

Radiative modification to a general $\overline{u'_j\theta'}$ model

Quantities requiring modeling:

$$\kappa' , \quad E'_m , \quad G' .$$

They all depend on temperature (Radiative heat transfer is an analytical equation), in particular we know that:

$$\kappa = c_0 + \frac{c_1}{T} + \frac{c_2}{T^2} + \frac{c_3}{T^3} + \frac{c_4}{T^4} + \frac{c_5}{T^5} ,$$

and

$$E_m = 4 \left(\frac{\theta}{T_0} + 1 \right)^4 ,$$

where:

$$T = \theta \Delta T + T_c , \quad \bar{T} = \bar{\theta} \Delta T + T_c , \quad T' = \theta' \Delta T , \quad T_0 = \frac{T_c}{\Delta T} .$$

Approximation of E'_m

It is possible to find the fluctuation of these two quantities by linearizing their analytical expressions, starting from E_m

$$E'_m = E_m - \bar{E}_m = 4 \left(\frac{\theta}{T_0} + 1 \right)^4 - 4 \left(\frac{\bar{\theta}}{T_0} + 1 \right)^4$$

$$E'_m = \frac{4\theta^4}{T_0^4} + \frac{16\theta^3}{T_0^3} + \frac{24\theta^2}{T_0^2} + \frac{16\theta}{T_0} - \frac{4\bar{\theta}^4}{T_0^4} - \frac{16\bar{\theta}^3}{T_0^3} - \frac{24\bar{\theta}^2}{T_0^2} - \frac{16\bar{\theta}}{T_0}$$

after substituting θ with $\theta' + \bar{\theta}$

$$E'_m = \frac{4(\bar{\theta}^4 + 4\bar{\theta}^3\theta' + 6\bar{\theta}^2\theta'^2 + 4\bar{\theta}\theta'^3 + \theta'^4)}{T_0^4} + \frac{16(\bar{\theta}^3 + 3\bar{\theta}^2\theta' + 3\bar{\theta}\theta'^2 + \theta'^3)}{T_0^3} + \frac{24(\bar{\theta}^2 + 2\bar{\theta}\theta' + \theta'^2)}{T_0^2} + \frac{16(\bar{\theta} + \theta')}{T_0} +$$

$$- \frac{4(\bar{\theta}^4 + 6\bar{\theta}^2\bar{\theta}'^2 + 4\bar{\theta}\bar{\theta}'^3 + \bar{\theta}'^4)}{T_0^4} - \frac{16(\bar{\theta}^3 + 3\bar{\theta}\bar{\theta}'^2 + \bar{\theta}'^3)}{T_0^3} - \frac{24\bar{\theta}^2 + \bar{\theta}'^2}{T_0^2} - \frac{16\bar{\theta}}{T_0} ,$$

$$E'_m = \frac{4(4\bar{\theta}^3\theta' + 6\bar{\theta}^2(\theta'^2 - \bar{\theta}'^2) + 4\bar{\theta}(\theta'^3 - \bar{\theta}'^3) + \theta'^4 - \bar{\theta}'^4)}{T_0^4} + \frac{16(3\bar{\theta}^2\theta' + 3\bar{\theta}(\theta'^2 - \bar{\theta}'^2) + \theta'^3 - \bar{\theta}'^3)}{T_0^3} +$$

$$+ \frac{24(2\bar{\theta}\theta' + \theta'^2 - \bar{\theta}'^2)}{T_0^2} + \frac{16(\theta')}{T_0} .$$

In the end, simplifying and underlying the dependency on temperature fluctuations θ' :

$$E'_m = \left(\frac{16\bar{\theta}^3}{T_0^4} + \frac{48\bar{\theta}^2}{T_0^3} + \frac{48\bar{\theta}}{T_0^2} + \frac{16}{T_0} \right) \theta' +$$

$$= \left(\frac{24\bar{\theta}^2}{T_0^4} + \frac{48\bar{\theta}}{T_0^3} + \frac{24}{T_0^2} \right) (\theta'^2 - \bar{\theta}'^2) +$$

$$= \left(\frac{16\bar{\theta}}{T_0^4} + \frac{16}{T_0^3} \right) (\theta'^3 - \bar{\theta}'^3) +$$

$$= \left(\frac{4}{T_0^4} \right) (\theta'^4 - \bar{\theta}'^4) .$$

Due to the linearization we can neglect all terms depending on higher order terms, therefore in the end a good approximation for E'_m is

$$E'_m \approx f_{E_m} \theta' ,$$

where f_{E_m} is the first model equation, equal to:

$$f_{E_m} = \frac{16\bar{\theta}^3}{T_0^4} + \frac{48\bar{\theta}^2}{T_0^3} + \frac{48\bar{\theta}}{T_0^2} + \frac{16}{T_0} .$$

