Notes on

Probability

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Contents

Chapter 1

Spazi Probabilizzati

Il nostro obiettivo è definire una funzione (quindi, un dominio e una legge) detta "probabilità" che associ a insiemi di configurazioni del sistema un indice della loro *likelyhood*[1].

Eventi

Formalmente, consideriamo l'insieme delle possibili configurazioni del sistema, lo chiamiamo *spazio campionario*, e lo indichiamo con Ω. Un suo sottoinsieme $A \subset \Omega$, e cioè un insieme di configurazioni, si dice *evento*. Quindi, $\mathcal{P}(\Omega)$ è l'insieme degli eventi, e in particolare $A = \Omega$ è detto evento *certo* e $A = \emptyset$ è detto evento *impossibile*.

1.1 Collezioni di eventi

Quindi la probabilità è una funzione su una collezione di eventi in $\mathbb{R}$. Poichè ci interessano solo i sottoinsiemi "sensati" dello spazio campionario, in primo luogo studiamo come strutturare il dominio della funzione.

Per costruire un insieme ci sono pochi metodi. I principali sono:

1. Fare una lista degli elementi,
2. Dato un insieme, definirne un sottoinsieme che rispetti una legge,
3. Dato un insieme, il suo insieme delle parti,
4. Dati due insiemi, il loro prodotto cartesiano,
5. Dati due insiemi, la loro unione, la loro intersezione, e la loro differenza.

Di questi è chiaro che l'unico adeguato alla costruzione di una probabilità è l'ultimo. Quindi, definiamo due collezioni di insiemi che siano *stabile* (o *chiusa*) rispetto a queste operazioni, rispettivamente nel contesto finito e in quello numerabile.

1.1.1 Algebra e σ-algebra

Definizione 1.1.1. La collezione $\mathcal{F} \subset \mathcal{P}(\Omega)$ si dice algebra di Ω se e solo se

(Normalization) $\emptyset, \Omega \in \mathcal{F}$

omplementation) $A \in \mathcal{F} \implies A^{\complement} \in \mathcal{F}$

n & Intersection) $\{A_1, \ldots, A_n\} \subset \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$

Questo insieme è stabile rispetto all'unione finita, all'intersezione finita e alla complementazione.

Definizione 1.1.2. La collezione $\mathcal{F} \subset \mathcal{P}(\Omega)$ si dice σ-algebra di Ω se e solo se

[1] In italiano, l'espresione più vicina sarebbe verosimiglianza.

(Normalization) $\emptyset, \Omega \in \mathcal{F}$

mplementation) $A \in \mathcal{F} \implies A^{\complement} \in \mathcal{F}$

n & Intersection) $\{A_1, \dots\} \subset \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Questo insieme è invece stabile rispetto all'unione numerabile, all'intersezione numerabile e alla complementazione.

Partiamo osservando che queste definizioni sono ridondanti grazie alla complementazione. Infatti, la prima condizione è equivalente a $\emptyset \in \mathcal{F}$ oppure a $\emptyset \in \mathcal{F}$, e la terza è equivalente alla sola stabilità rispetto all'unione o rispetto all'intersezione. Questa osservazione è operativamente comoda quando è necessario verificare che una collezione sia una σ-algebra.

1.1.2 Altre classi stabili

1.2 Misure di probabilità

Complements to Chapter 1

1.3 Probability construction

Operatively, there are two ways of furnishing a (measure of) probability on a measurable space $(\Omega, \mathcal{F})$: it may be claimed or inferred. In the first case, its law is given *a priori*, while in the second it is deducted *a posteriori*, from the knowledge of the probability values on some elementary events, that is.

The first is common in applications such as Bayesian Statistics, where you make an hypothesis on the distribution and then test it, while the second is also characteristic of measure theory. The key difference lies in the fact that we either have complete or minimal information about the probability.

In both cases some check are in order, as we need to ensure that the probability is *coherent*. Note that in the second case, we also need to ensure that the one generated is unique, the minimal information is *sufficient*, that is.

1.3.1 Discrete setting

Some results of this kind were produced already: we have dealt with the case of discrete partitions of the sample space Ω. We quote such theorems.

Theorem 1.3.1 (Existence and Uniqueness)**.** *Let $\mathcal{E} = \{E_k\}_{k \in I}$ be a discrete partition of Ω and let the function $\mathrm{p} : E_k \in \mathcal{E} \to p_k \in \mathbb{R}$ be such that*

$$\begin{cases} \displaystyle\sum_{k \in I} p_k = 1 & (1.1a) \\ p_k \geq 0 \quad \text{for all } k \in I. & (1.1b) \end{cases}$$

Then, there exist a unique probability $\mathbb{P}$ on $\sigma(\mathcal{E})$ such that $\mathbb{P}$ and p agree on $\mathcal{E}$. A function p with the properties (1.1a) and (1.1b) of is called a discrete probability density.

Morally, the discrete probability density on a partition describes a unique and consistent probability on the generated σ-algebra. Moreover, we have an explicit description of each event $E \in \sigma(\mathcal{E})$ and its probability $\mathbb{P}(E)$.

