Performance Modeling of Computer Systems and Networks

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Lehmer Generators
Implementation

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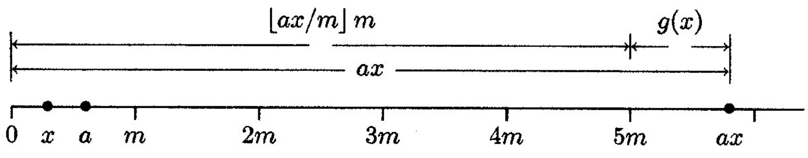
Overflow Is Possible

- Recall that $g(x) = ax \bmod m$
- The *ax* product can be as big as *a*(*m*-1)



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- If integers > *m* cannot be represented, integer overflow is possible!
- Not possible to evaluate g(x) in "obvious" way

Example 1: *m* decomposition

- consider $(a, m)=(48271, 2^{31}-1)$ 32 bit, 48271 considerato miglior generatore.

  $q=\lfloor m/a \rfloor=44488$ $r=m \bmod a=3399$ $< 44488 = q$

- consider $(a, m)=(16807, 2^{31}-1)$

  $q=\lfloor m/a \rfloor=127773$ $r=m \bmod a=2836$ $< 127773 = q$

- In both cases $r < q$ caratteristica "modulo compatibile".

  This characteristic is important!!
  (*modulus-compatibile*)

Rewriting g(x) to avoid overflow

$$\begin{aligned} g(x) &= ax \bmod m \\ &= ax - m\lfloor ax/m \rfloor \\ &= ax + [- m\lfloor x/q\rfloor + m\lfloor x/q\rfloor] - m\lfloor ax/m\rfloor \\ &= [ax - (aq+r)\lfloor x/q\rfloor] + [m\lfloor x/q\rfloor - m\lfloor ax/m\rfloor] \\ &= [a(x - q\lfloor x/q\rfloor) - r\lfloor x/q\rfloor] + [m\lfloor x/q\rfloor - m\lfloor ax/m\rfloor] \\ &= [a(x \bmod q) - r\lfloor x/q\rfloor] + m\,[\lfloor x/q\rfloor - \lfloor ax/m\rfloor] \\ &= \gamma(x) + m\,\delta(x) \end{aligned}$$

banalmente passiamo da un prodotto ad una somma.

where

viene fatto prima il modulo, dopo si moltiplica.

$\gamma(x) = a(x \bmod q) - r\lfloor x/q\rfloor$ and

$\delta(x) = \lfloor x/q\rfloor - \lfloor ax/m\rfloor$

questa seconda funzione non la calcolo proprio!

Note: mods are done before multiplications!!!

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Characterization of $\delta(x)$

$$g(x) = \gamma(x) + m\,\delta(x)$$

Theorem 2.2.1

If $m = aq+r$ is prime and $r < q$, for $x \in \chi_m$

ovvero sto in modulo compatibilità

$$\delta(x) = 0 \quad \text{or} \quad \delta(x) = 1$$

where

$$\delta(x) = \lfloor x/q\rfloor - \lfloor ax/m\rfloor$$

moreover

$$\delta(x) = 0 \ \text{ iff } \ \gamma(x) \in \chi_m$$
$$\delta(x) = 1 \ \text{ iff } \ -\gamma(x) \in \chi_m$$

devo vedere l'altra funzione gamma, se positivo allora ho delta(x) = 0, altrimenti vale 1.

where

$$\gamma(x) = a(x \bmod q) - r\lfloor x/q\rfloor$$

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Computing g(x)

- evaluates $g(x) = ax \bmod m$ with no values > m-1

Algorithm 1

```
t = a * (x % q) - r * (x / q);        /* t = γ(x) */
if (t > 0)
        return (t);                   /* δ(x) = 0 */
else
        return (t + m);               /*  δ(x) = 1 */
```

- returns $g(x) = \gamma(x) + m\,\delta(x)$
- the *ax* product is "trapped" in $\delta(x)$
- no overflow !!

