Performance Modeling of Computer Systems and Networks

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Memoryless property
and
probability distributions

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Memoryless property

Informally:
the RV does not "remember" the past,
it behaves as a new variable

L'esponenziale gode della memoryless property: si comporta come se fosse una variabile sempre nuova.

the future depends only on relevant information about the current time, not on information from further in the past

Example:
X is the time elapsed in a shop from 9 am on a certain day until the arrival of the first customer

X is the time a server waits for the first customer

The "memoryless" property makes a comparison between the probability distributions of the time a shop has to wait from 9 am onwards for his first customer, and the time that the shop still has to wait for the first customer on those occasions when no customer has arrived by any given later time:

the property of memorylessness is that these distributions of "time from now to the next customer" are exactly the same.

- exponential (continuo)
- geometric (discreto)

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The remaining service time

A post office has 2 clerks.
Customer B is being served by one clerk, and customer C is being served by the other clerk, when A walks in.
All service times are exponentially distributed.

What is *Pr* {A *is the last to leave*}?

Poiché tempi esponenziali prob:

½
Note that one of B and C will leave first.
Let us say B leaves first.
Then C and A will have the same distribution on their remaining service time.
It doesn't matter that C has been serving for a while.

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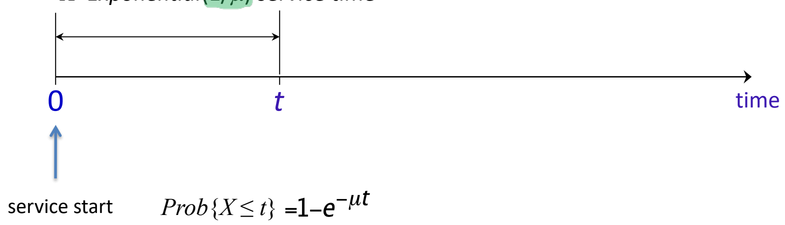
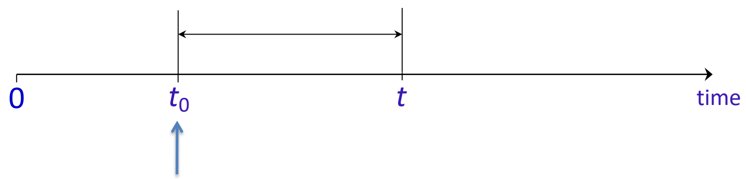
The remaining service time

media

X *Exponential*$(1/\mu)$ service time



$Prob\{X \le t\}$ $=1-e^{-\mu t}$



$Prob\{X \le t_0+t \mid X > t_0\}$?

$X - t_0$ is the *remaining service time*

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The remaining service time

$$Prob\{X \le t_0 + t | X > t_0\} = \frac{Prob\{t_0 < X \le t_0 + t\}}{Prob\{X > t_0\}} = \frac{Prob\{X \le t_0 + t\} - Prob\{X \le t_0\}}{Prob\{X > t_0\}}$$

$$= \frac{\cancel{1}-e^{-\mu(t_0+t)} - \left(\cancel{1}-e^{-\mu t_0}\right)}{\cancel{1}-\left(\cancel{1}-e^{-\mu t_0}\right)} = \frac{e^{-\mu t_0} - e^{-\mu(t_0+t)}}{e^{-\mu t_0}}$$

$$= 1 - \frac{\cancel{e^{-\mu t_0}} e^{-\mu t}}{\cancel{e^{-\mu t_0}}} = 1 - e^{-\mu t} = P(X \le t)$$

$$Prob\{X \le t_0 + t | X > t_0\} = Prob\{X \le t\}$$

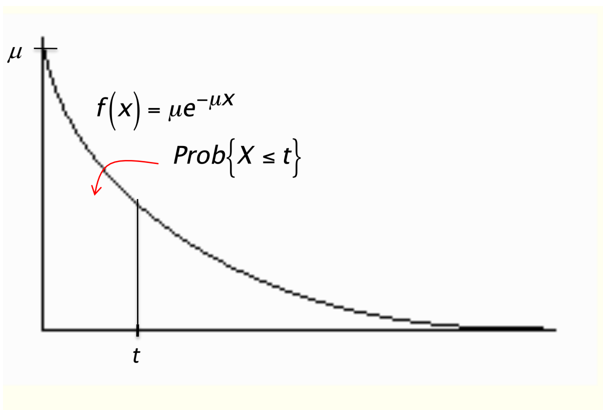
$$Prob\{X - t_0 \le t | X > t_0\} = Prob\{X \le t\}$$

remaining service time

service time

The two distributions are exactly the same.

