Notes on

Probability

from the lectures of
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Academic Year 2023-2024

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Contents

Complements to Chapter 1

0.1 Probability construction

Operatively, there are two ways of furnishing a (measure of) probability on a measurable space $(\Omega, \mathcal{F})$: we provide *a priori* the complete description of it, or we deduct it from the information we have. That is, key difference is how much information about the probability we are starting with. The first is common in applications such as Bayesian Statistics, where you make an hypothesis on the distribution and then update it with experiments, while the second is recurrent in modelisation.

In both cases some check are in order, as we need to ensure that the information we have is *coherent*, that is there is a probability which fits the given description, and *sufficient* to uniquely charachterize the probability.

We start with the complitely resolved case of discrete partitions, then move to theoretical results on general settings, and finally deal with the practical aspects of the matter.

0.1.1 Discrete setting

The case of discrete partitions of the sample space Ω is complitely resolved: that is we know everything about it with the bare minimal information. We state the theorems that allow and describe this kind of reasoning.

Remark 0.1.1. The reason why we care about partitions is that they model experiments where, at the end, only one of some states of the system is true. Think of rolling 4 dices, and being interested in the results of the first only. Then, only (and exactly) one the *states* "the first dice rolled to n" can suceed! Formally, that is equivalent to observing that even though $\omega \in \Omega$ describes the 4 results of the 4 rolls, that is it contains information we don't really care about (the other 3 rolls), we can partition the set according to the first result only. Importantly, the remaining structure of Ω is *not* garbage: for example, partitioning for the first roll could just be the first part of answering "how likely is it that the second dice rolls to 3, given the first rolled to 1?".

Theorem 0.1.2 (Existence and Uniqueness). *Let $\mathcal{E} = \{E_k\}_{k\in I}$ be a discrete[1] partition of Ω and suppose that the function $\mathrm{p} : E_k \in \mathcal{E} \to p_k \in \mathbb{R}$ satisfies*

$$\begin{cases} \displaystyle\sum_{k\in I} p_k = 1 & \text{(1a)} \\ p_k \geq 0 \quad \textit{for all } k \in I. & \text{(1b)} \end{cases}$$

Then, there exist a unique probability $\mathbb{P}$ on $\sigma(\mathcal{E})$ such that $\mathbb{P}$ and p agree on $\mathcal{E}$. A function p with the properties (1a) and (1b) of is called a discrete probability density.

Morally, if the probability p_k of the events of a discrete partition is known, we are guaranteed that there is a unique probability extending them with consistency to the generated σ-algebra.

In this setting, some very operative results are also available: we can describe explicitly each event $E \in \sigma(\mathcal{E})$ and its probability $\mathbb{P}(E)$.

[1] That is, it is at most countable: $I \subset \mathbb{N}$

Lemma 0.1.3. *Let $\mathcal{E} = \{E_k\}_{k\in I}$ be a discrete partition of Ω. Then,*

$$\sigma(\mathcal{E}) = \left\{ \bigcup_{k\in J} E_k \ \textit{ for } J \subset I \right\}.$$

Lemma 0.1.4. *Let $\mathrm{p} : E_k \in \mathcal{E} \to p_k \in \mathbb{R}$ and $\mathbb{P} : \sigma(\mathcal{E}) \to \mathbb{R}$ be as is Theorem 0.1.2. Then,*

$$\mathbb{P}\left(\bigcup_{k\in J} E_k \right) = \sum_{k\in J} \mathrm{p}_k \qquad \textit{for all } J \subset I.$$

Remark 0.1.5. A special case worth mentioning is that of the atomic partition on a discrete Ω. Here, the σ-algebra generated is $\mathcal{P}(\Omega)$ and the discrete probability density is commonly referred to as $p(\{\omega\}) = p_\omega$. By the precedent results, p defines a unique probability $\mathbb{P}$ consistent over $\mathcal{E}$ such that

$$\mathbb{P}(E) = \sum_{\omega\in E} p_\omega \qquad \text{for all } E \subset \Omega. \tag{2}$$

Now, we want to show by examples that $\mathbb{P}$ need not be given over just a single partition. Actually, since a probability is a very special kind of measure, it is not even necessary that the information is about its values! We will explain the matter thorugh examples, but first we state an handy result.

Corollary 0.1.6. *Let $\mathcal{E} = \{E_k\}_{k\in I}$ be a partition of Ω and suppose that the function $\mathrm{p} : E_k \in \mathcal{E} \to p_k \in \mathbb{R}$ satisfies $\sum_{k\in I} p_k = 1$ and $p_k \geq 0$ for all $k \in I$. Moreover, let $\mathcal{F} = \{F_h\}_{h\in J}$ be another partition and suppose for all k such that $p_k > 0$ the function $q^{(k)} : F_h \in \mathcal{F} \to q_{h\,|k} \in \mathbb{R}$ satisfies $\sum_{h\in J} q_{h\,|k} = 1$ and $q_{h\,|k} \geq 0$ for all $h \in J$.*

Then, there exist a unique probability $\mathbb{P}$ on $\sigma(\mathcal{E} \cup \mathcal{F})$ such that $\mathbb{P}$ and p agree on $\mathcal{E}$ and

$$\mathbb{P}(F_h|E_k) = q_{h|k} \quad \textit{for all } k \in I \textit{ and } h \in J.$$

The result benefits some explaining.

