**II Università di Roma, Tor Vergata**
**Dipartimento d'Ingegneria Civile e Ingegneria Informatica**
**LM in Ingegneria dell'Informazione e dell'Automazione**
**Complementi di Probabilità e Statistica - Advanced Statistics**
**Instructors: Roberto Monte & Massimo Regoli**
**Problems on Conditional Expectation with Solution 2022-12-08**

**Problem 1** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X, Y \in \mathcal{L}^2(\Omega; \mathbb{R})$ *such that*

$$\mathbf{E}[Y \mid X] = X \quad \text{and} \quad \mathbf{E}\left[Y^2 \mid X\right] = X^2.$$

*Prove that* $Y = X$, $\mathbf{P}$*-a.s. on* $\Omega$.

**Solution.** We have $Y = X$, $\mathbf{P}$-a.s. on $\Omega$ if and only if there exists an event $E \in \mathcal{E}$ such that $\mathbf{P}(E) = 0$ and $Y(\omega) = X(\omega)$ for every $\omega \in \Omega - E$. By virtue of the properties of the Lebesgue integral, we have

$$Y = X, \ \mathbf{P}\text{-a.s. on } \Omega \Leftrightarrow \int_\Omega (X - Y)^2 \, d\mathbf{P} = 0.$$

On the other hand,

$$\int_\Omega (X - Y)^2 \, d\mathbf{P} \equiv \mathbf{E}\left[(X - Y)^2\right].$$

Hence, we evaluate

$$\mathbf{E}\left[(X - Y)^2\right] = \mathbf{E}\left[X^2 - 2XY + Y^2\right] = \mathbf{E}\left[X^2\right] - 2\mathbf{E}[XY] + \mathbf{E}\left[Y^2\right]. \tag{1}$$

Now, by virtue of the properties of the conditional expectation operator, under our assumptions, we have

$$\mathbf{E}[XY] = \mathbf{E}[\mathbf{E}[XY \mid X]] = \mathbf{E}[X\mathbf{E}[Y \mid X]] = \mathbf{E}\left[X^2\right] \tag{2}$$

and

$$\mathbf{E}\left[Y^2\right] = \mathbf{E}\left[\mathbf{E}\left[Y^2 \mid X\right]\right] = \mathbf{E}\left[X^2\right]. \tag{3}$$

Combining (1)-(3) it follows

$$\mathbf{E}\left[(X - Y)^2\right] = 0,$$

which implies the desired result.

**Problem 2** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the real Borel state space. Let* $N \subseteq \mathbb{N}$, *let* $\{F_n\}_{n \in N}$ *be a complete system of mutually exclusive events of* $\Omega$ *and let* $\mathcal{F}$ *be the* $\sigma$*-algebra generated by* $\{F_n\}_{n \in N}$. *In symbols* $\mathcal{F} \equiv \sigma(\{F_n\}_{n \in N})$. *We know that a map* $Y : \Omega \to \mathbb{R}$ *is an* $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$*-random variable if and only if*

$$Y(\omega) = \sum_{n \in N} y_n 1_{F_n}(\omega), \quad \forall \omega \in \Omega,$$

*where* $(y_n)_{n \in N}$ *is a suitable sequence of real numbers.*

*Consider a random variable* $X \in L^2(\Omega; \mathbb{R})$ *and let* $L^2(\Omega_{\mathcal{F}}; \mathbb{R})$ *the subspace* $L^2(\Omega; \mathbb{R})$ *of space of all* $\mathcal{F}$*-random variables. Use the above claim to prove that*

$$\mathbf{E}[X \mid \mathcal{F}] = \underset{Y \in L^2(\Omega_{\mathcal{F}}; \mathbb{R})}{\arg\min} \, \mathbf{E}\left[(X - Y)^2\right]$$

*As a consequence, show that* $\mathbf{E}[X \mid \mathcal{F}]$ *is the orthogonal projection of* $X$ *on* $L^2(\Omega_{\mathcal{F}}; \mathbb{R})$.

**Solution.** The space $L^2(\Omega_{\mathcal{F}};\mathbb{R})$ of all real $\mathcal{F}$-random variables with finite moment of order 2 is a subspace of $L^2(\Omega;\mathbb{R})$ because it fulfills the conditions for a subset of Hilbert space to be a subspace of the Hilbert space. In fact, for all $X, Y \in L^2(\Omega_{\mathcal{F}};\mathbb{R})$ and all $\alpha, \beta \in \mathbb{R}$ the linear combination $\alpha X + \beta Y$ is also in $L^2(\Omega_{\mathcal{F}};\mathbb{R})$, to say $L^2(\Omega_{\mathcal{F}};\mathbb{R})$ is closed for the linear combination. In addition, if $(X_n)_{n\geq 1}$ is a sequence belonging to $L^2(\Omega_{\mathcal{F}};\mathbb{R})$ and such that $X_n \xrightarrow{L^2} X$, where $X \in L^2(\Omega;\mathbb{R})$, we also have $X \in L^2(\Omega_{\mathcal{F}};\mathbb{R})$, to say $L^2(\Omega_{\mathcal{F}};\mathbb{R})$ is a closed subset of $L^2(\Omega;\mathbb{R})$ in the topology induced by the norm.

Now, given $X \in L^2(\Omega_{\mathcal{F}};\mathbb{R})$, consider the functional $\Delta_X : L^2(\Omega_{\mathcal{F}};\mathbb{R}) \to \mathbb{R}_+$ given by

$$\Delta_X(Y) \stackrel{\text{def}}{=} \mathbf{E}\left[(X-Y)^2\right], \quad \forall Y \in L^2(\Omega_{\mathcal{F}};\mathbb{R}).$$

Since in the case under concern

$$Y \in L^2(\Omega_{\mathcal{F}};\mathbb{R}) \Leftrightarrow Y(\omega) = \sum_{n\in N} y_n 1_{F_n}(\omega), \quad \forall \omega \in \Omega,$$

we can write

$$\Delta_X(Y) = \mathbf{E}\left[\left(X - \sum_{n\in N} y_n 1_{F_n}\right)^2\right] \equiv \Delta_X(y_1,\dots,y_n,\dots).$$

Hence,

$$\begin{aligned}\Delta_X(y_1,\dots,y_n,\dots) &= \mathbf{E}\left[X^2 - 2\sum_{n\in N} y_n X 1_{F_n} + \sum_{m,n\in N} y_m y_n 1_{F_m} 1_{F_n}\right] \\ &= \mathbf{E}\left[X^2\right] - 2\sum_{n\in N} y_n \mathbf{E}\left[X 1_{F_n}\right] + \sum_{m,n\in N} y_m y_n \mathbf{E}\left[1_{F_m} 1_{F_n}\right].\end{aligned}$$

On the other hand,

$$1_{F_m} 1_{F_n} = \begin{cases} 1_{F_n} & \text{if } m = n \\ 1_{\varnothing} & \text{if } m \neq n \end{cases}.$$

Moreover,

$$\mathbf{E}[1_E] = \mathbf{P}(E), \quad \forall E \in \mathcal{E}$$

and

$$\mathbf{E}[X 1_E] = \int_\Omega X 1_E \, d\mathbf{P} = \int_E X \, d\mathbf{P}, \quad \forall E \in \mathcal{E}.$$

Therefore,

$$\Delta_X(Y) = \mathbf{E}\left[X^2\right] - 2\sum_{n\in N} y_n \int_{F_n} X \, d\mathbf{P} + \sum_{n\in N} y_n^2 \mathbf{P}(F_n).$$

As a consequence,

$$\partial_{y_m} \Delta_X(y_1,\dots,y_n,\dots) = -2\int_{Fm} X \, d\mathbf{P} + 2 y_m \mathbf{P}(F_m), \quad \forall m \in N,$$

which implies

$$\partial_{y_m} \Delta_X(y_1,\dots,y_n,\dots) = 0 \Leftrightarrow y_m = \frac{1}{\mathbf{P}(F_m)} \int_{F_m} X \, d\mathbf{P} = \mathbf{E}[X \mid F_m], \quad \forall m \in N.$$

Thus, a candidate minimum $Y$ for the functional $\Delta_X : L^2(\Omega_{\mathcal{F}};\mathbb{R}) \to \mathbb{R}_+$ takes the form

$$Y = \sum_{n\in N} \mathbf{E}[X \mid F_n] 1_{F_n} = \mathbf{E}[X \mid \mathcal{F}].$$

Now, we have

$$\partial_{y_m}^2 \Delta_X (y_1, \ldots, y_n, \ldots) = \mathbf{P}(F_m) > 0$$

and the functional $\Delta_X : L^2(\Omega_{\mathcal{F}}; \mathbb{R}) \to \mathbb{R}_+$ is known to be convex[1]. It then follow that

$$\mathbf{E}[X \mid \mathcal{F}] = \underset{Y \in L^2(\Omega_{\mathcal{F}}; \mathbb{R})}{\arg\min} \mathbf{E}\left[(X-Y)^2\right].$$

To complete the proof, it is sufficient to observe that in a Hilbert space the ortoghonal projection of a given vector onto a subspace determines the vector in the subspace of the minimum distance from the given vector.

