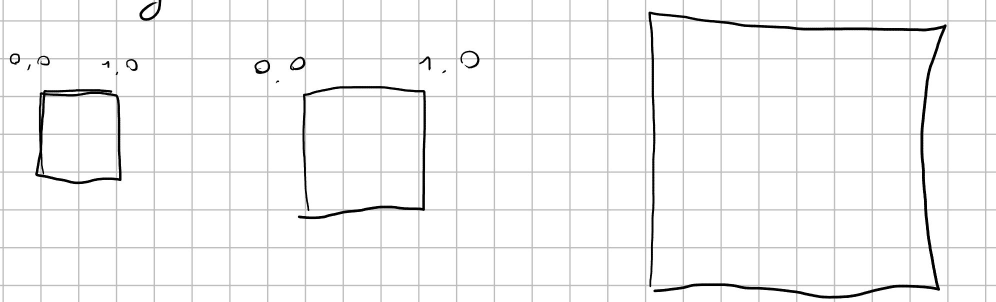
REVIEW OF COSMOLOGY

DISTANCE DEFINITION

On an expanding paper, we can define several notions of distance.



COMOVING DISTANCE → IS FIXED AND DOES NOT CHANGE IN TIME

$$d_c = \sqrt{(1-0)^2 + (0-0)^2} = 1$$

PHYSICAL DISTANCE → $d_p = a(t)\, d_c$

↳ Scale factor. Tells how much the universe is expanding

An expanding universe is described by an increasing scale factor. $a(t_2) > a(t_1) \ \forall\, t_2 > t_1$.

Doing cosmology is all about studying the evolution of the scale factor.

We define the Hubble parameter as

$$H(t) = \frac{\frac{da}{dt}}{a} = \frac{\dot{a}}{a}$$ $\rightarrow$ IT IS MEASURED IN s^{-1} and quantifies the metric expansion rate.

The Hubble parameter is fundamental for cosmology. Let's see why. If I am looking at galaxies in an expanding universe, I expect them to run away from me.

$$d_{PHYS} = a(t)\, d_{COM} \Rightarrow \dot{d}_{PHYS} = \dot{a}\, d_{COM} + a\, \dot{d}_{COM}.$$

$\dot{d}_{COM} = 0$ (NO PECULIAR MOTION) $\Rightarrow$ $\dot{d}_{PHYS} = \dot{a}\, d_{COM} = \frac{\dot{a}}{a} a d_{COM} = H(t) d_{PHYS}$