Remark 0.1.7. The difference with Theorem 0.1.2 is that the σ-algebra is *quite bigger*. To understands *how much*, it is sufficient to observe that it can also be charachterized as the σ-algebra generated by the partition $\mathcal{Q} = \{E_k \cap F_h\}_{k\in I, h\in J}$ of Ω.

Remark 0.1.8. Recall Remark 0.1.1. In proability, it is quite common to study the relationships between different states and this result guarantees that if we were given not only the probability of a state, but also the probability of another state *in relation with* the first, then we can furnish consistently and uniquely the probability over combinations of all sorts of these states.

A useful representation is that of a tree of events. Let us present it with an example.

Example 1. (??)(??)

Remark 0.1.9. By the precedent Remark, you understand that a special relationship correlating states is that of independence. In that case, we can do one of two things depending on the request of the problem. First, we can use the definition: that is we put $\mathbb{P}(F_h \cap E_k) = \mathbb{P}(F_h)\mathbb{P}(E_k)$, find the probabilities on the partition in Remark 0.1.7 and then, apply Theorem 0.1.2 and Lemmas 0.1.3,0.1.4 to find $\mathbb{P}$. Second, we make use of the fact that for $\mathbb{P}(E_k) > 0$ independence is equivalent to $\mathbb{P}(F_h|E_k) = \mathbb{P}(F_h)$. We establish all the conditional probabilities, and then apply Theorem 0.1.6 to show the existence of $\mathbb{P}$. Finding the values of $\mathbb{P}$ is then a matter of manipulating conditional probabilities.

Example 2. (??) (??)

Moreover, this theorem allows us to easy the question of probability modelisation for repeated experiments. (??)

And now a final Remark on *why* the discrete partition setting is solvable.

Remark 0.1.10. For the "partition" part, it all boils down to the fact that we only consider unions: the complementation is just the union of the other elements, and the intersection is the union of partition elements that are in all the sets. If now you add the "discrete" part, we not only describe everything as unions, but as at most countable unions: that is, complementation will be always well defined. Technically, if we considered an uncountable partition, even if S was a countable union of some of its elements, then the complementary S^{c} would be an uncountable union. That is because an uncountable collection without countably many elements is still uncountable. That would make S^{c} untractable under 0.1.4!

0.1.2 Carathéodory Theorem

For more general settings the short message is that Theorem 0.1.2 holds, provided some conditions, while no explicit characterization like that of Lemmas 0.1.3,0.1.4 is possible.

We present a powerful theorem of measure theory, that does just that. It will allow us to extend a *pre-measure*, a function with some coherence that is defined on a collection smaller than a σ-algebra, to a probability, in a unique fashion.

Definition 0.1.11. Let $\mathcal{A}$ be an algebra defined on Ω. Then, $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ is a *pre-measure* if

1. $\tilde{\mathbb{P}}(\Omega) = 1$ (Normalization)
2. $\tilde{\mathbb{P}}\left(\bigcup_{i=1}^n A_i\right) = \sum_i^n \tilde{\mathbb{P}}(A_i)$ for $A_i \in \mathcal{A}$ (Additivity)
3. if $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$, then $\tilde{\mathbb{P}}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \tilde{\mathbb{P}}(A_i)$.

Remark 0.1.12. The latter is the condition that ensure coherence with respect to the probability, and can be read as a *need-based σ-additivity.*

Theorem 0.1.13 (Carathéodory's Theorem)**.** *Let $\mathcal{A}$ be an algebra defined on Ω, and suppose that $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ is a pre-measure. Then, there exists a unique probability $\mathbb{P} : \sigma(\mathcal{A}) \to \mathbb{R}$ such that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ agree on $\mathcal{A}$.*

Remark 0.1.14. This version of the Carathéodory's Theorem is of theoretical interest and provides two results. First, it shows that **existence** of a consistent extension is guaranteed just by requiring the coherence of $\tilde{\mathbb{P}}$ with the conditions that the define a probability: these are encoded in Definition 0.1.11. Moreover, the theorem quantifies the idea that if $\tilde{\mathbb{P}}$, the information provided about $\mathbb{P}$ that is, is defined on a large enough collection, then its extension is **unique**. In particular, we require $\mathbb{P}$ to be given on an algebra, a much smaller collection than a σ-algebra.

We can refine the result for practical purposes by exploiting the equivalence between σ-additivity and continuity[2]. Better: we aim at simplifing the checks on $\tilde{\mathbb{P}}$ rather than changing the statement of 0.1.13 and we do so by furnishing an equivalent set of conditions.

Lemma 0.1.15 (Pre-measure, Continuity characterization)**.** *Let $\mathcal{A}$ be an algebra defined on Ω. Then $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ is a pre-measure if and only if*

1. $\tilde{\mathbb{P}}(\Omega) = 1$,
2. $\tilde{\mathbb{P}}\left(\bigcup_{i=1}^n A_i\right) = \sum_i^n \tilde{\mathbb{P}}(A_i)$,
3. *if* $A_i \downarrow \emptyset$, *then* $\tilde{\mathbb{P}}(A_i) \downarrow 0$,

where $A_n \in \mathcal{A}$.

Remark 0.1.16. The usefulness of this is that monotone continuity, that is showing a limit is 0, is generally much easier than working on countable unions to arbitrary sets.

[2]This result from measure theory is taken to be known and the details are out of the scope of these notes.

0.1.3 Practical construction, set structure

The following two resources cover the topic well (and probably better than I can).