Exponential$(1/\mu)$



Analytical models
Memoryless property

The remaining service time

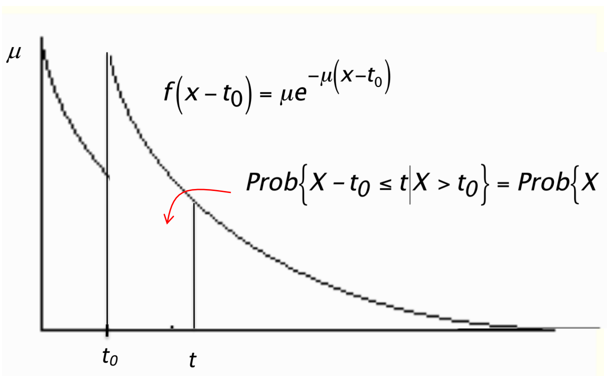
Exponential$(1/\mu)$

$$Prob\{X - t_0 \le t \mid X > t_0\}$$

Analytical models
Memoryless property

The remaining service time

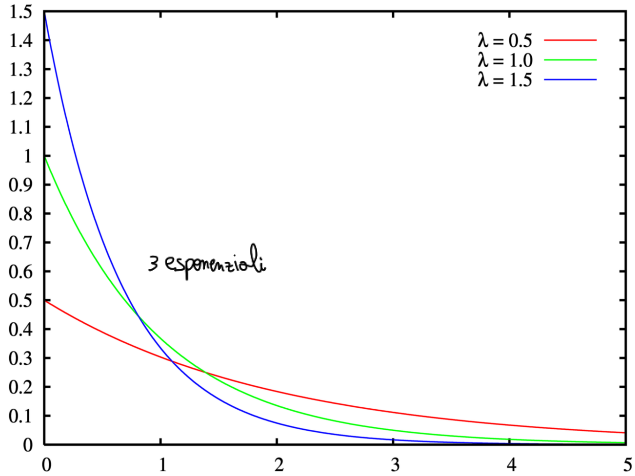
Exponential$(1/\mu)$



$$f(x - t_0) = \mu e^{-\mu(x - t_0)}$$

$$Prob\{X - t_0 \le t \mid X > t_0\} = Prob\{X \le t\}$$

The exponential is the only continuous distr memoryless *exponential distribution*



$f(x) = \lambda e^{-\lambda x}$

decadimento

1. For each unit step, the "quantity" is reduced of $e^{-\lambda}$

indipendente dal valore λ

λ is the *decay constant*

Media $\frac{1}{\lambda}$ cosa è ?

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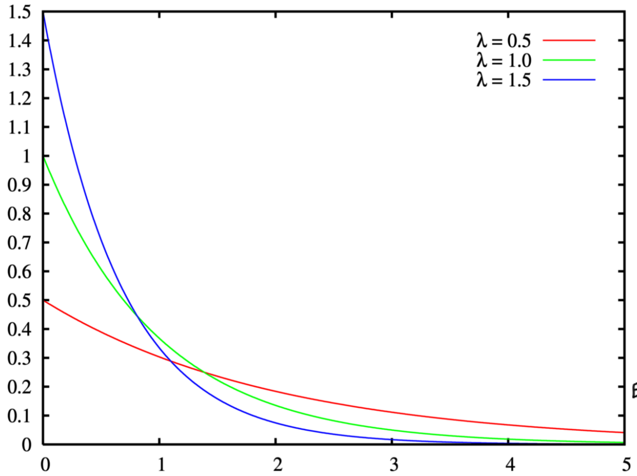
exponential distribution



Proprietà dell'esponenziale:

$f(x) = \lambda e^{-\lambda x}$

$f(n) = f(n-1)e^{-\lambda}$

ES: $f(1) = \lambda e^{-\lambda} = f(0)e^{-\lambda}$

$f(2) = \lambda e^{-2\lambda} = \lambda e^{-\lambda} \cdot e^{-\lambda} = f(1)e^{-\lambda}$

2. Independently of the λ value, 1/λ is called the exponential *time constant* : is the time to reduce the initial value of about 63.21%

$\int_0^{1/\lambda} f(t)\,dt = 1 - e^{-1}$

La media dell'esponenziale è $\frac{1}{\lambda}$, $f\left(\frac{1}{\lambda}\right) = \lambda e^{-1}$

