Report 4: Partial Differential Equations (PDEs)

Git: https://github.com/simonblaue/MCP-Ex4.git

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1 Laplace Equation

1.1 Iterator methods

First lets check which method converges the fastest:

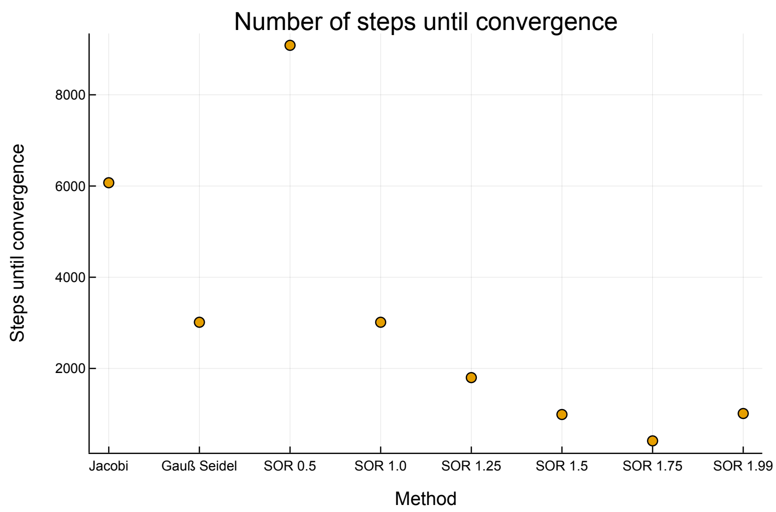


Figure 1: Number of steps until convergence concerning the Laplace error $\max \epsilon < 1 \times 10^{-3}$.

I observe that Gauß-Seidel and SOR with $\alpha = 1.0$ need the same number of time steps as expected. The fastest method is SOR with $\alpha = 1.75$, because with $\alpha = 1.99$ we get to close to unstable regimes. Obviously SOR with $\alpha = 0.5$ takes the longest, as the updating step is damped with the factor α. Now I will observe the evolution of the maximal error and the average error of the different methods.

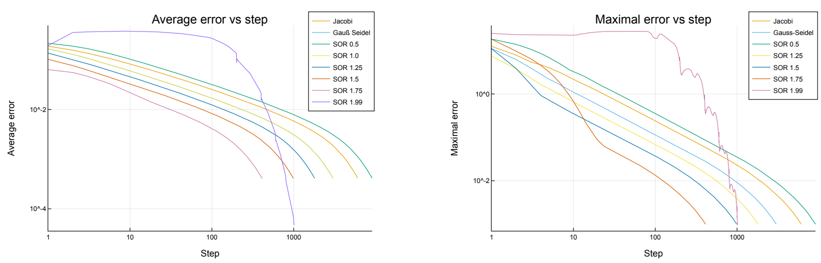


Figure 2: Maximal and average error for different iteration methods.

As discussed before, SOR with $\alpha = 1.0$ and Gauß-Seidel method are the same, hence I plotted only the latter. The error development for all methods besides SOR with$\alpha \geq 1.5$ seem qualitatively the same. However for higher α I observe indents in the curve, which seems to boost the algorithm. This is due to the algorithm taking bigger steps in the right direction once it found this direction. For $\alpha = 1.99$ the curve is very rigid, and I was not sure, if it is relay converged to the right solution, hence I plotted the result below. As it turns out the method also converges to the expected result.