- π-λ Theorem and Monotone Class Theorem: link
- Practical construction: Read the "extension.pdf" file in "assets" directory. In particular, recall continuity charchterization.

I will leave some operative remarks that integrate the resources, in the rest of the chapter.

Definition 0.1.17 (π-system)**.** Given a set Ω, a collection of subsets $\mathcal{C}$ is a π-system if stable it is under finite intersection. Explicitly, if $A_1, \ldots, A_n \in \mathcal{C}$ implies that $\bigcap_{i=1}^{n} A_i \in \mathcal{C}$.

Remark 0.1.18. Operatively, it suffices to show that $A \cap B \in \mathcal{C}$ whenever $A, B \in \mathcal{C}$, for $\mathcal{C}$ to be a π-system by inductive argument.

Definition 0.1.19 (λ-system)**.** Given a set Ω, a collection of subsets $\mathcal{C}$ containing Ω is a λ-system if it is stable under (pairwise) disjoint countable union and complementation. Explicitly, if $A_1, A_2, \dots \in \mathcal{C}$ and $A_i \cap A_j = \emptyset$ for all i,j implies that $\bigcup_{i=1} A_i \in \mathcal{C}$ and if .

Remark 0.1.20. Some textbooks require proper difference stability, that is $A, B \in \mathcal{C}$ implies $A \setminus B \in \mathcal{C}$, instead of complementation.

We show that they are equivalent. The question boils down to the identity

$$A \setminus B = A \cap B^{\mathsf{C}}, \tag{3}$$

where complementation is taken with respect to a space containing both A and B.

Since $B \subset A$ the above equation yields that $B \setminus A$ is the complementation of B with respect to A, then $\mathcal{C}$ being closed under proper difference is the same as $\mathcal{G}_A = \{B \in \mathcal{C} : B \subset A\}$, the collection of subsets of A in $\mathcal{C}$, being closed under complementation for all $A \in \mathcal{C}$. Importantly, this implies complementation stability with respect to Ω.

The converse, that complementation implies proper difference, holds as well. Suppose $B \subset A$. Then, $A \setminus B = A \cap B^{\mathsf{C}} = (A^{\mathsf{C}} \cup B)^{\mathsf{C}} \in \mathcal{C}$ by (3) and countable disjoint union.

Remark 0.1.21. The Monotone Class Theorem furnishes a tool to show that a certain property is satisfied by all sets in a σ-algebra. It is sufficient to show that the π-system generating the σ-algebra satisfies it, and that the set satysfing it is a λ-system. We can show the same if the property is satisfied over an algebra and the class satisfying the property constitues a monotone class. This patter of reasoning is termed a *monotone class argument*.

Bassetti Unbound

Exercise 1. *Let Ω be a set and let $A \subset \Omega$ a subset from it. Then, show that $\{A, A^{\mathsf{c}}, \emptyset, \Omega\}$ is a σ-algebra.*

Notes. This is the easiest nontrivial σ-algebra. It models a bet: the event may either happen or not (or nor could happen, that is the same as all the outcomes being realized). (??)

Exercise 2. *Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ be a colletion of σ-algebras. Is $\bigcap_{\alpha \in I} \mathcal{F}_\alpha$ a σ-algebra. What about $\bigcup_{\alpha \in I} \mathcal{F}_\alpha$?*

Notes. This[3] justifies minimality arguments on the function $\sigma(\cdot)$. Read this masterpiece, this essay and this very general and technical site .

Exercise 3. *Prove the well-definiteness of $\sigma(\mathcal{E})$ as the minimal σ-algebra containing $\mathcal{E}$.*

Notes (Sketch of proof). Let $\Sigma(\mathcal{E})$ be the collection of all the σ-algebras containing the collection $\mathcal{E}$ of subsets of Ω. (Prove that) $\Sigma(\mathcal{E})$ is not empty, and the family intersects to $\bigcap_{S \in \mathcal{E}} S = \sigma(\mathcal{E})$.

Exercise 4. *Let E_1,E_2 be events of Ω. Think of a sample space and construct a measure of probability $\mathbb{P} : \mathcal{P}(\Omega) \to \mathbb{R}$ such that the two events are independent and $\mathbb{P}(E_1) = \mathbb{P}(E_2) = \frac{1}{2}$.*

Notes. (??)

Exercise 5. *Let $\Omega = \mathbb{N}$. Show that $\mathrm{p}(\{n\}) = \theta^n(1-\theta)$ for all $n \in \mathbb{N}$ is a discrete probability density. That is, show that it is coherent enough for a probability extending it to $\mathcal{P}(\mathbb{N})$ to exist.*

Notes. (??)

Exercise 6. *Let E_1,E_2 be independent events on Ω such that $\mathrm{p}(E_1) = \mathrm{p}(E_2) = \frac{1}{2}$. Determine the sigma-algebra, and find the probability $\mathbb{P}$ on this space consistent with the two values of* p.

Exercise 7.

Remark 0.1.22. These are some examples of probability modelization Note that Ω is basically irrelevant. Here independence, a property *of the probability*[4], furnishes the necessary information for $\mathbb{P}$ to be defined uniquely. It allows us to work in a context of minimal information, by relating different partitions (that is, states).

Exercise 8 (Jacod Protter, 7.1)**.**

Remark 0.1.23. The idea is that only finitely many disjoint events can have probability $\mathbb{P}(E) \leq \alpha$. That is all infinite sequences (convergent or divergent doesn't really matter, as we can restrict ourselves to the lim sup) need to tend to zero.