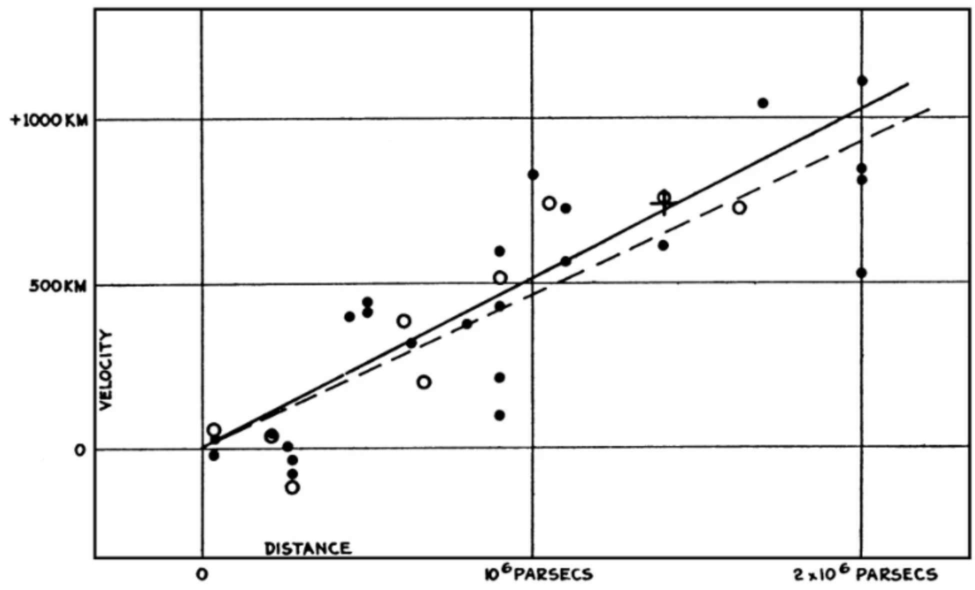
HUBBLE LAW

$$\boxed{\dot{d}_{PHYS} = v_{RECESSIONAL} = H(t)\, d_{PHYS}}$$

If the expansion rate is constant $H(t) = H_0$ Hubble constant

$$v_{REC} = H_0\, d_{PHYS}$$

Indeed these where the first observations made by Hubble in 1929. Hubble observed galaxies and discovered that the more distant galaxies had a faster recessional velocity.



FIRST EVIDENCE OF COSMIC EXPANSION BY HUBBLE

$$H_0 = \frac{\Delta y}{\Delta x} \simeq 500 \frac{km}{s\, Mpc}$$

10 TIMES HIGHER THAN NOWADAYS VALUES

The way in which Hubble was calibrating distances was wrong and therefore he got a wrong value.
Also Today, doing cosmology consists in

- Measure the distance of the source
- Measure its recessional velocity (or redshift) $z = \frac{v}{c}$
- Fit $H(t)$

WHY THE UNIVERSE IS EXPANDING?

WHAT IS H_0?

We need to use General Relativity (GR)

NOTE

FRLW METRIC

$$\boxed{ds^2 = -c^2 dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]}$$

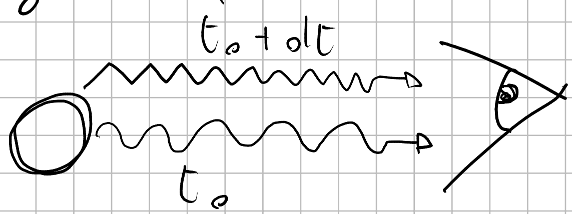
BEFORE GOING ON...

THE REDSHIFT

For a photon $ds^2=0 \Rightarrow c^2 dt^2 = a^2 dx^2$

$$dx = \frac{c\,dt}{a(t)}$$

Let us assume that we have a cosmological source emitting two photons at time t_o and $t_o + dt$. The two photons will travel the same distance r and reach the observer at t_E, $t_E + dt_E$



PHOTON 1 $\quad r = \int_{t_o}^{t_E} \frac{c\,dt}{a}$

PHOTON 2 $\quad r = \int_{t_o+dt_o}^{t_E+dt_E} \frac{c\,dt}{a} =$

$$\Rightarrow \int_{t_E}^{t_E+dt_E} \frac{c\,dt}{a(t)} = \int_{t_o}^{t_o+dt_o} \frac{c\,dt}{a(t)}$$

$$\frac{a(t_o)}{a(t_e)} = \frac{dt_e}{dt_o} = \frac{f_o}{f_e} = (1+z)$$

The expansion of the universe introduces a Doppler.

DISTANCES

Measure distances is very tricky in the universe.

COMOVING HORIZON = $\quad \Omega = c\eta = c\int_0^t \frac{dt'}{a(t')}$ COM DISTANCE TRAVELLED BY THE PHOTONS FROM THE START

COMOVING DISTANCE BETWEEN SOURCE - OBSERVER

$$d_c = \int_{t_e}^{t_0} c\,\frac{dt}{a(t)} = \int_a^1 \frac{da'}{a'^2 H(a)} =$$

$$= c\int_0^z \frac{dz'}{H(z)}$$

We can also define the luminosity distance of the source. If the source has luminosity L, I measure the flux. For a comoving observer

$$F = \frac{L}{4\pi d_c^2}$$

. But for an observer which is not comoving

$$F = \frac{La^2}{4\pi d_c^2} \Rightarrow d_L = \frac{d_c}{a} = c(1+z)\int_0^z \frac{dz'}{H(z')}$$

