Law of Iterated Logarithm for Extensions of Brownian motion

鍾世民
Simon Chung

Abstract

Let $B = \{B(t) : t \in \mathbb{R}_+\}$ be a standard Brownian motion. For $M(t) = \max\limits_{s \leqslant t} |B(s)|$, we can prove that the Law of Iterated Logarithm (LIL) can be applied to this stochastic process with the same function in LIL for B. Consequently, let $B_H = \{B_H(t) : t \in \mathbb{R}_+\}$ be a fractional Brownian motion and for the stationary storage process $Q_{B_H} = \sup\limits_{-\infty < s \leqslant t} (B_H(t) - B_H(s) - (t - s)), t \geq 0$, after we finish the proof of LIL for Q_{B_H}, we prove that the LIL for $\max\limits_{s \leqslant t} Q_{B_H}(s)$ use the same function as with Q_{B_H}, which is $(2 \ln t / A^2)^{\frac{1}{2(1-H)}}$.

Keywords— Brownian motion ,Fractional Brownian motion, Reflected fractional Brownian motion, Maximum process, Law of iterated logarithm, Borel-Cantelli lemma

1 Introduction and main results

The law of iterated logarithm(LIL) was firstly proposed by Khintchine [1] in 1924 in the purpose of describing the limit behavior of Bernoulli variables. From then on, LIL has been applied to more and more stochastic processes. In this project, we want to expend this study in the field of continuous stochastic processes. We start with standard Brownian motion.

DEFINITION 1. *A real-valued stochastic process* $B = \{B(t) : t \in \mathbb{R}_+\}$ *is a standard Brownian motion if it satisfies the following properties.*

(i) $B(0) = 0$ *almost surely;*

(ii) B *has stationary independent increments and* $B(t) - B(s) \sim \mathcal{N}(0, t - s)$ *for* $0 \leq s < t$*;*

(iii) the paths of B *are continuous almost surely.*

The LIL for standard Brownian motion has been considered as the classic LIL, which appears in many textbooks as the following form

$$\limsup_{t\to\infty} \frac{B(t)}{\sqrt{2t\ln\ln t}} = 1 \quad \text{a.s.} \tag{1}$$

Therefore, we take this question even further by considering the cumulative maximum process of the absolute value of a standard Brownian motion.

THEOREM 1. *Let* $M(t) = \max\limits_{s\leqslant t} |B(s)|$. *Then*

$$\limsup_{t\to\infty} \frac{M(t)}{\sqrt{2t\ln\ln t}} = 1 \quad \text{a.s.}$$

By the definition of $|B(t)|$, this Brownian motion will reflect immediately while it hits 0, which is an easy physics model in reality. However, we consider a relatively difficult physics model which will stay in 0 for some time before bouncing back. But first, we take a look at a more general version of Brownian motion.

DEFINITION 2. *Let $B_H = \{B_H(t) : t \in \mathbb{R}_+\}$ be a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$, which defined as a centered Gaussian process with covariance function*

$$Cov(B_H(t), B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Base on the standard fractional Brownian motion, a morerealistic model has been defined.

DEFINITION 3. *Consider a reflected (at 0) fractional Brownian motion with drift $Q_{B_H} = \{B_H(t) : t \in \mathbb{R}_+\}$, given by the following formula*

$$Q_{B_H}(t) = B_H(t) - ct + \max\left\{Q_{B_H}(0), -\inf_{s\in[0,t]}(B_H(s) - cs)\right\},$$

where $c > 0$.

The LIL for reflected fraction Brownian motion has been given by Dębicki and Kosiński [2] but without a clear proof.

THEOREM 2. *For any $H \in (0,1)$,*

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} = 1 \quad \text{a.s.},$$

where $A = \frac{1}{1-H}(\frac{H}{1-H})^{-H}$.

On the way of Dębicki and Kosiński proving the above theorem, they provided an alternative tool for not calculating the integral instead of summation.

LEMMA 1 (Dębicki and Kosiński (2017, Theorem 1)). *For all positive and nondecreasing functions $f(t)$ on some interval $[T, \infty)$,*

$$\mathbb{P}(Q_{B_H}(t) > f(t) \quad \text{i.o.}) = 0 \quad \text{or} \quad 1,$$

according as the integral

$$\mathcal{I}_f := \int_T^\infty \frac{1}{f(u)} \mathbb{P}\left(\sup_{t\in[0,f(u)]} Q_{B_H}(t) > f(u) \right) du$$

is finite or infinite.

