2. A Formal Framework

1. DISCUSSION

Three associated objects occupied the center of the preliminary survey

> E, the environment of the system undergoing adaptation,
> τ, the adaptive plan which determines successive structural modifications in response to the environment,
> μ, a measure of the performance of different structures in the environment.

Implicit in the discussion is a decomposition of the overall process into two disjoint parts—the adaptive system employing τ, and its environment E. This decomposition is usually fixed or strongly suggested by the particular emphasis of each study, but occasionally it can be arbitrary and, rarely, it can be a source of difficulty. Thus, in some biological studies the epidermis naturally serves as the adaptive system-environment boundary, while in other biological studies we deal with populations which have no fixed spatial boundaries, and in ecological settings the boundary shifts with every change in emphasis. Similarly, the emphasis of the study usually determines what notion of performance is relevant and how it is to be measured to yield μ. Because E, τ, and μ are central and can be regularly identified in problems of adaptation, the formal framework will be built around them.

In the basic formalism the adaptive plan τ will be taken to act at discrete instants of time, $t = 1, 2, 3, \ldots$, rather than continuously. The primary reason for adopting a discrete time-scale is the simpler form it confers on most of the important results. Also this formalism intersects smoothly with extant mathematical theories in several fields of interest where much of the development is based on a discrete time-scale, viz., mathematical economics, sequential sampling theory, the theory of self-reproducing automata, and major portions of population genetics. Where continuity is more appropriate, it is often straightforward to obtain continuous counterparts of definitions and theorems, though in some cases appropriate restatements are full-fledged research problems with the discrete results serving only as guidelines. In any case, the instants of time can be freely reinterpreted in different applications—they may be nanoseconds in one application (e.g., artificial intelligence), centuries in another (e.g., evolutionary theory). The properties and relations established with the formalism remain valid, only their durations will vary. Thus, at the outset, we come upon a major advantage of the formalism: Features or procedures easily observed in one process can be abstracted, set within the framework, and analyzed so that they can be interpreted in other processes where duration of occurrence, or other detail, obscures their role.

As our starting point for constructing the formalism let us take the domain of action of the adaptive plan, the set of *structures* $\mathcal{A}$. At the most abstract level $\mathcal{A}$ will simply be an arbitrary, nonempty set; when the theory is applied, $\mathcal{A}$ will designate the set of structures appropriate to the field of interest. Because the more general parts of the theory are valid for any nonempty set $\mathcal{A}$, we have great latitude in interpreting or applying the notion of structure in particular cases. Stated the other way around, the diversity of objects which can serve as elements of $\mathcal{A}$ assures flexibility in applying the theory. In practice, the elements of $\mathcal{A}$ can be the formal counterparts of objects much more complex than the basic structures (chromosomes, mixes of goods, etc.) of the preliminary survey. They may be sets, sequences, or probability distributions over the basic structures; moreover, portions of the adaptive system's past history may be explicitly represented as part of the structure. Often the basic structures themselves will exhibit additional properties, being presented as compositions of interacting components (chromosomes composed of alleles, programs composed of sets of instructions, etc.). Thus (referring to section 1.4), if the elements of $\mathcal{A}$ are to represent chromosomes with ℓ specified genes, where the ith gene has a set of k_i alleles $A_i = \{a_{i1}, \ldots, a_{ik_i}\}$, then the set of structures becomes the set of all combinations of alleles,

$$\mathcal{A} = A_1 \times A_2 \times \cdots \times A_\ell = \Pi_{i=1}^{\ell} A_i.$$

Finally, the set $\mathcal{A}$ will usually be potential rather than actual. That is, elements become available to the plan only by successive modification (e.g., by rearrangement of components or construction from primitive elements), rather than by selection from an extant set. We will examine all of these possibilities as we go along, noting that relevant elaborations of the elements of $\mathcal{A}$ provide a way of specializing the general parts of the theory for particular applications.