Approximation of κ' (only for variable κ)

To calculate κ' is necessary to calculate $\bar{\kappa}$ first as $\kappa' = \kappa - \bar{\kappa}$.

$$\bar{\kappa} = c_0 + \frac{\bar{c}_1}{\bar{T}} + \frac{\bar{c}_2}{\bar{T}^2} + \frac{\bar{c}_3}{\bar{T}^3} + \frac{\bar{c}_4}{\bar{T}^4} + \frac{\bar{c}_5}{\bar{T}^5} ,$$

taking into account the second term on the LHS, (remembering that c_1 is a constant):

$$\frac{1}{\bar{T}} = \frac{1}{\bar{T}(1 + \frac{T'}{\bar{T}})} ,$$

since $T'/\bar{T} \ll 1$ it is possible taking a taylor expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 \dots$ and linearize, therefore:

$$\frac{1}{\bar{T}} \approx \frac{1}{\bar{T}} \left(1 - \frac{T'}{\bar{T}}\right) = \frac{1}{\bar{T}} .$$

Proceeding with the same logic it is possible to show that, if $T'/\bar{T} \ll 1$, then:

$$\frac{1}{\bar{T}^2} \approx \frac{1}{\bar{T}^2} , \quad \frac{1}{\bar{T}^3} \approx \frac{1}{\bar{T}^3} , \quad \frac{1}{\bar{T}^4} \approx \frac{1}{\bar{T}^4} , \quad \frac{1}{\bar{T}^5} \approx \frac{1}{\bar{T}^5} .$$

Therefore

$$\bar{\kappa} \approx c_0 + \frac{c_1}{\bar{T}} + \frac{c_2}{\bar{T}^2} + \frac{c_3}{\bar{T}^3} + \frac{c_4}{\bar{T}^4} + \frac{c_5}{\bar{T}^5} .$$

To calculate κ' we start from the second term on the LHS ($c_0 - c_0 = 0$)

$$c_1 \left(\frac{1}{\bar{T}} - \frac{1}{\bar{T}} \right) = c_1 \left(\frac{\bar{T} - T}{T\bar{T}} \right) .$$

Substituting the expressions $T' = T - \bar{T} = \theta' \Delta T$ and linearizing the denominator ($T\bar{T} \approx \bar{T}^2$)

$$c_1 \left(\frac{1}{\bar{T}} - \frac{1}{\bar{T}} \right) \approx -c_1 \frac{\Delta T}{\bar{T}^2} \theta' .$$

For the third term, subtracting and expanding into $\bar{T} + T'$

$$c_2 \left(\frac{1}{\bar{T}^2} - \frac{1}{\bar{T}^2} \right) = c_2 \left(\frac{\bar{T}'^2 - 2\bar{T}T' - T'^2}{(\bar{T}^2 + 2\bar{T}T' + T'^2)(\bar{T}^2 + T'^2)} \right) .$$

Again linearizing the denominator ($\approx \bar{T}^4$) and the numerator ($\approx 2\bar{T}T'$) we reach

$$c_2 \left(\frac{1}{\bar{T}^2} - \frac{1}{\bar{T}^2} \right) \approx -c_2 \frac{2\Delta T}{\bar{T}^3} \theta' .$$

In the same fashion it is possible to demonstrate that

$$\begin{aligned} c_3 \left(\frac{1}{T^3} - \frac{1}{\overline{T^3}} \right) &\approx -c_3 \frac{3\Delta T}{\overline{T^4}} \theta' , \\ c_4 \left(\frac{1}{T^4} - \frac{1}{\overline{T^4}} \right) &\approx -c_4 \frac{4\Delta T}{\overline{T^5}} \theta' , \\ c_5 \left(\frac{1}{T^5} - \frac{1}{\overline{T^5}} \right) &\approx -c_5 \frac{5\Delta T}{\overline{T^6}} \theta' . \end{aligned}$$

