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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Solved Problems on Distribution Functions 2022-11-09

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *the real state space, and let* $F : \mathbb{R} \to \mathbb{R}_+$ *given by*

$$F(x) \stackrel{def}{=} ae^x 1_{\mathbb{R}_{--}}(x) - \left(\tfrac{1}{2}e^{-x} - b\right) 1_{\mathbb{R}_+}(x), \quad \forall x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$.

1. *Can you determine* $a, b \in \mathbb{R}$ *such that* $F : \mathbb{R} \to \mathbb{R}_+$ *is a distribution function of a random variable* $X : \Omega \to \mathbb{R}$.
2. *Is it possible to determine* $a, b \in \mathbb{R}$ *such that* $X : \Omega \to \mathbb{R}$ *is absolutely continuous? In this case, compute* $\mathbf{P}(-1 \leq X \leq 1)$.

Solution. We have

$$\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} ae^x = 0$$

and

$$\lim_{x \to +\infty} F(x) = \lim_{x \to +\infty} b - \tfrac{1}{2}e^{-x} = b.$$

Therefore, to make $F : \mathbb{R} \to \mathbb{R}_+$ a distribution function we need

$$b = 1.$$

Moreover, we have

$$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} ae^x = a \qquad \text{and} \qquad \lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \left(1 - \tfrac{1}{2}e^{-x}\right) = \frac{1}{2}.$$

To make $F : \mathbb{R} \to \mathbb{R}_+$ a distribution function we also need

$$a \leq \frac{1}{2}.$$

Under these conditions we have

$$F(x) = ae^x 1_{\mathbb{R}_{--}}(x) + \left(1 - \tfrac{1}{2}e^{-x}\right) 1_{\mathbb{R}_+}(x),$$

for every $x \in \mathbb{R}$. The latter is an increasing function on $\mathbb{R}$ which turns out to be a distribution function of a random variable $X : \Omega \to \mathbb{R}$. To make $X : \Omega \to \mathbb{R}$ absolutely continuous, we need that $F : \mathbb{R} \to \mathbb{R}_+$ is absolutely continuous. In particular, we need that $F : \mathbb{R} \to \mathbb{R}_+$ is continuous. Hence, we need

$$a = \lim_{x \to 0^-} F(x) = \lim_{x \to 0^+} F(x) = \frac{1}{2}.$$

Now, the function

$$F(x) \stackrel{\text{def}}{=} \tfrac{1}{2}e^x 1_{\mathbb{R}_{--}}(x) + \left(1 - \tfrac{1}{2}e^{-x}\right) 1_{\mathbb{R}_+}(x), \quad \forall x \in \mathbb{R},$$

is a continuous function on $\mathbb{R}$, which is differentiable everywhere in $\mathbb{R} - \{0\}$ with derivative

$$F'(x) = \tfrac{1}{2}e^{x}1_{\mathbb{R}_{--}}(x) + \tfrac{1}{2}e^{-x}1_{\mathbb{R}_{++}}(x),$$

for every $x \in \mathbb{R} - \{0\}$. Now, we have

$$\lim_{x\to 0^-} F'(x) = \tfrac{1}{2} = \lim_{x\to 0^+} F'(x).$$

It follows that F is actually differentiable everywhere in $\mathbb{R}$ with derivative

$$F'(x) = \tfrac{1}{2}e^{x}1_{\mathbb{R}_{--}}(x) + \tfrac{1}{2}e^{-x}1_{\mathbb{R}_{+}}(x),$$

or every $x \in \mathbb{R}$. Such derivative is clealy bounded. It then follows that $F : \mathbb{R} \to \mathbb{R}_+$ is absolutely continuous.

is a Lebesgue integrable function and we have

$$F(x) = \int'_{(-\infty,x]} \tilde{F}'(u)\ d\mu_L(u), \quad \forall x \in \mathbb{R}.$$

thanks to the positivity of both the functions $e^{x}1_{\mathbb{R}_{--}}(x)$ and $e^{-x}1_{\mathbb{R}_{++}}(x)$ on varying of $x \in \mathbb{R}$, we can write

$$\begin{aligned}
\int_{(-\infty,x]} \tilde{F}'(u)\ d\mu_L(u) &= \int_{(-\infty,x]} \left(\tfrac{1}{2}e^{u}1_{\mathbb{R}_{--}}(u) + \tfrac{1}{2}e^{-u}1_{\mathbb{R}_{+}}(u)\right)\ d\mu_L(u) \\
&= \tfrac{1}{2}\int_{(-\infty,x]} \left(e^{u}1_{\mathbb{R}_{--}}(u) + e^{-u}1_{\mathbb{R}_{+}}(u)\right)\ d\mu_L(u) \\
&= \tfrac{1}{2}\left(\int_{(-\infty,x]} e^{u}1_{\mathbb{R}_{--}}(u)\ d\mu_L(u) + \int_{(-\infty,x]} e^{-u}1_{\mathbb{R}_{+}}(u)\ d\mu_L(u)\right) \\
&= \tfrac{1}{2}\left(\int_{(-\infty,x]\cap\mathbb{R}_{--}} e^{u}\ d\mu_L(u) + \int_{(-\infty,x]\cap\mathbb{R}_{+}} e^{-u}\ d\mu_L(u)\right).
\end{aligned}$$

