PER LE STRUCT (variabili locali): ctrl+L sulla variabile nel codice del decompilatore e poi selezioni il tipo di dato desiderato.

Chiaramente questo potrai farlo solo dopo esserti costruito a mano la tua struct (vedere lezioni di Cesati a tal proposito).

CPS (2° file):

~~3)~~ URNA A: 5 palline bianche + 10 palline nere

URNA B: 3 palline bianche + 12 palline nere

MONETA EQUA → TESTA ⇒ ESTRAZ. URNA A; CROCE ⇒ ESTRAZ. URNA B

È stata estratta una pallina bianca

DATI:

$P(W|H) = 1/3$ $\quad$ $P(B|H) = 2/3$

$P(W|T) = 1/5$ $\quad$ $P(B|T) = 4/5$

$P(H) = P(T) = 1/2$

$P(H|W) = ?$ $\quad$ $P(T|W) = ?$

$$P(H|W) = \frac{P(H\cap W)}{P(W)} = \frac{P(W|H)P(H)}{P(W)}$$

$$P(W) = P(W|H)P(H) + P(W|T)P(T) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}\right) = \frac{4}{15}$$

$$\Rightarrow P(H|W) = \frac{1/3 \cdot 1/2}{4/15} = \frac{1}{6}\cdot\frac{15}{4} = \frac{5}{8}$$

$$P(T|W) = \frac{P(W\cap T)}{P(W)} = \frac{P(W|T)P(T)}{P(W)} = \frac{1/5 \cdot 1/2}{4/15} = \frac{1}{10}\cdot\frac{15}{4} = \frac{3}{8}$$

4) N palline → M bianche; N-M nere

~~estrazioni con reimbussolamento~~

~~5)~~ ~~R(X)~~ $P(F=1) = 4/5 \Rightarrow P(F=0) = 1/5$

$P(S=1|F=0) = 3/5$

$$P(F=1 \cup S=1) = P(F=1) + \underbrace{P(F=0)P(S=1|F=0)}_{P(F=0\cap S=1)} =$$

$$= 4/5 + 1/5 \cdot 3/5 = \frac{20}{25} + \frac{3}{25} = \frac{23}{25}$$

$$P(F=1 | F=1 \cup S=1) = \frac{P(F=1)}{P(F=1\cup S=1)} = \frac{4/5}{23/25} = \frac{4}{5}\cdot\frac{25}{23} = \frac{20}{23}$$

$P(\text{sum is } 7) = P(\{6,1\}) + P(\{5,2\}) + P(\{4,3\}) + P(\{3,4\}) + P(\{2,5\}) + P(\{1,6\}) = 6 \cdot 1/36 = 1/6$

$P(5 \text{ is observed}) = P(5 \text{ only on the 1st dice}) + P(5 \text{ only on the 2nd dice}) + P(\{5,5\}) = 5\cdot 1/36 + 5 \cdot 1/36 + 1/36 = 11/36$

data luogo

$P(\text{sum is } 7 \cap 5 \text{ is observed}) = P(\{2,5\}) + P(\{5,2\}) = \frac{2}{36} = \frac{1}{18}$

$P(\text{sum } 7 \mid 5 \text{ is observed}) = \frac{P(\text{sum is } 7 \cap 5 \text{ is observed})}{P(5 \text{ is observed})} = \frac{1}{18} \cdot \frac{36}{11} = \frac{2}{11}$

$\neq P(\text{sum } 7) \Rightarrow$ I due eventi NON sono indipendenti.

~~6~~

- Il 95% dei malati viene sgamato.
- Il 99% dei NON malati risulta non malato.
- Il 10% della popolazione è effettivamente malato.

DATI:

$P(Pos \mid Dis) = 0,95$

$P(Neg \mid NoDis) = 0,99$

$P(Dis) = 0,1 \Rightarrow P(NoDis) = 0,9$

- $P(Dis \mid Pos) = \frac{P(Pos \cap Dis)}{P(Pos)} = \frac{P(Pos \mid Dis)\, P(Dis)}{P(Pos)}$

$\rightsquigarrow P(Pos) = P(Pos \cap Dis) + P(Pos \cap NoDis) =$

$= P(Pos \mid Dis)\, P(Dis) + P(Pos \mid NoDis)\, P(NoDis) =$

$= 0,95 \cdot 0,1 + 0,01 \cdot 0,9 = 0,104 \Rightarrow P(Neg) = 0,896$

$\Rightarrow P(Dis \mid Pos) = \frac{0,95 \cdot 0,1}{0,104} \approx 0,913 = 91,3\%$

- $P(NoDis \mid Pos) = 1 - 0,913 = 0,087 = 8,7\%$

data

- $P(Dis \mid Neg) = \frac{P(Neg \cap Dis)}{P(Neg)} = \frac{P(Neg \mid Dis)\, P(Dis)}{P(Neg)}$

luogo

$= \frac{0,05 \cdot 0,1}{0,896} \approx 0,0056 \doteq 0,56\%$

- $P(NoDis \mid Neg) = 1 - 0,0056 = 0,9944 = 99,44\%$

~~8~~ $P(Spam) = 0,4$ $\quad P(NoSpam) = 0,6$

$P(CTO \mid Spam) = 0,01$

$P(CTO \mid NoSpam) = 0,004$

$\to P(Spam \mid CTO) = \frac{P(CTO \cap Spam)}{P(CTO)} = \frac{P(CTO \mid Spam)\, P(Spam)}{P(CTO)}$

$\to P(CTO) = P(CTO \cap Spam) + P(CTO \cap NoSpam) =$

$= P(CTO \mid Spam)\, P(Spam) + P(CTO \mid NoSpam)\, P(NoSpam) =$

$= 0,01 \cdot 0,4 + 0,004 \cdot 0,6 = 0,0064$

$\Rightarrow P(Spam \mid CTO) = \frac{0,01 \cdot 0,4}{0,0064} = 0,625 = 62,5\%$

Rifaccio gli esercizi del terzo foglio di CFS:

1) $X \sim U_{NIF}(-1,1)$, $g(x) = \alpha + \beta x$ $(\beta \neq 0)$

$Y = g(X)$

$\rightarrow$ $g(x)$ è una funzione continuamente differenziabile $\Rightarrow$ è boreliana

$\Rightarrow$ Y è una v.a.

$$\rightarrow F_Y(y) = P(Y \leq y) = P(\alpha + \beta X \leq y) = P\left(X \leq \frac{y-\alpha}{\beta}\right) = F_X\left(\frac{y-\alpha}{\beta}\right) =$$

$$= \frac{\frac{y-\alpha}{\beta} + 1}{2} \mathbb{1}_{(-1,1)}\left(\frac{y-\alpha}{\beta}\right) + \mathbb{1}_{(1,+\infty)}\left(\frac{y-\alpha}{\beta}\right) =$$

$$= \frac{y-(\alpha-\beta)}{2\beta} \mathbb{1}_{(-1,1)}\left(\frac{y-\alpha}{\beta}\right) + \mathbb{1}_{(1,+\infty)}\left(\frac{y-\alpha}{\beta}\right)$$

$\frac{y-\alpha}{\beta} > 1$ per $y > \alpha + \beta$

$\frac{y-\alpha}{\beta} > -1$ per $y > \alpha - \beta$

$$\Rightarrow F_Y(y) = \frac{y-(\alpha-\beta)}{2\beta} \mathbb{1}_{(\alpha-\beta,\, \alpha+\beta)}(y) + \mathbb{1}_{(\alpha+\beta,\, +\infty)}(y)$$

$\Rightarrow Y \sim U_{NIF}(\alpha - \beta, \alpha + \beta)$

$\rightarrow$ Le v.a. uniformi sono assolutamente continue $\Rightarrow$ Y è assolutamente continua.

$\rightarrow$ Per dimostrare che il momento del ~~secondo~~ primo ordine di Y è finito, devo provare che $\int_\Omega |Y| \, dP < +\infty \iff$

$\int_{\mathbb{R}} |y| f_Y(y) \, d\mu_L(y) < +\infty \iff \int_{\mathbb{R}} |\alpha + \beta x| f_X(x) \, d\mu_L(x) < +\infty$

$\beta > 1 \Rightarrow \int \frac{1}{2}(-\alpha - \beta x)\, d\mu_L(x) = -\infty \quad \ldots \ +\infty$ √

Per dimostrare che il momento del secondo ordine di Y è finito, devo provare che $\int_\Omega |Y|^2\, dP < +\infty \iff$

$$\int_{\mathbb{R}} |y|^2 f_Y(y)\, d\mu_L(y) < +\infty \iff \int_{\mathbb{R}} |\alpha+\beta x|^2 f_X(x)\, d\mu_L(x) < +\infty$$

$$\int_{\mathbb{R}} (\alpha^2+\beta^2x^2+2\alpha\beta x)\, \frac{1}{2}\, \mathbb{1}_{(-1,1)}(x)\, d\mu_L(x) =$$

$$= \frac{1}{2}\int_{(-1,1)} (\alpha^2+\beta^2x^2+2\alpha\beta x)\, d\mu_L(x) = \frac{1}{2}\int_{-1}^{1} (\alpha^2+\beta^2x^2+2\alpha\beta x)\, dx =$$

$$= \frac{1}{2}\left[\alpha^2 x + \frac{\beta^2}{3}x^3 + \alpha\beta x^2\right]_{-1}^{1} =$$

$$= \frac{1}{2}\left(\alpha^2 + \frac{\beta^2}{3} + \cancel{\alpha\beta} + \alpha^2 + \frac{\beta^2}{3} - \cancel{\alpha\beta}\right) = \alpha^2 + \frac{\beta^2}{3} < +\infty$$ √

~~$\rightarrow D^2[Y]$ è finalmente $\alpha^2 + \beta^2/3$~~ $\leftarrow$ IL CAZZO

~~VERIFICA: $\frac{(\alpha+\beta-\alpha+\beta)^2}{12}$~~

$$E[Y^2] =$$

$$\rightarrow \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_L(y)$$

Y ASS. CONTINUA

$$f_Y(y) = F_Y'(y) = \frac{1}{2\beta}\, \mathbb{1}_{(\alpha-\beta,\, \alpha+\beta)}(y)$$

$$\Rightarrow E[Y^2] = \int_{\mathbb{R}} y^2 \frac{1}{2\beta}\, \mathbb{1}_{(\alpha-\beta,\, \alpha+\beta)}(y)\, d\mu_L(y) = \frac{1}{2\beta}\int_{\alpha-\beta}^{\alpha+\beta} y^2\, dy =$$

$$= \frac{1}{2\beta}\left[\frac{y^3}{3}\right]_{\alpha-\beta}^{\alpha+\beta} = \frac{1}{\ldots}\left[\alpha^3+\beta^3+3\alpha^2\beta+3\alpha\beta^2 - (\alpha^3-\beta^3-3\alpha^2\beta+3\alpha\beta^2)\right] =$$

data luogo

$$= \frac{1}{6\beta}\left(2\beta^3+6\alpha^2\beta\right) = \frac{\beta^2}{3}+\alpha^2$$

~~Corretto~~

~~3 I due integrali sono uguali~~

Per calcolarmi la varianza, mi serve prima $E[Y]$

$$E[Y] = \int_{\mathbb{R}} y f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y \frac{1}{2\beta} \mathbb{1}_{(\alpha-\beta,\alpha+\beta)}(y)\, d\mu_L(y) =$$

$$= \frac{1}{2\beta}\int_{\alpha-\beta}^{\alpha+\beta} y\, dy = \frac{1}{4\beta}\left[y^2\right]_{\alpha-\beta}^{\alpha+\beta} = \frac{1}{4\beta}\left(\alpha^2+\beta^2+2\alpha\beta-\alpha^2-\beta^2+2\alpha\beta\right) =$$

$$= \frac{1}{4\beta}(4\alpha\beta) = \alpha$$

$$\Rightarrow E^2[Y] = \alpha^2$$

$$\Rightarrow D^2[Y] = E[Y^2]-E^2[Y] = \beta^2/3$$

VERIFICA:

$E[Y] = \frac{a+b}{2} = \frac{\alpha+\beta+\alpha-\beta}{2} = \alpha$ ✓

$D^2[Y] = \frac{(b-a)^2}{12} = \frac{(\alpha+\beta-\alpha+\beta)^2}{12} = \frac{4\beta^2}{12} = \beta^2/3$ ✓

Perfect

2) $X \sim U_{NIF}(-1,1)$ $\quad g(x)=|x|$ $\quad Y = g(X)$

data luogo

$\rightarrow g(x)$ è continua $\Rightarrow g(x)$ è boreliana $\Rightarrow Y$ è una v.a.

$$\rightarrow F_Y(y) = P(Y\le y) = P(|X|\le y) = \cancel{P(-y\le X\le y)}$$

CASO $y<0 \Rightarrow F_Y(y) = 0$

CASO $y\ge 0 \Rightarrow F_Y(y) = P(-y\le X\le y) = \int_{[-y,y]} f_X(u)\, d\mu_L(u) =$

$$= \int_{[-y,y]} \mathbb{1}_{(-1,1)}(u)\, \tfrac{1}{2}\, d\mu_L(u) =$$

SOTTOCASO $0\le y<1 \Rightarrow F_Y(y) = \frac{1}{2}\int_{[-y,y]} d\mu_L(u) = \frac{1}{2}\int_{-y}^{y} du =$

$$= \tfrac{1}{2}[u]_{-y}^{y} = \tfrac{1}{2}(y+y) = y$$

SOTTOCASO $y\ge 1 \Rightarrow F_Y(y) = \frac{1}{2}\int_{[-1,1]} d\mu_L(u) = \frac{1}{2}[u]_{-1}^{1} = 1$

$$\Rightarrow F_Y(y) = y\,\mathbb{1}_{(0,1)}(y) + \mathbb{1}_{(1,+\infty)}(y) = \begin{cases} 0 & \text{se } y<0 \\ y & \text{se } 0\le y<1 \\ 1 & \text{se } y\ge 1 \end{cases}$$

$\Rightarrow Y \sim U_{NIF}(0,1)$

$\rightarrow$ Poiché le v.a. uniformi sono ass. continue, Y è ass. continua.

$\rightarrow$ Verifico che $\int_{\Omega} |Y|^2\, dP < +\infty \iff \int_{\mathbb{R}} |y|^2 f_Y(y)\, d\mu_L(y) < +\infty$

dove $f_Y(y) = \mathbb{1}_{(0,1)}(y)$

$$\Rightarrow \int_{\mathbb{R}} |y|^2 \mathbb{1}_{(0,1)}(y)\, d\mu_L(y) = \int_{(0,1)} y^2\, d\mu_L(y) = \int_0^1 y^2\, dy = \frac{1}{3}\left[y^3\right]_0^1 = 1/3 < +\infty$$

$\Rightarrow \exists$ momento del secondo ordine di Y

$\Rightarrow \exists$ momento del primo ordine di Y

$$E[Y^2] = \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_L(y) = 1/3$$

$$E[Y] = \int_{\mathbb{R}} y f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y \mathbb{1}_{(0,1)}(y)\, d\mu_L(y) = \int_0^1 y\, dy = \frac{1}{2}[y^2]_0^1 = 1/2$$

$$D^2[Y] = E[Y^2] - E^2[Y] = 1/3 - 1/4 = 1/12$$

3) $X \sim \text{UNIF}(-1,1)$ $\quad g(x) = x^2 \quad Y = g(X)$

$\rightarrow g(x)$ è una funzione continuamente differenziabile

$\Rightarrow g(x)$ è boreliana $\Rightarrow Y$ è una v.a.

$\rightarrow F_Y(y) = P(Y \leq y) = P(X^2 \leq y) =$

CASO $y<0 \Rightarrow F_Y(y) = 0$

CASO $y \geq 0 \Rightarrow F_Y(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{(-\sqrt{y},\sqrt{y})} f_X(u)\, d\mu_L(u) =$

$$= \int_{(-\sqrt{y},\sqrt{y})} 1/2\, \mathbb{1}_{(-1,1)}(u)\, d\mu_L(u) =$$

SOTTOCASO $0 \leq y < 1 \Rightarrow \int_{(-\sqrt{y},\sqrt{y})} 1/2\, d\mu_L(u) = \int_{-\sqrt{y}}^{\sqrt{y}} 1/2\, du = 1/2[u]_{-\sqrt{y}}^{\sqrt{y}} =$

$$= 1/2(\sqrt{y}+\sqrt{y}) = \sqrt{y}$$

SOTTOCASO $y \geq 1 \Rightarrow \int_{(-1,1)} 1/2\, d\mu_L(u) = 1/2[u]_{-1}^{1} = 1$

$\Rightarrow F_Y(y) = \sqrt{y}\, \mathbb{1}_{(0,1)}(y) + \mathbb{1}_{(1,+\infty)}(y)$

$$F_Y'(y) = \frac{1}{2\sqrt{y}} \mathbb{1}_{(0,1)}(y)$$

$$\rightarrow \int_{(-\infty,y]} F_Y'(u)\, d\mu_L(u) = \int_{(-\infty,y]} \frac{1}{2\sqrt{y}} \mathbb{1}_{(0,1)}(y)\, d\mu_L(u) =$$

CASO $y<0 \Rightarrow 0$

CASO $0 \leq y < 1 \Rightarrow \int_{[0,y]} \frac{1}{2\sqrt{y}}\, d\mu_L(u) = \int_0^y \frac{1}{2} u^{-1/2}\, du = 1/2\left[\frac{u^{1/2}}{1/2}\right]_0^y = \sqrt{y}$

CASO $y \geq 1 \Rightarrow \int_{[0,1]} \frac{1}{2\sqrt{u}}\, d\mu_L(u) = 1/2\left[\frac{u^{1/2}}{1/2}\right]_0^1 = 1$

$\Rightarrow \int_{(-\infty,y]} F_Y'(u)\, d\mu_L(u) = F_Y(y) \Rightarrow Y$ è assolutamente continua.

$\rightarrow$ Verifico che $\int_\Omega |Y|^2\, dP < +\infty \Leftrightarrow \int_{\mathbb{R}} |y|^2 f_Y(y)\, d\mu_L(y) < +\infty \Leftrightarrow$

$$\int_{\mathbb{R}} |x^2|^2 f_X(x)\, d\mu_L(x) < +\infty$$

$$\int_{\mathbb{R}} |x^2|^2 \mathbb{1}_{(-1,1)}(x) \cdot 1/2\, d\mu_L(x) = \int_{(-1,1)} x^4 \cdot 1/2\, d\mu_L(x) = 1/2 \int_{-1}^{1} x^4\, dx =$$

$$= \frac{1}{10}[x^5]_{-1}^{1} = \frac{1}{10}(1+1) = 1/5 \quad < +\infty \quad \checkmark$$

$\Rightarrow \exists$ momento di ordine 2 di $Y \Rightarrow \exists$ momento di ordine 1 di Y

$$E[Y^2] = \int_{\mathbb{R}} (x^2)^2 f_X(x)\, d\mu_L(x) = 1/5$$

$$E[Y] = \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} x^2 \cdot 1/2\, \mathbb{1}_{(-1,1)}(x)\, d\mu_L(x) = 1/2 \int_{(-1,1)} x^2\, d\mu_L(x)$$

data

$$= \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{6}\left[x^3\right]_{-1}^{1} = \frac{1}{6}(1+1) = \frac{1}{3}$$

luogo

$$D^2[Y] = E[Y^2] - E^2[Y] = \frac{1}{5} - \frac{1}{9} = \frac{9-5}{45} = \frac{4}{45}$$

4) $X \sim \mathrm{Unif}(-1,1)$ $\qquad g(x) = \begin{cases} 0 & \text{se } x \le 0 \\ x^2 & \text{se } x > 0 \end{cases}$

$Y = g(X)$

→ $g(x)$ è una funzione continua in $\mathbb{R}$ ⇒ g è boreliana ⇒ Y è una v.a.

→ $P(Y \le y) =$

CASO $y<0 \Rightarrow P(Y\le y) = 0$

CASO $y \ge 0 \Rightarrow P(Y \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) =$

$$= \begin{cases} \frac{\sqrt{y}+1}{2} & \text{se } 0 \le y < 1 \\ 1 & \text{se } y \ge 1 \end{cases}$$

⇝ VERIFICA: $P(X \le \sqrt{y}) = \int_{(-\infty,\sqrt{y}]} f_X(u)\,d\mu_L(u) = \int_{(-\infty,\sqrt{y}]} \mathbb{1}_{(-1,1)}(u)\,\frac{1}{2}\,d\mu_L(u) =$

$0 \le y < 1$: $= \int_{[-1,\sqrt{y}]} \frac{1}{2}\,d\mu_L(u) = \frac{1}{2}\int_{-1}^{\sqrt{y}} du = \frac{1}{2}[u]_{-1}^{\sqrt{y}} = \frac{1}{2}(\sqrt{y}+1)$

$y \ge 1$: $= \int_{[-1,1]} \frac{1}{2}\,d\mu_L(u) = \frac{1}{2}[u]_{-1}^{1} = \frac{1}{2}(1+1) = 1$ ✓

⇒ $F_Y(y) = \begin{cases} 0 & \text{se } y<0 \\ \frac{1}{2}(\sqrt{y}+1) & \text{se } 0 \le y < 1 \\ 1 & \text{se } y \ge 1 \end{cases} = \frac{1}{2}(\sqrt{y}+1)\,\mathbb{1}_{(0,1)}(y) + \mathbb{1}_{(1,\infty)}(y)$

data

luogo

⇒ $f_Y(y) = \frac{1}{4\sqrt{y}}\,\mathbb{1}_{(0,1)}(y)$

→ $\int_{(-\infty,y]} \frac{1}{4\sqrt{y}}\,\mathbb{1}_{(0,1)}(y)\,d\mu_L(y) =$

$y<0$: $= 0$

$0 \le y < 1$: $\int_{[0,y]} \frac{1}{4\sqrt{y}}\,d\mu_L(y) = \int_0^y \frac{1}{4} u^{-1/2}\,du = \frac{1}{4}\left[\frac{\sqrt{u}}{1/2}\right]_0^y = \frac{1}{2}(\sqrt{y})$

$y \ge 1$: $\int_{[0,1]} \frac{1}{4\sqrt{u}}\,d\mu_L(u) = \frac{1}{4}\left[\frac{\sqrt{u}}{1/2}\right]_0^1 = \frac{1}{2}$

⇒ $\int_{(-\infty,y]} \frac{1}{4\sqrt{u}}\,\mathbb{1}_{(0,1)}(u)\,d\mu_L(u) \ne F_Y(y)$ ⇒ Y non è assolutam. continua.

Verifico se $\int_\Omega Y^2\,dP < +\infty \iff \int_{\mathbb{R}} y^2 f_Y(y)\,d\mu_L(y) < +\infty$

$\iff \int_{\mathbb{R}} |g(x)|^2 f_X(x)\,d\mu_L(x) < +\infty$

⇝ $\int_{\mathbb{R}} y^2 \frac{1}{4\sqrt{y}}\,\mathbb{1}_{(0,1)}(y)\,d\mu_L(y) = \int_{[0,1]} \frac{1}{4} y^{3/2}\,d\mu_L(y) = \frac{1}{4}\int_0^1 y^{3/2}\,dy =$

$= \frac{1}{4}\left[\frac{y^{5/2}}{5/2}\right]_0^1 = \frac{1}{10} < +\infty$ ✓

⇒ ∃ momento di ordine 1 per Y; ∃ momento di ordine 2 per Y

⇝ $E[Y^2] = \frac{1}{10}$

SI POTEVA FARE MA SFRUTTANDO g(X)

data

luogo

$$E[Y] = \int_{\mathbb{R}} y \frac{1}{4\sqrt{y}} 1\!\!1_{(0,1)}(y)\, d\mu_L(y) =$$

$$= \int_{[0,1]} \tfrac{1}{4} y^{1/2} d\mu_L(y) = \tfrac{1}{4}\int_0^1 y^{1/2} dy = \tfrac{1}{4}\left[\frac{y^{3/2}}{3/2}\right]_0^1 = \tfrac{1}{6}$$

$$D^2[Y] = E[Y^2] - E^2[Y] = \tfrac{1}{10} - \tfrac{1}{36} = \frac{18-5}{180} = \frac{13}{180}$$

5) $X \sim \mathrm{Unif}(-1,1)$ $\qquad g(x) := x^2 - 2x$ $\qquad Y := g(X)$

$\rightarrow g(x)$ è continuamente differenziabile $\Rightarrow g^{(x)}$ è boreliana

$\Rightarrow Y$ è una v.a.

$\rightarrow F_Y(y) = P(Y \leq y) = P(X^2 - 2X \leq y) = P(X^2 - 2X - y \leq 0) =$

$$X_1 = \frac{1 \pm \sqrt{1+y}}{1} = \left\langle \begin{matrix} 1+\sqrt{1+y} \\ 1-\sqrt{1+y} \end{matrix} \right. \qquad (\text{con } y \geq -1)!$$

$$= P(1-\sqrt{1+y} \leq X \leq 1+\sqrt{1+y}) =$$

$$= \int_{[1-\sqrt{1+y},\, 1+\sqrt{1+y}]} f_X(u)\, d\mu_L(u) = \int_{[1-\sqrt{1+y},\, 1+\sqrt{1+y}]} 1\!\!1_{(-1,1)}(u) \cdot \tfrac{1}{2}\, d\mu_L(u) =$$

$$\rightarrow 1-\sqrt{1+y} \leq -1 \Rightarrow \sqrt{1+y} \geq 2 \Rightarrow 1+y \geq 4 \Rightarrow y \geq 3$$

$y \geq 3$: $\displaystyle = \int_{[-1,1]} \tfrac{1}{2}\, d\mu_L(u) = \tfrac{1}{2}[u]_{-1}^{1} = \tfrac{1}{2}(1+1) = 1$

$-1 \leq y < 3$: $\displaystyle = \int_{[1-\sqrt{1+y},\,1]} \tfrac{1}{2}\, d\mu_L(u) = \tfrac{1}{2}[u]_{1-\sqrt{1+y}}^{1} = \tfrac{1}{2}(1-1+\sqrt{1+y}) = \tfrac{1}{2}\sqrt{1+y}$

$y < -1$: $= 0$

data

luogo

$$\Rightarrow F_Y(y) = \tfrac{1}{2}\sqrt{1+y}\, 1\!\!1_{(-1,3)}(y) + 1\!\!1_{(3,+\infty)}(y)$$

$$\Rightarrow F_Y'(y) = \frac{1}{4}\frac{1}{\sqrt{1+y}} 1\!\!1_{(-1,3)}(y)$$

$$\Rightarrow \int_{(-\infty,y]} F_Y'(y)\, d\mu_L(u) = \int_{(-\infty,y]} \tfrac{1}{4}(1+y)^{-1/2} 1\!\!1_{(-1,3)}(y)\, d\mu_L(u) =$$

$y < -1$: $= 0$

$-1 \leq y < 3$: $\displaystyle = \int_{[-1,y]} \tfrac{1}{4}(1+u)^{-1/2} d\mu_L(u) = \int_{-1}^{y} \tfrac{1}{4}(1+u)^{-1/2} du = \tfrac{1}{4}\left[\frac{(1+u)^{1/2}}{1/2}\right]_{-1}^{y} =$

$= \tfrac{1}{2}[\sqrt{1+y}]$

$y \geq 3$: $\displaystyle = \int_{[-1,3]} \tfrac{1}{4}(1+u)^{-1/2} d\mu_L(u) = \tfrac{1}{2}\left[(1+u)^{1/2}\right]_{-1}^{3} = \tfrac{1}{2}\sqrt{1+3} = 1$

$$\Rightarrow \int_{(-\infty,y]} F_Y'(y)\, d\mu_L(u) = F_Y(y) \Rightarrow Y \text{ è assolutamente continua.}$$

$\rightarrow$ Verifico se $\displaystyle \int_{\Omega} Y^2 dP < +\infty \Leftrightarrow \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_L(y) < +\infty$

$$\Leftrightarrow \int_{\mathbb{R}} (x^2-2x)^2 f_X(x)\, d\mu_L(x) < +\infty$$

$$\rightarrow \int_{\mathbb{R}} (x^2-2x)^2 \cdot \tfrac{1}{2} 1\!\!1_{(-1,1)}(x)\, d\mu_L(x) = \tfrac{1}{2}\int_{[-1,1]} (x^4+4x^2-4x^3)\, d\mu_L(x) =$$

$$= \tfrac{1}{2}\int_{-1}^{1} (x^4 - 4x^3 + 4x^2)\, dx = \tfrac{1}{2}\left[\frac{x^5}{5} - x^4 + \frac{4}{3}x^3\right]_{-1}^{1} =$$

$$= \tfrac{1}{2}\left(\tfrac{1}{5} - 1 + \tfrac{4}{3} + \tfrac{1}{5} + 1 + \tfrac{4}{3}\right) = \tfrac{1}{5} + \tfrac{4}{3} = \frac{3+20}{15} = \frac{23}{15} < +\infty$$

data

luogo

$\Rightarrow \exists$ momento di ordine 1 e di ordine 2 per Y.