Lemma 1.3.2. *Let $\mathcal{E} = \{E_k\}_{k \in I}$ be a discrete partition of Ω. Then*

$$\sigma(\mathcal{E}) = \left\{ \bigcup_{k \in J} E_k \ \text{ for } J \subset I \right\}.$$

Now suppose that $\mathrm{p} : E_k \in \mathcal{E} \to p_k \in \mathbb{R}$ is a discrete probability density. Then, if $\mathbb{P}$ is the probability defined in Theorem 1.3.1,

$$\mathbb{P}\left(\bigcup_{k \in J} E_k \right) = \sum_{k \in J} \mathrm{p}_k \quad \text{for all } J \subset I.$$

Remark 1.3.3. A special case worth mentioning is that of the atomic partition on a discrete Ω. Here, the σ-algebragenerated is $\mathcal{P}(\Omega)$ and the discrete probability density is commonly referred to as $p(\{\omega\}) = p_\omega$. By the precedent results, p defines a unique agreeing probability $\mathbb{P}$ such that

$$\mathbb{P}(E) = \sum_{\omega \in E} p_\omega \quad \text{for all } E \subset \Omega. \tag{1.2}$$

1.3.2 General setting

For more general settings the short message is that Theorem 1.3.1 holds, provided some conditions, while no explicit characterization like that of Lemma 1.3.2 is possible.

We present a powerful theorem of measure theory, that does just that. It will allow us to extend a *pre-probability*, a function with some coherence that is defined on a collection smaller than a σ-algebra, to a probability, in a unique fashion.

Definizione 1.3.4. Let $\mathcal{A}$ be an algebra defined on Ω. Then, $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ is a *pre-probability* if

1. $\tilde{\mathbb{P}}(\Omega) = 1$ (Normalization)
2. $\tilde{\mathbb{P}}\left(\bigcup_{i=1}^n A_i\right) = \sum_i^n \tilde{\mathbb{P}}(A_i)$ for $A_i \in \mathcal{A}$ (Additivity)
3. if $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$, then $\tilde{\mathbb{P}}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \tilde{\mathbb{P}}(A_i)$.

Remark 1.3.5. The latter is the condition that ensure coherence with respect to the probability, and can be read as a *need-based σ-additivity.*

Theorem 1.3.6 (Carathéodory's Theorem)**.** *Let $\mathcal{A}$ be an algebra defined on Ω, and suppose that $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ is a pre-probability. Then, there exists a unique probability $\mathbb{P} : \sigma(\mathcal{A}) \to \mathbb{R}$ such that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ agree on $\mathcal{A}$.*

Remark 1.3.7. This version of the Carathéodory's Theorem is of theoretical interest and provides two results. First, it shows that **existence** of a consistent extension is guaranteed just by requiring the coherence of $\tilde{\mathbb{P}}$ with the conditions that the define a probability: these are encoded in Definition 1.3.4. Moreover, the theorem quantifies the idea that if $\tilde{\mathbb{P}}$, the information provided about $\mathbb{P}$ that is, is defined on a large enough collection, then its extension is **unique**. In particular, we require $\mathbb{P}$ to be given on an algebra, a much smaller collection than a σ-algebra.

We can refine the result for practical purposes by exploiting the equivalence between σ-additivity and continuity[2].

Lemma 1.3.8 (Carathéodory's Theorem, continuity characterization)**.** *Let $\mathcal{A}$ be an algebra defined on Ω, and suppose that $\tilde{\mathbb{P}} : \mathcal{A} \to \mathbb{R}$ satisfies*

1. $\tilde{\mathbb{P}}(\Omega) = 1$,
2. $\tilde{\mathbb{P}}\left(\bigcup_{i=1}^n A_i\right) = \sum_i^n \tilde{\mathbb{P}}(A_i)$,
3. *if* $A_i \downarrow \emptyset$, *then* $\tilde{\mathbb{P}}(A_i) \downarrow 0$,

where $A_n \in \mathcal{A}$. Then, there exists a unique probability $\mathbb{P} : \sigma(\mathcal{A}) \to \mathbb{R}$ such that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ agree on $\mathcal{A}$.

Remark 1.3.9. The usefulness of this is that monotone continuity, that is showing a limit is 0, is generally much easier than working on countable unions to arbitrary sets. It is of theoretical interest that this only holds if we use an algebra: as we will see, if it wasn't for this characterization weaker conditions on the collection (namely, that it is a π-system instead of an algebra) could be used.

Another refinement is in order: the minimal information condition could be weaker. This is an application of the Monotone class theorem, and while we will not go into details, some more can be found here.

Definizione 1.3.10 (π-system)**.** A collection $\mathcal{C}$ is said to be a π-system if stable under finite intersection. Explicitly, if $A_1, \ldots, A_n \in \mathcal{C}$ then $\bigcap_{i=1}^n A_i \in \mathcal{C}$.

Remark 1.3.11. Operatively, it suffices to show that if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$ for finite intersection stability to hold, by inductive argument.

Lemma 1.3.12.

This theorem is not just about comparisons. It furnishes sharp[3] conditions for the uniqueness in Theorem 1.3.6: it requires the pre-probability to be defined just on a π-system containing Ω, something weaker than an algebra.

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[2]This result from measure theory is taken to be known and the details are out of the scope of these notes.

[3]Mathematical gibberish for "minimal"

Borel's σ-algebra

σ-algebra on Bernoulli Space