Hence,

$$\int_{(-\infty,x]} F'(u)\ d\mu_L(u) = \begin{cases} \tfrac{1}{2}\int_{(-\infty,x]} e^{u}\ d\mu_L(u) & \text{if } x < 0 \\ \tfrac{1}{2}\left(\int_{(-\infty,0]} e^{u}\ d\mu_L(u) + \int_{[0,x]} e^{-u}\ d\mu_L(u)\right) & \text{if } x \geq 0 \end{cases}.$$

Now, we have

$$\int_{(-\infty,x]} e^{u}\ d\mu_L(u) = \int_{-\infty}^{x} e^{u}\ du = e^{u}\big|_{-\infty}^{x} = e^{x}$$

and

$$\int_{[0,x]} e^{-u}\ d\mu_L(u) = \int_{0}^{x} e^{-u}\ du = -\ e^{-u}\big|_{0}^{x} = 1 - e^{-x}$$

These imply

$$\int_{(-\infty,x]} F'(u)\ d\mu_L(u) = \begin{cases} \tfrac{1}{2}e^{x} & \text{if } x < 0 \\ \tfrac{1}{2}(1 + 1 - e^{-x}) & \text{if } x \geq 0 \end{cases} = \tfrac{1}{2}e^{x}1_{\mathbb{R}_{--}}(x) + \left(1 - \tfrac{1}{2}e^{-x}\right)1_{\mathbb{R}_{+}}(x) = F(x)$$

for every $x \in \mathbb{R}$. It then follows that In the end, we have

$$\mathbf{P}(-1 \leq X \leq 1) = F(1) - F(-1) = 1 - \tfrac{1}{2}e^{-1} - \tfrac{1}{2}e^{-1} = 1 - e^{-1}.$$

Problem 2 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *the real state space, and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \stackrel{def}{=} \alpha + \beta x, \quad \forall x \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\beta \neq 0$.

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \stackrel{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution.

1. The function $g : \mathbb{R} \to \mathbb{R}$ is a Borel function. Therefore, $Y = g \circ X$ is a random variable.

2. Recall that $X \sim Unif(-1, 1)$ is absolutely continuous with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x),$$

   for every $x \in \mathbb{R}$. Hence, writing $F_X : \mathbb{R} \to \mathbb{R}$ for the distribution function of X, we have

$$\begin{aligned} F_X(x) &= \int_{(-\infty,x]} f_X(u)\, d\mu_L(u) = \int_{(-\infty,x]} \frac{1}{2} 1_{[-1,1]}(u)\, d\mu_L(u) \\ &= \frac{1}{2} \int_{(-\infty,x]\cap[-1,1]} d\mu_L(u) = \frac{1}{2} \mu_L((-\infty, x] \cap [-1, 1]). \end{aligned}$$

   On the other hand,

$$(-\infty, x] \cap [-1, 1] = \begin{cases} \varnothing, & \text{if } x < -1, \\ \{-1\}, & \text{if } x = -1, \\ [-1, x], & \text{if } x < -1. \end{cases}$$

   Therefore,

$$F_X(x) = \begin{cases} 0, & \text{if } x < -1, \\ \frac{x+1}{2}, & \text{if } -1 \leq x < 1, \\ 1, & \text{if } 1 \leq x. \end{cases}$$

   Now, since g is a continuously differentiable real function on $\mathbb{R}$, in particular a Borel function, then $Y \equiv g(X) = \alpha + \beta X$ is a real random variable. To compute the distribution function F_Y, we apply the definition

$$F_Y(y) \stackrel{\text{def}}{=} \mathbf{P}(Y \leq y), \quad \forall y \in \mathbb{R}.$$

On the other hand, considering that $\beta \neq 0$, we have

$$\begin{aligned}\mathbf{P}(Y \leq y) &= \mathbf{P}(\alpha + \beta X \leq y) = \mathbf{P}\left(X \leq \frac{y-\alpha}{\beta}\right) \\ &= F_X\left(\frac{y-\alpha}{\beta}\right) = \begin{cases} 0, & \text{if } \frac{y-\alpha}{\beta} < -1 \Leftrightarrow y < \alpha - \beta, \\ \frac{\frac{y-\alpha}{\beta}+1}{2} = \frac{y+\beta-\alpha}{2\beta}, & \text{if } -1 \leq \frac{y-\alpha}{\beta} < 1 \Leftrightarrow \alpha - \beta \leq y < \alpha + \beta, \\ 1, & \text{if } 1 \leq \frac{y-\alpha}{\beta} \Leftrightarrow \alpha + \beta \leq y. \end{cases}\end{aligned}$$

Summarizing,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < \alpha - \beta, \\ \frac{y+\beta-\alpha}{2\beta}, & \text{if } \alpha - \beta \leq y \leq \alpha + \beta, \\ 1, & \text{if } \alpha + \beta < y. \end{cases}$$

Therefore, the random variable Y turns out to be a uniformly distributed random variable on the interval $[\alpha - \beta, \alpha + \beta]$. In symbols, $Y \sim Unif(\alpha - \beta, \alpha + \beta)$. It then follows that Y is absolutely continuous with density $f_Y : \mathbb{R} \to \mathbb{R}$ given by

$$f_Y(y) = \frac{1}{2\beta} 1_{[\alpha-\beta, \alpha+\beta]}(y).$$

3. Since X is in the linear space $\mathcal{L}^2(\Omega; \mathbb{R})$, the random variable $Y = \alpha + \beta X$ is also in the linear space $\mathcal{L}^2(\Omega; \mathbb{R})$. Hence, Y has finite moments of order 1 and 2.