And therefore we found an expression for κ' as

$$\kappa' \approx f_\kappa \theta' ,$$

where

$$f_\kappa = - \left(c_1 \frac{\Delta T}{\overline{T^2}} + c_2 \frac{2\Delta T}{\overline{T^3}} + c_3 \frac{3\Delta T}{\overline{T^4}} + c_4 \frac{4\Delta T}{\overline{T^5}} + c_5 \frac{5\Delta T}{\overline{T^6}} \right)$$

Approximation of G'

This is the most complex part of the modeling and requires the knowledge of the spectral wavenumbers of the non-radiative channel flow. It is possible to know only the spectral dependency of the incident radiation fluctuations, and in homogeneous isotropic turbulence for grey gas is

$$\widehat{G'} = \frac{\kappa}{\omega} \text{atan} \left(\frac{\omega}{\kappa} \right) \widehat{E'_m} .$$

From this it is already possible to state, following from the previous considerations on E'_m , that

$$\widehat{G'} \approx \frac{\kappa}{\omega} \text{atan} \left(\frac{\omega}{\kappa} \right) f_{E_m} \widehat{\theta'} .$$

The above equation states that the modes of G' are connected to the modes of θ' depending on the wavenumber and on the absorption coefficient. This means that the higher the absorption coefficient the stronger G' gets and always on smaller ω . It is possible to translate this proportionality to physical space if there is the knowledge of a characteristic wavenumber for which it is possible to write

$$G' \approx f_G f_{E_m} \theta'$$

where

$$f_G = \frac{\kappa}{\omega_c} \text{atan} \left(\frac{\omega_c}{\kappa} \right)$$

and ω_c is a characteristic wavenumber that represents the integral wavenumber weighted on the energy spectrum at a certain y -location calculated as

$$\omega_c = \left\| \int_0^\infty \omega' E_\theta(\omega') d\omega' / \int_0^\infty E_\theta(\omega') d\omega' \right\|_2$$

This wavenumber should take into account the penalty of anisotropy on the growth of G' , i.e. there should be weights in the norm that promote the spectral direction which contains energy at the largest wavenumbers since radiative energy will most likely escape from those directions. For the present case, with DNS knowledge of the spectral redistribution of energy in the channel, it is identified that the spanwise spectral wavenumber is dominant over the streamwise (i.e., energy at larger wavenumbers) and we assume that the wall normal direction contains energy at the same wavenumbers that the spanwise direction. Therefore, following the spectral energy redistribution noticed in figure 1, it is possible to approximate the characteristic wavenumber as a parabola (shape of energy redistribution in spanwise direction) as

$$\omega_c = (c_{r33} - c_{r22})y^2 - 2(c_{r33} - c_{r22})y + c_{r33} ,$$

with c_{r33} and c_{r22} the wavenumber of maximum energy at the walls and in the middle of the channel respectively.

Modification of the $\overline{\theta'^2} - \varepsilon_\theta$ two equation model

From the expressions derived in the previous section it is possible to extend a general $\overline{\theta'^2} - \varepsilon_\theta$ model to fit radiative flows. The two additional equations read:

$$\frac{\partial \overline{\theta'^2}}{\partial t} + \overline{u_j} \frac{\partial \overline{\theta'^2}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\alpha \frac{\partial \overline{\theta'^2}}{\partial x_j} - \overline{u'_j \theta'^2} \right) - 2 \overline{u'_j \theta'} \frac{\partial \overline{\theta}}{\partial x_j} - 2 \alpha \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} - \boxed{2 \frac{\alpha}{\lambda} (\kappa Q)' \overline{\theta'}} \quad (1)$$

