Radiative modification to a general $\overline{u_j'\theta'}$ model

Quantities requiring modeling:

$$\kappa' \ , \quad E_m' \ , \quad G' \ .$$

They all depend on temperature (Radiative heat transfer is an analytical equation), in particular we know that:

$$\kappa = c_0 + \frac{c_1}{T} + \frac{c_2}{T^2} + \frac{c_3}{T^3} + \frac{c_4}{T^4} + \frac{c_5}{T^5} \ ,$$

and

$$E_m = 4\left(\frac{\theta}{T_0} + 1\right)^4 \ ,$$

where:

$$T = \theta \Delta T + T_c \ , \quad \overline{T} = \overline{\theta}\Delta T + T_c \ , \quad T' = \theta' \Delta T \ , \quad T_0 = \frac{T_c}{\Delta T} \ .$$

Approximation of E_m'

It is possible to find the fluctuation of these two quantities by linearizing their analytical expressions, starting from E_m

$$E_m' = E_m - \overline{E_m} = 4\left(\frac{\theta}{T_0} + 1\right)^4 - 4\overline{\left(\frac{\theta}{T_0} + 1\right)^4}$$

$$E_m' = \frac{4\theta^4}{T_0^4} + \frac{16\theta^3}{T_0^3} + \frac{24\theta^2}{T_0^2} + \frac{16\theta}{T_0} - \frac{4\overline{\theta^4}}{T_0^4} - \frac{16\overline{\theta^3}}{T_0^3} - \frac{24\overline{\theta^2}}{T_0^2} - \frac{16\overline{\theta}}{T_0}$$

after subtituting θ with $\theta' + \overline{\theta}$

$$E_m' = \frac{4(\overline{\theta}^4 + 4\overline{\theta}^3\theta' + 6\overline{\theta}^2\theta'^2 + 4\overline{\theta}\theta'^3 + \theta'^4)}{T_0^4} + \frac{16(\overline{\theta}^3 + 3\overline{\theta}^2\theta' + 3\overline{\theta}\theta'^2 + \theta'^3)}{T_0^3} + \frac{24(\overline{\theta}^2 + 2\overline{\theta}\theta' + \theta'^2)}{T_0^2} + \frac{16(\overline{\theta} + \theta')}{T_0} +$$
$$- \frac{4(\overline{\theta}^4 + 6\overline{\theta}^2\overline{\theta'^2} + 4\overline{\theta}\overline{\theta'^3} + \overline{\theta'^4})}{T_0^4} - \frac{16(\overline{\theta}^3 + 3\overline{\theta}\overline{\theta'^2} + \overline{\theta'^3})}{T_0^3} - \frac{24\overline{\theta}^2 + \overline{\theta'^2}}{T_0^2} - \frac{16\overline{\theta}}{T_0} \ ,$$

$$E_m' = \frac{4(4\overline{\theta}^3\theta' + 6\overline{\theta}^2(\theta'^2 - \overline{\theta'^2}) + 4\overline{\theta}(\theta'^3 - \overline{\theta'^3}) + \theta'^4 - \overline{\theta'^4})}{T_0^4} + \frac{16(3\overline{\theta}^2\theta' + 3\overline{\theta}(\theta'^2 - \overline{\theta'^2}) + \theta'^3 - \overline{\theta'^3})}{T_0^3} +$$
$$+ \frac{24(2\overline{\theta}\theta' + \theta'^2 - \overline{\theta'^2})}{T_0^2} + \frac{16(\theta')}{T_0} \ .$$

In the end, simplifying and underlying the dependency on temperature fluctuations θ':

$$\begin{aligned} E_m' &= \left(\frac{16\overline{\theta}^3}{T_0^4} + \frac{48\overline{\theta}^2}{T_0^3} + \frac{48\overline{\theta}}{T_0^2} + \frac{16}{T_0}\right)\theta' + \\ &= \left(\frac{24\overline{\theta}^2}{T_0^4} + \frac{48\overline{\theta}}{T_0^3} + \frac{24}{T_0^2}\right)(\theta'^2 - \overline{\theta'^2}) + \\ &= \left(\frac{16\overline{\theta}}{T_0^4} + \frac{16}{T_0^3}\right)(\theta'^3 - \overline{\theta'^3}) + \\ &= \left(\frac{4}{T_0^4}\right)(\theta'^4 - \overline{\theta'^4}) \ . \end{aligned}$$

Due to the linearization we can neglect all terms depending on higher order terms, therefore in the end a good approximation for E'_m is

$$E'_m \approx f_{E_m}\theta' \ ,$$

where f_{E_m} is the first model equation, equal to:

$$f_{E_m} = \frac{16\overline{\theta}^3}{T_0^4} + \frac{48\overline{\theta}^2}{T_0^3} + \frac{48\overline{\theta}}{T_0^2} + \frac{16}{T_0} \ .$$

