Static analysis and software verification

Lecture 2 - Mathematical background

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- Set $\approx$ Logics
  - subset $\approx$ implication
  - union $\approx$ disjunction (or)
  - intersection $\approx$ conjunction (and)

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- Example: $\leq$ in $\mathbb{Z}$ (informally)
  - Any integer is equal to itself, therefore it is less or equal
  - If an integer i_1 is less or equal than i_2, and i_2 is less or equal than i_1, then i_1 and i_2 are the same
  - i_1 is less or equal than i_2, i_2 is less or equal than i_3, then i_1 is less or equal than i_3

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  - $|\wp(\{0,1\})| = 4$

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- A set X equipped with a partial order $\sqsubseteq$ is a poset, denoted with $\langle X, \sqsubseteq \rangle$

Exercise

$\supseteq$ on $\wp(X)$ is a partial order?

Exercise

The inverse of a partial order is a partial order?

Exercise

Other examples of posets?

Hasse diagrams

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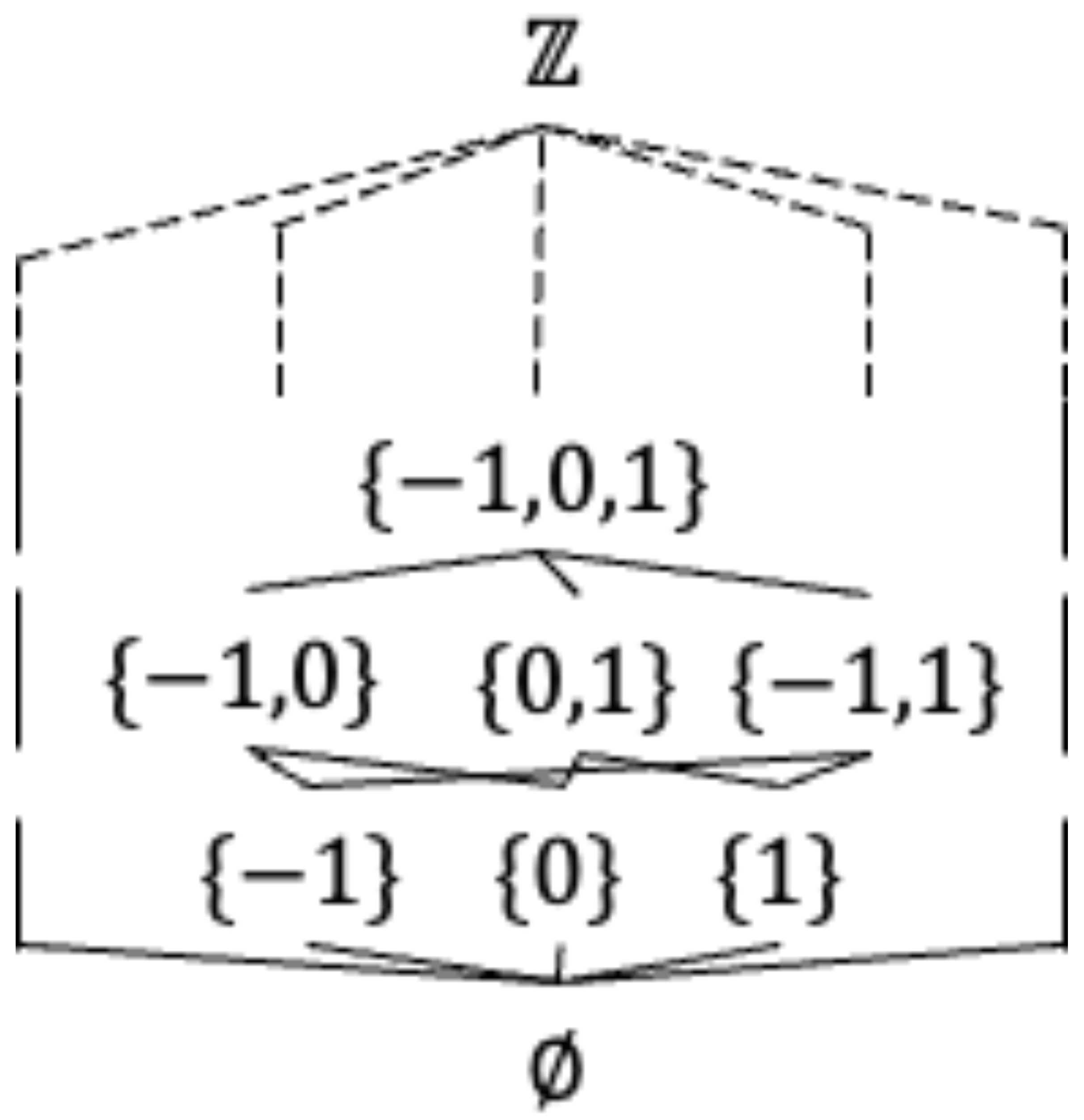
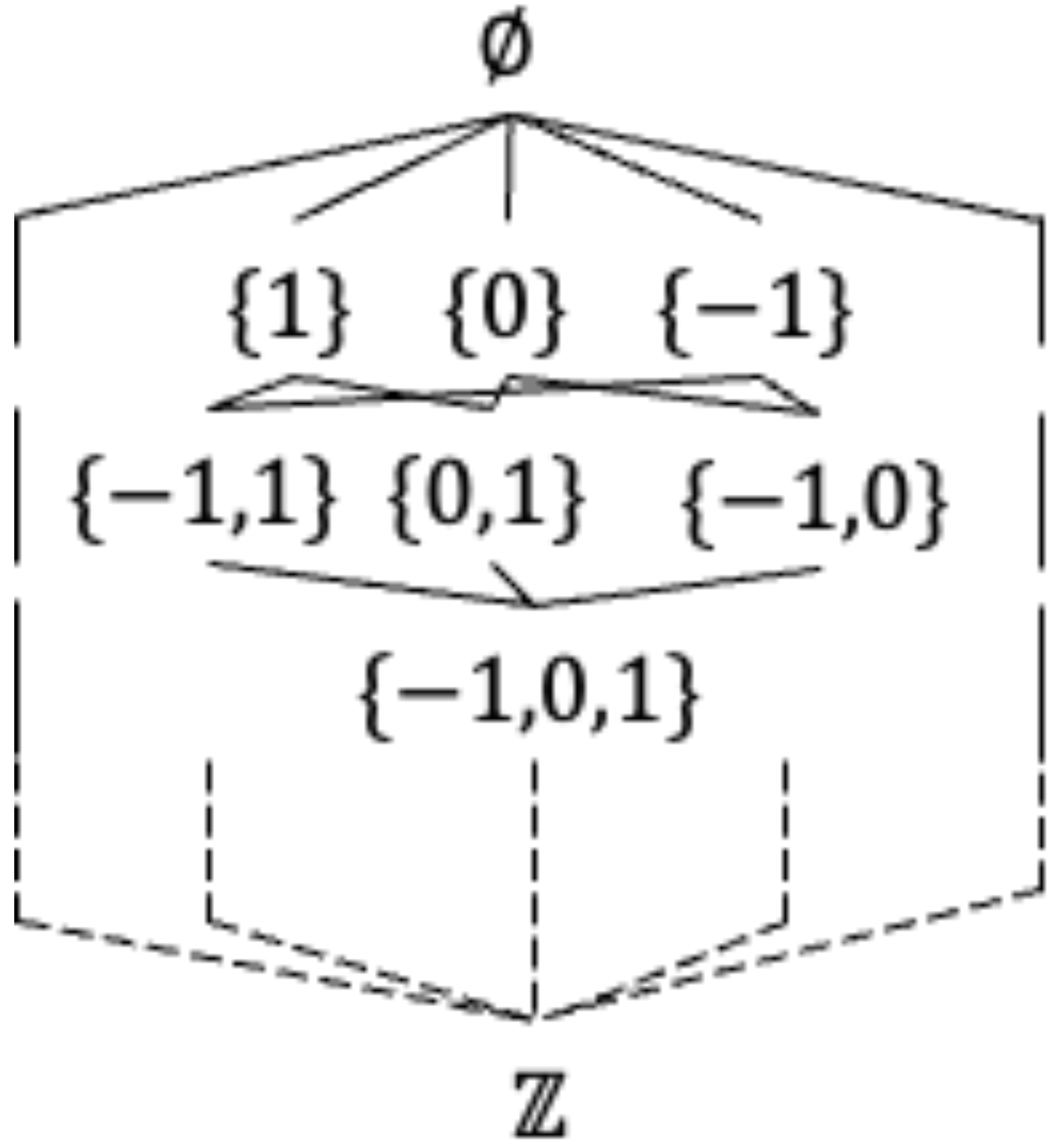
- Inverse poset: 180° rotation

Hasse diagrams

Examples





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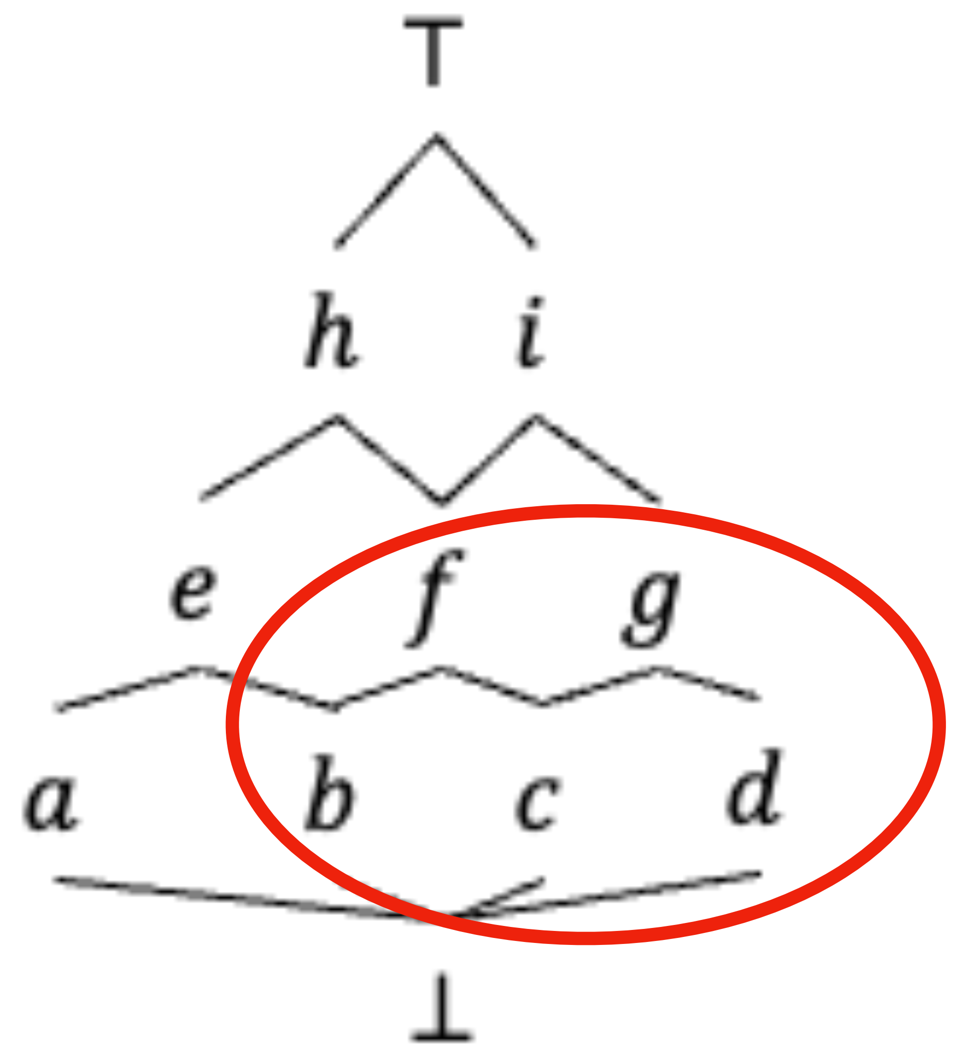

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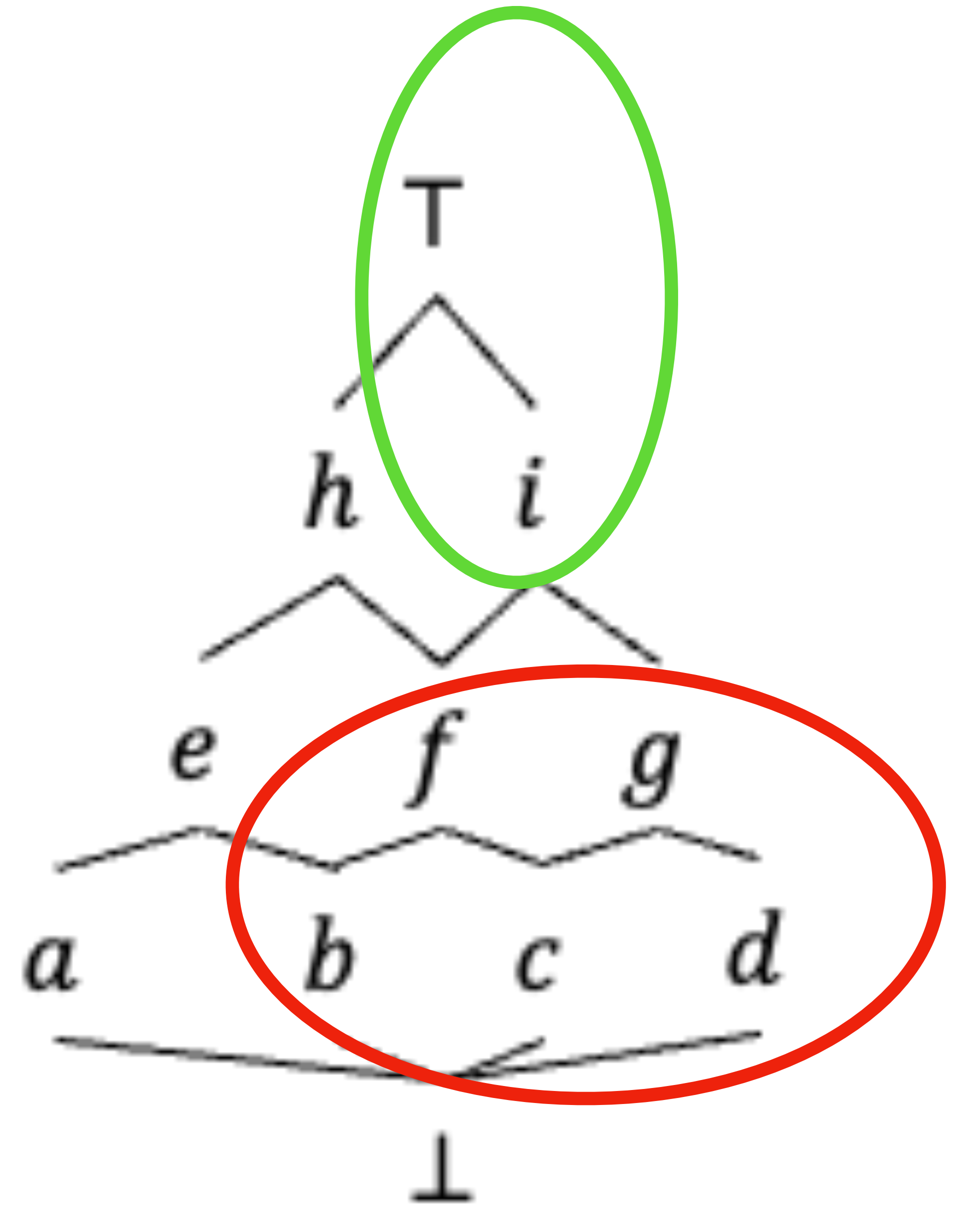

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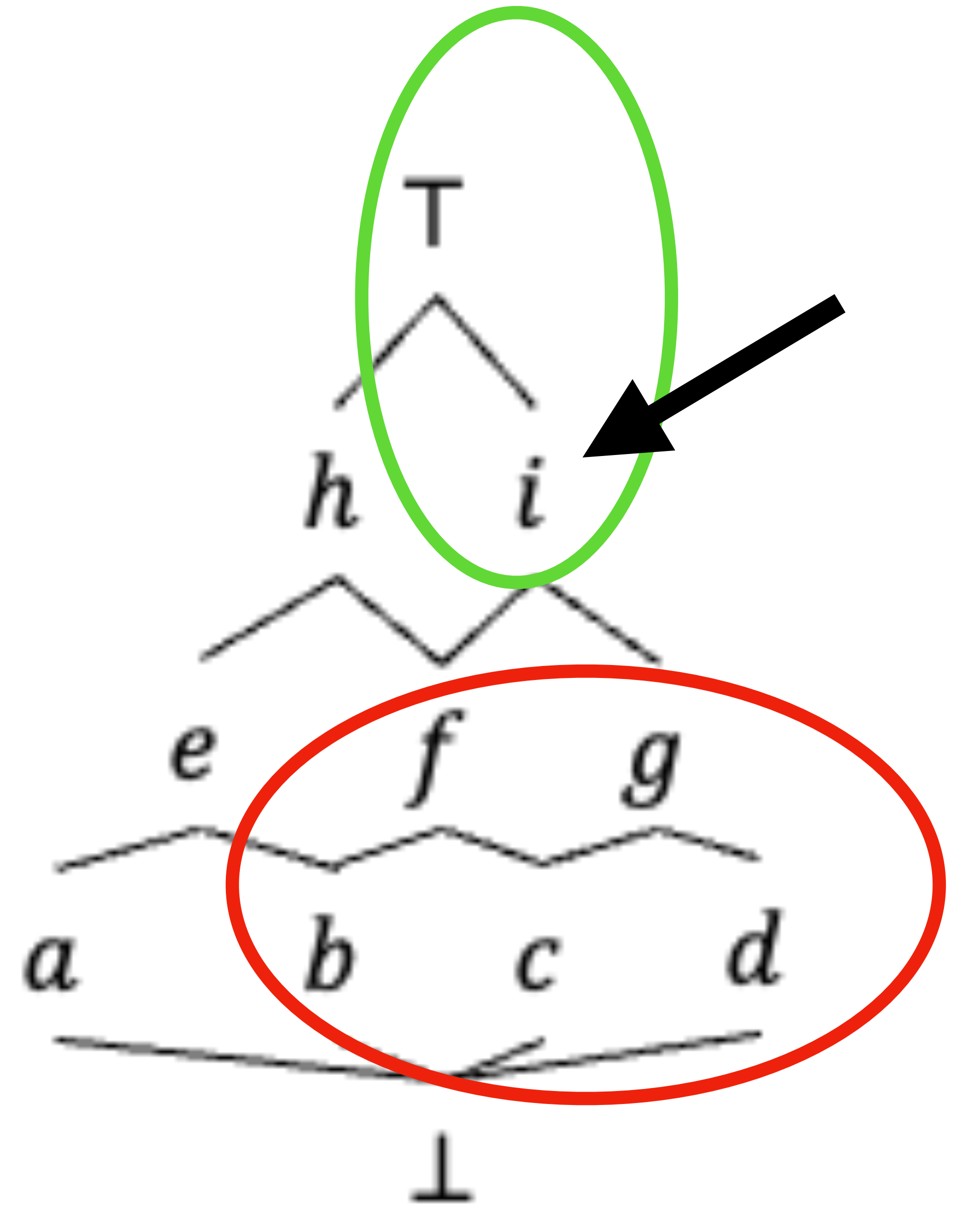

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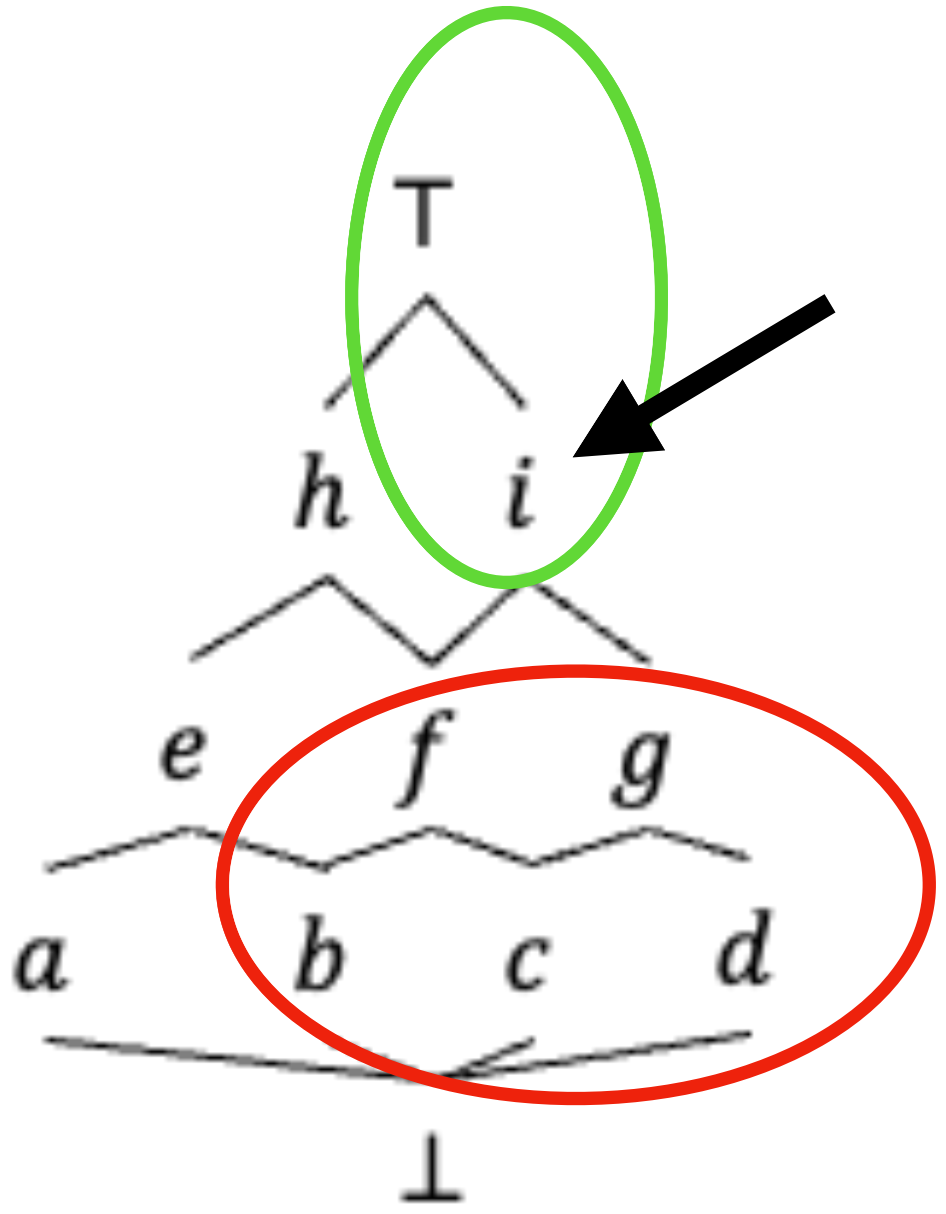