□

**Problem 3** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ be the real Borel state space. Let $X, Y \in L^2(\Omega; \mathbb{R})$.*

1. *Prove in all details that $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$ a.e. on $\Omega$ implies $Cov(X,Y) = 0$, but $X$ and $Y$ may not be independent.*
2. *Prove in all details that $Cov(X,Y) = 0$ does not imply $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$.*

*Hint: in the first case, to generate a suitable counterexample one may consider the random variables $X \sim Ber(p)$, $Z \sim N(0,1)$, independent of $X$, and $Y = XZ$. In the second case consider $X \sim N(0,1)$ and $Y = X^2$.*

---

[1] To prove the convexity of the functional $\Delta_X : L^2(\Omega_{\mathcal{F}}; \mathbb{R}) \to \mathbb{R}_+$, we may observe that thanks, to the Cauchy-Schwarz inequality and the convexity of the standard quadratic function $f(u) \stackrel{\text{def}}{=} u^2$, we have

$$\begin{aligned}
\Delta_X(\theta Y + (1-\theta)Z) &= \mathbf{E}\left[(X - (\theta Y + (1-\theta)Z))^2\right] \\
&= \mathbf{E}\left[(\theta(X-Y) + (1-\theta)(X-Z))^2\right] \\
&= \mathbf{E}\left[\theta^2(X-Y)^2 + 2\theta(1-\theta)(X-Y)(X-Z) + (1-\theta)^2(X-Z)^2\right] \\
&= \theta^2\mathbf{E}\left[(X-Y)^2\right] + 2\theta(1-\theta)\mathbf{E}[(X-Y)(X-Z)] + (1-\theta)^2\mathbf{E}\left[(X-Z)^2\right] \\
&\leq \theta^2\mathbf{E}\left[(X-Y)^2\right] + 2\theta(1-\theta)\left|\mathbf{E}[(X-Y)(X-Z)]\right| + (1-\theta)^2\mathbf{E}\left[(X-Z)^2\right] \\
&\leq \theta^2\mathbf{E}\left[(X-Y)^2\right] + 2\theta(1-\theta)\mathbf{E}\left[(X-Y)^2\right]^{1/2}\mathbf{E}\left[(X-Z)^2\right]^{1/2} + (1-\theta)^2\mathbf{E}\left[(X-Z)^2\right] \\
&= \left(\theta\mathbf{E}\left[(X-Y)^2\right]^{1/2} + (1-\theta)\mathbf{E}\left[(X-Z)^2\right]^{1/2}\right)^2 \\
&\leq \theta\mathbf{E}\left[(X-Y)^2\right] + (1-\theta)\mathbf{E}\left[(X-Z)^2\right],
\end{aligned}$$

for every $\theta \in [0,1]$.

To show the convexity of the standard quadratic function, $f(u) \stackrel{\text{def}}{=} u^2$, we may observe that the inequality

$$(u-v)^2 \geq 0,$$

which holds true for every $u, v \in \mathbb{R}$, implies

$$-\theta(1-\theta)(u-v)^2 \leq 0,$$

which holds true for every $u, v \in \mathbb{R}$ and $\theta \in [0,1]$. The latter can be rewritten as

$$-\theta(1-\theta)\left(u^2 - 2uv + v^2\right) \leq 0$$

or equivalently

$$\theta^2 u^2 - \theta u^2 + 2\theta(1-\theta)uv + (1-\theta)^2 v^2 - (1-\theta)v^2.$$

This implies

$$\theta^2 u^2 + 2\theta(1-\theta)uv + (1-\theta)^2 v^2 \leq \theta u^2 + (1-\theta)v^2.$$

Hence,

$$(\theta u + (1-\theta)v)^2 \leq \theta u^2 + (1-\theta)v^2,$$

which proves the desired result.

**Solution.**

1. Under the assumption $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$ a.e. on $\Omega$, by virtue of the properties of the conditional expectation operator, we can write

$$\mathbf{E}[XY] = \mathbf{E}[\mathbf{E}[XY \mid X]] = \mathbf{E}[X\mathbf{E}[Y \mid X]] = \mathbf{E}[X\mathbf{E}[Y]] = \mathbf{E}[X]\,\mathbf{E}[Y]$$

Therefore,

$$Cov(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = 0.$$

Now, if we consider the random $X \sim Ber(p)$, $Z \sim N(0,1)$, independent of $X$, and $Y = XZ$, we have

$$\begin{aligned} Cov(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = \mathbf{E}\left[X^2 Z\right] - \mathbf{E}[X]\,\mathbf{E}[XZ] \\ &= \mathbf{E}\left[X^2\right]\mathbf{E}[Z] - \mathbf{E}[X]^2\,\mathbf{E}[Z] \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\mathbf{P}(X \leq 0) = q,$$

and, on account that $X$ and $Z$ are independent,

$$\begin{aligned} \mathbf{P}(Y \leq 0) &= \mathbf{P}(XZ \leq 0) \\ &= \mathbf{P}(XZ \leq 0, X = 0) + \mathbf{P}(XZ \leq 0, X = 1) \\ &= \mathbf{P}(XZ \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) + \mathbf{P}(XZ \leq 0 \mid X = 1)\,\mathbf{P}(X = 1) \\ &= \mathbf{P}(0 \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) + \mathbf{P}(Z \leq 0 \mid X = 1)\,\mathbf{P}(X = 1) \\ &= \mathbf{P}(\Omega)\,\mathbf{P}(X = 0) + \mathbf{P}(Z \leq 0)\,\mathbf{P}(X = 1) \\ &= q + \frac{1}{2}p. \end{aligned}$$

Furthermore, the same arguments as above shows that

$$\begin{aligned} \mathbf{P}(X \leq 0, Y \leq 0) &= \mathbf{P}(X \leq 0, XZ \leq 0) \\ &= \mathbf{P}(X = 0, XZ \leq 0) \\ &= \mathbf{P}(XZ \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) \\ &= q. \end{aligned}$$

Hence, we have

$$\mathbf{P}(X \leq 0)\,\mathbf{P}(Y \leq 0) = q\left(q + \frac{1}{2}p\right) \neq q = \mathbf{P}(X \leq 0, Y \leq 0)$$

which shows that $X$ and $Y$ are not be independent.

2. To show that $Cov(X,Y) = 0$ does not imply $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$, we consider $X \sim N(0,1)$ and $Y = X^2$. We have

$$\begin{aligned} Cov(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] \\ &= \mathbf{E}\left[X^3\right] - \mathbf{E}[X]\,\mathbf{E}\left[X^2\right] \\ &= 0, \end{aligned}$$

but

$$\mathbf{E}[Y \mid X] = \mathbf{E}\left[X^2 \mid X\right] = X^2 \neq \mathbf{E}\left[X^2\right] = \mathbf{E}[Y].$$

□

**Problem 4** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X$ and $Y$ be independent standard Gaussian distributed random variables on $\Omega$. Set*

$$U \stackrel{def}{=} X + Y, \qquad V \stackrel{def}{=} X - Y.$$

1. *Compute the distributions of $U$ and $V$.*

2. *Prove that $U$ and $V$ are independent.*

3. *Compute $\mathbf{E}[X \mid U]$, $\mathbf{E}[X \mid V]$, $\mathbf{E}[Y \mid U]$, $\mathbf{E}[Y \mid V]$.*

4. *Compute $\mathbf{E}[XY \mid U]$.*

**Exercise 5** *Hint: First, concentrate your attention on the circumstance that $X$ and $Y$ are independent and standard Gaussian distributed. Second, it might be useful to consider $\mathbf{E}\left[X^2 \mid U\right]$ and $\mathbf{E}\left[Y^2 \mid U\right]$.*