4. Thanks to the linearity of the expectation operator, we have

$$\mathbf{E}[Y] = \mathbf{E}[\alpha + \beta X] = \alpha + \beta \mathbf{E}[X],$$

where

$$\mathbf{E}[X] = \int_{\mathbb{R}} \frac{1}{2} x 1_{[-1,1]}(x)\, d\mu_L(x) = \frac{1}{2} \int_{[-1,1]} x\, d\mu_L(x) = \frac{1}{2} \int_{-1}^{1} x\, dx = \frac{1}{4}\, x^2\big|_{-1}^{1} = 0.$$

Therefore,

$$\mathbf{E}[Y] = \alpha.$$

Moreover considering the properties of the variance operator, we have

$$\mathbf{D}^2[Y] = \mathbf{D}^2[\alpha + \beta X] = \beta^2 \mathbf{D}^2[X],$$

where

$$\mathbf{D}^2[X] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X] = \mathbf{E}\left[X^2\right]$$

and

$$\mathbf{E}\left[X^2\right] = \int_{\mathbb{R}} \frac{1}{2} x^2 1_{[-1,1]}(x)\, d\mu_L(x) = \frac{1}{2} \int_{[-1,1]} x^2\, d\mu_L(x) = \frac{1}{2} \int_{-1}^{1} x^2\, dx = \frac{1}{6}\, x^3\big|_{-1}^{1} = \frac{1}{3}.$$

Therefore,

$$\mathbf{D}^2[Y] = \frac{\beta^2}{3}.$$

Problem 3 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *the real state space, and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} |x|, \quad \forall x \in \mathbb{R},$$

where $|x|$ *is the absolute value of* x.

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. Recall that, since $X \sim Unif(-1, 1)$, the random variable X is absolutely continuous with density

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x),$$

for every $x \in \mathbb{R}$. Now, we have

$$F_Y(y) \overset{\text{def}}{=} \mathbf{P}(Y \le y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(|X| \le y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(-y \le X \le y), & \text{if } y \ge 0. \end{cases}$$

On the othe hand, under the assumption $y \ge 0$, we have

$$\begin{aligned} \mathbf{P}(-y \le X \le y) &= \int_{[-y,y]} f_X(x)\, d\mu_X(x) \\ &= \int_{[-y,y]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\ &= \frac{1}{2} \int_{[-y,y]\cap[-1,1]} d\mu_X(x) \\ &= \frac{1}{2} \mu_X([-y, y] \cap [-1, 1]), \end{aligned}$$

where

$$\mu_X([-y, y] \cap [-1, 1]) = \begin{cases} \mu_X([-y, y]) = 2y, & \text{if } y \le 1, \\ \mu_X([-1, 1]) = 2, & \text{if } y > 1. \end{cases}$$

It follows

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ y, & \text{if } 0 \le y \le 1, \\ 1, & \text{if } y > 1. \end{cases}$$

We can then recognize that $Y \sim Unif(0, 1)$, which implies that Y is absolutely continuous with density given by

$$f_Y(y) = 1_{[0,1]}(y),$$

for every $y \in \mathbb{R}$, and Y has finite first and second order moments. More specifically

$$\begin{aligned}
\mathbf{E}[Y] &= \int_{\mathbb{R}} y f_Y(y)\, d\mu_X(y) = \int_{\mathbb{R}} y 1_{[0,1]}(y)\, d\mu_X(y) \\
&= \int_{[0,1]} y d\mu_X(y) = \int_0^1 y dy = \frac{1}{2} y^2 \Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[Y^2\right] &= \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_X(y) = \int_{\mathbb{R}} y^2 1_{[0,1]}(y)\, d\mu_X(y) \\
&= \int_{[0,1]} y^2 d\mu_X(y) = \int_0^1 y^2 dy = \frac{1}{3} y^3 \Big|_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

It follows

$$\mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Note that, since $Y = |X|$ it would be possible to compute $\mathbf{E}[Y]$ and $\mathbf{E}\left[Y^2\right]$ by using the density of X. That is

$$\begin{aligned}
\mathbf{E}[Y] = \mathbf{E}[|X|] &= \int_{\mathbb{R}} |x| f_X(x)\, d\mu_X(x) \\
&= \int_{\mathbb{R}} |x| \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\
&= \frac{1}{2} \int_{[-1,1]} |x|\, d\mu_X(x) \\
&= \frac{1}{2} \left(\int_{[-1,0]} -x d\mu_X(x) + \int_{[0,1]} x d\mu_X(x) \right) \\
&= \frac{1}{2} \left(\int_{-1}^0 -x dx + \int_0^1 x dx \right) \\
&= \frac{1}{2} \left(- \int_{-1}^0 x dx + \int_0^1 x dx \right) \\
&= \frac{1}{2} \left(- \frac{1}{2} x^2 \Big|_{-1}^0 + \frac{1}{2} x^2 \Big|_0^1 \right) \\
&= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[Y^2\right] = \mathbf{E}\left[|X|^2\right] = \mathbf{E}\left[X^2\right] &= \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_X(x) \\
&= \int_{\mathbb{R}} x^2 \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\
&= \frac{1}{2}\int_{[-1,1]} x^2 d\mu_X(x) \\
&= \frac{1}{2}\int_{-1}^{1} x^2 dx \\
&= \frac{1}{2}\frac{1}{3}x^3\Big|_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) \\
&= \frac{1}{3}.
\end{aligned}$$