Like what had been done to LIL for Brownian motion and its extension, we also want to apply LIL to the extension of reflected fraction Brownian motion.

THEOREM 3. *For any* $H \in (0,1)$,

$$\limsup_{t\to\infty} \frac{\max\limits_{s\leqslant t} Q_{B_H}(s)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} = 1 \quad \text{a.s.},$$

where $A = \frac{1}{1-H}(\frac{H}{1-H})^{-H}$.

2 Proofs of the main results

Proof of Theorem 1. Let $f(t) = \sqrt{2t \ln \ln t}$, $t_n = \alpha^n$ for any $\alpha > 1$ and $E_n := \left\{ \max\limits_{t_n \leqslant t \leqslant t_{n+1}} \frac{\max_{s \leqslant t} B(s)}{(1+\delta) f(t)} > 1 \right\}$ for any $\delta > 0$.

Then we have

$$\begin{aligned}
\mathbb{P}(E_n) &\leqslant \mathbb{P}\left(\max_{t_n \leqslant t \leqslant t_{n+1}} \frac{\max\limits_{s \leqslant t} B(s)}{(1+\delta) f(t_n)} > 1 \right) \leqslant \mathbb{P}\left(\max_{0 \leqslant t \leqslant t_{n+1}} \frac{\max\limits_{s \leqslant t} B(s)}{(1+\delta) f(t_n)} > 1 \right) \\
&= \mathbb{P}\left(\max_{0 \leqslant t \leqslant t_{n+1}} B(t) > (1+\delta) f(t_n) \right) = 2\mathbb{P}\left(B(t_{n+1}) > (1+\delta) f(t_n) \right) \\
&\leqslant c n^{-\frac{(1+\delta)^2}{\alpha}},
\end{aligned}$$

where the last inequality holds because we already know the distribution of standard Brownian motion and c is a suitable constant. When $\alpha \in (1, (1+\delta)^2)$, by p-series test and comparison theorem, $\sum_{n=1}^{\infty} \mathbb{P}(E_n)$ converges. By Borel-Cantelli Lemma, $P(\limsup\limits_{n \to \infty} E_n) = 0$.

Since

$$\begin{aligned}
\mathbb{P}(\limsup_{n \to \infty} E_n) &= \mathbb{P}\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geqslant k}^{\infty} \left\{ \max_{t_n \leqslant t \leqslant t_{n+1}} \frac{\max\limits_{s \leqslant t} B(s)}{(1+\delta) f(t)} > 1 \right\} \right) \\
&\geqslant \mathbb{P}\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geqslant k}^{\infty} \left\{ \max_{t_n \leqslant t \leqslant t_{n+1}} \frac{\max\limits_{s \leqslant t} B(s)}{(1+\delta) f(t_{n+1})} > 1 \right\} \right) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geqslant k}^{\infty} \left\{ \frac{\max\limits_{0 \leqslant s \leqslant t_{n+1}} B(s)}{f(t_{n+1})} > 1 + \delta \right\} \right) \geqslant 0,
\end{aligned}$$

we have

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} \bigcap_{n \geqslant k}^{\infty} \left\{ \frac{\max\limits_{0 \leqslant s \leqslant t_{n+1}} B(s)}{f(t_{n+1})} \leqslant 1 + \delta \right\} \right) = 1.$$

Moreover,

$$\begin{aligned}
&\mathbb{P}\left(\bigcup_{k=1}^{\infty}\bigcap_{n\geqslant k}^{\infty}\left\{\frac{\max\limits_{0\leqslant s\leqslant t_{n+1}} B(s)}{f(t_{n+1})}\leqslant 1+\delta\right\}\right)\\
&=\mathbb{P}\left(\exists n_0\in\mathbb{N},\text{ s.t. if } n>n_0,\text{ then}\frac{\max\limits_{0\leqslant s\leqslant t_{n+1}} B(s)}{f(t_{n+1})}\leqslant 1+\delta\right)\\
&\leqslant\mathbb{P}\left(\limsup_{n\to\infty}\frac{\max\limits_{0\leqslant s\leqslant t_{n+1}} B(s)}{f(t_{n+1})}\leqslant 1+\delta\right)\leqslant 1.
\end{aligned}$$