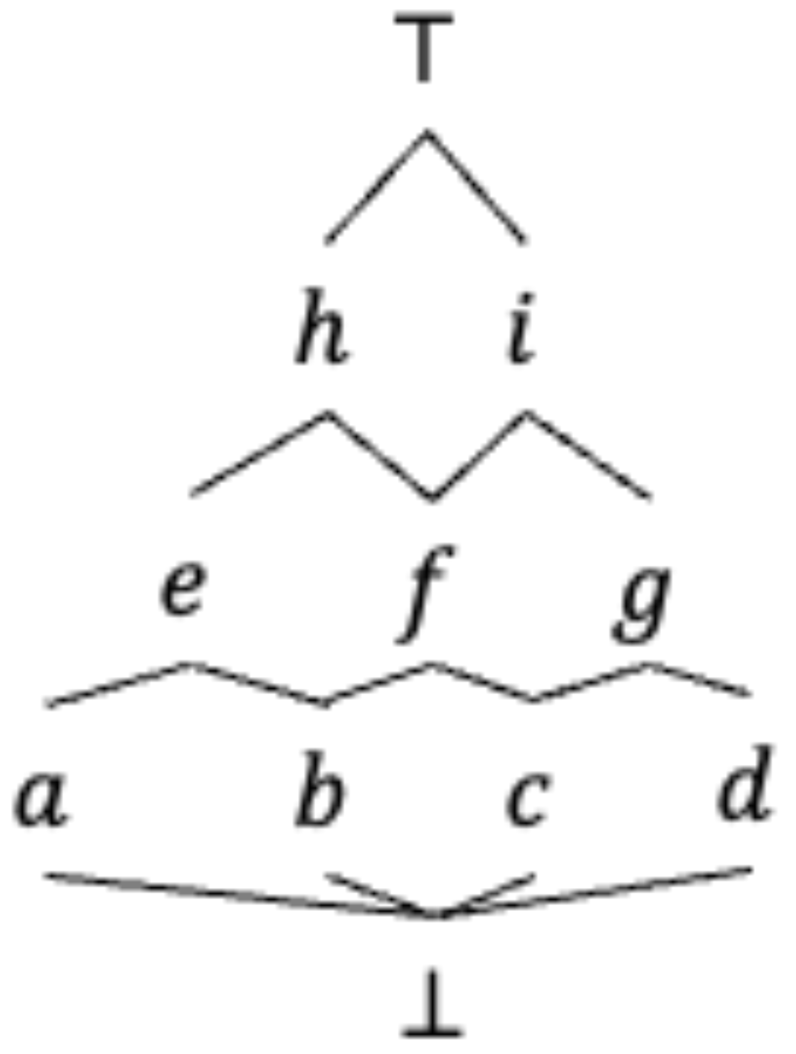

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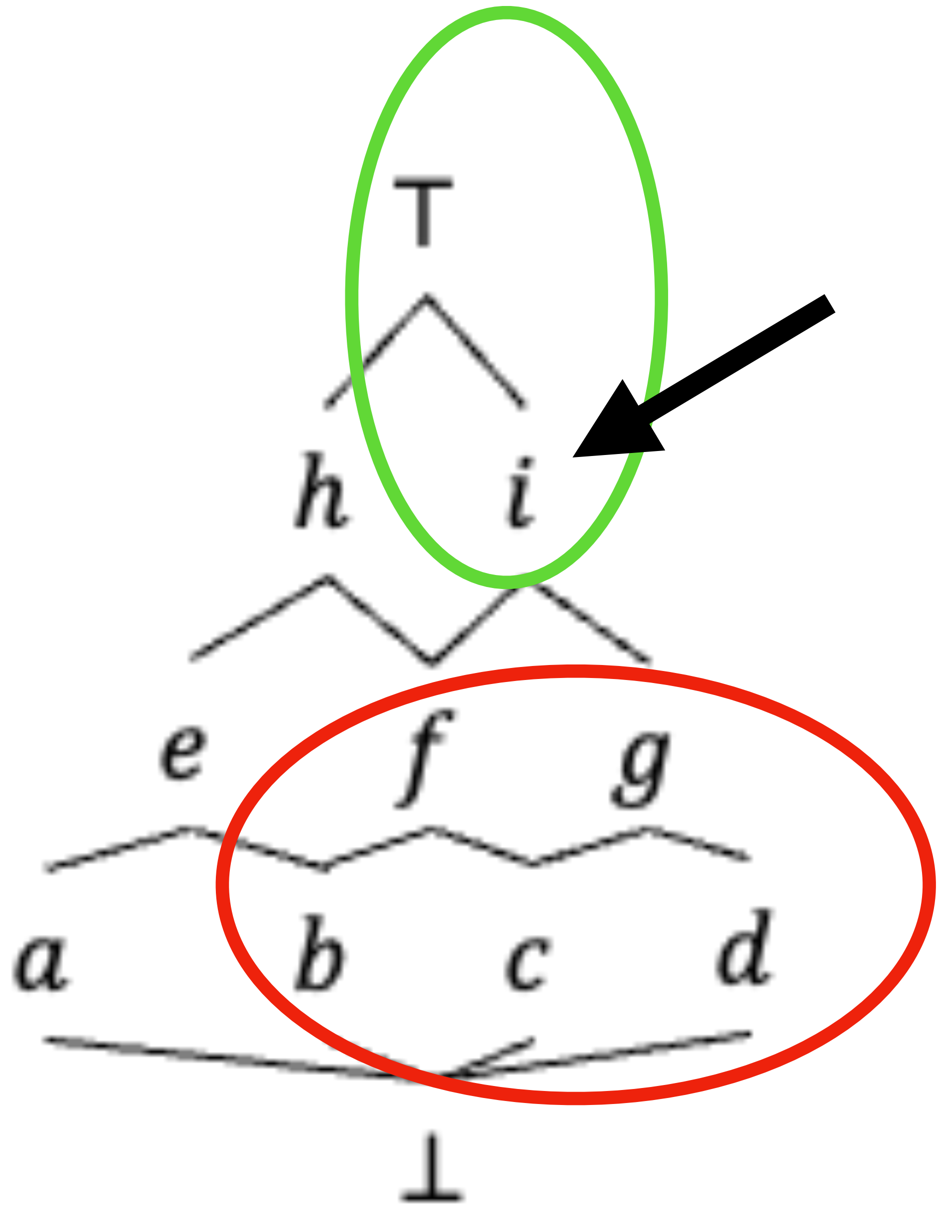

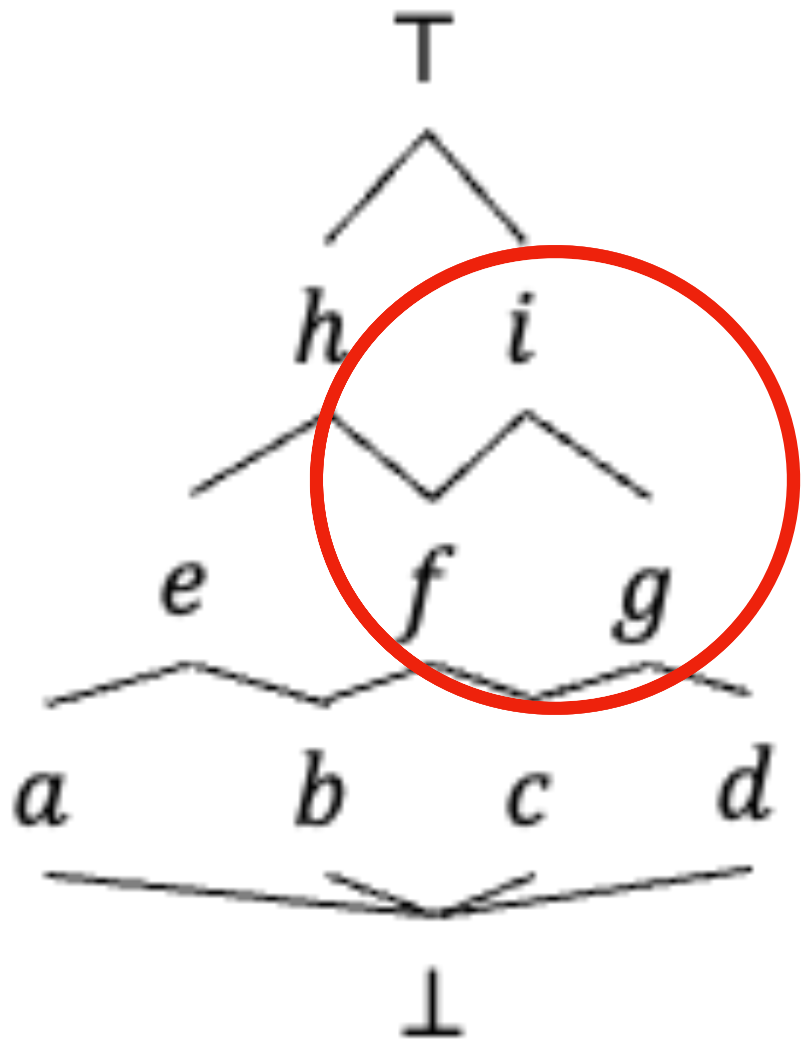

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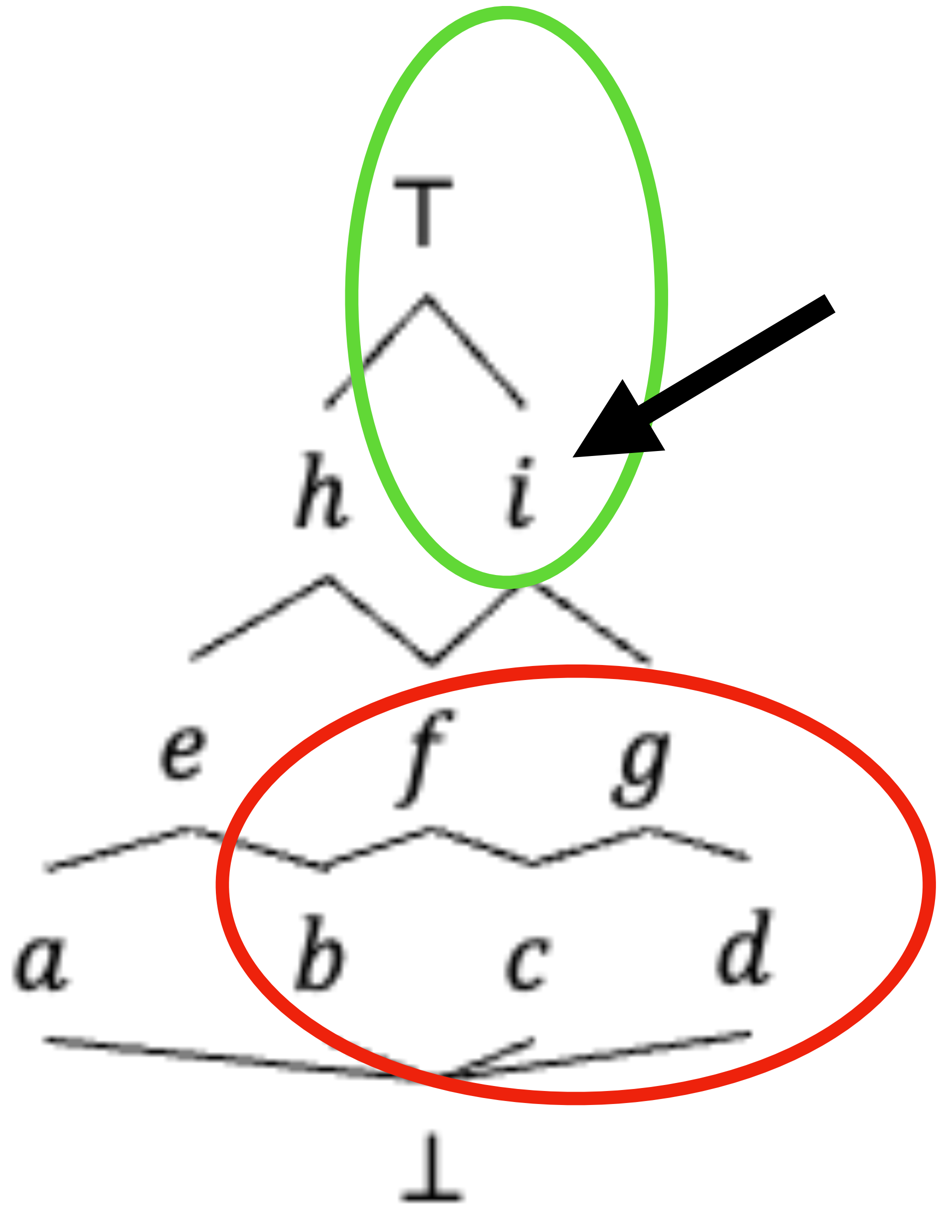

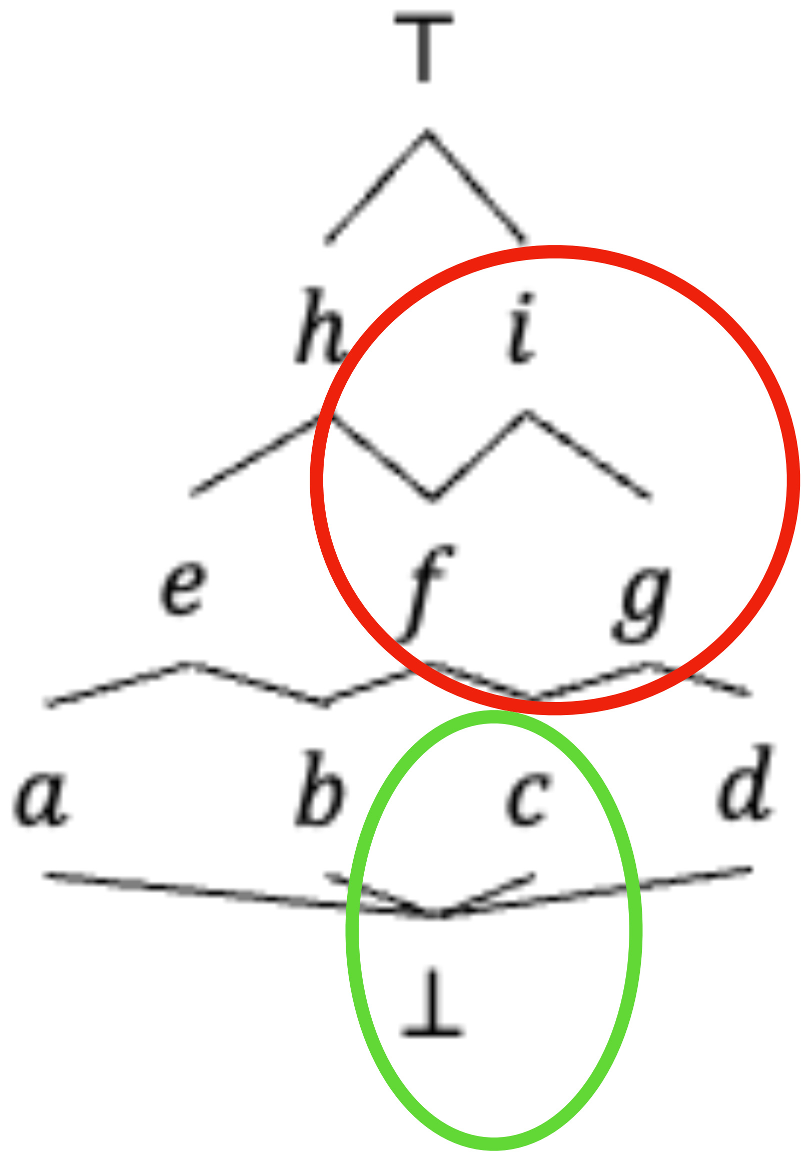

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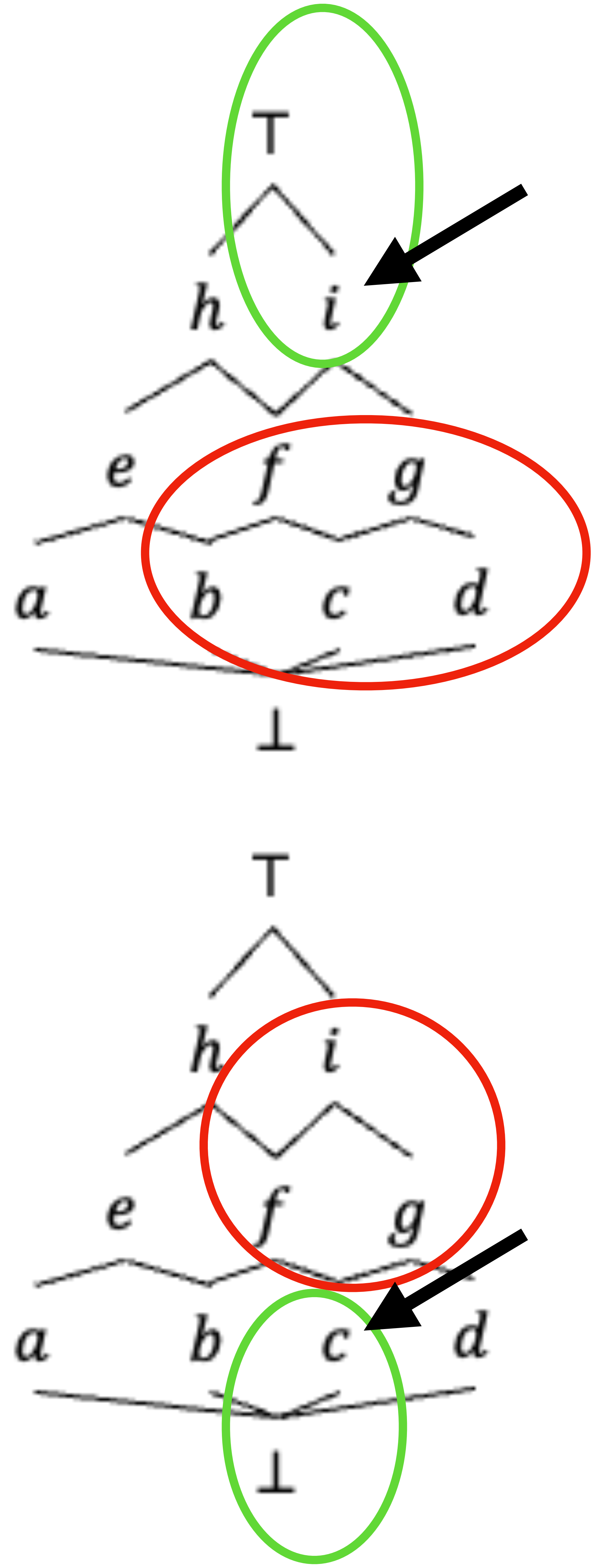

Exercise

- Given $\langle \wp(X), \subseteq \rangle$, let $S_1, S_2 \in \wp(X)$. Is $S_1 \cup S_2$ the lub of $\{S_1, S_2\}$?
- Given $\langle \wp(X), \subseteq \rangle$, let $S_1, S_2 \in \wp(X)$. Is $S_1 \cap S_2$ the glb of $\{S_1, S_2\}$?