The *adaptive plan* τ produces a sequence of structures, i.e., a trajectory through $\mathcal{A}$, by making successive selections from a set of *operators* Ω. The particular selections made are influenced by information obtained from the environment E, so that the plan τ typically generates different trajectories in different environments. The adaptive system's ability to discriminate among various environments is limited by the range I of stimuli or *signals* it can receive. More formally: Let the structure tried at time t be $\mathcal{A}(t) \in \mathcal{A}$. Then the particular environment E confronting the adaptive system reacts by producing a signal $I(t)$. Different structures may of course be capable of receiving different ranges of signals. That is, if I_A is the range of signals which A can receive, then for $A' \neq A$ it may be that $I_A \neq I_{A'}$. To keep the presentation simple, I is used to designate the *total* range of signals $\cup_{A \in \mathcal{A}} I_A$ receivable by structures in $\mathcal{A}$. The particular information $I(t)$ received by the adaptive system at time t will then be constrained to the subset of signals $I_{\mathcal{A}(t)} \subset I$ which the structure at time t, $\mathcal{A}(t)$, can receive. I may have many components corresponding, say, to different sensors. Thus, referring to the example of section 1.3, I consists of ab components $I_1 \times I_2 \times \cdots \times I_{ab} = \Pi_{i=1}^{ab} I_i$. In this case $I_i = \{0,1\}$ for all i since the ith component of I represents the range of values the ith sensor δ_i

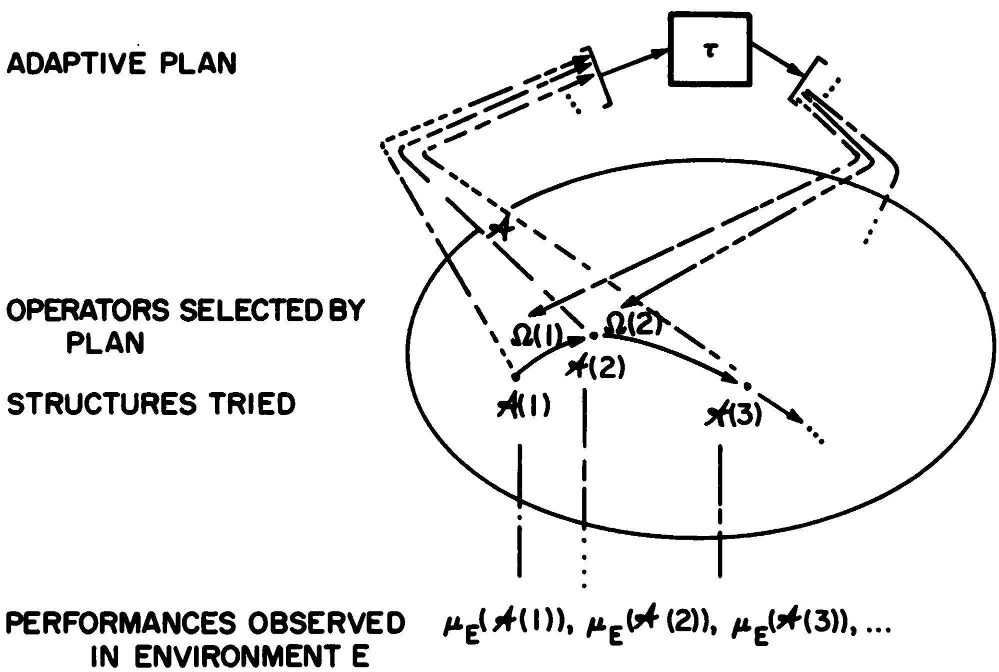

Fig. 2. Schematic of adaptive plan's operation

can transmit. That is, given a particular signal $I(t) \in I$ at time t, the ith component $I_i(t)$ is the value $\delta_i(t)$ of the ith sensor at time t. In general the sets I_i may be quite different, corresponding to different kinds of sensors or sensory modalities.

The formal presentation of an adaptive plan τ can be simplified by requiring that $\mathcal{A}(t)$ serve as the state of the plan at time t. That is, in addition to being the structure tried at time t, $\mathcal{A}(t)$ must summarize whatever accumulated information is to be available to τ. We have just provided that the total information received by τ up to time t is given by the sequence $\langle I(1), I(2), \ldots, I(t-1)\rangle$. Generally only part of this information is retained. To provide for the representation of the retained information we can make use of the latitude in specifying $\mathcal{A}$. Think of $\mathcal{A}$ as consisting of two components $\mathcal{A}_1$ and $\mathfrak{M}$, where $\mathcal{A}_1(t)$ is the structure tested against the environment at time t, and the *memory* $\mathfrak{M}(t)$ represents other retained parts of the *input history* $\langle I(1), I(2), \ldots, I(t-1)\rangle$. Then the plan can be represented by the two-argument function

$$\tau: I \times \mathcal{A} \to \mathcal{A}.$$

Here the structure to be tried at time $t+1$, $\mathcal{A}_1(t+1)$, along with the updated memory $\mathfrak{M}(t+1)$, is given by

$$(\mathcal{A}_1(t+1), \mathfrak{M}(t+1)) = \mathcal{A}(t+1) = \tau(I(t), \mathcal{A}(t)) = \tau(I(t), (\mathcal{A}_1(t), \mathfrak{M}(t))).$$