The inequality before the last one holds because the greatest cluster point of that fraction is smaller than $1+\delta$, which means all cluster point is smaller than $1+\delta$. And a point is a cluster point if and only if there are infinitely points in its neighbor. Therefore, we have

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{\max\limits_{0\leqslant s\leqslant t_{n+1}} B(s)}{f(t_{n+1})}\leqslant 1+\delta\right)=1.$$

Since δ is abitary, we have probability one that

$$\limsup_{t\to\infty}\frac{\max\limits_{0\leqslant s\leqslant t} B(s)}{f(t)}\leqslant 1. \tag{2}$$

On the other hand,

$$\limsup_{t\to\infty}\frac{\max\limits_{0\leqslant s\leqslant t} B(s)}{f(t)}\geqslant\limsup_{t\to\infty}\frac{B(s)}{f(t)}=1\quad\text{a.s.} \tag{3}$$

By (2) and (3), we can complete the first part of the proof

$$\limsup_{t\to\infty}\frac{\max\limits_{0\leqslant s\leqslant t} B(s)}{f(t)}=1\quad\text{a.s.}$$

Since $|B(t)|=B^{+}(t)\vee B^{-}(t)$, where $a\vee b:=\max\{a,b\}$, we have

$$\max_{0\leqslant s\leqslant t}|B(s)|=\max_{0\leqslant s\leqslant t}\left\{B^{+}(s)\vee B^{-}(s)\right\}=\max_{0\leqslant s\leqslant t}B^{+}(t)\vee\max_{0\leqslant s\leqslant t}B^{-}(t).$$

Moreover,

$$\max_{s\leqslant t} B^+(t) = \max_{s\leqslant t}(B(t)\vee 0) = \max_{s\leqslant t} B(t),$$

which means that

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B^+(s)}{f(t)} = \limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B(s)}{f(t)} = 1 \quad \text{a.s.}$$

By symmetry property, we have

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B^-(s)}{f(t)} = 1 \quad \text{a.s.}$$

Therefore,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} |B(s)|}{f(t)} &= \limsup_{t\to\infty} \left(\frac{\max\limits_{0\leqslant s\leqslant t} B^+(s)}{f(t)} \bigvee \frac{\max\limits_{0\leqslant s\leqslant t} B^-(s)}{f(t)} \right) \\
&\leqslant \left(\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B^+(s)}{f(t)} \right) \bigvee \left(\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B^-(s)}{f(t)} \right) \\
&= 1 \quad \text{a.s.}
\end{aligned}$$

As for another part,

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} |B(s)|}{f(t)} \geqslant \limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} B(s)}{f(t)} = 1.$$

Hence, we can finish the proof with

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} |B(s)|}{f(t)} = 1 \quad \text{a.s.}$$

$\square$

Proof of Theorem 2. Let

$$\tau_0 = \frac{H}{1-H}\ , a = \frac{1}{2\tau_0^{2H}}\ , B = H\left(\frac{H}{1-H}\right)^{-H-2}\ , b = \frac{B}{2A},$$

$$\mathcal{H}_{B_H} = \lim_{T\to\infty} T^{-1}\mathbb{E}\exp\left(\sup_{t\in[0,T]}(\sqrt{2}B_H(t) - t^{2H})\right) \in (0,\infty),$$

$\Psi(u) = \frac{1}{\sqrt{2\pi}}\int_u^\infty \exp(-\frac{1}{2}x^2)dx,\ f_p(s) = \left(\frac{2}{A^2}g(s)\right)^{\frac{1}{2(1-H)}}$ and $g(s) = \ln s + (1 + c_H - p)\ln\ln s$ for $p \in \mathbb{R}$, $c_H = \frac{1}{H} - 1 - \frac{1}{2(1-H)}$ and $H \in (0,1)$

Like the proof of Theorem 1, we can be separated it into two parts, which is

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \geqslant 1 \quad \text{a.s.} \tag{4}$$

and

$$\limsup_{t\to\infty} \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \leqslant 1 \quad \text{a.s..} \tag{5}$$

Since we need to calculate the integral in Lemma 1, we need the tail probability of Q_{B_H}, which has been given by Piterbarg in 2001 [3]. Namely, for any $T > 0$,

$$\mathbb{P}\left(\sup_{[0,Tu]} Q_{B_H}(t) > u\right) = \sqrt{\pi}a^{\frac{2}{H}}b^{-\frac{1}{2}}T\mathcal{H}^2_{B_H}(Au^{1-H})^{\frac{2}{H}-1}\Psi(Au^{1-H})(1+o(1)). \tag{6}$$

