CHAPTER 7

Vectors and Matrices

Matrices and vectors are unfamiliar but essential mathematical tools in the social sciences and in science in general. In this book they will be used extensively in the study of networks and in probability models. They will be introduced in this and the next chapter. Please be patient. Their usefulness will become apparent in ensuing chapters.

The familiar single number is a *scalar*. A column of numbers is a *column vector*, and a row of numbers is a *row vector*. A matrix is a rectangular array of numbers that may have more than one row (a horizontal set of numbers) and column (a vertical set of numbers).

A matrix has both rows (horizontal arrays) and columns (vertical arrays). The number of rows and the number of columns give the order of the matrix. For example, suppose we wanted to create a five by five matrix **A** of the total dollar value of trade among five countries. The five rows represent the five countries as exporters and the five columns the same countries as importers.

$$\mathbf{A} = \begin{pmatrix} 0 & 50 & 70 & 80 & 30 \\ 20 & 0 & 49 & 30 & 50 \\ 69 & 68 & 0 & 60 & 80 \\ 70 & 20 & 50 & 0 & 10 \\ 10 & 60 & 70 & 5 & 0 \end{pmatrix} \tag{7.1}$$

Every element of a matrix has two subscripts, for the element's row (counting from the top) and for its column (counting from the left). The rows and columns are numbered from the upper left corner of the matrix. Matrix **A** has five rows and five columns. Suppose we found out that exports from Country 3 to country 2 were 70 instead of 68. Then we would change $a_{3,2}$ from 68 to 70. Notice that matrices are referred to by capital letters but their elements lowercase letters.

After the change the matrix **A** would have the following form:

$$\mathbf{A} = \begin{pmatrix} 0 & 50 & 70 & 80 & 30 \\ 20 & 0 & 40 & 30 & 50 \\ 60 & 70 & 0 & 60 & 80 \\ 70 & 20 & 50 & 0 & 10 \\ 10 & 60 & 70 & 5 & 0 \end{pmatrix} \tag{7.2}$$

Notice that there are zeros on the *main diagonal* of the matrix. The main diagonal of a matrix runs from the upper left to the lower right. You might ask yourself why this is true.

Some countries, which import more than they export, have a trade deficit, and others, which export more than they import, have a trade surplus. Clearly, a country has a trade surplus if the sum of its row, the amount it exports to other countries, is greater than the sum of its column, the amount it imports from other countries. There is a way in mathematics of representing the sums of numbers using the capital Greek letter sigma. For example, suppose that we want to calculate the sum of the first row of the first **A**, which gives the amount exported by Country 1. We would first define a range variable j that runs from 1 to 5, for the columns of **A**, then sum across the column for the first row.

$$\sum_{j=1}^{5} a_{1j} = a_{11} + a_{12} + a_{13} + a_{14} + a_{15} = 230 \tag{7.3}$$

This says "Take the sum of first row of the matrix **A** across the columns referenced by j." The first subscript in the summation refers to the row of the matrix. The second subscript refers to the column. Since j is a range variable, the summation is over the range of values defined by j. To get the sum of the second row, we would write,

$$\sum_{j=1}^{5} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} + a_{25} = 140 \tag{7.4}$$

To get the sum of all the rows, we could create a new variable i that stands for the rows, and then form a row sum for each row i. The result is a vector of numbers. We could also create a vector **e** of exports in this fashion.

$$\mathbf{e}_i = \sum_{j=1}^{5} a_{ij} = a_{i1} + a_{i2} + a_{i3} + a_{i4} + a_{i5} \tag{7.5}$$

$$\mathbf{e} = \begin{pmatrix} 230 \\ 140 \\ 268 \\ 150 \\ 145 \end{pmatrix} \tag{7.6}$$

$\mathbf{e_i}$ gives the exports for country i. For example, $\mathbf{e}_3 = 268$. The same thing could be done with respect to imports. We could create a vector **m** of imports by summing down each column of **A**.

$$m_j = \sum_{i=1}^{5} a_{ij} \tag{7.7}$$

$$\mathbf{m} = \begin{pmatrix} 160 \\ 198 \\ 230 \\ 175 \\ 170 \end{pmatrix} \tag{7.8}$$

From these two vectors, we can calculate whether each country has a trade surplus or deficit. The surplus **s**, positive or negative, is the difference between elements of the two vectors.

$$\mathbf{s} = \mathbf{e} - \mathbf{m} = \begin{pmatrix} 70 \\ -58 \\ 38 \\ -25 \\ -25 \end{pmatrix} \tag{7.9}$$

SOCIOMETRIC MATRICES

Now let us look at another example, one in which a matrix tells us who influences whom among a set of 6 friends. In this matrix **P**, a one means that the row person influences the column person, and a zero means that he does not. **P** is an instance of an adjacency matrix. An adjacency matrix is a square matrix of ones and zeros. A one corresponds to the presence of a relationship, and a zero corresponds to its absence. For example, in the matrix **P**, $p_{ij} = 1$ means that iPj, or i influences j.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{7.10}$$

If we sum the rows of this matrix, we can find out how many people are influenced by each person.

$$\text{Influence} = \begin{pmatrix} 5 \\ 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{pmatrix} \tag{7.11}$$

On the other hand, the sum of the columns gives us how many others each person was influenced by

$$\text{Influenced by} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 3 \\ 3 \\ 4 \end{pmatrix}. \tag{7.12}$$

Notice how different the row and column sums are. The sum of each row is a possible measure of each person's *power*, since it tells how many people she influences. The sum of each column, however, tells us how many people each person lets influence her. It would be a measure of how subject someone is to influence from others. A person high on power but low on being influenced by others would be a trendsetter. A person high on power but subject to influence from others would be a transmitter of trends.