**Solution.** .

**Problem 6** *Let $N$ be a geometric random variable with success probability $p$, which models the first occurrence of success in $n$ independent trials, and let $(X_n)_{n\geq 1}$ be a sequence of independent and normally distributed random variables with mean $\mu$ and variance $\sigma^2$, which are also independent of $N$. Study the conditional expectation*

$$\mathbf{E}\left[\sum_{k=1}^{N} X_k \mid N\right].$$

*Use the properties of the conditional expectation to compute the expectation and the variance of the random sum*

$$S_N \stackrel{def}{=} \sum_{k=1}^{N} X_k.$$

**Solution.** Since the random variables of the sequence $(X_n)_{n\geq 1}$ are independent and are also independent of $N$, which is geometrically distributed, we can write

$$\begin{aligned}
\mathbf{E}\left[\sum_{k=1}^{N} X_k \mid N\right] &= \sum_{n=1}^{\infty} \mathbf{E}\left[\sum_{k=1}^{N} X_k \mid N=n\right] 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \int_{\Omega}\left(\sum_{k=1}^{N} X_k\right) 1_{\{N=n\}}\, d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \int_{\{N=n\}}\left(\sum_{k=1}^{N} X_k\right) d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \int_{\{N=n\}}\left(\sum_{k=1}^{n} X_k\right) d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \int_{\Omega}\left(\sum_{k=1}^{n} X_k\right) 1_{\{N=n\}}\, d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right) 1_{\{N=n\}}\right]\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \left(\sum_{k=1}^{n} \mathbf{E}[X_k]\right) \mathbf{E}\left[1_{\{N=n\}}\right]\right) 1_{\{N=n_m\}} \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\mathbf{P}(N=n)} \left(\sum_{k=1}^{n} \mu\right) \mathbf{P}(N=n)\right) 1_{\{N=n_m\}} \\
&= \sum_{n=1}^{\infty} n\mu 1_{\{N=n_m\}} \\
&= \mu \sum_{n=1}^{\infty} n 1_{\{N=n\}} \\
&\overset{\mathbf{P}\text{-a.s.}}{=} \mu N.
\end{aligned}$$

Now, we can write

$$\mathbf{E}\left[\sum_{k=1}^{N} X_k\right] = \mathbf{E}\left[\mathbf{E}\left[\sum_{k=1}^{N} X_k \mid N\right]\right] = \mathbf{E}[\mu N] = \mu \mathbf{E}[N] = \frac{\mu}{p}.$$

and

$$\mathbf{D}^2\left[\sum_{k=1}^{N} X_k\right] = \mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2\right] - \mathbf{E}\left[\sum_{k=1}^{N} X_k\right]^2 = \mathbf{E}\left[\mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2 \mid N\right]\right] - \frac{\mu^2}{p^2}.$$

Thus, we are left with computing

$$\mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2 \mid N\right].$$

A straightforward computation yields

$$
\begin{aligned}
\mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2 \mid N\right] &= \sum_{n=1}^{\infty} \mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2 \mid N=n\right] 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \int_{\Omega} \left(\sum_{k=1}^{N} X_k\right)^2 1_{\{N=n\}}\, d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \int_{\{N=n\}} \left(\sum_{k=1}^{N} X_k\right)^2 d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \int_{\{N=n\}} \left(\sum_{k=1}^{n} X_k\right)^2 d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \int_{\Omega} \left(\sum_{k=1}^{n} X_k\right)^2 1_{\{N=n\}}\, d\mathbf{P}\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)^2 1_{\{N=n\}}\right]\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] \mathbf{E}\left[1_{\{N=n\}}\right]\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\mathbf{P}(N=n)} \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] \mathbf{P}(N=n)\right) 1_{\{N=n\}} \\
&= \sum_{n=1}^{\infty} \mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] 1_{\{N=n\}},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{E}\left[\left(\sum_{k=1}^{n} X_k\right)^2\right] &= \mathbf{E}\left[\sum_{k=1}^{n} X_k^2 + \sum_{k,\ell=1}^{n} X_k X_\ell\right] \\
&= \sum_{k=1}^{n} \mathbf{E}\left[X_k^2\right] + \sum_{k,\ell=1}^{n} \mathbf{E}\left[X_k\right] \mathbf{E}\left[X_k\right] \\
&= \sum_{k=1}^{n} \left(\mu^2 + \sigma^2\right) + \sum_{k,\ell=1}^{n} \mu^2 \\
&= \left(\mu^2 + \sigma^2\right) n + \mu^2 (n-1)\, n \\
&= \sigma^2 n + \mu^2 n^2.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{k=1}^{N} X_k\right)^2 \mid N\right] &= \sum_{n=1}^{\infty}\left(\sigma^2 n+\mu^2 n^2\right) 1_{\{N=n\}} \\
&= \sigma^2 \sum_{n=1}^{\infty} n 1_{\{N=n\}}+\mu^2 \sum_{n=1}^{\infty} n^2 1_{\{N=n\}} \\
&= \sigma^2 N+\mu^2 N^2 .
\end{aligned}$$

It then follows

$$\begin{aligned}
\mathbf{E}\left[\sigma^2 N+\mu^2 N^2\right] &= \sigma^2 \mathbf{E}[N]+\mu^2 \mathbf{E}\left[N^2\right] \\
&= \frac{\sigma^2}{p}+\mu^2\left(\mathbf{D}^2[N]+\mathbf{E}[N]^2\right) \\
&= \frac{\sigma^2}{p}+\mu^2\left(\frac{2-p}{p^2}\right).
\end{aligned}$$

In the end,

$$\mathbf{D}^2\left[\sum_{k=1}^{N} X_k\right]=\frac{\sigma^2}{p}+\mu^2\left(\frac{2-p}{p^2}\right)-\frac{\mu^2}{p^2}=\frac{\sigma^2}{p}+\mu^2\left(\frac{1-p}{p^2}\right).$$

**Problem 7** *Let $N$ be a Poisson random variable with rate parameters $\lambda$ and let $(X_k)_{k=1}^n$ a finite sequence of independent standard Bernoulli random variables with success parameter $p$, which are also independent of $N$. Study the conditional expectation*

$$\mathbf{E}\left[\sum_{k=1}^{N} X_k \mid N\right].$$

*Use the properties of the conditional expectation to compute the expectation and the variance of the random sum*

$$S_N \stackrel{def}{=} \sum_{k=1}^{N} X_k .$$

**Solution.** .

**Problem 8** *Let $B$ be a binomial random variable with number of trials paramenter $n$ and success probability $p$, which models the number of successes in $n$ independent trials, and let $(X_k)_{k=1}^n$ be a finite sequence of independent and exponentially distributed random variables with rate parameter $\lambda$, which are also independent of $B$. Study the conditional expectation*

$$\mathbf{E}\left[\sum_{k=1}^{B} X_k \mid B\right].$$

*Use the properties of the conditional expectation to compute the expectation (and the variance) of the random sum*

$$S_B \stackrel{def}{=} \sum_{k=1}^{B} X_k .$$

**Solution.** .

**Problem 9** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $X$ and $Y$ be independent standard Rademacher random variables*[2] *on $\Omega$. Set $Z \stackrel{def}{=} X + Y$.*

1. *Compute $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$.*
2. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ uncorrelated?*
3. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with standard Rademacher random variables, can you compute $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ and $\mathbf{E}[XY \mid Z]$?*

**Solution.** Since $X$ and $Y$ be independent standard Rademacher random variables, we have

$$Z(\omega) = (X+Y)(\omega) = \begin{cases} -2, & \text{if } \omega \in \{X=-1, Y=-1\}, \\ 0, & \text{if } \omega \in \{X=-1, Y=1\} \cup \{X=1, Y=-1\}, \\ 2, & \text{if } \omega \in \{X=1, Y=1\}. \end{cases}$$

That is to say,

$$X + Y = -2 \cdot 1_{\{X=-1,Y=-1\}} + 2 \cdot 1_{\{X=1,Y=1\}} + 0 \cdot 1_{\{X=-1,Y=1\} \cup \{X=1,Y=-1\}},$$

equivalently

$$Z = -2 \cdot 1_{\{Z=-2\}} + 2 \cdot 1_{\{Z=2\}} + 0 \cdot 1_{\{Z=0\}}.$$

Furthermore,

$$\mathbf{P}(Z=-2) = \mathbf{P}(X+Y=-2) = \mathbf{P}(X=-1, Y=-1) = \mathbf{P}(X=-1)\,\mathbf{P}(Y=-1) = \frac{1}{4},$$

$$\mathbf{P}(Z=2) = \mathbf{P}(X+Y=2) = \mathbf{P}(X=1, Y=1) = \mathbf{P}(X=1)\,\mathbf{P}(Y=1) = \frac{1}{4},$$

and

$$\begin{aligned} \mathbf{P}(Z=0) &= \mathbf{P}(X+Y=0) = \mathbf{P}(\{X=-1, Y=1\} \cup \{X=1, Y=-1\}) \\ &= \mathbf{P}(X=-1, Y=1) + \mathbf{P}(X=1, Y=-1) \\ &= \mathbf{P}(X=-1)\,\mathbf{P}(Y=1) + \mathbf{P}(X=1)\,\mathbf{P}(Y=-1) \\ &= \frac{1}{2}. \end{aligned}$$