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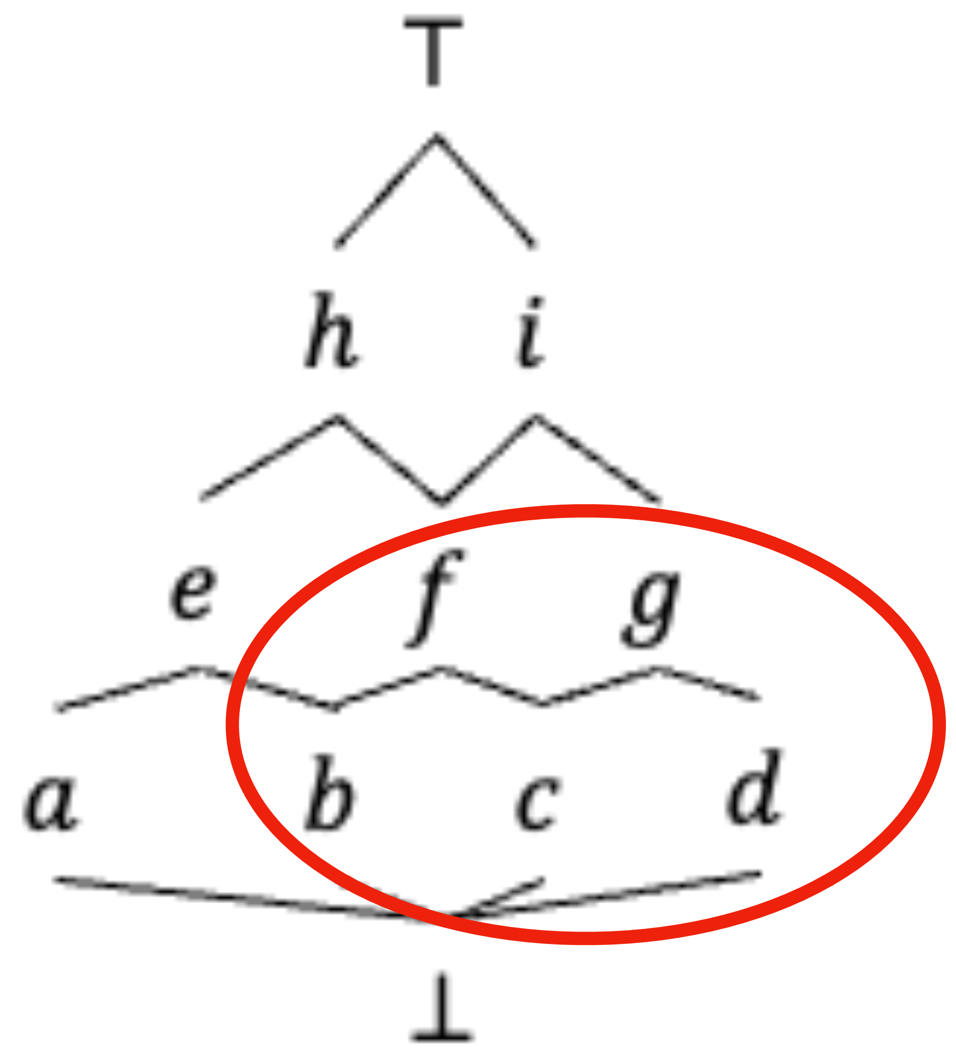

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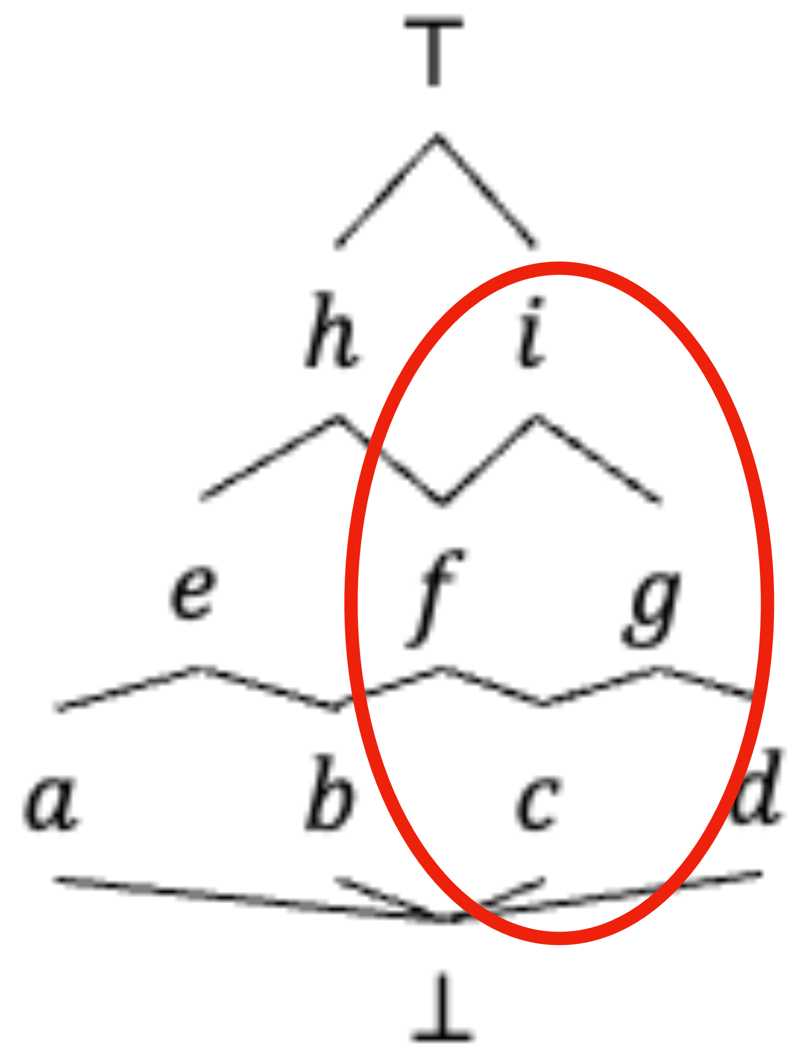

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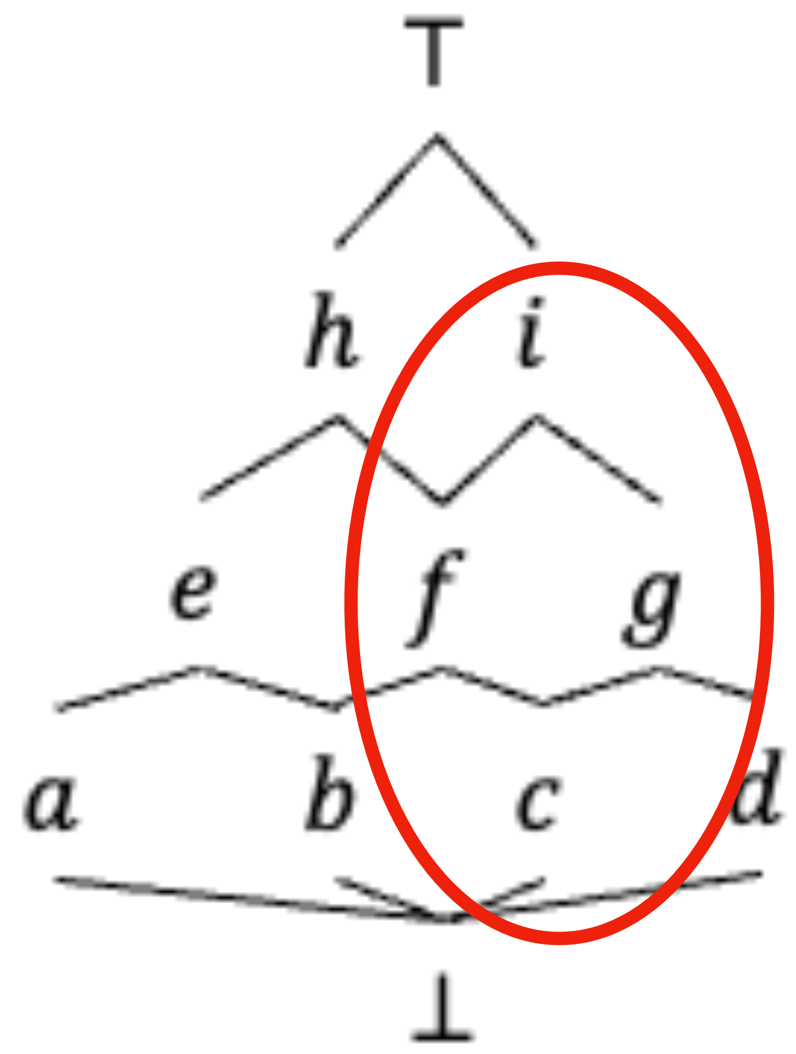

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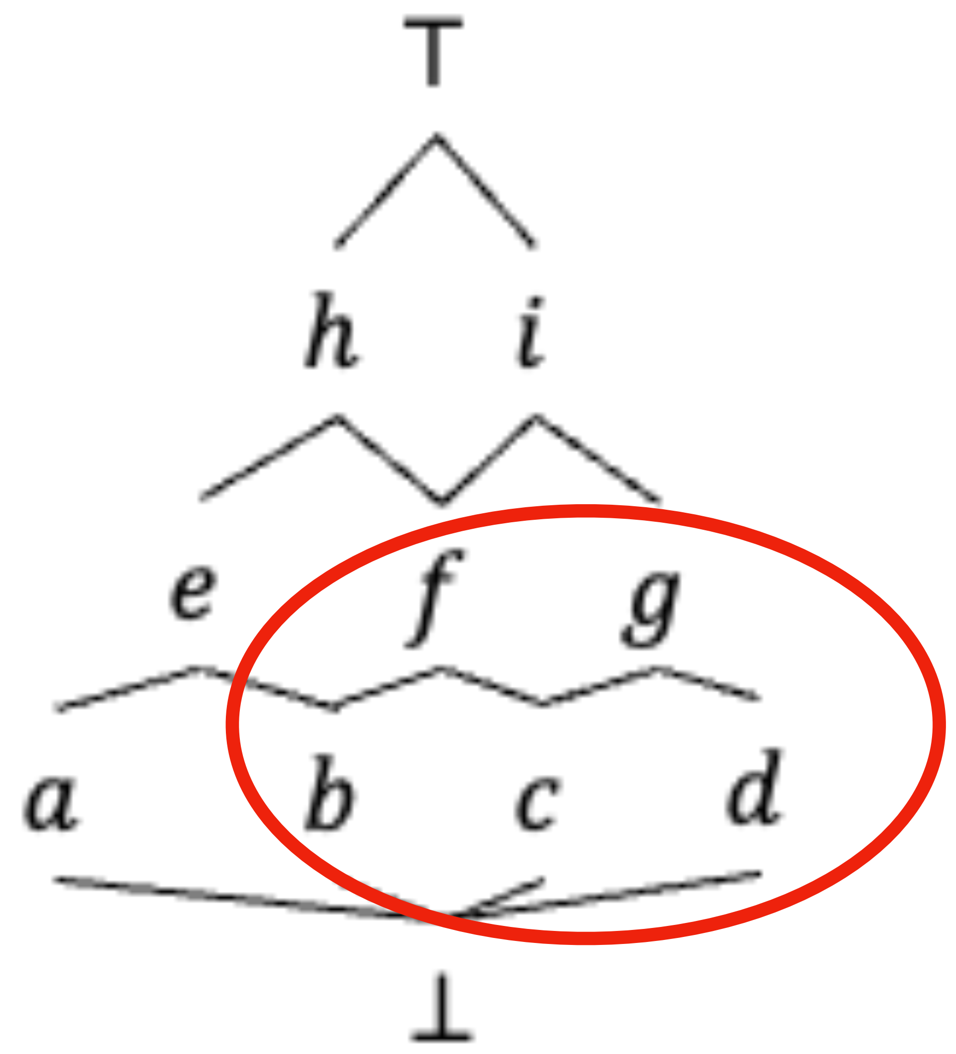

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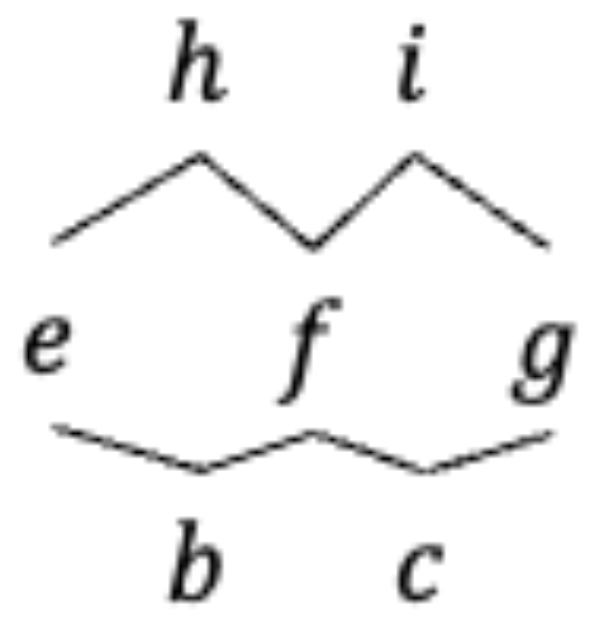

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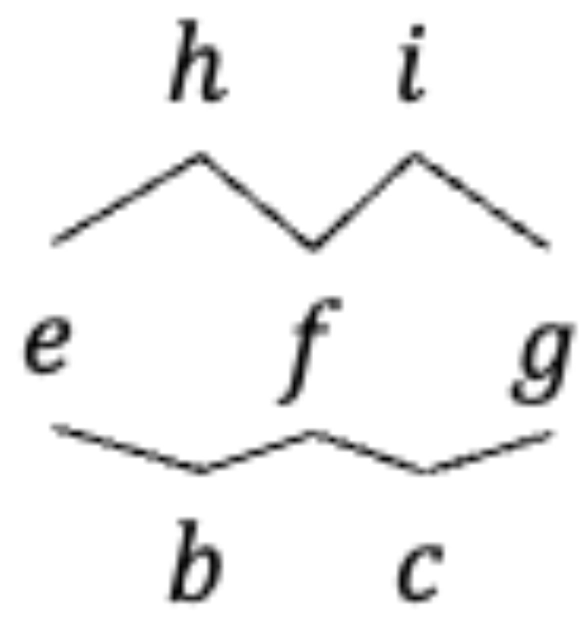

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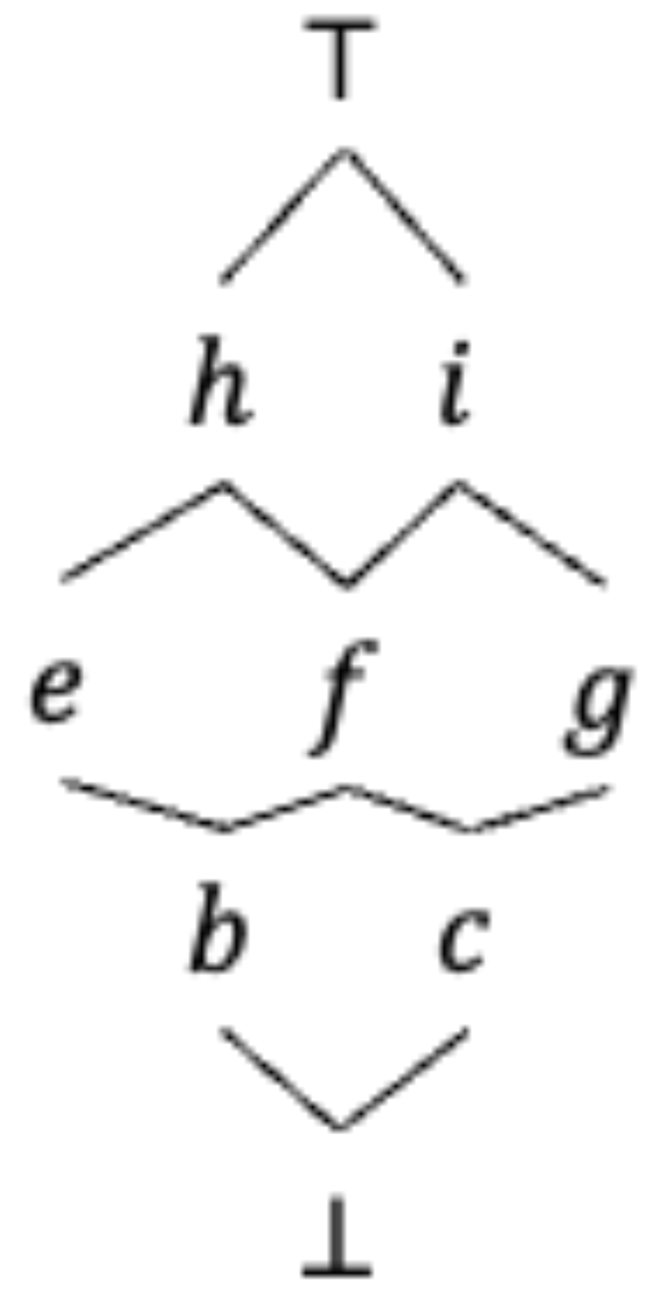

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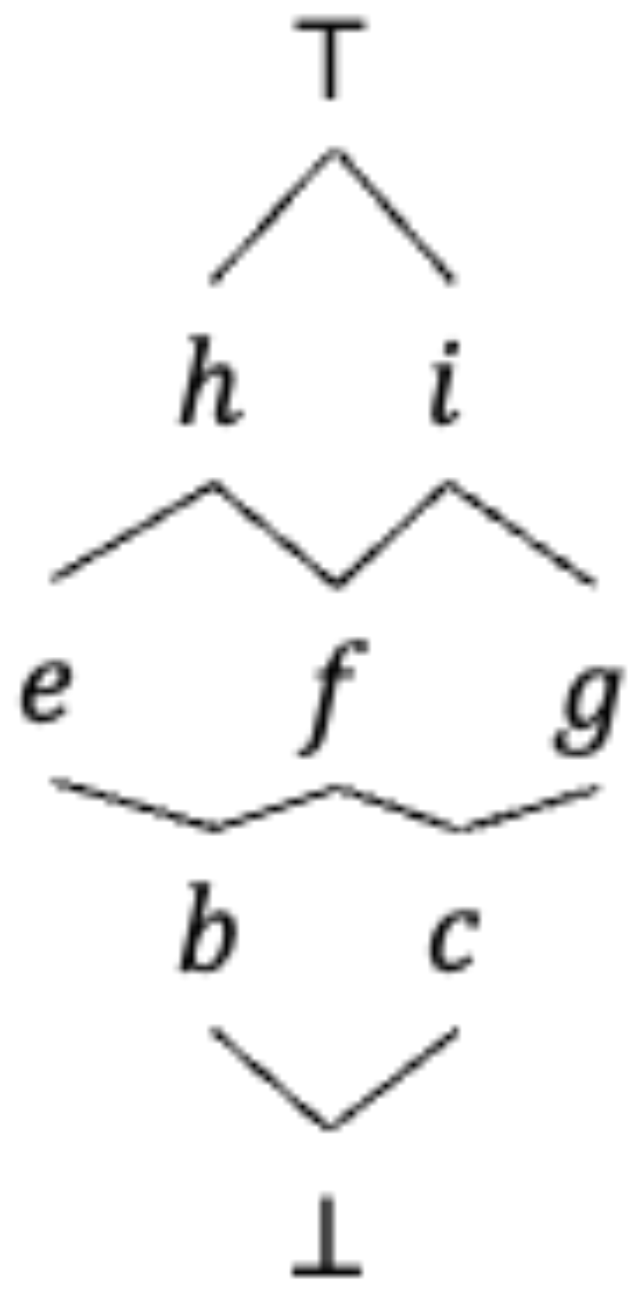

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  - $\forall x, y \in X \,.\, x \sqcup y$ exists (join semi lattice)
  - $\forall x, y \in X \,.\, x \sqcap y$ exists (meet semi lattice)
- In a lattice we have that $x \sqsubseteq y$ iff

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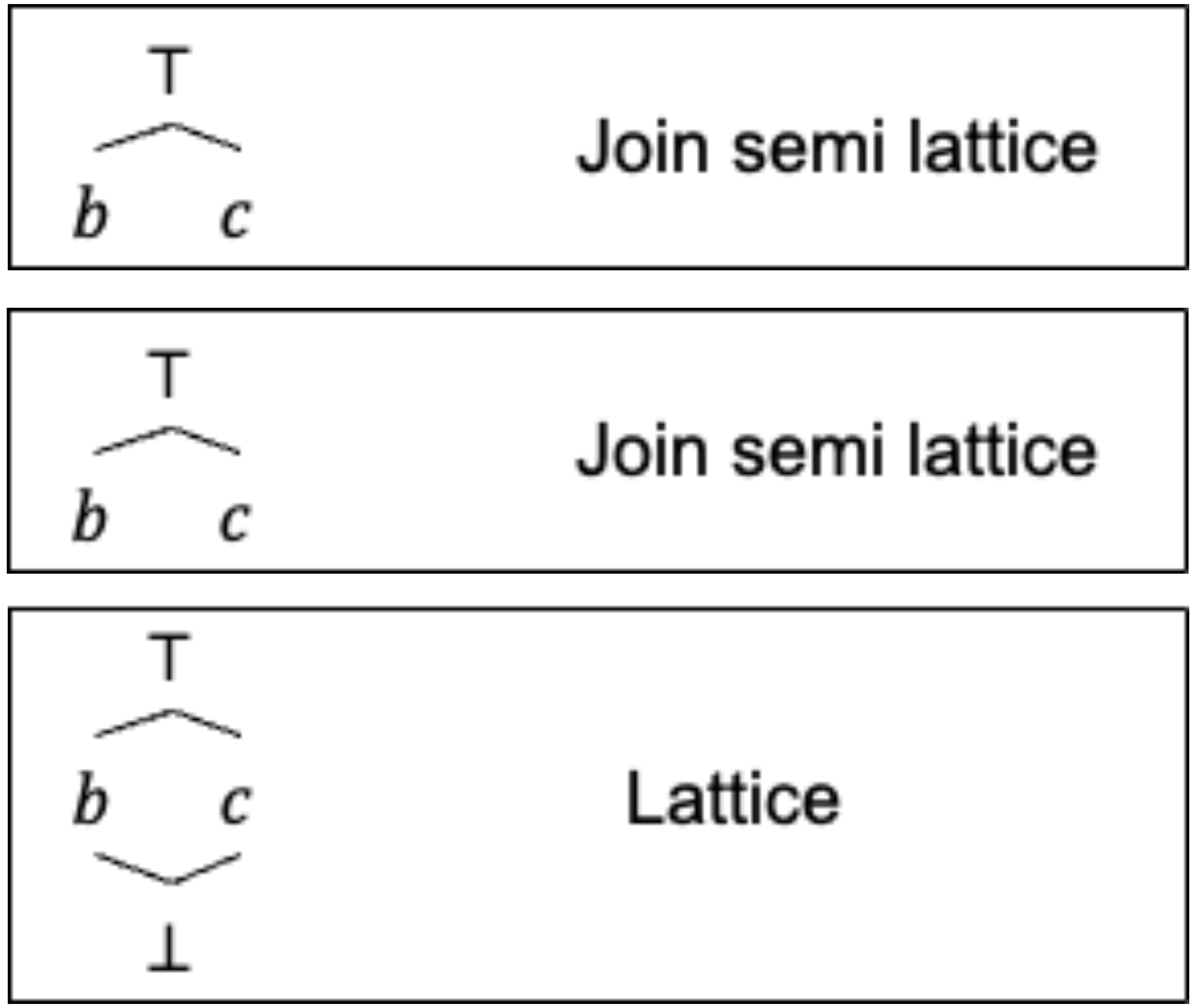

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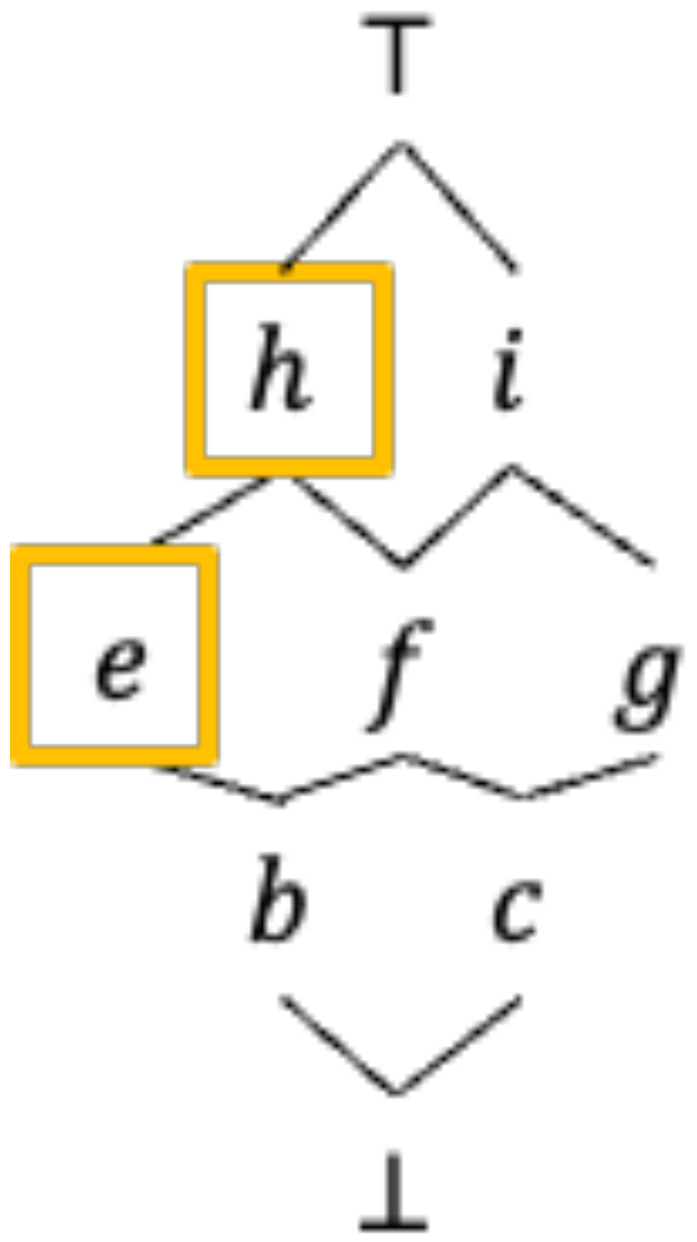