Exercise 9 (Jacod Protter, 7.2)**.**

Remark 0.1.24. Same idea as in 8, but the fact that here we also apply results about cardinality. That is we group events by having a probability larger than $\frac{1}{n}$ and then use 7.1 to show that their cardinality need be discrete, as the whole collection is countable union of finite collections.

[3] The answer is that the first is, in fact, a σ-algebra, while the second not so, as it does not contain crossed unions and intersections.

[4] Independence and conditioal probabilities are the charachteristic that really distinguish a probability from a measure.

Exercise 10 (Jacod Protter, 7.10)**.**

Remark 0.1.25. This is an analytical result, but shows that the definition of discrete random variables is coherent with its charachterization in terms of cumulative density function.

To prove the result, one could create a bijection between $\mathbb{N}$ and the set of jump discontinuities $\mathbb{D}$, by using monotonicity and order on reals. A more instructive approach, though, is that of 0.1.24. We consider for each n the set

$$D_n = \left\{ x_0 \in [0,1] : \text{ in } x_0 \text{ is located a jump discontinuity larger than } \frac{1}{n} \right\}.$$

By boundedness of $[0,1]$, D_n need be finite. Then, $\bigcup_{n\in\mathbb{N}} D_n$ is discrete.

Analitically, you could also show that removable discontinuities need be discrete. See here for further considerations. This does not have direct applications in probability.

Exercise 11. *Let*

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0. \end{cases}$$

Show that this is a CDF. Does there exist, and if so, is it unique, the distribution P that generates F? If it exists and is unique, find it.

This distribution is denoted by $\mathcal{E}(\lambda)$ and is called the negative exponential distribution.

Remark 0.1.26. For $0 < a < b < +\infty$,

$$P((a,b]) = e^{-a} - e^{-b}.$$

Exercise 12. *Find an example of X and Y random variables from a measurable space such that $P_X = P_Y$ but $P(X = Y) \neq 1$.*

Remark 0.1.27. Start by setting $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1) = (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Then, observe that

$$\mathbb{P}\{X = Y\} = \mathbb{P}\{\omega : X(\omega) = Y(\omega)\} = \mathbb{P}\{\omega : X(\omega) = t, Y(\omega) = t, \text{ for } t \in \mathbb{R}\}$$

and comparatively,

$$P_X = P_Y \implies \mathbb{P}\{X \in A\} = \mathbb{P}\{Y \in A\},$$

for every $A \in \mathcal{B}$.

In other words, the first condition requires that the random variables have indistinguishable (probabilistically) preimages, while the second condition only requires that the probability of preimage sets be equal: the identity in the first case is element-wise, in the second, it's about the law.

Idea: The symmetry of the law of X and Y produces structures indistinguishable by the law but with different relations.

Solution: For $\Omega = \{a,b,c\}$ with $\mathbb{P}(\{a\}) = \mathbb{P}(\{c\}) = \frac{1}{4}$ and $\mathbb{P}(\{b\}) = \frac{1}{2}$, and $X : (a,b,c) \to (1,0,-1)$, $Y : (a,b,c) \to (-1,0,1)$, the two random variables will be symmetric, fulfilling the required condition. Moreover, if Ω contains only 2 elements, then $P\{X = Y\} = 0$.

Exercise 13. *Let $U(0,1) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B})$ be the random variable with a uniform distribution between 0 and 1. Explicitly,*

$$F_U(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$$

Find the distribution of

$$X : \omega \in (\Omega, \mathcal{F}, \mathbb{P}) \to -\log(U(\omega))\mathbb{1}\{U(\omega) > 0\} \in (\mathbb{R}, \mathcal{B}).$$

Remark 0.1.28. The problem of finding the distribution of a random variable Y can be approached in two ways:

1. If Y is discrete, it suffices to find the the PMF, i.e. determine the support S first and then the value $\mathbb{P}(X = s)$ for each $s \in S$. P_X is the image law.

2. If Y is not discrete, it is not possible to describe P_X by finding its value on atoms. It is necessary to find its value on a π-system. Often, this means finding the CDF F_X.

In this case, we obviously adopt the second approach.

Idea: See the function as the transformation of U through a function t.

Idea: Work with the monotonically increasing and invertible composite function to determine the interval for which you want to find the preimage with respect to U, and then solve using F_U.

Exercise 14 (JP, 9.5)**.**

Remark 0.1.29. This shows that the role of e.v. on a class of functions is similar to that of the probability on a σ-algebra, a case which can be found for $X = 1$.

Exercise 15. *Consider the function $Q : \mathcal{F} \to \mathbb{R}$ such that*

$$Q(A) = \int_A f \, \mathrm{d}m,$$

where f is a PMF (that is, f is measurable, $f \leq 0$, $\int_{\mathbb{R}} f \, \mathrm{d}m$). Show that it is a probability.

Remark 0.1.30. The result is a particular case of the precedent exercise. Since we consider a PMF f, it is equivalent to restrict the X of exercise [JP, 9.5] to absolutely continuous r.v.'s.