1. Since $Z$ is a discrete random variable, to compute $\mathbf{E}[X \mid Z]$ we can apply the formula

$$\mathbf{E}[X \mid Z] = \mathbf{E}[X \mid Z=-2]\,1_{\{Z=-2\}} + \mathbf{E}[X \mid Z=2]\,1_{\{Z=2\}} + \mathbf{E}[X \mid Z=0]\,1_{\{Z=0\}},$$

where

$$\begin{aligned} \mathbf{E}[X \mid Z=-2] &= \frac{1}{\mathbf{P}(Z=-2)} \int_{\{Z=2\}} X d\mathbf{P} = 4 \int_{\{X=-1,Y=-1\}} X d\mathbf{P} \\ &= -4 \int_{\{X=-1,Y=-1\}} d\mathbf{P} = -4\mathbf{P}(X=-1, Y=-1) \\ &= -1, \end{aligned}$$

[2] A standard Rademacher random variable $R$ is given by

$$R \stackrel{\text{def}}{=} \begin{cases} 1, & \mathbf{P}(R=1) = 1/2, \\ -1, & \mathbf{P}(R=-1) = 1/2. \end{cases}$$

$$
\begin{aligned}
\mathbf{E}[X \mid Z=2] &= \frac{1}{\mathbf{P}(Z=2)} \int_{\{Z=2\}} X d\mathbf{P} = 4 \int_{\{X=1,Y=1\}} X d\mathbf{P} \\
&= 4 \int_{\{X=1,Y=1\}} d\mathbf{P} = 4\mathbf{P}(X=1, Y=1) \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}[X \mid Z=0] &= \frac{1}{\mathbf{P}(Z=0)} \int_{\{Z=0\}} X d\mathbf{P} = 2 \left( \int_{\{X=-1,Y=1\} \cup \{X=1,Y=-1\}} X d\mathbf{P} \right) \\
&= 2 \left( \int_{\{X=-1,Y=1\}} X d\mathbf{P} + \int_{\{X=1,Y=-1\}} X d\mathbf{P} \right) \\
&= 2(-1 \cdot \mathbf{P}(X=-1, Y=1) + 1 \cdot \mathbf{P}(X=1, Y=-1)) \\
&= 2 \left( -1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} \right) \\
&= 0.
\end{aligned}
$$

It follows

$$
\mathbf{E}[X \mid Z] = -1 \cdot 1_{\{Z=-2\}} + 1 \cdot 1_{\{Z=-2\}} + 0 \cdot 1_{\{Z=0\}} = \frac{1}{2} Z.
$$

In addition, since $X$ and $Y$ clearly play the same role,

$$
\mathbf{E}[Y \mid Z] = \frac{1}{2} Z.
$$

Another argument, based on the properties of the conditional expectation, is the following. Observe that

$$
Z = \mathbf{E}[Z \mid Z] = \mathbf{E}[X + Y \mid Z] = \mathbf{E}[X \mid Z] + \mathbf{E}[Y \mid Z].
$$

On the other hand, we know that

$$
\mathbf{E}[X \mid Z] = g_X(Z) \quad \text{and} \quad \mathbf{E}[Y \mid Z] = g_Y(Z)
$$

where $g_X : \mathbb{R} \to \mathbb{R}$ and $g_X : \mathbb{R} \to \mathbb{R}$ are suitable Borel functions. The structure of the function $g_X : \mathbb{R} \to \mathbb{R}$ [resp. $g_Y : \mathbb{R} \to \mathbb{R}$] depends on the joint distribution of $X$ and $Z$ [resp. $Y$ and $Z$] and on the distribution of $Z$. However, in our case, it is not difficult to show that

$$
F_{X,Z}(u, z) = F_{Y,Z}(u, z),
$$

for every $(u, z) \in \mathbb{R}^2$. In fact,

$$
\begin{aligned}
&F_{X,Z}(u, z) \\
&= \mathbf{P}(X \leq u, Z \leq z) \\
&= \mathbf{P}(X \leq u, X + Y \leq z) \\
&= \mathbf{P}(X \leq u, X + Y \leq z, X = 1) + \mathbf{P}(X \leq u, X + Y \leq z, X = -1) \\
&= \mathbf{P}(X \leq u, X + Y \leq z \mid X = 1) \mathbf{P}(X = 1) + \mathbf{P}(X \leq u, X + Y \leq z \mid X = -1) \mathbf{P}(X = -1) \\
&= \frac{1}{2} (\mathbf{P}(X \leq u, X + Y \leq z \mid X = 1) + \mathbf{P}(X \leq u, X + Y \leq z \mid X = -1)) \\
&= \frac{1}{2} (\mathbf{P}(1 \leq u, 1 + Y \leq z \mid X = 1) + \mathbf{P}(-1 \leq u, -1 + Y \leq z \mid X = -1)) \\
&= \frac{1}{2} (\mathbf{P}(1 \leq u, Y \leq z - 1 \mid X = 1) + \mathbf{P}(-1 \leq u, Y \leq z + 1 \mid X = -1)) \\
&= \begin{cases} 0, & \text{if } u < -1, \\ \frac{1}{2}\mathbf{P}(Y \leq z + 1 \mid X = -1) = \frac{1}{2}\mathbf{P}(Y \leq z + 1), & \text{if } -1 \leq u < 1, \\ \frac{1}{2}(\mathbf{P}(Y \leq z - 1 \mid X = 1) + \mathbf{P}(Y \leq z + 1 \mid X = -1)) = \frac{1}{2}(\mathbf{P}(Y \leq z - 1) + \mathbf{P}(Y \leq z + 1)), & \text{if } 1 \leq u. \end{cases}
\end{aligned}
$$

Similarly,

$$F_{Y,Z}(u,z)$$
$$= \begin{cases} 0, & \text{if } u < -1, \\ \frac{1}{2}\mathbf{P}(X \le z+1 \mid Y=-1) = \frac{1}{2}\mathbf{P}(X \le z+1), & \text{if } -1 \le u < 1, \\ \frac{1}{2}(\mathbf{P}(X \le z-1 \mid Y=1) + \mathbf{P}(X \le z+1 \mid Y=-1)) = \frac{1}{2}(\mathbf{P}(X \le z-1) + \mathbf{P}(X \le z+1)), & \text{if } 1 \le u. \end{cases}$$

Therefore, on account that $X$ and $Y$ have the same distribution, we obtain the desired result. As a consequence, we can asses that

$$g_X = g_Y,$$

which implies

$$\mathbf{E}[X \mid Z] = \mathbf{E}[Y \mid Z].$$

It then follows

$$2\mathbf{E}[X \mid Z] = 2\mathbf{E}[Y \mid Z] = Z,$$

which yields

$$\mathbf{E}[X \mid Z] = \mathbf{E}[Y \mid Z] = \frac{1}{2}Z,$$

as expected.

2. Thanks to what shown above, we have

$$\mathbf{E}[X \mid Z]\,\mathbf{E}[Y \mid Z] = \frac{1}{4}Z^2 \sim Ber\left(\frac{1}{2}\right).$$

Hence,

$$\mathbf{E}[\mathbf{E}[X \mid Z]\,\mathbf{E}[Y \mid Z]] = \frac{1}{2}.$$

On the other hand,

$$\mathbf{E}[\mathbf{E}[X \mid Z]] = \mathbf{E}[\mathbf{E}[Y \mid Z]] = \mathbf{E}\left[\frac{1}{2}Z\right] = \frac{1}{2}\mathbf{E}[Z] = 0.$$

Hence, $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ are not uncorrelated.