Modulus compatibility

- we must have r < q in $m = aq+r$
- multiplier *a* is *modulus-compatibile* (MC) with *m* iff r < q
- choose *a* MC with $m = 2^{31}-1$, then algorithm 1 can port to any 32-bit machine
- e.g.: $a=48271$ is MC with $m = 2^{31}-1$

  r = 3399 q = 44 488

Non bisogna mai usare generatori random senza saperne le specifiche.

Modulus-Compatible MC and Full-Period FP

- no MC multipliers $> (m-1)/2$

- more densely distributed on low end [0, m-1]

i modulo compatibili sono più vicini allo '0' rispetto ad 'm'.

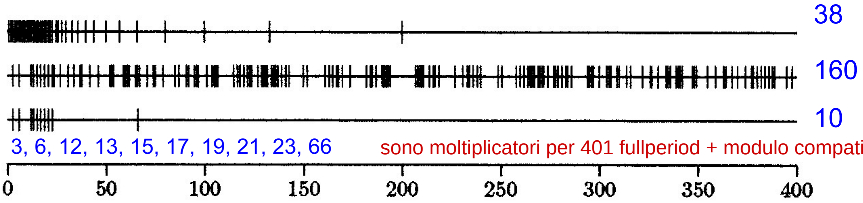
- consider a tiny modulus m= 401:

(row 1: MC; row 2: FP; row 3: MC & FP)

modulo compatibili --------->

full - period ---------------->

intersezione tra i due ----->



sono moltiplicatori per 401 fullperiod + modulo compatibili, no overflow!

Multipliers *a* between 1 and 400

MC and smallness

- multiplier a is "small" iff $a^2 < m$

- if a is small, then a is MC

  all multipliers from 1 to $\lfloor \sqrt{m} \rfloor$=46340 are MC

dettagli non troppo rilevanti

- if a is MC, a is not necessarily small

  a=48271 is MC with 2^{31}-1 but is not small

- start with a small (therefore MC) multiplier

  search until the first FP multiplier is found

Example: FPMC multipliers for m= 2^{31}-1

- For $m=2^{31}-1$ and FPMC $a=7$, there are 23093 FPMC multipliers

$$7^{1} \bmod 2147483647 = 7$$
$$7^{5} \bmod 2147483647 = 16807$$
$$7^{113039} \bmod 2147483647 = 41214$$
$$7^{188509} \bmod 2147483647 = 25697$$
$$7^{536035} \bmod 2147483647 = 63295$$

.
.

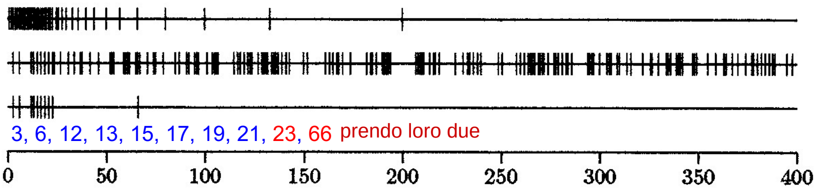
- a= 16807 is a "minimal" standard
- a= 48271 provides (slightly) more random sequences

Randomness

- choose the FPMC multiplier that gives "most random" sequences
- no universal definition of randomness
- in 2-space (x_0, x_1), (x_1, x_2), (x_2, x_3),…. form a lattice structure

Se rivediamo graficamente l'esempio di prima con m =13, abbiamo sempre una struttura geometrica detta "Lattice"