Now let us look at another example. The following matrices show patterns of friendship and helping among 14 workers in a section of a factory. The **H** matrix shows who helped whom, and the **P** matrix shows which pairs of workers were friends.

$$\mathbf{P} = \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&1&0&0&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&1&1&0&0&0&1\\ 0&0&1&0&1&1&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&0&0&0 \end{pmatrix} \tag{7.13}$$

$$\mathbf{H} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&1&0&0&0&0 \end{pmatrix} \tag{7.14}$$

The sums of the rows of **P** give us the number of friends each person had. The sum of the rows of **H** gives the number of others each person helped. The sum of the columns of **H** give the number of others who helped each person.

Notice that the row and column sums of **H** mean quite different things. A person whose row sum for **H** is large helps many others. He is generous and perhaps good at his job, so that others want his help. A person whose column sum is large is helped by others. He may seem helpless.

Remember that a symmetric relation is one in which aRb implies bRa. Therefore, the adjacency matrix **R** for a symmetric relation R is one in which $aRb = bRa$ for every a and b. This type of matrix is called a symmetric matrix. In a symmetric matrix the ith row is equal to the ith column for every i: $r_{ij} = r_{ji}$ for every i and j. You should satisfy yourself that the matrix **P** is symmetric while the matrix **H** is not.

PROBABILITY MATRICES

As we shall see in the later chapter, matrices are useful ways of analyzing probabilities of changes among sets of states. What you learn in these chapters might help you in your next trip to Las Vegas. Suppose, for example, that you are playing the roulette wheel in Las Vegas, making \$10 bets on red or black, and that you will quit once you have lost all your money or have \$100. There are 11 states you can be in, having \$0, \$10, up to \$100. The casinos have arranged things so that you are more likely to lose (10/19) each \$10 bet than to win (9/19; surprise!). An 11 by 11 matrix can be used to show the probabilities of moving from one state to another.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{19} & 0 & \frac{9}{19} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.15}$$

All but two of the main diagonal elements of the matrix are zeros because you either win or lose. The main diagonals in the rows for \$0 (the first) and \$100 (the last) are 1.00 because these are when you have decided to stop gambling.

In a later chapter we will examine the advantages of using matrices to answer questions like the probability of your ending up either broke or with $100 and how this depends on the amount of money you start with.

THE MATRIX, TRANSPOSED

We know that the element in the ith row and jth column of an adjacency matrix $\mathbf{R}$ gives us the relation between person i and person j.

$$r_{ij} = 1 \quad \text{when } iRj \tag{7.16}$$

The transpose of a matrix $\mathbf{R}$, $\mathbf{R}^T$, reverses the rows and columns.

$$(\mathbf{R}^T)_{ij} = 1 \quad \text{whenever } jRi \tag{7.17}$$

$$(\mathbf{R}^T)_{ij} = 1 \quad \text{whenever } r_{ij} = 1 \tag{7.18}$$

For example, the matrix $\mathbf{H}$ shows who (rows) helped whom (the columns), but the transpose of $\mathbf{H}$ would show who (the rows) was helped by whom (the columns).

We have been discussing the transpose of adjacency matrices, but all matrices, whether adjacency matrices or not, have transposes formed by reversing the rows and columns. The general definition of the transpose is as follows:

Definition. In the transpose $\mathbf{A}^T$ of the matrix $\mathbf{A}$, $(\mathbf{A}^T)_{ij} = \mathbf{A}_{ji}$.

For example,

$$\begin{pmatrix} 3 & -1 & 5 \\ 0 & 2 & 4 \\ 5 & 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 5 \\ -1 & 2 & 1 \\ 5 & 4 & 3 \end{pmatrix} \tag{7.19}$$

It should be clear that if a relationship R is symmetric, then $\mathbf{R}^T = \mathbf{R}$ for its adjacency matrix. Also, it should be clear that for any matrix $\mathbf{M}$, $(\mathbf{M}^T)^T = \mathbf{M}$.

Finally, the main diagonal of the $\mathbf{A}$ matrix on the first page of this chapter is filled with zeros because a country cannot export to itself.

Chapter Demonstrations

- *Graphs from Matrices*

EXERCISES

1. The demonstration *Graphs from Matrices* shows how zero-one adjacency matrices can represent networks. Why is only the bottom half of the

matrix (below the main diagonal) represented in the top figure, the one you click on and check?

2. Using this demonstration, create a network in which one person is liked by five others who are not friends of each other. Draw the matrix for this network.
3. Again using the demonstration, create a network in which there are five people, all of whom like each other. Draw the matrix for this network.