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Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ the real state space, and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \overset{def}{=} x^2, \quad \forall x \in \mathbb{R}.$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of the random variable Y?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 5 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ the real state space, and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \overset{def}{=} x^3, \quad \forall x \in \mathbb{R}.$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of the random variable Y?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. Recall that $X \sim Unif(-1,1)$ is absolutely continuous, with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x).$$

1. The function g is clearly continuous. In particular, g is a Borel function. Therefore, $Y = g \circ X$ is a random variable.

2. The distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of Y is given by

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(g(X) \leq y)$$

for every $y \in \mathbb{R}$. Now, due to the definition of g, we have

$$\{x \in \mathbb{R} : g(x) \leq y\} = \begin{cases} \left\{x \in \mathbb{R} : x \leq -\sqrt[3]{|y|}\right\}, & \text{if } y < 0, \\ \left\{x \in \mathbb{R} : x \leq \sqrt{y}\right\}, & \text{if } y \geq 0. \end{cases}$$

Hence,

$$\{g(X) \leq y\} = \begin{cases} \left\{X \leq -\sqrt[3]{|y|}\right\}, & \text{if } y < 0, \\ \left\{X \leq \sqrt{y}\right\}, & \text{if } y \geq 0. \end{cases}$$

It follows,

$$\mathbf{P}(g(X) \leq y)) = \begin{cases} \mathbf{P}\left(X \leq -\sqrt[3]{|y|}\right) & \text{if } y < 0, \\ \mathbf{P}\left(X \leq \sqrt{y}\right), & \text{if } y \geq 0. \end{cases}$$

On the other hand, since $X \sim Unif(-1,1)$, for every $y < 0$, we have

$$\begin{aligned} \mathbf{P}\left(X \leq -\sqrt[3]{|y|}\right) &= \int_{\left(-\infty, -\sqrt[3]{|y|}\right]} f_X(x)\, d\mu_L(x) \\ &= \int_{\left(-\infty, -\sqrt[3]{|y|}\right]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_L(x) \\ &= \frac{1}{2} \int_{\left(-\infty, -\sqrt[3]{|y|}\right] \cap [-1,1]} d\mu_L(x) \\ &= \frac{1}{2} \mu_L\left(\left(-\infty, -\sqrt[3]{|y|}\right] \cap [-1,1]\right), \end{aligned}$$

where

$$\left(-\infty, -\sqrt[3]{|y|}\right] \cap [-1,1] = \begin{cases} \varnothing, & \text{if } \; y < -1, \\ \left[-1, -\sqrt[3]{|y|}\right], & \text{if } \; -1 \leq y < 0, \end{cases}$$

Therefore,

$$\mathbf{P}\left(X \leq -\sqrt[3]{|y|}\right) = \begin{cases} 0, & \text{if } y < -1, \\ \frac{1}{2}\left(-\sqrt[3]{|y|} + 1\right), & \text{if } \; -1 \leq y < 0. \end{cases}$$

Similarly, for every $y > 0$, we have

$$\begin{aligned}\mathbf{P}\left(X \leq \sqrt[3]{y}\right) &= \int_{(-\infty,\,\sqrt[3]{y}]} f_X(x)\, d\mu_L(x) \\ &= \int_{(-\infty,\,\sqrt[3]{y}]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_L(x) \\ &= \frac{1}{2}\int_{(-\infty,\,\sqrt[3]{y}]\cap[-1,1]} d\mu_L(x) \\ &= \frac{1}{2}\mu_L\left((-\infty,\,\sqrt[3]{y}]\cap[-1,1]\right),\end{aligned}$$

where

$$(-\infty,\,\sqrt[3]{y}]\cap[-1,1] = \begin{cases} \left[-1,\,\sqrt[3]{y}\right], & \text{if } 0 \leq y \leq 1, \\ [-1,1], & \text{if } 1 < y < 1. \end{cases}$$

Therefore,

$$\mathbf{P}\left(X \leq \sqrt[3]{y}\right) = \begin{cases} \sqrt[3]{y} + 1, & \text{if } 0 \leq y \leq 1, \\ 1, & \text{if } 1 < y. \end{cases}$$

We can then write,

$$F_Y(y) = \frac{1}{2}\left(1 - \sqrt[3]{|y|}\right) 1_{[-1,0]}(y) + \frac{1}{2}\left(1 + \sqrt[3]{y}\right) 1_{[0,1]}(y) + 1_{(1,+\infty)}(y).$$

3. Note that $F_Y : \mathbb{R} \to \mathbb{R}$ is continuous in $\mathbb{R}$, it is differentiable in $\mathbb{R} - \{-1, 0, 1\}$ and we have

$$F_Y'(y) = \begin{cases} 0, & \text{if } y < -1, \\ \frac{1}{6}\frac{\sqrt[3]{|y|}}{|y|}, & \text{if } -1 < y < 0, \\ \frac{1}{6}\frac{\sqrt[3]{y}}{y}, & \text{if } 0 < y < 1, \\ 0, & \text{if } 1 < y. \end{cases}$$