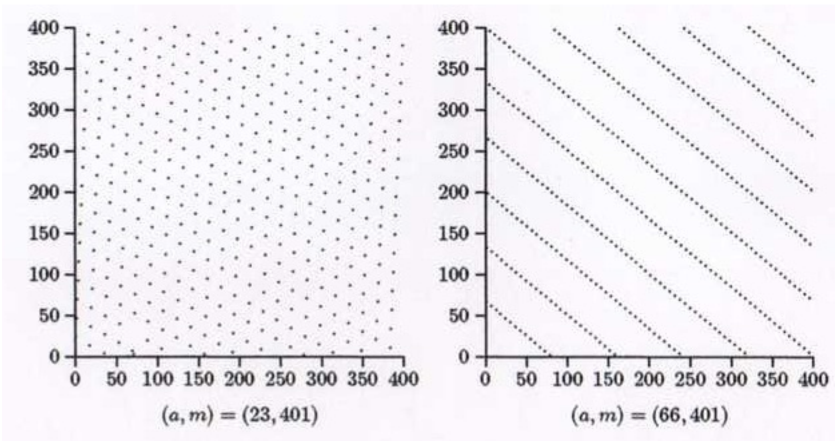
- the first row shows 38 multipliers MC
- the second row shows 160 multipliers FP
- the third row shows 10 multipliers MC and FP



multipliers *a* between 1 and 400

13

In entrambi i casi sono strutture a "lattice", ma la prima sembra coprire meglio l'area, la seconda ha più vuoti.



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Lehmer generator implementation with $(a, m) = (48271, 2^{31} - 1)$

implementazione "vera" in C.

```
Random(void) {
    static long state = 1;
    const long A = 48271;           /* multiplier*/
    const long M = 2147483647;      /* modulus */
    const long Q = M / A;           /* quotient */
    const long R = M % A;           /* remainder */
    long t = A * (state % Q) - R * (state / Q);
    if (t > 0)
        state = t;
    else
        state = t + M;
    return ((double) state / M);
}
```


A Not-As-Good RNG Library

- ANSI C library <stdlib.h> provides the function rand()
- simulates drawing from 1, 2, … m-1 with $m \geq 2^{15} - 1$
- value returned is not normalized; typical to use
  u = (double) rand() / RAND_MAX;
- ANSI C standard does not specify algorithm details
- for scientific work, avoid using rand() !!!

la rand() di stdlib non specifica nel dettaglio l'algoritmo, non essendo ben documentata è meglio evitarla per lavori scientifici.

http://www.cplusplus.com/reference/cstdlib/rand/

rand

<cstdlib>

`int rand (void);`

Generate random number

Returns a pseudo-random integral number in the range between 0 and RAND_MAX.

This number is generated by an algorithm that returns a sequence of apparently non-related numbers each time it is called. This algorithm uses a seed to generate the series, which should be initialized to some distinctive value using function srand.

RAND_MAX is a constant defined in <cstdlib>.

A typical way to generate trivial pseudo-random numbers in a determined range using rand is to use the modulo of the returned value by the range span and add the initial value of the range:

```
v1 = rand() % 100;         // v1 in the range 0 to 99
v2 = rand() % 100 + 1;     // v2 in the range 1 to 100
v3 = rand() % 30 + 1985;   // v3 in the range 1985-2014
```

Notice though that this modulo operation does not generate uniformly distributed random numbers in the span (since in most cases this operation makes lower numbers slightly more likely).

C++ supports a wide range of powerful tools to generate random and pseudo-random numbers (see <random> for more info).

Nostro generatore di Lehmer.

- defined in the source files rng.h and rng.c
- based on the implementation considered here
  - double Random(void) — estrae il nostro 'u'
  - void PutSeed(long seed) — mette il 'seme', se seed=0 il seme viene chiesto da tastiera, se metto seed = -1 viene scelto il clock del sistema.
  - void GetSeed(long *seed)
  - void TestRandom(void)

per replicare un esperimento, devo conoscere il seme, e questo lo faccio con getSeed, o anche per conoscere a quale punto della 'ruota' sono. Ogni volta che faccio 'random' vado avanti nella ruota.