Approximation of κ' (only for variable κ)

To calculate κ' is necessary to calculate $\overline{\kappa}$ first as $\kappa' = \kappa - \overline{\kappa}$.

$$\overline{\kappa} = c_0 + \overline{\frac{c_1}{T}} + \overline{\frac{c_2}{T^2}} + \overline{\frac{c_3}{T^3}} + \overline{\frac{c_4}{T^4}} + \overline{\frac{c_5}{T^5}} \ ,$$

taking into account the second term on the LHS, (remembering that c_1 is a constant):

$$\overline{\frac{1}{T}} = \overline{\frac{1}{\overline{T}(1+\frac{T'}{\overline{T}})}} \ ,$$

since $T'/\overline{T} \ll 1$ it is possible taking a taylor expansion $(1+x)^{-1} = 1 - x + x^2 - x^3...$ and linearize, therefore:

$$\overline{\frac{1}{T}} \approx \overline{\frac{1}{\overline{T}}(1-\frac{T'}{\overline{T}})} = \frac{1}{\overline{T}} \ .$$

Proceding with the same logic it is possible to show that, if $T'/\overline{T} \ll 1$, then:

$$\overline{\frac{1}{T^2}} \approx \frac{1}{\overline{T}^2} \ , \quad \overline{\frac{1}{T^3}} \approx \frac{1}{\overline{T}^3} \ , \quad \overline{\frac{1}{T^4}} \approx \frac{1}{\overline{T}^4} \ , \quad \overline{\frac{1}{T^5}} \approx \frac{1}{\overline{T}^5} \ .$$

Therefore

$$\overline{\kappa} \approx c_0 + \frac{c_1}{\overline{T}} + \frac{c_2}{\overline{T}^2} + \frac{c_3}{\overline{T}^3} + \frac{c_4}{\overline{T}^4} + \frac{c_5}{\overline{T}^5} \ .$$

To calculate κ' we start from the second term on the LHS ($c_0 - c_0 = 0$)

$$c_1\left(\frac{1}{T} - \frac{1}{\overline{T}}\right) = c_1\left(\frac{\overline{T} - T}{T\overline{T}}\right) \ .$$

Substituting the expressions $T' = T - \overline{T} = \theta'\Delta T$ and linearizing the denominator ($T\overline{T} \approx \overline{T}^2$)

$$c_1\left(\frac{1}{T} - \frac{1}{\overline{T}}\right) \approx -c_1\frac{\Delta T}{\overline{T}^2}\theta' \ .$$

For the third term, subtracting and expanding into $\overline{T} + T'$

$$c_2\left(\frac{1}{T^2} - \frac{1}{\overline{T^2}}\right) = c_2\left(\frac{\overline{T'^2} - 2\overline{T}T' - T'^2}{(\overline{T}^2 + 2\overline{T}T' + T'^2)(\overline{T}^2 + \overline{T'^2})}\right) \ .$$

Again linearizing the denominator ($\approx \overline{T}^4$) and the numerator ($\approx 2\overline{T}T'$) we reach

$$c_2\left(\frac{1}{T^2} - \frac{1}{\overline{T^2}}\right) \approx -c_2\frac{2\Delta T}{\overline{T}^3}\theta' \ .$$

In the same fashion it is possible to demonstrate that

$$c_3\left(\frac{1}{T^3}-\frac{1}{\overline{T}^3}\right)\approx -c_3\frac{3\Delta T}{\overline{T}^4}\theta' \ ,$$
$$c_4\left(\frac{1}{T^4}-\frac{1}{\overline{T}^4}\right)\approx -c_4\frac{4\Delta T}{\overline{T}^5}\theta' \ ,$$
$$c_5\left(\frac{1}{T^5}-\frac{1}{\overline{T}^5}\right)\approx -c_5\frac{5\Delta T}{\overline{T}^6}\theta' \ .$$

And therefore we found an expression for κ' as

$$\kappa' \approx f_\kappa \theta' \ ,$$

where

$$f_\kappa = -\left(c_1\frac{\Delta T}{\overline{T}^2}+c_2\frac{2\Delta T}{\overline{T}^3}+c_3\frac{3\Delta T}{\overline{T}^4}+c_4\frac{4\Delta T}{\overline{T}^4}+c_5\frac{5\Delta T}{\overline{T}^6}\right)$$

Approximation of G'