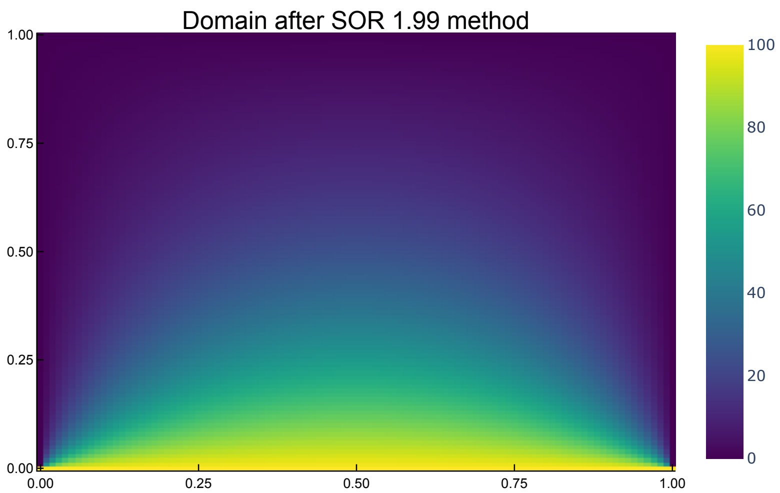
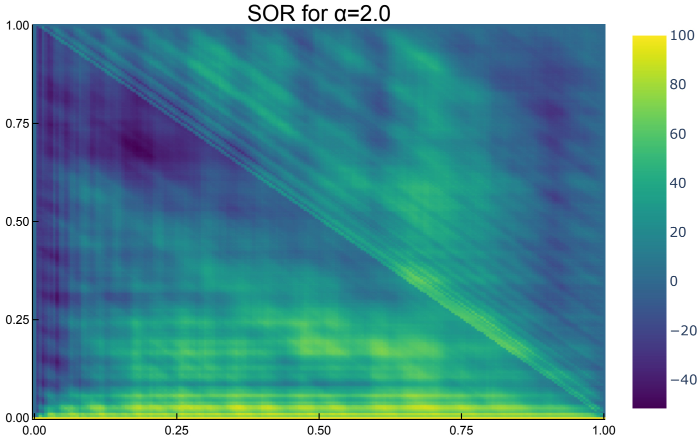
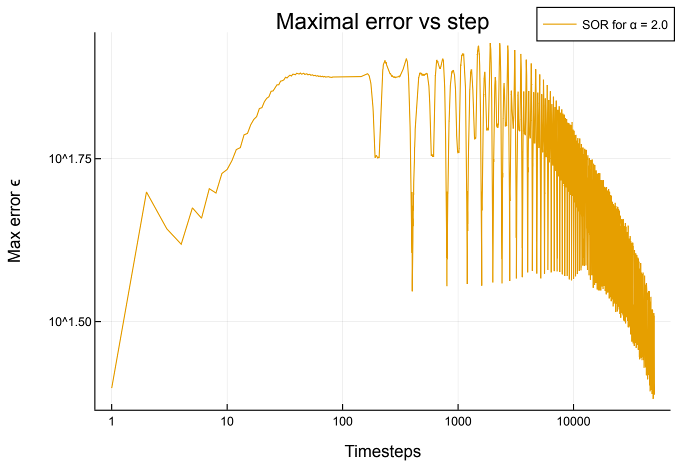


Figure 3: Domain after iterating with the SOR 1.99 method to verify right convergence.

1.2 SOR

The natural question to ask is what happens for an even further boosted SOR method? I found that for $\alpha = 2.0$ the algorithm does not converge in 50000 steps. It seems that it would not have converged to the right domain, as the result shows stripes and the maximal error starts to fluctuate a lot after 100 time steps. The system can not recover from that.





1.3 Infinite sum solution

Now I will cross verify the iterative solution with an aproximated analytical solution. First have a look at the analytical "infinite" sum solution for different numbers of summands.