$\Rightarrow E[Y^2] = \frac{23}{15}$

$\rightsquigarrow E[Y] = \int_{\mathbb{R}} (x^2-2x)\, \frac{1}{2}\, \mathbb{1}_{(-1,1)}(x)\, d\mu_1(x) =$

$= \frac{1}{2}\int_{-1}^{1} (x^2-2x)\, dx = \frac{1}{2}\left[\frac{x^3}{3} - x^2\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} - 1 + \frac{1}{3} + 1\right) = \frac{1}{3}$

$\rightsquigarrow D^2[Y] = E[Y^2] - E^2[Y] = \frac{23}{15} - \frac{1}{9} = \frac{69-5}{45} = \frac{64}{45}$

6) $X \sim Exp(1)$ $\qquad g(x) = 1 - e^{-x}$ $\qquad Y = g(X)$

$\rightarrow$ $1-e^{-x}$ è una funzione continuamente differenziabile $\Rightarrow$ è boreliana $\Rightarrow$ Y è una variabile aleatoria.

$\rightarrow F_Y(y) = P(Y \leq y) = P(1 - e^{-X} \leq y) = P(e^{-X} \geq 1-y) =$

$y \geq 1$: $= 1$

$y < 1$: $= P(-X \geq \ln(1-y)) = P\left(X \leq \ln\left(\frac{1}{1-y}\right)\right) =$

$= \int_{\left(-\infty, \ln\left(\frac{1}{1-y}\right)\right]} f_X(u)\, d\mu_1(u) = \int_{\left(-\infty, \ln\left(\frac{1}{1-y}\right)\right]} \mathbb{1}_{(0,+\infty)}(u)\, e^{-u}\, d\mu_1(u) =$

$1-y<1 \Leftrightarrow y>0$: $\int_{\left[0, \ln\left(\frac{1}{1-y}\right)\right]} e^{-u}\, d\mu_1(u) = -\int_0^{\ln\left(\frac{1}{1-y}\right)} -e^{-u}\, du = -\left[e^{-u}\right]_0^{\ln\left(\frac{1}{1-y}\right)} =$

$= -e^{\ln(1-y)} + 1 = -1 + y + 1 = y$

$y \leq 0$: $= 0$

$\Rightarrow F_Y(y) = \begin{cases} 0 & \text{se } y<0 \\ y & \text{se } 0 \leq y < 1 \\ 1 & \text{se } y \geq 1 \end{cases}$

$\Rightarrow Y \sim U_{NIF}(0,1) \Rightarrow f_Y(y) = \mathbb{1}_{(0,1)}(y)$

$\rightarrow$ Poiché le v.a. uniformi sono assolutamente continue, anche Y è assolutamente continua.

$\rightarrow$ Verifico che $\int_{\Omega} Y^2\, dP < +\infty \Leftrightarrow \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_1(y) < +\infty$

$\Leftrightarrow \int_{\mathbb{R}} (1-e^{-x})^2 f_X(x)\, d\mu_1(x) < +\infty$

$\rightsquigarrow \int_{\mathbb{R}} y^2\, \mathbb{1}_{(0,1)}(y)\, d\mu_1(y) = \int_0^1 y^2\, dy = \frac{1}{3}\left[y^3\right]_0^1 = \frac{1}{3} < +\infty$ ✓

$\Rightarrow \exists$ momento di ordine 2 di Y; $\exists$ momento di ordine 1 di Y.

$E[Y^2] = \frac{1}{3}$

$\rightsquigarrow E[Y] = \int_{\mathbb{R}} y f_Y(y)\, d\mu_1(y) = \int_{\mathbb{R}} y\, \mathbb{1}_{(0,1)}(y)\, d\mu_1(y) = \int_0^1 y\, dy = \frac{1}{2}\left[y^2\right]_0^1 = \frac{1}{2}$

$\rightsquigarrow D^2[Y] = E[Y^2] - E^2[Y] = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

IV FOGLIO

4) $X \sim Geom\left(\frac{1}{10}\right)$

Devo innanzitutto provare che $\sum_{n=0}^{+\infty} |x_n|^2 p_n < +\infty$

$\sum_{n=0}^{+\infty} n^2 \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^n \underset{\substack{m=n+1 \\ n=m-1}}{=} \frac{1}{10}\sum_{m=1}^{+\infty} (m-1)^2 \left(\frac{9}{10}\right)^{m-1} = \frac{1}{9}\sum_{m=1}^{+\infty} (m^2-2m+1)\left(\frac{9}{10}\right)^m$

data $= \frac{1}{9}\sum_{m=1}^{+\infty}\left(\frac{9}{10}\right)^m - \frac{2}{9}\sum_{m=1}^{+\infty} m\left(\frac{9}{10}\right)^m + \frac{1}{9}\sum_{m=1}^{+\infty} m^2\left(\frac{9}{10}\right)^m =$

luogo

$= \frac{1}{9}\sum_{m=0}^{+\infty}\left(\frac{9}{10}\right)^m - \frac{1}{9}\left(\frac{9}{10}\right)^0 - \frac{2}{9}\frac{9/10}{1/100} + \frac{1}{9}\frac{9/10 \cdot 19/10}{1/1000} =$

$= \frac{1}{9}\frac{1}{1/10} - \frac{1}{9} - \frac{2}{10}\cdot 100 + \frac{19}{100}\cdot 1000 =$

$= 1 - 20 + 190 = 171 < +\infty$ √

$\rightsquigarrow E[X^2] = 171$

$\rightsquigarrow E[X] = \sum_{n=0}^{+\infty} x_n p_n = \sum_{m=0}^{+\infty} m \cdot \frac{1}{10}\left(\frac{9}{10}\right)^m \overset{\substack{m=m+1\\ m=m-1}}{=} \frac{1}{10}\sum_{m=1}^{+\infty}(m-1)\left(\frac{9}{10}\right)^{m-1} =$

$= \frac{1}{9}\sum_{m=1}^{+\infty} m\left(\frac{9}{10}\right)^m - \frac{1}{9}\sum_{m=0}^{+\infty}\left(\frac{9}{10}\right)^m + \frac{1}{9}\left(\frac{9}{10}\right)^0 =$

$= \frac{1}{9}\frac{9/10}{1/100} - \frac{1}{9}\cdot\frac{1}{1/10} + \frac{1}{9} = \frac{1}{10}\cdot 100 - \frac{10}{9} + \frac{1}{9} = 9$

$\rightsquigarrow \mathcal{D}^2[X] = E[X^2] - E^2[X] = 171 - 81 = 90$

VERIFICA:

$E[X] = \frac{1}{p} - 1 = 10 - 1 = 9$

$\mathcal{D}^2[X] = \frac{1-p}{p^2} = \frac{9/10}{1/100} = \frac{9}{10}\cdot 100 = 90$ √

→ Tchebischev per $P(0<X<23)$

$0 < E[X] < 23 \Rightarrow P(0<X<23) \geq 1 - \frac{\mathcal{D}^2[X]}{K^2}$

dove $K = \min\{E[X], 23 - E[X]\} = \min\{9, 14\} = 9$

$\Rightarrow P(0<X<23) \geq 1 - \frac{90}{81}$ Grazie al cazzo

data

luogo

5) X = somma dei valori usciti in 100 lanci di un dado.

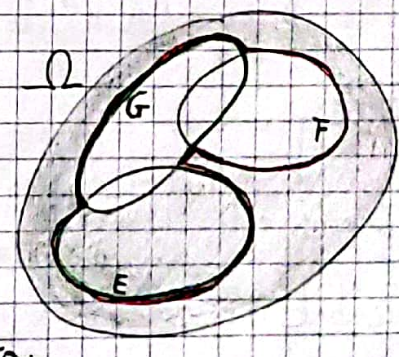
$E[X]$ dovrebbe essere 350

$Var(X) = E[(X-E[X])^2] = \sum_{K=100}^{600}(K - E[X])^2 \cdot P(X=K) =$

$= \sum_{K=100}^{600}(K-350)^2 \cdot p_K$ Per ora metto da parte

22/11/2019 1)

- $P(E\cap G) = P(E)P(G)$ ← sfruttato
- $P(F\cap G) = P(F)P(G)$ ← sfruttato
- $P(E\cap F) = 0$
- $P(E\cup G) = a \Rightarrow P(E) + P(G) - P(E)P(G) = a$ $1 - P(F) - P(G) + P(F)P(G)$
- $P(F^c\cap G^c) = b \xrightarrow{F,G \text{ indip.}} P(F^c)P(G^c) = b \Rightarrow (1-P(F))(1-P(G)) = b$
- $P(E\cup F\cup G) = c \Rightarrow P((E\cup F)\cup G) = c \Rightarrow; P(F\cup(E\cup G)) = c$ X
- $a - b - c \neq 0$



- $E\cup G$
- $F^c\cap G^c$
- $E\cup F\cup G$

$P(E\cup F\cup G) = P(G) + P(E) + P(F) - P(E\cap G) - P(E\cap$

data

luogo

$$\begin{cases} P(E)+P(G)-P(E)P(G)=a \\ P(F)P(G)-P(F)-P(G)=b-1 \\ P(E)+P(F)+P(G)-P(E)P(G)-P(F)P(G)=c \end{cases}$$

$$\begin{cases} P(E)+P(G)-P(E)P(G)=a \\ P(F)P(G)-P(F)-P(G)=b-1 \\ -P(G)=c+b-1-a \end{cases}$$

$$\begin{cases} \cancel{P(E)}+1+\cancel{a}-b-c-\cancel{P(E)}-aP(E)+bP(E)+cP(E)-\cancel{a}=0 \\ \cancel{P(F)}+aP(F)-bP(F)-cP(F)-\cancel{P(F)}\cancel{-1}-a+\cancel{b}+c-\cancel{b}+\cancel{1}=0 \\ P(G)=1+a-b-c \end{cases}$$

$$\begin{cases} -P(E)(a-b-c)=-1+b+c \\ P(F)(a-b-c)=a-c \\ P(G)=1+a-b-c \end{cases}$$

$$\begin{cases} P(E)=\dfrac{1-b-c}{a-b-c} \\ P(F)=\dfrac{a-c}{a-b-c} \\ P(G)=1+a-b-c \end{cases}$$

Molto bene :)

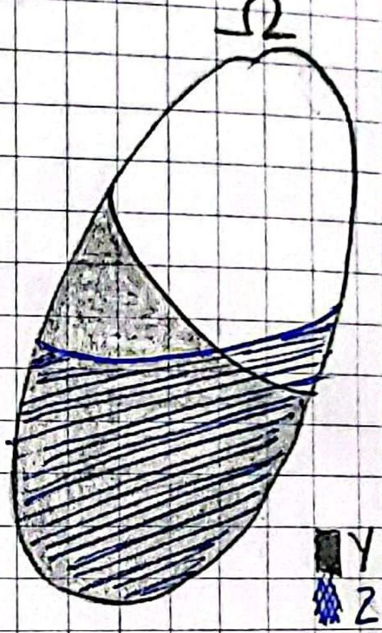
data ______

luogo ______

9) "The cab is yellow" $= Y$
"The cab is not yellow" $= Y^c$

"The eyewitness says the cab is yellow" $= Z$

"The eyewitness says the cab is not yellow" $= Z^c$



$P(Y) = 1/5$ $\qquad$ $P(Y^c) = 4/5$

$\cancel{P((Y\cap Z)\cup(Y^c\cap Z^c)) = 4/5}$

$P(Y|Z) = ?$ $\qquad$ $P((Y\cap Z)\cup(Y^c\cap Z^c)) = 4/5$

$$P(Y|Z) = \frac{P(Z\cap Y)}{P(Z)} = \frac{P(Z|Y)\,P(Y)}{P(Z)}$$

$$P(Z) = P(Z\cap Y) + P(Z\cap Y^c) = P(Z|Y)P(Y) + P(Z|Y^c)P(Y^c)$$

$$P((Y\cap Z)\cup(Y^c\cap Z^c)) = P(Y\cap Z) + P(Y^c\cap Z^c) = \frac{4}{5}$$

$$\begin{cases} P(Y) = 1/5 = P(Y\cap Z) + P(Y\cap Z^c) \\ P(Y\cap Z) + P(Y^c\cap Z^c) = \frac{4}{5} \\ P(Z) = P(Y\cap Z) + P(Y^c\cap Z) \\ P(Y\cap Z) + P(Y\cap Z^c) + P(Y^c\cap Z) + P(Y^c\cap Z^c) = 1 \\ \cancel{P(Y\cap Z) + P(Y^c\cap Z) + P(Y\cap Z^c) = P(Y) + P(Z) - P(Y\cap Z)} \end{cases}$$

$$\begin{cases} P(Y \cap Z) + P(Y \cap Z^c) = 1/5 \\ P(Y \cap Z) + P(Y^c \cap Z^c) = 4/5 \\ P(Y \cap Z) + P(Y^c \cap Z) = P(Z) \\ P(Y \cap Z) + P(Y^c \cap Z) + P(Y \cap Z^c) + P(Y^c \cap Z^c) = \\ \cancel{2P(Y \cap Z) + P(Y^c \cap Z) + P(Y \cap Z^c) = P(Z) + 1/5} \end{cases}$$

$$\begin{cases} -2P(Y \cap Z) = -P(Z) \Rightarrow P(Y \cap Z) = \frac{1}{2}P(Z) \\ P(Y \cap Z^c) = 1/5 - 1/2\, P(Z) \\ P(Y^c \cap Z) = 1/2\, P(Z) \\ P(Y^c \cap Z^c) = 4/5 - 1/2\, P(Z) \end{cases}$$

$$\Downarrow$$

$$P(Y|Z) = \frac{P(Y \cap Z)}{P(Z)} = \frac{1/2\, P(Z)}{P(Z)} = \frac{1}{2}$$

No, l'informazione non è ~~~~ utile, dato che il testimone ha ragione con probabilità del 50%.

---

10) $P(H_1) = 1$ $\qquad P(D_k) = 1/5 \qquad K = 1, \dots, 5$

$P(H_2) = 2/3$

$P(H_3) = 1/2$ $\qquad P(D_k | H_{tot} = 5) = ?$ per $K = 1, \dots, 5$

$P(H_4) = 1/3$

$P(H_5) = 0$

data ____________

luogo ____________

$$P(D_2 \mid A_{TOT}=5) = \frac{P(A_{TOT}=5 \mid D_2)\, P(D_2)}{P(A_{TOT}=5)} = \frac{\frac{32}{243} \cdot \frac{1}{5}}{0,2334} \simeq 0,1128$$

$$P(D_1 \mid A_{TOT}=5) = \frac{P(A_{TOT}=5 \mid D_1)\, P(D_1)}{P(A_{TOT}=5)} = \frac{1 \cdot \frac{1}{5}}{0,2334} \simeq 0,8569$$

data $\rightarrow x \geq 1 \wedge y \geq 0 \Rightarrow 1-e^{-2y}$

luogo ↳ OK, ho preso l'esempio sbagliato

~~LA MIA VITA È PROPRIO TUTTA SBAGLIATA QUA~~

[redacted]

Tornando all'esercizio del 24/11/2019...

$$f(x,y) = Kxe^{-(x+y)} \mathbb{1}_{\mathbb{R}^2_+}(x,y) \quad \forall (x,y) \in \mathbb{R}^2$$

$\rightarrow$ Devo imporre che $\int_{\mathbb{R}^2} f(x,y)\, d\mu_L^2(x,y) = 1$

$$\int_{\mathbb{R}^2} f(x,y)\, d\mu_L^2(x,y) = \int_{\mathbb{R}^2} Kxe^{-(x+y)} \mathbb{1}_{\mathbb{R}^2_+}(x,y)\, d\mu_L^2(x,y) =$$

$$= \int_{\mathbb{R}^2_+} Kxe^{-(x+y)} d\mu_L^2(x,y) = K\int_0^{+\infty} xe^{-x} dx \int_0^{+\infty} e^{-y} dy =$$

$$= K\left[-e^{-y}\right]_0^{+\infty} \left\{ \left[-xe^{-x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-x} dx \right\} =$$

$$= K\left\{0 + \left[e^{-x}\right]_0^{+\infty}\right\} = K = 1 \quad \checkmark$$

data

luogo

$$\Rightarrow f(x,y) = xe^{-(x+y)} \mathbb{1}_{\mathbb{R}^2_+}(x,y)$$

$$\rightarrow F_{X,Y}(x,y) = \int_{(-\infty,x]\times(-\infty,y]} \hat{x}e^{-(\hat{x}+\hat{y})} \mathbb{1}_{\mathbb{R}^2_+}(\hat{x},\hat{y})\, d\mu_L(\hat{x},\hat{y}) =$$

- Se $x \leq 0 \vee y \leq 0 \Rightarrow = 0$
- Se $x > 0 \wedge y > 0 \Rightarrow = \int_{[0,x]\times[0,y]} \hat{x}e^{-(\hat{x}+\hat{y})} d\mu_L(\hat{x},\hat{y}) =$

$$= \int_0^x \hat{x}e^{-\hat{x}} d\hat{x} \int_0^y e^{-\hat{y}} d\hat{y} = \left[-e^{-\hat{y}}\right]_0^y \left\{ \left[-\hat{x}e^{-\hat{x}}\right]_0^x + \left[-e^{-\hat{x}}\right]_0^x \right\}$$

$$= (1-e^{-y})(-xe^{-x} + 1 - e^{-x}) = -xe^{-x} + 1 - e^{-x} + xe^{-(x+y)} - e^{-y} + e^{-(x+y)}$$

$$\Rightarrow F_{X,Y}(x,y) = \mathbb{1}_{\mathbb{R}^2_+}(x,y)\left(xe^{-(x+y)} - xe^{-x} + e^{-(x+y)} - e^{-x} - e^{-y} + 1\right)$$

$$\rightarrow \frac{\partial F^2}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(-xe^{-x}e^{-y} - e^{-x}e^{-y} + e^{-y}\right) \mathbb{1}_{\mathbb{R}^2_+}(x,y)$$

$$= \left[e^{-y}(-\cancel{e^{-x}} + xe^{-x}) + \cancel{e^{-y}e^{-x}}\right] \mathbb{1}_{\mathbb{R}^2_+}(x,y)$$

$$= xe^{-(x+y)} \mathbb{1}_{\mathbb{R}^2_+}(x,y) = f(x,y) \quad \checkmark$$

$$\rightarrow F_X(x) = \lim_{y\to+\infty} \mathbb{1}_{\mathbb{R}^2_+}(x,y)\left(xe^{-(x+y)} - xe^{-x} + e^{-(x+y)} - e^{-x} - e^{-y} + 1\right)$$

$$= (-xe^{-x} - e^{-x} + 1)\mathbb{1}_{\mathbb{R}_+}(x)$$

$$\rightarrow F_Y(y) = \lim_{x\to+\infty} \mathbb{1}_{\mathbb{R}^2_+}(x,y)\left(xe^{-(x+y)} - xe^{-x} + e^{-(x+y)} - e^{-x} - e^{-y} + 1\right)$$

$$= (1-e^{-y})\mathbb{1}_{\mathbb{R}_+}(y)$$

Verifico se data $F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$:

luogo $F_X(x)F_Y(y) = 1_{\mathbb{R}_+}(x)\,(-xe^{-x}-e^{-x}+1)\,1_{\mathbb{R}_+}(y)\,(1-e^{-y})$

$$= 1_{\mathbb{R}^2_+}(x,y)\left[-xe^{-x}+xe^{-(x+y)}-e^{-x}+e^{-(x+y)}+1-e^{-y}\right]$$

$\Rightarrow X,Y$ sono v.a. indipendenti

$$\Rightarrow f(x,y) = f_X(x)f_Y(y) \Rightarrow \begin{cases} f_X(x) = xe^{-x}1_{\mathbb{R}^+}(x) \\ f_Y(y) = e^{-y}1_{\mathbb{R}^+}(y) \end{cases}$$

(*) Qui userei la formula di $f_X(x), f_Y(y)$

$$\rightsquigarrow \frac{dF_X(x)}{dx} = \frac{d}{dx}(1-xe^{-x}-e^{-x})1_{\mathbb{R}_+}(x) =$$

$$= 1_{\mathbb{R}^+}(x)(-e^{-x}+xe^{-x}+e^{-x}) = xe^{-x}1_{\mathbb{R}^+}(x) = f_X(x) \checkmark$$

$$\rightsquigarrow \frac{dF_Y(y)}{dy} = \frac{d}{dy}(1-e^{-y})1_{\mathbb{R}_+}(y) = e^{-y}1_{\mathbb{R}_+}(y) = f_Y(y) \checkmark$$

$\exists f_X(x), f_Y(y) \Rightarrow X,Y$ sono ass. continue

$$\to E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} x^3 e^{-x} 1_{\mathbb{R}_+}(x)\, d\mu_L(x) =$$

$$= \int_{\mathbb{R}_+} x^3 e^{-x}\, d\mu_L(x) = \int_0^{+\infty} x^3 e^{-x}\, dx = \left[x^3e^{-x}\right]_0^{+\infty} + \int_0^{+\infty} 3x^2 e^{-x}\, dx$$

$$= \left[x^3e^{-x}\right]_{+\infty}^0 + \left[3x^2e^{-x}\right]_{+\infty}^0 + \int_0^{+\infty} 6xe^{-x}\, dx =$$

$$= \left[x^3e^{-x}\right]_{+\infty}^0 + \left[3x^2e^{-x}\right]_{+\infty}^0 + \left[6xe^{-x}\right]_{+\infty}^0 + \int_0^{+\infty} 6e^{-x}\, dx =$$

data luogo

$$= \left[x^3e^{-x}\right]_{+\infty}^0 + \left[3x^2e^{-x}\right]_{+\infty}^0 + \left[6xe^{-x}\right]_{+\infty}^0 + \left[6e^{-x}\right]_{+\infty}^0 =$$

$$= 0+0+0+6 = 6 \quad (<+\infty \ \checkmark)$$

$$E[X] = \int_{\mathbb{R}} x f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}_+} x^2 e^{-x}\, d\mu_L(x) = \int_0^{+\infty} x^2 e^{-x}\, dx =$$

$$= \left[x^2e^{-x}\right]_{+\infty}^0 + \int_0^{+\infty} 2xe^{-x}\, dx = \left[-2xe^{-x}\right]_0^{+\infty} + \int_0^{+\infty} 2e^{-x}\, dx =$$

$$= \left[2e^{-x}\right]_{+\infty}^0 = 2$$

$$D^2[X] = E[X^2] - E^2[X] = 6-4 = 2$$

$$\to E[Y^2] = \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y^2 e^{-y} 1_{\mathbb{R}^+}(y)\, d\mu_L(y) = \ldots = 2$$

$$\to E[Y] = \int_{\mathbb{R}} y f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y e^{-y} 1_{\mathbb{R}^+}(y)\, d\mu_L(y) =$$

$$= \int_{\mathbb{R}_+} y e^{-y}\, d\mu_L(y) = \int_0^{+\infty} y e^{-y}\, dy = \left[ye^{-y}\right]_{+\infty}^0 + \int_0^{+\infty} e^{-y}\, dy =$$

$$= \left[-e^{-y}\right]_0^{+\infty} = 1$$

$$D^2[Y] = E[Y^2] - E^2[Y] = 2-1 = 1$$

$\to Cov(X,Y) = E[XY] - E[X]E[Y] = 0$ perché X,Y sono indipendenti

$\to E[(X,Y)] = (E[X], E[Y])^T = (2,1)^T$

$\cancel{\to Var(X,Y) = \begin{bmatrix} D^2[X_1] & Cov(X_2,X_1) \\ Cov(X_1,X_2) & D^2[X_2] \end{bmatrix} = }$

$$\to Var(X,Y) = \begin{bmatrix} D^2[X] & Cov(Y,X) \\ Cov(X,Y) & D^2[Y] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Esercizio 1 del 27/01/2020:

$P(T_1) = 7/10$

$P(T_2 | T_1^c) = 4/10$

$$P(T_1 \cup T_2) = P(T_1) + P(T_2) - \overbrace{P(T_1 \cap T_2)}^{=0} =$$

$$= P(T_1) + [P(T_2 \cap T_1) + P(T_2 \cap T_1^c)] =$$

$$= P(T_1) + P(T_2 \cap T_1^c) = P(T_1) + P(T_2 | T_1^c) P(T_1^c) =$$

$$= 7/10 + 4/10 \cdot 3/10 = \frac{82}{100} = 0,82$$

$$P(T_1 | T_1 \cup T_2) = \frac{P(T_1 \cap (T_1 \cup T_2))}{P(T_1 \cup T_2)} = \frac{P(T_1)}{P(T_1 \cup T_2)} = \frac{7}{10} \cdot \frac{100}{82} = \frac{70}{82}$$

A = "The number observed on the upper face of a die is 4"

$P(A) = \frac{11}{36}$ $\quad$ $P(Sum=8) = \frac{5}{36}$

$\Rightarrow P(A)P(Sum=8) = \frac{55}{36^2}$

$P(A \cap Sum=8) = P((4,4)) = \frac{1}{36} \neq P(A)P(Sum=8)$

$\Rightarrow$ I 2 eventi non sono indipendenti.

Esercizio 2:

$f(x,y) = K e^{-(x^2 - xy + y^2/2)} \quad \forall (x,y) \in \mathbb{R}^2$

$\to$ Devo imporre $\int_{\mathbb{R}^2} f(x,y)\, d\mu_2(x,y) = 1$

$$\int_{\mathbb{R}^2} K e^{-(x^2-xy+y^2/2)} d\mu_2(x,y) = K \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} e^{-1/2(y-x)^2} e^{-x^2/2} dy =$$

$$= K \int_{-\infty}^{+\infty} e^{-x^2/2} dx \int_{-\infty}^{+\infty} e^{-1/2(y-x)^2} dy = K \int_{-\infty}^{+\infty} e^{-x^2/2} \sqrt{2\pi}\, dx =$$

$$= K \sqrt{2\pi}\sqrt{2\pi} = 2\pi K = 1 \Rightarrow K = \frac{1}{2\pi}$$

$$\to f(x,y) = \frac{1}{2\pi} e^{-(x^2-xy+y^2/2)} \quad \forall (x,y) \in \mathbb{R}^2$$

To be continued...

$$\to f_X(x) = \int_{\mathbb{R}} f(x,y)\, d\mu_1(y) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{-(x^2-xy+y^2/2)} d\mu_1(y) =$$

$$\cancel{= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-x^2} e^{+xy} e^{-y^2/2} dy = \frac{1}{2\pi} e^{-x^2} \int_{-\infty}^{+\infty} e^{xy - y^2/2} dy =}$$

$$\cancel{= \frac{1}{2\pi} e^{-x^2} [}$$

data

luogo

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(x^2-xy+y^2/2)}\,dy =$$

$$= \frac{1}{2\pi} e^{-x^2/2}\int_{-\infty}^{+\infty} e^{-1/2(y-x)^2}\,dy = \frac{1}{2\pi} e^{-x^2/2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} e$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,d\mu_L(x) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{-(x^2-xy+y^2/2)}\,d\mu_L(x) =$$

$$= \boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(x^2-xy+y^2/2)}\,dx} = \cancel{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-x^2/2} e^{-1/2(x-y)^2}\,dx}$$

$= \frac{1}{2\pi} e^{-y^2/2} \int_{-\infty} e \; e \; du =$

data

luogo

$$= \frac{1}{2\pi} e^{-y^2/2} \left\{ \left[-\frac{1}{y} e^{-uy} e^{-u^2} \right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{1}{y} e^{-uy} (-2u) e^{-u^2} du \right\} =$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(x^2 - xy + y^2/2)} dx = \frac{1}{2\pi} e^{-y^2/4} \int_{-\infty}^{+\infty} e^{-(x^2 - xy + y^2/4)} dx =$$

$$= \frac{1}{2\pi} e^{-y^2/4} \int_{-\infty}^{+\infty} e^{-(x - y/2)^2} dx = \frac{1}{2\pi} e^{-y^2/4} \cdot \sqrt{\pi} = \frac{1}{2\sqrt{\pi}} e^{-y^2/4}$$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2} \frac{y^2}{(\sqrt{2})^2}}$$

$X \sim N(0,1)$ $\qquad$ $Y \sim N(0,2)$ $\Rightarrow$ Si, X e Y sono gaussiane.

$\Rightarrow E[X] = 0 \qquad E[Y] = 0$

$D^2[X] = 1 \qquad D^2[Y] = 2$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\to E[XY] = \int_{\mathbb{R}^2} xy f(x,y) \, d\mu_L^2(x,y) = \int_{\mathbb{R}^2} \frac{1}{2\pi} xy \, e^{-(x^2 - xy + y^2/2)} d\mu_L^2(x,y)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} x \, dx \int_{-\infty}^{+\infty} y e^{-1/2(y-x)^2} e^{-x^2/2} dy = \frac{-1}{2\pi} \int_{-\infty}^{+\infty} x e^{-x^2/2} dx \int_{-\infty}^{+\infty} \left[(x-y) e^{-1/2(y-x)^2} + - x e^{-1/2(y-x)^2} \right] dy$$

$$= + \frac{1}{2\pi}\int_{-\infty}^{+\infty} x e^{-x^2/2}\, dx \left[\int_{-\infty}^{+\infty} (y-x) e^{-\frac{1}{2}(y-x)^2} dy + \int_{-\infty}^{+\infty} x e^{-\frac{1}{2}(y-x)^2} dy\right] =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, x e^{-x^2/2}\left\{\left[-e^{-\frac{1}{2}(y-x)^2}\right]_{-\infty}^{+\infty} + x\sqrt{2\pi}\right\} =$$

$$= \frac{1}{2\pi}\sqrt{2\pi}\int_{-\infty}^{+\infty} x^2 e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}}\left\{\left[-xe^{-x^2/2}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\sqrt{2\pi}\right) = 1$$

$\Rightarrow Cov(X,Y) = E[XY] - E[X]E[Y] = 1 - 0 = 1$

$\rightarrow$ X, Y son v.a. NON indipendenti.