(The projection of τ on $\mathfrak{M}$,

$$\tau_{\mathfrak{M}}: I \times \mathcal{A}_1 \times \mathfrak{M} \to \mathfrak{M}$$

defined so that

$$\tau_{\mathfrak{M}}(I(t), \mathcal{A}_1(t), \mathfrak{M}(t)) = \text{proj}_2\,[\tau(I(t), \mathcal{A}(t))] = \mathfrak{M}(t+1)$$

is that part of τ which determines how the plan's memory is updated.) It is clear that any theorems or interpretations established for the simple form

$$\tau: I \times \mathcal{A} \to \mathcal{A}$$

can at once be elaborated, without loss of generality or range of application, to the form

$$\tau: I \times (\mathcal{A}_1 \times \mathfrak{M}) \to (\mathcal{A}_1 \times \mathfrak{M}).$$

Thus the framework can be developed in terms of the simple, two-argument form of τ, elaborating it whenever we wish to study the mechanisms of trial selection or memory update in greater detail.

In what follows it will often be convenient to treat the adaptive plan τ as a stochastic process; instead of determining a unique structure $\mathcal{A}(t+1)$ from $I(t)$ and $\mathcal{A}(t)$, τ assigns probabilities to a range of structures and then selects accordingly. That is, given $I(t)$, $\mathcal{A}(t)$ may be transformed into any one of several structures $A'_1, A'_2, \ldots, A'_j, \ldots$, the structure A'_j being selected with probability P'_j. More formally: Let $\mathcal{P}$ be a set of admissible probability distributions over $\mathcal{A}$. Then

$$\tau : I \times \mathcal{A} \rightarrow \mathcal{P}$$

will be interpreted as assigning to each pair $(I(t), \mathcal{A}(t))$ a particular distribution over $\mathcal{A}$, $\mathcal{P}(t+1) \in \mathcal{P}$. The structure $\mathcal{A}(t+1)$ to be tried at time $t+1$ will then be determined by drawing a random sample from $\mathcal{A}$ according to the probability distribution $\mathcal{P}(t+1) = \tau(I(t), \mathcal{A}(t))$. For those cases where the plan τ is to determine the next structure $\mathcal{A}(t+1)$ uniquely, the distribution $\mathcal{P}(t+1)$ simply becomes a degenerate, one-point distribution where a single structure in $\mathcal{A}$ is assigned probability 1. Hence the form

$$\tau : I \times \mathcal{A} \rightarrow \mathcal{P}$$

includes the previous

$$\tau : I \times \mathcal{A} \rightarrow \mathcal{A}$$

as a special case.

In practice the transformation of $\mathcal{A}(t)$ to $\mathcal{A}(t+1)$ is usually accomplished by the application of an operator from some specified set of operators Ω. Thus the detailed operation of the adaptive plan τ is given by a function

$$\tau' : I \times \mathcal{A} \rightarrow \Omega$$

and the set of operators

$$\Omega = \{\omega : \mathcal{A} \rightarrow \mathcal{P}\}$$

where the stochastic aspect is now embodied in the operators. If

$$\omega_t = \tau'(I(t), \mathcal{A}(t))$$

designates the particular operator selected by τ' at time t, then

$$\mathcal{P}(t+1) = \omega_t(\mathcal{A}(t)) = [\tau'(I(t), \mathcal{A}(t))](\mathcal{A}(t))$$

gives the resulting distribution over $\mathcal{A}$. Hence τ' determines τ once the functions in Ω are specified:

$$\tau(I(t), \mathcal{A}(t)) = [\tau'(I(t), \mathcal{A}(t))](\mathcal{A}(t)) = \mathcal{P}(t+1).$$

That is, the range of τ' can be changed from Ω to $\mathcal{P}$ with τ' being redefined so that

$$\tau'(I(t), \mathcal{A}(t)) = [\tau'(i(t), \mathcal{A}(t))](\mathcal{A}(t)) = \mathcal{P}(t+1).$$

With this extension τ' and τ become identical; for this reason one symbol "τ" will be used to designate both functions, the range being specified whenever the distinction is important.