Radiative term $\mathcal{R}_{\overline{\theta'^2}}$

$$\begin{aligned} \frac{\partial \varepsilon_\theta}{\partial t} + \overline{u_j} \frac{\partial \varepsilon_\theta}{\partial x_j} &= \frac{\partial}{\partial x_j} \left(\alpha \frac{\partial \varepsilon_\theta}{\partial x_j} - \overline{u'_j \varepsilon_\theta} \right) - 2 \alpha \frac{\partial \overline{\theta'}}{\partial x_k} \frac{\partial \overline{u'_j}}{\partial x_k} \frac{\partial \overline{\theta}}{\partial x_j} - 2 \alpha \overline{u'_j} \frac{\partial \overline{\theta'}}{\partial x_k} \frac{\partial^2 \overline{\theta}}{\partial x_j \partial x_k} \\ &\quad - 2 \alpha \frac{\partial \overline{\theta'}}{\partial x_k} \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{u_j}}{\partial x_k} - 2 \alpha \frac{\partial \overline{\theta'}}{\partial x_k} \frac{\partial \overline{u'_j}}{\partial x_k} \frac{\partial \overline{\theta'}}{\partial x_j} - 2 \alpha \frac{\partial^2 \overline{\theta'}^2}{\partial x_j \partial x_k} - \boxed{2 \frac{\alpha}{\lambda} \frac{\partial (\kappa Q)'}{\partial x_k} \frac{\partial \overline{\theta'}}{\partial x_k}} \end{aligned} \quad (2)$$

Radiative term $\mathcal{R}_{\varepsilon_\theta}$

where $\varepsilon_\theta = \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j}$ and $Q = E_m - G$.

The two equations contain the two additional ‘‘Radiative terms’’ which are unclosed terms and need to be modelled following the considerations done in the above section regarding κ' , E'_m and G' . Starting from equation (??), the additional term reads

$$\mathcal{R}_{\overline{\theta'^2}} = 2 \frac{\alpha}{\lambda} (\overline{\kappa} (E'_m \overline{\theta'} - G' \overline{\theta'}) + (\overline{E_m} - \overline{G}) \overline{\kappa' \theta'} + \overline{\kappa' E'_m \theta'} - \overline{\kappa' G' \theta'})$$

The last two terms on the LHS are third order terms and much smaller than the others, and are therefore neglected. Following from the modelling of radiative fluctuations

$$\mathcal{R}_{\overline{\theta'^2}} = 2 \frac{\alpha}{\lambda} (\overline{\kappa} f_{E_m} (1 - f_G) + (\overline{E_m} - \overline{G}) f_\kappa) \overline{\theta'^2}$$

If κ is constant $f_\kappa = 0$.

For equation (??) the radiative term can be expanded as

$$\begin{aligned} \mathcal{R}_{\varepsilon_\theta} &= 2 \frac{\alpha}{\lambda} \underbrace{\left(\frac{\partial \overline{\kappa' Q}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} + \frac{\partial \overline{\kappa' Q'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \right)}_{\text{fluctuating } \kappa} \overset{\text{linearize}}{\rightarrow} + 2 \frac{\alpha}{\lambda} \underbrace{\left(\frac{\partial \overline{\kappa Q'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \right)}_{\text{average } \kappa} \\ \mathcal{R}_{\varepsilon_\theta} &= 2 \frac{\alpha}{\lambda} \left(\frac{\partial \overline{\kappa'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \overline{Q} + \overline{\kappa'} \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{Q}}{\partial y} \right) + 2 \frac{\alpha}{\lambda} \left(\overline{Q'} \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{\kappa}}{\partial y} + \frac{\partial \overline{Q'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \overline{\kappa} \right) \end{aligned}$$

By substituting $\kappa' = f_\kappa \theta'$, $\overline{Q} = \overline{E_m} - \overline{G}$ and $Q' = f_{E_m} (1 - f_G) \theta'$ we obtain

$$\begin{aligned} \frac{\partial \overline{\kappa'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \overline{Q} &= (\overline{E_m} - \overline{G}) \left(f_\kappa \varepsilon_\theta + \frac{1}{2} \frac{\partial f_\kappa}{\partial y} \frac{\partial \overline{\theta'^2}}{\partial y} \right) \\ \overline{\kappa'} \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{Q}}{\partial y} &= \frac{\partial (\overline{E_m} - \overline{G})}{\partial y} \frac{f_\kappa}{2} \frac{\partial \overline{\theta'^2}}{\partial y} \\ \overline{Q'} \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{\kappa}}{\partial y} &= \frac{\partial \overline{\kappa}}{\partial y} \frac{f_{E_m} (1 - f_G)}{2} \frac{\partial \overline{\theta'^2}}{\partial y} \\ \frac{\partial \overline{Q'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \overline{\kappa} &= \overline{\kappa} \left(f_{E_m} (1 - f_G) \varepsilon_\theta + \frac{1}{2} \frac{\partial f_{E_m} (1 - f_G)}{\partial y} \frac{\partial \overline{\theta'^2}}{\partial y} \right) \end{aligned}$$