The metric should be put in Einstein's equation.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \overset{+\Lambda g_{\mu\nu}?}{=} \frac{8\pi G}{c^4} T_{\mu\nu}$$

Where $T_{\mu\nu}$ is the stress energy tensor

$$T_{\mu\nu} = \begin{pmatrix} -\rho \xrightarrow{} \text{DENSITY} & & & \\ & p \xrightarrow{} \text{PRESSURE} & & \\ & & p & \\ & & & p \end{pmatrix} \Rightarrow \text{PERFECT FLUID}$$

If you put these two quantities in Einstein's equation, and do a very long math

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \qquad \text{I FRIEDMANN EQUATION}$$

Let me define $\rho_{CRIT} = \frac{8\pi G}{H_0^2}$

CURVATURE ENERGY — DARK ENERGY

$$\Rightarrow \left(\frac{H}{H_0}\right)^2 = \frac{\rho}{\rho_{CRIT}} - \frac{kc^2}{8\pi G\,\rho_{CRIT}}\frac{1}{a^2} + \frac{\Lambda c^2}{3(8\pi G)}\frac{1}{\rho_{CRIT}}$$

$$\boxed{\left(\frac{H}{H_0}\right)^2 = \frac{1}{\rho_{CRIT}}\left(\rho(a) - \frac{\rho_{CURV}}{a^2} + \rho_\Lambda\right)}$$

The equation above is telling us several results:

1) If the Universe has energy inside, it is natural that it expands

2) The ρ_Λ is a constant and does not decrease in density as the universe expand

3) Even curvature has its energy density.

NOTE FROM $T^{\mu\nu}{}_{;\nu} = 0$ we can also derive

$$\dot{\rho} = -3H\left(\rho + \frac{P}{c^2}\right) \Rightarrow \text{CONTINUITY equation}$$

If I chose an equation of state

$$P = w\rho c^2$$

$w = 0 \rightarrow$ MATTER $\Rightarrow$ $\rho \propto a^{-3}$

$w = 1/3 \rightarrow$ RADIATIONS $\rho \propto a^{-4}$

$w = -1 \rightarrow$ COSMO. COSTANT $\rho \propto a^{0} = \text{COST}.$

The matter energy density decrease since the universe is expanding (like normal diluition). Photons goes like a^{-4} since $E = h\nu$ and you also get a factor from here. While Λ does not change with the scale factor.

$$\boxed{\left(\frac{H}{H_0}\right)^2 = \Omega_m a^{-3} + \Omega_R a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda}$$

$$\Omega_x = \frac{\rho_x}{\rho_{CRIT}}$$

Some examples:

$\Omega_R = \Omega_k = \Omega_\Lambda = 0$

$\Omega_m \neq 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_m a^{-3} \Rightarrow \frac{\dot{a}}{a} = H_0 \Omega_m^{1/2} a^{-3/2}$$

$$\dot{a} = H_0 \Omega_m^{1/2} a^{-1/2}$$

$$\int da\, a^{1/2} = H_0 \Omega_m^{1/2} t$$

$$\frac{2}{3} a^{3/2} = H_0 \Omega_m^{1/2} t$$

$$\rightarrow a \propto t^{2/3}$$

WHAT IS THE SOLUTION IF $\Omega_\Lambda \neq 0$?

SUMMARY

1) In an expanding Universe $v_{REC} = H(t)\, d_{PHYS}$

2) The first Friedmann equation models the expansion of the Universe with several physical parameters.
   $H_0 \rightarrow$ CRITICAL DENSITY.

3) To measure $H(t)$ and hence cosmology we need to measure distance and redshift of sources.