Per verificare se $Z = (X,Y)$ è un vettore aleatorio gaussiano, considero $W = aX + bY$ e calcolo $f_W(w)$: $(a, b \neq 0)$

$$f_W(w) = \int_{\mathbb{R}} f\left(x, \frac{w}{b} - \frac{ax}{b}\right) \cdot \frac{1}{|b|}\, d\mu_L(x) =$$

$$= \frac{1}{|b|}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(x^2 - \frac{xw}{b} + \frac{ax^2}{b} + \frac{w^2 + a^2x^2 - 2axw}{2b^2}\right)}\, d\mu_L(x) =$$

$$= \frac{1}{2\pi|b|}\int_{-\infty}^{+\infty} e^{-\left[x^2\left(1+\frac{a}{b}+\frac{a^2}{2b^2}\right) - x\left(\frac{w}{b}+\frac{2aw}{b^2}\right) + \frac{w^2}{2b^2}\right]}\, dx =$$

$$= \frac{1}{2\pi|b|}\int_{-\infty}^{+\infty} e^{-\left[x^2\frac{2b^2+2ab+a^2}{2b^2} - x\frac{2bw+2aw}{2b^2} + \frac{w^2}{2b^2}\right]}\, dx =$$

$$= \frac{1}{2\pi|b|}\int_{-\infty}^{+\infty} e^{-\frac{1}{2b^2}\left[x^2(\overbrace{2b^2+2ab+a^2}^{\neq (a+b)^2}) - 2wx(a+b) + w^2\right]}\, dx$$

$a=1, b=1$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(5x^2 - 4wx + w^2)}\, dx = \qquad \sqrt{5}\cdot\frac{2}{5} \quad 2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(5x^2 - 4wx + \frac{4}{5}w^2\right)}\, e^{-\frac{1}{2}\cdot\frac{1}{5}w^2}\, dx =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{5}x - \frac{2}{\sqrt{5}}w\right)^2}\, e^{-\frac{1}{10}w^2}\, dx =$$

$$= \frac{1}{2\pi} e^{-\frac{1}{10}w^2}\int_{-\infty}^{+\infty} e^{-\frac{5}{2}(x-2w)^2}\, dx = \frac{1}{2\pi} e^{-\frac{1}{10}w^2}\sqrt{\frac{2}{5}\pi} =$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{5}}\, e^{-\frac{1}{2}\left(\frac{w}{\sqrt{5}}\right)^2} \quad \rightarrow \text{Sembra una Gaussiana}$$

$\hookrightarrow$ Temo che si tratti di un vett. aleatorio gaussiano ma non voglio perderci altro tempo.

Torno a 24/11/2019 | 3) $X \sim N(0,1)$ $\quad B \sim Ber(1/2)$

$Y = BX$ $\quad X, B$ indipendenti

(I) $P(Y \leq y) = P(BX \leq y) = P(BX \leq y \cap B=0) + P(BX \leq y \cap B=1)$

$= P(BX \leq y | B=0) P(B=0) + P(BX \leq y | B=1) P(B=1) =$

$= P(0 \leq y | B=0) \frac{1}{2} + P(X \leq y | B=1) \frac{1}{2} \underset{\uparrow B,X \text{ indip}}{=}$

$= P(y \geq 0 | B=0) \frac{1}{2} + P(X \leq y) \frac{1}{2} =$

$= P(y \geq 0) \frac{1}{2} + P(X \leq y) \frac{1}{2}$

→ Se $y < 0 \Rightarrow = \frac{1}{2} P(X \leq y)$

→ Se $y \geq 0 \Rightarrow = \frac{1}{2} + \frac{1}{2} P(X \leq y)$

$\Rightarrow P(Y \leq y) = \frac{1}{2} P(X \leq y) \mathbb{1}_{(-\infty,0)}(y) + \left[\frac{1}{2} + \frac{1}{2} P(X \leq y)\right] \mathbb{1}_{[0,+\infty)}(y)$

⇝ $P(Y \leq 0) = 0 + \frac{1}{2} + \frac{1}{2} P(X \leq 0) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

⇝ $\lim_{y \to 0^-} P(Y \leq y) = \lim_{y \to 0^-} \frac{1}{2} P(X \leq y) = \frac{1}{2} P(X \leq 0) = \frac{1}{4}$

$\Rightarrow F_Y(y) = P(Y \leq y)$ NON è continua $\Rightarrow$ NON può essere assolutam. continua

(II) $Cov(X,Y) = E[XY] - E[X]E[Y] = E[BX^2] - E[X]E[BX] =$

$\underset{\substack{\uparrow \\ B,X \text{ indip.} \\ B,X^2 \text{ indip.}}}{=} E[B]E[X^2] - E[B]E^2[X] = E[B](D^2[X]) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\Rightarrow X, Y$ sono correlate $\Rightarrow X, Y$ sono dipendenti

(III) $Cov(B,Y) = E[BY] - E[B]E[Y] = E[B^2 X] - E[B]E[BX] =$

$\underset{\substack{\uparrow \\ B,X \text{ indip.} \\ B^2,X \text{ indip.}}}{=} E[B^2]E[X] - E^2[B]E[X] =$

$= E[X] D^2[B] = 0 \quad \Rightarrow B, Y$ sono scorrelate

⇝ Provo a mostrare che $P(B \leq x_1, BX \leq x_2) \neq P(B \leq x_1) P(BX \leq x_2)$ per qualche x_1, x_2.

• $x_1 = x_2 = 0 \quad P(B \leq 0, BX \leq 0) = \frac{1}{2}$

$P(B \leq 0) P(BX \leq 0) = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} \neq P(B \leq 0, BX \leq 0)$

$\Rightarrow B, Y$ NON sono indipendenti.

(IV) No, $(X,Y)^T$ non è una distribuzione Gaussiana bivariata perché Y non è una v.a. Gaussiana.

Salto il punto (V) per ora.

Homework 16/11/2021: 1)

$f_{X_1,X_2}(x_1,x_2) := \mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2) \quad \forall (x_1,x_2) \in \mathbb{R}^2$

$Y = \min(X_1, X_2) \qquad Z = \max(X_1, X_2)$

~~$\rightarrow F_Y(y) = P(Y \leq y) = 1 - P(Y > y)$~~

Verifico se X_1, X_2 sono indipendenti:

$f_{X_1}(x_1) = \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_2) = \int_{\mathbb{R}} \mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2)\, d\mu_L(x_2) =$

data

luogo

$$= \int_{[0,1]} \mathbb{1}_{[0,1]}(x_1)\, d\mu_L(x_2) = \mathbb{1}_{[0,1]}(x_1) \int_0^1 dx_2 = \mathbb{1}_{[0,1]}(x_1)$$

$\rightsquigarrow$ Analogam. $f_{X_2}(x_2) = \mathbb{1}_{[0,1]}(x_2) \Rightarrow$ $X_1 \sim \text{UNIF}(0,1)$, $X_2 \sim \text{UNIF}(0,1)$

$$\Rightarrow f_{X_1}(x_1) f_{X_2}(x_2) = f_{X_1,X_2}(x_1,x_2) \implies X_1, X_2 \text{ sono indip.}$$

$$\rightsquigarrow F_Y(y) = P(Y \le y) = 1 - P(Y > y) \overset{X_1,X_2 \text{ indip.}}{=} 1 - P(X_1 > y)P(X_2 > y) =$$

$$= \begin{cases} 0 & \text{se } y < 0 \\ 1-(1-P(X_1 \le y))(1-P(X_2 \le y)) = 1-(1-y)^2 = 2y - y^2 & \text{se } 0 \le y \le 1 \\ 1 & \text{se } y > 1 \end{cases}$$

$$\Rightarrow F_Y(y) = (2y - y^2)\mathbb{1}_{[0,1]}(y) + \mathbb{1}_{(1,+\infty)}(y)$$

$$\rightsquigarrow F_Z(z) = P(Z \le z) \overset{X_1,X_2 \text{ indip.}}{=} P(X_1 \le z)P(X_2 \le z) =$$

$$= \begin{cases} 0 & \text{se } z < 0 \\ z^2 & \text{se } 0 \le z \le 1 \\ 1 & \text{se } z > 1 \end{cases}$$

$$\Rightarrow F_Z(z) = z^2 \mathbb{1}_{[0,1]}(z) + \mathbb{1}_{(1,+\infty)}(z)$$

$$F_{Y,Z}(y,z) = P(Y \le y, Z \le z) = P(\min\{X_1,X_2\} \le y, \max\{X_1,X_2\} \le z) =$$

$$= \begin{cases} 0 & \text{se } y < 0 \lor z < 0 \\ 1 & \text{se } y > 1 \land z > 1 \\ P(\min\{X_1,X_2\} \le y) & \text{se } 0 \le y \le 1 \land z > 1 \\ P(\max\{X_1,X_2\} \le z) & \text{se } y > 1 \land 0 \le z \le 1 \\ \cancel{P(\max\{X_1,X_2\} \le z)} & \cancel{\text{se } 0 \le y \le 1 \land 0 \le z \le 1 \land y \ge z} \\ P(\max\{X_1,X_2\} \le z \cap \min\{X_1,X_2\} \le \max\{X_1,X_2\}) & \text{se } 0 \le y \le 1 \land 0 \le z \le 1 \land y \ge z \\ P(\max\{X_1,X_2\} \le z \cap \min\{X_1,X_2\} \le y \cap \min\{X_1,X_2\} \le \max\{X_1,X_2\}) & \text{se } 0 \le y \le 1 \land 0 \le z \le 1 \land y < z \end{cases}$$

ASPE' CHE STO A FA' UN CASINO

$$F_{Y,Z}(y,z) = P(Y \le y, Z \le z) = P(X_1 \le \min\{y,z\}, X_2 \le \min\{y,z\}) + P(y < X_1 \le z, X_2 \le \min\{y,z\}) + P(X_1 \le \min\{y,z\}, y < X_2 \le z)$$

$$\underset{X_1,X_2 \text{ indip.}}{=} P(X_1 \le \min\{y,z\})P(X_2 \le \min\{y,z\}) + P(y < X_1 \le z)P(X_2 \le \min\{y,z\}) + P(X_1 \le \min\{y,z\})P(y < X_2 \le z) \overset{0 \le y \le 1,\ 0 \le z \le 1}{=}$$

$\to \boxed{y < z}$

$$= P(X_1 \le y)P(X_2 \le y) + P(y < X_1 \le z)P(X_2 \le y) + P(X_1 \le y)P(y < X_2 \le z) =$$

$$= y^2 + y \cdot (z-y) + y(z-y) = y^2 + 2yz - 2y^2 = 2yz - y^2 \ (*)$$

$\to \boxed{y \ge z}$

$$= P(X_1 \le z)P(X_2 \le z) + 0 + 0 = z^2$$

(*) Se avessimo avuto $z > 1 \Rightarrow F_{Y,Z}(y,z) = P(X_1 \le y)P(X_2 \le y) + P(X_1 > y)P(X_2 \le y) + P(X_1 \le y)P(X_2 > y) =$

$$= y^2 + y(1-y) + y(1-y) = 2y - y^2$$

data

luogo

data

luogo

$$\Rightarrow F_{Y,Z}(y,z) = \begin{cases} 0 & \text{se } y<0 \vee z<0 \\ 2yz-y^2 & \text{se } 0\le y\le z\le 1 \\ z^2 & \text{se } 0\le z<y \wedge z\le 1 \\ 2y-y^2 & \text{se } 0\le y\le 1<z \\ 1 & \text{se } y>1 \wedge z>1 \end{cases}$$

$$= \begin{cases} F_X(y)\left(2F_X(z)-F_X(y)\right) & \text{se } y\le z \\ F_X(z)^2 & \text{se } y>z \end{cases}$$

$$\Rightarrow F_{Y,Z}(y,z) = F_X(y)\left(2F_X(z)-F_X(y)\right)\mathbb{1}_{\{y\le z\}}(y,z) + F_X(z)^2\mathbb{1}_{\{y>z\}}(y,z)$$

DISTRIBUZIONI MARGINALI:

$$F_Y(y) = \lim_{z\to+\infty} F_{Y,Z}(y,z) = \lim_{z\to+\infty} F_X(y)\left(2F_X(z)-F_X(y)\right) =$$

$$= F_X(y)\left(2-F_X(y)\right) = 2F_X(y)-F_X(y)^2 =$$

$$= 2\left[\mathbb{1}_{[0,1]}(y)\cdot y+\mathbb{1}_{(1,+\infty)}(y)\right] - \left[\mathbb{1}_{[0,1]}(y)\cdot y+\mathbb{1}_{(1,+\infty)}(y)\right]^2$$

$$F_Z(z) = \lim_{y\to+\infty} F_{Y,Z}(y,z) = \lim_{y\to+\infty} F_X(z)^2 = F_X(z)^2 = \left[\mathbb{1}_{[0,1]}(z)\cdot z+\mathbb{1}_{(1,+\infty)}(z)\right]$$

$$\rightsquigarrow f_Y(y) = (2-2y)\mathbb{1}_{[0,1]}(y)$$

$$f_Z(z) = 2z\,\mathbb{1}_{[0,1]}(z)$$

$$E[Y] = \int_{\mathbb{R}} y f_Y(y)\,d\mu_L(y) = \int_{\mathbb{R}} (2y-2y^2)\mathbb{1}_{[0,1]}(y)\,d\mu_L(y) = \int_0^1 (2y-2y^2)\,dy =$$

$$= \left[y^2-\frac{2}{3}y^3\right]_0^1 = 1-\frac{2}{3} = \frac{1}{3}$$

data

luogo

$$E[Z] = \int_{\mathbb{R}} z f_Z(z)\,d\mu_L(z) = \int_{\mathbb{R}} 2z^2\mathbb{1}_{[0,1]}(z)\,d\mu_L(z) =$$

$$= \int_0^1 2z^2\,dz = \left[\frac{2}{3}z^3\right]_0^1 = \frac{2}{3}$$

---

2) $F(x_1,x_2) := \left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right)\mathbb{1}_{\mathbb{R}_+}(x_1)\mathbb{1}_{\mathbb{R}_+}(x_2) \quad \forall (x_1,x_2)\in\mathbb{R}^2$

- $\lim_{x_1\to+\infty}\lim_{x_2\to+\infty} F(x_1,x_2) = 1-0-0+0 = 1$ ✓
- $\lim_{x_1\to-\infty} F(x_1,x_2) = 0$ ✓
- $\lim_{x_2\to-\infty} F(x_1,x_2) = 0$ ✓
- $1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)} \geq 0 \quad \forall x_1,x_2\geq 0$

$\rightarrow$ FISSO $x_2\geq 0 \Rightarrow e^{-x_2}e^{-x_1}-e^{-x_1} \geq e^{-x_2}-1$

$\Rightarrow e^{-x_1}(e^{-x_2}-1) \geq e^{-x_2}-1 \Rightarrow$ ok se $x_2=0$

$\overset{x_2\neq 0}{\Rightarrow} e^{-x_1} \leq 1 \quad \forall x_1\geq 0$ ✓

$\rightarrow$ ANALOGAM. FISSANDO $x_1\geq 0$ ✓

- $F(x_1,x_2)$ NON decrescente

$\rightarrow$ FISSO $x_2\geq 0 \Rightarrow \frac{\partial F}{\partial x_1} = e^{-x_1}-e^{-x_2}e^{-x_1} = e^{-x_1}(1-e^{-x_2}) \geq 0 \;\forall x_2\geq 0$ ✓

$\rightarrow$ ANALOGAM. FISSANO $x_1\geq 0$

In definitiva, $F(x_1,x_2)$ è la funz. distribuz. di un vettore aleatorio (X_1,X_2)

$$F_{X_1}(x_1) = \lim_{x_2\to+\infty} \left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right)\mathbb{1}_{\mathbb{R}_+}(x_1)\mathbb{1}_{\mathbb{R}_+}(x_2) \overset{\text{data luogo}}{=} (1-e^{-x_1})\mathbb{1}_{\mathbb{R}_+}(x_1) \Rightarrow X_1 \sim esp(1)$$

$$F_{X_2}(x_2) = \lim_{x_1\to+\infty} \left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right)\mathbb{1}_{\mathbb{R}_+}(x_1)\mathbb{1}_{\mathbb{R}_+}(x_2) =$$

$$= \lim_{x_1\to+\infty} (1-e^{-x_2})\mathbb{1}_{\mathbb{R}_+}(x_2) \Rightarrow X_2 \sim esp(1)$$

Devo verificare se $F(x_1,x_2) = \int_{(-\infty,x_1]\times(-\infty,x_2]} \frac{\partial^2 F(u,v)}{\partial u \partial v}\, d\mu_L^2(u,v)$

$$\frac{\partial^2 F(u,v)}{\partial u \partial v} = \frac{\partial}{\partial v}\left(e^{-u}-e^{-v}e^{-u}\right)\mathbb{1}_{\mathbb{R}_+}(u)\mathbb{1}_{\mathbb{R}_+}(v) = e^{-(u+v)}\mathbb{1}_{\mathbb{R}_+}(u)\mathbb{1}_{\mathbb{R}_+}(v)$$

$$\int_{(-\infty,x_1]\times(-\infty,x_2]} e^{-(u+v)}\mathbb{1}_{\mathbb{R}_+}(u)\mathbb{1}_{\mathbb{R}_+}(v)\, d\mu_L^2(u,v) =$$

- $x_1<0 \vee x_2<0 \Rightarrow 0$
- $x_1\geq 0 \wedge x_2 \geq 0 \Rightarrow \int_{[0,x_1]\times[0,x_2]} e^{-(u+v)}\,d\mu_L^2(u,v) = \int_0^{x_1} e^{-u}\,du \int_0^{x_2} e^{-v}\,dv =$

$$= \left[-e^{-u}\right]_0^{x_1}\left[-e^{-v}\right]_0^{x_2} = (1-e^{-x_1})(1-e^{-x_2}) = 1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}$$

$\Rightarrow$ L'uguaglianza è effettivamente verificata $\Rightarrow F(x_1,x_2)$ è assolutamente continua.

$$F_{X_1}(x_1)F_{X_2}(x_2) = \mathbb{1}_{\mathbb{R}_+}(x_1)\mathbb{1}_{\mathbb{R}_+}(x_2)(1-e^{-x_1})(1-e^{-x_2}) = F(x_1,x_2)$$

$\Rightarrow X_1, X_2$ sono variabili aleatorie indipendenti.

$\rightarrow$ Poiché $F(x_1,x_2)$ è assolutam. continua, allora X_1, X_2 sono ass. continue

$Z = \min\{X_1, X_2\}$

$$F_Z(z) = P(Z\leq z) = 1-P(Z> z) \overset{X_1,X_2 \text{ indip.}}{=} 1-P(X_1> z)P(X_2> z) \overset{\text{data luogo}}{=}$$

$$= 1-(1-F_{X_1}(z))(1-F_{X_2}(z)) = 1-\left[1-(1-e^{-z})\mathbb{1}_{\mathbb{R}_+}(z)\right]\left[1-(1-e^{-z})\mathbb{1}_{\mathbb{R}_+}(z)\right]$$

$$= \begin{cases} 0 & \text{se } z<0 \\ 1-(e^{-z})(e^{-z}) = 1-e^{-2z} & \text{se } z\geq 0 \end{cases}$$

$\Rightarrow F_Z(z) = (1-e^{-2z})\mathbb{1}_{\mathbb{R}_+}(z) \Rightarrow Z \sim esp(2)$

$\rightarrow$ Poiché le v.a. esponenziali sono ass. continue $\Rightarrow Z$ è ass. continua.

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3) (X,Y) ass. continuo

$$f_{X,Y}(x,y) := \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathbb{1}_{(0,1)\times(0,2)}(x,y) \qquad \forall (x,y)\in\mathbb{R}^2$$

Verifico che $\int_{\mathbb{R}^2} f_{X,Y}(x,y)\,d\mu_L^2(x,y) = 1$

$$\int_{\mathbb{R}^2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathbb{1}_{(0,1)\times(0,2)}(x,y)\,d\mu_L^2(x,y) = \frac{6}{7}\int_{(0,1)\times(0,2)}\left(x^2+\frac{xy}{2}\right)d\mu_L^2(x,y) =$$

$$= \frac{6}{7}\int_0^1 dx \int_0^2 \left(x^2+\frac{xy}{2}\right)dy = \frac{6}{7}\int_0^1 dx\left[x^2y+\frac{xy^2}{4}\right]_0^2 =$$

$$= \frac{6}{7}\int_0^1 dx\,(2x^2+x) = \frac{6}{7}\left[\frac{2}{3}x^3+\frac{1}{2}x^2\right]_0^1 = \frac{6}{7}\left(\frac{2}{3}+\frac{1}{2}\right) =$$

$$= \frac{6}{7}\cdot\frac{4+3}{6} = 1 \quad \checkmark$$

data
luogo

X,Y sono assolutam. continue perché, per ipotesi il vettore (X,Y) è assolutam. continuo.

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(y) = \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right) 1\!\!1_{(0,1)\times(0,2)}(x,y)\, d\mu_L(y) =$$

$$= \int_{(0,2)} \frac{6}{7} 1\!\!1_{(0,1)}(x) \left[x^2+\frac{xy}{2}\right] d\mu_L(y) = \int_0^2 \frac{6}{7} 1\!\!1_{(0,1)}(x) \left[x^2+\frac{xy}{2}\right] dy =$$

$$= \frac{6}{7} 1\!\!1_{(0,1)}(x) \left[x^2 y + \frac{xy^2}{4}\right]_0^2 = 1\!\!1_{(0,1)}(x) \cdot \frac{6}{7}(2x^2+x)$$

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(x) = \int_{(0,1)} \frac{6}{7} 1\!\!1_{(0,2)}(y) \left[x^2+\frac{xy}{2}\right] d\mu_L(x) =$$

$$= \frac{6}{7} 1\!\!1_{(0,2)}(y) \int_0^1 \left(x^2+\frac{xy}{2}\right) dx = 1\!\!1_{(0,2)}(y) \cdot \frac{6}{7}\left[\frac{1}{3}x^3+\frac{x^2 y}{4}\right]_0^1 =$$

$$= 1\!\!1_{(0,2)}(y) \frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y\right)$$

$$\rightsquigarrow f_X(x) f_Y(y) = 1\!\!1_{(0,1)}(x) 1\!\!1_{(0,2)}(y) \frac{6}{7}(2x^2+x)\frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y\right) =$$

$$= 1\!\!1_{(0,1)\times(0,2)}(x,y) \frac{36}{49}\left(\frac{2}{3}x^2+\frac{1}{2}x^2 y+\frac{1}{3}x+\frac{1}{4}xy\right) \neq f_{X,Y}(x,y)$$

$\Rightarrow$ X,Y NON sono indipendenti.

$$\rightarrow P(X>Y) = P(Y<X) = \int_{\mathbb{R}} P(Y<x) f_X(x)\, d\mu_L(x) =$$

$$= \int F_Y(x) f_X(x)\, d\mu_L(x)$$

Devo prima calcolare $F_Y(y)$

data
luogo

$$F_Y(y) = \int_{(-\infty,y]} f_Y(\hat{y})\, d\mu_L(\hat{y}) = \int_{(-\infty,y]} 1\!\!1_{(0,2)}(\hat{y}) \frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}\hat{y}\right) d\mu_L(\hat{y})$$

- $y<0 \Rightarrow 0$
- $0\le y<2 \Rightarrow \frac{6}{7}\int_{(0,y]}\left(\frac{1}{3}+\frac{1}{4}\hat{y}\right) d\mu_L(\hat{y}) = \frac{6}{7}\int_0^y \left(\frac{1}{3}+\frac{1}{4}\hat{y}\right) d\hat{y} = \frac{6}{7}\left[\frac{1}{3}\hat{y}+\frac{1}{8}\hat{y}^2\right]_0^y =$

$$= \frac{6}{7}\left(\frac{1}{3}y+\frac{1}{8}y^2\right)$$

- $y\ge 2 \Rightarrow \frac{6}{7}\int_{(0,2)}\left(\frac{1}{3}+\frac{1}{4}\hat{y}\right) d\mu_L(\hat{y}) = \frac{6}{7}\left[\frac{1}{3}\hat{y}+\frac{1}{8}\hat{y}^2\right]_0^2 = \frac{6}{7}\left(\frac{2}{3}+\frac{4}{8}\right) =$

$$= \frac{6}{7}\cdot\frac{4+3}{6} = 1 \quad \checkmark$$

$$\Rightarrow F_Y(y) = \frac{6}{7}\left(\frac{1}{3}y+\frac{1}{8}y^2\right) 1\!\!1_{(0,2)}(y) + 1\!\!1_{(2,+\infty)}(y)$$

$$\Rightarrow P(X>Y) = \int_{\mathbb{R}} \frac{6}{7}\left(\frac{1}{3}x+\frac{1}{8}x^2\right) 1\!\!1_{(0,2)}(x) \cdot 1\!\!1_{(0,1)}(x) \cdot \frac{6}{7}(2x^2+x)\, d\mu_L(x) =$$

$$= \int_{(0,1)} \frac{36}{49}\left(\frac{2}{3}x^3+\frac{1}{3}x^2+\frac{1}{4}x^4+\frac{1}{8}x^3\right) d\mu_L(x) =$$

$$= \frac{36}{49}\int_0^1 \left(\frac{1}{4}x^4+\frac{19}{24}x^3+\frac{1}{3}x^2\right) dx =$$

$$= \frac{36}{49}\left[\frac{1}{20}x^5+\frac{19}{96}x^4+\frac{1}{9}x^3\right]_0^1 = \frac{36}{49}\cdot\left(\frac{1}{20}+\frac{19}{96}+\frac{1}{9}\right) =$$

$$= \frac{36}{49}\cdot\frac{144+570+320}{2880} = \frac{36}{49}\cdot\frac{1034}{2880} = \frac{517}{1960}$$

$$\approx \boxed{0{,}2638}$$

Esercizio 7 del foglio 2:

$\mathbb{1}_s : \Omega \to \{0,1\}$ $\qquad$ $\mathbb{1}_c : \Omega \to \{0,1\}$

$P(\mathbb{1}_s) = 0,60473$ $\qquad$ $P(\mathbb{1}_c) = 0,17802$

$$F_{(\mathbb{1}_s,\mathbb{1}_c)}(x,y) = P(\mathbb{1}_s \leq x, \mathbb{1}_c \leq y) =$$

- $x<0 \vee y<0 \Rightarrow 0$
- $x \geq 1 \wedge y \geq 1 \Rightarrow 1$
- $0 \leq x < 1 \wedge y \geq 1 \Rightarrow P(\mathbb{1}_s \leq x) = 0,39527$
- $x \geq 1 \wedge 0 \leq y < 1 \Rightarrow P(\mathbb{1}_c \leq y) = 0,82198$
- $0 \leq x < 1 \wedge 0 \leq y < 1 \Rightarrow P(\mathbb{1}_s \leq x, \mathbb{1}_c \leq y) = 0,32120$

$$\Rightarrow F_{(\mathbb{1}_s,\mathbb{1}_c)}(x,y) = 0,39527\, \mathbb{1}_{[0,1)\times[1,+\infty)}(x,y) + 0,82198\, \mathbb{1}_{[1,+\infty)\times[0,1)}(x,y) +$$
$$+ 0,32120\, \mathbb{1}_{[0,1)\times[0,1)}(x,y) + \mathbb{1}_{[1,+\infty)\times[1,+\infty)}(x,y)$$

$$F_{\mathbb{1}_s}(x) = \lim_{y\to+\infty} F_{(\mathbb{1}_s,\mathbb{1}_c)}(x,y) = 0,39527\, \mathbb{1}_{[0,1)}(x) + \mathbb{1}_{[1,+\infty)}(x)$$

$$F_{\mathbb{1}_c}(y) = \lim_{x\to+\infty} F_{(\mathbb{1}_s,\mathbb{1}_c)}(x,y) = 0,82198\, \mathbb{1}_{[0,1)}(y) + \mathbb{1}_{[1,+\infty)}(y)$$

$$\to F_{\mathbb{1}_s}(x) F_{\mathbb{1}_c}(y) = 0,32490\, \mathbb{1}_{[0,1)\times[0,1)}(x,y) + 0,39527\, \mathbb{1}_{[0,1)\times[1,+\infty)}(x,y) +$$
$$+ 0,82198\, \mathbb{1}_{[1,+\infty)\times[0,1)}(x,y) + \mathbb{1}_{[1,+\infty)\times[1,+\infty)}(y) \neq F_{(\mathbb{1}_s,\mathbb{1}_c)}(x,y)$$

$\Rightarrow \mathbb{1}_s, \mathbb{1}_c$ NON sono indipendenti.

data $P(C|S) = \frac{P(C\cap S)}{P(S)} = \frac{0,10395}{0,60473} \approx 0,17189$

luogo $P(S|C) = \frac{P(S\cap C)}{P(C)} = \frac{0,10395}{0,17802} \approx 0,58392$

$P(S) = P(S\cap C) + P(S\cap C^c) = P(S|C)P(C) + P(S|C^c)P(C^c)$

$\rightsquigarrow P(S|C^c) = \frac{P(S\cap C^c)}{P(C^c)} = \frac{0,50078}{0,82198} \approx 0,60924$

$\Rightarrow P(S) = 0,58392 \cdot 0,17802 + 0,60924 \cdot 0,82198 \approx 0,60473$ ✓

$P(C|S) = \frac{P(S\cap C)}{P(S)} = \frac{P(S|C)P(C)}{P(S)} = \frac{0,58392 \cdot 0,17802}{0,60473} \approx$

$\approx 0,17189$ ✓

$f_X(x) = \int_{\mathbb{R}} f(x,y)\, d\mu_L(y) = \int_{\mathbb{R}} x e^{-(x+y)} 1\!\!1_{\mathbb{R}^2_+}(x,y)\, d\mu_L(y) = 1\!\!1_{\mathbb{R}_+}(x) x e^{-x} \int_{\mathbb{R}_+} e^{-y} dy$

$= 1\!\!1_{\mathbb{R}_+}(x)\, x e^{-x} \int_0^{+\infty} e^{-y} dy = 1\!\!1_{\mathbb{R}_+}(x)\, x e^{-x} \left[-e^{-y}\right]_0^{+\infty} = 1\!\!1_{\mathbb{R}^+}(x)\, x e^{-x}$

$f_Y(y) =$

→ Terzo esercizio di 12/02/2020 → $\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

→ Primi 3 esercizi di 25/02/2020

Terzo esercizio di 12/02/2020:

data

luogo

$U \sim V \sim N(0,1)$ (U,V) gaussiano

$\rho := corr(U,V) = \frac{cov(U,V)}{\sigma_U \sigma_V}$

$X = U - \rho V$ $\qquad Y = \sqrt{1-\rho}\, V$

Provo a trovare A, b t.c.

