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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Random Variables 2021-11-10

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the Euclidean real line endowed with the Borel* σ*-algebra. Let* $F : \mathbb{R} \to \mathbb{R}_+$ *given by*

$$F(x) \overset{def}{=} ae^x 1_{\mathbb{R}_{--}}(x) - \left(\tfrac{1}{2}e^{-x} - b\right) 1_{\mathbb{R}_+}(x), \quad \forall x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$.

1. *Determine* $a, b \in \mathbb{R}$ *such that* $F : \mathbb{R} \to \mathbb{R}_+$ *is a distribution function of a random variable* $X : \Omega \to \mathbb{R}$.
2. *Is it possible to determine* $a, b \in \mathbb{R}$ *such that* $X : \Omega \to \mathbb{R}$ *is absolutely continuous? In this case, compute* $\mathbf{P}(-1 \leq X \leq 1)$.

Solution. .

Problem 2 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X, Y \in \mathcal{L}^2(\Omega; \mathbb{R})$. *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(a, b) \overset{def}{=} \mathbf{E}\left[(X - (a + bY))^2\right].$$

Prove that there exists

$$(a^*, b^*) \equiv \underset{(a,b)\in\mathbb{R}^2}{\arg\min}\{f(a, b)\}$$

and compute it.

Solution. .

Problem 3 *The decoration of a Christmas tree in a mall is made by* 1000 *small light bulbs. The life-time of each light bulb is exponentially distributed with an average life time of* 20 *days (rather cheap bulbs indeed!). The mall manager decides to turn on the lights of the Christmas tree on the midnight of the* 15*-th of November. Estimate the probability that at least* 800 *bulbs are still working on the midnight of the* 25*-th of December.*

Hints: write X *for the random variable compute representing the life time of each bulb and compute the probability that each bulb will last until the midnight of the* 25*-th of December; write* Y *for the random variable counting the number of light bulbs out of* 1.000 *which are still on at the midnight of the* 25*-th of December and guess how it is distributed; use the Markov inequality to make the estimate.*

Solution. .

Problem 4 *Let* X *be a geometrically distributed random variable with success probability* $p = 0.1$. *Compute* $\mathbf{E}[X]$ *and* $\mathbf{D}^2[X]$. *Use the Tchebychev inequality to estimate* $\mathbf{P}(0 < X < 23)$.

Solution. .

Problem 5 *Suppose that we roll a standard fair die* 100 *times. Let* X *be the sum of the numbers that appear over the* 100 *rolls. Use the Tchebychev inequality to bound* $\mathbf{P}(200 < X < 400)$.

Solution. .