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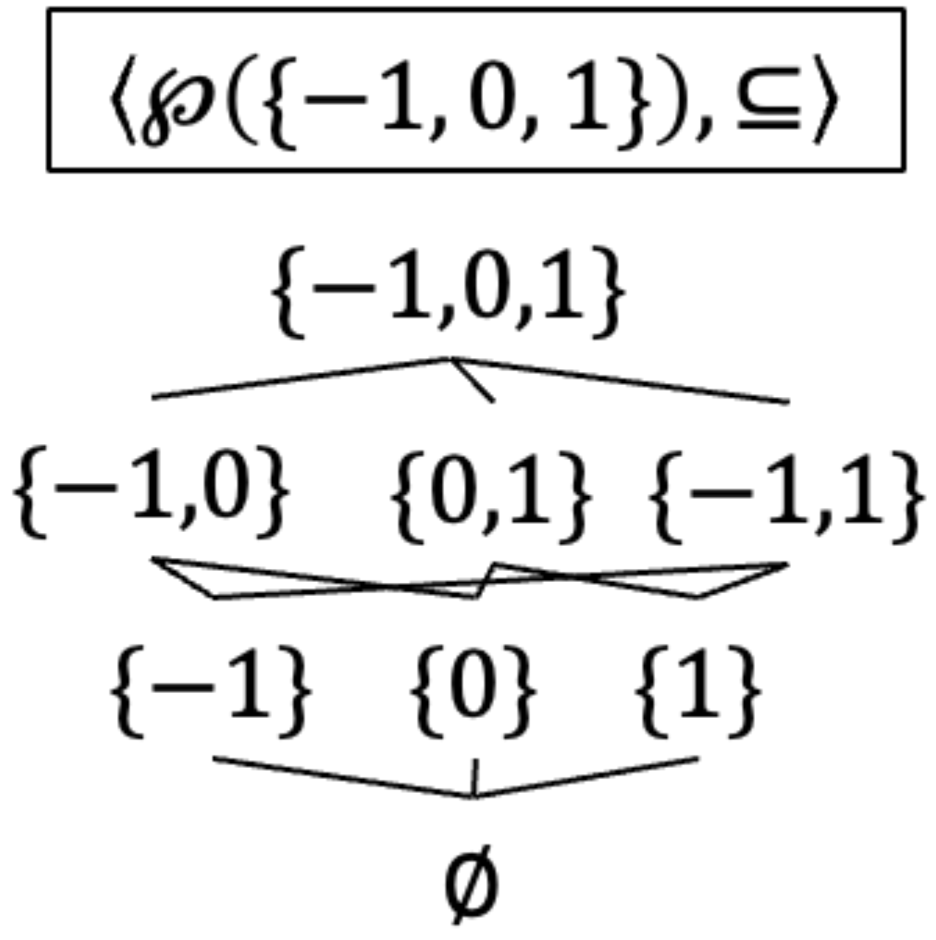

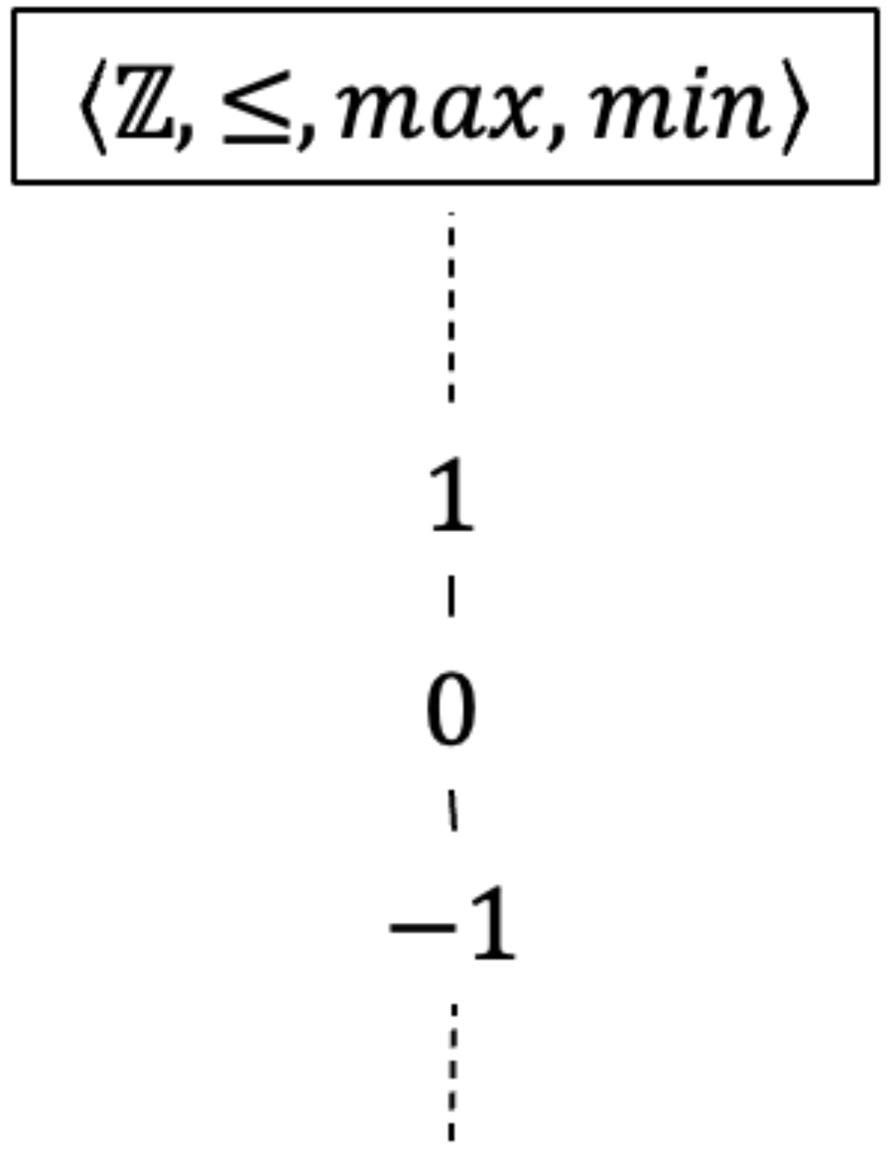

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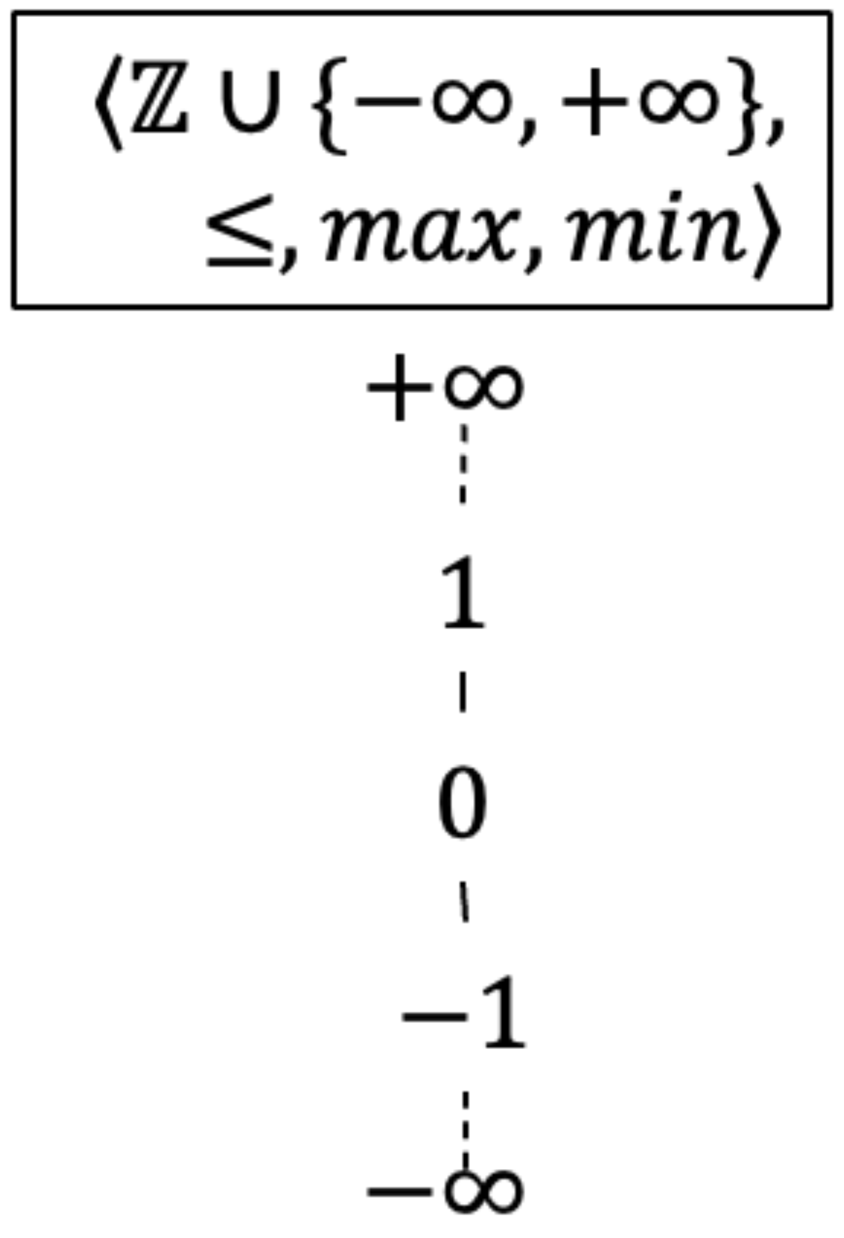

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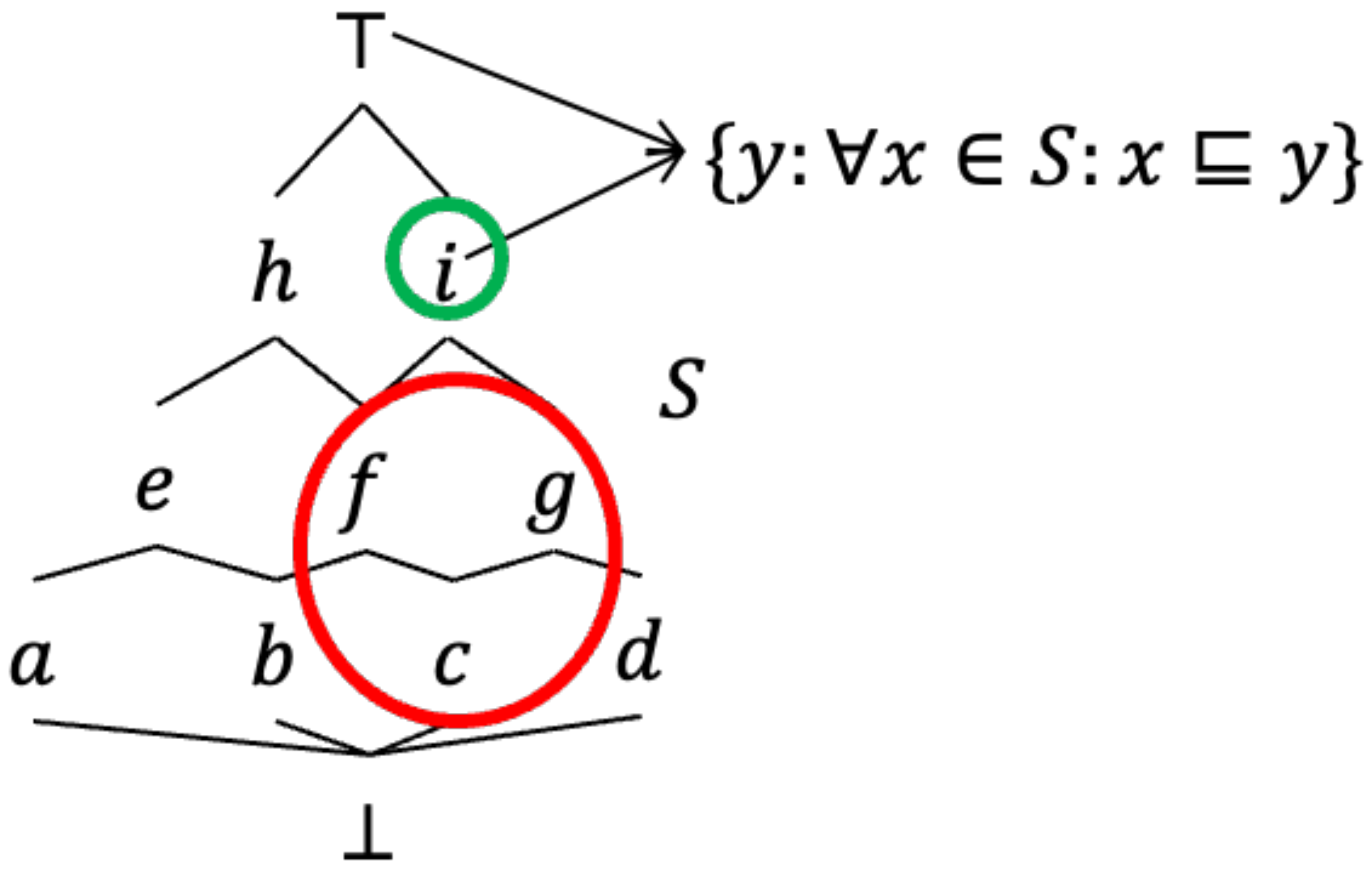

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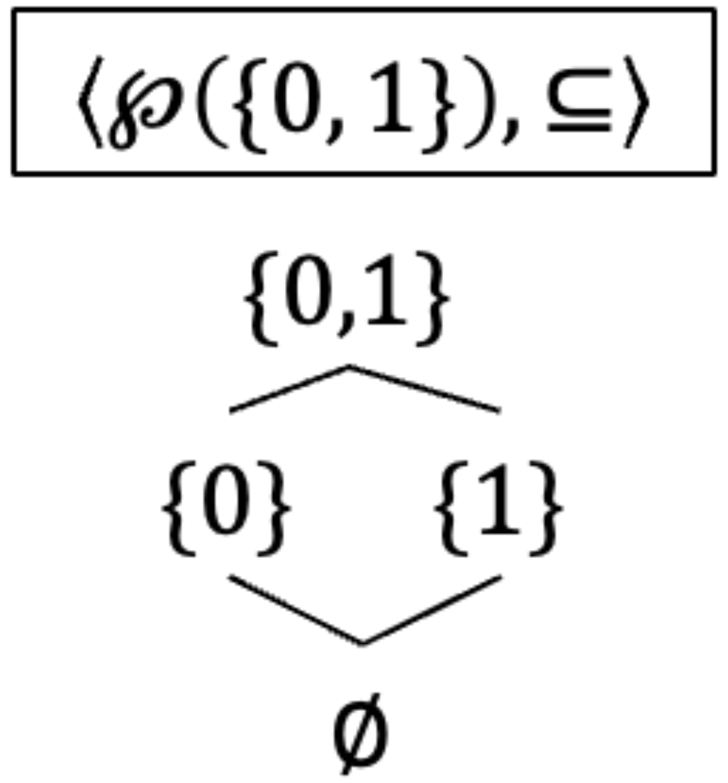
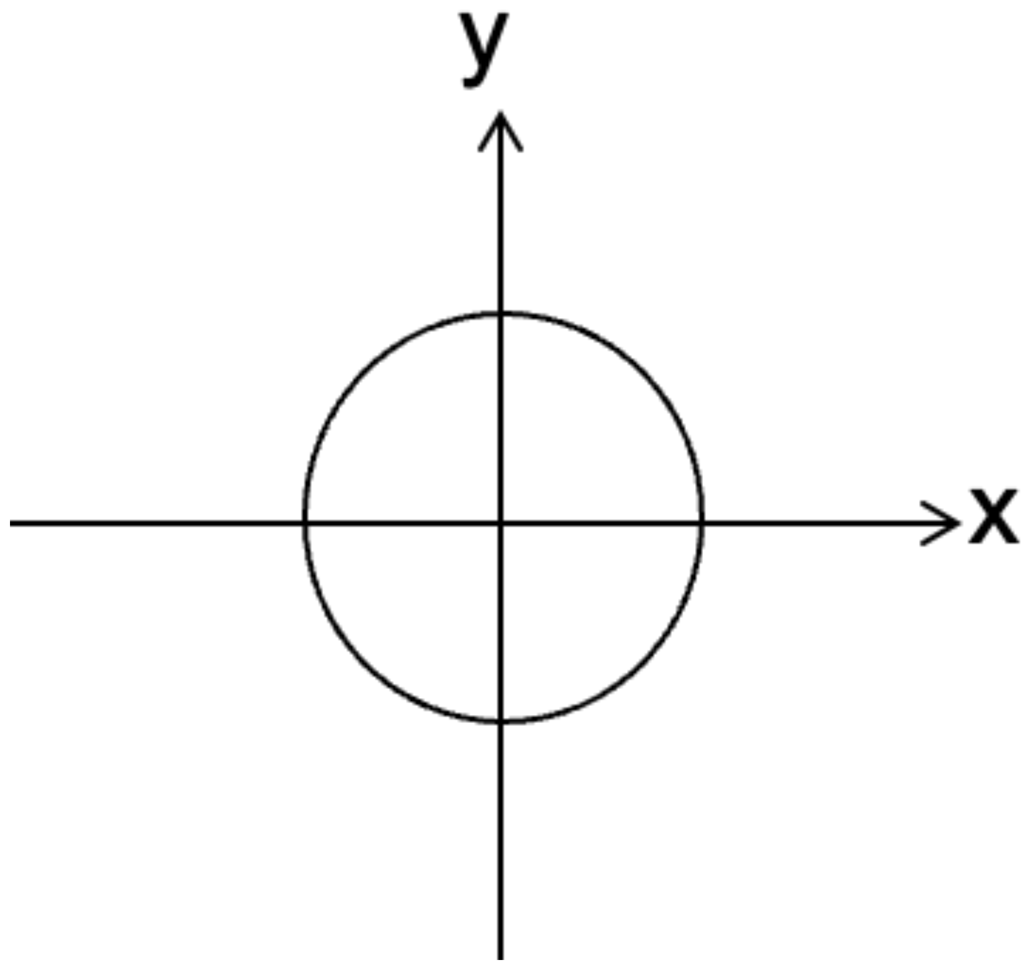
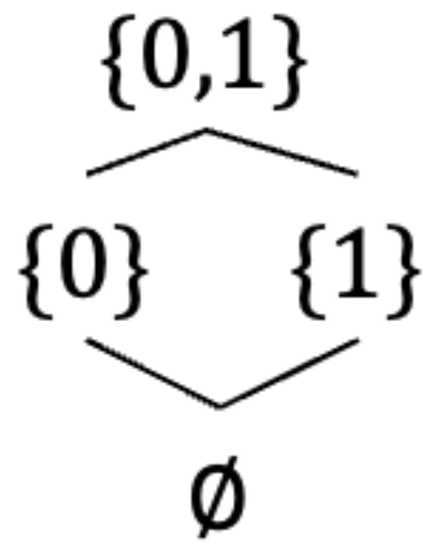
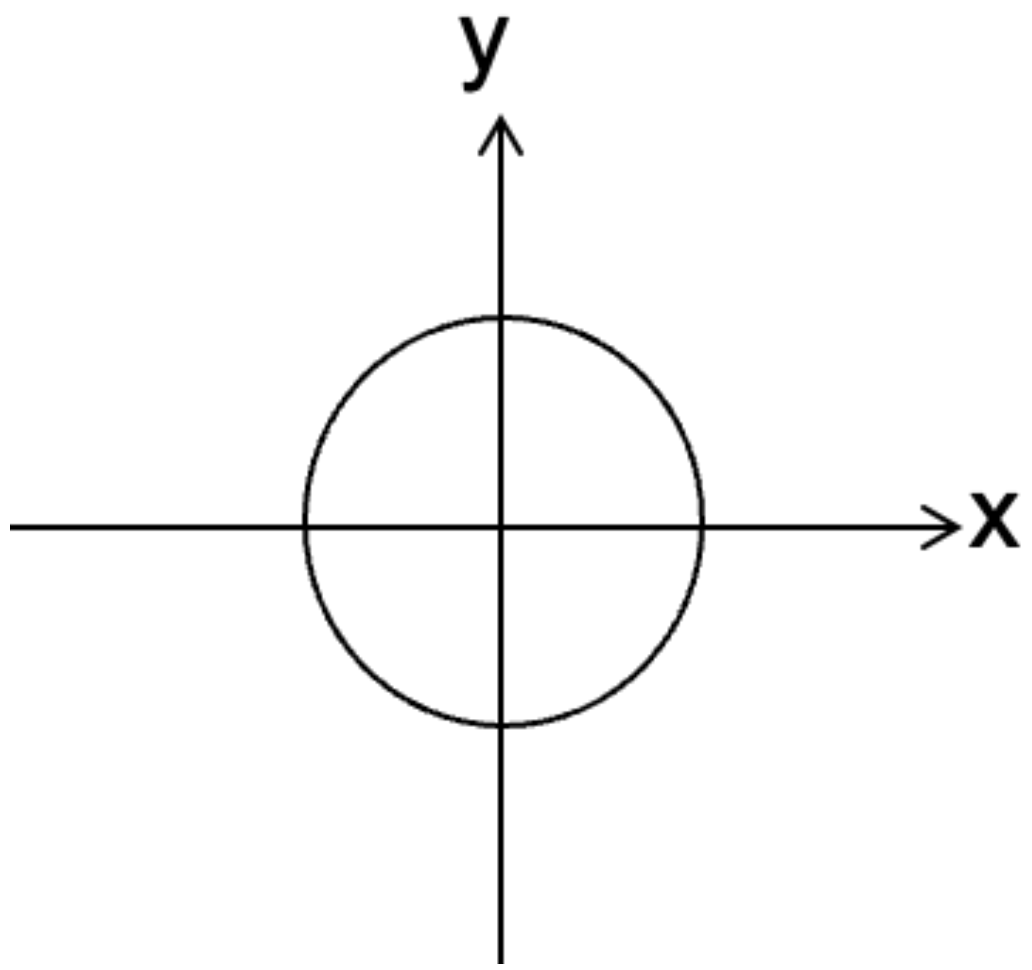




