Radiative modification to a general $\overline{\theta'^2} - \epsilon_\theta$ model

Quantities requiring modeling:

$$\kappa' \ , \quad E'_m \ , \quad G' \ .$$

They all depend on temperature (Radiative heat transfer is an analytical equation), in particular we know that:

$$\kappa = c_0 + \frac{c_1}{T} + \frac{c_2}{T^2} + \frac{c_3}{T^3} + \frac{c_4}{T^4} + \frac{c_5}{T^5} \ ,$$

and

$$E_m = 4\left(\frac{\theta}{T_0} + 1\right)^4 \ ,$$

where:

$$T = \theta \Delta T + T_c \ , \quad \overline{T} = \overline{\theta}\Delta T + T_c \ , \quad T' = \theta' \Delta T \ , \quad T_0 = \frac{T_c}{\Delta T} \ .$$

Approximation of E'_m

It is possible to find the fluctuation of these two quantities by linearizing their analytical expressions, starting from E_m

$$E'_m = E_m - \overline{E_m} = 4\left(\frac{\theta}{T_0} + 1\right)^4 - 4\overline{\left(\frac{\theta}{T_0} + 1\right)^4}$$

$$E'_m = \frac{4\theta^4}{T_0^4} + \frac{16\theta^3}{T_0^3} + \frac{24\theta^2}{T_0^2} + \frac{16\theta}{T_0} - \frac{4\overline{\theta^4}}{T_0^4} - \frac{16\overline{\theta^3}}{T_0^3} - \frac{24\overline{\theta^2}}{T_0^2} - \frac{16\overline{\theta}}{T_0}$$

after subtituting θ with $\theta' + \overline{\theta}$

$$E'_m = \frac{4(\overline{\theta}^4 + 4\overline{\theta}^3\theta' + 6\overline{\theta}^2\theta'^2 + 4\overline{\theta}\theta'^3 + \theta'^4)}{T_0^4} + \frac{16(\overline{\theta}^3 + 3\overline{\theta}^2\theta' + 3\overline{\theta}\theta'^2 + \theta'^3)}{T_0^3} + \frac{24(\overline{\theta}^2 + 2\overline{\theta}\theta' + \theta'^2)}{T_0^2} + \frac{16(\overline{\theta} + \theta')}{T_0} +$$
$$- \frac{4(\overline{\theta}^4 + 6\overline{\theta}^2\overline{\theta'^2} + 4\overline{\theta}\overline{\theta'^3} + \overline{\theta'^4})}{T_0^4} - \frac{16(\overline{\theta}^3 + 3\overline{\theta}\overline{\theta'^2} + \overline{\theta'^3})}{T_0^3} - \frac{24\overline{\theta}^2 + \overline{\theta'^2}}{T_0^2} - \frac{16\overline{\theta}}{T_0} \ ,$$

$$E'_m = \frac{4(4\overline{\theta}^3\theta' + 6\overline{\theta}^2(\theta'^2 - \overline{\theta'^2}) + 4\overline{\theta}(\theta'^3 - \overline{\theta'^3}) + \theta'^4 - \overline{\theta'^4})}{T_0^4} + \frac{16(3\overline{\theta}^2\theta' + 3\overline{\theta}(\theta'^2 - \overline{\theta'^2}) + \theta'^3 - \overline{\theta'^3})}{T_0^3} +$$
$$+ \frac{24(2\overline{\theta}\theta' + \theta'^2 - \overline{\theta'^2})}{T_0^2} + \frac{16(\theta')}{T_0} \ .$$