The general objective of this formalism is comparison of adaptive plans, either as hypotheses about natural phenomena or as algorithms for artificial systems. The comparison naturally centers on the efficiency of different plans in locating high performance structures under a variety of environmental conditions. For a comparison to be made there must be a set of plans, given either explicitly or implicitly, which are candidates for comparison. This set will be formally designated $\mathfrak{T}$. Often $\mathfrak{T}$ will be the set of all possible plans employing the operators in Ω, but in some cases there will be constraints restricting $\mathfrak{T}$, while in others $\mathfrak{T}$ will be enlarged to include all possible plans over $\mathcal{A}$ (i.e., all possible functions of the form $\tau: I \times \mathcal{A} \rightarrow \mathcal{P}$). $\mathfrak{T}$, however defined, represents the set of technical or feasible options for the adaptive system under consideration.

As indicated in the survey, a nontrivial problem of adaptation exists only when the adaptive plan is faced with an initial uncertainty about its environment. This uncertainty is formalized by designating the set $\mathcal{E}$ of alternatives corresponding to characteristics of the environment unknown to the adaptive plan. The dependence of the plan's action upon the environment finds its formal counterpart in the dependence of the input $I(t)$ upon which *environment* $E \in \mathcal{E}$ actually confronts the plan. One case of particular importance is that in which the adaptive plan receives a direct indication of the performance of each structure it tries. That is, a part of the input $I(t)$ will be the *payoff* $\mu_E(\mathcal{A}(t))$ determined by the function

$$\mu_E: \mathcal{A} \rightarrow \text{Reals}$$

which measures the performance of each structure in the given environment.

Sometimes, when the performance of a structure in the environment E depends upon random factors, it is useful to treat the utility function as assigning a random variable from some predetermined set $\mathcal{U}$ to each structure in $\mathcal{A}$. Thus

$$\mu_E: \mathcal{A} \rightarrow \mathcal{U}$$

and the payoff assigned to $\mathcal{A}(t)$ is determined by a trial of the random variable $\mu_E(\mathcal{A}(t)) = \mathcal{U}(t)$. This extension does not add any generality to the framework (and hence is unnecessary at the abstract level) because any randomness involved in the interaction between the adaptive system and its environment can be subsumed in the stochastic action of the operators. (See chapter 5 and section 7.2, however.)

Much can be learned about adaptive plans in general by studying plans which act *only* in terms of payoff, plans for which

$$I(t) = \mu_E(\mathcal{A}(t)).$$

In particular, plans which receive information in addition to payoff should do at least as well as plans which receive only payoff information. Thus, the efficiency of payoff-only plans with respect to $\mathcal{E}$ sets a nontrivial lower bound on the efficiency of other plans.

To pose a problem in adaptation unambiguously one more element is required: a *criterion* χ for comparing the efficiency of different plans $\tau \in \mathcal{T}$ under the uncertainty represented by $\mathcal{E}$. Such a criterion must of necessity be fairly sophisticated since it must somehow take into account the varying efficiency of a plan in different environments. Thus, even with a definite measure of efficiency such as the average rate of increase of payoff, there is still the problem of variations across the environments $\mathcal{E}$. How is a plan which is highly efficient only in some subset of $\mathcal{E}$ to be compared with a plan which is moderately efficient in all the environments in $\mathcal{E}$? It should be clear that the plan favored will often depend upon the particular application. In spite of this there are some broadly based criteria which have quite general applicability. The simplest of these requires that a plan accumulate payoff in each $E \in \mathcal{E}$ more rapidly than an enumerative plan which has the same domain of action $\mathcal{A}$. The intuitive content of this criterion is clear: A plan which does not accumulate payoff at least as rapidly as the extremely inefficient enumerative plans should, except in simple situations, be eliminated as a hypothesis (about natural systems) or an algorithm (for artificial systems). Because it is often useful to smooth out short-term variations in judging a plan, several broadly based criteria are stated in terms of the long-term average rate of payoff. When the adaptive plan has the deterministic form $\tau : I \times \mathcal{A} \rightarrow \mathcal{A}$, other, more general criteria are based on the cumulative payoff function