This is the most complex part of the modeling and requires the knowledge of the spectral wavenumbers of the non-radiative channel flow. It is possible to know only the spectral dependency of the incident radiation fluctuations, and in homogeneous isotropic turbulence for grey gas is

$$\widehat{G'} = \frac{\kappa}{\omega}\mathrm{atan}\left(\frac{\omega}{\kappa}\right)\widehat{E'_m} \ .$$

From this it is already possible to state, following from the previous considerations on E'_m, that

$$\widehat{G'} \approx \frac{\kappa}{\omega}\mathrm{atan}\left(\frac{\omega}{\kappa}\right) f_{E_m}\widehat{\theta'} \ .$$

The above equation states that the modes of G' are connected to the modes of θ' depending on the wavenumber and on the absorption coefficient. This means that the higher the absorption coefficient the stronger G' gets and always on smaller ω. It is possible to translate this proportionality to physical space if there is the knowledge of a charateristic wavenumber for which it is possible to write

$$G' \approx f_G f_{E_m}\theta'$$

where

$$f_G = \frac{\kappa}{\omega_c}\mathrm{atan}\left(\frac{\omega_c}{\kappa}\right)$$

and ω_c is a charateristic wavenumber that represents the integral wavenumber weighted on the energy spectrum at a certain $y-$location calculated as

$$\omega_c = \left\| \int_0^\infty \boldsymbol{\omega'} E_\theta(\boldsymbol{\omega'})d\boldsymbol{\omega'} / \int_0^\infty E_\theta(\boldsymbol{\omega'})d\boldsymbol{\omega'} \right\|_2$$

This wavenumber should take into account the penalty of anisotropicity on the growth of G', i.e. there should be weights in the norm that promote the spectral direction which contains energy at the largest wavenumbers since radiative energy will most likely escape from those directions. For the present case, with DNS knowledge of the spectral redistribution of energy in the channel, it is identified that the spanwise spectral wavenumber is dominant over the streamwise (i.e., energy at larger wavenumbers) and we assume that the wall normal direction contains energy at the same wavenumbers that the spawise direction. Therefore, following the spectral energy redistribution noticed in figure 1, it is possible to approximate the charateristic wavenumber as a parabola (shape of energy redistribution in spanwise direction) as

$$\omega_c = (c_{r33}-c_{r22})y^2 - 2(c_{r33}-c_{r22})y + c_{r33} \ ,$$

with c_{r33} and c_{r22} the wavenumber of maximum energy at the walls and in the middle of the channel respectively.

Modification of the $\overline{\theta'^2} - \epsilon_\theta$ two equation model

From the expressions derived in the previous section it is possible to extend a general $\overline{\theta'^2} - \epsilon_\theta$ model to fit radiative flows. The two additional equations read:

$$\frac{\partial\overline{\theta'^2}}{\partial t} + \overline{u}_j \frac{\partial\overline{\theta'^2}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\alpha\frac{\partial\overline{\theta'^2}}{\partial x_j} - \overline{u'_j\theta'^2}\right) - 2\overline{u'_j\theta'}\frac{\partial\overline{\theta}}{\partial x_j} - 2\alpha\overline{\frac{\partial\theta'}{\partial x_j}\frac{\partial\theta'}{\partial x_j}} - \overline{(\kappa Q)'\theta'}$$

$$\frac{\partial\overline{\theta'^2}}{\partial t} + \overline{u}_j \frac{\partial\overline{\theta'^2}}{\partial x_j} =$$