In the end, simplifying and underlying the dependency on temperature fluctuations θ':

$$\begin{aligned} E'_m &= \left(\frac{16\overline{\theta}^3}{T_0^4} + \frac{48\overline{\theta}^2}{T_0^3} + \frac{48\overline{\theta}}{T_0^2} + \frac{16}{T_0}\right)\theta' + \\ &= \left(\frac{24\overline{\theta}^2}{T_0^4} + \frac{48\overline{\theta}}{T_0^3} + \frac{24}{T_0^2}\right)(\theta'^2 - \overline{\theta'^2}) + \\ &= \left(\frac{16\overline{\theta}}{T_0^4} + \frac{16}{T_0^3}\right)(\theta'^3 - \overline{\theta'^3}) + \\ &= \left(\frac{4}{T_0^4}\right)(\theta'^4 - \overline{\theta'^4}) \ . \end{aligned}$$

Due to the linearization we can neglect all terms depending on higher order terms, therefore in the end a good approximation for E'_m is

$$E'_m \approx f_{E_m}\theta' \ ,$$

where f_{E_m} is the first model equation, equal to:

$$f_{E_m} = \frac{16\overline{\theta}^3}{T_0^4} + \frac{48\overline{\theta}^2}{T_0^3} + \frac{48\overline{\theta}}{T_0^2} + \frac{16}{T_0} \ .$$

Approximation of κ' (only for variable κ)

To calculate κ' is necessary to calculate $\overline{\kappa}$ first as $\kappa' = \kappa - \overline{\kappa}$.

$$\overline{\kappa} = c_0 + \overline{\frac{c_1}{T}} + \overline{\frac{c_2}{T^2}} + \overline{\frac{c_3}{T^3}} + \overline{\frac{c_4}{T^4}} + \overline{\frac{c_5}{T^5}} \ ,$$

taking into account the second term on the LHS, (remembering that c_1 is a constant):

$$\overline{\frac{1}{T}} = \overline{\frac{1}{\overline{T}(1 + \frac{T'}{\overline{T}})}} \ ,$$

since $T'/\overline{T} \ll 1$ it is possible taking a taylor expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 ...$ and linearize, therefore:

$$\overline{\frac{1}{T}} \approx \overline{\frac{1}{\overline{T}}(1 - \frac{T'}{\overline{T}})} = \frac{1}{\overline{T}} \ .$$

Proceding with the same logic it is possible to show that, if $T'/\overline{T} \ll 1$, then:

$$\overline{\frac{1}{T^2}} \approx \frac{1}{\overline{T}^2} \ , \quad \overline{\frac{1}{T^3}} \approx \frac{1}{\overline{T}^3} \ , \quad \overline{\frac{1}{T^4}} \approx \frac{1}{\overline{T}^4} \ , \quad \overline{\frac{1}{T^5}} \approx \frac{1}{\overline{T}^5} \ .$$

Therefore

$$\overline{\kappa} \approx c_0 + \frac{c_1}{\overline{T}} + \frac{c_2}{\overline{T}^2} + \frac{c_3}{\overline{T}^3} + \frac{c_4}{\overline{T}^4} + \frac{c_5}{\overline{T}^5} \ .$$

To calculate κ' we start from the second term on the LHS ($c_0 - c_0 = 0$)

$$c_1\left(\frac{1}{T} - \frac{1}{\overline{T}}\right) = c_1\left(\frac{\overline{T} - T}{T\overline{T}}\right) \ .$$

Substituting the expressions $T' = T - \overline{T} = \theta'\Delta T$ and linearizing the denominator ($T\overline{T} \approx \overline{T}^2$)

$$c_1\left(\frac{1}{T} - \frac{1}{\overline{T}}\right) \approx -c_1\frac{\Delta T}{\overline{T}^2}\theta' \ .$$

For the third term, subtracting and expanding into $\overline{T} + T'$

$$c_2\left(\frac{1}{T^2} - \frac{1}{\overline{T}^2}\right) = c_2\left(\frac{\overline{T'^2} - 2\overline{T}T' - T'^2}{(\overline{T}^2 + 2\overline{T}T' + T'^2)(\overline{T}^2 + \overline{T'^2})}\right) \ .$$