3. Since $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ are not not uncorrelated, they cannot be independent.

4. By virtue of the properties of the conditional expectation, we have

$$\mathbf{E}\left[(X+Y)^2 \mid Z\right] = \mathbf{E}\left[Z^2 \mid Z\right] = Z^2.$$

On the other hand,

$$\begin{aligned} \mathbf{E}\left[(X+Y)^2 \mid Z\right] &= \mathbf{E}\left[X^2 + 2XY + Y^2 \mid Z\right] \\ &= \mathbf{E}\left[X^2 \mid Z\right] + 2\mathbf{E}[XY \mid Z] + \mathbf{E}\left[Y^2 \mid Z\right]. \end{aligned}$$

Now, since $X \sim Y \sim Rad(1/2)$, we have $X^2 \sim Y^2 \sim Dir(1)$. We then obtain

$$Z^2 = \mathbf{E}\left[(X+Y)^2 \mid Z\right] = \mathbf{E}[1 \mid Z] + 2\mathbf{E}[XY \mid Z] + \mathbf{E}[1 \mid Z] = 2 + 2\mathbf{E}[XY \mid Z].$$

The latter yields

$$\mathbf{E}[XY \mid Z] = \frac{1}{2}Z^2 - 1.$$

**Problem 10** *Let $Z$ [resp. $R$] be a standard Gaussian [Rademacher] random variable on a probability space $\Omega$. In symbols, $X \sim N(0,1)$ and $R \sim Rad(1/2)$. Assume that $X$ and $R$ are independent and define $Y \equiv R \cdot X$.*

1. *Is the random variable $Y$ Gaussian?*
2. *Are the random variables $X$ and $Y$ uncorrelated? Are $X$ and $Y$ independent?*
3. *Are the random variables $R$ and $Y$ uncorrelated? Are $R$ and $Y$ independent?*
4. *Does the random vector $(X,Y)^\intercal$ have a bivariate Gaussian distribution? Hint: consider the possibility that $(X,Y)^\intercal$ has a bivariate Gaussian distribution; how the random variable $Z \equiv X + Y$ should be distributed?*
5. *Can you compute $\mathbf{E}[Y \mid X]$ and $\mathbf{E}[X \mid Y]$?*

**Solution.** .

**Problem 11** *Let $X$ [resp. $B$] be a standard Gaussian [Bernoulli] random variable on a probability space $\Omega$. In symbols, $X \sim N(0,1)$ and $B \sim Ber(1/2)$. Assume that $X$ and $B$ are independent and define $Y \equiv B \cdot X$.* ***Specifying carefully the properties used****, answer the following questions:*

1. *Is the random variable $Y$ Gaussian? Is $Y$ absolutely continuous?*
2. *Are the random variables $X$ and $Y$ uncorrelated? Are $X$ and $Y$ independent?*
3. *Are the random variables $B$ and $Y$ uncorrelated? Are $B$ and $Y$ independent?*
4. *Does the random vector $(X,Y)^\intercal$ have a bivariate Gaussian distribution?*
5. *Can you compute $\mathbf{E}[Y \mid X]$? What about $\mathbf{E}[X \mid Y]$?*

**Solution.** /.

**Problem 12** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $X$ and $Y$ be independent standard Rademacher random variables*[3] *on $\Omega$. Set $Z \stackrel{\text{def}}{=} X + Y$.*

1. *Compute $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$.*
2. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ uncorrelated?*
3. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with standard Rademacher random variables, can you compute $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ and $\mathbf{E}[XY \mid Z]$?*

**Solution.** .

[3] A standard Rademacher random variable $R$ is given by

$$R \stackrel{\text{def}}{=} \begin{cases} 1, & \mathbf{P}(R=1) = 1/2, \\ -1, & \mathbf{P}(R=-1) = 1/2. \end{cases}$$

**Problem 13** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *a probability space, let* $X$ *and* $Y$ *be independent standard Bernoulli random variables on* $\Omega$. *Define* $Z \stackrel{def}{=} X + Y$.

1. *Compute* $\mathbf{E}[X \mid Z]$ *and* $\mathbf{E}[Y \mid Z]$.
2. *Are the random variables* $\mathbf{E}[X \mid Z]$ *and* $\mathbf{E}[Y \mid Z]$ *uncorrelated?*
3. *Are the random variables* $\mathbf{E}[X \mid Z]$ *and* $\mathbf{E}[Y \mid Z]$ *independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with Bernoulli random variables, can you compute* $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ *and* $\mathbf{E}[XY \mid Z]$*?*

**Solution.** .

**Problem 14** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *a probability space, let* $R$ *be a standard Rademacher random variable on* $\Omega$, *and let* $X$ *be a real random variable on* $\Omega$ *symmetric about* $0$ *with finite second order moment. Assume that* $X$ *and* $R$ *are independent and define* $Y \stackrel{def}{=} R \cdot X$.

1. *Has the random varable* $Y$ *the same distribution of* $X$*?*
2. *Are the random variables* $X$ *and* $Y$ *uncorrelated?*
3. *Are the random variables* $X$ *and* $Y$ *independent?*
4. *Can you compute* $\mathbf{E}[Y \mid X]$*?*

**Solution.** .

**Problem 15** *Let* $X$ *[resp.* $R$*] be a standard Gaussian [Rademacher] random variable on a probability space* $\Omega$. *In symbols,* $X \sim N(0,1)$ *and* $R \sim Rad(1/2)$. *Assume that* $X$ *and* $R$ *are independent and define* $Y \equiv R \cdot X$.

1. *Is the random varable* $Y$ *Gaussian?*
2. *Are the random variables* $X$ *and* $Y$ *independent?*
3. *Does the random vector* $(X, Y)^\intercal$ *have a bivariate Gaussian distribution? Hint: consider the possibility that* $(X, Y)^\intercal$ *has a bivariate Gaussian distribution; how the random variable* $Z \equiv X + Y$ *should be distributed?*
4. *Can you compute* $\mathbf{E}[Y \mid X]$ *and* $\mathbf{E}[X \mid Y]$*?*

**Solution.** .

**Problem 16** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $\mathcal{F}$ *be a sub-*$\sigma$*-algebra of* $\mathcal{E}$, *and let* $X, Y$ *be real random variables on* $\Omega$ *with finite second order moment.*

1. *Show that*
$$\mathbf{E}\left[(X - \mathbf{E}[X \mid \mathcal{F}])^2\right] \leq \mathbf{E}\left[(X - \mathbf{E}[X])^2\right].$$
2. *Show that*
$$\mathbf{E}[XY \mid \mathcal{F}]^2 \leq \mathbf{E}\left[X^2 \mid \mathcal{F}\right] \mathbf{E}\left[Y^2 \mid \mathcal{F}\right]. \tag{4}$$

**Solution.** .

1. In the space $\mathcal{L}^2(\Omega_{\mathcal{F}};\mathbb{R})$ of the real $\mathcal{F}$-random variables having fnite moment of the second order the conditional expectation of $X$ given $\mathcal{F}$ is characterized as

$$\mathbf{E}[X \mid \mathcal{F}] = \underset{Y\in\mathcal{L}^2(\Omega_{\mathcal{F}};\mathbb{R})}{\arg\min}\ \mathbf{E}\left[(X-Y)^2\right].$$

   This means that
$$\mathbf{E}\left[(X-\mathbf{E}[X\mid\mathcal{F}])^2\right] \leq \mathbf{E}\left[(X-Y)^2\right],$$

   for every $Y\in\mathcal{L}^2(\Omega_{\mathcal{F}};\mathbb{R})$. In particular, since the deterministic random variable $\mathbf{E}[X]\equiv\mathbf{E}[X]\ 1_{\Omega}$ is clearly in $\mathcal{L}^2(\Omega_{\mathcal{F}};\mathbb{R})$, setting $Y\equiv\mathbf{E}[X]$ we obtain the desired inequality.

2. Note first that for all real random variables $X,Y$ on $\Omega$ we have

$$|XY| \leq \frac{1}{2}\left(X^2+Y^2\right).$$

   Therefore, the assumption that $X$ and $Y$ have finite second moment implies that $XY$ has finite first order moment. Hence, both the sides of (5) are well defined. Now, given any $z\in\mathbb{R}$, the random variable $X+zY$ has finite second order moment and, by virtue of the positivity of the conditional expectation operator, we have

$$\mathbf{E}\left[(X+zY)^2 \mid \mathcal{F}\right] \geq 0.$$

   On the other hand, the linearity of the conditional expectation operator implies

$$\mathbf{E}\left[(X+zY)^2\mid\mathcal{F}\right] = \mathbf{E}\left[X^2\mid\mathcal{F}\right] + 2z\mathbf{E}[XY\mid\mathcal{F}] + z^2\mathbf{E}\left[Y^2\mid\mathcal{F}\right].$$

   As a consequence, we can write

$$\mathbf{E}\left[X^2\mid\mathcal{F}\right] + 2z\mathbf{E}[XY\mid\mathcal{F}] + z^2\mathbf{E}\left[Y^2\mid\mathcal{F}\right] \geq 0$$

   for every $z\in\mathbb{R}$. It follows that

$$\Delta \equiv \mathbf{E}[XY\mid\mathcal{F}]^2 - \mathbf{E}\left[X^2\mid\mathcal{F}\right]\mathbf{E}\left[Y^2\mid\mathcal{F}\right] \leq 0,$$

   which is the desired (5).