Exercise 16. *Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}, \mathcal{E}(\lambda))$ for $\lambda > 0$, where we recall that $f_{\mathcal{E}}(x) = \lambda e^{-\lambda x}$.*

1. *Consider $X : \omega \in \Omega \to \lfloor \omega \rfloor \in \mathbb{R}$. Show that $X \sim \mathcal{G}(1 - e^{-\lambda})$, that is that the image law is a geometric distribution.*

2. *Compute $\mathbb{E}[X]$ directly, that is without making use of the expectation rule.*

3. *Compute $\mathbb{E}[X]$ using the expectation rule, and compare the proceedings with the above.*

Remark 0.1.31. First, observe that since $X(\Omega) = \mathbb{Z}$, X is discrete since $P_X(\mathbb{Z}) = 1$. Moreover, the support is just $\mathbb{N}$, since the exponential distribution is null for negatives and so we can consider instead $\tilde{X}$, a.s. agreeing, that is null for $x < 0$. Then, it suffices to find the PMF:

$$p_x(n) = \mathbb{P}\left\{X = n\right\} = \mathbb{P}\left\{X = n\right\} = \mathbb{P}\left((n, n+1]\right) = \int_{(n,n+1]} f \, \mathrm{d}m = \int_n^{n+1} \lambda e^{-\lambda t} \, \mathrm{d}t = \left(1 - e^{-\lambda}\right) e^{-\lambda n}.$$

This is, in fact, a geometric distribution. For the second point, we first verify that the expected value exists: by the above discussion, since $\tilde{X} = X$ a.s. and since $\tilde{X} \leq 0$, it follows that X admitts e.v. We can find it with the following equialities:

$$\mathbb{E}[X] = \int_\Omega \lfloor \omega \rfloor \, \mathrm{d}\mathbb{P}(\omega) = \int_\Omega \sum^\infty k \mathbb{1}_{(n,n+1]} \, \mathrm{d}\mathbb{P}(\omega) = \sum^\infty k \int_\Omega \mathbb{1}_{(n,n+1]} \, \mathrm{d}\mathbb{P}(\omega) =$$

$$= \sum^\infty k \mathbb{P}\left((n, n+1]\right) = \frac{1}{1 - e^{-\lambda}}.$$

We have used the corollary for series to the MCT in order to commute integration and summation (note that the r.v. is positive and so this is allowed). The last passage is motivated by the known result

$$\sum^\infty k \, t^{k-1} = \frac{1}{(1-t)^2}.$$

With the expectation rule, this gets notably shortened.

$$\mathbb{E}[X] = \int_{\mathbb{R}} \mathrm{id} \, \mathrm{d}P_X = \int_{\mathbb{R}} \lfloor x \rfloor \, \mathrm{d}P_X(x) = \sum_{k \in \mathbb{N}} k p_X(k) = \frac{1}{1 - e^{-\lambda}}.$$

Thus, the expectation rule, known as e.r., can make computations much easier. Not only that, but if we were just given the image law of X and not the probabiltiy on $(\Omega, \mathcal{F})$, we would have still been able to compute the expected value.

Exercise 17. *Find the e.v. and var. of the discrete uniform distribution.*

Remark 0.1.32. Let $X \sim \mathrm{d}\mathcal{U}(\{1, \dots, n\})$. Then, the e.v. exists because the X is positive, and $\mathbb{E}[X] = \frac{n+1}{2}$. Moreover, $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = \frac{n^2-1}{12}$. Where the computation of the second moment comes from the closed form for the sum

$$\sum_{k=0}^{n} k^2 = \frac{n(2n+1)(n+1)}{6}.$$

Exercise 18. *By computing the e.v. of the distribution $p(k) = \frac{6}{\pi^2}\frac{1}{k^2}$ of $\mathbb{N}$, show that the expected value need not be finite. Otherwise said, show that there are r.v. that admitt the e.v., but not variance.*

Remark 0.1.33. As the distribution is positive, one has that the e.v. exists. Moreover, X is discrete, and so

$$\mathbb{E}[X] = \sum^{\infty} kp(k) = \frac{6}{\pi^2}\sum^{\infty}\frac{1}{k} = \infty.$$

Obviusly, $X \notin L^1$ and so Var is not defined.

Chapter 1

Esperimenti congiunti

Supponiamo di svolgere due esperimenti aleatori diversi, rappresentati dai due spazi $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ e $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. Per fare previsioni, è del tutto lecito cercare di modellizzarli complessivamente come un solo esperimento, detto "congiunto".

1.0.1 Spazio prodotto

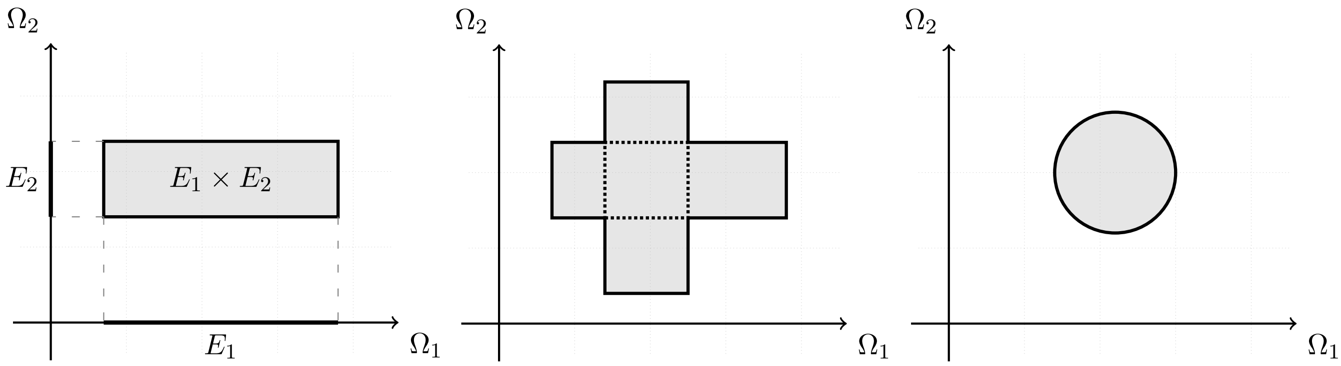
Partiamo dall'insieme degli esiti Ω: che nel primo esperimento si verifichi un certo esito e nel secondo un altro è del tutto equivalente al verificarsi di un esito "bidimensionale" che contenga come prima coordinata l'esito del primo esperimento e come seconda coordinata l'esito del secondo. In altri termini, Ω è in bigezione con $\Omega_1 \times \Omega_2$: quindi, scegliamo proprio il prodotto per rappresentare l'esperimento congiunto:

$$\Omega = \Omega_1 \times \Omega_2.$$

A questo punto, stabiliamo una struttura σ-additiva che dia concretezza ai predicati sugli esiti: imponiamo su $\Omega_1 \times \Omega_2$ una σ-algebra $\mathcal{A}$. Ovviamente, vogliamo poterci ridurre a studiare i due esperimenti singolarmente, e quindi è necessario che per $E \in \mathcal{F}_1$ si abbia che $E \times \Omega_2 \in \mathcal{A}$. Non solo: stiamo considerando l'esperimento congiunto proprio per studiare le relazioni tra i due esperimenti, quindi un altro requisito è che se $E_1 \in \mathcal{F}_1$ e $E_2 \in \mathcal{F}_2$ allora $E_1 \times E_2 \in \mathcal{A}$. Poichè la collezione $\{E_1 \times E_2 : E_i \in \mathcal{F}_i \text{ for } i = 1, 2\}$, detta "collezione dei rettangoli", non è una σ-algebra, considereremo come struttura dell'esperimento congiunto

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\left(\{E_1 \times E_2 : E_i \in \mathcal{F}_i \text{ for } i = 1, 2\}\right),$$

e cioè la più piccola[1] σ-algebra che contiene gli esiti di interesse. Diamo un'intuizione considerando il caso $(\Omega_i, \mathcal{F}_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ illustrato in Figura 1.1.



(a) Un insieme della collezione dei rettangoli.

(b) L'unione di due rettangoli che non sia un rettangolo.

(c) Un evento che non sia un rettangolo.

Figure 1.1: Esempi di eventi in $\mathcal{F}_1 \otimes \mathcal{F}_2$, per $\Omega_i = \mathbb{R}$ e $\mathcal{F}_i = \mathcal{B}(\mathbb{R})$.

[1] Per quelli che non hanno ancora bevuto il caffè mattutino, questo è desiderabile perchè più spazzatura aggiungiamo, più si restringe la classe di probabilità che potremo definirci sopra. Un esempio: la probabilità uniforme può essere definita su $\mathcal{B}([0, 1])$, ma non su $\mathcal{P}([0, 1])$.

Il generico insieme della collezione dei rettangoli $E_1 \times E_2$ può essere rappresentato come in Figura 1.1a, in accordo con l'idea di rettangolo[2]. Pertanto, i due insiemi in Figura 1.1b sono entrambi appartenenti alla collezione, ma la loro unione evidentemente no: questo esemplifica come la collezione dei rettangoli può fallire ad essere una σ-algebra. Ma $\mathcal{F}_1 \otimes \mathcal{F}_2$ contiene anche insiemi più complessi: la Figura 1.1c mostra un evento di questa σ-algebra.

Infine, alcune considerazioni. In primo luogo, quanto visto può essere facilmente esteso a più di due spazi procedendo iterativamente.

Inoltre, un risultato notevole è che definendo la σ-algebra di Borel di $\mathbb{R}^n$ come quella generata dalla topologia, e cioè $\mathcal{B}(\mathbb{R}^n) \overset{\text{def}}{=} \sigma(\mathcal{T}_{\mathbb{R}^n})$, si ottiene che $\otimes^n \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)$. Questo conferma la buona definizione dello spazio prodotto e, per le prossime sezioni, motiva il fatto che comunemente i vettori di variabili aleatorie siano definiti su $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

1.0.2 Misura di Probabilità prodotto

Finora, le scelte sono state fatte in maniera obbligata: se avessimo considerato oggetti diversi da $\times\Omega_i$ e $\otimes\mathcal{F}_i$ la descrizione dello spazio misurabile sarebbe stata logicamente equivalente[3], o avrebbe perso di flessibilità o di informazione.

Per la misura di probabilità $\mathbb{P}$ su questo spazio, non avremo lo stesso lusso: infatti, l'unico requisito che possiamo imporre è quello di consistenza con gli spazi di partenza. Formalmente, che per $E_1 \in \mathcal{F}_1$ si abbia

$$\mathbb{P}_1(E_1) = \mathbb{P}(E_1 \times \Omega_2),$$

e analogamente per $E_2 \in \mathcal{F}_2$. Si può dimostrare che esistono più misure di probabilità con questa caratteristica: possiamo garantire l'unicità solo con ulteriori vincoli modellistici e, in particolare, è sufficiente che gli esperimenti marginali siano indipendenti[4], e cioè valga la fattorizzazione

$$\mathbb{P}(E_1 \times \Omega_2 \cap \Omega_1 \times E_2) = \mathbb{P}(E_1 \times \Omega_2)\mathbb{P}(\Omega_1 \times E_2).$$

Poichè $E_1 \times \Omega_2 \cap \Omega_1 \times E_2 = E_1 \times E_2$, possiamo imporre il requisito come segue.