$(X,Y)^T = b + A (U,V)^T$

$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$

$\Rightarrow \begin{bmatrix} U-\rho V \\ \sqrt{1-\rho}\, V \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$

$\Rightarrow \begin{bmatrix} U-\rho V \\ \sqrt{1-\rho}\, V \end{bmatrix} = \begin{bmatrix} 1 & -\rho \\ 0 & \sqrt{1-\rho} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$

$\begin{bmatrix} 1 & -\rho \\ 0 & \sqrt{1-\rho} \end{bmatrix}$ ha rango max $\Rightarrow (X,Y)$ è un vett. gaussiano

$Cov(X,Y) = E[XY] - E[X]E[Y] = E[(U-\rho V)\sqrt{1-\rho}\, V] +$
$- E[U-\rho V]\, E[\sqrt{1-\rho}\, V] = E[\sqrt{1-\rho}\, UV] - E[\rho\sqrt{1-\rho}\, V^2] +$
$-(E[U] - E[\rho V])\, E[\sqrt{1-\rho}\, V] =$

data

luogo

$= \sqrt{1-\rho}\, E[UV] - \rho\sqrt{1-\rho}\, E[V^2] + \sqrt{1-\rho}\, E[V]\left(\rho E[V] - E[U]\right)$

$\cancel{= \sqrt{1-\rho}\, E[UV]}$

$Cov(U,V) = \sqrt{D^2[U]}\sqrt{D^2[V]}\,\rho = \rho$

$\Rightarrow E[UV] - E[U]E[V] = \rho \Rightarrow E[UV] - 0 = \rho$

$\Rightarrow E[UV] = \rho$

$D^2[V] = E[V^2] - E^2[V] \Rightarrow E[V^2] = D^2[V] + E^2[V] =$

$= 1 + 0 = 1$

$\Rightarrow \sqrt{1-\rho}\cdot\rho - \sqrt{1-\rho}\cdot\rho + 0 = 0$

$\Rightarrow Cov(X,Y) = 0$; poiché $(X,Y)^T$ è un vettore gaussiano, X, Y sono indip.

$\rightarrow E[X] = E[U - \rho V] = E[U] - \rho E[V] = 0$

$\rightarrow E[Y] = E[\sqrt{1-\rho}\, V] = \sqrt{1-\rho}\, E[V] = 0$

$\rightarrow D^2[X] = E[X^2] - E^2[X] = E[X^2] = E[U^2 + \rho^2 V^2 - 2\rho UV] =$

$= E[U^2] + \rho^2 E[V^2] - 2\rho E[UV] =$

$\rightarrow E[U^2] = D^2[U] + E^2[U] = 1 + 0 = 1$

$\rightarrow E[V^2] = D^2[V] + E^2[V] = 1 + 0 = 1$

data

luogo

$\Rightarrow D^2[X] = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2$

$D^2[Y] = E[Y^2] - E^2[Y] = E[(1-\rho)V^2] - 0 =$

$= (1-\rho)E[V^2] = 1 - \rho$

$\Rightarrow X \sim N(0, 1-\rho^2) \qquad Y \sim N(0, 1-\rho)$

- $E[X^2Y^2] \overset{X,Y \text{ indip.} \Rightarrow X^2, Y^2 \text{ indip.}}{=} E[X^2]E[Y^2] = \left(D^2[X] + \underbrace{E^2[X]}_{0}\right)\left(D^2[Y] + \underbrace{E^2[Y]}_{0}\right)$

$= D^2[X]\, D^2[Y] = (1+\rho)(1-\rho)^2$

- $E[XY^3] = E[X]E[Y^3] = 0$
- $E[Y^4] = 3$ (è la curtosi)
- $E[U^2V^2]$

$\begin{cases} X = U - \rho V \\ Y = \sqrt{1-\rho}\, V \end{cases} \Rightarrow \begin{cases} X = U - \frac{\rho}{\sqrt{1-\rho}} Y \\ V = \frac{Y}{\sqrt{1-\rho}} \end{cases} \Rightarrow \begin{cases} U = X + \frac{\rho}{\sqrt{1-\rho}} Y \\ V = \frac{Y}{\sqrt{1-\rho}} \end{cases}$

$\Rightarrow E[U^2V^2] = E\left[\left(X^2 + \frac{\rho^2}{1-\rho} Y^2 + \frac{2\rho}{\sqrt{1-\rho}} XY\right)\left(\frac{Y^2}{1-\rho}\right)\right] =$

data
luogo

$$= \frac{1}{1-\rho}\left\{E[X^2Y^2] + \frac{\rho^2}{1-\rho}E[Y^4] + \frac{2\rho}{\sqrt{1-\rho}}E[XY^3]\right\} =$$

$$= \frac{1}{1-\rho}\left\{(1+\rho)(1-\rho)^2 + \frac{\rho^2}{1-\rho}3 + 0\right\} =$$

$$= \frac{1+\rho}{1-\rho} + \frac{3\rho^2}{(1-\rho)^2}$$

Esercizio 3 del 25/02/2020:

$Z_1 \sim Z_2 \sim Z_3 \sim N(0,1)$ $\qquad Z_1, Z_2, Z_3$ indipendenti

$X_1 = Z_1 + Z_2 + Z_3$ $\qquad X_2 = Z_1 - Z_2 + Z_3$

$X_3 = Z_1 - Z_3$

→ $Z := (Z_1, Z_2, Z_3)$ è un vettore gaussiano poiché Z_1, Z_2, Z_3 sono v.a. gaussiane standard indipendenti tra loro.

Provo a trovare $b \in \mathbb{R}^3$, $A \in \mathbb{R}^3 \times \mathbb{R}^3$ t.c.

$$X = b + AZ \Rightarrow \begin{bmatrix} Z_1+Z_2+Z_3 \\ Z_1-Z_2+Z_3 \\ Z_1-Z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Z_1+Z_2+Z_3 \\ Z_1-Z_2+Z_3 \\ Z_1-Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

data
luogo

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

⇒ A ha rango massimo ⇒ X è un vettore gaussiano (*)

(*) con $\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\Sigma^2 = AA^T$

$$AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

→ X è un vettore gaussiano con $\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\Sigma^2 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

⇒ $D^2[X_1] = 3$, $D^2[X_2] = 3$, $D^2[X_3] = 2$

$Cov(X_1, X_2) = 1$, $Cov(X_1, X_3) = 0$, $Cov(X_2, X_3) = 0$

$E[X_1] = 0$, $E[X_2] = 0$, $E[X_3] = 0$

→ $F_X(x_1, x_2, x_3) = P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3) =$

$= P(Z_1+Z_2+Z_3 \le x_1, Z_1-Z_2+Z_3 \le x_2, Z_1-Z_3 \le x_3)$

Ci torno dopo

→ Poiché le v.a. che compongono un vettore aleatorio, se sono scorrelate, sono anche indipendenti, possiamo concludere che:

X_1, X_2 NON sono indipendenti tra loro.

X_1, X_3 sono indipendenti tra loro.

X_2, X_3 sono indipendenti tra loro.

$\rightarrow X_1 \sim N(0,3) \qquad X_2 \sim N(0,3) \qquad X_3 \sim N(0,$

$\rightarrow E[X_1X_2^2] = E[(Z_1+Z_2+Z_3)(Z_1^2+Z_2^2+Z_3^2-2Z_1Z_2+2Z_1Z_3-2Z_2Z_3)] =$

$= E[Z_1^3+Z_1Z_2^2+Z_1Z_3^2-2Z_1^2Z_2+2Z_1^2Z_3-\cancel{2Z_1Z_2Z_3}+$

$+Z_1^2Z_2+Z_2^3+Z_2Z_3^2-2Z_1Z_2^2+\cancel{2Z_1Z_2Z_3}-2Z_2^2Z_3+$

$+Z_1^2Z_3+Z_2^2Z_3+Z_3^3-2Z_1Z_2Z_3+2Z_1Z_3^2-2Z_2Z_3^2] =$

$= E[Z_1^3]+E[Z_2^3]+E[Z_3^3] \cancel{-E[Z_1]E[Z_2^2]+3E[Z_1]E[Z_3^2]}$

$\cancel{-E[Z_1^2]E[Z_2]+3E[Z_1^2]E[Z_3]} \cancel{-E[Z_2]E[Z_3^2]-E[Z_2^2]E[Z_3]}$

$\cancel{-2E[Z_1]E[Z_2]E[Z_3]} =$

$= E[Z_1^3]+E[Z_2^3]+E[Z_3^3] = 0$

Le v.a. gaussiane hanno momento di ordine dispari nullo.

$E[X_1^2X_2^2] = E[(Z_1^2+Z_2^2+Z_3^2+2Z_1Z_2+2Z_1Z_3+2Z_2Z_3)(Z_1^2+Z_2^2+Z_3^2+$
$-2Z_1Z_2+2Z_1Z_3-2Z_2Z_3)] =$

$= E[Z_1^4]+E[Z_1^2]E[Z_2^2]+E[Z_1^2]E[Z_3^2]+E[Z_1^2]E[Z_2^2]+$

$+E[Z_2^4]+E[Z_2^2]E[Z_3^2]+E[Z_1^2]E[Z_3^2]+E[Z_2^2]E[Z_3^2]+$

$+E[Z_3^4]-4E[Z_1^2]E[Z_2^2]+4E[Z_1^2]E[Z_3^2]-4E[Z_2^2]E[Z_3^2]$

$= E[Z_1^4]+E[Z_2^4]+E[Z_3^4]-2E[Z_1^2]E[Z_2^2]+6E[Z_1^2]E[Z_3^2]+$

$-2E[Z_2^2]E[Z_3^2]$

▸ $D^2[Z_1] = E[Z_1^2] - E^2[Z_1] \Rightarrow E[Z_1^2] = 1$

▸ $E[Z_2^2] = 1$ ▸ $E[Z_3^2] = 1$

$\Rightarrow E[X_1^2X_2^2] = 3+3+3-2+6-2 = 11$

$E[X_2X_3^2] = E[(Z_1-Z_2+Z_3)(Z_1^2+Z_3^2-2Z_1Z_3)] = 0$

(anche qui verrebbero tutti momenti di ordine dispari di Z_1, Z_2, Z_3)

$E[X_2^2X_3^2] = E[(Z_1^2+Z_2^2+Z_3^2-2Z_1Z_2+2Z_1Z_3-2Z_2Z_3)(Z_1^2+Z_3^2-2Z_1Z_3)]$

$= E[Z_1^4]+E[Z_1^2]E[Z_3^2]+E[Z_2^2]E[Z_3^2]+E[Z_1^2]E[Z_2^2]+$

$+E[Z_1^2]E[Z_3^2]+E[Z_3^4]-4E[Z_1^2]E[Z_3^2] =$

$= E[Z_1^4]+E[Z_3^4]-2E[Z_1^2]E[Z_3^2]+E[Z_2^2]E[Z_3^2]+E[Z_1^2]E[Z_2^2]$

$= 3+3-2+1+1 = 6$

Torniamo a $f_X(x_1,x_2,x_3)$...

$$f_X(x_1,x_2,x_3) = \frac{1}{\sqrt{8\pi^3 \det(\Sigma^2)}}\, e^{-\frac{1}{2}(x_1,x_2,x_3)^T(\Sigma^2)^{-1}(x_1,x_2,x_3)}$$

$\rightarrow \det(\Sigma^2) = 2(9-1) = 16$

$$\rightarrow (\Sigma^2)^{-1} = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \cdot \frac{1}{16} = \begin{bmatrix} 3/8 & -1/8 & 0 \\ -1/8 & 3/8 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

data
luogo

$$\frac{1}{2}[x_1\ x_2\ x_3]\begin{bmatrix}3/8 & -1/8 & 0\\ -1/8 & 3/8 & 0\\ 0 & 0 & 1/2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=$$

$$=\left[\tfrac{3}{8}x_1-\tfrac{1}{8}x_2,\ -\tfrac{1}{8}x_1+\tfrac{3}{8}x_2\ \ \tfrac{1}{2}x_3\right]\begin{bmatrix}\frac{1}{2}x_1\\ \frac{1}{2}x_2\\ \frac{1}{2}x_3\end{bmatrix}=$$

$$=\frac{3}{16}x_1^2-\frac{1}{16}x_1x_2-\frac{1}{16}x_1x_2+\frac{3}{16}x_2^2+\frac{1}{4}x_3^2=$$

$$=\frac{1}{16}\left(3x_1^2+3x_2^2+4x_3^2-2x_1x_2\right)$$

$$\Rightarrow f_X(x_1,x_2,x_3)=\frac{1}{8\pi\sqrt{2\pi}}e^{-\frac{1}{16}(3x_1^2+3x_2^2+4x_3^2-2x_1x_2)}$$

$$\Rightarrow F_X(x_1,x_2,x_3)=\int_{(-\infty,x_1]\times(-\infty,x_2]\times(-\infty,x_3]}\frac{1}{8\pi\sqrt{2\pi}}e^{-\frac{1}{16}(3u_1^2+3u_2^2+4u_3^2-2u_1u_2)}\,d\mu_L^3(u_1,u_2,u_3)$$

$$=\frac{1}{8\pi\sqrt{2\pi}}\int_{-\infty}^{x_1}du_1\int_{-\infty}^{x_2}du_2\,e^{-\frac{1}{16}(3u_1^2+3u_2^2-2u_1u_2)}\int_{-\infty}^{x_3}du_3\,e^{-\frac{1}{4}u_3^2}$$

$$=\frac{1}{4\sqrt{2\pi}}\left[\frac{1}{2}\left(1+erf\left(\frac{x_3}{2}\right)\right)\right]\int_{-\infty}^{x_1}du_1\int_{-\infty}^{x_2}du_2\,e^{-\frac{1}{16}(3u_2^2-2u_1u_2+\frac{1}{3}u_1^2)}e^{-\frac{1}{6}u_1^2}$$

$$=\frac{1}{4\sqrt{\pi}\sqrt{2\pi}}\left[\frac{1}{2}\left(1+erf\left(\frac{x_3}{2}\right)\right)\right]\int_{-\infty}^{x_1}du_1\,e^{-\frac{1}{6}u_1^2}\int_{-\infty}^{x_2}du_2\,e^{-\frac{3}{16}(u_2-\frac{1}{3}u_1)^2}=$$

$$=\frac{1}{\sqrt{2\pi}\sqrt{3}}\left[\frac{1}{2}\left(1+erf\left(\frac{x_3}{2}\right)\right)\right]\int du_1\,e^{-\frac{1}{6}u_1^2}\left[\frac{1}{2}\left(1+erf\left(\frac{x_2-\frac{1}{3}u_1}{4}\sqrt{3}\right)\right)\right]=$$

$$=\frac{1}{\sqrt{3}\sqrt{2\pi}}\left[\frac{1}{2}\left(1+erf\left(\frac{x_3}{2}\right)\right)\right]\left\{\frac{1}{2}\int_{-\infty}e^{-\frac{1}{6}u_1^2}du_1-\frac{1}{2}\int_{-\infty}^{x_1}e^{-\frac{1}{6}u_1^2}erf\left(\frac{x_2-\frac{1}{3}u_1}{4}\sqrt{3}\right)du_1\right.$$

data
luogo

$$=\frac{1}{4\sqrt{3}\sqrt{2\pi}}\left[1+erf\left(\frac{x_3}{2}\right)\right]\left[\int_{-\infty}^{x_1}e^{-\frac{1}{6}u_1^2}du_1-\int_{-\infty}^{x_1}e^{-\frac{1}{6}u_1^2}\cdot\frac{2}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^k\left[\frac{\sqrt{3}}{4}(x_2-\frac{1}{3}u_1)\right]^{2k+1}}{(2k+1)k!}du_1\right]$$

Facciamo che sparisci dalla mia vista :)

Esercizio 1 del 25/02/2020:

$g(x):=\sqrt{x}\quad\forall x\in\mathbb{R}_+$ $\qquad h(x):=\ln(x)\quad\forall x\in\mathbb{R}_{++}$

$\rightarrow Y(\omega):=g(X(\omega))$ è una variabile aleatoria $\Leftrightarrow P(X<0)=0$

$\rightarrow Z(\omega):=h(X(\omega))$ è una variabile aleatoria $\Leftrightarrow P(X\leq 0)=0$

$F_Y(y)=P(Y\leq y)=P(\sqrt{X}\leq y)=$

- $y<0\Rightarrow 0$
- $y\geq 0\Rightarrow P(X\leq y^2)=F_X(y^2)$

$\Rightarrow F_Y(y)=F_X(y^2)\,\mathbb{1}_{[0,+\infty)}(y)$

$F_Z(z)=P(Z\leq z)=P(\ln(X)\leq z)=P(X\leq e^z)=F_X(e^z)$

$\lambda\in(0,1)\ \rightsquigarrow\ \lambda F_Y+(1-\lambda)F_Z$

- $\lim_{u\to-\infty}\lambda F_Y(u)+(1-\lambda)F_Z(u)=\lambda\lim_{u\to-\infty}F_Y(u)+(1-\lambda)\lim_{u\to-\infty}F_Z(u)=0+0$ ✓
- $\lambda F_Y(u)+(1-\lambda)F_Z(u)\geq 0\ \ \forall u$ perché $\lambda\geq 0,(1-\lambda)\geq 0,F_Y\geq 0,F_Z\geq 0$ ✓

data

luogo

- $\lim_{u\to+\infty} \lambda F_Y(u)+(1-\lambda)F_Z(u) = \lambda \lim_{u\to+\infty} F_Y(u)+(1-\lambda)\lim_{u\to+\infty} F_Z(u) =$
$= \lambda\cdot 1+(1-\lambda)\cdot 1 = \lambda+1-\lambda = 1$ ✓
- $\frac{d}{du}\left(\lambda F_Y(u)+(1-\lambda)F_Z(u)\right) = \lambda F_Y'(u)+(1-\lambda)F_Z'(u) \geq 0 \quad \forall u$
poiché $\lambda\geq 0$, $1-\lambda\geq 0$, $F_Y'(u)\geq 0$, $F_Z'(u)\geq 0$ ✓

$\boxed{F_Y^2}$

- $\lim_{u\to-\infty} F_Y^2(u) = \lim_{u\to-\infty} F_Y(u)\cdot F_Y(u) = 0$ ✓
- $F_Y^2(u)$ è chiaramente non negativa $\forall u$ ✓
- $\lim_{u\to+\infty} F_Y^2(u) = \lim_{u\to+\infty} F_Y(u)\cdot F_Y(u) = 1$ ✓
- $\frac{d}{du}\left(F_Y^2(u)\right) = 2F_Y(u)\cdot F_Y'(u) \geq 0 \ \forall u$ poiché $F_Y(u)\geq 0$, $F_Y'(u)\geq 0$ ✓

$\to$ Analogamente per F_X^2

$\boxed{F_Y F_Z}$

- $\lim_{u\to\mp\infty} F_Y(u)F_Z(u) = 0$ ✓
- $F_Y(u)F_Z(u)\geq 0 \ \forall u$ perché $F_Y(u)\geq 0$, $F_Z(u)\geq 0 \ \forall u$ ✓
- $\lim_{u\to+\infty} F_Y(u)F_Z(u) = 1$ ✓
- $\frac{d}{du}\left(F_Y(u)F_Z(u)\right) = F_Y'(u)F_Z(u)+F_Y(u)F_Z'(u) \geq 0 \ \forall u$ poiché
$F_Y'(u)\geq 0$, $F_Z(u)\geq 0$, $F_Y(u)\geq 0$, $F_Z'(u)\geq 0 \ \forall u$ ✓

Esercizio 2 del 25/02/2020:

$F(x,y) = \left(1-e^{-y}-\frac{1}{2}ye^{-x}\right)\mathbb{1}_{\mathbb{R}^2_+\cap\{x>y\}}(x,y)+\left(1-e^{-x}-\frac{1}{2}xe^{-y}\right)\mathbb{1}_{\mathbb{R}^2_+\cap\{x\leq y\}}(x,y) \quad \forall (x,y)\in\mathbb{R}^2$

- $\lim_{x\to+\infty}\lim_{y\to+\infty} 1-e^{-y}-\frac{1}{2}ye^{-x} = 1$
- $\lim_{x\to+\infty}\lim_{y\to+\infty} 1-e^{-x}-\frac{1}{2}xe^{-y} = 1$
- $\lim_{x\to-\infty} F(x,y) = 0$ perché entrambe le funzioni indicatrici valgono 0.
- Analogam. $\lim_{y\to-\infty} F(x,y) = 0$
- Devo provare che $F(x,y)\geq 0 \quad \forall (x,y)\in\mathbb{R}^2$
  - $\to$ Se $x<0 \vee y<0 \Rightarrow F(x,y)=0$ ✓
  - $\to$ Se $0\leq y<x \Rightarrow F(x,y) = 1-e^{-y}-\frac{1}{2}ye^{-x}\geq 0$

  $\Rightarrow \frac{1}{2}ye^{-x}\leq 1-e^{-y} \Rightarrow e^{-x}\leq\frac{2}{y}(1-e^{-y})$

  e^{-x} è sicuram. minore di e^{-y}

Provo a dim. che $e^{-y}\leq\frac{2}{y}(1-e^{-y})$

$\Rightarrow 1\leq\frac{2}{y}(e^{y}-1) \Rightarrow y\leq 2(e^{y}-1)$

$y=0 \Rightarrow 0\leq 2(1-1) \Rightarrow 0\leq 0$ ✓ $\leftarrow$ questo è l'unico pto di intersez.

Dimostro ora che $2(e^{y}-1)$ cresce + velocem. di $y \quad \forall y\geq 0$

$\Rightarrow \frac{d}{dy}\left(2(e^{y}-1)\right) = 2e^{y}$

$\frac{d}{dy}(y) = 1$

$\Rightarrow y\leq 2(e^{y}-1) \quad \forall y\geq 0$

Scansionato con CamScanner

$\Rightarrow F(x,y) \geq 0 \quad \forall\ 0 \leq y < x$

$\rightarrow$ Se $0 \leq x \leq y \Rightarrow F(x,y) = 1 - e^{-x} - \frac{1}{2} x e^{-y}$

↳ Questo si dimostra allo stesso modo del caso precedente ✓

- Devo provare che $F'(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

$\rightarrow$ Se $x<0, y<0 \Rightarrow F'(x,y) = 0$ ✓

$\rightarrow$ Se $0 \leq y < x \Rightarrow F(x,y) = 1 - e^{-y} - \frac{1}{2} y e^{-x}$

$\Rightarrow \frac{\partial F(x,y)}{\partial y} = e^{-y} - \frac{1}{2} e^{-x} \geq 0 \Rightarrow e^{-y} \geq \frac{1}{2} e^{-x}$

$e^{-x} < e^{-y}$ poiché $x > y$

↳ BASTEREBBE DUNQUE PROVARE CHE $\frac{1}{2} e^{-y} \leq e^{-y}$

MA È OVVIO!

~~$\rightarrow 0 \leq x \leq y \Rightarrow F(x,y) = 1 - e^{-x} - \frac{1}{2} x e^{-y}$~~

$\frac{\partial F(x,y)}{\partial x} = \frac{1}{2} y e^{-x} \geq 0 \quad \forall\ x,y \geq 0$ ✓

$\rightarrow 0 \leq x \leq y \Rightarrow F(x,y) = 1 - e^{-x} - \frac{1}{2} x e^{-y}$

$\frac{\partial F(x,y)}{\partial y} = \frac{1}{2} x e^{-y} \geq 0 \quad \forall\ x,y \geq 0$ ✓

$\frac{\partial F(x,y)}{\partial x} = e^{-x} - \frac{1}{2} e^{-y}$ ← si dimostra che è ≥ 0 come prima

In definitiva, $F(x,y)$ è una funz. di distribuz. ✓

STAMPA ANCHE LE SOLUZIONI DEGLI ESERCIZI 1 & 5 DEL FILE "CPS - Homework - 2021-11-16 ..." (versione nuova)

data
luogo
Art-ware bellissime arti

Esercizio 4:

$f(x,y) := K e^{-\frac{x^2 - xy + y^2}{2}} \quad \forall (x,y) \in \mathbb{R}^2$

$\int_{\mathbb{R}^2} K e^{-\frac{x^2-xy+y^2}{2}} d\mu_L^2(x,y) = K \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2 - xy + y^2)} dy =$

$= K \int_{-\infty}^{+\infty} e^{-\frac{3}{8}x^2} dx \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y - \frac{1}{2}x)^2} dy = K \int_{-\infty}^{+\infty} e^{-\frac{3}{8}x^2} dx \sqrt{2\pi} =$

$= K \cdot \sqrt{2\pi} \cdot \sqrt{\frac{8}{3}\pi} = \frac{4\sqrt{3}}{3} \pi K = 1 \Rightarrow K = \frac{\sqrt{3}}{4\pi}$

$\rightarrow f_X(x) = \int_{\mathbb{R}} f(x,y)\, d\mu_L(y) = \int_{\mathbb{R}} \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2-xy+y^2)} d\mu_L(y)$

$= \frac{\sqrt{3}}{4\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2-xy+y^2)} dy = \frac{\sqrt{3}}{4\pi} \int_{-\infty}^{+\infty} e^{-\frac{3}{8}x^2} e^{-\frac{1}{2}(y-\frac{1}{2}x)^2} dy =$

$= \frac{\sqrt{6}}{4\sqrt{\pi}} e^{-\frac{3}{8}x^2} = \sqrt{\frac{3}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{3}{8}x^2}$

$\rightarrow f_Y(y) = \int_{\mathbb{R}} f(x,y)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2-xy+y^2)} d\mu_L(x) = \sqrt{\frac{3}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{3}{8}y^2}$

$\Rightarrow X \sim Y \sim N(0, 4/3)$

$\rightarrow f_X(x) f_Y(y) = \sqrt{\frac{3}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{3}{8}x^2} \sqrt{\frac{3}{4}} \frac{1}{\sqrt{2\pi}} e^{-\frac{3}{8}y^2} = \frac{3}{8\pi} e^{-\frac{3}{8}(x^2+y^2)} \neq f(x,y)$

$\Rightarrow X, Y$ NON sono indip.