**Problem 17** *Let* $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ *be a probability space and let* $(\mathbb{R},\mathcal{B}(\mathbb{R}))\equiv\mathbb{R}$ *be the real Borel state space. Let* $X,Y\in L^2(\Omega;\mathbb{R})$.

1. *Prove in all details that* $\mathbf{E}[Y\mid X]=\mathbf{E}[Y]$ *a.e. on* $\Omega$ *implies* $Cov(X,Y)=0$, *but* $X$ *and* $Y$ *may not be independent.*

2. *Prove in all details that* $Cov(X,Y)=0$ *does not imply* $\mathbf{E}[Y\mid X]=\mathbf{E}[Y]$.

Hint: in the first case, to generate a suitable counterexample one may consider the random variables $X\sim Ber(p)$, $Z\sim N(0,1)$, independent of $X$, and $Y=XZ$. In the second case consider $X\sim N(0,1)$ and $Y=X^2$.

**Solution.**

1. Under the assumption $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$ a.e. on $\Omega$, by virtue of the properties of the conditional expectation operator, we can write

$$\mathbf{E}[XY] = \mathbf{E}[\mathbf{E}[XY \mid X]] = \mathbf{E}[X\mathbf{E}[Y \mid X]] = \mathbf{E}[X\mathbf{E}[Y]] = \mathbf{E}[X]\,\mathbf{E}[Y]$$

Therefore,

$$Cov(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = 0.$$

Now, if we consider the random $X \sim Ber(p)$, $Z \sim N(0,1)$, independent of $X$, and $Y = XZ$, we have

$$\begin{aligned} Cov(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = \mathbf{E}\left[X^2 Z\right] - \mathbf{E}[X]\,\mathbf{E}[XZ] \\ &= \mathbf{E}\left[X^2\right]\mathbf{E}[Z] - \mathbf{E}[X]^2\,\mathbf{E}[Z] \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\mathbf{P}(X \leq 0) = q,$$

and, on account that $X$ and $Z$ are independent,

$$\begin{aligned} \mathbf{P}(Y \leq 0) &= \mathbf{P}(XZ \leq 0) \\ &= \mathbf{P}(XZ \leq 0, X = 0) + \mathbf{P}(XZ \leq 0, X = 1) \\ &= \mathbf{P}(XZ \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) + \mathbf{P}(XZ \leq 0 \mid X = 1)\,\mathbf{P}(X = 1) \\ &= \mathbf{P}(0 \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) + \mathbf{P}(Z \leq 0 \mid X = 1)\,\mathbf{P}(X = 1) \\ &= \mathbf{P}(\Omega)\,\mathbf{P}(X = 0) + \mathbf{P}(Z \leq 0)\,\mathbf{P}(X = 1) \\ &= q + \frac{1}{2}p. \end{aligned}$$

Furthermore, the same arguments as above shows that

$$\begin{aligned} \mathbf{P}(X \leq 0, Y \leq 0) &= \mathbf{P}(X \leq 0, XZ \leq 0) \\ &= \mathbf{P}(X = 0, XZ \leq 0) \\ &= \mathbf{P}(XZ \leq 0 \mid X = 0)\,\mathbf{P}(X = 0) \\ &= q. \end{aligned}$$

Hence, we have

$$\mathbf{P}(X \leq 0)\,\mathbf{P}(Y \leq 0) = q\left(q + \frac{1}{2}p\right) \neq q = \mathbf{P}(X \leq 0, Y \leq 0)$$

which shows that $X$ and $Y$ are not be independent.

2. To show that $Cov(X,Y) = 0$ does not imply $\mathbf{E}[Y \mid X] = \mathbf{E}[Y]$, we consider $X \sim N(0,1)$ and $Y = X^2$. We have

$$\begin{aligned} Cov(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] \\ &= \mathbf{E}\left[X^3\right] - \mathbf{E}[X]\,\mathbf{E}\left[X^2\right] \\ &= 0, \end{aligned}$$

but

$$\mathbf{E}[Y \mid X] = \mathbf{E}\left[X^2 \mid X\right] = X^2 \neq \mathbf{E}\left[X^2\right] = \mathbf{E}[Y].$$

□

**Problem 18** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the Euclidean real line endowed with the Borel* $\sigma$*-algebra, and let* $X, Y \in L^2(\Omega; \mathbb{R})$*. Fixed any sub-*$\sigma$*-algebra* $\mathcal{F}$ *of* $\mathcal{E}$*, we call conditional variance of* $X$ *given* $\mathcal{F}$ *the random variable*

$$\mathbf{D}^2[X \mid \mathcal{F}] \overset{def}{=} \mathbf{E}\left[(X - \mathbf{E}[X \mid \mathcal{F}])^2 \mid \mathcal{F}\right].$$

*Prove that:*

1. *we have*
$$\mathbf{D}^2[X \mid \mathcal{F}] = \mathbf{E}\left[X^2 \mid \mathcal{F}\right] - \mathbf{E}[X \mid \mathcal{F}]^2;$$

2. *if* $X$ *is an* $\mathcal{F}$*-random variable, then we have*
$$\mathbf{D}^2[X \mid \mathcal{F}] = 0;$$

3. *if* $X$ *is* $\mathcal{F}$*-independent, then we have*
$$\mathbf{D}^2[X \mid \mathcal{F}] = \mathbf{D}^2[X];$$

4. *if* $X$ *is an* $\mathcal{F}$*-random variable and* $Y$ *is* $\mathcal{F}$*-independent, then we have*
$$\mathbf{D}^2[X + Y \mid \mathcal{F}] = \mathbf{D}^2[Y].$$

**Solution.** .

**Problem 19** *Let* $X$ *[resp.* $B$*] be a standard Gaussian [Bernoulli] random variable on a probability space* $\Omega$*. In symbols,* $X \sim N(0,1)$ *and* $B \sim Ber(1/2)$*. Assume that* $X$ *and* $B$ *are independent and define* $Y \equiv B \cdot X$*.* ***Specifying carefully the properties used****, answer the following questions:*

1. *Is the random variable* $Y$ *Gaussian? Is* $Y$ *absolutely continuous?*
2. *Are the random variables* $X$ *and* $Y$ *uncorrelated? Are* $X$ *and* $Y$ *independent?*
3. *Are the random variables* $B$ *and* $Y$ *uncorrelated? Are* $B$ *and* $Y$ *independent?*
4. *Does the random vector* $(X, Y)^\intercal$ *have a bivariate Gaussian distribution?*
5. *Can you compute* $\mathbf{E}[Y \mid X]$*? What about* $\mathbf{E}[X \mid Y]$*?*

**Solution.**

1. To check whether $Y$ is Gaussian we compute $\mathbf{P}(Y \leq y)$, for every $y \in \mathbb{R}$. On account that $\{B = 0\}$, $\{B = 1\}$ constitute a partition of $\Omega$, the random variables $B$ and $X$ are independent, and $X \sim N(0,1)$, we can write

$$\begin{aligned}
\mathbf{P}(Y \leq y) &= \mathbf{P}(BX \leq y) \\
&= \mathbf{P}(BX \leq y, B = 0) + \mathbf{P}(BX \leq y, B = 1) \\
&= \mathbf{P}(BX \leq y \mid B = 0)\,\mathbf{P}(B = 0) + \mathbf{P}(BX \leq y \mid B = 1)\,\mathbf{P}(B = 1) \\
&= \frac{1}{2}(\mathbf{P}(0 \leq y \mid R = 0) + \mathbf{P}(X \leq y \mid R = 1)) \\
&= \frac{1}{2}(\mathbf{P}(0 \leq y) + \mathbf{P}(X \leq y)) \\
&= \begin{cases} \frac{1}{2}\mathbf{P}(X \leq y) = \frac{1}{2}F_X(x), & \text{if } y < 0, \\ \frac{1}{2}(1 + \mathbf{P}(X \leq y)) = \frac{1}{2}(1 + F_X(x)) & \text{if } 0 \leq y. \end{cases}
\end{aligned}$$

Now, we have

$$\lim_{y\to 0^-} \mathbf{P}(Y \leq y) = \lim_{y\to 0^-} \frac{1}{2} F_X(x) = \frac{1}{4}$$

and

$$\lim_{y\to 0^+} \mathbf{P}(Y \leq y) = \lim_{y\to 0^+} \frac{1}{2}(1 + F_X(x)) = \frac{3}{4}$$

This proves that $Y$ is not Gaussian.