Both $\overline{\theta'^2}$ and ε_θ are required for the calculation of α_t which approximates $\overline{v'\theta'}$ as

$$\overline{v'\theta'} = -\alpha_t \frac{\partial \bar{\theta}}{\partial y}$$

In a standard two equation turbulent flux model, α_t is calculated with a damping function f_λ which takes into account low Reynolds number effects near the walls and a mixed time scale for temperature

$$\alpha_t = C_\lambda f_\lambda k \sqrt{\frac{k}{\varepsilon} \frac{\overline{\theta'^2}}{\varepsilon_\theta}}$$

This because the temperature time scale is calculated as a mix of the velocity time scale $\tau_u = k/\varepsilon$ and the scalar time scale $\tau_s = \overline{\theta'^2}/\varepsilon_\theta$. In case of a radiative flow, radiative fluctuations act as a direct dissipation of temperature fluctuation, reducing significantly the scalar time scale. Therefore the new scalar time scale has to account for radiation as

$$\tau_s = \frac{\overline{\theta'^2}}{\varepsilon_\theta + \varepsilon_r}$$

The new radiative dissipation in a strict sense is the dissipation part of the radiative term, which can be decomposed as

$$\mathcal{R}_{\overline{\theta'^2}} = 2 \left(\underbrace{\frac{\partial q'_{ry} \theta'}{\partial y}}_{\phi_r} - \underbrace{q'_{rj} \frac{\partial \theta'}{\partial x_j}}_{\varepsilon_r} \right)$$

On the other hand, in the center of the channel $\varepsilon_r \gg \phi_r$, therefore it is possible to assume $\varepsilon \approx 0.5 \mathcal{R}_{\overline{\theta'^2}}$. Therefore the new formulation for the eddy diffusivity is

$$\alpha_t = C_\lambda f_\lambda k \sqrt{\frac{k}{\varepsilon} \frac{\overline{\theta'^2}}{\varepsilon_\theta + 0.5 \mathcal{R}_{\overline{\theta'^2}}}}$$