$\rightarrow P(X=Y) = \int_{\{(x,y) \in \mathbb{R}^2 : x=y\}} f(x,y)\, d\mu_L^2(x,y) = 0$ perché $\{(x,y) \in \mathbb{R}^2 : x=y\}$ è un insieme ...

$$\rightarrow P(X\geq Y)=\int_{\{(x,y)\in\mathbb{R}^2:\,x\geq y\}} f(\hat{x},\hat{y})\,d\mu_L^2(\hat{x},\hat{y})=$$

$$=\int_{\{(x,y)\in\mathbb{R}^2:\,x\geq y\}}\frac{\sqrt{3}}{4\pi}e^{-\frac{1}{2}(\hat{x}^2-\hat{x}\hat{y}+\hat{y}^2)}\,d\mu_L^2(\hat{x},\hat{y})=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}d\hat{x}\int_{-\infty}^{\hat{x}}e^{-\frac{1}{2}(\hat{x}^2-\hat{x}\hat{y}+\hat{y}^2)}\,d\hat{y}=$$

$$=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}e^{-\frac{3}{8}\hat{x}^2}\,d\hat{x}\int_{-\infty}^{x}e^{-\frac{1}{2}(\hat{y}-\frac{1}{2}\hat{x})^2}\,d\hat{y}=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}d\hat{x}\,e^{-\frac{3}{8}\hat{x}^2}$$

! Me so' incasinato...

$$\rightarrow P(X\geq Y)=\int_{\{(x,y)\in\mathbb{R}^2:\,x\geq y\}} f(x,y)\,d\mu_L^2(x,y)=\int_{\{(x,y)\in\mathbb{R}^2:\,x\geq y\}}\frac{\sqrt{3}}{4\pi}e^{-\frac{1}{2}(x^2-xy+y^2)}\,d\mu_L^2(x,y)=$$

$$=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}e^{-\frac{1}{2}(x^2-xy+y^2)}\,dy=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}e^{-\frac{3}{8}x^2}\,dx\int_{-\infty}^{x}e^{-\frac{1}{2}(y-\frac{1}{2}x)^2}\,dy=$$

$$=\frac{\sqrt{3}}{4\pi}\int_{-\infty}^{+\infty}e^{-\frac{3}{8}x^2}\,dx\cdot\sqrt{2\pi}\cdot\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{1}{2}\frac{x}{\sqrt{2}}\right)\right]=$$

$$=\frac{\sqrt{6}}{8\sqrt{\pi}}\left[\int_{-\infty}^{+\infty}e^{-\frac{3}{8}x^2}\,dx+\underbrace{\int_{-\infty}^{+\infty}e^{-\frac{3}{8}x^2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{2}}\right)dx}_{0}\right]=$$

$$=\frac{\sqrt{6}}{8\sqrt{\pi}}\cdot\sqrt{2\pi}\cdot\frac{2}{\sqrt{3}}=\frac{1}{2}$$

$$\rightarrow\frac{\sqrt{3}}{4\pi}\cdot\frac{2\pi}{\sqrt{3}}=\frac{1}{2}\ \checkmark$$

Esercizio 6:

$$f(x,y):=\frac{4x+2y}{3}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y)\quad\forall(x,y)\in\mathbb{R}^2$$

- $\frac{4x+2y}{3}\geq 0\ \forall(x,y):\ 0\leq x\leq 1\ \land\ 0\leq y\leq 1$

- $$\int_{\mathbb{R}^2}\frac{4x+2y}{3}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y)\,d\mu_L^2(x,y)=\int_{[0,1]\times[0,1]}\frac{4x+2y}{3}\,d\mu_L^2(x,y)=$$

$$=\int_0^1dx\int_0^1\frac{4x+2y}{3}\,dy=\frac{1}{3}\int_0^1dx\left[4xy+y^2\right]_0^1=\frac{1}{3}\int_0^1(4x+1)\,dx=$$

$$=\frac{1}{3}\left[2x^2+x\right]_0^1=\frac{1}{3}(2+1)=1\ \checkmark$$

$$\rightarrow f_X(x)=\int_{\mathbb{R}}f(x,y)\,d\mu_L(y)=\int_{\mathbb{R}}\frac{4x+2y}{3}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y)\,d\mu_L(y)=$$

$$=\mathbb{1}_{[0,1]}(x)\int_{[0,1]}\frac{4x+2y}{3}\,d\mu_L(y)=\mathbb{1}_{[0,1]}(x)\int_0^1\frac{4x+2y}{3}\,dy=$$

$$=\frac{1}{3}\mathbb{1}_{[0,1]}(x)\left[4xy+y^2\right]_0^1=\mathbb{1}_{[0,1]}(x)\left(\frac{4}{3}x+\frac{1}{3}\right)$$

$$\rightarrow f_Y(y)=\int_{\mathbb{R}}f(x,y)\,d\mu_L(x)=\int_{\mathbb{R}}\frac{4x+2y}{3}\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y)\,d\mu_L(x)=$$

$$=\frac{1}{3}\mathbb{1}_{[0,1]}(y)\int_{[0,1]}(4x+2y)\,d\mu_L(x)=\frac{1}{3}\mathbb{1}_{[0,1]}(y)\int_0^1(4x+2y)\,dx=$$

$$=\frac{1}{3}\mathbb{1}_{[0,1]}(y)\left[2x^2+2xy\right]_0^1=\mathbb{1}_{[0,1]}(y)\left(\frac{2}{3}+\frac{2}{3}y\right)$$

- Sicuramente $f_X(x)\geq 0\ \forall x$ $\quad f_Y(y)\geq 0\ \forall y$

data luogo $\int_{\mathbb{R}} f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \left(\tfrac{1}{3}+\tfrac{4}{3}x\right) \mathbb{1}_{[0,1]}(x)\, d\mu_L(x) =$

$$= \int_0^1 \left(\tfrac{1}{3}+\tfrac{4}{3}x\right) dx = \left[\tfrac{1}{3}x + \tfrac{2}{3}x^2\right]_0^1 = \tfrac{1}{3}+\tfrac{2}{3} = 1 \quad \checkmark$$

$$\blacktriangleright \int_{\mathbb{R}} f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} \left(\tfrac{2}{3}+\tfrac{2}{3}y\right)\mathbb{1}_{[0,1]}(y)\, d\mu_L(y) = \int_0^1 \left(\tfrac{2}{3}+\tfrac{2}{3}y\right) dy =$$

$$= \left[\tfrac{2}{3}y + \tfrac{1}{3}y^2\right]_0^1 = \tfrac{2}{3}+\tfrac{1}{3} = 1 \quad \checkmark$$

$$\rightarrow f_X(x) f_Y(y) = \left(\tfrac{1}{3}+\tfrac{4}{3}x\right)\left(\tfrac{2}{3}+\tfrac{2}{3}y\right)\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y) =$$

$$= \left(\tfrac{2}{9}+\tfrac{2}{9}y+\tfrac{8}{9}x+\tfrac{8}{9}xy\right)\mathbb{1}_{[0,1]}(x)\mathbb{1}_{[0,1]}(y) \neq f(x,y)$$

$\rightarrow X, Y$ <u>NON</u> sono indipendenti.

Riprendo il problema 2 di 25/02/2020...

$$F(x,y) = \left(1-e^{-y}-\tfrac{1}{2}ye^{-x}\right)\mathbb{1}_{\mathbb{R}^2_+(x>y)}(x,y) + \left(1-e^{-x}-\tfrac{1}{2}xe^{-y}\right)\mathbb{1}_{\mathbb{R}^2_+(x\le y)}(x,y)$$

$$F_X(x) = \lim_{x\to+\infty} F(x,y) = \lim_{x\to+\infty}\left(1-e^{-y}-\tfrac{1}{2}ye^{-x}\right)\mathbb{1}_{\mathbb{R}_+}(y) = (1-e^{-y})\mathbb{1}_{\mathbb{R}_+}(y)$$

$$F_Y(y) = \lim_{y\to+\infty} F(x,y) = \lim_{y\to+\infty}\left(1-e^{-x}-\tfrac{1}{2}xe^{-y}\right)\mathbb{1}_{\mathbb{R}^*_+}(x) = (1-e^{-x})\mathbb{1}_{\mathbb{R}_+}(x)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\left(e^{-y}-\tfrac{1}{2}e^{-x}\right)\mathbb{1}_{\mathbb{R}^2_+(x>y)}(x,y) + \left(\tfrac{1}{2}xe^{-y}\right)\mathbb{1}_{\mathbb{R}^2_+(x\le y)}(x,y)\right] =$$

$$= \tfrac{1}{2}e^{-x}\mathbb{1}_{\mathbb{R}^2_+(x>y)}(x,y) + \tfrac{1}{2}e^{-y}\mathbb{1}_{\mathbb{R}^2_+(x\le y)}(x,y)$$

$$\rightarrow \int_{(-\infty,x]\times(-\infty,y]} \left(\tfrac{1}{2}e^{-u}\mathbb{1}_{\mathbb{R}^2_+(u>v)}(u,v) + \tfrac{1}{2}e^{-v}\mathbb{1}_{\mathbb{R}^2_+(u\le v)}(u,v)\right) d\mu_L^2(u,v)$$ data luogo

$$= \frac{1}{2}\int_{[0,x]\times[0,y]} \left(e^{-u}\mathbb{1}_{(u>v)}(u,v) + e^{-v}\mathbb{1}_{(u\le v)}(u,v)\right) d\mu_L^2(u,v) =$$

$$= \frac{1}{2}\left[\int_{[0,x]\times[0,y]} e^{-u}\mathbb{1}_{(u>v)}(u,v)\, d\mu_L^2(u,v) + \int_{[0,x]\times[0,y]} e^{-v}\mathbb{1}_{(u\le v)}(u,v)\, d\mu_L^2(u,v)\right] =$$

$$= \frac{1}{2}\left[\int_{u=0}^{x} du \int_{v=0}^{u} e^{-u}\, dv + \int_{v=0}^{y} dv \int_{u=0}^{v} e^{-v}\, du\right] =$$

$$= \frac{1}{2}\left[\int_{u=0}^{x} e^{-u}\, du\, [v]_0^u + \int_{v=0}^{y} e^{-v}\, dv\, [u]_0^v\right] =$$

$$= \frac{1}{2}\left[\int_{u=0}^{x} u e^{-u}\, du + \int_{v=0}^{y} v e^{-v}\, dv\right] =$$

$$\cancel{\tfrac{1}{2}} \int_{u=0}^{x} u e^{-u}\, du = \left[-ue^{-u}\right]_0^x + \int_0^x e^{-u}\, du =$$

$$= (-xe^{-x}) + \left[-e^{-u}\right]_0^x = -xe^{-x} + (-e^{-x}+1) =$$

$$= 1 - e^{-x} - xe^{-x}$$

$$\rightsquigarrow \frac{1}{2}\left(1-e^{-x}-xe^{-x}+1-e^{-y}-ye^{-y}\right) \neq F(x,y)$$

Riprendiamo l'esercizio 6 di prima...

$$f(x,y) = \frac{4x+2y}{3} \mathbb{1}_{[0,1]}(x) \mathbb{1}_{[0,1]}(y)$$

$$f_X(x) = \left(\frac{4}{3}x+\frac{1}{3}\right)\mathbb{1}_{[0,1]}(x)$$

X, Y NON indipendenti

$$f_Y(y) = \left(\frac{2}{3}y+\frac{2}{3}\right)\mathbb{1}_{[0,1]}(y)$$

$\rightarrow$ ~~$E[X|Y=y] = 0$ perché l'evento $\{Y=y\}$ ha probabilità nulla.~~

$\rightarrow$ ~~$E[X|Y] =$~~ $\rightarrow E[X|Y=y] = \int_{\mathbb{R}} x \, f_{X|Y}(x,y)\, dx$

dove $f_{X|Y}(x,y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

Per quanto riguarda $E[X|Y]$ proprio boh...

Esercizio 7:

$$f(x_1,x_2,x_3) = K(x_1+x_2^2+x_3^2)\mathbb{1}_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)$$

$$\int_{\mathbb{R}^3} K(x_1+x_2^2+x_3^2)\mathbb{1}_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)\, d\mu_L^3(x_1,x_2,x_3) =$$

$$= K\int_{[0,1]\times[0,1]\times[0,1]} (x_1+x_2^2+x_3^2)\, d\mu_L^3(x_1,x_2,x_3) =$$

$$= K\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\, (x_1+x_2^2+x_3^2) =$$

$$= K\int_0^1 dx_1 \int_0^1 dx_2 \left[x_1x_3 + x_2^2x_3 + \frac{1}{3}x_3^3\right]_0^1 =$$

$$= K\int_0^1 dx_1 \int_0^1 dx_2 \left(x_1+x_2^2+\frac{1}{3}\right) =$$

$$= K\int_0^1 dx_1 \left[x_1x_2 + \frac{1}{3}x_2^3 + \frac{1}{3}x_2\right]_0^1 =$$

$$= K\int_0^1 dx_1 \left(x_1+\frac{1}{3}+\frac{1}{3}\right) = K\int_0^1 \left(\frac{2}{3}+x_1\right)dx_1 = K\left[\frac{2}{3}x_1+\frac{1}{2}x_1^2\right]_0^1 =$$

$$= K\left(\frac{2}{3}+\frac{1}{2}\right) = \frac{4+3}{6}K = \frac{7}{6}K = 1 \Rightarrow K = \frac{6}{7}$$

$$\Rightarrow f(x_1,x_2,x_3) = \frac{6}{7}(x_1+x_2^2+x_3^2)\mathbb{1}_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)$$

$$\rightarrow P\left(X_2 < \tfrac{1}{2},\ X_3 > \tfrac{1}{2}\right) = \int_{\mathbb{R}^3\cap\{X_2<1/2\}\cap\{X_3>1/2\}} f(x_1,x_2,x_3)\, d\mu_L^3(x_1,x_2,x_3) =$$

$$= \frac{6}{7}\int_{[0,1]\times[0,1/2]\times(1/2,1]} (x_1+x_2^2+x_3^2)\, d\mu_L^3(x_1,x_2,x_3) =$$

$$= \frac{6}{7}\int_0^1 dx_1 \int_0^{1/2} dx_2 \int_{1/2}^1 dx_3\, (x_1+x_2^2+x_3^2) =$$

$$= \frac{6}{7}\int_0^1 dx_1 \int_0^{1/2} dx_2 \left[x_1x_3 + x_2^2x_3 + \frac{1}{3}x_3^3\right]_{1/2}^1 =$$

$$= \frac{6}{7}\int_0^1 dx_1 \int_0^{1/2} dx_2 \left(x_1+x_2^2+\frac{1}{3}-\frac{1}{2}x_1-\frac{1}{2}x_2^2-\frac{1}{24}\right) =$$

$$= \frac{6}{7}\int_0^1 dx_1 \int_0^{1/2} dx_2 \left(\frac{1}{2}x_1+\frac{1}{2}x_2^2+\frac{7}{24}\right) =$$

$$= \frac{6}{7}\int_0^1 dx_1 \left[\frac{1}{2}x_1x_2+\frac{1}{6}x_2^3+\frac{7}{24}x_2\right]_0^{1/2} =$$

$$= \frac{6}{7}\int_0^1 dx_1 \left(\tfrac{1}{4}x_1 + \tfrac{1}{48} + \tfrac{7}{48}\right) =$$

$$= \frac{6}{7}\int_0^1 dx_1 \left(\tfrac{1}{4}x_1 + \tfrac{1}{6}\right) = \frac{6}{7}\left[\frac{1}{8}x_1^2 + \frac{1}{6}x_1\right]_0^1 =$$

$$= \frac{6}{7}\left(\frac{1}{8}+\frac{1}{6}\right) = \frac{6}{7}\,\frac{6+8}{48} = \frac{6}{7}\cdot\frac{7}{24} = \frac{1}{4}$$

$$\rightarrow f_{X_1,X_2}(x_1,x_2) = \int_{\mathbb{R}} f(x_1,x_2,x_3)\, d\mu_L(x_3) =$$

$$= \int_{\mathbb{R}} \frac{6}{7}(x_1+x_2^2+x_3^2)\,\mathbb{1}_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)\, d\mu_L(x_3) =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2)\int_{[0,1]}(x_1+x_2^2+x_3^2)\, d\mu_L(x_3) =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2)\int_0^1 (x_1+x_2^2+x_3^2)\, dx_3 =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2)\left[x_1x_3 + x_2^2x_3 + \frac{1}{3}x_3^3\right]_0^1 =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]\times[0,1]}(x_1,x_2)\left(x_1+x_2^2+\tfrac{1}{3}\right)$$

$$E[(X_1,X_2)^T] = (E[X_1], E[X_2])$$

$$\rightarrow f_{X_1}(x_1) = \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_2) = \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_1)\int_{[0,1]}\left(x_1+x_2^2+\tfrac{1}{3}\right) d\mu_L(x_2)$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_1)\int_0^1 \left(x_1+x_2^2+\tfrac{1}{3}\right) dx_2 =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_1)\left[x_1x_2 + \frac{1}{3}x_2^3 + \frac{1}{3}x_2\right]_0^1 =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_1)\left(x_1+\frac{1}{3}+\frac{1}{3}\right) = \frac{6}{7}\left(x_1+\frac{2}{3}\right)\mathbb{1}_{[0,1]}(x_1)$$

$$\rightarrow f_{X_2}(x_2) = \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_1) = \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_2)\int_{[0,1]}\left(x_1+x_2^2+\tfrac{1}{3}\right) d\mu_L(x_1) =$$

$$= \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_2)\int_0^1\left(x_1+x_2^2+\tfrac{1}{3}\right) dx_1 = \frac{6}{7}\,\mathbb{1}_{[0,1]}(x_2)\left[\frac{1}{2}x_1^2 + x_1x_2^2 + \frac{1}{3}x_1\right]_0^1 =$$

$$= \frac{6}{7}\left(\frac{5}{6}+x_2^2\right)\mathbb{1}_{[0,1]}(x_2)$$

$$\rightarrow E[X_1] = \int_{\mathbb{R}} x_1 f_{X_1}(x_1)\, d\mu_L(x_1) = \int_{\mathbb{R}} x_1 \frac{6}{7}\left(x_1+\frac{2}{3}\right)\mathbb{1}_{[0,1]}(x_1)\, d\mu_L(x_1) =$$

$$= \frac{6}{7}\int_{[0,1]}\left(x_1^2+\frac{2}{3}x_1\right) d\mu_L(x_1) = \frac{6}{7}\int_0^1\left(x_1^2+\frac{2}{3}x_1\right) dx_1 = \frac{6}{7}\left[\frac{1}{3}x_1^3+\frac{1}{3}x_1^2\right]_0^1$$

$$= \frac{6}{7}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{6}{7}\cdot\frac{2}{3} = \frac{4}{7}$$

$$\rightarrow E[X_2] = \int_{\mathbb{R}} x_2 f_{X_2}(x_2)\, d\mu_L(x_2) = \cancel{\frac{6}{7}}\int_{\mathbb{R}} x_2 \frac{6}{7}\left(\frac{5}{6}+x_2^2\right)\mathbb{1}_{[0,1]}(x_2)\, d\mu_L(x_2)$$

$$= \frac{6}{7}\int_{[0,1]}\left(\frac{5}{6}x_2+x_2^3\right) d\mu_L(x_2) = \frac{6}{7}\int_0^1\left(\frac{5}{6}x_2+x_2^3\right) dx_2 =$$

$$= \frac{6}{7}\left[\frac{5}{12}x_2^2+\frac{1}{4}x_2^4\right]_0^1 = \frac{6}{7}\left(\frac{5}{12}+\frac{1}{4}\right) = \frac{6}{7}\,\frac{8}{12} = \frac{4}{7}$$

$$\Rightarrow E[(X_1,X_2)^T] = \left[\tfrac{4}{7}, \tfrac{4}{7}\right]$$

Ultimissimo hw pre-esonero:

Esercizio 4:

$E[Y|X] = E[Y] \Rightarrow Cov(X,Y)=0$ con X,Y non necessariam. indipendenti

$X \sim Ber(p)$ $\quad Z \sim N(0,1)$ $\quad Y = XZ$ $\quad (X,Z$ indip.$)$

$Cov(X,Y) = E[XY]-E[X]E[Y] = E[X^2Z]-E[X]E[XZ] =$
$= E[X^2]E[Z]-E^2[X]E[Z] = E[Z]D^2[X] = 0$

$E[Y] = E[XZ] = E[X]E[Z] = 0$

$E[Y|X] = E[XZ|X] = XE[Z|X] = XE[Z] = 0$

Verifico se X,Y sono indip:

$P(X \le x, Y \le y) = P(X \le x, Y \le y, X=0) + P(X \le x, Y \le y, X=1)$
$= P(X \le x, Y \le y | X=0)P(X=0) + P(X \le x, Y \le y | X=1)P(X=1) =$
$= (1-p)\, P(0 \le x, 0 \le y) + p\, P(1 \le x, Z \le y) =$

- $x<0 \wedge y<0 \Rightarrow 0$
- $0 \le x < 1 \wedge y<0 \Rightarrow 0$
- $x \ge 1 \wedge y<0 \Rightarrow p\, P(Z \le y)$
- $x<0 \wedge y \ge 0 \Rightarrow 0$
- $0 \le x < 1 \wedge y \ge 0 \Rightarrow 1-p$

$P(X \le x)P(Y \le y) = P(X \le x)P(XZ \le y) = P(X \le x)[P(XZ \le y | X=1)P(X=1)$
$+ P(XZ \le y | X=0)P(X=0)] =$
$= P(X \le x)[P(Z \le y)P(X=1) + P(0 \le y)P(X=0)] =$
$= P(X \le x)[p\, P(Z \le y) + (1-p)\, P(0 \le y)]$

$\rightsquigarrow$ Se $0 \le x < 1 \wedge y<0 \Rightarrow (1-p)[p\, P(Z \le y)] \neq 0$

↳ Già qui si vede che $P(X \le x, Y \le y) \neq P(X \le x)P(Y \le y)$
$\Rightarrow X,Y$ non indipendenti.

$Cov(X,Y) = 0 \not\Rightarrow E[Y|X] = E[Y]$

$X \sim N(0,1)$, $\quad Y = X^2$

$Cov(X,Y) = Cov(X,X^2) = E[X^3] - E[X]E[X^2] = 0$

$E[Y|X] = E[X^2|X] = X^2 \neq E[X^2]$ ✓

Prima o poi fatti gli esercizi 8, 9.

↳ Domani, se hai tempo, fatti anche quelli sulle prob. condiz. degli esami passati.

$$\Rightarrow \begin{cases} E[X|Z] = \frac{1}{2}(X+Y) = \frac{1}{2} Z \\ E[Y|Z] = \frac{1}{2}(X+Y) = \frac{1}{2} Z \end{cases}$$

Non ne sono sicuro però

Ricominciamo

$X \sim Y \sim Rad(\frac{1}{2})$ indipendenti $\qquad Z = X+Y$

$$E[X|Z] = \sum_{n \in \{1,2,3\}} E[X|Z=z_n] \, \mathbb{1}_{\{Z=z_n\}} =$$

$$= E[X|Z=-2] \, \mathbb{1}_{\{Z=-2\}} + E[X|Z=0] \, \mathbb{1}_{\{Z=0\}} + E[X|Z=2] \, \mathbb{1}_{\{Z=2\}}$$

$$= -1 \, \mathbb{1}_{\{Z=-2\}} + 0 \cdot \mathbb{1}_{\{Z=0\}} + 1 \cdot \mathbb{1}_{\{Z=2\}} = \frac{1}{2} Z$$

Analogamente per $E[Y|Z]$

Allora boh ci fidiamo

$$Cov(E[X|Z], E[Y|Z]) = Cov(\tfrac{1}{2} Z, \tfrac{1}{2} Z) = D^2[\tfrac{1}{2} Z] = \tfrac{1}{4} D^2[Z]$$

$$= \tfrac{1}{4} D^2[X+Y] \underset{X,Y \text{ indip.}}{=} \tfrac{1}{4}[D^2[X] + D^2[Y]] = \tfrac{1}{4}[E[X^2] + E[Y^2]] =$$

$$= \tfrac{1}{4}(1+1) = \tfrac{1}{2}$$ (ho riciclato i calcoli della pag. preced.)

$\Rightarrow E[X|Z], E[Y|Z]$ NON sono correlate $\Rightarrow$ non possono essere neanche indipendenti.

$\rightarrow E[(X+Y)^2|Z] \underset{\text{data}}{=} \cancel{E[\ldots]}\; E[(X+Y)^2|X+Y] = (X+Y)^2$

luogo

$\rightarrow E[XY|Z] = E[XY|X+Y]$

~~Considera~~

Considera invece $E[(X+Y)^2|X+Y] = E[X^2+2XY+Y^2|X+Y]$

$= E[X^2|X+Y] + 2E[XY|X+Y] + E[Y^2|X+Y] = (X+Y)^2$

$= \cancel{E[X^2]} \Rightarrow E[X^2] + 2E[XY|X+Y] + E[Y^2] = (X+Y)^2$

X^2, Y^2 sono v.a. Dirac $\Rightarrow$ sono indipendenti da chiunque

$\Rightarrow 1 + 2E[XY|X+Y] + 1 = (X+Y)^2$

$\Rightarrow 2E[XY|X+Y] = (X+Y)^2 - 2$

$\Rightarrow E[XY|X+Y] = \frac{1}{2}(X+Y)^2 - 1$ ✓

Esercizio 9:

$X \sim Y \sim Ber(1/2)$ indipendenti $\qquad Z = X+Y$

$E[X|Z] = \sum_{n \in \{1,2,3\}} E[X|Z=z_n] \mathbb{1}_{\{Z=z_n\}} =$

$= E[X|Z=0]\mathbb{1}_{\{Z=0\}} + E[X|Z=1]\mathbb{1}_{\{Z=1\}} + E[X|Z=2]\mathbb{1}_{\{Z=2\}}$

$= 0 + 1/2\, \mathbb{1}_{\{Z=1\}} + 1\,\mathbb{1}_{\{Z=2\}} = 1/2\, Z$

$\rightsquigarrow$ Analogam.. $E[Y|Z] = 1/2\, Z$

$Cov(E[X|Z], E[Y|Z]) = Cov(1/2\, Z, 1/2\, Z) \underset{\text{data}}{=} D^2[1/2\, Z]$

luogo

$= 1/4\, D^2[Z] = 1/4\, D^2[X+Y] \underset{X,Y \text{ indip.}}{=}$

$= 1/4\, [D^2[X] + D^2[Y]] = 1/4\, [1/4 + 1/4] = 1/8$

$\Rightarrow E[X|Z], E[Y|Z]$ NON sono scorrelate

$\Rightarrow$ NON sono indipendenti.

$\rightarrow E[(X+Y)^2|Z] = E[(X+Y)^2|X+Y] = (X+Y)^2$

$\rightarrow E[XY|Z] = E[XY|X+Y]$

$\rightsquigarrow E[(X+Y)^2|X+Y] = \cancel{E[X^2] + 2E[XY|X+Y]}$

$= E[X^2|X+Y] + 2E[XY|X+Y] + E[Y^2|X+Y] = (X+Y)^2$

$= E[X|X+Y] + 2E[XY|X+Y] + E[Y|X+Y]$

$\Rightarrow 1/2\, Z + 2E[XY|X+Y] + 1/2\, Z = Z^2$

$\Rightarrow 2E[XY|Z] = Z^2 - Z$

$\Rightarrow E[XY|Z] = 1/2\, (Z^2 - Z)$

Esercizio Demaggo: $R_1 \sim R_2 \sim Rad(1/2)$ indipendenti

$X = R_1 - R_2$, $\quad Y = -R_1 R_2$

$\rightarrow E[R_1|X] = E[R_1|R_1 - R_2] =$

$= \sum_{n\in\{1,2,3\}} E[R_1|X = x_n]\, \mathbb{1}_{\{X=x_n\}} =$

$= E[R_1|X=-2]\,\mathbb{1}_{\{X=-2\}} + E[R_1|X=0]\,\mathbb{1}_{\{X=0\}} +$

$+ E[R_1|X=2]\,\mathbb{1}_{\{X=2\}} = -1\,\mathbb{1}_{\{X=-2\}} + 0\cdot\mathbb{1}_{\{X=0\}} + 1\,\mathbb{1}_{\{X=2\}} = \frac{1}{2}X$

$\rightarrow E[R_2|X] = \sum_{n\in\{1,2,3\}} E[R_2|X=x_n]\,\mathbb{1}_{\{X=x_n\}} =$

$= E[R_2|X=-2]\,\mathbb{1}_{\{X=-2\}} + E[R_2|X=0]\,\mathbb{1}_{\{X=0\}} + E[R_2|X=2]\,\mathbb{1}_{\{X=2\}}$

$= 1\,\mathbb{1}_{\{X=-2\}} + 0\,\mathbb{1}_{\{X=0\}} - 1\,\mathbb{1}_{\{X=2\}} = -\frac{1}{2}X$

$\rightarrow E[R_1|Y] = E[R_1|-R_1R_2] = \sum_{m\in\{1,2\}} E[R_1|Y=y_m]\,\mathbb{1}_{\{Y=y_m\}} =$

$= E[R_1|Y=1]\,\mathbb{1}_{\{Y=1\}} + E[R_1|Y=-1]\,\mathbb{1}_{\{Y=-1\}} =$

$= 0\,\mathbb{1}_{\{Y=1\}} + 0\,\mathbb{1}_{\{Y=-1\}} \Rightarrow E[R_1|Y] \sim Dirac(0)$

$\rightarrow E[R_2|Y] = \sum_{m\in\{1,2\}} E[R_2|Y=y_m]\,\mathbb{1}_{\{Y=y_m\}} = E[R_2|Y=1]\,\mathbb{1}_{\{Y=1\}} +$

$+ E[R_2|Y=-1]\,\mathbb{1}_{\{Y=-1\}} = 0\,\mathbb{1}_{\{Y=1\}} + 0\,\mathbb{1}_{\{Y=-1\}}$

$\Rightarrow E[R_2|Y] \sim Dirac(0)$

$Cov(E[R_1|X], E[R_2|X]) = Cov(\frac{1}{2}X, -\frac{1}{2}X) =$

$= E[-\frac{1}{4}X^2] - E[\frac{1}{2}X]E[-\frac{1}{2}X] =$

$= -\frac{1}{4}E[X^2] + \frac{1}{4}E^2[X]$

$E[X^2] = \sum_{i=1}^{3} x_i^2 P(X=x_i) = (-2)^2\cdot\frac{1}{4} + 0^2\cdot\frac{1}{2} + 2^2\cdot\frac{1}{4} =$

$= 4\cdot\frac{1}{4} + 0 + 4\cdot\frac{1}{4} = 2$

$E[X] = 0$

$\Rightarrow Cov(E[R_1|X], E[R_2|X]) = -\frac{1}{4}\cdot 2 + 0 = -\frac{1}{2}$

$\Rightarrow E[R_1|X], E[R_2|X]$ NON sono ~~indipendenti~~ scorrelate

$\Rightarrow$ NON sono manco indipendenti.