Suppose u_1 is large enough such that equation (6) holds when $u > u_1$. Let $T = 1$, we have

$$\begin{aligned} I(u) &:= \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{[0,f_p(u)]} Q_{B_H(t)} > f_p(u)\right) \\ &= \sqrt{\pi}a^{\frac{2}{H}}b^{-\frac{1}{2}}\mathcal{H}^2_{B_H}(Af_p(u)^{1-H})^{\frac{2}{H}-1}\Psi(Af_p(u)^{1-H})f_p(u)^{-1}(1+o(1)) \\ &= \kappa\frac{\int_{\sqrt{2g(u)}}^\infty \exp\left(-\frac{1}{2}x^2\right)dx}{g(u)^{-(c_H+\frac{1}{2})}}, \end{aligned}$$

where $\kappa = a^{\frac{2}{H}}b^{-\frac{1}{2}}\mathcal{H}^2_{B_H}2^{c_H}A^{-\frac{1}{2(1-H))}}(1+o(1))$.

Since

$$\lim_{t\to\infty}\frac{\Psi\left(\sqrt{2t}\right)}{\frac{1}{\sqrt{4\pi t}}\exp(-t)}=\lim_{t\to\infty}\frac{t^{-\frac{1}{2}}}{\frac{1}{2}t^{-\frac{3}{2}}+t^{-\frac{1}{2}}}=1,$$

there exists $u_2 \geqslant u_1$ large enough such that for $u \geqslant u_2$,

$$I(u)=\kappa\frac{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2g(u)}}\exp(-g(u))\,(1+o(1))}{g(u)^{-(c_H+\frac{1}{2})}}=\kappa'\frac{(\ln u+(1+c_H-p)\ln\ln u)^{c_H}}{u(\ln u)^{1+c_H-p}},$$

where $\kappa'=\frac{\kappa}{2\sqrt{\pi}}(1+o(1))$.

Hence, we can conclude that

$$\mathcal{I}_f=\int_T^\infty I(u)du=\infty \text{ if }\begin{cases} H\in(0,\frac{2}{3}] \text{ and } p=1+c_H;\\ H\in(\frac{2}{3},1) \text{ and } p=1;\end{cases}$$

By Lemma 1,

$$\mathbb{P}\left(Q_{B_H}(t)>f_p(t)\quad \text{i.o.}\right)=1\quad\text{for all } H\in(0,1) \text{ for some } p.$$

Then

$$\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{f_p(t)}\geqslant 1\quad\text{a.s.}$$

Since

$$\lim_{t\to\infty}\frac{f_p(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}=\lim_{t\to\infty}\frac{\left(\frac{2}{A^2}(\ln t+(1+c_H-p)\ln\ln t)\right)^{\frac{1}{2(1-H)}}}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}=1,$$

we have

$$\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}=\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{f_p(t)}\frac{f_p(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}\geqslant 1\quad\text{a.s.}\tag{7}$$

Similarly,

$$\mathcal{I}_f=\int_T^\infty I(u)du<\infty, \text{ if }\begin{cases} H\in(0,\frac{2}{3}] \text{ and } p=-1;\\ H\in(\frac{2}{3},1] \text{ and } p=1+c_H;\end{cases}$$

Also, by Lemma 1,

$$\mathbb{P}\left(Q_{B_H}(t)>f_p(t)\quad \text{i.o.}\right)=0\quad\text{for all } H\in(0,1) \text{ and for some } p.$$

So we have

$$\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{f_p(t)}\leqslant 1 \quad \text{a.s.}$$

Then,

$$\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}=\limsup_{t\to\infty}\frac{Q_{B_H}(t)}{f_p(t)}\frac{f_p(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}\leqslant 1 \quad \text{a.s.} \tag{8}$$

By (7) and (8), the proof is completed. □

Proof of Theorem 3. Let $S > 0$ be any fixed number,

$$a_0 = S,\ y_0 = f(a_0),\ b_0 = a_0 + y_0.$$

For $i > 0$,

$$a_i = b_{i-1},\ y_i = f(a_i),\ b_i = a_i + y_i.$$

Like we did in the proof of Theorems 1 and Theorem 2, we can divide the proof into two steps,

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} Q_{B_H}(s)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \geqslant 1 \quad \text{a.s.} \tag{9}$$

and

$$\limsup_{t\to\infty} \frac{\max\limits_{0\leqslant s\leqslant t} Q_{B_H}(s)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \leqslant 1 \quad \text{a.s.} \tag{10}$$

Since $\frac{\max\limits_{0\leqslant s\leqslant t} Q_{B_H}(s)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}} \geqslant \frac{Q_{B_H}(t)}{(\frac{2}{A^2}\ln t)^{\frac{1}{2(1-H)}}}$, by Theorem 2, (9) holds.