Results

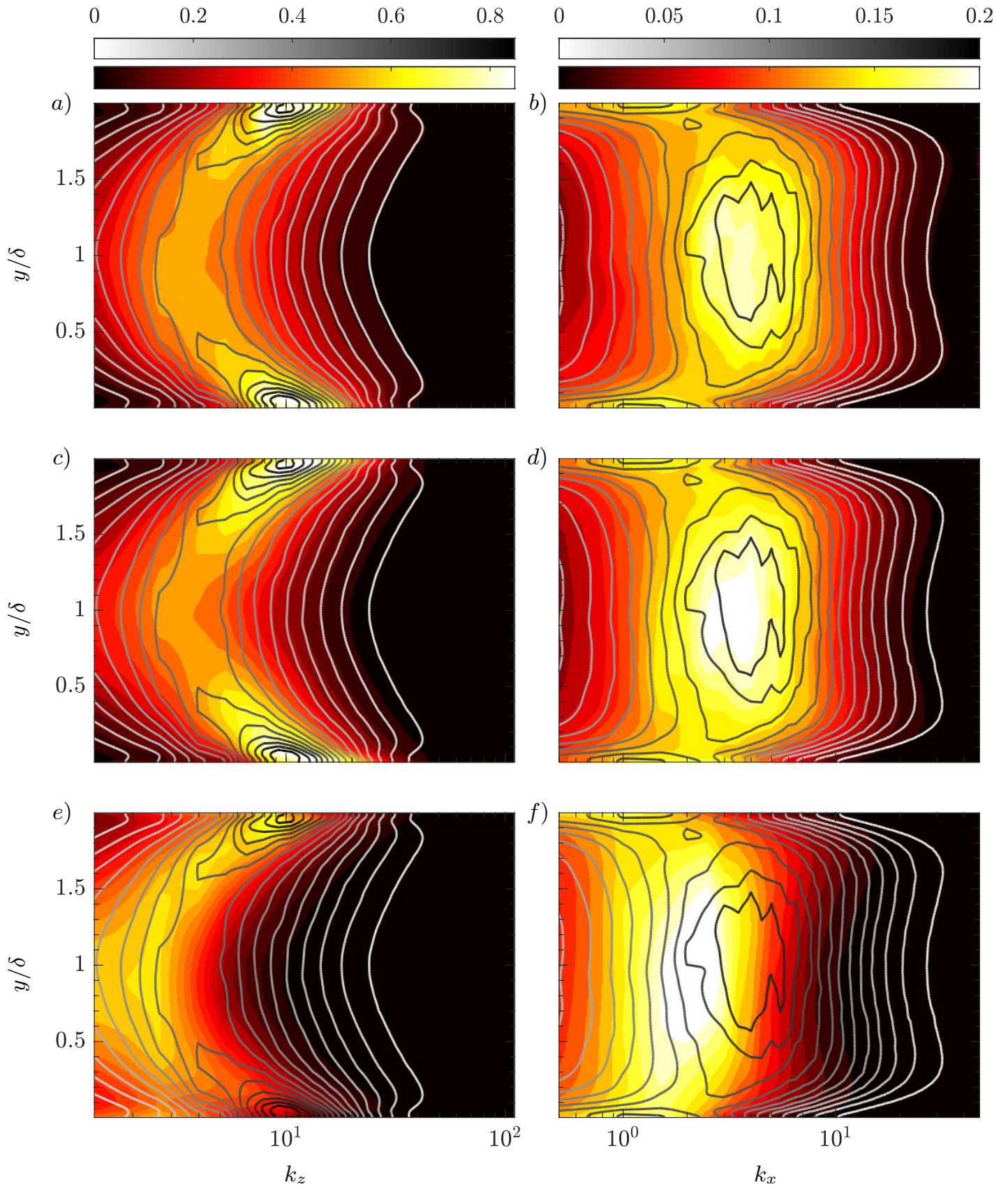


Figure 1: RANS simulation with different turbulent heat flux models for different τ cases