Essendo due Dirac, $E[R_1|Y], E[R_2|Y]$ sono indipendenti e, quindi, scorrelate.

$E[X|Y] = E[R_1 - R_2|Y] = E[R_1|Y] - E[R_2|Y]$

$\Rightarrow E[X|Y] \sim Dirac(0)$

$E[Y|X] = E[-R_1R_2|R_1 - R_2]$

$\rightsquigarrow E[(X-Y)^2|X] = E[X^2|X] - 2E[XY|X] + E[Y^2|X]$

$= X^2 - 2X\,E[Y|X] + E[Y^2|X]$

~~E[(X+Y)²] X≠Y~~

data

luogo Boh provo a calcolare $Cov(X,Y) =$

$= Cov(R_1 - R_2, -R_1R_2) = E[-(R_1-R_2)R_1R_2] - E[R_1-R_2]E[-R_1R_2]$

$= -E[R_1^2R_2] + E[R_1R_2^2] + (E[R_1]E[-R_1R_2] - E[R_2]E[-R_1R_2])$

$= -E[R_1^2]E[R_2] + E[R_1]E[R_2^2] + E^2[R_1]E[R_2] - E[R_1]E^2[R_2]$

$= -1 \cdot 0 + 0 \cdot 1 + 0 - 0 = 0$

$P(R_1 - R_2 \le x, -R_1R_2 \le y) = P(X \le x, Y \le y)$

- $x < -2 \vee y < -1 \Rightarrow 0$
- $x \ge 2 \wedge y \ge 1 \Rightarrow 1$

$R_1 = -1 \quad R_2 = 1$

- $-2 \le x < 0 \wedge -1 \le y < 1 \Rightarrow P(R_1 - R_2 \le x, -R_1R_2 \le y) = 0$

$~~= P(R_1 = 1, R_2 = 1) = 1/4~~$

- $-2 \le x < 0 \wedge y \ge 1 \Rightarrow P(R_1 = -1, R_2 = 1) = 1/4$
- $0 \le x < 2 \wedge -1 \le y < 1 \Rightarrow P((R_1 = -1, R_2 = -1) \cup (R_1 = 1, R_2 = 1))$

$= 1/2$

data

- $0 \le x < 2 \wedge y \ge 1 \Rightarrow P((R_1 = -1, R_2 = -1) \cup (R_1 = 1, R_2 = 1) \cup (R_1 = -1, R_2 = 1))$

$= 3/4$

luogo

- $x \ge 2 \wedge -1 \le y < 1 \Rightarrow P((R_1 = -1, R_2 = -1) \cup (R_1 = 1, R_2 = 1)) = 1/2$

$P(R_1 - R_2 \le x)\, P(-R_1R_2 \le y) =$

- $x < -2 \vee y < -1 \Rightarrow 0$
- $x \ge 2 \wedge y \ge 1 \Rightarrow 1$
- $-2 \le x < 0 \wedge -1 \le y < 1 \Rightarrow 1/4 \cdot 1/2 = 1/8$

$\Rightarrow$ Già si vede che X, Y non sono indip.

Tornando a $E[Y|X]$...

$$E[Y|X] = \sum_{n \in \{1,2,3\}} E[Y|X = x_n] \mathbb{1}_{\{X = x_n\}} =$$

$$= E[Y|X = -2] \mathbb{1}_{\{X = -2\}} + E[Y|X = 0] \mathbb{1}_{\{X = 0\}} + E[Y|X = 2] \mathbb{1}_{\{X = 2\}}$$

$R_1 = -1, R_2 = 1 \qquad R_1 = -1, R_2 = -1 \mid R_1 = 1, R_2 = 1 \qquad R_1 = 1, R_2 = -1$

$$= 1 \mathbb{1}_{\{X = 2\}} - 1 \mathbb{1}_{\{X = 0\}} + 1 \mathbb{1}_{\{X = 2\}} = |X| - 1$$

Poiché $E[X|Y]$ è una Dirac $\Rightarrow E[X|Y], E[Y|X]$ sono indipendenti e, quindi, scorrelate.

data $E[X^2|Y] = E[(R_1-R_2)^2|Y] =$

luogo $= E[R_1^2|Y] - 2E[R_1R_2|Y] + E[R_2^2|Y] =$

$$= E[R_1^2|-R_1R_2] + 2E[R_1R_2|-R_1R_2] + E[R_2^2|-R_1R_2] =$$

$$= E[R_1^2] + 2(-R_1R_2) + E[R_2^2] =$$

$$= 1 - 2R_1R_2 + 1 = 2+2Y$$

$$E[Y^2|X] = E[R_1^2R_2^2|R_1-R_2] = E[R_1^2R_2^2] = 1$$

CPS - es. successioni di variabili aleatorie

data

luogo

$\cancel{5}$ $(X_n)_{n\geq 1}$ identicamente distribuite

$$f_X(x) := \frac{2}{x^3} \mathbb{1}_{(1,+\infty)}(x)$$

$$Y_n := \frac{X_n}{n^\alpha} \quad \forall n \geq 1 \quad , \text{ dove } \alpha > 0$$

→ CONVERGENZA IN DISTRIBUZIONE

$$F_{Y_n}(y) = P(Y_n \leq y) = P\left(\frac{X_n}{n^\alpha} \leq y\right) = P(X_n \leq y n^\alpha) =$$

$$= F_{X_n}(y n^\alpha) =: F_X(y n^\alpha) = \int_{(-\infty, y n^\alpha]} f_X(x)\, d\mu_L(x) =$$

$n^\alpha \geq 1$ SEMPRE

$$= \int_{(-\infty, y n^\alpha)} \frac{2}{x^3} \mathbb{1}_{(1,+\infty)}(x)\, d\mu_L(x) =$$

- $y n^\alpha \leq 1$ (ovvero $y < \frac{1}{n^\alpha}$) $\Rightarrow F_{Y_n}(y) = 0$
- $y n^\alpha > 1$ (ovvero $y \geq \frac{1}{n^\alpha}$) $\Rightarrow F_{Y_n}(y) = \int_{(1, y n^\alpha)} \frac{2}{x^3}\, d\mu_L(x) =$

$$= \int_1^{y n^\alpha} 2x^{-3}\, dx = 2\left[\frac{x^{-2}}{-2}\right]_1^{y n^\alpha} = -\left[(y n^\alpha)^{-2} - (1)^{-2}\right] =$$

$$= 1 - \frac{1}{y^2 n^{2\alpha}}$$

$$\Rightarrow F_{Y_n}(y) = \left(1 - \frac{1}{y^2 n^{2\alpha}}\right) \mathbb{1}_{\left(\frac{1}{n^\alpha}, +\infty\right)}(y)$$

- $y \leq 0 \Rightarrow F_{Y_n}(y) = 0 \Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = 0$
- $y > 0 \Rightarrow F_{Y_n}(y) = 1 - \frac{1}{y^2 n^{2\alpha}}$ DEFIN. $\Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = 1$

$\Rightarrow F_Y(y) = \begin{cases} 0 & \text{se } y<0 \\ 1 & \text{se } y>0 \end{cases}$

$\Rightarrow F_Y(y)$ differisce da $H(y)$ solo nel suo punto di discontinuità $(y=0)$ $\Rightarrow Y_n \xrightarrow{w} Dirac(0)$

→ CONVERGENZA IN PROBABILITÀ

Se una successione di v.a. converge debolmente a una v.a. di Dirac, allora converge anche in probabilità alla v.a. di Dirac $\Rightarrow Y_n \xrightarrow{P} Dirac(0)$

→ CONVERGENZA IN MEDIA

$$E[|Y_n - Dirac(0)|] = E[Y_n] = E\left[\frac{X_n}{n^\alpha}\right] = \frac{1}{n^\alpha} E[X_n] =$$

$$= \frac{1}{n^\alpha} \int_{\mathbb{R}} x f_X(x)\, d\mu_L(x) = \frac{1}{n^\alpha} \int_{\mathbb{R}} \frac{2}{x^2} \mathbb{1}_{(1,+\infty)}(x)\, d\mu_L(x) =$$

$$= \frac{1}{n^\alpha} \int_1^{+\infty} \frac{2}{x^2}\, dx = \frac{2}{n^\alpha} \int_1^{+\infty} x^{-2}\, dx = \frac{2}{n^\alpha} \left[\frac{x^{-1}}{-1}\right]_1^{+\infty} =$$

$$= -\frac{2}{n^\alpha} \left[\frac{1}{x}\right]_1^{+\infty} = \frac{2}{n^\alpha} \xrightarrow{n\to+\infty} 0$$

$\Rightarrow Y_n \xrightarrow{L^1} Dirac(0)$

→ CONVERGENZA IN MEDIA QUADRATICA

$$E[|Y_n - Dirac(0)|^2] = E[Y_n^2] = E\left[\frac{X_n^2}{n^{2\alpha}}\right] = \frac{1}{n^{2\alpha}} E[X_n^2] =$$

$$= \frac{1}{n^{2\alpha}} \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_L(x) = \frac{1}{n^{2\alpha}} \int_1^{+\infty} \frac{2}{x}\, dx =$$

$$= \frac{2}{n^{2\alpha}} [\ln|x|]_1^{+\infty} =$$

$\Rightarrow X_n$ non ammette momento del 2° ordine (e quindi nemmeno i successivi)

$\Rightarrow Y_n$ non ammette momento del secondo ordine (e quindi nemmeno i successivi)

Possiamo dunque dire che $Y_n \xrightarrow{L^p} Dirac(0) \Leftrightarrow p=1$

{Per ora salto la 2° richiesta}

4

6 $(X_n)_{n\geq 1}$, $X_n \sim Ber\left(\frac{1}{n^\alpha}\right)$, $\alpha>0$ → X_n tutte indip. tra loro

$Y_n := \min\{X_1, \dots, X_n\}$

$\boxed{\text{PARTIAMO DALLE } X_n}$

→ CONVERGENZA IN DISTRIBUZIONE

$$F_{X_n}(x) = \begin{cases} 0 & \text{se } x<0 \\ 1-\frac{1}{n^\alpha} & \text{se } 0\leq x<1 \\ 1 & \text{se } x\geq 1 \end{cases}$$

$\Rightarrow \lim_{n\to+\infty} F_{X_n}(x) =$

- Se $x<0 \Rightarrow = 0$
- Se $x\geq 1 \Rightarrow$ ~~[illegible]~~ 1
- Se $0\leq x<1 \Rightarrow = \lim_{n\to+\infty} \left(1-\frac{1}{n^\alpha}\right) = 1$

$$\Rightarrow F_X(x) = \begin{cases} 0 & \text{se } x<0 \\ 1 & \text{se } x\ge 0 \end{cases}$$

$$\Rightarrow F_X(x) = \mathcal{H}(x) \quad \forall x \in \mathbb{R} \Rightarrow X_n \xrightarrow{w} Dirac(0)$$

→ CONVERGENZA IN PROBABILITÀ

Dalla teoria sappiamo che

$$X_n \xrightarrow{w} Dirac(0) \Longrightarrow X_n \xrightarrow{P} Dirac(0)$$

→ CONVERGENZA IN MEDIA

$$E[|X_n - Dirac(0)|] = E[X_n] = 1\cdot\frac{1}{n^\alpha} + 0\left(1-\frac{1}{n^\alpha}\right) =$$

$$= \frac{1}{n^\alpha} \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow X_n \xrightarrow{L^1} Dirac(0)$$

$$E[|X_n - Dirac(0)|^p] = E[X_n^p] =$$

$$= \sum_{i=0}^{1} x_i^p P(X_n = x_i) = 0^p P(X_n=0) + 1^p P(X_n=1) = \frac{1}{n^\alpha}$$

$$\xrightarrow{n\to+\infty} 0 \Longrightarrow X_n \xrightarrow{L^p} Dirac(0) \quad \forall p\ge 1$$

$\boxed{\text{PASSIAMO ALLE } Y_n}$

→ CONVERGENZA IN DISTRIBUZIONE

$$F_{Y_n}(y) = P(Y_n \le y) = 1 - P(Y_n > y) = \cancel{1-P(}$$

$$= 1 - P(\min\{X_1,\dots,X_n\} > y) =$$

$$= 1 - P(X_1>y)\dots P(X_n>y) =$$

$$\cancel{=1-[1-P(X} = 1 - \prod_{i=1}^{n}(1 - P(X_i \le y)) =$$

$$\cancel{= 1 - [1 - P(X_n \le y)]^n = 1 - [1 - F_{X_n}(y)]^n}$$

- Se $y<0 \Rightarrow \cancel{1-[1-0]^n} = 0$
- Se $y\ge 1 \Rightarrow \cancel{1-[1-1]^n} = 1$
- Se $0\le y<1 \Rightarrow 1 - \left[1 - \left(1-\frac{1}{n^\alpha}\right)\right]^n = 1 - \left[\frac{1}{n^\alpha}\right]^n =$

$$= 1 - \frac{1}{n^{\alpha n}} \qquad \Rightarrow Y_n \sim Ber\left(\frac{1}{n^{\alpha n}}\right)$$

$$\Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = \qquad (F_Y(y) =)$$

- Se $y<0 \Rightarrow\ = 0$
- Se $y\ge 1 \Rightarrow\ = 1$
- Se $0\le y<1 \Rightarrow \lim_{n\to+\infty}\left(1-\frac{1}{n^{\alpha n}}\right) = 1$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & \text{se } y<0 \\ 1 & \text{se } y\ge 0 \end{cases}$$

$$\Rightarrow F_Y(y) = \mathcal{H}(y) \quad \forall y\in\mathbb{R} \Rightarrow Y_n \xrightarrow{w} Dirac(0)$$

→ CONVERGENZA IN PROBABILITÀ

Dalla teoria sappiamo che

$$Y_n \xrightarrow{w} Dirac(0) \Rightarrow Y_n \xrightarrow{P} Dirac(0)$$

→ CONVERGENZA IN MEDIA

$E[|Y_n - Dirac(0)|^p] = E[Y_n^p] =$

$$= \sum_{i=0}^{1} y_i^p\, P(Y_n = y_i) = 0^p \cdot P(Y_n=0) + 1^p\, P(Y_n=1) =$$

$$= 0\left(1-\frac{1}{n^{\alpha n}}\right) + 1\cdot\frac{1}{n^{\alpha n}} = \frac{1}{n^{\alpha n}} \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow Y_n \xrightarrow{L^p} Dirac(0) \qquad \forall p\geq 1$$

$\boxed{\text{CONVERGENZA QUASI CERTA}}$

X_n

$$P\left(\bigcap_{m\geq n}\{|X_m - Dirac(0)|<\varepsilon\}\right) \leq P\left(\bigcap_{m=n}^{2n}\{X_n<\varepsilon\}\right) \overset{X_n \text{ indip}}{=}$$

$$= \prod_{m=n}^{2n} P(X_n<\varepsilon) = \prod_{m=n}^{2n} P(X_n=0) = \prod_{n=m}^{2n}\left(1-\frac{1}{m^\alpha}\right) \leq$$

Poiché $X_n \sim Ber(\frac{1}{n^\alpha})$, questo è vero $\forall \varepsilon: 0<\varepsilon\leq 1$

$$\leq \prod_{m=n}^{2n}\left(1-\frac{1}{(2n)^\alpha}\right) = \left(1-\frac{1}{(2n)^\alpha}\right)^n$$

~~→ Se $\alpha = 1 \Rightarrow \lim_{n\to+\infty}\left(1-\frac{1}{2n}\right)^n = \frac{1}{\sqrt{e}}$ $\boxed{<1}$~~

~~(e per $\alpha>1$ il limite è ancora + piccolo)~~

$$\lim_{n\to+\infty}\left(1-\frac{1}{(2n)^\alpha}\right)^n = \cancel{\lim_{n\to+\infty}\left[\left(1-\frac{1}{(2n)^\alpha}\right)^{(2n)^\alpha}\right]} = \frac{1}{2}$$

Dovrebbe essere minore di 1 $\forall \alpha>0$ ma non ci scommetterei

Rifacciamo

$$P\left(\bigcap_{m\geq n}\{|X_m - Dirac(0)|<\varepsilon\}\right) \leq P\left(\bigcap_{n=m}^{[m^\alpha+m]}\{X_n<\varepsilon\}\right) \overset{X_n \text{ indip}}{=}$$

$$= \prod_{n=m}^{[m^\alpha+m]} P(X_n<\varepsilon) = \prod_{n=m}^{[m^\alpha+m]} P(X_n=0) = \prod_{n=m}^{[m^\alpha+m]}\left(1-\frac{1}{n^\alpha}\right) \leq$$

$$\leq \prod_{n=m}^{[m^\alpha+m]}\left(1-\frac{1}{([m^\alpha]+m)^\alpha}\right)$$

TUA MADRE

Terzo tentativo

$$P\left(\bigcup_{m\geq n}\{|X_n - X|\geq\varepsilon\}\right)$$

MA CHI TE SENCULA

Dopo aver dato una sbirciata alle soluzioni...

X_n

$$\to P(\{|X_n - \overset{Dirac(0)}{X}|\geq\varepsilon\}) = P(X_n\geq\varepsilon) =$$

$$= \begin{cases} P(X_n=1) & \text{se } 0<\varepsilon\leq 1 \\ 0 & \text{se } \varepsilon>1 \end{cases}$$

$$\sum_{n=1}^{+\infty} P(\{|X_n-X|\geq\varepsilon\}) = \sum_{n=1}^{+\infty} P(X_n\geq\varepsilon) =$$

$$= \begin{cases} \sum_{n=1}^{+\infty} P(X_n=1) & \text{se } 0<\varepsilon\leq 1 \\ 0 & \text{se } \varepsilon>0 \end{cases} =$$

$$= \begin{cases} \sum_{n=1}^{+\infty} \left(\frac{1}{n^\alpha}\right) & \text{se } 0<\varepsilon\le 1 \\ 0 & \text{se } \varepsilon > 1 \end{cases} =$$

$$= \begin{cases} 0 & \text{se } \varepsilon > 1 \\ \zeta(\alpha) & \text{se } 0<\varepsilon\le 1 \wedge \alpha > 1 \\ \text{DIVERGE} & \text{se } 0<\varepsilon \le 1 \wedge 0<\alpha\le 1 \end{cases}$$

~~CASO GENERALE~~: CASO GENERALE RISP. ε:

X_n ammette $\sum_{n=1}^{+\infty} P(X_n \geq \varepsilon)$ finito se $\alpha > 1$

Ma in generale è vero che $\sum_{n=1}^{+\infty} P(X_n \geq \varepsilon) \geq P\left(\bigcup_{n=1}^{+\infty} \{X_n \geq \varepsilon\}\right)$

$\Rightarrow \forall \alpha > 1 \ \exists$ finito $\lim_{n\to+\infty} X_n(\omega)$

O meglio:

X_n ammette $\sum_{n=1}^{+\infty} P(X_n \geq \varepsilon)$ finito se $\alpha > 1$

~~Ma in generale~~ $\Rightarrow X_n$ ammette $\lim_{m\to+\infty} \sum_{n\geq m} P(X_n \geq \varepsilon)$ finito se $\alpha > 1$

$$\lim_{m\to+\infty} \sum_{n\geq m} P(X_n \geq \varepsilon) = \lim_{m\to+\infty} \sum_{n\geq m} P(X_n = 1) = \lim_{m\to+\infty} \sum_{n\geq m} \left(\frac{1}{n^\alpha}\right)$$

Cioè semplicemente qua Monte ha detto che, nel momento in cui $\sum_{n=1}^{+\infty} P(|X_n - \text{Dirac}(0)| \geq \varepsilon)$ converge, allora $\exists$ finito $\lim_{n\to+\infty} X_n(\omega)$, e la successione $(X_n)_{n\geq 1}$ converge a.s. a Dirac(0)

24/11/2019 $X \sim Exp(1)$

$$Y_n := \begin{cases} n & \text{se } 0 \leq X < \frac{1}{n} \\ 0 & \text{se } X \geq \frac{1}{n} \end{cases}$$

$\rightarrow P(X \geq \frac{1}{n}) = 1 - P(X < \frac{1}{n}) = e^{-1/n}$

$$\Rightarrow Y_n := \begin{cases} n & \text{con } P(Y=n) = 1-e^{-1/n} \\ 0 & \text{con } P(Y=0) = e^{-1/n} \end{cases} \Rightarrow Y_n \sim Ber(1-e^{-1/n})$$

$\rightarrow$ CONVERGENZA DEBOLE:

$$F_{Y_n}(y) = \begin{cases} 0 & \text{se } y<0 \\ e^{-1/n} & \text{se } 0\le y<n \\ 1 & \text{se } y \geq n \end{cases}$$

- Se $y<0 \Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = 0$
- Se $y \geq 0 \Rightarrow \exists n_y \in \mathbb{N} : \forall n > n_y \quad n > y$

$\Rightarrow F_{Y_n}(y) = e^{-1/n}$ DEFINITIVAMENTE $\Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = 1$

$$\Rightarrow \lim_{n\to+\infty} F_{Y_n}(y) = F_Y(y) = \begin{cases} 0 & \text{se } y<0 \\ 1 & \text{se } y\geq 0 \end{cases}$$

$\Rightarrow F_Y(y) = H(y) \ \forall y \in \mathbb{R} \Rightarrow Y_n \xrightarrow{W} \text{Dirac}(0)$

$\rightarrow$ CONVERGENZA IN PROBABILITÀ:

Dalla teoria sappiamo che:

$Y_n \xrightarrow{W} \text{Dirac}(0) \Rightarrow Y_n \xrightarrow{P} \text{Dirac}(0)$

→ CONVERGENZA QUASI CERTA:

data

Sia $E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\} \Rightarrow P(E_0) = P(X=0) = 0$

luogo

Sia $\omega \in \Omega \setminus E_0 \Rightarrow X(\omega) > 0 \Rightarrow X(\omega) > 1/n$ DEFINITIVAMENTE

$\Rightarrow Y_n(\omega) = 0$ DEFINITIVAMENTE $\Rightarrow \lim_{n \to +\infty} Y_n(\omega) = 0$

$\Rightarrow Y_n \xrightarrow{a.s.} \text{Dirac}(0)$

→ CONVERGENZA IN MEDIA

$$E[|Y_n - \text{Dirac}(0)|] = E[Y_n] = 0 \cdot e^{-1/n} + n(1 - e^{-1/n}) =$$

$$= n(1 - e^{-1/n})$$

$$\lim_{n \to +\infty} n(1 - e^{-1/n}) = \lim_{n \to +\infty} n\left[1 - \left(1 - \frac{1}{n}\right)\right] =$$

$$= \lim_{n \to +\infty} n\left[\frac{1}{n}\right] = 1 \neq 0 \Rightarrow$$

$\Rightarrow$ Non c'è convergenza in media

$\Rightarrow$ Non può esserci convergenza in media quadratica.

12/02/2020 $\check{X}_n := \max\{X_1, \dots, X_n\}$, dove $X_1, \dots, X_n$ sono estratti da $X \sim \text{Unif}(0, \theta)$ (nella beata speranza che siano indipendenti)

→ CONVERGENZA DEBOLE

$$F_{\check{X}_n}(x) = P(\check{X}_n < x) = \prod_{i=1}^{n} P(X_i < x)$$

- Se $x \le 0 \Rightarrow F_{\check{X}_n}(x) = 0$
- Se $x > \theta \Rightarrow F_{\check{X}_n}(x) = 1$
- Se $0 < x \le \theta \Rightarrow F_{\check{X}_n}(x) = \left(\frac{x}{\theta}\right)^n$

data

luogo

$$\Rightarrow \lim_{n \to +\infty} F_{\check{X}_n}(x) = F_{\check{X}}(x) =$$

- Se $x \le 0 \Rightarrow 0$
- Se $x > \theta \Rightarrow 1$
- Se $0 < x \le \theta \Rightarrow \lim_{n \to +\infty} \left(\frac{x}{\theta}\right)^n = 0$

$$\Rightarrow F_{\check{X}}(x) = \begin{cases} 0 & \text{se } x \le \theta \\ 1 & \text{se } x > \theta \end{cases}$$

$$\Rightarrow F_{\check{X}}(x) = H(x - \theta) \quad \forall x \in \mathbb{R} \setminus \{\theta\}$$

$$\Rightarrow \check{X}_n \xrightarrow{W} \text{Dirac}(\theta)$$

→ CONVERGENZA IN PROBABILITÀ

~~Lo faccio senza sfruttare la teoria:~~

$\cancel{\lim_{n \to +\infty} P(\{|\check{X}_n - \text{Dirac}(\theta)| < \varepsilon\}) = \lim_{n \to +\infty} P(}$

Dalla teoria:

$$\check{X}_n \xrightarrow{W} \text{Dirac}(\theta) \Longrightarrow \check{X}_n \xrightarrow{P} \text{Dirac}(\theta)$$

→ CONVERGENZA IN MEDIA

$$E[|\check{X}_n - \text{Dirac}(\theta)|] =$$

→ CONVERGENZA IN PROBABILITÀ

Si mostra sfruttando la teoria oppure

$\lim_{n\to+\infty} P(\{|\check{X}_n - Dirac(\theta)| < \varepsilon\}) =$

$= \lim_{n\to+\infty} P(\{\theta - \check{X}_n < \varepsilon\}) = \lim_{n\to+\infty} P(\check{X}_n > \theta - \varepsilon) =$

$= \lim_{n\to+\infty} (1 - F_{\check{X}_n}(\theta-\varepsilon)) \overset{0<\theta-\varepsilon\le\theta}{=}$

$= \lim_{n\to+\infty} \left(1 - \left(\frac{\theta-\varepsilon}{\theta}\right)^n\right) = 1$

$\Rightarrow \check{X}_n \xrightarrow{P} Dirac(\theta)$

→ CONVERGENZA IN MEDIA

$E[|\check{X}_n - Dirac(\theta)|] = E[Dirac(\theta) - \check{X}_n] =$

$= \theta - E[\check{X}_n] = \theta - \int_{\mathbb{R}} x f_{\check{X}_n}(x)\, d\mu_L(x)$

Dove $f_{\check{X}_n}(x) = \frac{1}{\theta^n}(n) x^{n-1} \mathbb{1}_{[0,\theta]}(x)$

$\Rightarrow \theta - \int_{\mathbb{R}} n \left(\frac{x}{\theta}\right)^n \mathbb{1}_{[0,\theta]}(x)\, d\mu_L(x) = \theta - \int_0^\theta n \left(\frac{x}{\theta}\right)^n dx =$

$= \theta - \frac{n}{\theta^n}\left[\frac{x^{n+1}}{n+1}\right]_0^\theta = \theta - \frac{n}{\theta^n}\cdot\frac{\theta^{n+1}}{n+1} = \theta\left(1 - \frac{n}{n+1}\right)$

$\xrightarrow{n\to+\infty} 0 \quad \Rightarrow \check{X}_n \xrightarrow{L^1} Dirac(\theta)$

→ CONVERGENZA IN MEDIA QUADRATICA

$E[|\check{X}_n - Dirac(\theta)|^2] =$

$= E[\check{X}_n^2 - 2\theta \check{X}_n + \theta^2] = E[\check{X}_n^2] - 2\theta E[\check{X}_n] + \theta^2$.

$\leadsto E[\check{X}_n^2] = \int_{\mathbb{R}} x^2 f_{\check{X}_n}(x)\, d\mu_L(x) = \int_{\mathbb{R}} n \cdot \frac{1}{\theta^n} x^{n+1} \mathbb{1}_{[0,\theta]}(x)\, d\mu_L(x) =$

$= \frac{n}{\theta^n}\int_0^\theta x^{n+1}\, dx = \frac{n}{\theta^n}\left[\frac{x^{n+2}}{n+2}\right]_0^\theta = \frac{n}{\theta^n}\left(\frac{\theta^{n+2}}{n+2}\right) = \theta^2 \cdot \frac{n}{n+2}$

$\Rightarrow E[|\check{X}_n - Dirac(\theta)|^2] = \theta^2 \cdot \frac{n}{n+2} - 2\theta^2 \frac{n}{n+1} + \theta^2 =$

$= \theta^2\left(\frac{n}{n+2} - \frac{2n}{n+1} + 1\right) \xrightarrow{n\to+\infty} \theta^2(1-2+1) = \theta^2 \cdot 0 = 0$

$\Rightarrow \check{X}_n \xrightarrow{L^2} Dirac(\theta)$

→ CONVERGENZA QUASI CERTA

$F_{\check{X}_n}(x) = \left(\frac{x}{\theta}\right)^n \mathbb{1}_{[0,\theta]}(x) + \mathbb{1}_{[\theta,+\infty)}(x)$

Anche questo punto lo lascio in sospeso.

