\omega}} \cos\phi \frac{\partial}{\partial \phi} + \sqrt{\frac{2J}{m\omega}} \sin\phi \frac{\partial}{\partial J} \end{pmatrix}$$

$$H(\phi, J) = \omega J$$

$$\mathcal{L} = \frac{p}{m}\partial_r - m\omega^2 x\, \partial_p$$

$$= \frac{\sqrt{2Jm\omega}\sin\phi}{m}\left(-\sqrt{\frac{m\omega}{2J}}\sin\phi\frac{\partial}{\partial\phi} + \sqrt{2Jm\omega}\cos\phi\frac{\partial}{\partial J}\right)$$

$$-\underbrace{m\omega^2\sqrt{\frac{2J}{m\omega}}}_{\sqrt{2Jm\omega^3}}\cos\phi\left(\frac{1}{\sqrt{2Jm\omega}}\cos\phi\frac{\partial}{\partial\phi} + \sqrt{\frac{2J}{m\omega}}\sin\phi\frac{\partial}{\partial J}\right)$$

$$= -\sqrt{\cancel{\frac{2J\omega}{m}\frac{m\omega}{2J}}\,\omega^2}\,\sin^2\phi\,\partial_\phi + \sqrt{\cancel{\frac{2J\omega}{m}2Jm\omega}}\,\sin\phi\cos\phi$$

$$\underbrace{-\sqrt{\frac{2Jm\omega^3}{\cancel{2Jm\omega}}\,\omega^2}\,\cos^2\phi\,\partial_\phi}_{-(\cos^2+\sin^2)\omega\,\partial_\phi} - \sqrt{\cancel{\frac{2Jm\omega^3\,2J}{m\omega}}}\,\sin\phi\cos\phi$$

$$= -\omega\,\partial_\phi$$

c)

$$\mathcal{L} f = \lambda f$$

$$-\omega\partial_\phi f = \lambda f$$

$$= \partial_\phi f = -\frac{\lambda}{\omega} f$$

$$\Rightarrow f = e^{-\frac{\lambda}{\omega}\phi} = e^{-\frac{\lambda}{\omega}(\phi+2\pi n)} = e^{-\frac{\lambda}{\omega}\phi}\underbrace{e^{-\frac{\lambda}{\omega}2\pi n}}_{=1}$$

$$\Rightarrow 1 = e^{-\frac{\lambda}{\omega}2\pi n} = e^{2\pi i}$$

$$\Leftrightarrow -\frac{\lambda}{\omega}\cancel{2\pi} n = \cancel{2\pi} i$$

$$\Leftrightarrow \lambda = -\omega i n$$

$$\Rightarrow f(\phi) = e^{i\phi n} \quad n \in \mathbb{N}$$

d) $\bar{f} \mathcal{L} f = -(\mathcal{L} f)^+ f$

$\Leftrightarrow \bar{f} \lambda \cancel{f} = -(\lambda f)^+ \cancel{f}$

$\Leftrightarrow \cancel{\bar{f}} \lambda = -\bar{\lambda} \cancel{\bar{f}}$

$\Leftrightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda \in \mathbb{C}$

$\mathcal{L}$ is antisemetic because there are always two eigenvalues λ & $\bar{\lambda}$ occuring pairwise

<u>A4</u>

a) $P(a) = \langle \delta(\frac{A}{N} - a) \rangle = N \langle \delta(A - Na) \rangle$

$$= N \langle \int \frac{d\lambda}{2\pi} e^{i\lambda(A-Na)} \rangle = N \int \frac{d\lambda}{2\pi} e^{i\lambda Na} \langle e^{i\lambda A} \rangle$$

with $$\langle e^{i\lambda A} \rangle = \frac{\mathrm{Tr}\left(e^{-\beta(H - h_0 A) + i\lambda A}\right)}{\mathrm{Tr}\left(e^{-\beta(H - h_0 A)}\right)} = \frac{Z(i\frac{\lambda}{\beta} + h_0)}{Z(h_0)}$$

$$f(i\tfrac{\lambda}{\beta} + h_0) = -\frac{1}{\beta N} \ln\left(Z(i\tfrac{\lambda}{\beta} + h_0)\right) \Leftrightarrow Z(i\tfrac{\lambda}{\beta} + h_0) = e^{-\beta N f(i\frac{\lambda}{\beta} + h_0)}$$

$$f(h_0) = -\frac{1}{\beta N} \ln\left(Z(h_0)\right) \Leftrightarrow Z(h_0) = e^{-\beta N f(h_0)}$$

$$\Rightarrow P(a) = N \int \frac{d\lambda}{2\pi} e^{N[i\lambda a - \beta f(i\frac{\lambda}{\beta} + h_0) + \beta f(h_0)]}$$

$\rightarrow$ saddle point and only in order of exp:

$$P(a) \asymp e^{N \operatorname{extr}_h [i\lambda a - \beta f(i\frac{\lambda}{\beta} + h_0) + \beta f(h_0)]} \quad \text{with } h = i\lambda/\beta$$

$$P(a) = e^{-\beta N \operatorname{extr}_h [ha - f(h + h_0) + f(h_0)]}$$

$$I_{h_0}(a) = \beta \operatorname{extr}_h \left[ha + f(h + h_0) - f(h_0)\right]$$

b) Shifting h by h_0 to $\tilde{h} = h - h_0$

$$I_{h_0}(a) = \beta \sup_{\tilde{h}} \left[\tilde{h}a + f(\tilde{h}+h_0) - f(h_0)\right] \qquad I_0(a) = \beta \sup_h \left[ha + f(h) - f(0)\right]$$

$$= \beta \sup_h \left[(h-h_0)a + f(h-h_0+h_0) - f(h_0)\right]$$

$$= \beta \sup_h \left[h a - h_0 a + f(h) - f(h_0) + f(0) - f(0)\right]$$

$$= \beta \left[\sup_h \left[h a + f(h) - f(0)\right] - h_0 a - f(h_0) + f(0)\right]$$

$$= \beta \left[\quad \cdots \quad \right] - \beta h_0 a + \beta\left[f(0) - f(h_0)\right]$$

$$= I_0(a) - \beta h_0 a + \beta\left[f(0) - f(h_0)\right]$$

c) $$P_{h_0}(A) \propto P_0(A)\, e^{\beta h_0 A}$$

$$P_{h_0}(A) = e^{N\left(I_0(a) - \beta h_0 a + \beta\left[f(0) - f(h_0)\right]\right)}$$

$$= \underbrace{e^{N I_0(a)}}_{P_0(a)}\, e^{N\left(-\beta h_0 a + \beta\left[f(0) - f(h_0)\right]\right)}$$

$$= P_0(a)\, e^{-N\beta h_0 a}\, e^{\overbrace{\beta\left[f(0) - f(h_0)\right]}^{\text{const}}}$$

$$\propto P_0(a)\, e^{-\beta h_0 a}$$