Functions (or maps)

$\boxed{\langle \wp(\{0,1\}), \subseteq \rangle}$





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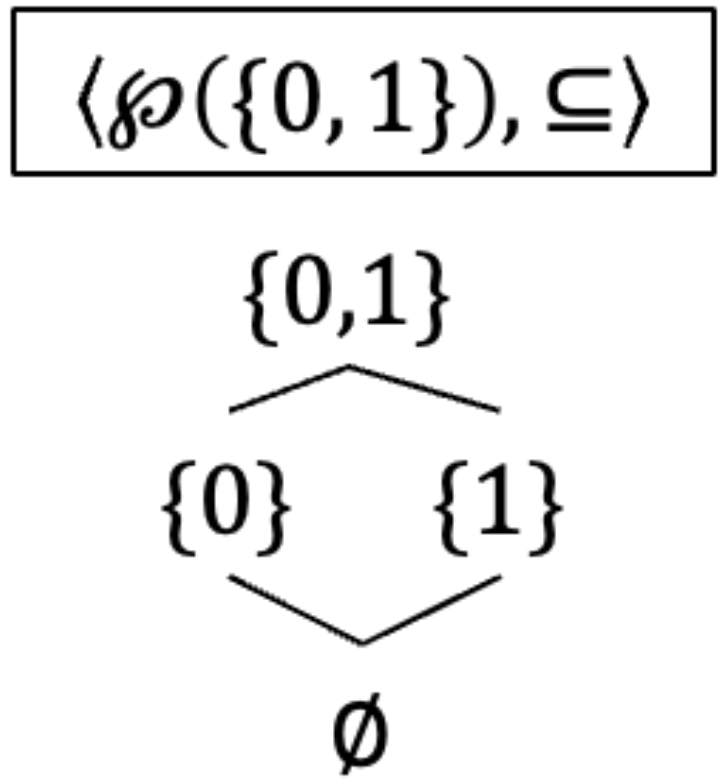

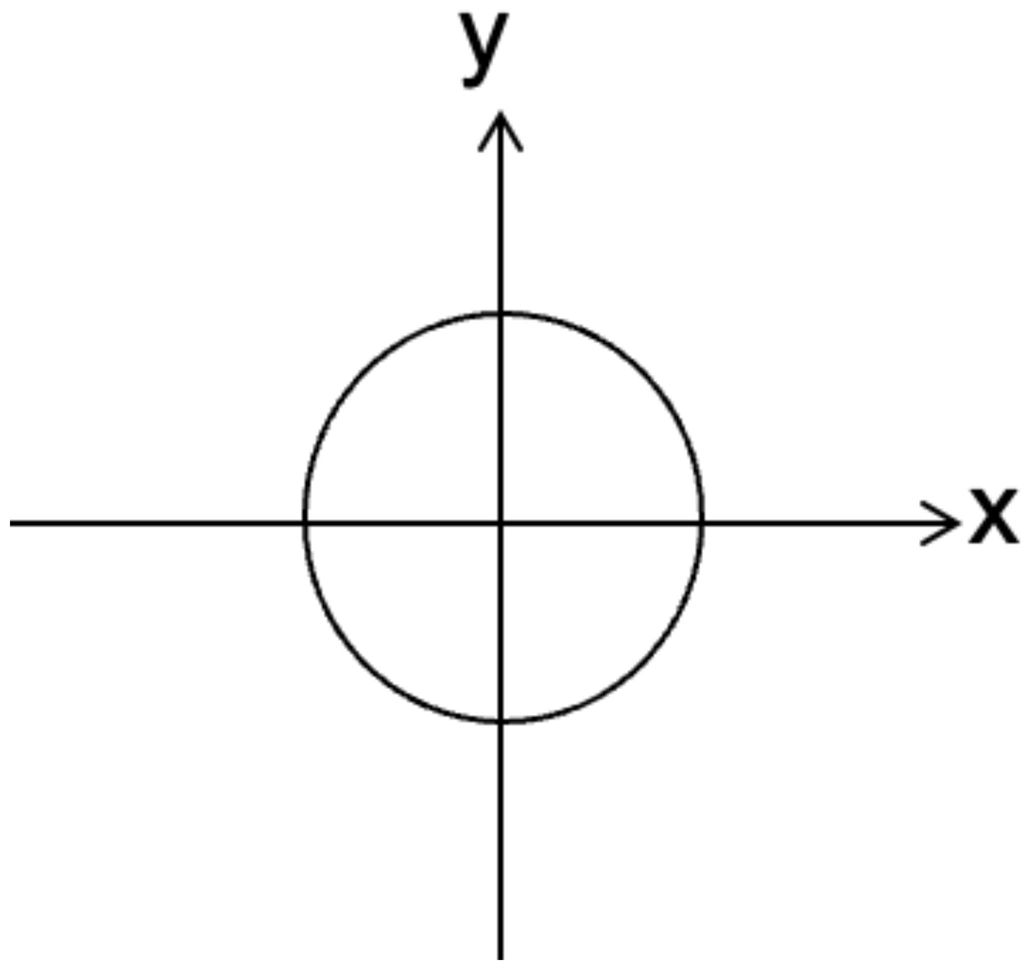

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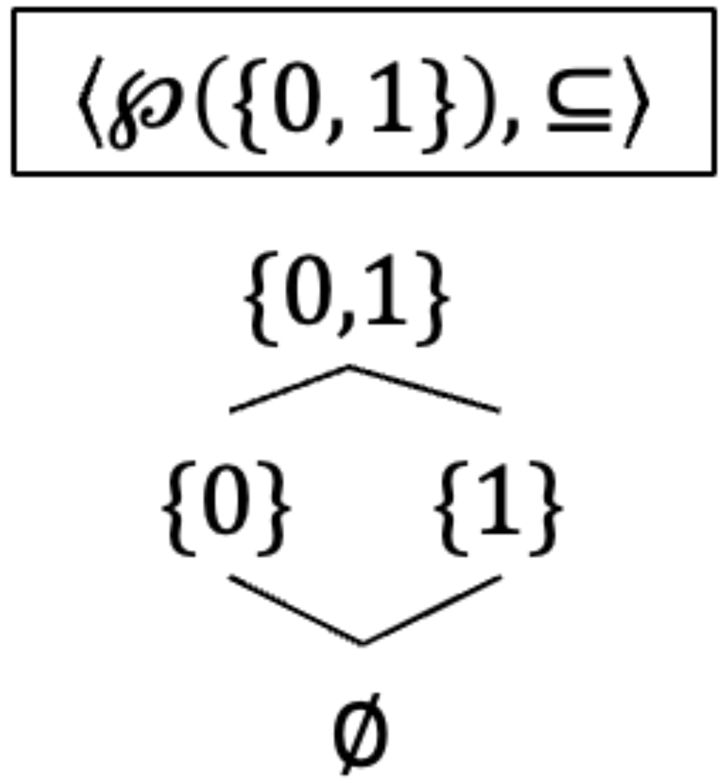

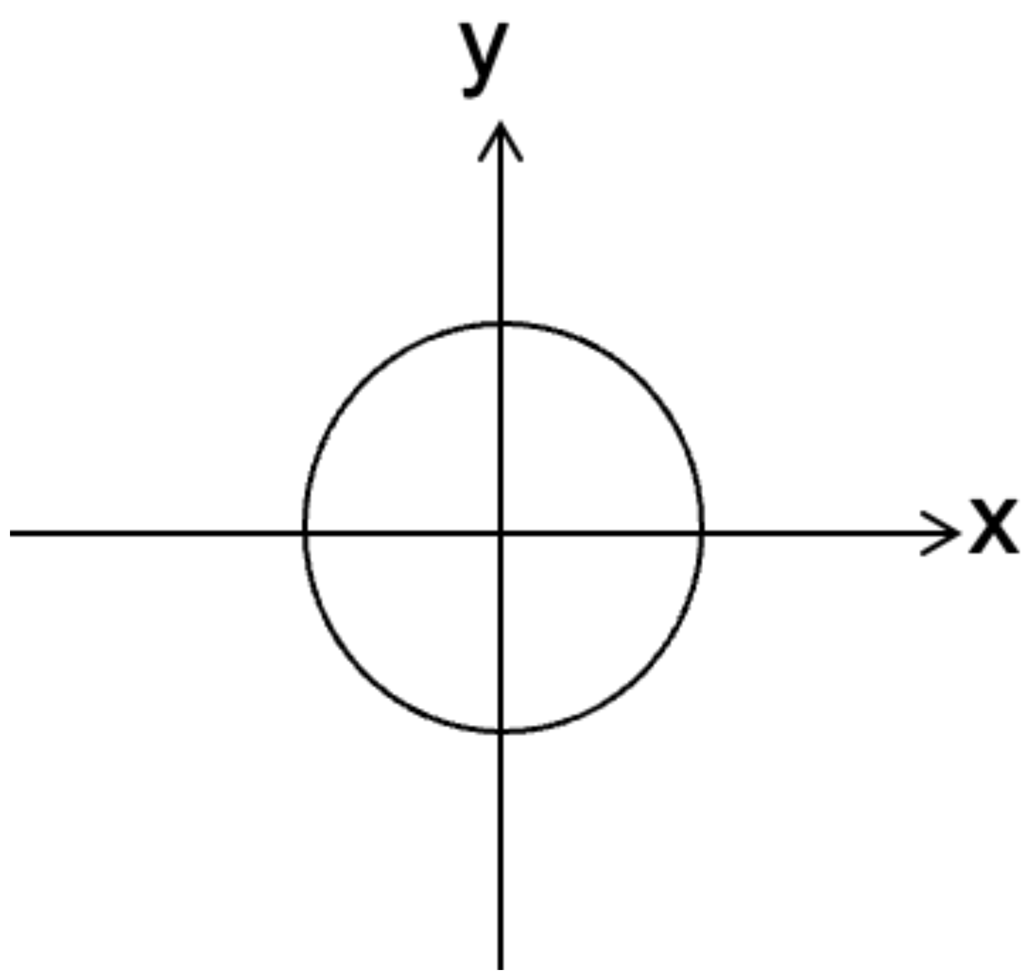

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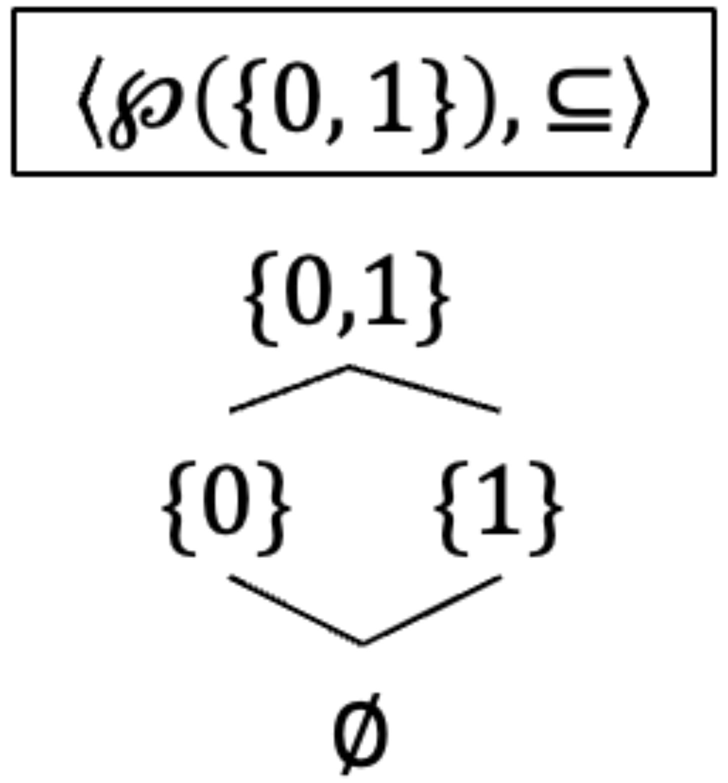

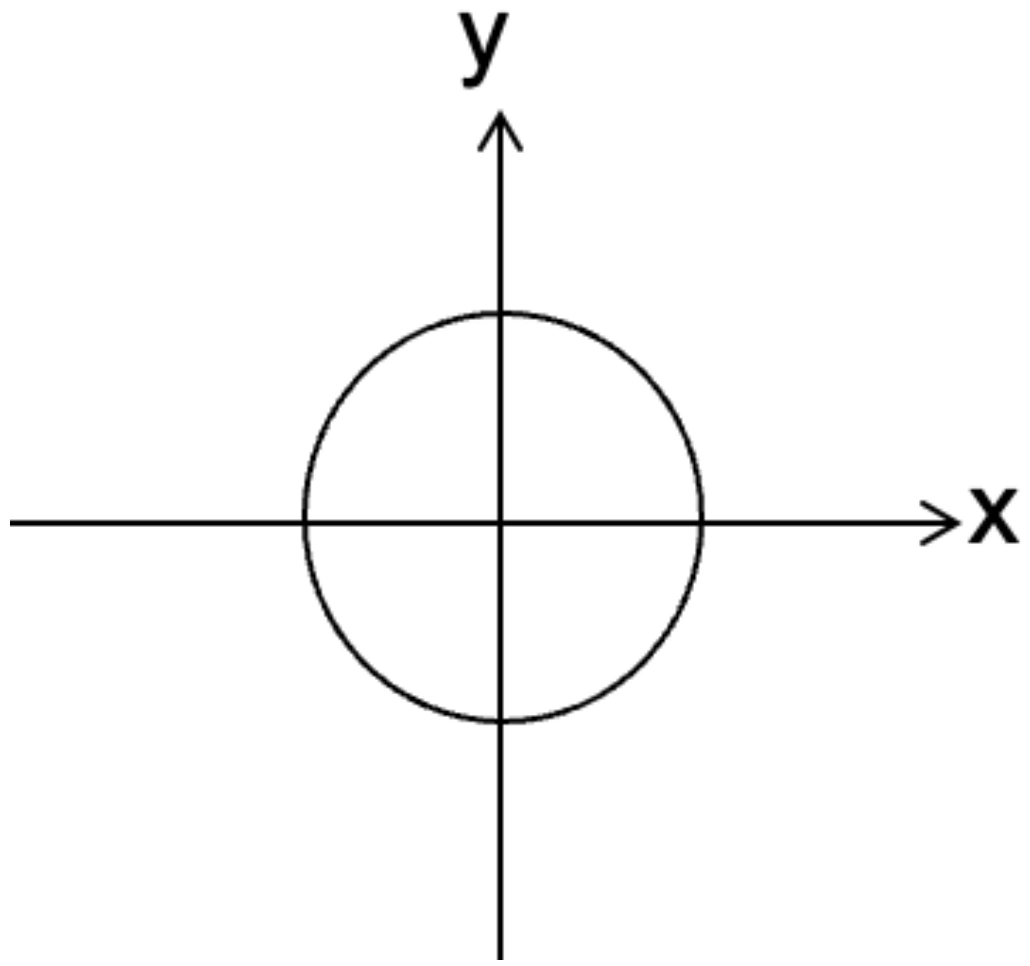

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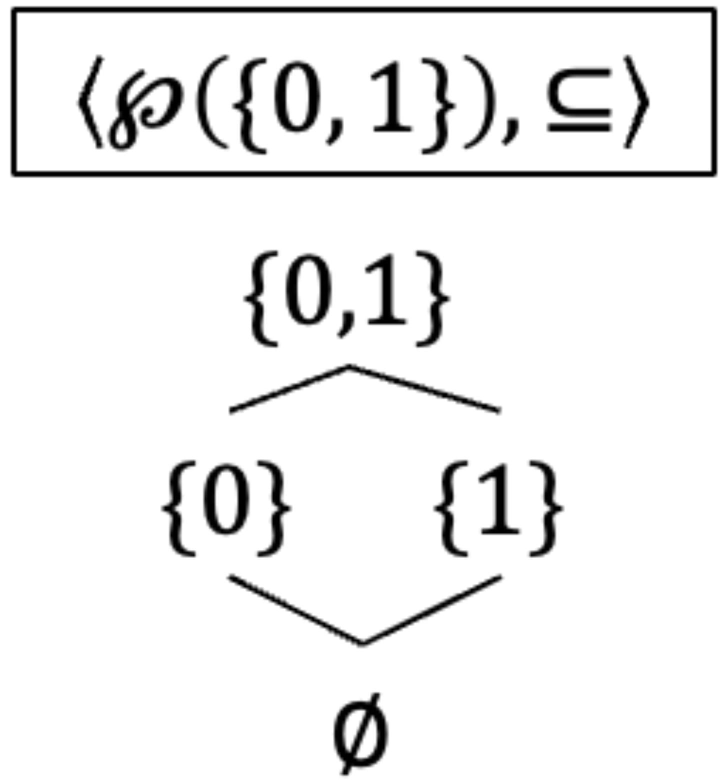

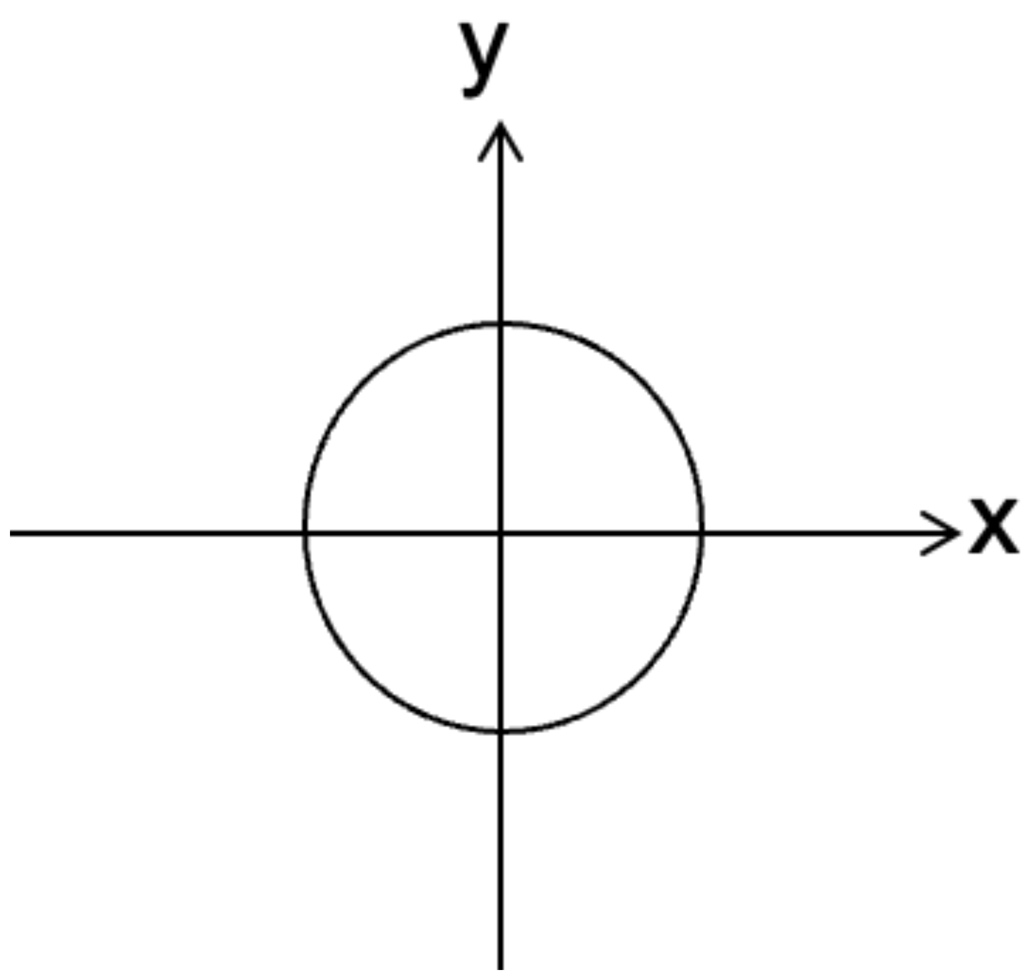

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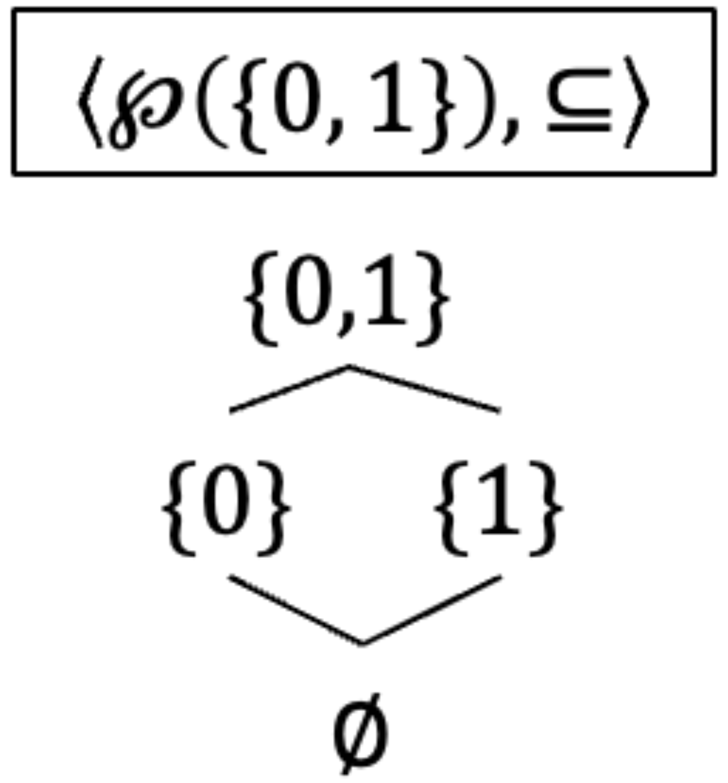

