# A Law of Iterated Logarithm for Reflected Fractional Brownian Motion

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# Introduction

# Gaussian Process

The stochastic process $\{X(t),\ t \in \mathbb{T}\}$ is called a Gaussian process if for any $m \geqslant 1$ and any points $\{t_1, ..., t_m\} \subset \mathbb{T}$, the vector $(X(t_1), ..., X(t_m))$ follows a multivariate normal distribution.

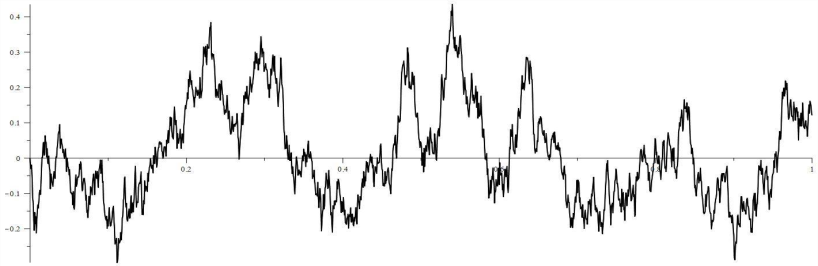
# Brownain Motion

Let $B = \{B(t) : t \in \mathbb{R}_+\}$ be a standard Brownain motion, which satisfied the following properties.

(i) $B(0) = 0$ almost surely;

(ii) $B$ has stationary independent increments and $B(t) - B(s) \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$;

(iii) almost surely, the paths of $B$ are continuous.



# Fractional Brownian Motion

Let $B_H = \{B_H(t) : t \in \mathbb{R}_+\}$ be a standard fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, which defined as a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_H(t), B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

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# Fractional Brownian Motion

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Note that
(i) if $H = \frac{1}{2}$, then $B_H$ is the standard Brownian motion $B$;
(ii) fBm is selfsimilar, i.e., $\{B_H(t),\ t \in \mathbb{R}_+\}$ and $\{s^{-H} B_H(st),\ t \in \mathbb{R}_+\}$ have the sane distribution for any $s > 0$.

# Reflected Fractional Brownian Motion

Consider a reflected (at 0) fractional Brownian motion (rfBm) with drift $Q_{B_H} = \{B_H(t) : t \in \mathbb{R}_+\}$, given by the following formula

$$Q_{B_H}(t) = B_H(t) - ct + \max\left(Q_{B_H}(0), -\inf_{s\in[0,t]}(B_H(s) - cs)\right),$$

where $c > 0$. To simplify the notation, we assume $c = 1$ in this talk.

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Note that if $Q_{B_H}(0) = 0$, then

$$Q_{B_H}(t) = B_H(t) - ct - \inf_{s\in[0,t]}(B_H(s) - cs) = \sup_{-\infty<s\leq t}(B_H(t) - B_H(s) - c(t-s))$$

# Literature Review

# Literature Review

| Form | $\frac{B(t)}{\sqrt{2t \ln\ln t}}$ | $\frac{\max_{0\le s\le t}\lvert B(s)\rvert}{\sqrt{2t \ln\ln t}}$ | $\frac{\max_{0\le s\le t}\lvert B(s)\rvert}{\sqrt{t/\ln\ln t}}$ |
|---|---|---|---|
| $\limsup_{t\nearrow\infty}$ | 1 | 1 | $\times$ |
| $\liminf_{t\nearrow\infty}$ | -1 | $\times$ | $\frac{\pi}{\sqrt{8}}$ |
| Form | $\frac{B_H(t)}{\sqrt{2t^{2H}\ln\ln\frac{1}{t}}}$ | $\frac{\max_{0\le s\le t}\lvert B_H(s)\rvert}{\sqrt{2t^{2H}\ln\ln\frac{1}{t}}}$ | $\frac{\max_{0\le s\le t}\lvert B_H(s)\rvert}{\sqrt{(2t/\ln\ln t)^{2H}}}$ |
| $\limsup_{t\nearrow 0}$ | 1 | 1 | $\times$ |
| $\liminf_{t\nearrow 0}$ | -1 | $\times$ | $c_\alpha$ |
| Form | $\frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}$ | $\frac{\max_{0\le s\le t} Q_{B_H}(s)}{g(t)}$ | $\frac{\max_{0\le s\le t} Q_{B_H}(s)}{h(t)}$ |
| $\limsup_{t\nearrow\infty}$ | 1 | ? | $\times$ |
| $\liminf_{t\nearrow\infty}$ | 0 | $\times$ | ? |

# LIL for $B(t)$

The Classic Law of Iterated Logarithm (LIL) is used to describe the the limit behavior of the fluctuations of standard Brownian motion, which has the form

$$\limsup_{t\to 0} \frac{B(t)}{\sqrt{2t \ln \ln \frac{1}{t}}} = 1 \text{ and } \limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t \ln \ln t}} = 1 \quad \text{a.s.}$$

Note that the denominator in the above equation is the maximum height of the fluctuation of $B$ above 0 as $t \to \infty$.