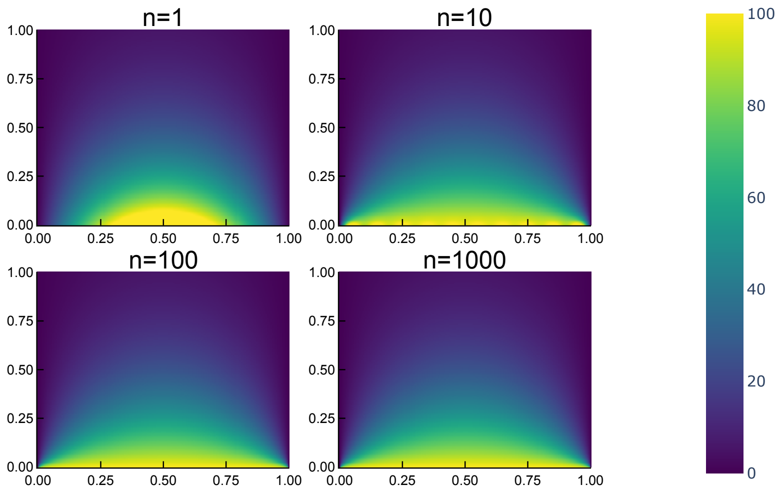
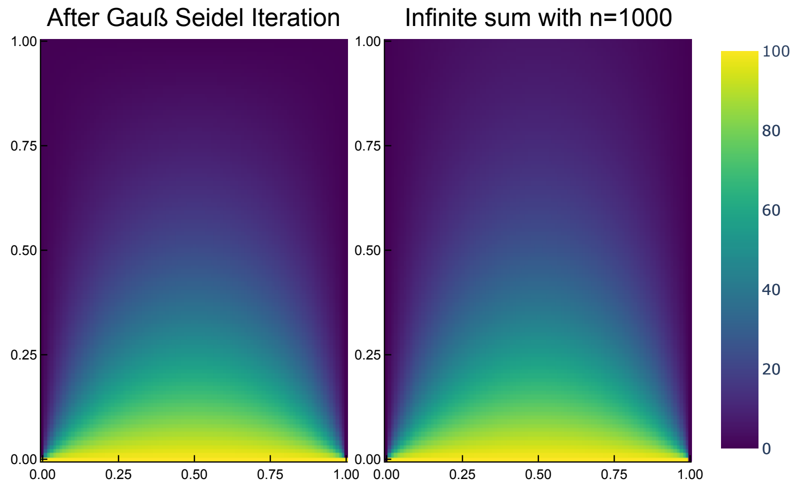
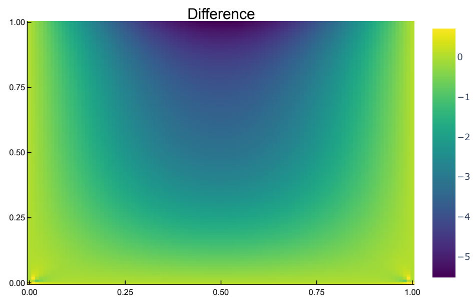


Figure 4: Results for the "infinite" sum solution to the Laplace equation for different numbers of terms n.

As I increase the number of terms more and more sine functions are added left and right to the first one, and towards the top an exponential decay fills in. Comparing the "infinite" sum solution for 1000 terms with the Gauß-Seidel method reveals that the exponential decay is to weak, or the Gauß-Seidel method did not propagate long enough. Subtracting both results displays that at the edges the solutions are identical (yellow) in the top they do not fit together as the iterative solution did not propergate that far.



(a) Comparison of the infinite sum solution with 1000 terms and the Gauß Seidel iterator solution.



(b) Difference between the infinite sum solution with 1000 terms and the Gauß Seidel iterator solution.

2 Diffusion

In this task I implemented four different method for solving the diffusion equation. To varyfiy they all converged to the same solution I plotted a time evolution of all of them. The time domain is in the y-axis, the rod domain in the x-axis, displayed in Figure 6. As expected, th rod cools down, but holds the qualitative temperature profile.

2.1 Integration methods

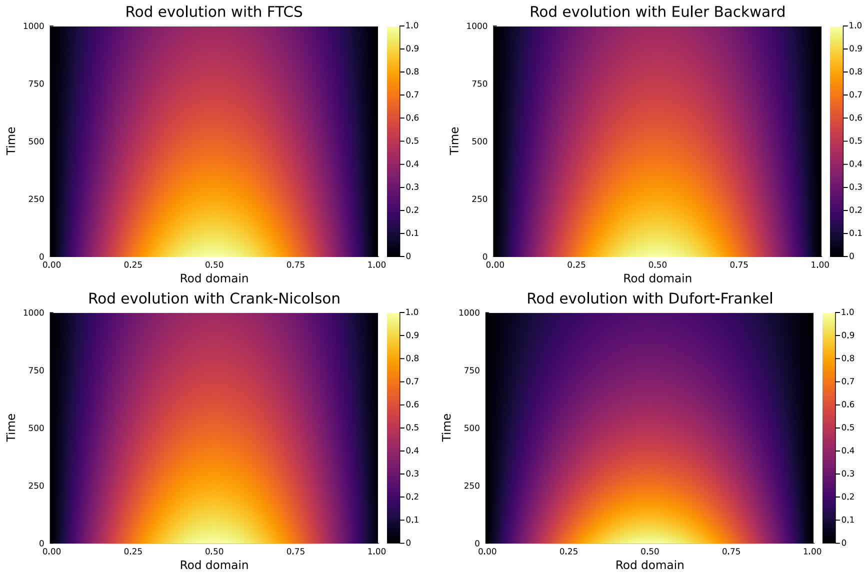


Figure 6: Temporal evolution of the rod's temperature along the y-axis. The evolution seems to be the same for all four integration methods.