$$U_{\tau,E}(T) = \sum_{t=1}^{T} \mu_E(\mathcal{A}(\tau, t))$$

where $\mathcal{A}(\tau, t)$ is the structure selected by τ in E at time t, $\mu_E(\mathcal{A}(\tau, t))$ is the corresponding payoff, and $U_{\tau,E}(T)$ is the total payoff received by τ in E to time T. (The average rate of payoff is just the function $U_{\tau,E}(T)/T$ based on the cumulative payoff function $U_{\tau,E}(T)$.) When the adaptive plan is stochastic, $\tau : I \times \mathcal{A} \rightarrow \mathcal{P}$, it is natural to substitute the expected payoff under $\mathcal{P}(t)$, $\bar{\mu}_E(\tau, t)$, for $\mu_E(\mathcal{A}(\tau, t))$. (If $\mathcal{A}$ is countable, $\bar{\mu}_E(\tau, t)$ is simply given by $\bar{\mu}_E(\tau, t) = \sum_j \mathcal{P}(A_j, t)\mu_E(A_j)$ where $\mathcal{P}(A_j, t)$ is the probability of selecting $A_j \in \mathcal{A}$ when the distribution over $\mathcal{A}$ is $\mathcal{P}(t)$.) Thus, for stochastic adaptive plans,

$$U_{\tau,E}(T) = \textstyle\sum_{t=1}^{T} \bar{\mu}_E(\tau, t).$$

Following this line, a useful performance target can be formulated in terms of the greatest possible cumulative payoff in the first T time-steps,

$$U_E^*(T) = \operatorname*{lub}_{\tau \in \mathfrak{I}} U_{\tau,E}(T).$$

An important criterion, appearing frequently in the literature of control theory and mathematical economics (see chapter 3, "Illustrations"), can be concisely formulated in terms of U_E^*: τ accumulates payoff at an *asymptotic optimal rate* if

$$\lim_{T\to\infty} [(U_{\tau,E}(T)/T)/(U_E^*(T)/T)] = \lim_{T\to\infty} [U_{\tau,E}(T)/U_E^*(T)] = 1.$$

In other words, the rate at which τ accumulates payoff is, in the limit, the same as the best possible rate. Often it is desirable to have a much stronger criterion setting standards on *interim* behavior. That is, even though the payoff rate approaches the optimum, it may take an intolerably long time before it is reasonably close. Thus, the stronger criterion sets a lower bound on the rate of approach to the optimum. For example, the criterion would designate a sequence $\langle c_T \rangle$ approaching 0 (such as $\langle (k/(T+k))^j \rangle$ or $\langle k/(k+e^{jT}) \rangle$, for $0 < j < \infty$) and then require for all T

$$[U_{\tau,E}(T)/U_E^*(T)] > (1 - c_T).$$

Clearly the plan τ satisfies the asymptotic optimal rate criterion when it satisfies this criterion and, in addition, τ can approach that rate no more slowly than c_T approaches 0.

The simplest way to extend these criteria to $\mathcal{E}$ is to require that a plan $\tau \in \mathfrak{I}$ meet the given criterion in each $E \in \mathcal{E}$.

τ is *robust in* $\mathcal{E}$ *with respect to the asymptotic optimal rate criterion for* $\mathfrak{I}$ when

$$\operatorname*{glb}_{E\in\mathcal{E}} \lim_{T\to\infty} [U_{\tau,E}(T)/U_E^*(T)] = 1.$$

τ is *robust in* $\mathcal{E}$ *with respect to the interim behavior criterion* $\langle c_T \rangle$ *for* $\mathfrak{I}$ when, for all T,

$$\operatorname*{glb}_{E\in\mathcal{E}} [U_{\tau,E}(T)/U_E^*(T)] > (1 - c_T).$$

Each criterion in effect classifies the plans in $\mathfrak{J}$ as "good" or "bad" according to whether or not it is satisfied. The first of these criteria is commonly met in a wide range of applications, while the second proves to be relevant to questions of survival under competition. (Once again, a plan satisfying the second criterion automatically meets the first but not vice versa.) Other criteria can be based on the cumulative payoff function and indeed criteria of a quite different kind can be useful in particular situations. Nevertheless the criteria given are representative and of general use; they will play a prominent role later.