By the reflection principle for Brownian motion, we can easily acquire that

$$\liminf_{t\to 0} \frac{B(t)}{\sqrt{2t \ln \ln \frac{1}{t}}} = -1 \text{ and } \liminf_{t\to\infty} \frac{B(t)}{\sqrt{2t \ln \ln t}} = -1 \quad \text{a.s.}$$

table

# LIL for $\max_{0\leq s\leq t}|B(s)|$

Next consider the cumulative maximum process of absolute Brownian motion. The $\liminf_{t\to 0}$ version was proved by Chung [1948] and we can derivate the $\limsup_{t\to 0}$ result.

$$\limsup_{t\to 0}\frac{\max\limits_{s\leqslant t}|B(s)|}{\sqrt{2t\ln\ln\frac{1}{t}}}=1 \text{ and } \liminf_{t\to 0}\frac{\max\limits_{s\leqslant t}|B(s)|}{\sqrt{t/\ln\ln t}}=\frac{\pi}{\sqrt{8}} \quad \text{a.s.}$$

table

# LIL for $\max_{0\leq s\leq t}|B_H(s)|$

Next consider the cumulative maximum process of $|B_H|$. The $\limsup_{t\to 0}$ result is shown in Li and Shao [2001] and the $\liminf_{t\to 0}$ result is proved by Monrad and Rootzén [1995].

$$\limsup_{t\to 0}\frac{\max\limits_{s\leqslant t}|B_H(t)|}{\sqrt{2t^{2H}\ln\ln\frac{1}{t}}}=1 \text{ and } \liminf_{t\to 0}\frac{\max\limits_{s\leqslant t}|B_H(t)|}{t^H/(\ln\ln t)^H}=c_\alpha \quad \text{a.s.,}$$

where $c_\alpha$ is a positive constant.

table

# LIL for $Q_{B_H}(t)$

Recently, the LIL for rfBm is given in Dębicki and Kosiński [2017] without a clear proof.

*Theorem 1.*

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} = 1 \quad \text{a.s.},$$

*where* $A = \frac{1}{1-H}\left(\frac{H}{1-H}\right)^{-H}$.

# The Proof of LIL for Reflected Fraction Brownian Motion

**Proof of Theorem 1.**

The proof can be separated into two parts, which is

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \geqslant 1\right) = 1 \tag{1}$$

and

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \leqslant 1\right) = 1. \tag{2}$$

If we can prove that $\mathbb{P}\left(Q_{B_H}(t) < f(t) \quad \text{i.o.}\right) = 0$, then (1) will hold. Similarly, (2) will hold if we can have $\mathbb{P}\left(Q_{B_H}(t) < f(t) \quad \text{i.o.}\right) = 1$.

Therefore, we need the following two Theorems.

*Theorem 2. (Dębicki and Kosiński [2017]) For all positive and nondecreasing functions $f(t)$ on some interval $[T,\infty)$,*

$$\mathbb{P}(Q_{B_H}(t) > f(t) \quad \text{i.o.}) = 0 \quad \text{or} \quad 1,$$

*according as the integral*

$$\mathcal{I}_f := \int_T^{\infty} \frac{1}{f(u)} \mathbb{P}\left( \sup_{t\in[0,f(u)]} Q_{B_H}(t) > f(u) \right) du$$

*is finite or infinite.*

*Theorem 3. (Piterbarg [2001]) For any $H \in (0,1)$, as $u \to \infty$*

$$\mathbb{P}\left(\sup_{[0,Tu]} Q_{B_H}(t) > u\right) = \sqrt{\pi} a^{\frac{2}{H}} b^{-\frac{1}{2}} T \mathcal{H}^2_{B_H} (Au^{1-H})^{\frac{2}{H}-1} \Psi(Au^{1-H})(1+o(1)), \tag{3}$$

*where*

$$\tau_0 = \frac{H}{1-H}, a = \frac{1}{2\tau_0^{2H}}, B = H\left(\frac{H}{1-H}\right)^{-H-2}, b = \frac{B}{2A},$$

$$\mathcal{H}_{B_H} = \lim_{T\to\infty} T^{-1} \mathbb{E} \exp\left(\sup_{t\in[0,T]} (\sqrt{2} B_H(t) - t^{2H})\right) \in (0,\infty)$$

*and* $\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp(-\frac{1}{2}x^2) dx$.

# The Proof of LIL for rfBm

Let $T = 1$ and replace $u$ with $f_p(t)$ in (3), we see that as $u \to \infty$,

$$\mathbb{P}\left(\sup_{[0,f_p(u)]} Q_{B_H(t)} > f_p(u)\right) = \sqrt{\pi} a^{\frac{2}{H}} b^{-\frac{1}{2}} \mathcal{H}_{B_H}^2 (Af_p(u)^{1-H})^{\frac{2}{H}-1} \Psi(Af_p(u)^{1-H})(1+o(1)). \tag{4}$$

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To calculate (4), let

$$g(u) = \ln u + (1 + c_H - p) \ln \ln u \ \text{ and } \ f_p(u) = (2A^{-2} g(u))^{\frac{1}{2(1-H)}},$$

where $c_H = \frac{1}{H} - 1 - \frac{1}{2(1-H)}$.
Notice that for all $p \in \mathbb{R}$, $g(u) \to \infty$ implies $f(u) \to \infty$ as $u \to \infty$.