2.2 Error comparison

To investigate the stability of the different method, I plotted the error against the analytical solution for different time step widths. I expect unconditionally stability for the implicit methods Euler Backwards and Crank-Nicolson and the Dufort-Frankel method. The FCTS method is not unconditionally stable and should result in a huge error at some large enough Δt, depending on Δx ($\Delta x = 0.01$). However also the other methods haev a time step dependent truncation error. I displayed their evolution individually in Figure 8. For comparing the other erros I will cut off the instability of the FCTS scheme at an error of $\epsilon = 0.007$. I observe that the other schemes seem stable in the tested regime.

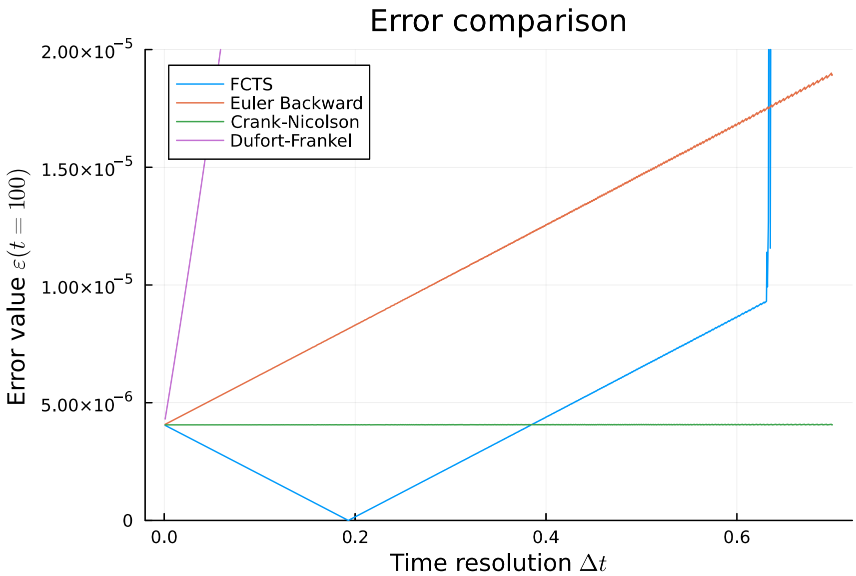


Figure 7: Error comparison for the different methods

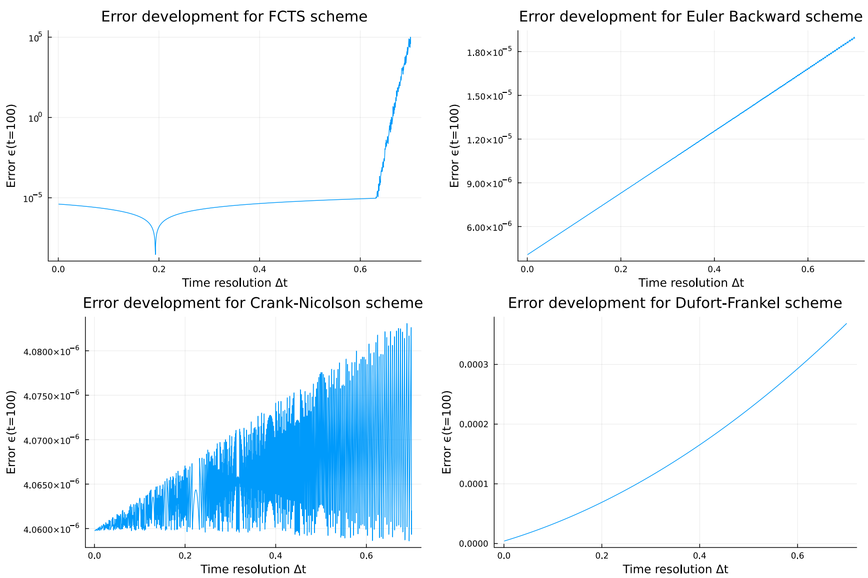


Figure 8

3 Solitons

3.1 Numerical method

We write $u(x,t) \rightarrow u_j^n(j\Delta x, n\Delta t) = U_j^n$. For the time derivative we have

$$\frac{\partial u}{\partial t} = \frac{u_{j-1}^n - u_{j+1}^n}{2\Delta t} + \mathcal{O}(\Delta t^2). \tag{1}$$