Again linearizing the denominator ($\approx \overline{T}^4$) and the numerator ($\approx 2\overline{T}T'$) we reach

$$c_2\left(\frac{1}{T^2} - \frac{1}{\overline{T}^2}\right) \approx -c_2\frac{2\Delta T}{\overline{T}^3}\theta' \ .$$

In the same fashion it is possible to demonstrate that

$$c_3\left(\frac{1}{T^3}-\frac{1}{\overline{T}^3}\right)\approx -c_3\frac{3\Delta T}{\overline{T}^4}\theta' \ ,$$
$$c_4\left(\frac{1}{T^4}-\frac{1}{\overline{T}^4}\right)\approx -c_4\frac{4\Delta T}{\overline{T}^5}\theta' \ ,$$
$$c_5\left(\frac{1}{T^5}-\frac{1}{\overline{T}^5}\right)\approx -c_5\frac{5\Delta T}{\overline{T}^6}\theta' \ .$$

And therefore we found an expression for κ' as

$$\kappa' \approx f_\kappa \theta' \ ,$$

where

$$f_\kappa = -\left(c_1\frac{\Delta T}{\overline{T}^2}+c_2\frac{2\Delta T}{\overline{T}^3}+c_3\frac{3\Delta T}{\overline{T}^4}+c_4\frac{4\Delta T}{\overline{T}^4}+c_5\frac{5\Delta T}{\overline{T}^6}\right)$$

Approximation of G'