2. PRESENTATION

A problem in adaptation will be said to be well posed once $\mathfrak{J}$, $\mathcal{E}$, and χ have been specified within the foregoing framework. An *adaptive system* is specified within this framework by the set of objects $(\mathcal{A}, \Omega, I, \tau)$ where

$\mathcal{A} = \{A_1, A_2, \ldots\}$ is the set of attainable structures, the domain of action of the adaptive plan,
$\Omega = \{\omega_1, \omega_2, \ldots\}$ is the set of operators for modifying structures with $\omega \in \Omega$ being a function $\omega : \mathcal{A} \to \mathcal{P}$, where $\mathcal{P}$ is some set of probability distributions over $\mathcal{A}$,
I is the set of possible inputs to the system from the environment, and
$\tau : I \times \mathcal{A} \to \Omega$ is the adaptive plan which, on the basis of the input and structure at time t, determines what operator is to be applied at time t.

Under the intended interpretation

$$\tau(I(t), \mathcal{A}(t)) = \omega_t \in \Omega \quad \text{and} \quad \omega_t(\mathcal{A}(t)) = \mathcal{P}(t+1),$$

where $\mathcal{P}(t+1)$ is a particular distribution over $\mathcal{A}$. $\mathcal{A}(t+1)$ is determined by drawing a random sample from $\mathcal{A}$ according to the distribution $\mathcal{P}(t+1)$. Given the input sequence $\langle I(1), I(2), \ldots \rangle$, τ completely determines the stochastic process. (Occasionally, when the adaptive system is to be deterministic with $\mathcal{A}(t+1)$ being uniquely determined once $I(t)$ and $\mathcal{A}(t)$ are given, τ will be defined without the use of operators so that $\tau: I \times \mathcal{A} \to \mathcal{A}$.) The structure of the adaptive system at time t, $\mathcal{A}(t)$, will be required to summarize whatever aspects of the input history are to be available to the plan. Hence it will often be useful to represent $\mathcal{A}$ as $\mathcal{A}_1 \times \mathfrak{M}$, where $\mathcal{A}_1$ is the set of structures to be directly tested and $\mathfrak{M}$ is the set of possible memory configurations, for retaining past history not directly incorporated in the tested structures.

$\mathfrak{J}$ is the set of feasible or possible plans of the form $\tau: I \times \mathcal{A} \to \Omega$ (or $\tau: I \times \mathcal{A} \to \mathcal{A}$) appropriate to the problem being investigated.

$\mathcal{E}$ represents the range of possible environments or, equivalently, the initial uncertainty of the adaptive system about its environment. When the plan τ tries a structure $\mathcal{A}(t) \in \mathcal{A}$ at time t, the particular environment $E \in \mathcal{E}$ confronting the adaptive system signals a response $I(t) \in I$. The performance or payoff $\mu_E(\mathcal{A}(t))$, given by the function $\mu_E: \mathcal{A} \to$ *Reals*, is generally an important part of the information $I(t)$. Given $E \neq E'$ for $E, E' \in \mathcal{E}$, the corresponding functions $\mu_E, \mu_{E'}$ are generally not identical so that a major part of the uncertainty about the environment is just about how well a structure will perform therein. When a plan employs, or receives, only information about payoff so that $I(t) = \mu_E(\mathcal{A}(t))$ it will be called a *payoff-only* plan.

Finally, the various plans in $\mathfrak{J}$ are to be compared over $\mathcal{E}$ according to a criterion χ. Comparisons will often be based on the cumulative payoff functions $U_{\tau,E}(T) = \sum_{t=1}^{T} \bar{\mu}_E(\tau, t)$, where $\bar{\mu}_E(\tau, t)$ is the expected payoff under $\mathcal{P}(t)$, and the "target" function $U_E^*(T) = \operatorname{lub}_{\tau \in \mathfrak{J}} U_{\tau,E}(T)$. An *interim behavior* criterion, based on a selected sequence $\langle c_T \rangle \to 0$ and of the form

$$\operatorname*{glb}_{E \in \mathcal{E}} [U_{\tau,E}(T)/U_E^*(T)] > (1 - c_T),$$

will be important in the sequel.