For the first space derivative we have

$$\frac{\partial u}{\partial x} = \frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2). \tag{2}$$

To get the third order space derivative I expand $u(x,t)$ around $(x \pm \Delta x, t)$ and $(x \pm 2\Delta x, t)$. With $\frac{\partial u}{\partial x} = u_x$ this reads as follows

$$u(x \pm \Delta x, t) = u \pm u_x \Delta x + u_{xx}\frac{\Delta x^2}{2} \pm u_{xxx}\frac{\Delta x^3}{6} + u_{xxxx}\frac{\Delta x^4}{24} + \mathcal{O}(\Delta x^5)$$
$$u(x \pm 2\Delta x, t) = u \pm u_x 2\Delta x + u_{xx}2\Delta x^2 \pm u_{xxx}\frac{4\Delta x^3}{3} + u_{xxxx}\frac{2\Delta x^4}{3} + \mathcal{O}(\Delta x^5)$$

Rewrite $u(x \pm \Delta x, t) = u_{j\pm1}^n$ and $u(x \pm 2\Delta x, t) = u_{j\pm2}^n$ and calculate

$$\begin{aligned}
&u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n \\
&= u + u_x 2\Delta x + u_{xx}2\Delta x^2 + u_{xxx}\frac{4\Delta x^3}{3} + u_{xxxx}\frac{2\Delta x^4}{3} + \mathcal{O}(\Delta x^5) \\
&\quad - 2u - 2u_x\Delta x - 2u_{xx}\frac{\Delta x^2}{2} - 2u_{xxx}\frac{\Delta x^3}{6} - 2u_{xxxx}\frac{\Delta x^4}{24} - 2\mathcal{O}(\Delta x^5) \\
&\quad + 2u - 2u_x\Delta x + 2u_{xx}\frac{\Delta x^2}{2} - 2u_{xxx}\frac{\Delta x^3}{6} + 2u_{xxxx}\frac{\Delta x^4}{24} + 2\mathcal{O}(\Delta x^5) \\
&\quad - u + u_x 2\Delta x - u_{xx}2\Delta x^2 + u_{xxx}\frac{4\Delta x^3}{3} - u_{xxxx}\frac{2\Delta x^4}{3} - \mathcal{O}(\Delta x^5) \\
&= 2u_{xxx}\frac{4}{3}\Delta x^3 - 4u_{xxx}\frac{1}{6}\Delta x^3 \\
&= u_{xxx}\left(\frac{8-2}{3}\Delta x^3\right) \\
&\Leftrightarrow \frac{\partial^3 u}{\partial x^3} = \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3}
\end{aligned}$$

The first term $u(x,t)$ is approximated by

$$u(x,t) = \frac{u_{j-1}^n + u_j^n + u_{j+1}^n}{3}$$

Now we can put the four terms together to finally get

$$\frac{u_j^{n-1} - u_j^{n+1}}{2\Delta t} + \epsilon \frac{u_{j-1}^n + u_j^n + u_{j+1}^n}{3} \frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} + \mu \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n}{2\Delta x^3} = 0$$

$$\begin{aligned} u_j^{n+1} = u_j^{n-1} &- \frac{\epsilon}{3} \frac{\Delta t}{\Delta x} [u_{j-1}^n + u_j^n + u_{j+1}^n] [u_{j-1}^n - u_{j+1}^n] \\ &- \mu \frac{\Delta t}{\Delta x^3} [u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n] \end{aligned} \tag{3}$$

3.2 Simulation of solitons

I did an animation, which can be found in the git repo in the folder anim. When only plotting every 250th time step I get Figure 9. I observe that in this time range 4 solitary waves form. If I run the simulation longer, I get more and more waves. As the new waves are smaler at first, they should move slower as we now from the analytical solution (amplitude $\propto$ velocity). This is hard to observe in this representation, but could hold true. If we focus at time step 1001 and 2000 we can verify that the distance between the first amplitude and the second indeed grew.