Therefore, $F_Y : \mathbb{R} \to \mathbb{R}$ is not differentiable in $y = -1$, $y = 0$, and $y = 1$. On the other hand, consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(y) \stackrel{\text{def}}{=} \frac{1}{6}\left(\frac{\sqrt[3]{|y|}}{|y|} 1_{(-1,0)}(y) + \frac{\sqrt[3]{y}}{y} 1_{(0,1)}(y)\right), \quad \forall y \in \mathbb{R}.$$

we have

$$\int_{(-\infty,y)} f(v)\, d\mu_L(v) = \frac{1}{6}\left(\int_{(-\infty,y)} \frac{\sqrt[3]{|v|}}{|v|} 1_{(-1,0)}(v)\, d\mu_L(v) + \int_{(-\infty,y)} \frac{\sqrt[3]{v}}{v} 1_{(0,1)}(v)\, d\mu_L(v)\right),$$

where

$$\int_{(-\infty,y)} \frac{\sqrt[3]{|v|}}{|v|} 1_{(-1,0)}(v)\, d\mu_L(v) = \int_{(-\infty,y)\cap(-1,0)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v) = \begin{cases} 0, & \text{if } y \leq -1, \\ \int_{(-1,y)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v), & \text{if } -1 < y < 0, \\ \int_{(-1,0)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v), & \text{if } 0 \leq y, \end{cases}$$

and

$$\int_{(-\infty,y)} \frac{\sqrt[3]{v}}{v} 1_{(0,1)}(v)\, d\mu_L(v) = \int_{(-\infty,y)\cap(0,1)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v) = \begin{cases} 0, & \text{if } y \leq 0, \\ \int_{(0,y)} \frac{\sqrt[3]{v}}{v} d\mu_L(v), & \text{if } 0 < y < 1, \\ \int_{(0,1)} \frac{\sqrt[3]{v}}{v} d\mu_L(v), & \text{if } 1 \leq y. \end{cases}$$

On the other hand,

$$\int_{(-1,y)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v) = \int_{-1}^{y} \frac{\sqrt[3]{|v|}}{|v|} dv = -3\sqrt[3]{|v|}\Big|_{-1}^{y} = 3\left(1-\sqrt[3]{|y|}\right),$$

for every $y \in (-1,0)$,

$$\int_{(-1,0)} \frac{\sqrt[3]{|v|}}{|v|} d\mu_L(v) = \lim_{y\to 0^-} \int_{-1}^{y} \frac{\sqrt[3]{|v|}}{|v|} dv = \lim_{y\to 0^-} 3\left(1-\sqrt[3]{|y|}\right) = 3.$$

Furthermore,

$$\int_{(0,y)} \frac{\sqrt[3]{v}}{v} d\mu_L(v) = \lim_{x\to 0^+} \int_{x}^{y} \frac{\sqrt[3]{v}}{v} dv = \lim_{x\to 0^+} 3\sqrt[3]{v}\Big|_{x}^{y} = \lim_{x\to 0^+} 3\left(\sqrt[3]{y}-\sqrt[3]{x}\right) = 3\sqrt[3]{y},$$

and

$$\int_{(0,1)} \frac{\sqrt[3]{v}}{v} d\mu_L(v) = \lim_{x\to 0^+} \int_{x}^{1} \frac{\sqrt[3]{v}}{v} dv = \lim_{x\to 0^+} 3\sqrt[3]{v}\Big|_{x}^{1} = \lim_{x\to 0^+} 3\left(1-\sqrt[3]{x}\right) = 3.$$

It then follows,

$$\int_{(-\infty,y)} f(v)\, d\mu_L(v) = \begin{cases} 0, & \text{if } y \le -1 \\ \frac{1}{2}\left(1-\sqrt[3]{|y|}\right), & \text{if } -1 < y \le 0, \\ \frac{1}{2}\left(1+\sqrt[3]{y}\right), & \text{if } 0 < y \le 1 \\ 1, & \text{if } 1 < y. \end{cases}$$

Hence,

$$\int_{(-\infty,y)} f(v)\, d\mu_L(v) = \frac{1}{2}\left(1-\sqrt[3]{|y|}\right) 1_{(-1,0]}(y) + \frac{1}{2}\left(1+\sqrt[3]{y}\right) 1_{(0,1]}(y) + 1_{(0,+\infty)}(y) = F_Y(y)$$

almost everywhere in $\mathbb{R}$. Therefore, Y is absolutely continuous in $\mathbb{R}$ and a density for Y is given by $f : \mathbb{R} \to \mathbb{R}$.