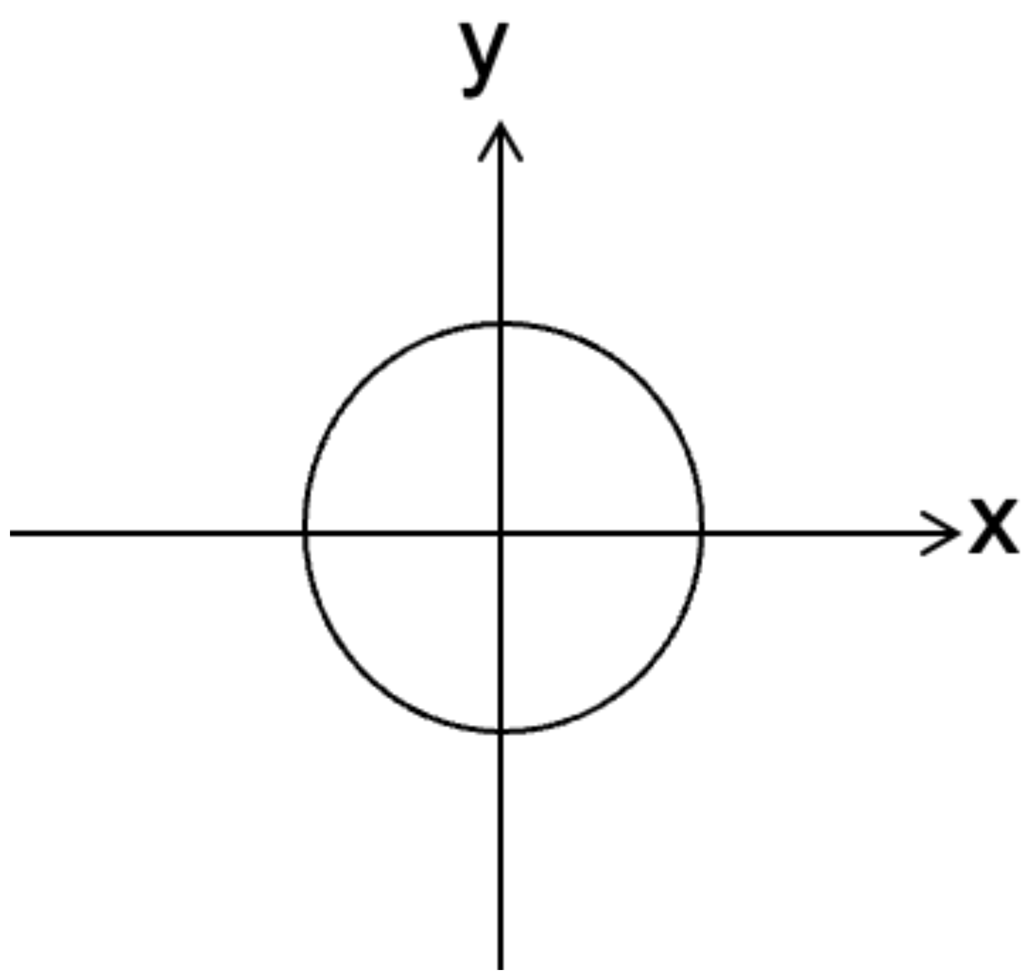

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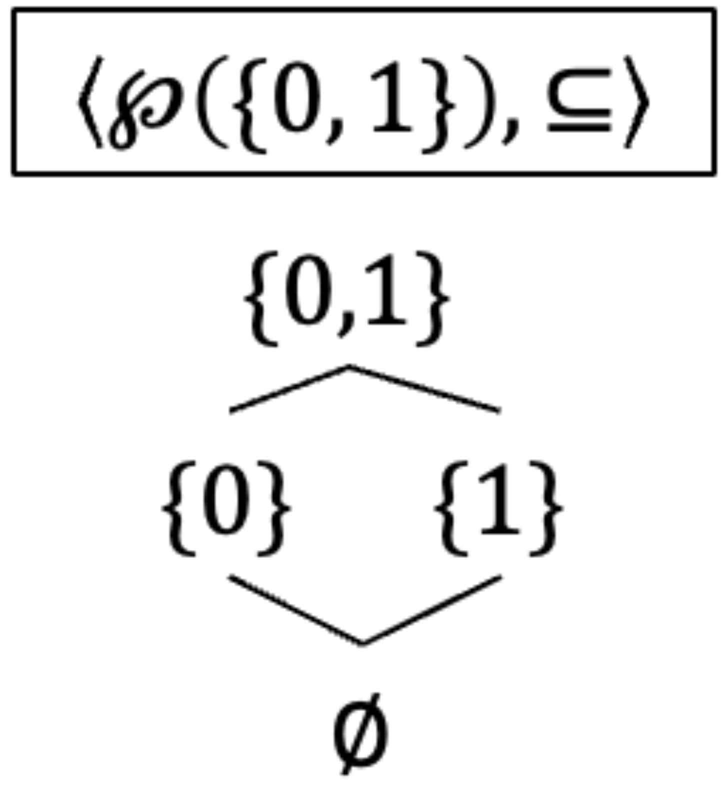

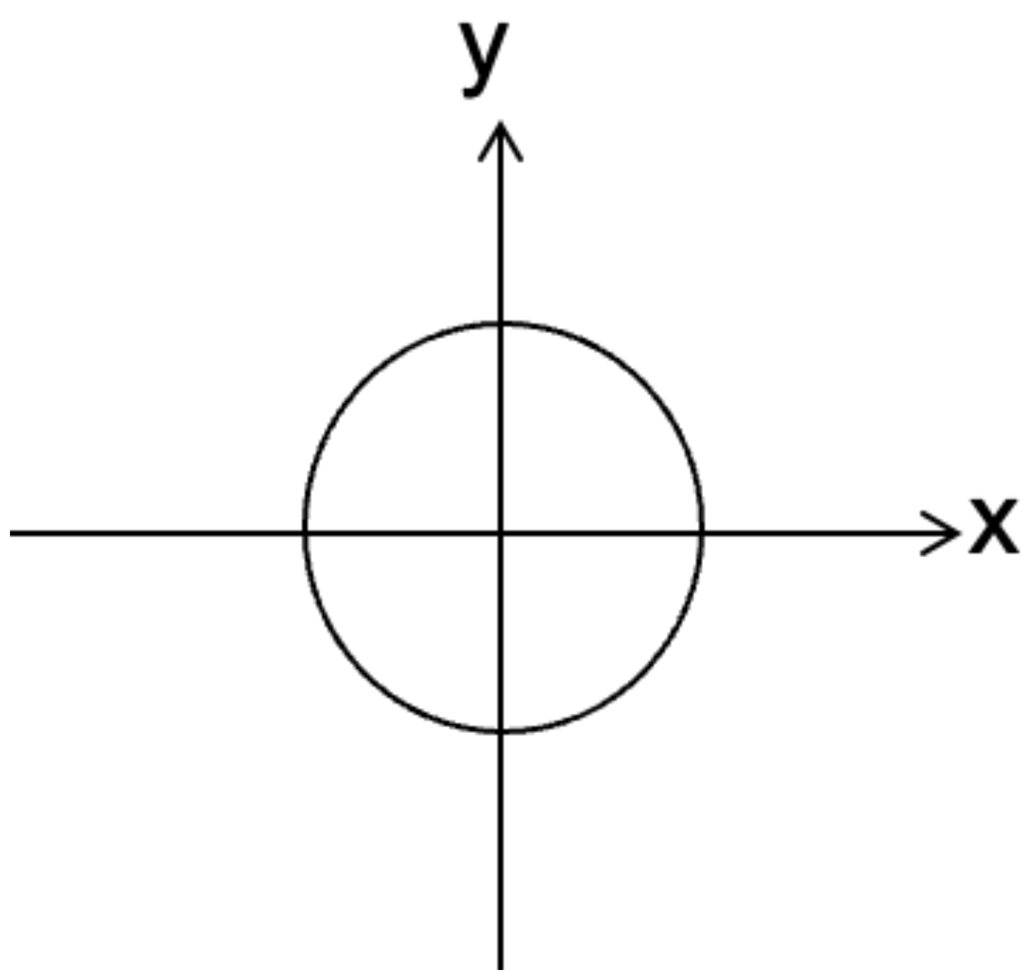

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$f' = f[2 \mapsto 2]$
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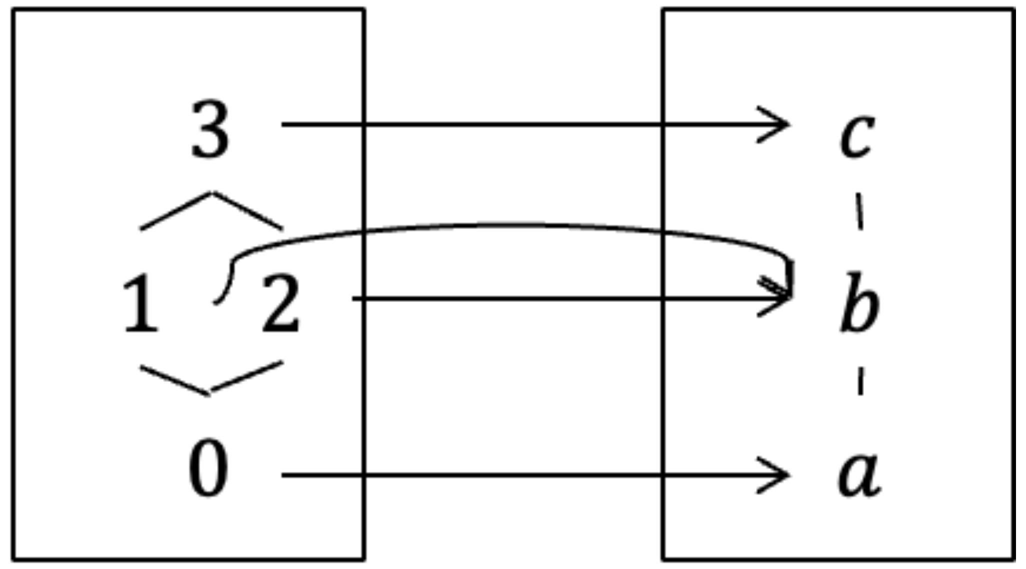
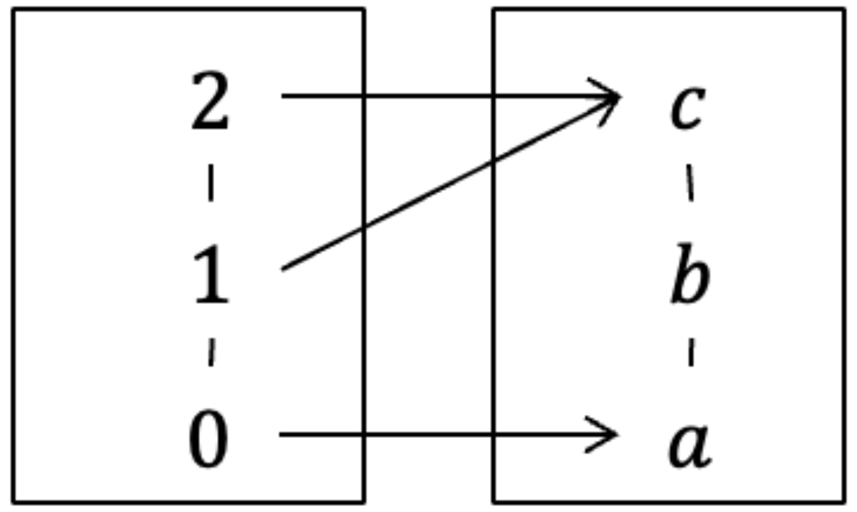
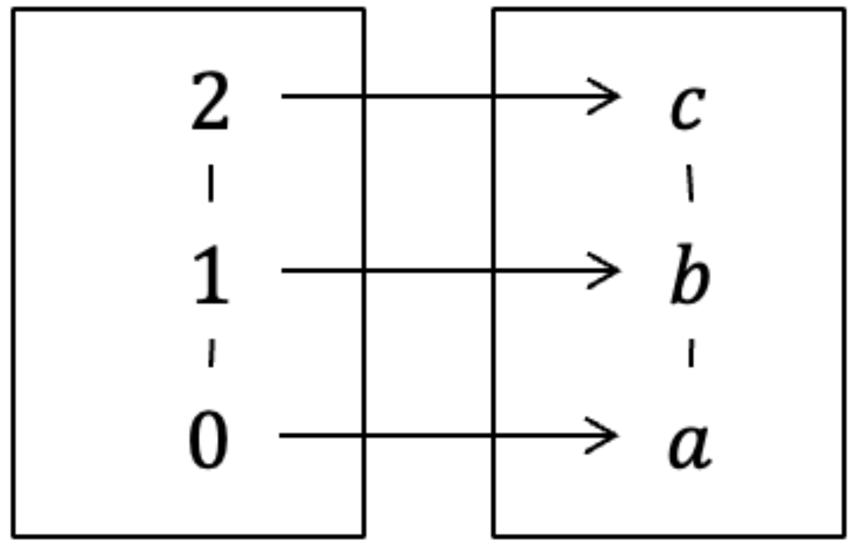
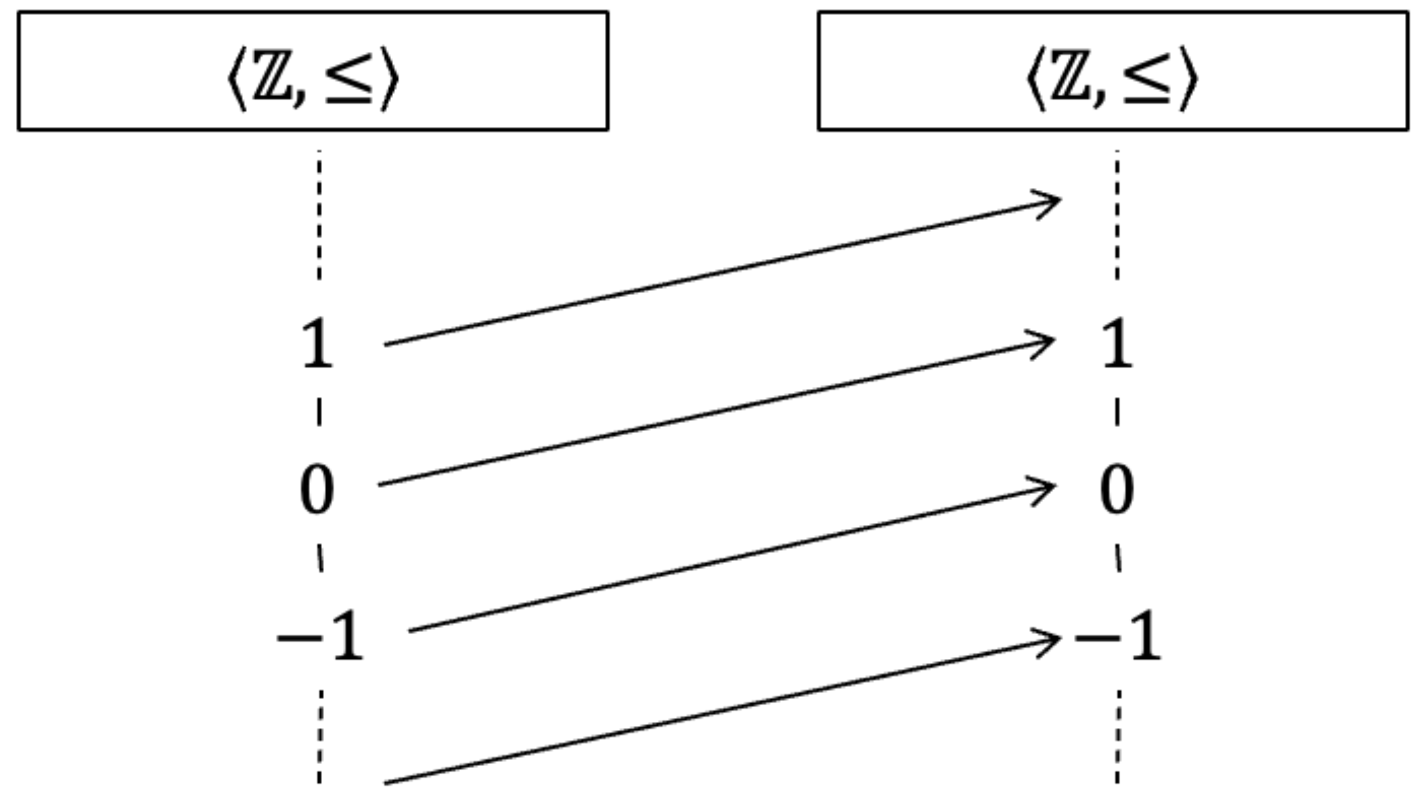