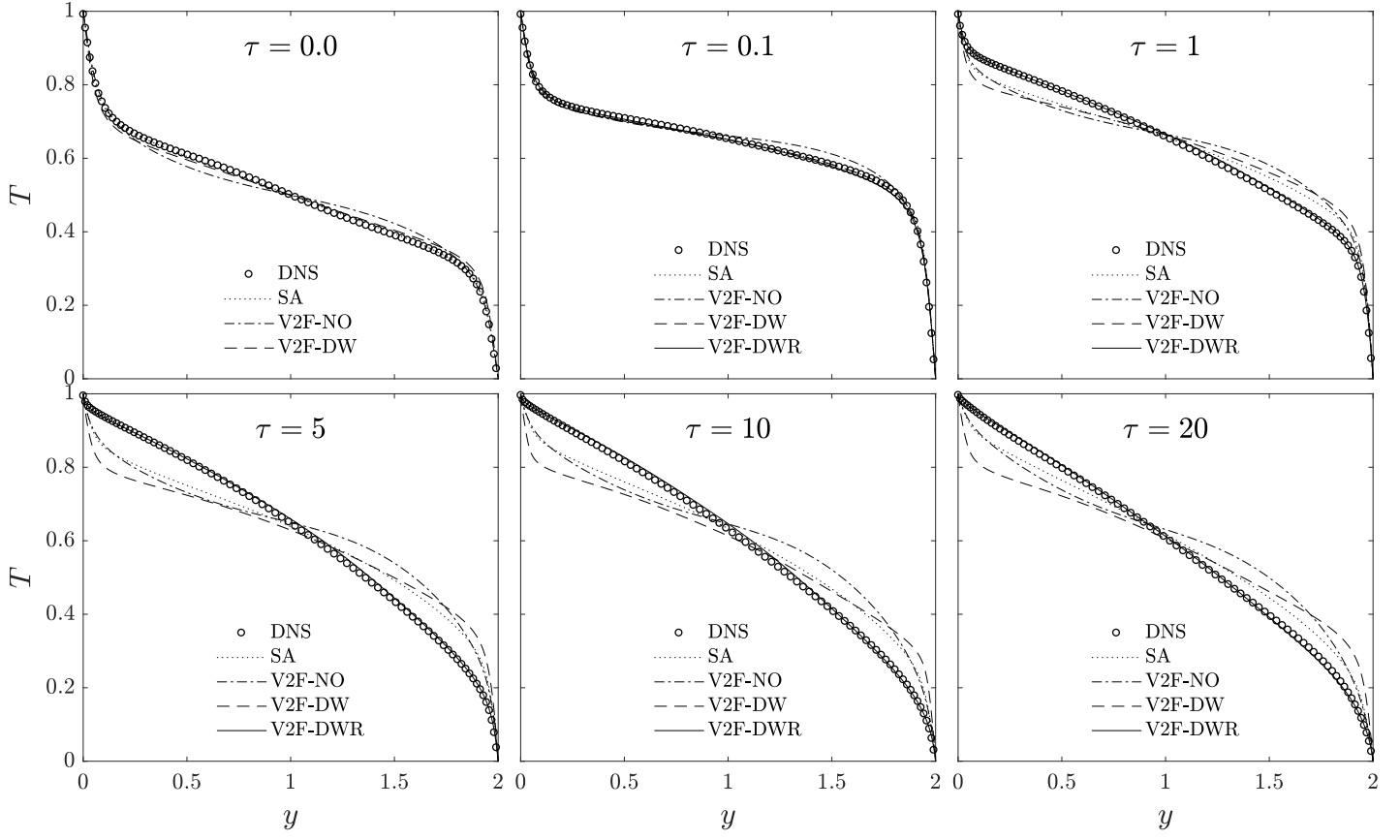


Figure 2: RANS simulation with different turbulent heat flux models for different τ cases

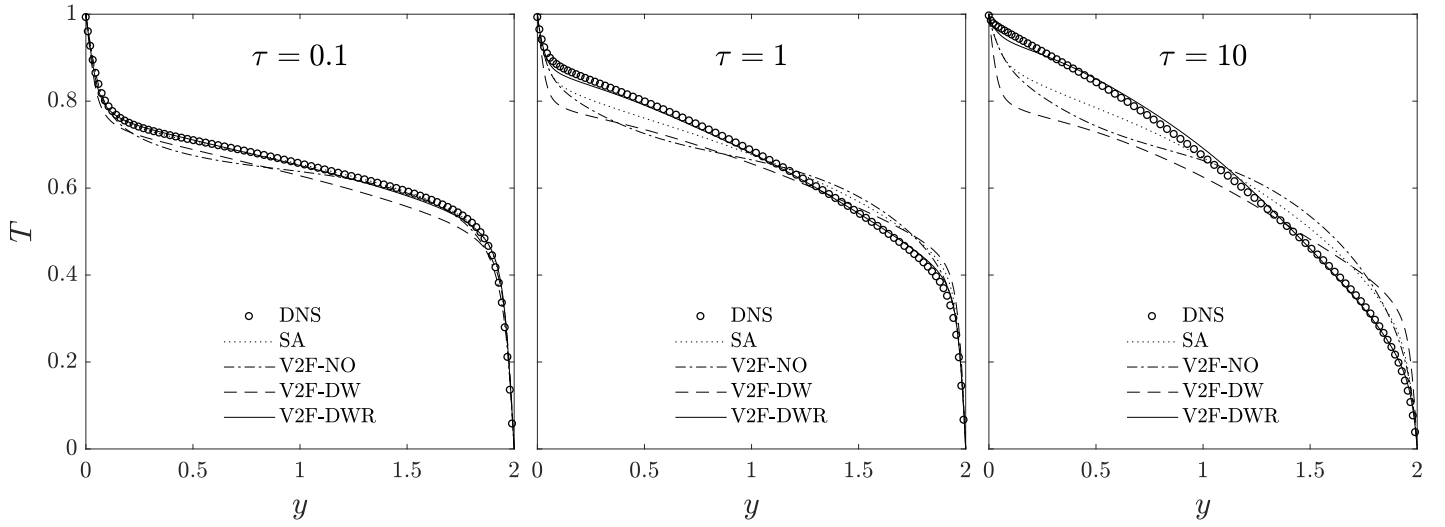


Figure 3: RANS simulation with different turbulent heat flux models for different τ cases