Santa Fe Institute
2016 Complex Systems Summer School

Exploring Nonlinear Dynamics

1 Logistic Map: Cobweb Plots and the Time Domain

In this exercise, you'll be exploring the dynamics of the logistic map, which was covered in the lecture Monday:

$$x_{k+1} = f(x_k) = rx_k(1 - x_k) \tag{1}$$

Point your browser to:

`http://tuvalu.santafe.edu/~jgarland/LogisticTools.html`

For now, we will be using the first section of this page which has two plots: the time series on the right and the "cobweb plot" on the left, which plots x_{k+1} versus x_k. Here is a brief explanation about the controls of this application.

Cobweb and Time Domain Plot Controls:

Inputs:

The "Initial Condition x_0" text field changes the point from which the trajectory begins. This input is only well defined between 0 and 1. The "Parameter (r)" text field changes the logistic map's r parameter. Alternatively, you can also change r by clicking your mouse on the cobweb plot. The top of the parabola–defined by Equation 1—will be moved to the point you clicked, effectively adjusting the r parameter. The r parameter currently selected will appear in the cobweb plot's legend, as well as in the "Parameter (r)" text field. The "Number of Initial Iterates" field changes how many iterates of the logistic map are plotted initially, equivalently, how long the initial trajectory is.

Plot Controls: After manually changing the text fields press "Restart Simulation" to update the plots. You can add single iterates to the end of the trajectory or remove a single iterate from the beginning of the trajectory using the respective buttons; these two buttons can be helpful for understanding the mechanics of the cobweb plot and for removing transient behavior. Finally, you can animate both the cobweb plot and time series by clicking “Start Animation”. This animation process is stopped by clicking “Stop Animation”.

1.1 Exercises

(1) Begin by building some intuition about the cobweb plot and its correspondence to the time domain. Start by creating a zero-point trajectory from $x_0 = 0.2$ $r = 3.5$. (Remember to click “Restart Simulation” after changing text fields to update the simulation.) Click the “Add Iterate” button, this will add a single iteration of the logistic map to the cobweb plot and time series. Continue adding iterates one at a time until you feel comfortable with how iterates are added to a cobweb plot and the relationship between the cobweb and time series plot.

(2) As you may have just discovered these cobweb plots can get very cluttered. Sometimes initial behavior—known as the “transient behavior”—can hide what is really going on. Create a 50-point trajectory from $x_0 = 0.2$ with $r = 3.5$. Here the transient behavior is cluttering the cobweb plot. Clean up the cobweb, by clicking “Remove Iterate” until the dynamics in both the cobweb plot and time domain are clear.

(3) Generate a ten-point trajectory of the logistic map with parameter value $r = 2.5$. What kind of dynamics is this? Now change $r = 2.99$. What kind of dynamics is this? What if you increase the number of iterates—do the dynamics look the same? This question can be explored by (a) manually adjusting the “Number of Initial Iterates” using the text field, (b) adding and removing single iterates, or (c) by simply clicking “Start Animation”. Vary the initial condition $x_0 \in [0, 1]$. Does this change the dynamics? What is happening here?

(4) Generate a 50-point trajectory of the logistic map starting at $x_0 = 0.2$ using parameter value $r = 3.68725$. What kind of dynamics is this? What if you click “Start Animation” and watch for a while, does your conclusion change? What is the take away here?

(5) Set $r = 3.828$ and plot 50 iterates, now click ”Start Animation”. The cobweb plot will be a mess, but the time-domain plot should show some interesting

patterns. Raise r slowly to 3.8285. For $r \in (3.828, 3.8285)$ the dynamics are very deceiving—be patient! Describe & explain what you see, if you don't see anything interesting you aren't being patient enough. How does this relate or not relate to problem 4?

(6) Find a two-cycle and the onset of chaos. Now find an n-cycle, where n is as large as you can find. Post your r and n values to the CSSS wiki for comparison and discussion.

(7) There appears to be a period 15 orbit at $r = 3.8521738$. What do you think? Can you verify or disprove this?

2 Exploration of Map Dynamics with Bifurcation Diagrams

Point your browser to:

`http://tuvalu.santafe.edu/~jgarland/LogisticTools.html`

On this page scroll down to "Logistic Map Bifurcation Exploration".

2.1 Bifurcation Diagrams

An effective way to explore a map's parameter space is through the use of a *bifurcation diagram*. A bifurcation diagram shows the behavior of a map's iterates x_k as the parameter r is changed. To explore the Logistic Map bifurcation we will use the second application on this page.

2.2 Exercises

(1) Starting with initial condition $x_0 = 0.5$, skip plotting the first 100 iterations (this will remove the transient behavior), then plot the next 200 iterations for each r value. Press "Update Plot". The blue points you see in the plot window make up the so-called "bifurcation diagram" for the logistic map. The horizontal axis ranges over the r parameter values, from r_{min} to r_{max} and should be a subset of [0,4]. The vertical axis is the range of the iterates of the logistic map,

ranging from $[x_{min}, x_{max}]$ and should be a subset of [0,1]. This diagram will allow you to explore a large range of r simultaneously.