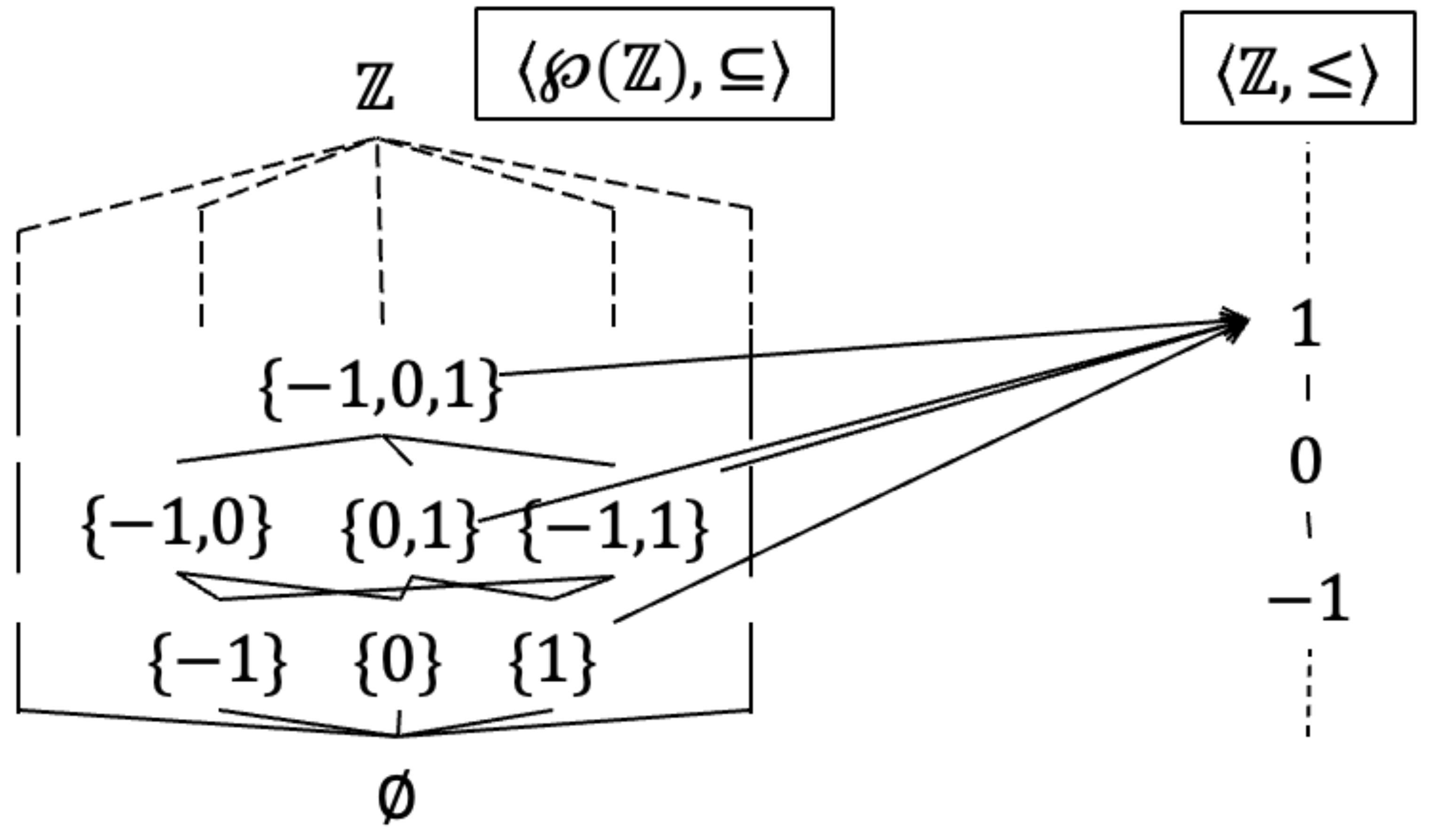




Functions

Example





Functions

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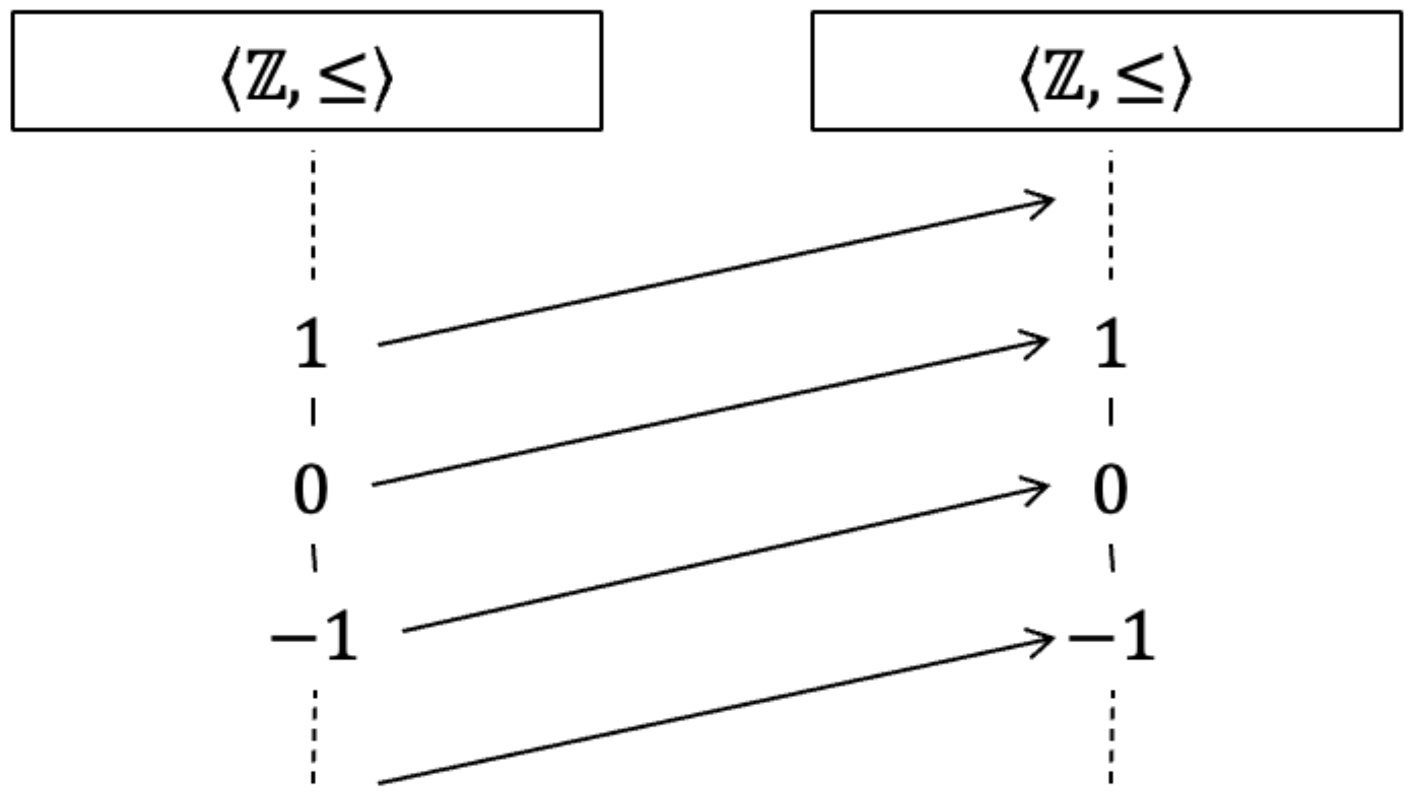
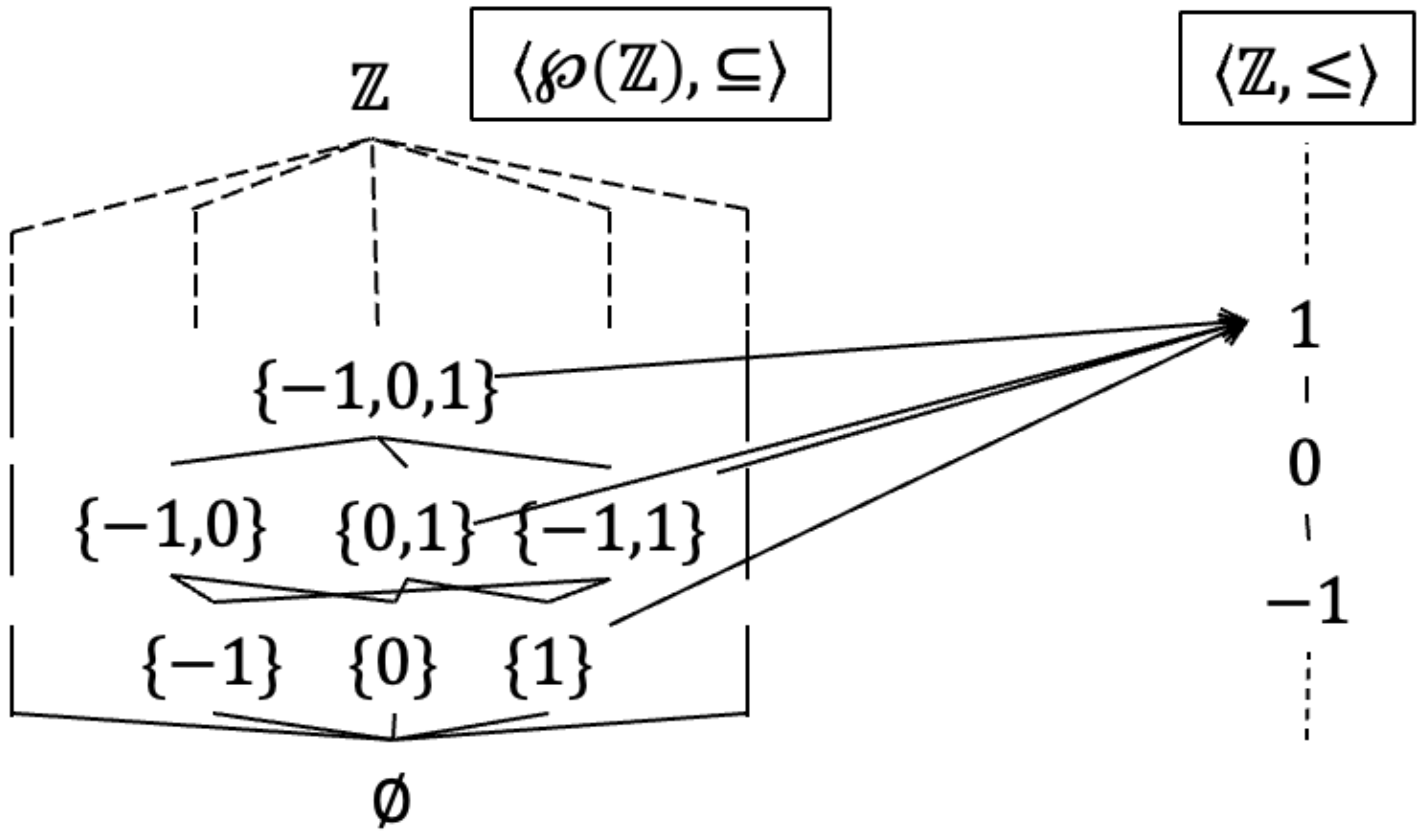
- $f : \langle \mathbb{Z}, \leq \rangle \to \langle \mathbb{Z}, \leq \rangle$





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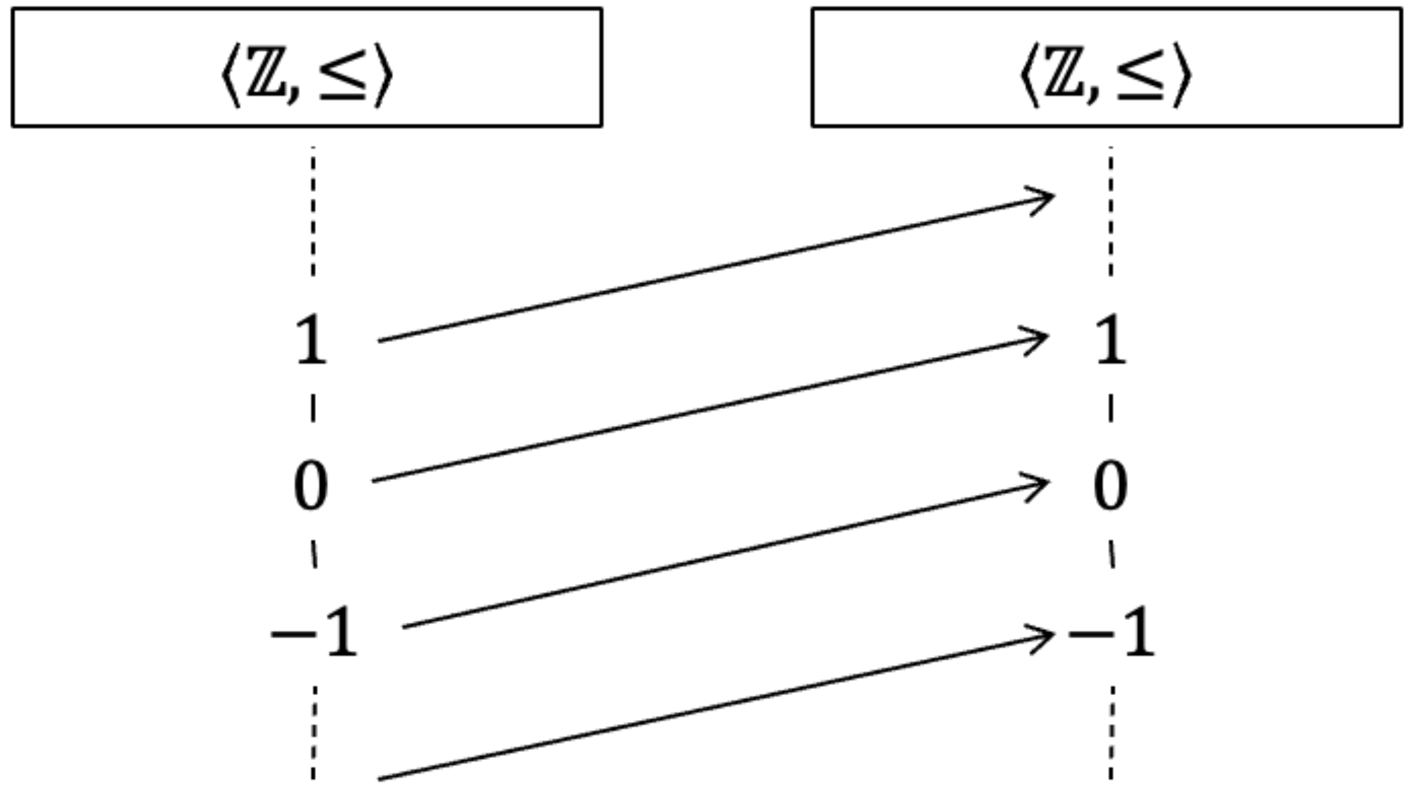
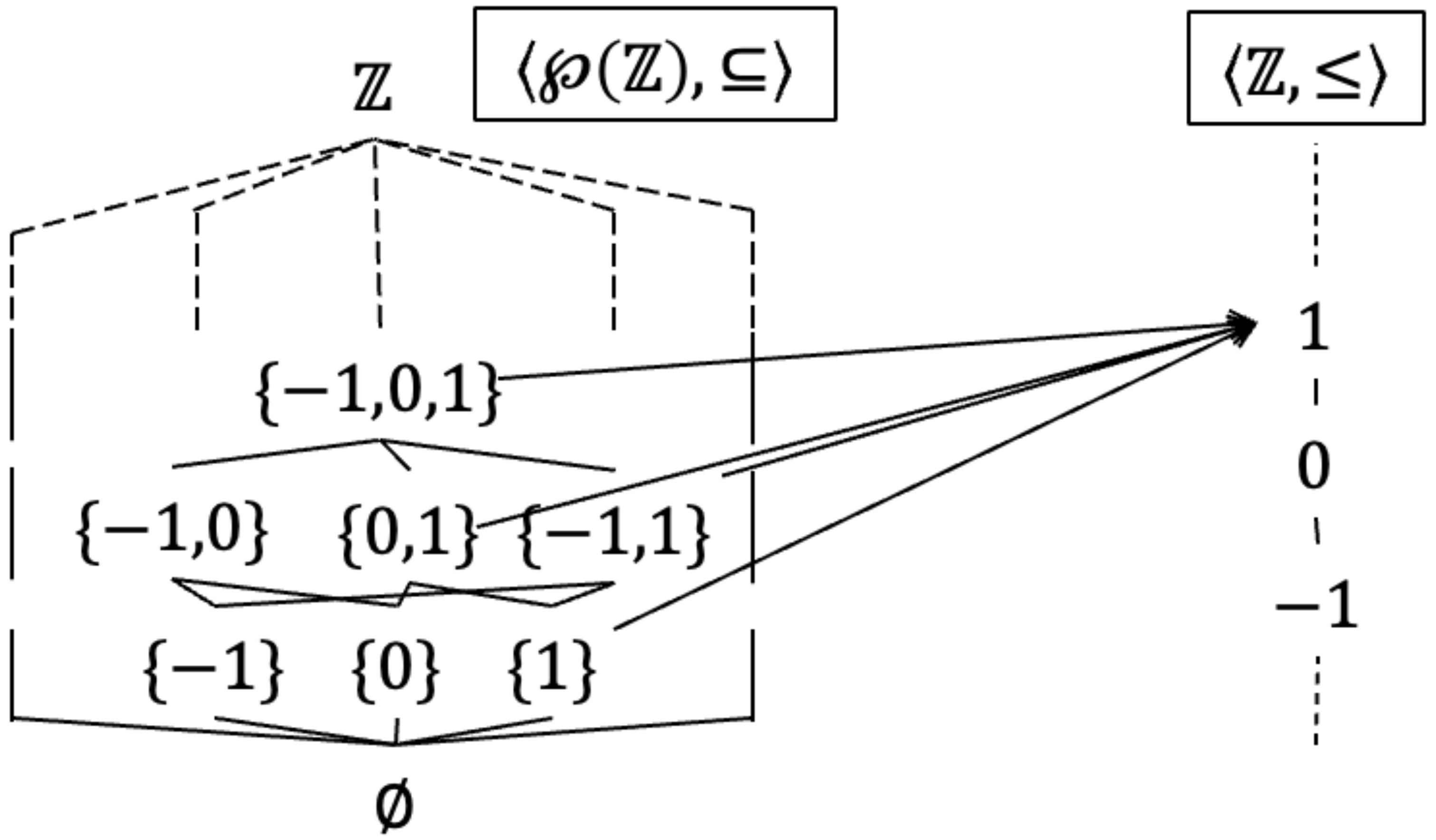
    - $f(x) = x + 1$, monotone, order embedding, surjective





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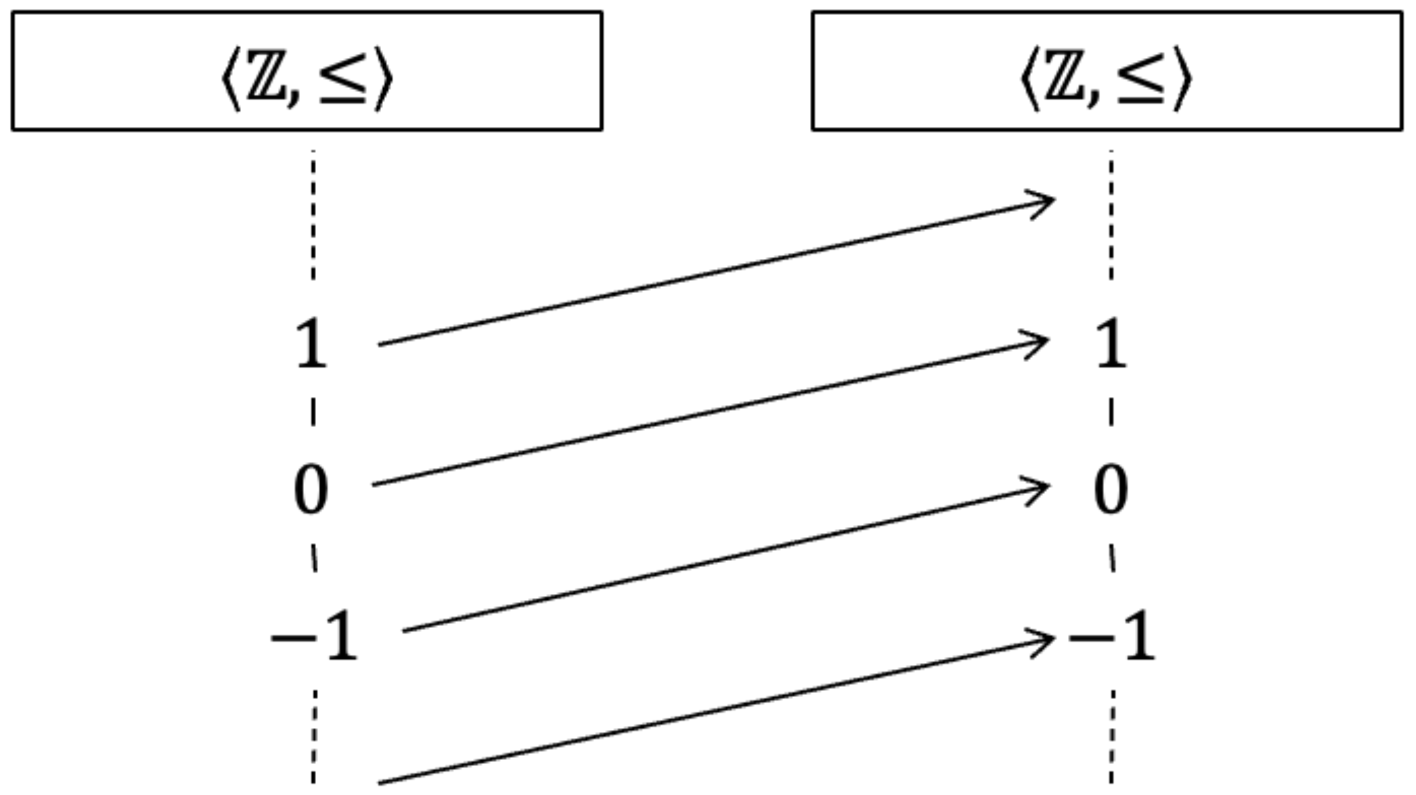