4. We have

$$\int_\Omega Y^2 d\mathbf{P} = \int_\Omega g(X)^2\, d\mathbf{P}.$$

Therefore, Y has finite moment of order 2 or not according to whether

$$\int_\Omega g(X)^2\, d\mathbf{P} < \infty.$$

Now, since X is absolutely continuous, we can write

$$\begin{aligned} \int_\Omega g(X)^2\, d\mathbf{P} &= \int_\mathbb{R} g(x)^2 f_X(x)\, d\mu_L(x) \\ &= \int_\mathbb{R} x^4 1_{(0,+\infty)}(x)\, 1_{(-1,1)}(x)\, d\mu_L(x) \\ &= \frac{1}{2}\int_{(0,1)} x^4 d\mu_L(x) \\ &= \frac{1}{2}\int_0^1 x^4 dx \\ &= \frac{1}{10}\, x^5\Big|_0^1 \\ &= \frac{1}{10}. \end{aligned}$$

It follows, that Y has finite moment of order 2 and

$$\mathbf{E}\left[Y^2\right] = \int_\Omega Y^2 d\mathbf{P} = \frac{1}{10}.$$

A fortiori Y has finite moment of order 1 and

$$\begin{aligned}
\mathbf{E}[Y] = \mathbf{E}[g(X)] &= \int_{\mathbb{R}} g(x) f_X(x)\, d\mu_L(x) \\
&= \int_{\mathbb{R}} x^2 1_{(0,+\infty)}(x)\, 1_{(-1,1)}(x)\, d\mu_L(x) \\
&= \frac{1}{2}\int_{(0,1)} x^2 d\mu_L(x) \\
&= \frac{1}{2}\int_0^1 x^2 dx \\
&= \frac{1}{6}\, x^3\Big|_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

In the end,

$$\mathbf{D}^2[Y] = \mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = \frac{1}{10} - \frac{1}{36} = \frac{13}{180}.$$

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Problem 6 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ the real state space, and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \overset{def}{=} \begin{cases} 0, & \text{if } x \leq 0. \\ x^2, & \text{if } x > 0. \end{cases}$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the random variable Y?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. Recall that $X \sim Unif(-1, 1)$ is absolutely continuous, with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x).$$

Note also that we can write

$$g(x) = x^2 1_{(0,+\infty)}(x),$$

for every $x \in \mathbb{R}$.

1. The function g is clearly continuous. In particular, g is a Borel function. Therefore, $Y = g \circ X$ is a random variable.

2. The distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of Y is given by

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(g(X) \leq y)$$

for every $y \in \mathbb{R}$. Now, due to the definition of g, we have

$$\{x \in \mathbb{R} : g(x) \leq y\} = \begin{cases} \varnothing, & \text{if } y < 0, \\ \left\{x \in \mathbb{R} : x \leq \sqrt{y}\right\}, & \text{if } y \geq 0. \end{cases}$$

Hence,

$$\{g(X) \leq y\} = \begin{cases} \varnothing, & \text{if } y < 0, \\ \left\{X \leq \sqrt{y}\right\}, & \text{if } y \geq 0. \end{cases}$$

It follows,

$$\mathbf{P}(g(X) \leq y)) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}\left(X \leq \sqrt{y}\right), & \text{if } y \geq 0. \end{cases}$$

On the other hand, since $X \sim Unif(-1, 1)$, we have

$$\begin{aligned} \mathbf{P}(X \leq \sqrt{y}) &= \int_{(-\infty, \sqrt{y}]} f_X(x)\, d\mu_L(x) \\ &= \int_{(-\infty, \sqrt{y}]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_L(x) \\ &= \frac{1}{2} \int_{(-\infty, \sqrt{y}] \cap [-1,1]} d\mu_L(x) \\ &= \frac{1}{2} \mu_L\left((-\infty, \sqrt{y}] \cap [-1, 1]\right), \end{aligned}$$

where

$$(-\infty, \sqrt{y}] \cap [-1, 1] = \begin{cases} \left[-1, \sqrt{y}\right], & \text{if } 0 \leq y < 1, \\ [-1, 1], & \text{if } y \geq 1. \end{cases}$$

Therefore,

$$\mathbf{P}(X \leq \sqrt{y}) = \begin{cases} \frac{1}{2}\left(\sqrt{y} + 1\right), & \text{if } y < 1, \\ 1, & \text{if } y \geq 1. \end{cases}$$

We can then write,

$$F_Y(y) = \frac{1}{2}(\sqrt{y} + 1)\, 1_{[0,1]}(y) + 1_{(1,+\infty)}(y).$$

Note that

$$\mathbf{P}(Y < 0) = F_Y(0) = 0.$$

Hence, Y is a non negative random variable.

3. Note that $F_Y : \mathbb{R} \to \mathbb{R}$ is not continuous since

$$\lim_{y \to 0-^-} F_Y(y) = 0 \quad \text{and} \quad \lim_{y \to 0^+} F_Y(y) = \frac{1}{2}.$$

A fortiori it is not absolutely continuous.