2. Since $Y \equiv B \cdot X$, the intuition is that the observation of the values taken by $X$ transmits information on the values taken by $Y$. That is $X$ and $Y$ are not independent. In fact, thanks to the independence of $X$ and $B$ and on account that $X \sim N(0,1)$, we have

$$\mathbf{E}[XY] = \mathbf{E}[XBX] = \mathbf{E}\left[BX^2\right] = \mathbf{E}[B]\,\mathbf{E}\left[X^2\right] = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

On the other hand,

$$\mathbf{E}[X]\,\mathbf{E}[Y] = 0.$$

Hence,

$$\mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = \frac{1}{2}$$

This shows that $X$ and $Y$ are correlated, which prevents that $X^2$ and $Y^2$ are independent.

3. Still thanks to the independence of $X$ and $B$ and on account that $X \sim N(0,1)$ and $B^2 \sim$ *Ber* $(1/2)$, we have

$$\mathbf{E}[BY] = \mathbf{E}[BBX] = \mathbf{E}\left[B^2X\right] = \mathbf{E}\left[B^2\right]\mathbf{E}[X] = \frac{1}{2}\mathbf{E}[X] = 0.$$

On the other hand,

$$\mathbf{E}[B]\,\mathbf{E}[Y] = \mathbf{E}[B]\,\mathbf{E}[BX] = \mathbf{E}[B]\,\mathbf{E}[B]\,\mathbf{E}[X] = \frac{1}{4}\mathbf{E}[X] = 0.$$

This shows that $B$ and $Y$ are uncorrelated. Here, the intuition is that the observation of the values taken by $B$ transmits information on the values taken by $Y$. Hence, the intuition is that $B$ and $Y$ are not independent. To prove this, we show that $B^2$ and $Y^2$ are correlated. In fact, thanks to the independence of $X$ and $B$ and on account that $B^4 \sim B^2 \sim$ *Ber* $(1/2)$, we have

$$\mathbf{E}\left[B^2Y^2\right] = \mathbf{E}\left[B^2B^2X^2\right] = \mathbf{E}\left[B^4X^2\right] = \mathbf{E}[B]\,\mathbf{E}\left[X^2\right] = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

and

$$\mathbf{E}\left[B^2\right]\mathbf{E}\left[Y^2\right] = \mathbf{E}\left[B^2\right]\mathbf{E}\left[B^2X^2\right] = \mathbf{E}\left[B^2\right]\mathbf{E}\left[B^2\right]\mathbf{E}\left[X^2\right] = \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{4}.$$

This shows that $B^2$ and $Y^2$ are correlated, which prevents that $B^2$ and $Y^2$ are independent. Eventually, $B$ and $Y$ cannot be independent.

4. Since $B \sim$ *Ber* $(1/2)$ is independent of $X$, we have

$$\mathbf{E}[Y \mid X] = \mathbf{E}[BX \mid X] = X\mathbf{E}[B \mid X] = \mathbf{E}[B]\,X = \frac{1}{2}X$$

$$\mathbf{E}[X \mid Y] = \mathbf{E}[X \mid BX]$$

By virtue of what shown above and the properties of the conditional expectation, on account that $B$ and $X$ are independent, we have,

$$\mathbf{E}\left[Y \mid X\right] = \mathbf{E}\left[BX \mid X\right] = X\mathbf{E}\left[B \mid X\right] = X\mathbf{E}\left[B\right] = \frac{1}{2}X.$$

Now, to compute $\mathbf{E}\left[X \mid Y\right]$, observe preliminarly that

$$B\left(\omega\right) = 1_{\{B=1\}}\left(\omega\right)$$

for every $\omega \in \Omega$. In fact,

$$1_{\{B=1\}}\left(\omega\right) = \begin{cases} 1 & \text{if } B\left(\omega\right) = 1 \\ 0 & \text{if } B\left(\omega\right) = 0 \end{cases}$$

Hence,

$$Y = BX = X1_{\{B=1\}}.$$

As a consequence, on account that

$$1_{\{B=1\}} + 1_{\{B=0\}} = 1_{\Omega},$$

we have

$$\begin{aligned} \mathbf{E}\left[X \mid Y\right] &= \mathbf{E}\left[X\left(1_{\{B=1\}} + 1_{\{B=0\}}\right) \mid Y\right] \\ &= \mathbf{E}\left[X1_{\{B=1\}} \mid Y\right] + \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] \\ &= \mathbf{E}\left[Y \mid Y\right] + \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] \\ &= Y + \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] \\ &= BX + \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right]. \end{aligned}$$

Thus, we are left with computing

$$\mathbf{E}\left[X1_{\{B=0\}} \mid Y\right].$$

To this goal, observe that

$$\int_{\{Y\in C\}} \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] d\mathbf{P}_{|\sigma(Y)} = \int_{\{Y\in C\}} X1_{\{B=0\}} d\mathbf{P} = \int_{\{Y\in C\}\cap\{B=0\}} X d\mathbf{P},$$

where

$$\{Y \in C\} \cap \{B = 0\} = \{BX \in C\} \cap \{B = 0\} = \begin{cases} \{B = 0\}, & \text{if } 0 \in C, \\ \varnothing, & \text{if } 0 \notin C. \end{cases}$$

Hence,

$$\int_{\{Y\in C\}} \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] d\mathbf{P}_{|\sigma(Y)} = \begin{cases} \int_{\{B=0\}} X d\mathbf{P}, & \text{if } 0 \in C, \\ 0, & \text{if } 0 \notin C, \end{cases}$$

for every $C \in \mathcal{B}\left(\mathbb{R}\right)$. On the other hand,

$$\int_{\{B=0\}} X d\mathbf{P} = \mathbf{E}\left[X1_{\{B=0\}}\right] = \mathbf{E}\left[X\right]\mathbf{E}\left[1_{\{B=0\}}\right] = \mathbf{E}\left[X\right]\mathbf{P}\left(B = 0\right) = \frac{1}{2}\mathbf{E}\left[X\right].$$

It then follows,

$$\int_{\{Y\in C\}} \mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] d\mathbf{P}_{|\sigma(Y)} = \begin{cases} \frac{1}{2}\mathbf{E}\left[X\right], & \text{if } 0 \in C, \\ 0, & \text{if } 0 \notin C. \end{cases}$$

We then claim that

$$\mathbf{E}\left[X1_{\{B=0\}} \mid Y\right] = \mathbf{E}\left[X\right]1_{\{Y=0\}}, \ \mathbf{P}_{|\sigma(Y)}\text{-a.s. on } \Omega.$$

In fact, we have

$$\{Y=0\}=\{XB=0\}=\{X\in\mathbb{R},B=0\}\cup\{X=0,B=1\},$$

where

$$\{X\in\mathbb{R},B=0\}\cap\{X=0,B=1\}=\varnothing.$$

This, on account of the independence of $X$ and $B$ and that $X$ is a continuous random variable, implies

$$\begin{aligned}\mathbf{P}(Y=0)&=\mathbf{P}(X\in\mathbb{R},B=0)+\mathbf{P}(X=0,B=1)\\&=\mathbf{P}(X\in\mathbb{R})\,\mathbf{P}(B=0)+\mathbf{P}(X=0)\,\mathbf{P}(B=1)\\&=\mathbf{P}(B=0)\\&=\frac{1}{2}.\end{aligned}$$

In addition, since

$$\{Y\in C\}\cap\{Y=0\}=\begin{cases}\{Y=0\} & \text{if } 0\in C\\ \varnothing & \text{if } 0\notin C\end{cases},$$

for every $C\in\mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned}\int_{\{Y\in C\}}\mathbf{E}[X]\,1_{\{Y=0\}}d\mathbf{P}_{|\sigma(Y)}&=\mathbf{E}[X]\int_{\{Y\in C\}\cap\{Y=0\}}d\mathbf{P}_{|\sigma(Y)}\\&=\mathbf{E}[X]\,\mathbf{P}(\{Y\in C\}\cap\{Y=0\})\\&=\begin{cases}\mathbf{E}[X]\,\mathbf{P}(Y=0) & \text{if } 0\in C\\ 0 & \text{if } 0\notin C\end{cases}\\&=\begin{cases}\frac{1}{2}\mathbf{E}[X] & \text{if } 0\in C\\ 0 & \text{if } 0\notin C\end{cases},\end{aligned}$$

for every $C\in\mathcal{B}(\mathbb{R})$. From what shown above, we can write

$$\int_{\{Y\in C\}}\mathbf{E}\left[X1_{\{B=0\}}\mid Y\right]d\mathbf{P}_{|\sigma(Y)}=\int_{\{Y\in C\}}\mathbf{E}[X]\,1_{\{Y=0\}}d\mathbf{P}_{|\sigma(Y)},$$

for every $C\in\mathcal{B}(\mathbb{R})$. This implies our claim. Summarizing we obtain

$$\mathbf{E}[X\mid Y]=BX+\mathbf{E}[X]\,1_{\{Y=0\}},\quad \mathbf{P}_{|\sigma(Y)}\text{-a.s. on }\Omega,$$

which completes the solution of 5.