- initial seed can be set directly, via prompt or by system clock
- PutSeed() and GetSeed() often used together
- *a*=48271 is the default multiplier

Example using the RNG

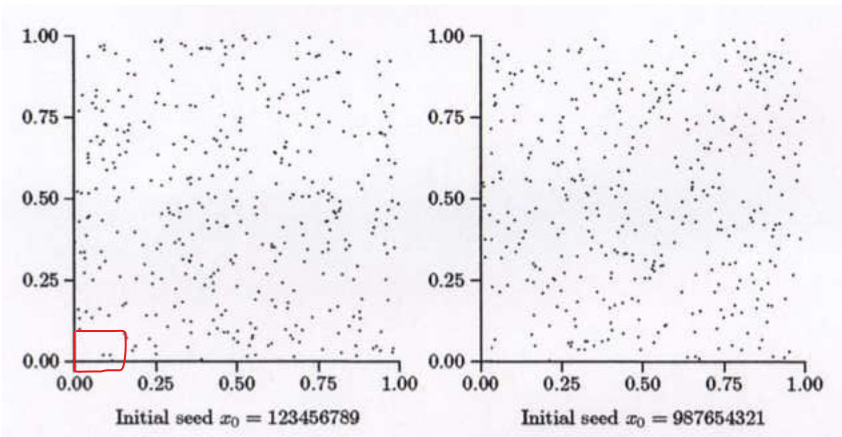
- generates 400 2-space points at random

```
seed = 123456789;          /* or 987654321 */
PutSeed(seed);
x0 = Random();
for (i = 0; i < 400; i++) {
        xi+1 = Random();
        Plot(xi, xi+1);     /* grafics function */
}
```

inizializzo generatore con tale seme
inizio generazione a partire da quel seme.

genero 400 valori
e li plotto, ognuno col successivo.

Sto generando 'solo' 400 punti, sembra randomico.
Non vedo struttura lattice.



Se zoommassi in un "quadratino" ?

Observations on Randomness

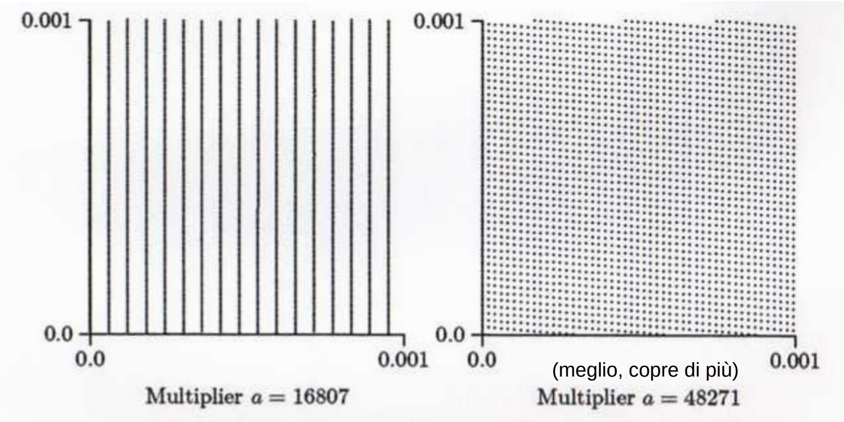
- no lattice structure is evident
- appearance of randomness is an illusion
- if all $m - 1 = 2^{31} - 2$ points were generated, lattice would be evident
- herein lies distinction between *ideal* and *good* generator !!

Example

- plotting all pairs (x_i, x_{i+1}) for $m = 2^{31} - 1$ would give a black square
- any tiny square should appear approximately the same
- zoom in the square with opposite corners (0, 0) and (0.001, 0.001)

```
seed = 123456789;
PutSeed(seed);
x0 = Random();
for (i = 0; i < 2147483646; i++) {      stavolta li genero tutti in un piccolo "quadratino"
        xi+1 = Random();
        if ((xi < 0.001) and (xi+1 < 0.001))
                Plot(xi, xi+1);
}
```

E' lo stesso generatore, STESSO SEME, solo che prima ho generato solo '400' punti,
qui li genero tutti in un "quadratino" 0.01 x 0.01



• further justification for using *a*=48271 over *a*=16807

considerations

• only 20 random numbers were needed
• seed x_0= 109.869.724
• resulting 20 random numbers

0.64 0.72 0.77 0.93 0.82 0.88 0.67 0.76 0.84 0.84
0.74 0.76 0.80 0.75 0.63 0.94 0.86 0.63 0.78 0.67

not discard outliers

→ Replicating simulation many times!!!!
So averaging the unusual cases