(2) Use the bifurcation diagram to further investigate question (5) from Section 1.1. Set "r_{min}" to 3.5 and "r_{max}" to 3.9. Then press "Update Plot". This will zoom in on the x-axis around the interval $[3.5, 3.9]$ allowing you to explore this subset of parameters in more detail. Continue to zoom in around [3.8, 3.875], this can be done by manually changing the text fields or by selecting a region of the bifurcation diagram with your mouse and pressing "Update Plot". Note, after zooming you may need to increase the density of points on the plot to have better resolution. This can be accomplished by clicking "darken". What is happening in this region? Does this match up with your explanation from question (5) in Section 1.1? Is this the same as question (4)?

(3)Set "r_{min}" to 3.8 and "r_{max}" to 3.875. Then press "Update Plot". This will zoom in on the x-axis around the interval $[3.8, 3.875]$ allowing you to explore this subset of parameters in more detail. On the left hand side of the bifurcation diagram there will be 3 vertical lines of missing pixels. What is happening here? Is this a side effect of the computational limits or your computer or is something interesting happening here with the dynamics? After you decide what is happening test your hypothesis by zooming in to one of these regions. Use your mouse to select one of these regions and zoom in by clicking update plot after each region selection. Remember that you may need to darken the plot to see what is going on in these small regions. When you are satisfied with how deep you have zoomed you can click the "Back" button in the control panel (not in the browser!) several times which will zoom out and leave boxes showing you the location of each zoom stage. This will give you some perspective on where you were in parameter space.

(4) Setting $r = 3.8521738$ results in a period 15 orbit, whereas setting $r = 3.85$ is period 30. But wait! Shouldn't we experience period doubling bifurcations as r *increases* not decreases? How can lowering r double the period? If this bothers you, use a combination of this application and the previous application to explore this.

(5) Use the zoom feature to investigate other windows of r values. Do you see any interesting structure?

If you have more time after completing the next section continue to play around with these plots. If you have any questions please call one of us over.

3 Pendulum

In this problem, you'll explore the damped, sinusoidally driven pendulum. This system is governed by the following system of ODEs:

$$\begin{aligned} mL\dot{\omega} &= -\beta\omega - mg\sin(\theta) - A\sin(\Omega t) \\ \dot{\theta} &= \omega \end{aligned}$$

In these equations, θ is angular position, ω is angular velocity, m is mass, L is pendulum length, g is acceleration due to gravity, β is the drag[1] coefficient, A is the (drive) amplitude and Ω is the (drive) frequency. For simplicity we fix $m = L = 1$:

$$\begin{aligned} \dot{\omega} &= -\beta\omega - g\sin(\theta) - A\sin(\Omega t) \\ \dot{\theta} &= \omega \end{aligned}$$

Now point your browser to:

`http://tuvalu.santafe.edu/~jgarland/pendulum.html`

To use this application, which captures the dynamics of the driven, damped pendulum, set the parameter values and initial conditions in the text fields located in the bottom-right panel, then click "Setup". This should be done any time you change the text field values. To begin the simulation you just setup, either click "Start Animation", which will generate a trajectory one step at a time, or simply click "Generate Full Trajectory" which will generate an n-long trajectory, where n is the "Max Trajectory Length" input.

The Time panel (bottom left) shows time-series plots of θ (angular position, shown in yellow) and ω (angular velocity, shown in blue). The Phase panel (top right) shows a state-space portrait, with ω on the y-axis and θ on the x-axis.

(a) Set the drive amplitude(A) and frequency(Ω) to 0 and the drag coefficient (β) to 0.1. Start the pendulum at $[\theta_0] = [\frac{\pi}{4}] = 0.78539$, and $[\omega_0] = 0$, then click "Setup" and then "Start Animation" and watch the pendulum in the top left

[1] This is also referred to commonly as the damping coefficient.

pane. What kind of attractor is this? Repeat this, but instead of watching the pendulum, watch the Time and Phase panels. Repeat for different initial conditions. What kind of attractor is this?

(b) Now set the drive frequency to 0.667 and drag coefficient to 0.5. Raise the amplitude slowly from zero to 1.5 and describe what you see. Examine both the Phase panel as well as the pendulum! Be sure to try 0.2, 0.9, 1.07, 1.15, 1.35, 1.45, and 1.47 during your explorations. Remember that transient behavior can be deceiving, so you should allow enough time for the dynamics to settle down.

Homework

- Let b_i be the i^{th} value of the parameter r for which a bifurcation occurs in the first period doubling cascade of the logistic map. The R value for which the dynamics changes from a fixed point to a periodic orbit is b_1, for example; the r value where the logistic map switches from period 2 to period 4 is b_2, and so on.

  The Feigenbaum number δ is defined as

  $$\lim_{k\to\infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k} = \delta.$$

  Use the "Bifurcation Diagram" application from Section 2 to estimate the Feigenbaum number.

- (for experts) Code up the logistic map yourself and plot x_n vs n for some of the interesting r values that you explored in problem 1.

- (for experts) Generate a bifurcation diagram of the logistic map for $1 < r < 4$. Obviously here do not use the "Bifurcation Diagram" application.

- Parameter values that produce a few periodic orbits are coded into the "Predefined Dynamics" dropdown menu in the pendulum application. Select some of these and explore the state space. Are all initial conditions attracted to these periodic orbits? Can you find higher period orbits?