This is the most complex part of the modeling and

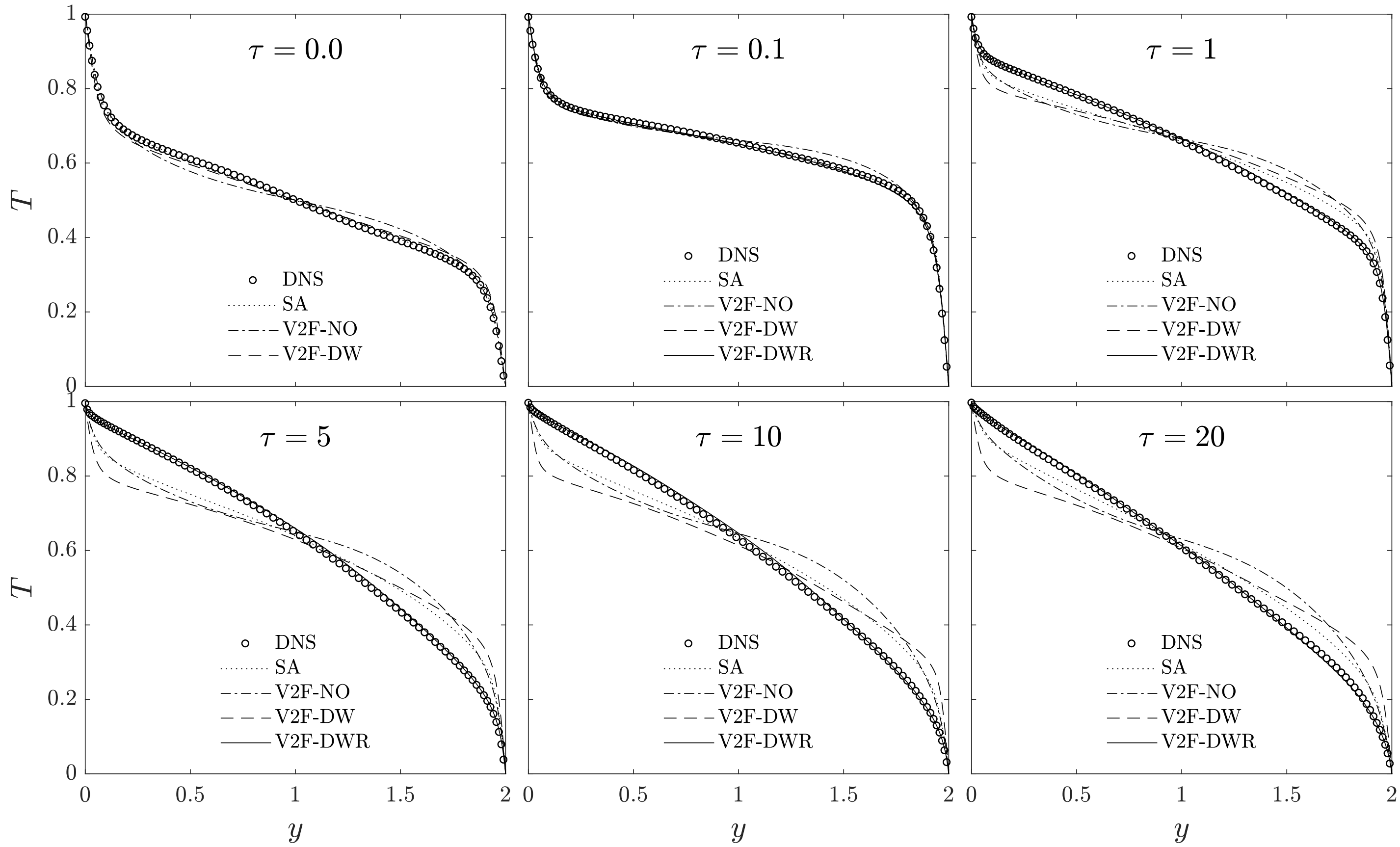

Figure 1: RANS simulation with different turbulent heat flux models for different τ cases

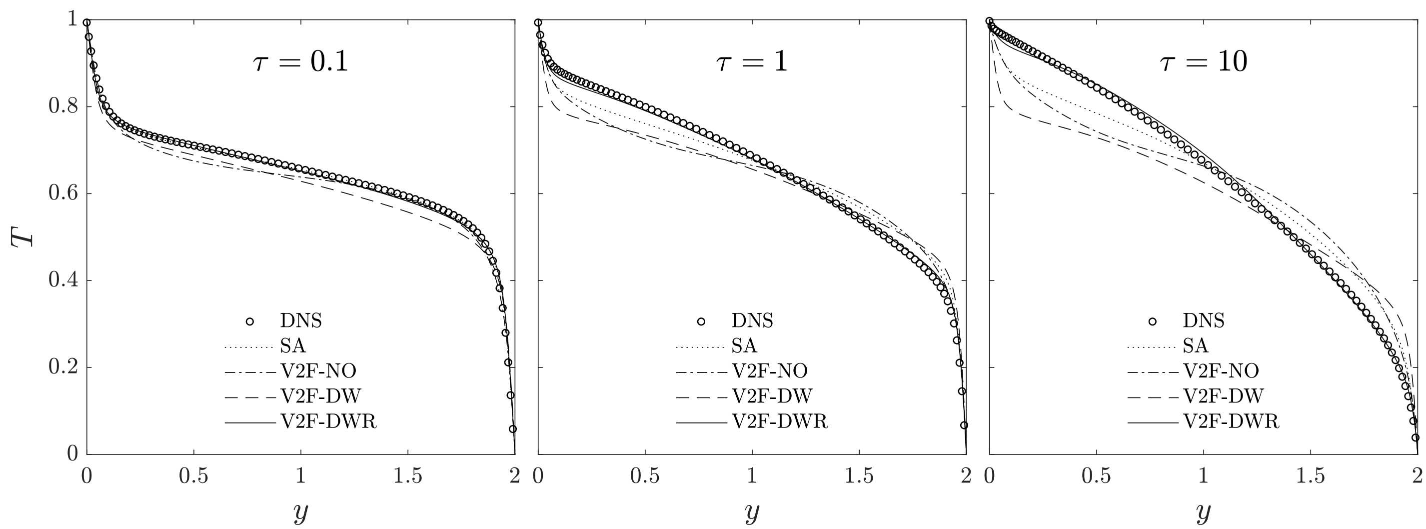

Figure 2: RANS simulation with different turbulent heat flux models for different τ cases