Results

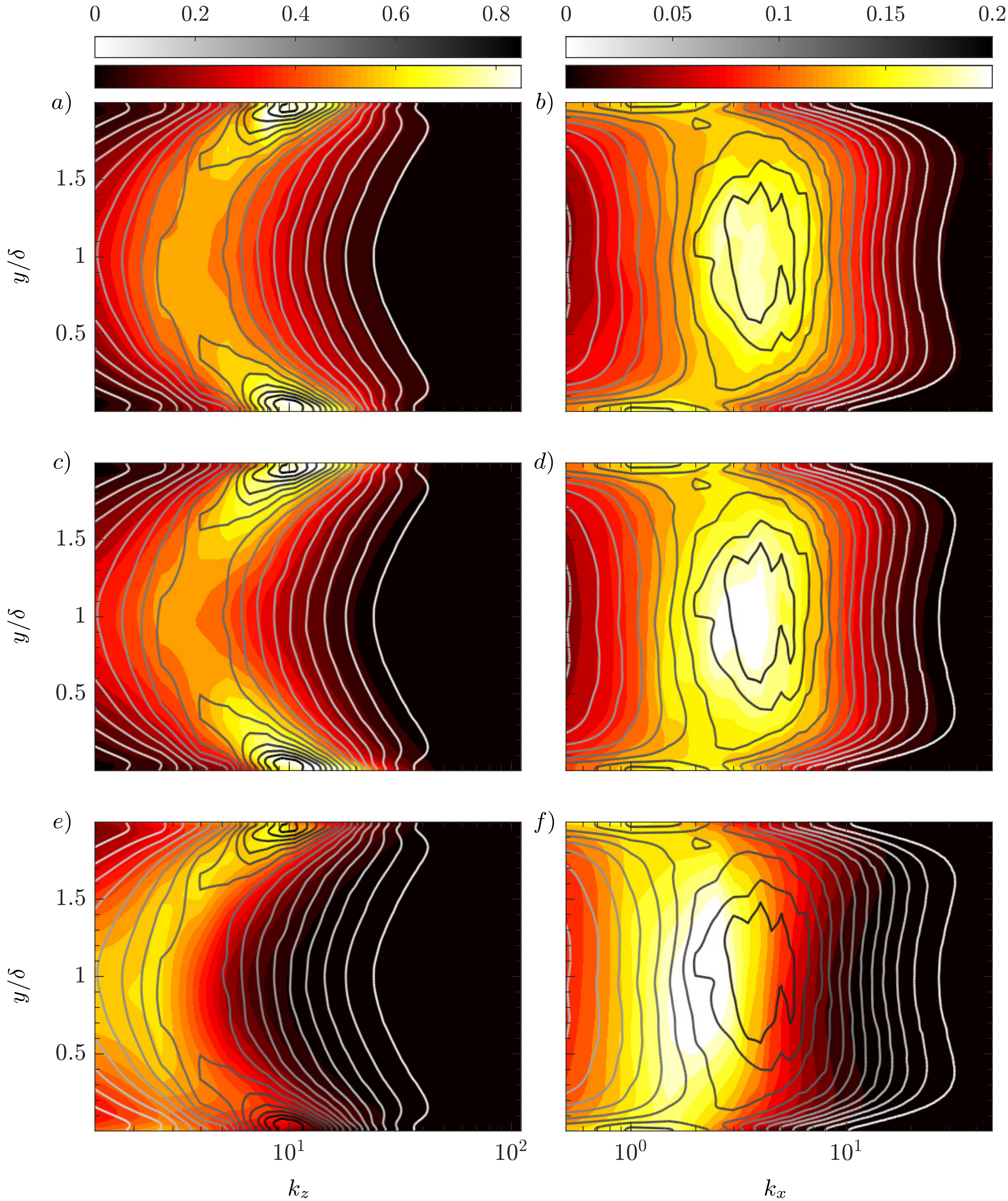

Figure 1: RANS simulation with different turbulent heat flux models for different τ cases