$(X_n)_{n\geq 1}$ 25/02/2020 data luogo $X_n(\omega) = \begin{cases} 1 & \text{se } 0\leq\omega\leq\frac{n+1}{2n} \quad (\text{con prob. } \frac{n+1}{2n}) \\ 0 & \text{altrimenti} \quad (\text{con prob. } 1-\frac{n+1}{2n}) \end{cases}$

- Prendo $B\in\mathcal{B}[0,1]$ t.c. $B\cap\{0\}\neq\emptyset \Rightarrow X_n(\omega)\in B$ è un evento: infatti, si verifica per $\forall\omega\in\Omega$.
- Prendo $B\in\mathcal{B}[0,1]$ t.c. $B\cap\{0\}\neq\emptyset \wedge B\cap\{1\}=\emptyset \Rightarrow X_n(\omega)\in B$ è un evento: infatti, si verifica per $\frac{n+1}{2n}<\omega\leq 1$
- Prendo $B\in\mathcal{B}[0,1]$ t.c. $B\cap\{0\}=\emptyset \wedge B\cap\{1\}\neq\emptyset \Rightarrow X_n(\omega)\in B$ è un evento: infatti, si verifica per $0\leq\omega\leq\frac{n+1}{2n}$
- Prendo $B\in\mathcal{B}[0,1]$ t.c. $B\cap\{0\}=\emptyset \wedge B\cap\{1\}=\emptyset \Rightarrow X_n(\omega)\in B$ è un evento: infatti, non si verifica per alcun ω.

→ CONVERGENZA IN DISTRIBUZIONE

$F_{X_n}(x) = P(X_n\leq x) =$

- Se $x<0 \Rightarrow F_{X_n}(x)=0$
- Se $x\geq 1 \Rightarrow F_{X_n}(x)=1$
- Se $0\leq x<1 \Rightarrow F_{X_n}(x) = P(X_n=0) = 1-\frac{n+1}{2n}$

$\Rightarrow F_X(x) = \lim_{n\to+\infty} F_{X_n}(x) =$

- Se $x<0 \Rightarrow F_X(x)=0$
- ~~Se ...~~
- Se $x\geq 1 \Rightarrow F_X(x) = 1$
- Se $0\leq x<1 \Rightarrow F_X(x) = 1/2$ $\Rightarrow X\sim Ber(1/2)$

$\Rightarrow X_n(\omega) \xrightarrow{w} X$, dove $X\sim Ber(1/2)$ data luogo

→ CONVERGENZA IN PROBABILITÀ

$\lim_{n\to+\infty} P(\{|X_n-X|<\varepsilon\})$ PANICO

Cose che per ora hai lasciato in sospeso:

- Esercizio 5 homework (2° richiesta) 3
- Ultimo punto di 12/02/2020 1
- 25/02/2020 2

↳ PROVO A RIPRENDERE QUESTO ESERCIZIO

→ CONVERGENZA IN PROBABILITÀ

Provo a considerare la v.a. $X_n - X$:

$$X_n - X = \begin{cases} 1 & \text{con probabilità } \frac{n+1}{4n} \\ 0 & \text{con probabilità } \frac{1}{2} \\ -1 & \text{con probabilità } \frac{1}{2}\left(1-\frac{n+1}{2n}\right) \end{cases}$$

Ora considero la v.a. $|X_n-X|$:

$$|X_n-X| = \begin{cases} 1 & \text{con probabilità } \frac{1}{2} \\ 0 & \text{con probabilità } \frac{1}{2} \end{cases}$$ → È questo a prescindere da n

Mi sembra un po' difficile che $\lim_{n\to+\infty} P(\{|X_n-X|<\varepsilon\})=1$

e che $\lim_{n\to+\infty} P(\{|X_n-X|\geq\varepsilon\})=0$

$\Rightarrow$ Non si direbbe che $X_n \xrightarrow{P} X$

In tal caso, non sarebbe neanche vero che $X_n \xrightarrow{a.s.} X$

→ CONVERGENZA IN MEDIA

data $E[|X_n - X|] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} \neq 0 \Rightarrow$

luogo

non c'è convergenza in media $\Rightarrow$

non può esserci nemmeno convergenza in media quadratica

NOT SURE

ORALE MAI

CPS - HW STIMATORI PUNTUALI 4)

data

luogo $f_X(x) = \frac{1}{\theta} e^{-\frac{x-3}{\theta}} \mathbb{1}_{[3,+\infty)}(x) \quad \forall x \in \mathbb{R}, \; \theta > 0$

METODO DEI MOMENTI:

$$E[X] = \int_{\mathbb{R}} x f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{x}{\theta} e^{-\frac{x-3}{\theta}} \mathbb{1}_{[3,+\infty)}(x)\, d\mu_L(x) =$$

$$= -e^{\frac{3}{\theta}} \int_3^{+\infty} -\frac{x}{\theta} e^{-\frac{x}{\theta}}\, dx = -e^{\frac{3}{\theta}} \left\{ \left[e^{-\frac{x}{\theta}} \cdot x \right]_3^{+\infty} + \theta \int_3^{+\infty} -\frac{1}{\theta} e^{-\frac{x}{\theta}}\, dx \right\} =$$

$$= -e^{\frac{3}{\theta}} \left\{ -e^{-\frac{3}{\theta}} \cdot 3 + \theta \left[e^{-\frac{x}{\theta}} \right]_3^{+\infty} \right\} = -e^{\frac{3}{\theta}} \left(-3e^{-\frac{3}{\theta}} - \theta e^{-\frac{3}{\theta}} \right) =$$

$$= 3 + \theta$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n} X_k = 3 + \hat{\theta}_M \Rightarrow \hat{\theta}_M = \overline{X}_n - 3$$

METODO DELLA ~~SOMIGLIANZA~~ MASSIMA VEROSIMIGLIANZA:

$$L_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;\theta) = \prod_{k=1}^{n} \frac{1}{\theta} e^{-\frac{x_k-3}{\theta}} \mathbb{1}_{[3,+\infty)}(x_k)$$

$$= \frac{1}{\theta^n} e^{\frac{3n}{\theta} - \sum_{k=1}^{n} \frac{x_k}{\theta}} \mathbb{1}_{[3,+\infty)^n}(x_1,\dots,x_n) =$$

$$\Rightarrow \log_e \left[L_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n) \right] = \left(-n \log_e \theta + \frac{3n}{\theta} - \sum_{k=1}^{n} \frac{x_k}{\theta} \right) \mathbb{1}_{[3,+\infty)^n}(x_1,\dots,x_n)$$

$$\Rightarrow \frac{d}{d\theta} \left\{ \log_e \left[L_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n) \right] \right\} = \left(-\frac{n}{\theta} - \frac{3n}{\theta^2} + \sum_{k=1}^{n} \frac{x_k}{\theta^2} \right) \mathbb{1}_{[3,+\infty)^n}(x_1,\dots,x_n) = 0$$

data

luogo

$$\Rightarrow \frac{1}{\theta} \left(\frac{1}{\theta} \sum_{k=1}^{n} x_k - \frac{3n}{\theta} - n \right) \mathbb{1}_{[3,+\infty)^n}(x_1,\dots,x_n) = 0$$

$$\Rightarrow \frac{1}{\theta} \sum_{k=1}^{n} x_k - \frac{3n}{\theta} = n \Rightarrow \sum_{k=1}^{n} x_k - 3n = n\theta$$

$$\Rightarrow \theta = \frac{1}{n} \sum_{k=1}^{n} x_k - 3 = \overline{x}_n - 3 \Rightarrow \hat{\theta}_{ML} = \overline{X}_n - 3$$

▸ $E[X] = 3 + \theta \Rightarrow \overline{X}_n = 3 + \hat{\theta}_M = 3 + \hat{\theta}_{ML}$ → stimatore per $E[X]$

▸ $E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{x^2}{\theta} e^{-\frac{x-3}{\theta}} \mathbb{1}_{[3,+\infty)}(x)\, d\mu_L(x) =$

$$= -e^{\frac{3}{\theta}} \int_3^{+\infty} -\frac{x^2}{\theta} e^{-\frac{x}{\theta}}\, dx = -e^{3/\theta} \left\{ \left[x^2 e^{-\frac{x}{\theta}} \right]_3^{+\infty} + 2\theta \int_3^{+\infty} -\frac{x}{\theta} e^{-\frac{x}{\theta}}\, dx \right\} =$$

$$= e^{3/\theta} \left[+9e^{-3/\theta} + 2\theta \left(+3e^{-3/\theta} + \theta e^{-3/\theta} \right) \right] =$$

$$= 9 + 2\theta(3+\theta) = 2\theta^2 + 6\theta + 9$$

$$\Rightarrow D^2[X] = E[X^2] - E^2[X] = 2\theta^2 + 6\theta + 9 - \theta^2 - 6\theta - 9 = \theta^2$$

$$\Rightarrow S^2_{X,n} = \hat{\theta}_M^2 = \hat{\theta}_{ML}^2$$ → stimatore per $D^2[X]$

↯

Anche se non ne sono sicurissimo

5)

data $f_X(x) = \theta x^{\theta-1} \mathbb{1}_{[0,1]}(x) \quad \forall x \in \mathbb{R},\ \theta > 0$

luogo METODO DEI MOMENTI:

$$E[X] = \int_{\mathbb{R}} x f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \theta x^{\theta} \mathbb{1}_{[0,1]}(x)\, d\mu_L(x) = \theta \int_0^1 x^{\theta}\, dx =$$

$$= \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \frac{\theta}{\theta+1}$$

$$\Rightarrow \frac{1}{n}\sum_{k=1}^{n} X_k = \frac{\hat{\theta}_n^M}{\hat{\theta}_n^M + 1} \Rightarrow \hat{\theta}_n^M = \hat{\theta}_n^M \cdot \frac{1}{n}\sum_{k=1}^{n} X_k + \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$\Rightarrow \hat{\theta}_n^M \left(1 - \frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n}\sum_{k=1}^{n} X_k \Rightarrow \hat{\theta}_n^M (1 - \overline{X}_n) = \overline{X}_n$$

$$\Rightarrow \hat{\theta}_n^M = \frac{\overline{X}_n}{1 - \overline{X}_n}$$

$$\cancel{E[\hat{\theta}_n^M] = E\left[\frac{\overline{X}_n}{1-\overline{X}_n}\right] = E\left[\frac{\overline{X}_n - 1 + 1}{1 - \overline{X}_n}\right] = E\left[\frac{\overline{X}_n - 1}{1-\overline{X}_n}\right] + E\left[\frac{1}{1-\overline{X}_n}\right] =}$$

$$\cancel{= -1 + E\left[\frac{1}{1-\overline{X}_n}\right]}$$

Oppure potrei osservare che:

$$E[\hat{\theta}_n^M] = E\left[\frac{\overline{X}_n}{1-\overline{X}_n}\right] \neq \frac{E[\overline{X}_n]}{E[1-\overline{X}_n]} = \frac{E[X]}{1 - E[X]} = \frac{\theta}{\theta+1} \cdot \frac{1}{1 - \frac{\theta}{\theta+1}}$$

$$= \frac{\theta}{\theta+1} \cdot \frac{\theta+1}{\theta+1-\theta} = \frac{\theta}{\theta+1} \cdot \frac{\theta+1}{1} = \theta$$

$\Rightarrow \hat{\theta}_n^M$ è distorto.

$\cancel{P(|\hat{\theta}_n^M - \theta| \geq \varepsilon) = P\left(\left|\frac{X_n}{1-\overline{X}_n} - \theta\right| \geq \varepsilon\right)}$ data

luogo

$$P(|\overline{X}_n - E[\overline{X}_n]| \geq \varepsilon) \leq \frac{D^2[\overline{X}_n]}{\varepsilon^2} = \frac{\frac{1}{n} D^2[X]}{\varepsilon^2}$$

$$\rightsquigarrow E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \theta x^{\theta+1} \mathbb{1}_{[0,1]}(x)\, d\mu_L(x) =$$

$$= \theta \int_0^1 x^{\theta+1}\, dx = \theta \left[\frac{x^{\theta+2}}{\theta+2} \right]_0^1 = \frac{\theta}{\theta+2}$$

$$\Rightarrow D^2[X] = E[X^2] - E^2[X] = \frac{\theta}{\theta+2} - \frac{\theta^2}{(\theta+1)^2} = \frac{\theta(\theta^2+2\theta+1) - \theta^2(\theta+2)}{(\theta+1)^2(\theta+2)}$$

$$= \frac{\theta^3 + 2\theta^2 + \theta - \theta^3 - 2\theta^2}{(\theta+1)^2(\theta+2)} = \frac{\theta}{(\theta+1)^2(\theta+2)}$$

$$\Rightarrow P(|\overline{X}_n - E[\overline{X}_n]| \geq \varepsilon) \leq \frac{\theta}{n(\theta+1)^2(\theta+2)\varepsilon^2} \xrightarrow{n \to +\infty} 0$$

$$\Rightarrow \overline{X}_n \xrightarrow{P} E[\overline{X}_n] = E[X] = \frac{\theta}{\theta+1}$$

$$\Rightarrow \overline{X}_n \xrightarrow{P} \frac{\theta}{\theta+1}$$

$$P(|1 - \overline{X}_n - E[1-\overline{X}_n]| \geq \varepsilon) \leq \frac{D^2[1-\overline{X}_n]}{\varepsilon^2} = \frac{D^2[\overline{X}_n]}{\varepsilon^2}$$

$$= \frac{\theta}{n(\theta+1)^2(\theta+2)\varepsilon^2} \xrightarrow{n \to +\infty} 0$$

$$\Rightarrow 1 - \overline{X}_n \xrightarrow{P} E[1 - \overline{X}_n] = 1 - E[X] = 1 - \frac{\theta}{\theta+1} = \frac{\theta+1-\theta}{\theta+1} = \frac{1}{\theta+1}$$

data $\rightsquigarrow$ $1-\overline{X}_n = 1-\frac{1}{n}\sum_{k=1}^{n} X_k$

luogo

è pari a 0 se $\sum_{k=1}^{n} X_k = n$

$$P\left(\sum_{k=1}^{n} X_k = n\right) = P(X_1=1, \dots, X_n=1) = 0 \quad ✓$$

$$\Rightarrow \frac{\overline{X}_n}{1-\overline{X}_n} \xrightarrow{P} \frac{\theta}{\theta+1}\cdot \theta+1 = \theta$$

$\Rightarrow \hat{\theta}_n^* = \frac{\overline{X}_n}{1-\overline{X}_n}$ è consistente in probabilità.

$$E[(\hat{\theta}_n^* - \theta)^2] = E\left[\frac{\overline{X}_n^2}{\overline{X}_n^2 - 2\overline{X}_n + 1}\right] - 2\theta E\left[\frac{\overline{X}_n}{1-\overline{X}_n}\right] + \theta^2 \to E\left[\frac{\overline{X}_n}{1-\overline{X}_n}\right] \neq \theta$$

$$\to E\left[\frac{\overline{X}_n^2}{\overline{X}_n^2 - 2\overline{X}_n + 1}\right] \neq \frac{E[\overline{X}_n^2]}{E[\overline{X}_n^2] - 2E[\overline{X}_n] + 1}$$

Dove $E[\overline{X}_n^2] = D^2[\overline{X}_n] + E^2[\overline{X}_n] = \frac{1}{n} D^2[X] + E^2[X] =$

$$= \frac{\theta}{n(\theta+1)^2(\theta+2)} + \frac{\theta^2}{(\theta+1)^2} = \frac{\theta + n\theta^2(\theta+2)}{n(\theta+1)^2(\theta+2)}$$

$$\Rightarrow E\left[\frac{\overline{X}_n^2}{\overline{X}_n^2 - 2\overline{X}_n + 1}\right] \neq \frac{\frac{\theta+n\theta^2(\theta+2)}{n(\theta+1)^2(\theta+2)}}{\frac{\theta+n\theta^2(\theta+2)}{n(\theta+1)^2(\theta+2)} - \frac{2\theta}{\theta+1} + 1} =$$

$$= \frac{\theta[1+n\theta(\theta+2)]}{n(\theta+1)^2(\theta+2)} \cdot \frac{n(\theta+1)^2(\theta+2)}{\theta[1+n\theta(\theta+2)] - 2\theta n(\theta+1)(\theta+2) + n(\theta+1)^2(\theta+2)}$$

$$= \frac{\theta[1+n\theta(\theta+2)]}{\theta + n\theta^3 + 2n\theta^2 - 2n\theta(\theta^2+3\theta+2) + (n\theta+2n)(\theta^2+2\theta+1)} =$$

data luogo

$$= \frac{\theta[1+n\theta(\theta+2)]}{\theta + n\theta^3 + 2n\theta^2 - 2n\theta^3 - 6n\theta^2 - 4n\theta + n\theta^3 + 2n\theta^2 + n\theta + 2n\theta^2 + 4n\theta + 2n} =$$

$$= \frac{\theta[1+n\theta(\theta+2)]}{\theta + n\theta + 2n} =$$

$$= \frac{\theta^2(1/\theta + n\theta + 2n)}{\theta + n\theta + 2n} = \frac{\theta(1+n\theta^2+2n\theta)}{\theta+n\theta+2n}$$

$$\Rightarrow E[(\hat{\theta}_n^* - \theta)^2] \neq \frac{\theta(1+n\theta^2+2n\theta)}{\theta+n\theta+2n} - \theta^2$$ ← Non mi rappresenta un cazzo

↳ Boh è letteralmente uno schifo, questo lo chiediamo a Monte.

METODO DELLA MASSIMA VEROSOMIGLIANZA:

$$\mathcal{L}_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)=\prod_{k=1}^{n} f_X(x_k;\theta)=$$

$$=\prod_{k=1}^{n}\theta x_k^{\theta-1}\mathbb{1}_{[0,1]}(x)=\theta^n\left(\prod_{k=1}^{n}x_k^{\theta-1}\right)\mathbb{1}_{[0,1]^n}(x_1,\dots,x_n)$$

$$\to \log\left[\mathcal{L}_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)\right]=\left[n\log_e\theta+\sum_{k=1}^{n}(\theta-1)\log_e(x_k)\right]\mathbb{1}_{[0,1]^n}(x_1,\dots,x_n)$$

$$=\left[n\log_e(\theta)+(\theta-1)\sum_{k=1}^{n}\log_e(x_k)\right]\mathbb{1}_{[0,1]^n}(x_1,\dots,x_n)$$

$$\to\frac{d}{d\theta}\left\{\log_e\left[\mathcal{L}_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)\right]\right\}=\left(\frac{n}{\theta}+\sum_{k=1}^{n}\log(x_k)\right)\mathbb{1}_{[0,1]^n}(x_1,\dots,x_n)$$

$$\leadsto \frac{n}{\theta}+\sum_{k=1}^{n}\log(x_k)=0\Rightarrow -n=\theta\sum_{k=1}^{n}\log(x_k)$$

$$\Rightarrow\theta=-\frac{n}{\sum_{k=1}^{n}\log(x_k)}$$ ← Jo basito

$$\Rightarrow\hat{\theta}_n^{ML}=-n\cdot\frac{1}{\sum_{k=1}^{n}\log X_k}$$

Intanto passiamo ai successivi (con un po' di timore...)

6) $f_X(x)=\frac{1}{2}e^{-|x-\theta|}\quad\forall x\in\mathbb{R},\ \theta>0$

METODO DEI MOMENTI:

$$E[X]=\int_{\mathbb{R}}x f_X(x)\,d\mu_L(x)=\int_{\mathbb{R}}\frac{1}{2}xe^{-|x-\theta|}\,d\mu_L(x)=$$

$$=\int_{\mathbb{R}\cap\{x<\theta\}}\frac{1}{2}xe^{x-\theta}\,d\mu_L(x)+\int_{\mathbb{R}\cap\{x\geq\theta\}}\frac{1}{2}xe^{\theta-x}\,d\mu_L(x)=$$

$$=\frac{1}{2}e^{-\theta}\int_{-\infty}^{\theta}xe^{x}\,dx+\frac{1}{2}e^{\theta}\int_{\theta}^{+\infty}xe^{-x}\,dx=$$

$$=\frac{1}{2}e^{-\theta}\left\{\left[xe^{x}\right]_{-\infty}^{\theta}-\int_{-\infty}^{\theta}e^{x}\,dx\right\}+\frac{1}{2}e^{\theta}\left\{\left[-xe^{-x}\right]_{\theta}^{+\infty}-\int_{\theta}^{+\infty}-e^{-x}\,dx\right\}=$$

$$=\frac{1}{2}e^{-\theta}\left(\theta e^{\theta}-\left[e^{x}\right]_{-\infty}^{\theta}\right)+\frac{1}{2}e^{\theta}\left(\theta e^{-\theta}-\left[e^{-x}\right]_{\theta}^{+\infty}\right)=$$

$$=\frac{1}{2}e^{-\theta}\left(\theta e^{\theta}-e^{\theta}\right)+\frac{1}{2}e^{\theta}\left(\theta e^{-\theta}+e^{-\theta}\right)=$$

$$=\frac{1}{2}(\theta-1)+\frac{1}{2}(\theta+1)=\frac{1}{2}\theta-\frac{1}{2}+\frac{1}{2}\theta+\frac{1}{2}=\theta$$

$$\Rightarrow\hat{\theta}_n^{M}=\overline{X}_n$$

$$\leadsto E[\hat{\theta}_n^{M}]=E[\overline{X}_n]=E[X]=\theta\Rightarrow\hat{\theta}_n^{M}$$ è NON distorto.

data
luogo

$$P(|\hat{\theta}_n^M-\theta|\geq\varepsilon)=P(|\hat{\theta}_n^M-E[\hat{\theta}_n^M]|\geq\varepsilon)\leq\frac{D^2[\hat{\theta}_n^M]}{\varepsilon^2}=\frac{D^2[\bar{X}_n]}{\varepsilon^2}=\frac{\frac{1}{n}D^2[X]}{\varepsilon^2}$$

$$\Rightarrow E[X^2]=\int_{\mathbb{R}}x^2 f_X(x)\,d\mu_L(x)=\int_{\mathbb{R}}\frac{1}{2}x^2e^{-|x-\theta|}\,d\mu_L(x)=$$

$$=\int_{\mathbb{R}\cap\{x<\theta\}}\frac{1}{2}x^2e^{x-\theta}\,d\mu_L(x)+\int_{\mathbb{R}\cap\{x\geq\theta\}}\frac{1}{2}x^2e^{\theta-x}\,d\mu_L(x)=$$

$$=\frac{1}{2}e^{-\theta}\int_{-\infty}^{\theta}x^2e^{x}\,dx+\frac{1}{2}e^{\theta}\int_{\theta}^{+\infty}x^2e^{-x}\,dx=$$

$$=\frac{1}{2}e^{-\theta}\left\{\left[x^2e^{x}\right]_{-\infty}^{\theta}-2\int_{-\infty}^{\theta}xe^{x}\,dx\right\}+\frac{1}{2}e^{\theta}\left\{\left[-x^2e^{-x}\right]_{\theta}^{+\infty}+2\int_{\theta}^{+\infty}xe^{-x}\,dx\right\}$$

$$=\frac{1}{2}e^{-\theta}\left\{\theta^2e^{\theta}-2\left[\theta e^{\theta}-e^{\theta}\right]\right\}+\frac{1}{2}e^{\theta}\left\{\theta^2e^{-\theta}+2\left(\theta e^{-\theta}+e^{-\theta}\right)\right\}=$$

$$=\frac{1}{2}\left(\theta^2-2\theta+2\right)+\frac{1}{2}\left(\theta^2+2\theta+2\right)=\theta^2+2$$

$$\Rightarrow D^2[X]=E[X^2]-E^2[X]=\theta^2+2-\theta^2=2$$

$$\Rightarrow P(|\hat{\theta}_n^M-\theta|\geq\varepsilon)\leq\frac{2}{\varepsilon^2 n}\xrightarrow{n\to+\infty}0$$

data
luogo

$$E[(\hat{\theta}_n^M-\theta)^2]=E[(\bar{X}_n-\theta)^2]=$$

$$=E[\bar{X}_n^2-2\theta\bar{X}_n+\theta^2]=$$

$$=E[\bar{X}_n^2]-2\theta E[\bar{X}_n]+\theta^2=E[\bar{X}_n^2]-\theta^2=$$

$$=D[\bar{X}_n^2]+E^2[\bar{X}_n]-\theta^2=\frac{1}{n}D^2[X]+\theta^2-\theta^2=$$

$$=\frac{1}{n}\cdot 2\xrightarrow{n\to+\infty}0$$

$\Rightarrow\hat{\theta}_n^M$ è consistente in media quadratica.

METODO DELLA MASSIMA VEROSIMIGLIANZA:

$$L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)=\prod_{k=1}^{n}f_X(x_k;\theta)=\prod_{k=1}^{n}\frac{1}{2}e^{-|x_k-\theta|}=\frac{1}{2^n}\prod_{k=1}^{n}e^{-|x_k-\theta|}$$

$$\log(L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n))=-n\log(2)-\sum_{k=1}^{n}|x_k-\theta|$$

$\rightarrow$ Supponendo di conoscere gli n elementi della sommatoria, saremmo in grado di risolvere i valori assoluti e di "eliminarli". In tal modo, otterremmo una somma di tanti termini indipendenti da θ e una somma di termini pari a θ o a $-\theta$. Di conseguenza, se n è fissato, ~~la der~~ $\frac{d}{d\theta}L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)$ è pari a un numero che cade tra $-n$ e n. Affinché sia pari a 0, è necessario che la metà degli x_k sia minore di θ e l'altra metà sia maggiore di θ. ~~... è necessario che si verifi~~

data luogo

Ciò ci induce a prendere uno stimatore che sia pari ~~a $\sum$~~ alla media di $\sum_{k=1}^{n} X_k$, che non è altro che $\frac{1}{n}\sum_{k=1}^{n} X_k = \overline{X}_n$.

$\hookrightarrow \hat{\Theta}_n^{ML} = \overline{X}_n$

Per giunta, come avevamo dimostrato precedentemente, $\overline{X}_n$ è uno stimatore non distorto, consistente in probabilità e consistente in media quadratica.

7) $X \sim Unif(0,\theta)$

METODO DELLA MASSIMA VEROSIMIGLIANZA:

$$L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;\theta) = \prod_{k=1}^{n} \frac{1}{\theta}\, 1_{[0,\theta]}(x_k) =$$

$$= \frac{1}{\theta^n}\, 1_{[0,\theta]^n}(x_1,\dots,x_n)$$

$$\frac{\partial}{\partial\theta} L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n) = -n\theta^{-n-1}\, 1_{[0,\theta]^n}(x_1,\dots,x_n) = -\frac{n}{\theta^{n+1}}\, 1_{[0,\theta]^n}(x_1,\dots,x_n)$$

E se passassimo al logaritmo?

$$\log(L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)) = -n\log(\theta)\, 1_{[0,\theta]^n}(x_1,\dots,x_n)$$

$$\frac{\partial}{\partial\theta}(-n\log(\theta)) = -\frac{n}{\theta}\, 1_{[0,\theta]^n}(x_1,\dots,x_n) \to \text{Sto come prima}$$

↳ Sta densità congiunta non ha punti di massimo

data luogo

C'è un però: $L_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n)$ assume il valore massimo se $x_k \in [0,\theta]\ \forall k=1,\dots,n$ ma, al contempo, θ è il più piccolo possibile

↓

Sarei tentato di prendere $\hat{\Theta}_n^{ML} = \max\{X_1,\dots,X_n\} \equiv \check{X}_n$

$\to E[\check{X}_n] = ?$

$$F_{\check{X}_n}(x) = P(\check{X}_n \le x) = P(X_1 \le x,\dots,X_n \le x) = \prod_{k=1}^{n} P(X_k \le x) =$$

$$= \prod_{k=1}^{n} P(X \le x) = \prod_{k=1}^{n} F_X(x) = F_X^n(x) = \frac{x^n}{\theta^n}\, 1_{[0,\theta]}(x) + 1_{(\theta,+\infty)}(x)$$

$$\to F'_{\check{X}_n}(x) = \frac{1}{\theta^n}\cdot n x^{n-1}\, 1_{[0,\theta]}(x)$$

$$\to \int_{(-\infty,x]} \frac{1}{\theta^n} n u^{n-1}\, 1_{[0,\theta]}(u)\, d\mu_L(u) =$$

$\to x \le 0 \Rightarrow = 0$

$$\to 0 \le x < \theta \Rightarrow \int_0^x \frac{n}{\theta^n} u^{n-1}\, du = \frac{n}{\theta^n}\left[\frac{u^n}{n}\right]_0^x = \frac{n}{\theta^n}\cdot\frac{x^n}{n} = \frac{x^n}{\theta^n}$$

$$\to x > \theta \Rightarrow \int_0^\theta \frac{n}{\theta^n} u^{n-1}\, du = \frac{n}{\theta^n}\left[\frac{u^n}{n}\right]_0^\theta = \frac{n}{\theta^n}\cdot\frac{\theta^n}{n} = 1$$

$$\Rightarrow \int_{(-\infty,x]} \frac{1}{\theta^n} n u^{n-1}\, 1_{[0,\theta]}(u)\, d\mu_L(u) = F_{\check{X}_n}(x) \Rightarrow \check{X}_n \text{ è ass. continua.}$$

$$\to E[\check{X}_n] = \int_{\mathbb{R}} x\, f_{\check{X}_n}(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{n}{\theta^n} x^n\, 1_{[0,\theta]}(x)\, d\mu_L(x) =$$

$$= \frac{n}{\theta^n}\int_0^\theta x^n\, dx = \frac{n}{\theta^n}\left[\frac{x^{n+1}}{n+1}\right]_0^\theta = \frac{n}{\theta^n}\cdot\frac{\theta^{n+1}}{n+1} = \theta\cdot\frac{n}{n+1} \ne \theta \Rightarrow \check{X}_n \text{ è distorto}$$

METODO DEI MOMENTI: $E[X] = \frac{\theta}{2} \Rightarrow \bar{X}_n = \frac{\hat{\theta}_n^M}{2} \Rightarrow \hat{\theta}_n^M = 2\bar{X}_n$

$E[\hat{\theta}_n^M] = E[2\bar{X}_n] = 2E[X] = 2\frac{\theta}{2} = \theta$

$\Rightarrow \hat{\theta}_n^M$ è non distorto.