4. We have

$$\int_{\Omega} Y^2 d\mathbf{P} = \int_{\Omega} g(X)^2 \, d\mathbf{P}.$$

Therefore, Y has finite moment of order 2 or not according to whether

$$\int_{\Omega} g(X)^2 \, d\mathbf{P} < \infty.$$

Now, since X is absolutely continuous, we can write

$$\begin{aligned}
\int_{\Omega} g(X)^2 \, d\mathbf{P} &= \int_{\mathbb{R}} g(x)^2 f_X(x) \, d\mu_L(x) \\
&= \int_{\mathbb{R}} x^4 1_{(0,+\infty)}(x) \, 1_{(-1,1)}(x) \, d\mu_L(x) \\
&= \frac{1}{2} \int_{(0,1)} x^4 d\mu_L(x) \\
&= \frac{1}{2} \int_0^1 x^4 dx \\
&= \frac{1}{10} \left. x^5 \right|_0^1 \\
&= \frac{1}{10}.
\end{aligned}$$

It follows, that Y has finite moment of order 2 and

$$\mathbf{E}\left[Y^2\right] = \int_{\Omega} Y^2 d\mathbf{P} = \frac{1}{10}.$$

A fortiori Y has finite moment of order 1 and

$$\begin{aligned}
\mathbf{E}[Y] = \mathbf{E}[g(X)] &= \int_{\mathbb{R}} g(x) f_X(x) \, d\mu_L(x) \\
&= \int_{\mathbb{R}} x^2 1_{(0,+\infty)}(x) \, 1_{(-1,1)}(x) \, d\mu_L(x) \\
&= \frac{1}{2} \int_{(0,1)} x^2 d\mu_L(x) \\
&= \frac{1}{2} \int_0^1 x^2 dx \\
&= \frac{1}{6} \left. x^3 \right|_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

In the end,

$$\mathbf{D}^2[Y] = \mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = \frac{1}{10} - \frac{1}{36} = \frac{13}{180}.$$

Problem 7 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \stackrel{def}{=} x^2 - 2x, \quad \forall x \in \mathbb{R},$$

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 8 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, and let* $X : \Omega \to \mathbb{R}$ *be an exponentially distributed random variable with rate parameter* $\lambda = 1$*. In symbols,* $X \sim Exp(1)$*. Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} 1 - \exp(-x), \quad \forall x \in \mathbb{R},$$

where $\exp : \mathbb{R} \to \mathbb{R}$ *is the Neper exponential function.*

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 9 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $(0, 1)$*. In symbols,* $X \sim Unif(0, 1)$*. Consider the function* $g : \mathbb{R}_{++} \to \mathbb{R}$ *given by*

$$g(y) \overset{def}{=} -\frac{1}{\lambda} \ln(y), \quad \forall \in \mathbb{R}_{++},$$

where $\ln : \mathbb{R}_{++} \to \mathbb{R}$ *is the natural logarithm function and* $\lambda > 0$*.*

1. *Can you state that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a real random variable on Ω*?*

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}$ *of* $Y : \Omega \to \mathbb{R}$*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

   Hint: recall the properties of the ***logarithm*** *and* ***exponential*** *function.*

Solution. .

Problem 10 *Consider the function $f : \mathbb{R} \to \mathbb{R}_+$ given by*

$$f(x) \overset{def}{=} \begin{matrix} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{matrix}.$$

Prove that f is a density function (it could be helpful to draw the graph of f). Assume that $X : \Omega \to \mathbb{R}$ is a real random variable on some probability space with density f. Determine the distribution function $F_X : \mathbb{R} \to \mathbb{R}$ of X. Compute $\mathbf{E}[X]$ and $\mathbf{D}^2[X]$. In the end, consider the function $Y : \Omega \to \mathbb{R}$ given by

$$Y(\omega) \overset{def}{=} e^{X(\omega)}, \quad \forall \omega \in \Omega.$$

Is Y a random variable? In case it is, determine the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of Y. Can you say that Y is absolutely continuous? In the affirmative case, can you compute the density function of Y? Does Y have finite expectation and variance? What about computing $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?

Solution. .

Problem 11 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X \sim Exp(1)$ be an exponentially distributed random variable with rate parameter $\lambda = 1$. Consider the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} \ln(X(\omega)), \quad \forall \omega \in \Omega.$$

Is Y a random variable? In the affirmative case, determine the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of Y. Can you say that Y is absolutely continuous? In the affirmative case, can you compute the density function of Y? Does Y have finite expectation and variance? What about computing $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?

Solution. We have

$$f_X(x) = e^{-x} 1_{\mathbb{R}_+}(x),$$

for every $x \in \mathbb{R}$. Therefore,

$$\mathbf{P}(X \leq 0) = \int_{\mathbb{R}_-} f_X(x) \, d\mu_L(x) = 0.$$

Hence, the function Y, composition of the strictly positive random variable $X : \Omega \to \mathbb{R}$ and the continuous function $\ln : \mathbb{R}_+ \to \mathbb{R}$ turns out to be well defined as a random variable. Now, according to the definition and considering that the exponential function is increasing, we have

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) = \mathbf{P}(\ln(X) \leq y) \\ &= \mathbf{P}(\exp(\ln(X)) \leq \exp(y)) = \mathbf{P}(X \leq e^y) \\ &= \int_{(-\infty, e^y]} f_X(u) \, d\mu_L(u), \end{aligned}$$

for every $y \in \mathbb{R}$. On the other hand, since $e^y > 0$, we can write

$$\begin{aligned}
\int_{(-\infty,e^y]} f_X(u)\ d\mu_L(u) &= \int_{(-\infty,e^y]} e^{-x} 1_{\mathbb{R}_+}(x)\ d\mu_L(u) = \int_{(-\infty,e^y]\cap\mathbb{R}_+} e^{-x}\ d\mu_L(u) \\
&= \int_{(0,e^y]} e^{-x}\ d\mu_L(u) = \int_0^{e^y} e^{-u}\ du \\
&= 1 - e^{-e^y}.
\end{aligned}$$