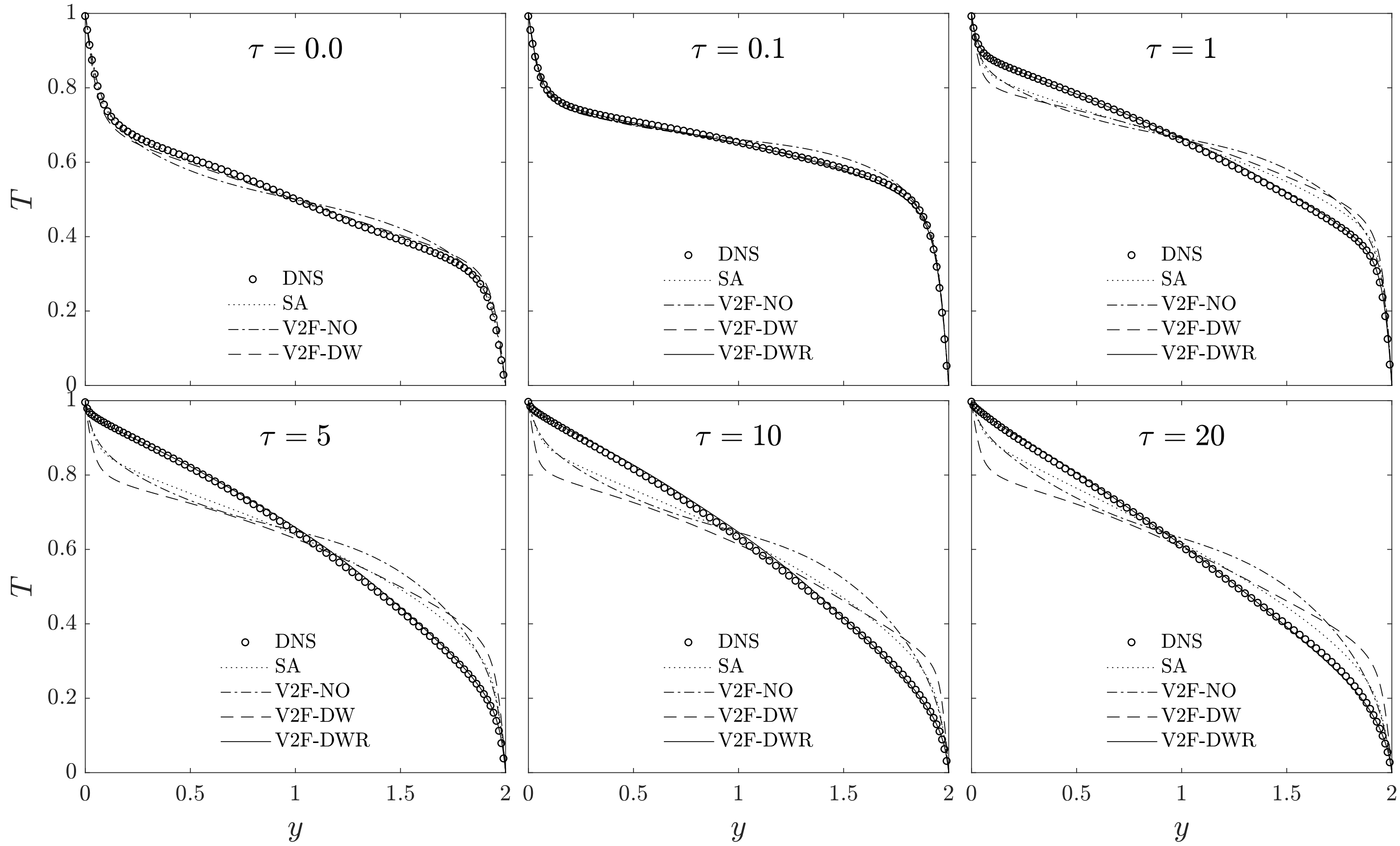

Figure 2: RANS simulation with different turbulent heat flux models for different τ cases

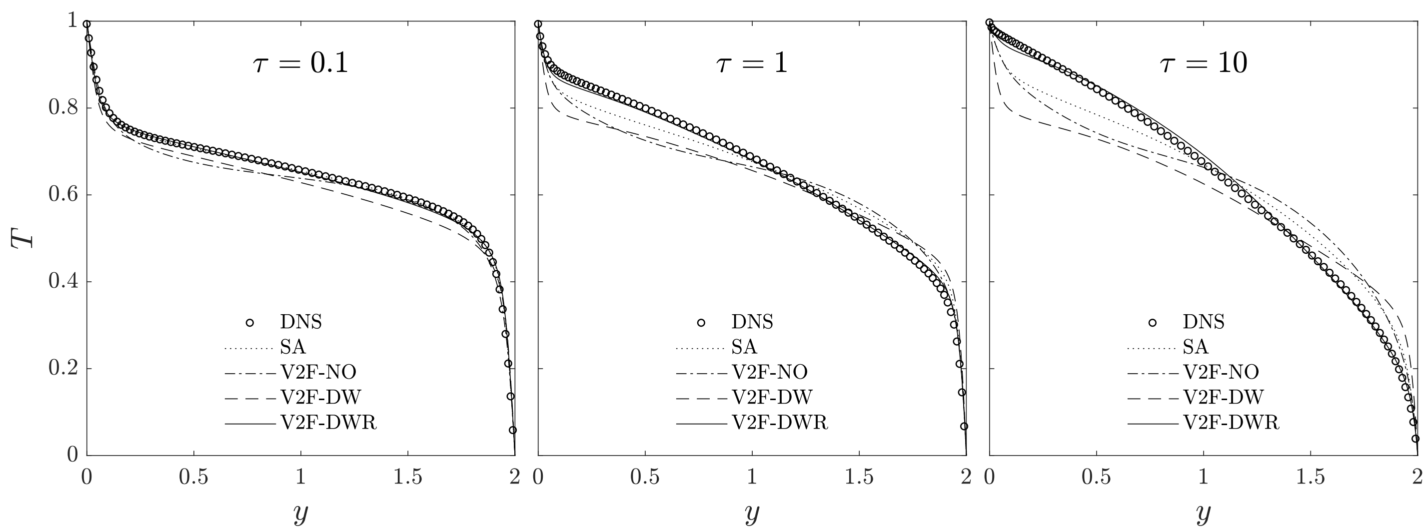

Figure 3: RANS simulation with different turbulent heat flux models for different τ cases