3.2.1 Colliding solitons

I again have a simulation of this in mit git repo. Plotting every 1500th time step is shown in Figure 10. We observe that after collision the smaller wave suddenly moves in the same direction as the bigger wave. The bigger waves moves in the same direction as before but slower.

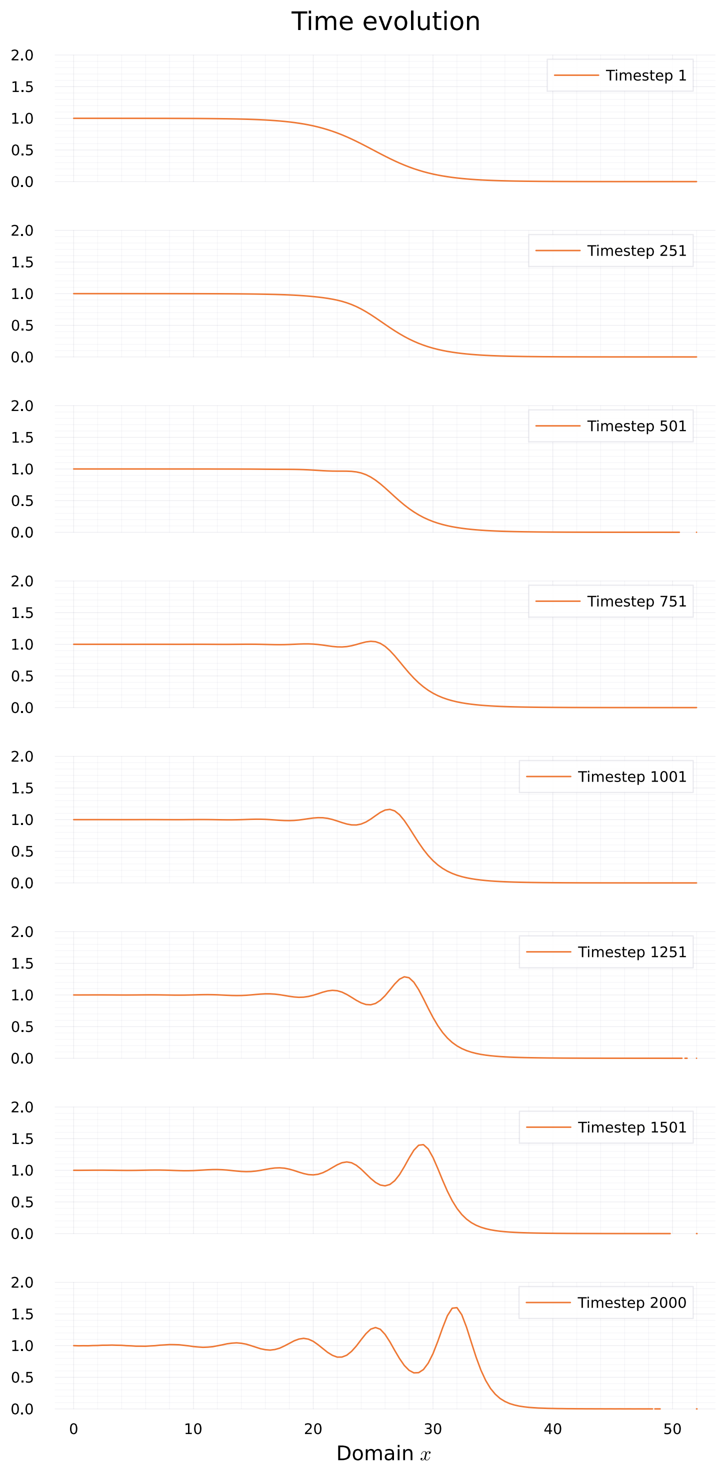


Figure 9: Time evolution beginning with one solitary wave, which splits in multiple.

Figure 10: Time evolution of two colliding solitary waves.