Theorem 1.0.1. *Siano $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ e $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ due spazi di probabilità. Allora esiste ed è unica la probabilità $\mathbb{P}$ su $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ tale che*

$$\mathbb{P}(E_1 \times E_2) = \mathbb{P}_1(E_1)\mathbb{P}_2(E_2)$$

per $E_1 \in \mathcal{F}_1$ e $E_2 \in \mathcal{F}_2$. Chiamiamo $\mathbb{P}$ la "probabilità prodotto" e la indichiamo con $\mathbb{P}_1 \otimes \mathbb{P}_2$.

Una proprietà desiderabile di $\mathbb{P}_1 \otimes \mathbb{P}_2$ è la computabilità. Sia $C \in \mathcal{F}_1 \otimes \mathcal{F}_2$, e siano le "sezioni" di C definite come segue:

$$C_1(\omega_2) = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in C\} \qquad C_2(\omega_1) = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in C\}$$

Allora, le funzioni "probabilità della sezione"

$$\omega_1 \in \Omega_1 \longmapsto \mathbb{P}_2(C_2(\omega_1)) \in \mathbb{R}$$
$$\omega_2 \in \Omega_2 \longmapsto \mathbb{P}_1(C_1(\omega_2)) \in \mathbb{R}$$

sono misurabili e limitate, e quindi integrabili. Questo conferma la buona posizione della regola

$$(\mathbb{P}_1 \otimes \mathbb{P}_2)(C) = \int\limits_{\Omega_1 \times \Omega_2} \mathbb{1}_C(\boldsymbol{\omega})\, \mathbb{P}_1 \otimes \mathbb{P}_2(\mathrm{d}\boldsymbol{\omega}) =$$
$$= \int\limits_{\Omega_1} \mathbb{P}_2(C_2(\omega_1))\, \mathbb{P}_1(\mathrm{d}\omega_1).$$

[2]Achtung: anche l'insieme $\mathbb{Q}^2$, per esempio, appartiene a questa collezione, ma la sua rappresentazione geometricamente non è un rettangolo.

[3]E cioè tutto sarebbe stato identico ai fini modellistici, "modulo" una bigezione.

[4]Ricordiamo che $A \perp B$ se e solo se $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

Proof. Proveremo un solo caso. Sia $C = E_1 \times E_2$ con $E_i \in \mathcal{F}_i$. Si dimostra che $\pi_2 : (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \to \omega_2 \in \Omega_2$ è misurabile ed integrabile. Allora,

$$\begin{aligned}\mathbb{P}_1 \otimes \mathbb{P}_2(E_1 \times E_2) &= \mathbb{P}_1(E_1)\mathbb{P}_2(E_2) = \\ &= \mathbb{P}_1(E_1)\mathbb{E}_{\mathbb{P}_2}[\mathbb{1}_{E_2}] = \mathbb{P}_1(E_1)\mathbb{E}_{\otimes\mathbb{P}}[\mathbb{1}_{E_2} \circ \pi_2] = \\ &= \int_{\Omega_2} \mathbb{P}_1(E_1)\mathbb{1}_{E_2}(\omega_2)\mathbb{P}_2(\mathrm{d}\omega_2).\end{aligned}$$

Poichè se $\omega_2 \in E_2$ allora $C_1(\omega_2) = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in C = E_1 \times E_2\} = E_1$, si ha che $t_2(\omega_2) = \mathbb{P}(E_1)$. Similmente, se $\omega_2 \notin E_2$, allora $C_1(\omega_2) = \emptyset$ e $t_2(\omega_2) = 0$. Quindi, $t_2 = \mathbb{P}_1(E_1)\mathbb{1}_{E_2}$, da cui segue la tesi. Per linearità, segue il caso di $C = \bigcup^n E_{1,i} \times E_{2,i}$ con $E_{2,i}$ disgiunti (se non lo sono, si può recastare il set in una sommatoria dove lo sono). Infine, per C generico, è sufficiente trovare un'approssimante $C^{(n)} = \bigcup^n E_{1,i}^{(n)} \times E_{2,i}^{(n)}$ dal basso (?? esplicitarla ??), e osservare che per continuità monotona di $\otimes\mathbb{P}$ e MCT applicato alla sequenza $t_2^{(n)} \uparrow t_2$ segue la tesi in generale. □

Ovviamente, anche in questo caso tutto si può estendere a più di due spazi iterativamente.

1.0.3 Vettori aleatori prodotto

Rimangono da introdurre le variabili aleatorie su o da spazi prodotto, e per farlo cerchiamo di stabilire dei criteri di misurabilità.

La situazione più semplice, quella in cui date delle variabili aleatorie definite sullo stesso spazio, ci chiediamo come si comporta il loro vettore, è stata già incontrata nel caso di vettori aleatori reali: il vettore era misurabile se e solo lo erano le componenti. A conferma[5] della buona definizione dello spazio prodotto, questa proprietà continua a valere.

Lemma 1.0.2 (Misurabilità delle componenti). *Per $X : \Omega \to E$, $Y : \Omega \to E$, si ha che $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ e $Y : (\Omega, \mathcal{A}) \to (E, \mathcal{F})$ sono misurabili se e solo se lo è $(X, Y) : (\Omega, \mathcal{A}) \to (E \times F, \mathcal{E} \otimes \mathcal{F})$.*

La seconda casistica, quella in cui il dominio è uno spazio prodotto, è più delicata e l'implicazione vale solo in un verso. Per fissare le idee, consideriamo solo un caso.