It then follows

$$F_Y(y) = 1 - e^{-e^y},$$

for every $y \in \mathbb{R}$. The distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}$ and

$$F_Y'(y) = e^y e^{-e^y}$$

In addition, $F_Y'(y)$ is clearly bounded. In fact, we have

$$F_Y''(y) = e^y e^{-e^y}(1 - e^y).$$

Hence, $F_Y'(y)$ takes a unique maximum at the point $y = 0$ with value $F_Y'(0) = e^{-1}$. As a consequence, Y is absolutely continuous and has a density $f_Y : \mathbb{R} \to \mathbb{R}$ given by

$$f_Y(y) = F_Y'(y) = e^y e^{-e^y}.$$

To check whether Y has finite first order moment, we study

$$\begin{aligned}
\int_\Omega |Y|\ d\mathbf{P} &= \int_{\mathbb{R}} |y|\, f_Y(y)\ d\mu_L(y) = \int_{\mathbb{R}} |y|\, e^y e^{-e^y}\ d\mu_L(y) \\
&= \int_{\mathbb{R}_-} -y e^y e^{-e^y}\ d\mu_L(y) + \int_{\mathbb{R}_+} y e^y e^{-e^y}\ d\mu_L(y) \\
&= -\int_{-\infty}^0 y e^y e^{-e^y}\ dy + \int_0^{+\infty} y e^y e^{-e^y}\ dy.
\end{aligned}$$

Consider $\int_0^{+\infty} y e^y e^{-e^y}\ dy$. Setting $e^y = z$ we have

$$y = \ln(z), \quad dy = \frac{1}{z} dz$$

and

$$\int_0^{+\infty} y e^y e^{-e^y}\ dy = \int_1^{+\infty} \ln(z)\, e^{-z}\ dz,$$

where

$$\begin{aligned}
\int_1^{+\infty} \ln(z)\, e^{-z}\ dz \le \int_1^{+\infty} z e^{-z}\ dz &= -\int_1^{+\infty} z\ de^{-z} \\
&= -\left(z e^{-z}\Big|_1^{+\infty} - \int_1^{+\infty} e^{-z}\ dz \right) \\
&= -\left(z e^{-z}\Big|_1^{+\infty} + e^{-z}\Big|_1^{+\infty} \right) \\
&= 2e^{-1}.
\end{aligned}$$

More precisely,

$$\begin{aligned}
\int_0^{+\infty} ye^y e^{-e^y}\ dy &= \int_1^{+\infty} \ln(z)\, e^{-z}\ dz = -\int_1^{+\infty} \ln(z)\ de^{-z} \\
&= -\left[\ln(z)\, e^{-z}\Big|_1^{+\infty} - \int_1^{+\infty} \frac{e^{-z}}{z}\ dz\right] \\
&= \int_1^{+\infty} \frac{e^{-z}}{z}\ dz \\
&= \Gamma(0,1) \simeq 0.21939.
\end{aligned}$$

Similarly,

$$\int_{-\infty}^0 ye^y e^{-e^y}\ dy = \int_0^1 \ln(z)\, e^{-z}\ dz,$$

where

$$\begin{aligned}
\int_0^1 \ln(z)\, e^{-z}\ dz &\geq \int_0^1 \ln(z)\ dz \\
&= \left(z\ln(z)|_0^1 - \int_0^1 z\frac{1}{z}\ dz\right) \\
&= -1
\end{aligned}$$

More precisely,

$$\int_{-\infty}^0 ye^y e^{-e^y}\ dy = \int_0^1 \ln(z)\, e^{-z}\ dz \simeq -0.7966.$$

As a consequence,

$$\int_\Omega |Y|\ d\mathbf{P} < \infty.$$

More precisely,

$$\int_\Omega |Y|\ d\mathbf{P} \simeq 0.2194 + 0.7966 = 1.016\,0.$$

It follows that Y has finite expectation given by

$$\begin{aligned}
\int_\Omega Y\ d\mathbf{P} &= \int_{\mathbb{R}_-} ye^y e^{-e^y}\ d\mu_L(y) + \int_{\mathbb{R}_+} ye^y e^{-e^y}\ d\mu_L(y) \\
&= \int_{-\infty}^0 ye^y e^{-e^y}\ dy + \int_0^{+\infty} ye^y e^{-e^y}\ dy \\
&= -0.7966 + 0.2194 \\
&= -0.5772.
\end{aligned}$$

Now, we have

$$\int_\Omega Y^2\ d\mathbf{P} = \int_{\mathbb{R}} y^2 f_X(y)\ d\mu_L(y) = \int_{\mathbb{R}} y^{2y} e^{-e^y}\ d\mu_L(y) = \int_{-\infty}^{+\infty} y^2 e^y e^{-e^y}\ dy$$

and, with a similar argument as above, it is possible to prove that

$$\int_{-\infty}^{+\infty} y^2 e^y e^{-e^y}\ dy < \infty.$$

More precisely

$$\int_{-\infty}^{+\infty} y^2 e^y e^{-e^y}\ dy = \gamma^2 + \frac{\pi^2}{6},$$

where γ is the Euler gamma constant. $\square$