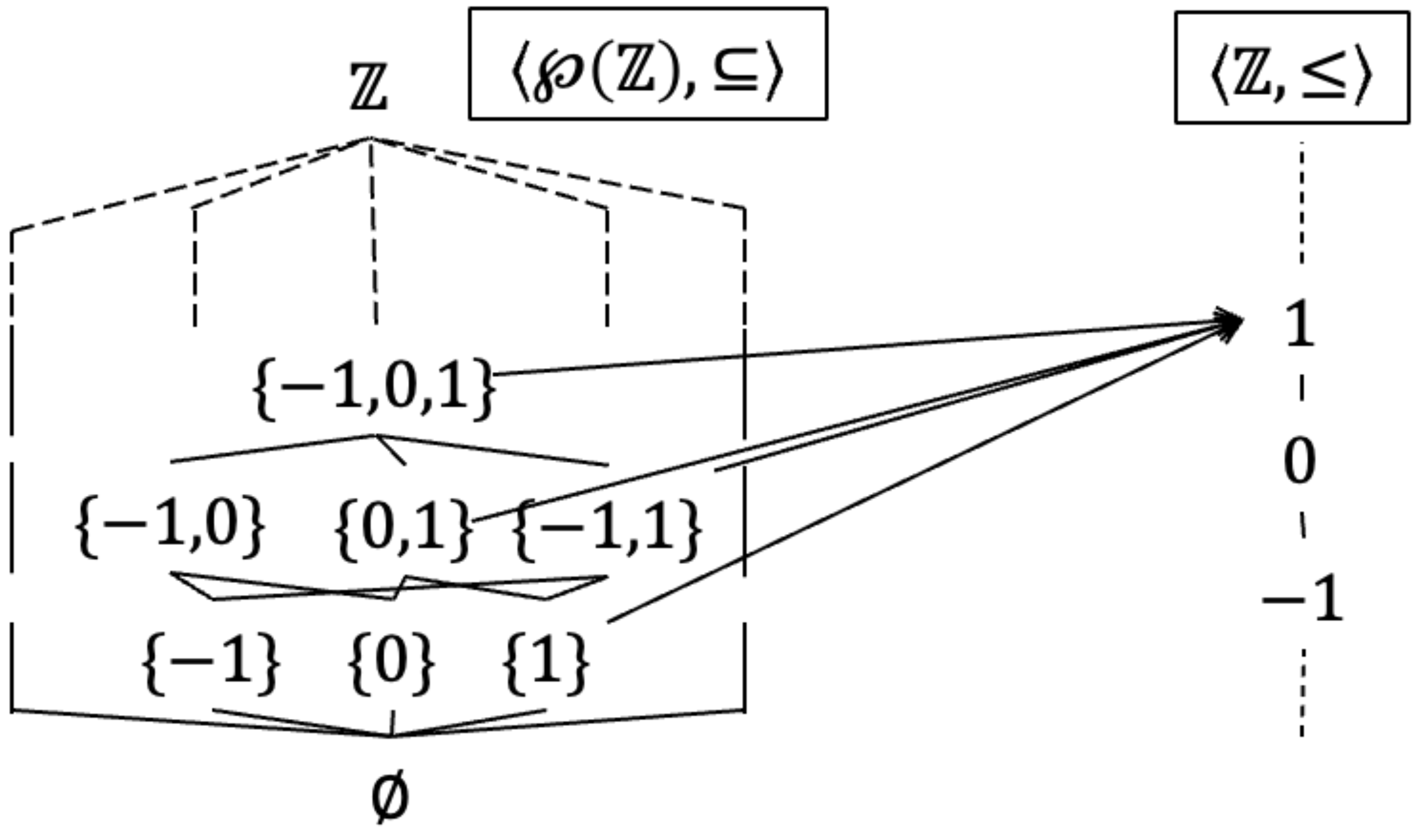

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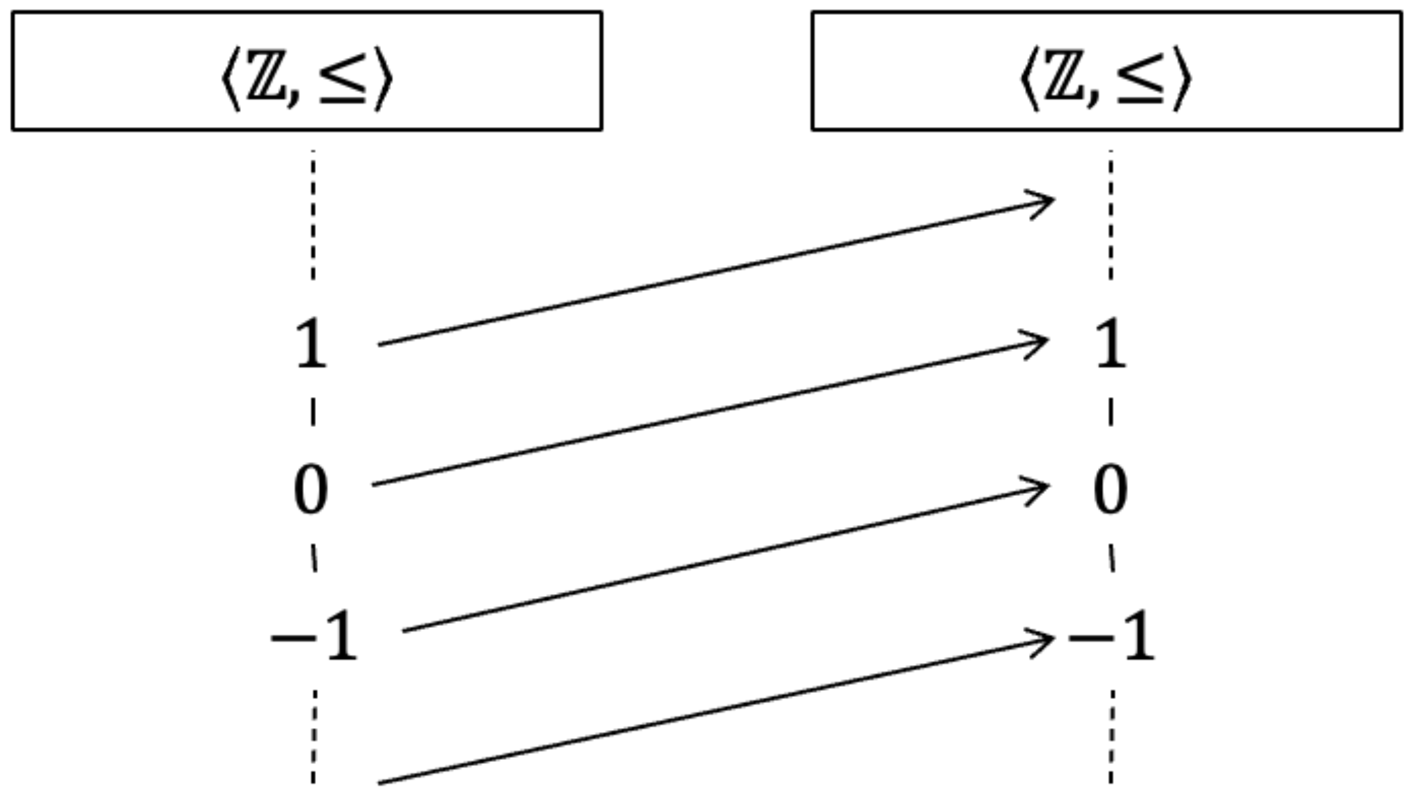
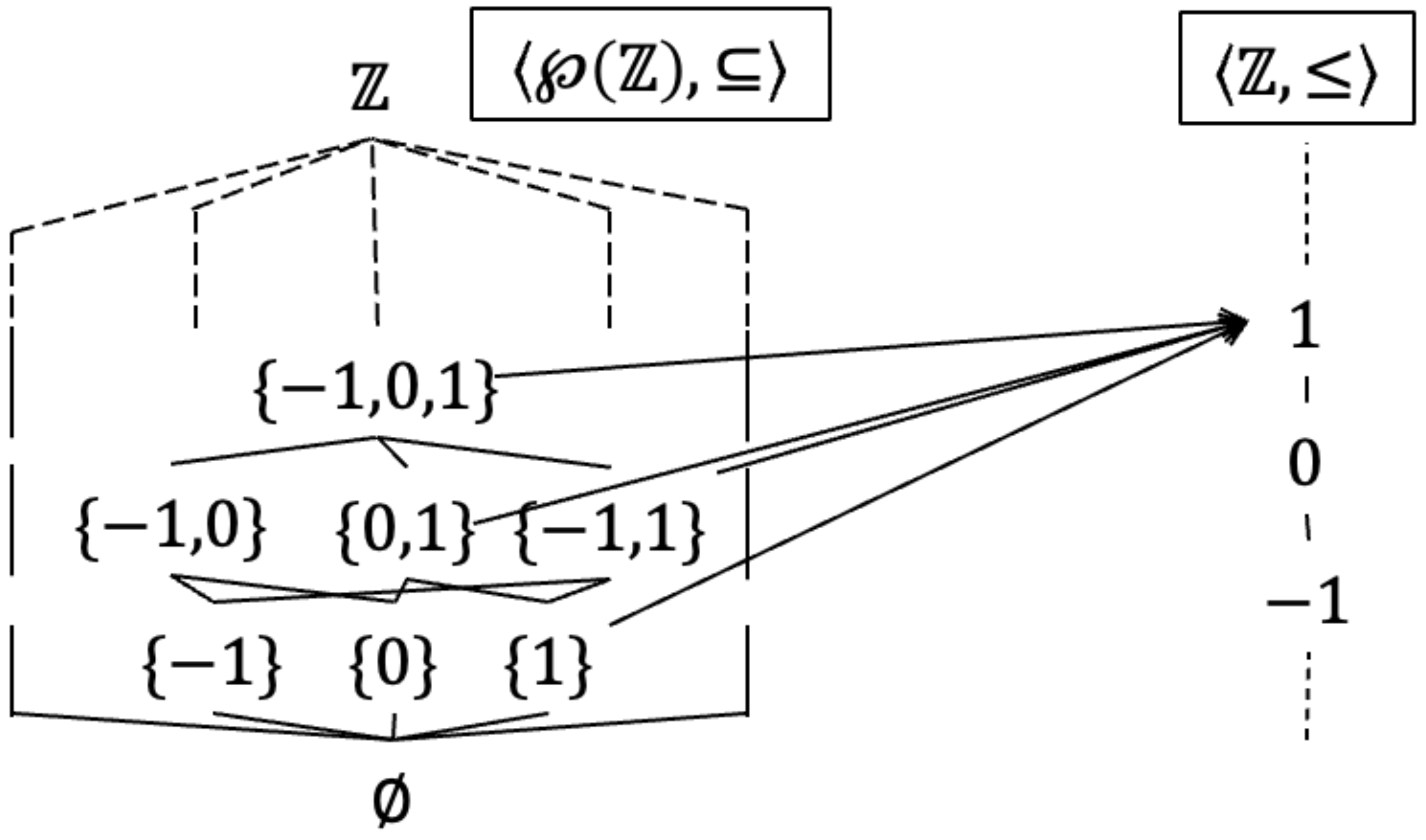
    - $f(X) = \max(X)$, monotone, but not an embedding





Join/meet preserving functions

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- A join or meet preserving function is monotone, but the opposite does not necessarily hold

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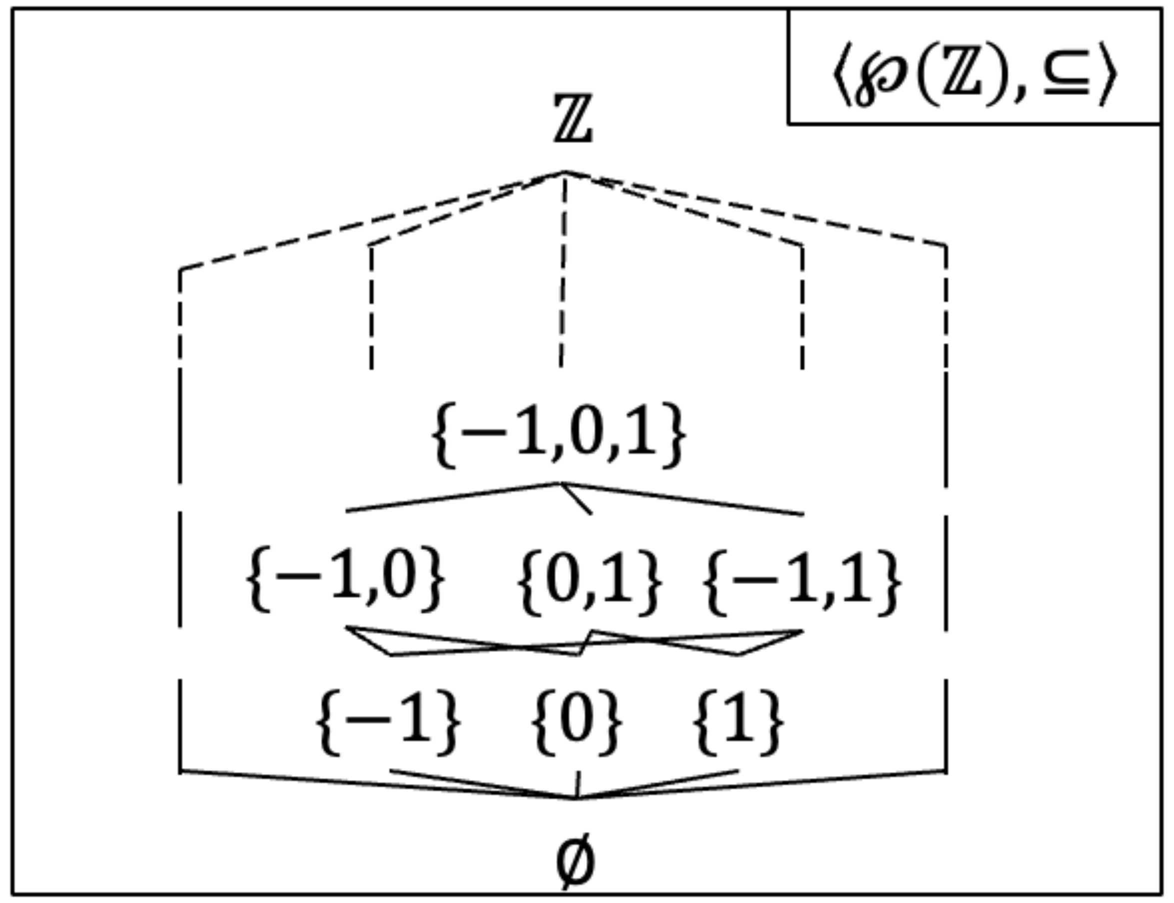

- Examples of chains: $\{\emptyset\}, \{\{\emptyset\}, \{0,1\}\}, \{\{0\}, \{-1,0,1\}\}$

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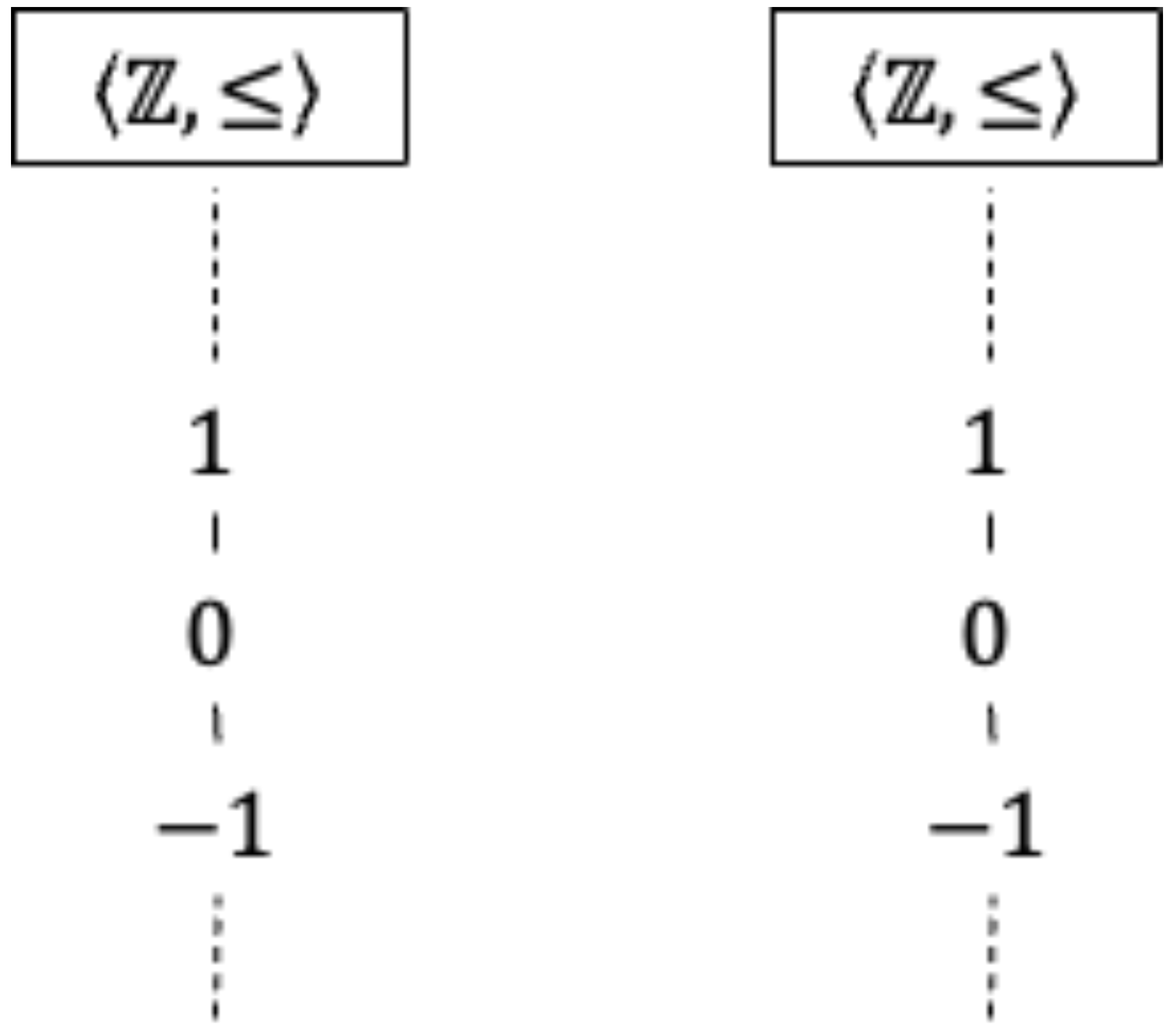

$f = 2 * x$

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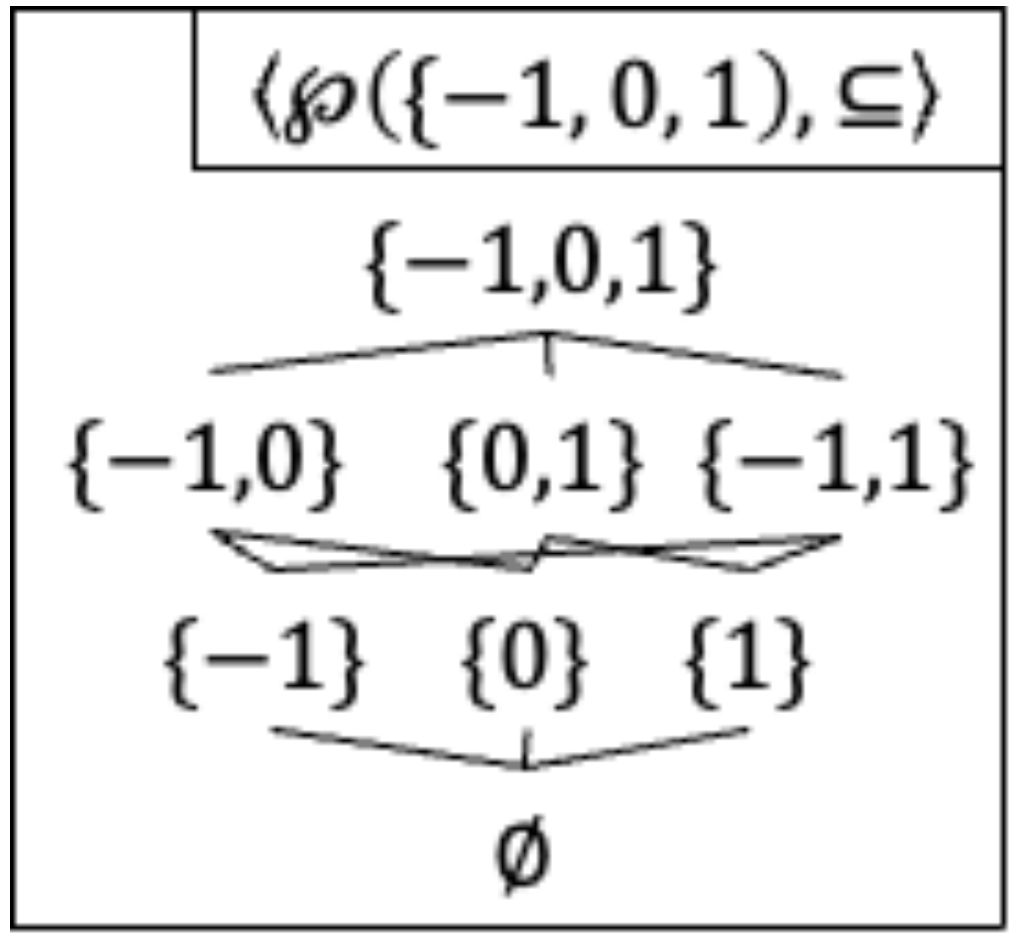

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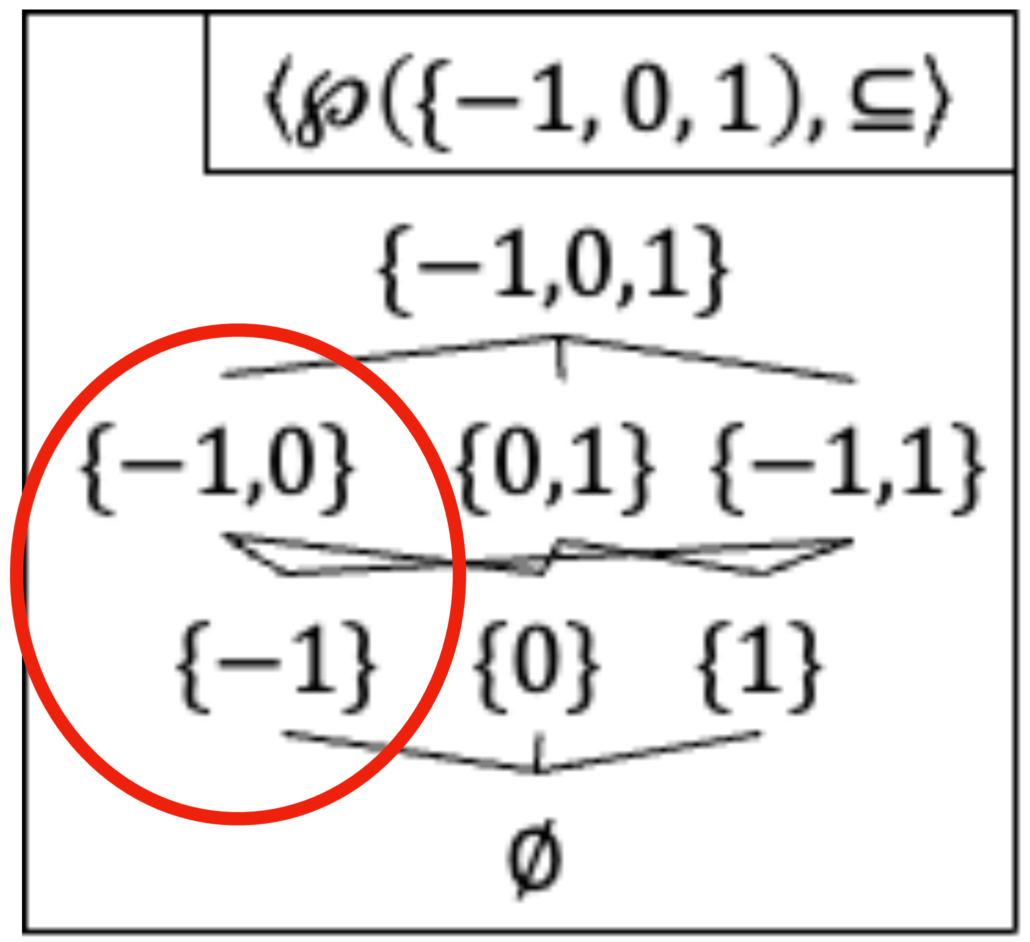

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- If X is infinite, the iterations could be infinite

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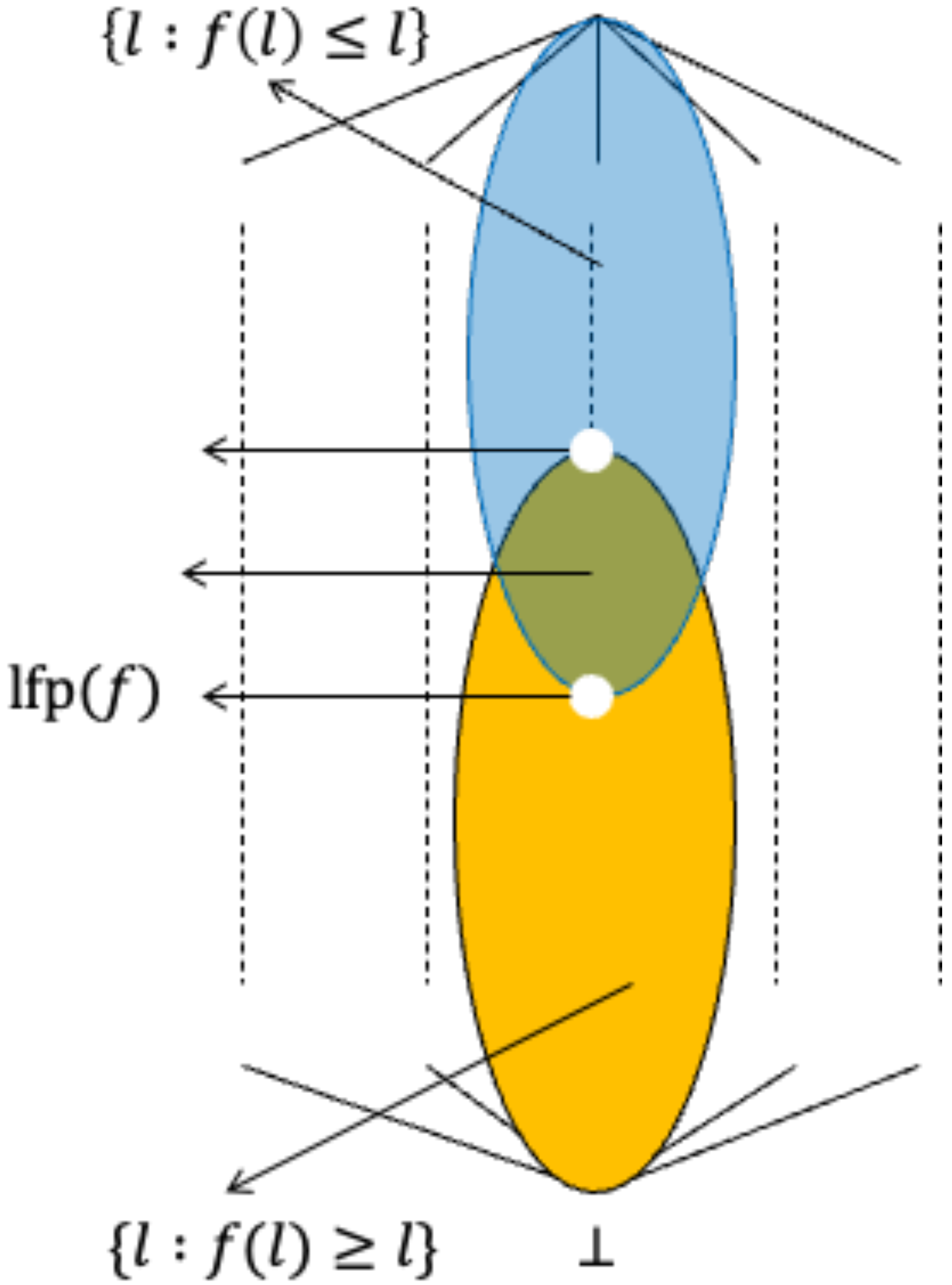

Kleene fixpoint theorem

- Let $\langle X, \sqsubseteq, \perp, \top, \sqcup, \sqcap \rangle$ be a lattice, let $f : X \to X$ be a continuous function. f has a fixpoint and can be computed as $\sqcup_{n \geq 0} f^n(\perp)$