Lemma 1.0.3 (Misurabilità delle restrizioni). *Sia $X : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ un vettore aleatorio. Allora, la restrizione*

$$X(\cdot, \omega_2) : \Omega_1 \to \mathbb{R}$$

è misurabile per ogni $\omega_2 \in \Omega_2$ rispetto a $\mathcal{F}_1$.

Questa è solo una condizione necessaria e non un criterio per la misurabilità: non vale l'inverso. Esplicitamente, la misurabilità di $X(\cdot, \omega_2)$ e $X(\omega_1, \cdot)$ rispetto ad ogni $\omega_1 \in \Omega_1$ e $\omega_2 \in \Omega_2$ non implica quella del vettore (come ci si potrebbe aspettare...).

Ciò detto il risultato caratterizza gli insiemi misurabili dello spazio prodotto: se $C \in \otimes\mathcal{F}_i$ allora $X = \mathbb{1}_C$ è misurabile, la restrizione $\mathbb{1}_C(\cdot, \omega_2)$ è misurabile, e la sezione $C_1(\omega_2)$ è un evento[6]. In altri termini, tutte le "sezioni" di un insieme misurabile nello spazio prodotto sono misurabili rispetto agli spazi marginali.

1.0.4 Criterio di Indipendenza

Supponiamo di considerare gli esperimenti $(E_i, \mathcal{E}_i, \mathbb{P}_i)$ congiuntamente, ed imponiamo che siano indipendenti: per quanto visto, consideriamo cioè la misura prodotto $\otimes\mathbb{P}_i$ su $(\times E_i, \otimes\mathcal{E}_i)$. Ora *accendiamo* le variabili aleatorie: se ognuno di questi spazi fosse lo spazio immagine di un solo esperimento $(\Omega, \mathcal{F}, \mathbb{P})$ attraverso $X_i : \Omega \to E_i$, allora avremmo che

$$X = (X_1, \dots, X_n) : \Omega \to E_1 \times \cdots \times E_n.$$

[5] Che il criterio valesse per i vettori reali lo rende *desiderabile* in generale

[6] Questo prova quanto visto nella regola per computare $\otimes\mathbb{P}$

Ora facciamo un ragionamento modellistico: che gli spazi iniziali considerati congiuntamente fossero indipendenti significa che la conoscenza di un evento su uno non influenza le predizioni di un evento sull'altro. Introducendo le variabili aleatorie, lasciamo la struttura delle dipendenze inalterata, quindi sarebbe desiderabile che la conoscenza di un evento su uno spazio immagine non influenzasse le previsioni degli eventi sugli altri spazi immagine. In altri termini, se le definizioni fossero ben poste dovremmo avere l'indipendenza delle variabili aleatorie, e cioè (nel caso $n = 2$)

$$P_{(X_1,X_2)}(E_1 \times E_2) = \mathbb{P}\{X_1 \in E_1, X_2 \in E_2\} = \mathbb{P}\{X_1 \in E_1\}\mathbb{P}\{X_2 \in E_2\} = P_{X_1}(E_1)P_{X_2}(E_2).$$

Questo è confermato dal seguente risultato, per cui vale anche l'inverso: se le variabili aleatorie sono indipendenti, allora il loro vettore è definito sullo spazio prodotto.

Lemma 1.0.4. *Le variabili aleatorie $X : \Omega \to E$ e $Y : \Omega \to F$ sono indipendenti se e solo se*

$$P_{(X,Y)} = P_X \otimes P_Y. \tag{1.1}$$

A livello operativo, questo risultato ha due conseguenze.

In primo luogo, è un criterio per stabilire se due variabili aleatorie sono indipendenti: è necessario e sufficiente che la probabilità congiunta fattorizzi nelle probabilità marginali per ogni scelta di eventi negli spazi di arrivo.

La seconda è che permette di provare il seguente risultato, che a sua volta conferma che è sufficiente informare sulla legge di ogni variabile X_i e dire che sono indipendenti perchè esista una variabile aleatoria nello spazio prodotto con queste caratteristiche.

Lemma 1.0.5. *Siano $X_i : \Omega \to (E_i, \mathcal{E}_i, P_i)$ variabili aleatorie. Allora, il vettore $X = (X_1, \dots, X_n) : \Omega \to (\times E_i, \otimes \mathcal{E}_i, \otimes P_i)$ è tale che*

- *La sua i-esima componente è distribuita come X_i: in simboli, $(X)_i \sim X_i$,*
- *Le sue componenti sono una famiglia di variabili aleatorie indipendenti.*

Lasciamo in appendice al capitolo la dimostrazione di 1.0.4.

?? da sistemare. Osserviamo che 1.1 vale se e solo se $P_{(X,Y)}(A \times B) = P_X(A)P_Y(B)$ per ogni $A \in \mathcal{E}$ e $B \in \mathcal{F}$. Consideriamo $P_{(X,Y)}(A \times B) = \mathbb{P}(\{X \in A\}, \{Y \in B\}) = \mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\}$. Se le r.v. sono indipendenti, percorrerla da sinistra dimostra la tesi; se vale 1.1 è sufficiente osservare che si ottine la def di indipendendenza stocastica. □

1.0.5 Teorema di Fubini-Tonelli