4) For EM observations we can measure redshift with spectroscopy. We can measure d_L assuming intrinsic luminosity (STANDARD CANDLE)

GRAVITATIONAL WAVE SOURCES AT COSMOLOGICAL DISTANCE

Let us recall a couple of fundamental results from the previous paragraph.

$$\frac{dt_{SOU}}{dt_{OBS}} = \frac{f_{OBS}}{f_{SOU}} = (1+z) \longrightarrow \text{REDSHIFT}$$

For GW signals you have seen that

$$\frac{df_s}{dt_s} = \frac{96}{5} \pi^{8/3} (G M_c)^{5/3} f_s^{11/3}$$

using the relation between OBS and source.

$$(1+z)\frac{d}{dt_0}\left[f_0 (1+z)\right] = \frac{96}{5} \pi^{8/3} (G M_c)^{5/3} f_0^{11/3}$$

$$\text{If } \frac{da(t_0)}{dt_0} \simeq 0 \Rightarrow \frac{df_0}{dt_0} = \frac{96}{5} \pi^{8/3} (G \mathcal{M}_c)^{5/3} f_0^{11/3}$$

$$\mathcal{M}_c \equiv (1+z) M_c$$

NOTE The frequency evolution of a GW signal for an observer at cosmological distance has the same form (CHIRP) for an observer at source BUT $M_{OBS} = (1+z)M$.

$\Rightarrow$ Also the GW phase evolution is the same

$$\phi_s(t_s) = \int \frac{df_s}{dt_s} dt_s = \int \frac{df_0}{dt_0} dt_0 = \phi_0(t_0)$$

What about the amplitude? We have seen that:

$$h \propto \frac{\mathcal{M}_s^{5/3}}{r} f_s^{2/3} \sin(\phi_s)$$

$f_s = f_o(1+z)$
$\phi_s = \phi_o$

$$\Rightarrow \quad h \propto \frac{\overbrace{\mathcal{M}_s^{5/3}(1+z)^{5/3}}^{\to\ \mathcal{M}_o^{5/3}}}{\underbrace{r(1+z)}_{\to\ d_L}} f_o^{2/3} \sin(\phi_o)$$

$$h \propto \frac{\mathcal{M}_o^{5/3}}{d_L} f_o^{2/3} \sin(\phi_o)$$

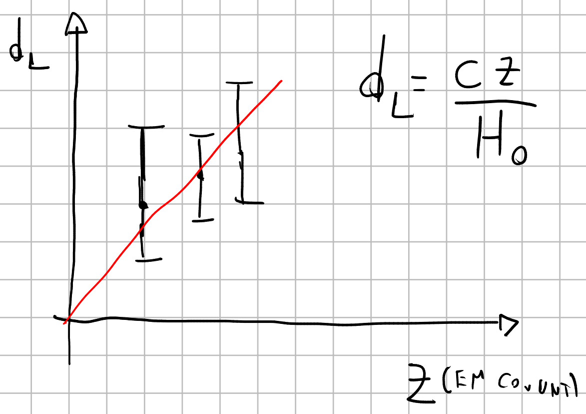
In summary: 1) The phase of GW is always the same BUT

2) The distance is replaced by the luminosity distance

3) Masses are redshifted, i.e. $\mathcal{M}_o = (1+z)\mathcal{M}_s$

Example: COSMOLOGY WITH GWs BRIGHT SIRENS

To fit a cosmological model we just need d_L (from GWs) and a redshift z estimation (not from GWs)



$$d_L = \frac{cz}{H_0}$$

If we are lucky we can get the redshift from an EM counterpart. (GRB/KILONA). Then we can put the GW on Hubble diagram and fit cosmology

$$p(H_0|GW,GRB) \propto p(GW,GRB|H_0)\overbrace{p(H_0)}^{\text{prior}} = p(H_0)\int p(GW,GRB|H_0,z)\,p(z)\,dz$$

$$= p(H_0)\int p(GW|H_0,z)\,p(\overbrace{GRB|z}^{\delta(z-z_{EM})})\,p(z)\,dz = p(H_0)\,p(GW|d_L(H_0,z_{EM}))$$