**Problem 20** *Let $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ a probability space and let $B\sim Ber(1/2)$ [resp. $R\sim Rad(1/2)$] a standard Bernoulli [resp. Rademacher] random variable on $\Omega$. Assume that $B$ and $R$ are independent and define $X\overset{def}{=}B+R$.*

1. *Compute $\mathbf{E}[X\mid B]$, $\mathbf{E}[X\mid R]$, $\mathbf{E}[B\mid X]$, and $\mathbf{E}[R\mid X]$. In addition,* ***specifying carefully the properties used****, answer the following questions:*
2. *Are the random variables $\mathbf{E}[X\mid B]$, $\mathbf{E}[X\mid R]$ uncorrelated? Are $\mathbf{E}[X\mid B]$, $\mathbf{E}[X\mid R]$ independent?*
3. *Are the random variables $\mathbf{E}[B\mid X]$ and $\mathbf{E}[R\mid X]$ uncorrelated? Are $\mathbf{E}[B\mid X]$ and $\mathbf{E}[R\mid X]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with a Bernoulli and a Rademacher random variable, can you compute $\mathbf{E}[BR\mid X]$?*

**Solution.** .

**Problem 21** *Let $U, V$ real random variables on a probability space $\Omega$ such that such that $U \sim V \sim N(0,1)$, the vector $(U,V)^{\intercal}$ is Gaussian, and $Corr(U,V) \equiv \rho < 1$. Consider the real random variables*

$$X \stackrel{def}{=} U - \rho V \quad \text{and} \quad Y \stackrel{def}{=} \sqrt{1-\rho} V.$$

1. *Can you prove that the vector $(X,Y)^{\intercal}$ Gaussian?*
2. *Are the random variables $X$ and $Y$ independent?*
3. *Compute the distributions of $X$ and $Y$;*
4. *Compute $\mathbf{E}\left[X^2Y^2\right]$, $\mathbf{E}\left[XY^3\right]$, $\mathbf{E}\left[Y^4\right]$.*
5. *Compute $\mathbf{E}\left[U^2V^2\right]$.*

**Solution.** .

**Problem 22** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space and let $B_1$ and $B_2$ be standard Bernoulli random variables on $\Omega$. In symbols, $B_k \sim Ber(1/2)$, for $k = 1, 2$. Assume that $B_1$ and $B_2$ are independent and set*

$$X \stackrel{def}{=} B_1 + B_2, \quad Y \stackrel{def}{=} B_1 \cdot B_2$$

1. *Compute $\mathbf{E}[B_k \mid X]$ and $\mathbf{E}[B_k \mid Y]$ for $k = 1, 2$.*
2. *Are the random variables $\mathbf{E}[B_1 \mid X]$ and $\mathbf{E}[B_2 \mid X]$ uncorrelated? Are they independent?*
3. *Are the random variables $\mathbf{E}[B_1 \mid Y]$ and $\mathbf{E}[B_2 \mid Y]$ uncorrelated? Are they independent?*
4. *Compute $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$.*
5. *Are the random variables $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$ uncorrelated? Are they independent?*
6. *Compute $\mathbf{E}\left[X^2 \mid Y\right]$ and $\mathbf{E}\left[Y^2 \mid X\right]$.*

**Solution.** .

**Problem 23** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $X$ and $Y$ be independent standard Bernoulli random variables on $\Omega$. Define $Z \stackrel{def}{=} X + Y$.*

1. *Compute $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$.*
2. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ uncorrelated?*
3. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with Bernoulli random variables, can you compute $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ and $\mathbf{E}[XY \mid Z]$?*

**Solution.** .

**Problem 24** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space and let $R_1$ and $R_2$ be standard Rademacher random variables on $\Omega$. In symbols, $R_k \sim Rad\,(1/2)$, for $k = 1, 2$. Assume that $R_1$ and $R_2$ are independent and set*

$$X \stackrel{def}{=} R_1 - R_2, \quad Y \stackrel{def}{=} -R_1 \cdot R_2$$

1. *Compute $\mathbf{E}[R_k \mid X]$ and $\mathbf{E}[R_k \mid Y]$ for $k = 1, 2$.*
2. *Are the random variables $\mathbf{E}[R_1 \mid X]$ and $\mathbf{E}[R_2 \mid X]$ uncorrelated? Are they independent?*
3. *Are the random variables $\mathbf{E}[R_1 \mid Y]$ and $\mathbf{E}[R_2 \mid Y]$ uncorrelated? Are they independent?*
4. *Compute $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$.*
5. *Are the random variables $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$ uncorrelated? Are they independent?*
6. *Compute $\mathbf{E}\left[X^2 \mid Y\right]$ and $\mathbf{E}\left[Y^2 \mid X\right]$.*

**Solution.** .

**Problem 25** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $X$ and $Y$ be independent standard Rademacher random variables*[4] *on $\Omega$. Set $Z \stackrel{def}{=} X + Y$.*

1. *Compute $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$.*
2. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ uncorrelated?*
3. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with standard Rademacher random variables, can you compute $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ and $\mathbf{E}[XY \mid Z]$?*

**Solution.** .

**Problem 26** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $X$ and $Y$ be independent standard Bernoulli random variables on $\Omega$. Define $Z \stackrel{def}{=} X + Y$.*

1. *Compute $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$.*
2. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ uncorrelated?*
3. *Are the random variables $\mathbf{E}[X \mid Z]$ and $\mathbf{E}[Y \mid Z]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with Bernoulli random variables, can you compute $\mathbf{E}\left[(X+Y)^2 \mid Z\right]$ and $\mathbf{E}[XY \mid Z]$?*

**Solution.** .

---

[4] A standard Rademacher random variable $R$ is given by

$$R \stackrel{\text{def}}{=} \begin{cases} 1, & \mathbf{P}(R=1) = 1/2, \\ -1, & \mathbf{P}(R=-1) = 1/2. \end{cases}$$

**Problem 27** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space, let $R$ be a standard Rademacher random variable on $\Omega$, and let $X$ be a real random variable on $\Omega$ symmetric about $0$ with finite second order moment. Assume that $X$ and $R$ are independent and define $Y \stackrel{def}{=} R \cdot X$.*

1. *Has the random varable $Y$ the same distribution of $X$?*
2. *Are the random variables $X$ and $Y$ uncorrelated?*
3. *Are the random variables $X$ and $Y$ independent?*
4. *Can you compute $\mathbf{E}[Y \mid X]$?*

**Solution.** .

**Problem 28** *Let $X$ [resp. $R$] be a standard Gaussian [Rademacher] random variable on a probability space $\Omega$. In symbols, $X \sim N(0,1)$ and $R \sim Rad(1/2)$. Assume that $X$ and $R$ are independent and define $Y \equiv R \cdot X$.*

1. *Is the random varable $Y$ Gaussian?*
2. *Are the random variables $X$ and $Y$ independent?*
3. *Does the random vector $(X, Y)^{\intercal}$ have a bivariate Gaussian distribution? Hint: consider the possibility that $(X, Y)^{\intercal}$ has a bivariate Gaussian distribution; how the random variable $Z \equiv X + Y$ should be distributed?*
4. *Can you compute $\mathbf{E}[Y \mid X]$ and $\mathbf{E}[X \mid Y]$?*

**Solution.** .

**Problem 29** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, let $\mathcal{F}$ be a sub-$\sigma$-algebra of $\mathcal{E}$, and let $X, Y$ be real random variables on $\Omega$ with finite second order moment.*

1. *Show that*
$$\mathbf{E}\left[(X - \mathbf{E}[X \mid \mathcal{F}])^2\right] \leq \mathbf{E}\left[(X - \mathbf{E}[X])^2\right].$$
2. *Show that*
$$\mathbf{E}[XY \mid \mathcal{F}]^2 \leq \mathbf{E}\left[X^2 \mid \mathcal{F}\right] \mathbf{E}\left[Y^2 \mid \mathcal{F}\right]. \tag{5}$$

**Solution.** .