# The Proof of LIL for rfBm

From (2), there exists $u_1$ large enough such that for all $u > u_1$,

$$I(u) = \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{[0,f_p(u)]} Q_{B_H}(t) > f_p(u)\right) = C\frac{\Psi(\sqrt{2g(u)})}{g(u)^{-(c_H+\frac{1}{2})}}$$

where $C = 2^{c_H+\frac{1}{2}}\sqrt{\pi}A^{\frac{1}{1-H}}a^{\frac{2}{H}}b^{-\frac{1}{2}}\mathcal{H}_{B_H}^2(1+o(1))$.

# The Proof of LIL for rfBm

Since

$$\lim_{t\to\infty} \frac{\Psi\left(\sqrt{2t}\right)}{\frac{1}{\sqrt{4\pi t}}\exp(-t)} \overset{\text{L'H}}{=} \lim_{t\to\infty} \frac{t^{-\frac{1}{2}}}{\frac{1}{2}t^{-\frac{3}{2}} + t^{-\frac{1}{2}}} = 1,$$

there exists $u_2 \geqslant u_1$ large enough such that for $u \geqslant u_2$,

$$\Psi(\sqrt{2t}) = \frac{1}{\sqrt{4\pi t}}\exp(-t)\left(1 + o(1)\right).$$

# The Proof of LIL for rfBm

Recall $g(u) = \ln u + (1 + c_H - p) \ln \ln u$.
For any $p \in \mathbb{R}$ and $u > u_2$,

$$I(u) = C \frac{\Psi(\sqrt{2g(u)})}{g(u)^{-(c_H + \frac{1}{2})}} = \mathcal{C} \frac{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2g(u)}} \exp(-g(u)) (1 + o(1))}{g(u)^{-(c_H + \frac{1}{2})}}$$

$$= \mathcal{C}' \frac{(\ln u + (1 + c_H - p) \ln \ln u)^{c_H}}{u (\ln u)^{1 + c_H - p}}$$

$$\Rightarrow \mathcal{I}_f = \int_T^\infty I(u) du = \infty \,\text{if}\, \begin{cases} H \in (0, \frac{2}{3}] \text{ and takes } p = 1 + c_H; \\ H \in (\frac{2}{3}, 1) \text{ and takes } p = 1; \end{cases}$$

# The Proof of LIL for rfBm

Therefore, by Theorem 2,

$$\mathbb{P}\left(Q_{B_H}(t) > f_p(t) \quad \text{i.o.}\right) = 1 \quad \text{for all } H \in (0,1) \text{ for some } p.$$

$$\Rightarrow \limsup_{t\to\infty} \frac{Q_{B_H}(t)}{f_p(t)} \geqslant 1 \quad \text{a.s.}$$

Since

$$\lim_{t\to\infty} \frac{f_p(t)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} = \lim_{t\to\infty} \frac{\left(\frac{2}{A^2}\left(\ln t + (1 + c_H - p)\ln\ln t\right)\right)^{\frac{1}{2(1-H)}}}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} = 1,$$

we have

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} = \limsup_{t\to\infty} \frac{Q_{B_H}(t)}{f_p(t)} \frac{f_p(t)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} \geqslant 1 \quad \text{a.s.} \tag{5}$$

# The Proof of LIL for rfBm

Similarly,

$$\mathcal{I}_f = \int_T^\infty I(u)du < \infty, \text{ if } \begin{cases} H \in (0, \frac{2}{3}] \text{ and takes } p = -1; \\ H \in (\frac{2}{3}, 1] \text{ and takes } p = 1 + c_H; \end{cases}$$

Also, by Theorem 2,

$$\mathbb{P}\left(Q_{B_H}(t) > f_p(t) \quad \text{i.o.}\right) = 0 \quad \text{for all } H \in (0,1) \text{ for some } p.$$

$$\Rightarrow \limsup_{t\to\infty} \frac{Q_{B_H}(t)}{f_p(t)} \leqslant 1 \quad \text{a.s.}$$

$$\Rightarrow \limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} = \limsup_{t\to\infty} \frac{Q_{B_H}(t)}{f_p(t)} \frac{f_p(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \leqslant 1 \quad \text{a.s.} \quad (6)$$

# The Proof of LIL for rfBm

By (5) and (6), we can finish the proof

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}} = 1 \quad \text{a.s.}$$

$\square$

# Future Work

# Future Work

We are focusing on the LIL for the maximum process of rfBm, which is

$$\limsup_{t\to\infty} \frac{\sup_{s\leqslant t} Q_{B_H}(s)}{g(t)} = 1 \text{ and } \liminf_{t\to\infty} \frac{\sup_{s\leqslant t} Q_{B_H}(s)}{h(t)} = 1 \quad \text{a.s.}$$

Our goal is to find the exact form of $g(t)$ and $h(t)$.

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Thanks for Your Attention