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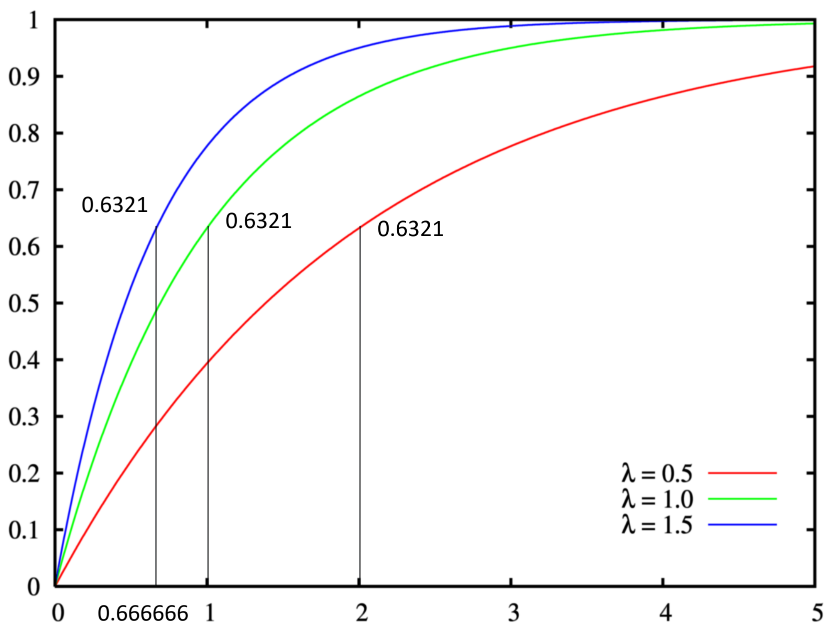

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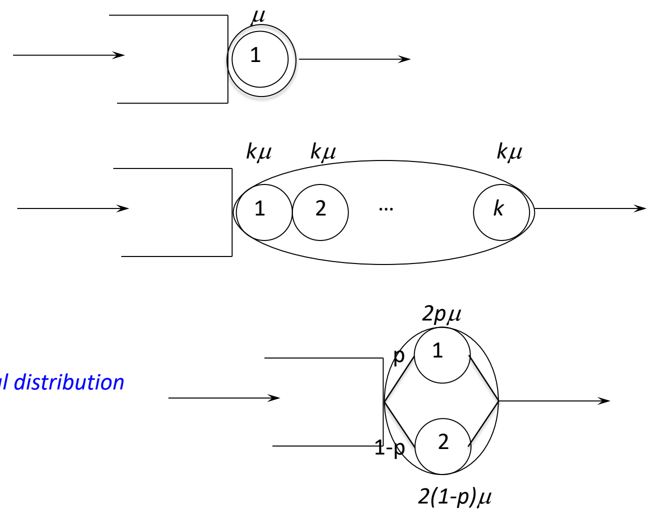
Phase-type distributions

Exponential

k-Erlang

hyperexponential distribution



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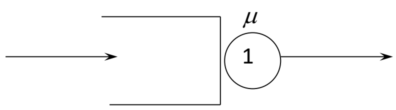
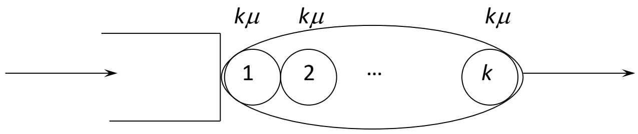
Phase-type distributions

Exponential



k-Erlang



- a job that requires the execution of k programs in series
- I/O unit that needs of a series of operations to serve a requirement (search for a cylinder, sector, read/write etc.)

Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

k-Erlang $f_i(x) = k\mu e^{-k\mu x}$ $E[X_i] = \frac{1}{k\mu}$ $\sigma^2(X_i) = \left(\frac{1}{k\mu}\right)^2$

$$f(x) = (k\mu)^k \frac{e^{-k\mu x}}{(k-1)!} x^{k-1} \qquad k \geq 1$$

$$E[X] = \sum_{i=1}^{k} E[X_i] = k\frac{1}{k\mu} = \frac{1}{\mu}$$

as the exponential!