With the help of this framework each of the fundamental questions about adaptation posed in chapter 1, section 1, can be translated into a formal counterpart:

Original	*Formal*
To what parts of its environment is the organism (system, organization) adapting?	What is $\mathcal{E}$?
How does the environment act upon the adapting organism (system, organization)?	What is I?
What structures are undergoing adaptation?	What is $\mathcal{A}$?
What are the mechanisms of adaptation?	What is Ω?
What part of the history of its interaction with the environment does the organism (system, organization) retain in addition to that summarized in the structure tested?	What is $\mathcal{M}$?
What limits are there to the adaptive process?	What is $\mathfrak{J}$?
How are different (hypotheses about) adaptive processes to be compared?	What is χ?

3. COMPARISON WITH THE DUBINS-SAVAGE FORMALIZATION OF THE GAMBLER'S PROBLEM

> . . . much of the mathematical essence of a theory of gambling consists of the discovery and demonstration of sharp inequalities for stochastic processes . . . this theory is closely akin to dynamic programming and Bayesian statistics. In the reviewer's opinion, [*How to Gamble If You Must*] is one of the most original books published since World War II.
>
> M. Iosifescu. *Math. Rev. 38*, 5, Review 5276 (1969).

For those who have read, or can be induced to read, Dubins and Savage's influential book, this section (which requires special knowledge not essential for subsequent development) shows how to translate their formulation of the abstract gambler's problem to the present framework and vice versa. Briefly, their formulation is based on a progression of *fortunes* $f_0, f_1, f_2, \ldots$ which the gambler attains by a sequence of gambles. A *gamble* is naturally given as a probability distribution over the set of all possible fortunes F. The gambler's range of choice at any time t depends directly and only upon his current fortune f_t so that, as Dubins and Savage remark, the word "state" might be more appropriate than "fortune." The gambler's range of choice for each fortune f is dictated by the *gambling house* Γ. The *strategy* σ for confronting the house is a function which at each time t selects a gamble in Γ on the basis of the sequence or *partial history* of fortunes to that time $(f_0, f_1, \ldots, f_t)$. Finally the *utility* of a given fortune f to the gambler is specified by a utility function u. Thus an abstract gambler's problem is well posed when the objects (F, Γ, u) have been specified; the gambler's response to the problem is given by his strategy σ.

The objects of the Dubins-Savage framework can be put in a one-to-one correspondence with formally equivalent objects in the present framework. With the help of this correspondence any theorem proved in one framework can automatically be translated to a statement which is a valid theorem in the other framework. The relation between the intended interpretations of corresponding objects is in itself enlightening, but the real advantage accrues from the ability to transfer results from one framework to the other with a guarantee of validity.

The following table presents the formal correspondence with an indication of the intended interpretation of each formal object. In this table the superscript "*" on a set will indicate the set of all *finite* sequences (or strings) which can be formed from that set; thus F^* is the set of all partial histories.

Dubins-Savage	*Adaptive Systems*
F, *fortunes*	$\mathcal{A}_1$, basic structures (see τ below).
γ, a probability distribution over fortunes or a *gamble*.	P, a probability distribution over structures, i.e., $P \in \mathcal{P}$.
Γ, a function assigning a set of gambles to each $f \in F$, the *house*.	The (induced) function which assigns to each $A \in \mathcal{A}$ the set of distributions $\mathcal{P}_A = \{\omega(A), \omega \in \Omega\}$.
$\sigma: F^* \to \{\gamma\}$, a *strategy* which assigns to each *partial history* $p \in F^*$ a gamble $\Gamma(f)$, where f is the latest fortune in the sequence p.	$\tau: \mathcal{A} \to \mathcal{P}$, an *adaptive plan*; τ uses only the *retained history* $\mathfrak{M}$ in $\mathcal{A} = \mathcal{A}_1 \times \mathfrak{M}$, but τ has the same generality as σ if $\mathcal{A}_1 = F$ and $\mathfrak{M} = F^*$.
$u: F \to$ Reals, *utility*.	$\mu_E: \mathcal{A} \to$ Reals, performance.

As implied by their terminology, Dubins and Savage treat situations wherein the *expectation* for any strategy σ, given an initial fortune F, is less than F. That is, the strategies are operating in environments wherein continued operation makes degraded performance ever more likely. (This is similar to adaptation in an environment having only nonreplaceable resources, so that performance can only decline in the long run.) In contrast, the present work is primarily concerned with complex environments wherein performance can be permanently improved, if only the right information can be acquired and exploited. Despite the differences, or more likely because of them, theorems from one framework have interesting, and sometimes surprising, translations in the other framework.