On the other hand, in order to prove (10), we use the same structure as in the proof of Theorem 1 in Dębicki and Kosiński (2017) and the result of Theorem 2 in this article. There exist $N \in \mathbb{N}$ such that when $T \geqslant N$,

$$\sum_{k=\lfloor T\rfloor}^{\infty} \mathbb{P}\left(\sup_{t\in[0,f_p(b_k)]} Q_{B_H}(t) \geqslant f_p(b_k)\right) < \int_T^{\infty} \frac{1}{f_p(u)} \mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_{B_H}(t) > f_p(u)\right) du < \infty,$$

where p is in a suitable range.

Since

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\sup_{t\in[0,f_p(b_k)]} Q_{B_H}(t) \geqslant f_p(b_k)\right) \leqslant T + \sum_{k=\lfloor T\rfloor}^{\infty} \mathbb{P}\left(\sup_{t\in[0,f_p(b_k)]} Q_{B_H}(t) \geqslant f_p(b_k)\right) < \infty,$$

by Borel-Cantelli Lemma,

$$\mathbb{P}\left(\limsup_{k\to\infty}\left\{\sup_{t\in[0,f_p(b_k)]} Q_{B_H}(t) \geqslant f_p(b_k)\right\}\right) = 0.$$

However, since this partition can change with the value of S, which means that the event with continuous time has the same probability measure as events applied with the partition in the begining, we can acquire that

$$\mathbb{P}\left(\limsup_{t\to\infty}\left\{\max_{0\leqslant s\leqslant t} Q_{B_H}(t)\geqslant f_p(t)\right\}\right)=0.$$

The above equation is equivalent to

$$\mathbb{P}\left(\limsup_{t\to\infty}\left\{\max_{0\leqslant s\leqslant t} Q_{B_H}(t)\geqslant (1+\epsilon)\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}\right\}\right)=0$$

for any $\epsilon>0$ since $f_p(s)=\left(\frac{2}{A^2}(\ln s+(1+c_H-p)\ln\ln s)\right)^{\frac{1}{2(1-H)}}$. If we take the complement set of the above equation, it follows that

$$\mathbb{P}\left(\liminf_{t\to\infty}\left\{\max_{0\leqslant s\leqslant t} Q_{B_H}(t)\leqslant (1+\epsilon)\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}\right\}\right)=1.$$

Moreover,

$$\begin{aligned}
&\mathbb{P}\left(\liminf_{t\to\infty}\left\{\max_{0\leqslant s\leqslant t} Q_{B_H}(t)\leqslant (1+\epsilon)\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}\right\}\right)\\
&=\mathbb{P}\left(\exists t_0\in\mathbb{R}^+,\text{ s.t. if } t\geqslant t_0,\text{ then } \max_{0\leqslant s\leqslant t} Q_{B_H}(t)\leqslant (1+\epsilon)\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}\right)\\
&\leqslant \mathbb{P}\left(\limsup_{t\to\infty}\frac{\max_{0\leqslant s\leqslant t} Q_{B_H}(s)}{\left(\frac{2}{A^2}\ln t\right)^{\frac{1}{2(1-H)}}}\leqslant 1\right)\leqslant 1.
\end{aligned}$$

The inequality before the last one holds for the same reason in the proof of Theorem 1.

Hence, we finish our proof by having (10) holds. $\square$

References

[1] A. Khintchine. Über einen satz der wahrscheinlichkeitsrechnung. *Fundamenta Mathematicae*, 6(1):9–20, 1924.

[2] K. Dębicki and K. M. Kosiński. An Erdös–Révész type law of the iterated logarithm for reflected fractional brownian motion. *Extremes*, 20(4):729–749, 2017.

[3] V. I. Piterbarg. Large deviations of a storage process with fractional brownian motion as input. *Extremes*, 4(2):147–164, 2001.