SOME REMARKS ON REMOVABLE SINGULARITIES

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For a fixed $x_0 \in [a, b]$, we define

$$\omega_\delta(x_0) = \sup_{x \in \mathcal{U}_\delta(x_0)} f(x) - \inf_{x \in \mathcal{U}_\delta(x_0)} f(x),$$

where $\mathcal{U}_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$. Clearly, ω_δ is a bounded and decreasing function of δ. Therefore, it is well defined the limit

$$\omega(x_0) = \lim_{\delta \to 0} \omega_\delta(x_0).$$

We call $\omega(x_0)$ the *oscillation of f at x_0*. We have the following equivalence:

$$f \text{ is continuous at } x_0 \iff \omega(x_0) = 0. \tag{1.1}$$

In particular, it follows that the oscillation at every point of discointinuity is strictly positive.

Remark 1.1. If x_0 is a removable singularity of f, then

$$\omega(x_0) = \Big| f(x_0) - \lim_{x \to x_0} f(x) \Big|.$$

The following theorem holds.

Theorem 1.2. *Let*

$$\mathcal{A} = \{x \in [a, b] : x \textit{ is a removable or jump discontinuity of } f\}$$

Then $\mathcal{A}$ is at most countable.

Proof. Let

$$\mathcal{A}_n = \Big\{ x \in \mathcal{A} : \omega(x) > \frac{1}{n}. \Big\}.$$

We claim that every $\mathcal{A}_n$ is a finite set. Indeed, assume by contradiction that there exists $\overline{n}$ such that $\mathcal{A}_{\overline{n}}$ is infinite. Then, there is at least one accumulation point $x_0 \in \mathcal{A}_{\overline{n}}$. But then, it is not difficult to show that the limit of f for $x \to x_0$ does not exist. Hence, x_0 is an essential discontinuity, contradicting the fact that $\mathcal{A}$ contains only removable or jump discontinuities. Since

$$\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n,$$

we have the thesis. $\square$

In what follows, we will assume that f has only removable singularities. We can now state our main result.

Theorem 1.3. *Let $\widehat{f}$ be the function defined by*

$$\widehat{f}(x_0) = \lim_{x \to x_0} f(x).$$

Then $\widehat{f}$ is continuous.

The proof of this theorem is based on the following lemma.

Lemma 1.4. *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that*

$$\sup_{x\in[a,b]} |f(x) - g(x)| < \varepsilon,$$

for a certain $\varepsilon > 0$. Then

$$\sup_{x\in[a,b]} |\omega^{(f)}(x) - \omega^{(g)}(x)| < 2\varepsilon,$$

where $\omega^{(f)}, \omega^{(g)}$ are, respectively, the oscillations of f and g at x.

Proof. Let x_0 be fixed. By definition

$$|\omega^{(f)}(x_0) - \omega^{(g)}(x_0)| = \lim_{\delta\to 0} |\omega_\delta^{(f)}(x_0) - \omega_\delta^{(g)}(x_0)|.$$

For a fixed $\delta > 0$, we have

$$\omega_\delta^{(f)}(x_0) - \omega_\delta^{(g)}(x_0) = \sup f(x) - \sup g(x) - (\inf f(x) - \inf g(x)),$$

where it is understood that every supremum and infimum is computed over a neighborhood $\mathcal{U}_\delta(x_0)$ of x_0. Since

$$|\sup f(x) - \sup g(x)| + |\inf f(x) - \inf g(x)| \leq 2\varepsilon,$$

the proof is concluded. □

We are now in position to prove the main result. Define the sequence of functions

$$f_n(x) = \begin{cases} f(x) & \omega(x) \leq \frac{1}{n}, \\ \widehat{f}(x) & \omega(x) > \frac{1}{n} \end{cases}$$

Then, recalling Remark 1.1,

$$\sup_{x\in[a,b]} |f_n(x) - \widehat{f}(x)| \leq \frac{1}{n}.$$

Accordingly, by Lemma 1.4 we obtain

$$|\omega^n(x_0) - \omega(x_0)| < \frac{2}{n} \quad \forall x_0 \in [a, b],$$

where ω^n is the oscillation of f_n, and ω the oscillation of $\widehat{f}$. Finally, since by definition

$$\omega^n(x_0) \leq \frac{1}{n} \quad \forall x_0 \in [a, b],$$

we can collect the two inequalities to obtain

$$\omega(x_0) \leq \omega^n(x_0) + |\omega(x_0) - \omega^n(x_0)| \leq \frac{3}{n}$$

for every n, implying that $\widehat{f}$ is continuous at x_0.