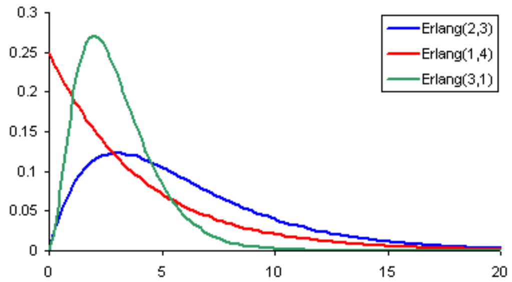
$$\sigma^2(X) = \sum_{i=1}^{k} \sigma^2(X_i) = k\left(\frac{1}{k\mu}\right)^2 = \frac{1}{k}\left(\frac{1}{\mu}\right)^2$$

k times less than the exponential!

Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

k-Erlang $f_i(x) = k\mu e^{-k\mu x}$ $E[X_i] = \frac{1}{k\mu}$ $\sigma^2(X_i) = \left(\frac{1}{k\mu}\right)^2$



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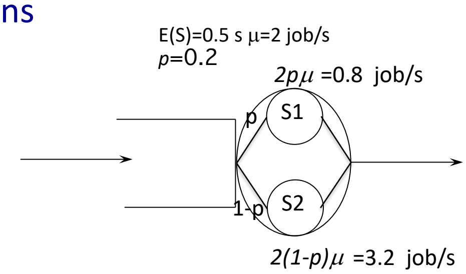
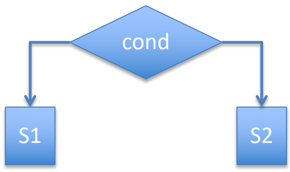
Phase-type distributions

hyperexponential distribution





- a job that requires the execution of 2 programs as an alternative

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Phase-type distributions

Exponential $f(x) = \mu e^{-\mu x}$ $E[X] = \frac{1}{\mu}$ $\sigma^2(X) = \left(\frac{1}{\mu}\right)^2$

hyperexponential distribution

$$f_1(x) = 2p\mu e^{-2p\mu x} \quad f_2(x) = 2(1-p)\mu e^{-2(1-p)\mu x}$$

$$f(x) = pf_1(x) + (1-p)f_2(x)$$

$$E[X] = pE[X_1] + (1-p)E[X_2] = p\frac{1}{2p\mu} + (1-p)\frac{1}{2(1-p)\mu} = \frac{1}{\mu}$$ as the exponential!

$$\sigma^2(X) = g(p)\left(\frac{1}{\mu}\right)^2$$ where $$g(p) = \frac{1}{2p(1-p)} - 1$$ $g(p)$ times more than the exponential!

$p=0.5 \longrightarrow g(p)=1$, hyperexpo=expo

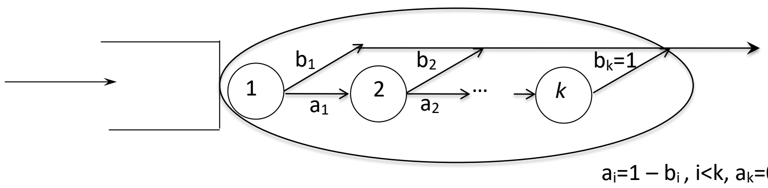
$\boxed{p=0.2} \longrightarrow \boxed{g(p)=2.125}$

$p \to 0.5 \longrightarrow g(p) \to 1$ variance decreases
$p \to 0$ or $1 \longrightarrow g(p) \to \infty$ variance grows

Cox distribution

∃ sempre distr. Cox che approssima bene la funzione arbitraria

How can we model a service demand with a different law, that is an arbitrary distribution?



$a_i = 1 - b_i$, $i<k$, $a_k=0$, $b_k=1$

- each stage is expo with mean $1/\mu_i$ (μ_i può essere diverso da μ_j)
- if $t_1, t_2, \ldots, t_k$, are the time spent in each stage the total time spent t is:
  - $t = t_1$ with probability b_1 (esco subito)
  - $t = t_1+t_2$ with probability a_1b_2 (esco al 2° stadio)
  - $t = t_1+t_2+t_3$ with probability $a_1a_2b_3$ (esco al 3° stadio)

  ...

  - $t = t_1+\ldots+t_k$ with probability $a_1a_2\ldots a_{k-1}b_k$

Arbitrary distribution

Case a) Arbitrary $f(t)$ with rational Laplace transform

$\longrightarrow$ $C_k(t) = f(t)$ for a given k,

exact, or with known precision

Case b) Arbitrary $g(t)$ without rational Laplace transform

$\longrightarrow$ $f(t) \approx g(t)$ approximate, with known precision

$C_k(t) \approx g(t)$

ovvero: $f(t)$ non razionale, cerco $g(t)$ simile (quindi l'errore è NOTO) e con Laplace razionale, riconducendomi al caso a)