$P(|\hat{\theta}_n^M - \theta| \geq \varepsilon) = P(|\hat{\theta}_n^M - E[\hat{\theta}_n^M]| \geq \varepsilon) \leq \frac{D^2[\hat{\theta}_n^M]}{\varepsilon^2} =$

$= \frac{D^2[2\bar{X}_n]}{\varepsilon^2} = \frac{4D^2[\bar{X}_n]}{\varepsilon^2} = \frac{4}{n} \cdot \frac{D^2[X]}{\varepsilon^2} = \frac{4}{n} \cdot \frac{\theta^2}{12} \cdot \frac{1}{\varepsilon^2} \xrightarrow{n \to +\infty} 0$

$\Rightarrow \hat{\theta}_n^M$ è consistente in probabilità.

$E[(\hat{\theta}_n^M - \theta)^2] = E[(2\bar{X}_n - \theta)^2] = E[4\bar{X}_n^2 - 4\theta\bar{X}_n + \theta^2] =$

$= 4E[\bar{X}_n^2] - 4\theta E[X] + \theta^2 = 4(D^2[\bar{X}_n] + E^2[\bar{X}_n]) - 4\theta \cdot \frac{\theta}{2} + \theta^2$

$= 4\left(\frac{1}{n} D^2[X] + \frac{\theta^2}{4}\right) - \theta^2 = \frac{4}{n} \cdot \frac{\theta^2}{12} + \theta^2 - \theta^2 = \frac{\theta^2}{3n} \xrightarrow{n \to +\infty} 0$

$\Rightarrow \hat{\theta}_n^M$ è consistente in media quadratica.

---

8) $\lambda > 0$, $X \sim Poiss(\lambda)$

$Z_n = \sum_{k=1}^{n} X_k \Rightarrow Z_n \sim Poiss(n\lambda)$

$E[Z_n] = n\lambda$ $\quad D^2[Z_n] = n\lambda$

$\bar{X}_n = \frac{1}{n} Z_n$ $\quad S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2$

$\rightarrow E[\bar{X}_n] = E[X] = \lambda$

$\rightarrow E[S_n^2] = D^2[X] = \lambda$

$\Rightarrow$ Sì, $\bar{X}_n$, S_n^2 possono entrambi essere usati per stimare λ; tra l'altro, si tratta di stimatori non distorti.

$\rightarrow D^2[\bar{X}_n] = \frac{1}{n} D^2[X] = \frac{\lambda}{n}$

$\rightarrow D^2[S_n^2] = D^2\left[\frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2\right] =$

$= \frac{1}{(n-1)^2} D^2\left[\sum_{k=1}^{n} (X_k^2 - 2X_k\bar{X}_n + \bar{X}_n^2)\right] =$

$= \frac{1}{(n-1)^2} D^2\left[\sum_{k=1}^{n} X_k^2 - 2\bar{X}_n \sum_{k=1}^{n} X_k + n\bar{X}_n^2\right] =$

$= \frac{1}{(n-1)^2} D^2\left[\sum_{k=1}^{n} X_k^2 - n\bar{X}_n^2\right] =$

$= \frac{1}{(n-1)^2} \left[E\left[\left(\sum_{k=1}^{n} X_k^2\right)^2 - 2n\bar{X}_n^2 \sum_{k=1}^{n} X_k^2 + n^2\bar{X}_n^4\right] - E^2\left[\sum_{k=1}^{n} X_k^2 - n\bar{X}_n^2\right]\right]$

$= \frac{1}{(n-1)^2} \left\{E\left[\left(\sum_{k=1}^{n} X_k^2\right)^2\right] - 2n E\left[\bar{X}_n^2 \sum_{k=1}^{n} X_k^2\right] + n^2 E[\bar{X}_n^4] - E^2\left[\sum_{k=1}^{n} X_k^2 - n\bar{X}_n^2\right]\right\}$

È orribile

$\sum_{k=0}^{+\infty} k^4 \frac{e^{-\lambda}\lambda^k}{k!}$ (fatto sta che non dipende solo da n)

Ma esiste la formula!

$D^2[S_n^2] = \frac{D^4[X]}{n}\left(\frac{E[X^4]}{D^4[X]} - \frac{n-3}{n-1}\right) = \frac{\lambda^2}{n}\left(\frac{E[X^4]}{\lambda^2} - \frac{n-3}{n-1}\right)$

$\rightarrow$ Tende a 0 allo stesso modo di $D^2[\bar{X}_n]$ $\Rightarrow$ non avrei preferenze tra i

9) $\lambda > 0$ $\quad X \sim Exp(\lambda)$ $\quad T_n = \min(X_1, \dots, X_n)$

data

luogo

$\Rightarrow T_n \sim Exp(n\lambda)$

$E[T_n] = \frac{1}{n\lambda}$ $\qquad D^2[T_n] = \frac{1}{n^2\lambda^2}$

Ponendo $\theta := \frac{1}{\lambda}$, ottengo che $f_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \mathbb{1}_{[0,+\infty)}(x)$

$\Rightarrow E[T_n] = \frac{\theta}{n}$ $\qquad D^2[T_n] = \frac{\theta^2}{n^2}$

$\Rightarrow$ Uno stimatore non distorto per il parametro θ è: nT_n

$Z_n = \sum_{k=1}^{n} X_k \Rightarrow Z_n \sim \Gamma(n, \lambda)$

$E[Z_n] = n/\lambda = n\theta$ $\qquad D^2[Z_n] = n/\lambda^2 = n\theta^2$

Uno stimatore non distorto per il parametro θ è: $\frac{Z_n}{n}$

METODO DELLA MASSIMA VEROSIMIGLIANZA:

$$\mathcal{L}_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;\theta) = \prod_{k=1}^{n} \frac{1}{\theta} e^{-\frac{x_k}{\theta}} \mathbb{1}_{[0,+\infty)}(x) =$$

$$= \mathbb{1}_{[0,+\infty)^n}(x_1,\dots,x_n) \cdot \frac{1}{\theta^n} \prod_{k=1}^{n} e^{-\frac{x_k}{\theta}}$$

$$\log(\mathcal{L}_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n)) = \mathbb{1}_{[0,+\infty)^n}(x_1,\dots,x_n)\left[(-n)\log(\theta) - \sum_{k=1}^{n}\frac{x_k}{\theta}\right] =$$

$$= \mathbb{1}_{[0,+\infty)^n}(x_1,\dots,x_n)\left[-n\log\theta - \frac{1}{\theta}\sum_{k=1}^{n} x_k\right]$$

$$\frac{\partial}{\partial\theta}\left[\log(\mathcal{L}_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n))\right] = \mathbb{1}_{[0,+\infty)^n}(x_1,\dots,x_n)\left[-\frac{n}{\theta} + \frac{1}{\theta^2}\overbrace{\sum_{k=1}^{n} x_k}^{n\bar{x}_n}\right]$$

data

luogo

$$-\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{k=1}^{n} x_k = 0 \Leftrightarrow$$

$$\frac{1}{\theta^2}\sum_{k=1}^{n} x_k = \frac{n}{\theta} \Leftrightarrow \frac{n\bar{x}_n}{\theta^2} = \frac{n}{\theta} \Leftrightarrow$$

$$\frac{\theta^2}{n\bar{x}_n} = \frac{\theta}{n} \Leftrightarrow \frac{\theta^2}{\theta} = \frac{n\bar{x}_n}{n} \Leftrightarrow \theta = \bar{x}_n$$

$\rightsquigarrow \hat{\theta}_{MLE} = \bar{X}_n = \frac{Z_n}{n}$ $\rightsquigarrow$ Già sappiamo che è non distorto

$\rightarrow D^2[\hat{\theta}_{MLE}] = D^2\left[\frac{Z_n}{n}\right] = \frac{1}{n^2} D^2[Z_n] = \frac{\theta^2}{n}$

$\rightarrow D^2[nT_n] = n^2 D^2[T_n] = \theta^2$

$\Rightarrow$ Tra $\hat{\theta}_{MLE}$ e nT_n, è preferibile lo stimatore $\hat{\theta}_{MLE}$

---

10) $f_X(x) = \frac{1}{2}\mathbb{1}_{\{-1\}}(x) + \frac{1-\theta}{2}\mathbb{1}_{\{0\}}(x) + \frac{\theta}{2}\mathbb{1}_{\{1\}}(x)$ $\qquad \forall\theta\in[0,1]$

METODO DELLA MASSIMA VEROSIMIGLIANZA:

$$\mathcal{L}_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;\theta) = \prod_{k=1}^{n}\left(\frac{1}{2}\mathbb{1}_{\{-1\}}(x) + \frac{1-\theta}{2}\mathbb{1}_{\{0\}}(x) + \frac{\theta}{2}\mathbb{1}_{\{1\}}(x)\right)$$

$$\log(\mathcal{L}_{X_1,\dots,X_n}(\theta; x_1,\dots,x_n)) = \sum_{k=1}^{n}\log\left(\frac{1}{2}\mathbb{1}_{\{-1\}}(x) + \frac{1-\theta}{2}\mathbb{1}_{\{0\}}(x) + \frac{\theta}{2}\mathbb{1}_{\{1\}}(x)\right)$$

↳ È troppo brutto, magari io procederei col metodo dei momenti (ma non so se è quello che vuole l'esercizio).

METODO DEI MOMENTI:

data

$$E[X] = \frac{1}{2}(-1) + \frac{1-\theta}{2}\cdot 0 + \frac{\theta}{2}\cdot 1 = \frac{\theta}{2} - \frac{1}{2}$$

luogo

$$\Rightarrow \overline{X}_n = \frac{\hat{\theta}_n^M}{2} - \frac{1}{2} \Rightarrow \hat{\theta}_n^M = 2\overline{X}_n + 1$$

(così era troppo facile :))

Confidence Intervals for the Mean of a Population

✓ GAUSSIANA CON VARIANZA σ^2 NOTA (pag. 352)

Ricominciamo: 1) $X \sim N(\mu, \sigma^2)$ con μ, σ^2 IGNOTE

$x_1 = -1{,}5$ $\quad x_2 = -0{,}5$
$x_3 = 1{,}5$ $\quad x_4 = 2{,}0$
$x_5 = 2{,}5$

→ INTERVALLO DI CONFIDENZA PER μ AL LIVELLO DI CONFIDENZA $1-\alpha = 0{,}99$:

$$\left(\overline{X}_n - t^{+}_{\frac{\alpha}{2}, n-1} \frac{S_n}{\sqrt{n}} \; ; \; \overline{X}_n + t^{+}_{\frac{\alpha}{2}, n-1} \frac{S_n}{\sqrt{n}} \right)$$

Realizzazione di quest'intervallo:

$$\left(\bar{x}_n - t^{+}_{\frac{\alpha}{2}, n-1} \frac{s_n}{\sqrt{n}} \; ; \; \bar{x}_n + t^{+}_{\frac{\alpha}{2}, n-1} \frac{s_n}{\sqrt{n}} \right)$$

$\rightarrow \sqrt{n} = \sqrt{5} \approx 2{,}24$

$\rightarrow \bar{x}_n = \frac{1}{n} \sum_{k=1}^{n} x_k = \frac{1}{5}(-1{,}5 - 0{,}5 + 1{,}5 + 2{,}0 + 2{,}5) = 0{,}8$

$\rightarrow s_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x}_n)^2 = \frac{1}{4}\left[(-1{,}5 - 0{,}8)^2 + (-0{,}5 - 0{,}8)^2 + (1{,}5 - 0{,}8)^2 + (2{,}0 - 0{,}8)^2 + (2{,}5 - 0{,}8)^2\right] = 2{,}95$

$\Rightarrow s_n = \sqrt{2{,}95} \approx 1{,}72$

$\rightarrow t^{+}_{\frac{\alpha}{2}, n-1} = t^{-}_{1-\frac{\alpha}{2}, n-1} = t^{-}_{0{,}995, 4} = 4{,}604$

$\Rightarrow$ L'intervallo è:

$$\left(0{,}8 - 4{,}604 \cdot \frac{1{,}72}{2{,}24} \; ; \; 0{,}8 + 4{,}604 \cdot \frac{1{,}72}{2{,}24}\right) \equiv (-2{,}74 \; ; \; 4{,}34)$$

Viene inspiegabilmente diverso dal prof

$\rightarrow$ AMPIEZZA $= 2\, t^{+}_{\frac{\alpha}{2}, n-1} \frac{S_n}{\sqrt{n}} = 0{,}1$

$\Rightarrow t^{-}_{1-\frac{\alpha}{2}, 4} = \frac{0{,}1}{2} \cdot \frac{2{,}24}{1{,}72} \approx 0{,}065 \rightsquigarrow$ Questo dovrebbe voler dire che $\nexists\, \alpha$: l'intervallo sia ampio 0,1 con $n=5$ (ci serve un n più grande)

INTERVALLO DI CONFIDENZA PER σ^2 AL LIVELLO DI CONFIDENZA $1-\alpha = 0{,}9$:

$$\rightarrow \left(\frac{(n-1) S^2_{x,n}}{\chi^{2+}_{n-1, \alpha/2}} \; ; \; \frac{(n-1) S^2_{x,n}}{\chi^{2-}_{n-1, \alpha/2}} \right)$$

Realizzazione di quest'intervallo:

$$\left(\frac{(n-1) s^2_{x,n}}{\chi^{2+}_{n-1, \alpha/2}} \; ; \; \frac{(n-1) s^2_{x,n}}{\chi^{2-}_{n-1, \alpha/2}} \right)$$

$\chi^{2+}_{n-1, \alpha/2} = \chi^{2-}_{4, 0{,}95} = 9{,}488$

$\chi^{2-}_{n-1, \alpha/2} = \chi^{2-}_{4, 0{,}05} = 0{,}711$

$\Rightarrow$ L'intervallo è:

$$\left(\frac{4 \cdot 2{,}95}{9{,}488} \; ; \; \frac{4 \cdot 2{,}95}{0{,}711} \right) = (1{,}24 \; ; \; 16{,}60)$$

2) $n = 100$ $\quad \bar{x}_{100} = 18d$ $\quad s^2_{x,100} = 8d^2$

→ INTERVALLO DI CONFIDENZA PER μ AL LIVELLO DI CONFIDENZA $1-\alpha = 0{,}99$

$$\left(\overline{X}_n - z^{+}_{\alpha/2} \frac{S_n}{\sqrt{n}} \; ; \; \overline{X}_n + z^{+}_{\alpha/2} \frac{S_n}{\sqrt{n}} \right)$$

Realizzazione di quest'intervallo:

$$\left(18d - z^{+}_{\alpha/2} \frac{\sqrt{8}\, d}{10} \; ; \; 18d + z^{+}_{\alpha/2} \frac{\sqrt{8}\, d}{10} \right)$$

$z^{+}_{\alpha/2} = z^{-}_{1-\alpha/2} = z^{-}_{0{,}995} = 2{,}58$

Lo non cade in quest'intervallo (mi di

$\Rightarrow$ L'intervallo è:

$$\left(18d - \frac{2{,}58\sqrt{8}}{10} d \; ; \; 18d + \frac{2{,}58\sqrt{8}}{10} d \right) \equiv (17{,}27 \; ; \; 18{,}73)$$

3) $n=60$ $\quad \bar{x}_{60}=9,4$ $\quad S_{x,60}=1,5$

data $\rightarrow$ INTERVALLO DI CONFIDENZA PER μ AL LIVELLO DI CONFIDENZA $1-\alpha=0,99$:

luogo

$$\left(\bar{X}_n - z^+_{\alpha/2}\frac{S_n}{\sqrt{n}} \,;\, \bar{X}_n + z^+_{\alpha/2}\frac{S_n}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(9,4 - z^+_{\alpha/2}\frac{1,5}{\sqrt{60}} \,;\, 9,4 + z^+_{\alpha/2}\frac{1,5}{\sqrt{60}}\right)$$

$z^+_{\alpha/2} = z^-_{1-\alpha/2} = z^-_{0,995} = 2,58$ $\Rightarrow$ l'intervallo è:

$$\left(9,4-2,58\cdot\frac{1,5}{\sqrt{60}} \,;\, 9,4+2,58\cdot\frac{1,5}{\sqrt{60}}\right) \equiv (8,90 \,;\, 9,90)$$

$\hookrightarrow$ Non è stato necessario effettuare alcuna assunzione sulla distribuzione

ORA ASSUMIAMO CHE $X \sim N(\mu, 2)$

$\rightarrow$ INTERVALLO DI CONFIDENZA PER μ AL LIVELLO DI CONFIDENZA $1-\alpha=0,99$:

$$\left(\bar{X}_n - z^+_{\alpha/2}\frac{\sigma}{\sqrt{n}} \,;\, \bar{X}_n + z^+_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(\bar{x}_n - z^+_{\alpha/2}\frac{\sqrt{2}}{\sqrt{n}} \,;\, \bar{x}_n + z^+_{\alpha/2}\frac{\sqrt{2}}{\sqrt{n}}\right)$$

$z^+_{\alpha/2} = z_{0,995} = 2,58$

$\Rightarrow$ Ampiezza dell'intervallo $= 2\cdot 2,58\cdot\frac{\sqrt{2}}{\sqrt{n}} \leq 0,25$

$\Rightarrow \frac{2\cdot 2,58\cdot\sqrt{2}}{0,25} \leq \sqrt{n} \Rightarrow n \geq 128\cdot 2,58^2 \Rightarrow n \geq 852,019$

$\Rightarrow n \geq 853$

CPS – Altri esercizi sugli intervalli di confidenza | 4)

data

luogo

$X \sim N(\mu_X, \sigma_X^2)$ $Y \sim N(\mu_Y, \sigma_Y^2)$

$D \equiv Y - X$ è ragionevolmente gaussiana

→ INTERVALLO DI CONFIDENZA PER $\mu_Y - \mu_X$ AL LIVELLO DI CONFIDENZA $1-\alpha$:

Nella beata speranza che la media di D sia ragionevolmente $\mu_Y - \mu_X$:

$$\left(\bar{D}_n - t^{+}_{\frac{\alpha}{2}, n-1} \frac{S_{D,n}}{\sqrt{n}} ; \bar{D}_n + t^{+}_{\frac{\alpha}{2}, n-1} \frac{S_{D,n}}{\sqrt{n}}\right)$$

È così:

$E[D] = E[Y-X] = E[Y] - E[X]$

Realizzazione di quest'intervallo:

$$\left(d_n - t^{+}_{\frac{\alpha}{2}, n-1} \frac{s_{D,n}}{\sqrt{n}} ; d_n + t^{+}_{\frac{\alpha}{2}} \frac{s_{D,n}}{\sqrt{n}}\right)$$

→ ORA ABBIAMO $x_1, \dots, x_7$; $y_1, \dots, y_7$; $1-\alpha = 0{,}95$

$$d_7 = \frac{1}{7} \sum_{k=1}^{7} (y_k - x_k) = \frac{1}{7}\left[5{,}73 - 3{,}85 + 3{,}84 - 2{,}82 + 4{,}78 - 3{,}44 + 4{,}40 - 3{,}48 + 1{,}91 - 1{,}92 + 4{,}98 - 4{,}39 + 4{,}94 - 3{,}12\right] = 1{,}08$$

$$s^2_{D,7} = \frac{1}{6} \sum_{k=1}^{7} (y_k - x_k - d_7)^2 = \frac{1}{6}\left[(5{,}73 - 3{,}85 - 1{,}08)^2 + (3{,}84 - 2{,}82 - 1{,}08)^2 + (4{,}78 - 3{,}44 - 1{,}08)^2 + (4{,}40 - 3{,}48 - 1{,}08)^2 + (1{,}91 - 1{,}92 - 1{,}08)^2 + (4{,}98 - 4{,}39 - 1{,}08)^2 + (4{,}94 - 3{,}12 - 1{,}08)^2\right] =$$

$$= \frac{1}{6}(0{,}64 + 0{,}0036 + 0{,}0676 + 0{,}0256 + 1{,}1881 + 0{,}2401 + 0{,}5476) \simeq 0{,}45$$

$\Rightarrow s_{D,7} = \sqrt{0{,}45} \approx 0{,}67$ ⟶ Monte qua non ha estratto la radice

$t^{+}_{\alpha/2, n-1} = t^{-}_{1-\alpha/2, n-1} = t^{-}_{0{,}975, 6} = 2{,}447$ ⟹ L'intervallo è:

$$\left(1{,}08 - 2{,}447 \cdot \frac{0{,}67}{\sqrt{7}} ; 1{,}08 + 2{,}447 \cdot \frac{0{,}67}{\sqrt{7}}\right) \equiv (0{,}46 ; 1{,}70)$$

5) $m = n = 50$ data luogo $\bar{p}_A = \frac{45}{50} = 0,9$ $\bar{p}_B = \frac{30}{50} = 0,6$

A, B indipendenti

→ INTERVALLO DI CONFIDENZA PER $p_A - p_B$ AL LIVELLO DI CONFIDENZA $1-\alpha$:

$$\left(\bar{X}_m - \bar{Y}_n - z^+_{\frac{\alpha}{2}} \sqrt{\frac{S^2_{X,m}}{m} + \frac{S^2_{Y,n}}{n}} \; ; \; \bar{X}_m - \bar{Y}_n + z_{\frac{\alpha}{2}} \sqrt{\frac{S^2_{X,m}}{m} + \frac{S^2_{Y,n}}{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(0,9 - 0,6 - z^+_{\alpha/2} \sqrt{\frac{0,9 \cdot 0,1}{50} + \frac{0,6 \cdot 0,4}{50}} \; ; \; 0,9 - 0,6 + z^+_{\alpha/2} \sqrt{\frac{0,9 \cdot 0,1}{50} + \frac{0,6 \cdot 0,4}{50}}\right)$$

$\rightarrow 1 - \alpha = 0,95$

$\Rightarrow z^+_{\alpha/2} = z^-_{1-\alpha/2} = z^-_{0,975} = 1,96$ $\Rightarrow$ l'intervallo è:

$$\left(0,3 - 1,96\sqrt{0,0066} \; ; \; 0,3 + 1,96\sqrt{0,0066}\right) \equiv (0,14 \; ; \; 0,46)$$

$\Rightarrow$ 0 NON appartiene all'intervallo $\Rightarrow$ la droga ha un'efficacia non nulla con probabilità 0,95.

6) $X \sim Ber(p)$

$Z_n = \sum_{k=1}^{n} X_k \Rightarrow Z_n \sim Bin(n, p)$

⇝ Se $np \geq 10 \wedge n(1-p) \geq 10 \Rightarrow Z_n$ è approssimativam. una normale (o meglio una gaussiana con media np e con varianza $np(1-p)$)

→ INTERVALLO DI CONFIDENZA PER np AL LIVELLO DI CONFIDENZA $1-\alpha$:

$$\left(\bar{X}_n - t^+_{\frac{\alpha}{2}, n-1} \frac{S_n}{\sqrt{n}} \; ; \; \bar{X}_n + t^+_{\frac{\alpha}{2}, n-1} \frac{S_n}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(n\bar{p} - t^+_{\alpha/2, n-1} \frac{n\bar{p}(1-\bar{p})}{\sqrt{n}} \; ; \; n\bar{p} + t^+_{\frac{\alpha}{2}, n-1} \frac{n\bar{p}(1-\bar{p})}{\sqrt{n}}\right)$$

data luogo

→ Se invece vogliamo stimare $\bar{p}$, avremo:

$$\left(\bar{p} - t^+_{\alpha/2, n-1} \frac{\bar{p}(1-\bar{p})}{\sqrt{n}} \; ; \; \bar{p} + t^+_{\alpha/2, n-1} \frac{\bar{p}(1-\bar{p})}{\sqrt{n}}\right)$$

→ FISSATI α, W, ~~DEVO~~ TROVARE IL MINIMO n:

$$\text{AMPIEZZA} = 2 t^+_{\alpha/2, n-1} \frac{\bar{p}(1-\bar{p})}{\sqrt{n}} \leq W$$

Come faccio a esplicitarmi n se ce l'ho pure qua?

7) $X \sim (\mu, \sigma^2)$ μ, σ^2 IGNOTE

Grazie all'indip. e alla gaussianità di $X_1, ..., X_n$

→ La statistica $\bar{X}_n$ ha la seguente distribuzione: $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

→ La statistica $\frac{X_{n+1} - \bar{X}_n}{\sigma\sqrt{1 + 1/n}}$ ha una distribuzione normale standard.

Infatti: $E\left[\frac{X_{n+1} - \bar{X}_n}{\sigma \sqrt{1 + 1/n}}\right] = \frac{1}{\sigma\sqrt{1+\frac{1}{n}}}\left[E[X_{n+1}] - E[\bar{X}_n]\right] = 0$

$D^2\left[\frac{X_{n+1} - \bar{X}_n}{\sigma\sqrt{1+\frac{1}{n}}}\right] = \frac{1}{\sigma^2(1+\frac{1}{n})}\left[D^2[X_{n+1}] + D^2[\bar{X}_n]\right] = \frac{1}{\sigma^2(\frac{n+1}{n})} \cdot (\sigma^2 + \frac{1}{n}\sigma^2) = 1$

→ Sappiamo che: X gaussiana $\Rightarrow \frac{(n-1)S^2_{X,n}}{\sigma^2} \sim \chi^2_{n-1}$

$\Rightarrow S^2_{X,n} \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$

→ Sappiamo che:

- X gaussiana $\Rightarrow \bar{X}_n, S^2_{X,n}$ sono indipendenti
- X_{n+1} indipendente da $X_1, ..., X_n \Rightarrow X_{n+1}$ indipendente da $S^2_{X,n}$

$\Rightarrow X_{n+1} - \bar{X}_n$ e $S^2_{X,n}$ sono INDIPENDENTI

data
luogo

$\rightarrow$ La statistica $\dfrac{X_{n+1}-\overline{X}_n}{S_{X,n}\sqrt{1+1/n}}$ ha la distribuzione di Student con $n-1$ gradi di libertà.

Intervallo di confidenza per il valore di X_{n+1} con livello di confid. $1-\alpha$:

$$\left(\overline{X}_n - t^+_{\alpha/2,n-1} S_{X,n}\sqrt{1+\tfrac{1}{n}} \; ; \; \overline{X}_n + t^+_{\alpha/2,n-1} S_{X,n}\sqrt{1+\tfrac{1}{n}}\right)$$

La realizzazione di quest'intervallo è:

$$\left(\overline{x}_n - t^+_{\alpha/2,n-1} S_{x,n}\sqrt{1+\tfrac{1}{n}} \; ; \; \overline{x}_n + t^+_{\alpha/2,n-1} S_{x,n}\sqrt{1+1/n}\right)$$

$\rightarrow x_1 = 7005$ $\quad x_2 = 7432$ $\quad x_3 = 7420$ $\quad x_4 = 6822$

$x_5 = 6752$ $\quad x_6 = 5333$ $\quad x_7 = 6552$ $\quad 1-\alpha = 0,95$

~~L'intervallo diventa:~~ $\bullet\ \overline{x}_n = \frac{1}{7}\sum_{k=1}^{7} x_k =$

$= \frac{1}{7}(7005+7432+7420+6822+6752+5333+6552) \cong 6759$

$\bullet\ S^2_{x,n} = \frac{1}{6}\sum_{k=1}^{7}(x_k - x_n)^2 = \frac{1}{6}\left[(7005-6759)^2+(7432-6759)^2+(7420-6759)^2+\right.$

$\left.+(6822-6759)^2+(6752-6759)^2+(5333-6579)^2+(6552-6579)^2\right] =$

$= \frac{1}{6}[60\,516+452\,929+436\,921+3969+49+1\,552\,516+729] =$

$\cong 417\,938 \qquad \Rightarrow S_{x,n} = \sqrt{417\,938} \approx 646$

$\bullet\ t^+_{\alpha/2,n-1} = t^-_{1-\alpha/2,n-1} = t^-_{0,975,6} = 2,447$

$\Rightarrow$ L'intervallo diventa:

$(6759 - 2,447\cdot 646\cdot 1,069 \; ; \; 6759+2,447\cdot 646\cdot 1,069) \equiv (5069 \; ; \; 8449)$

data
luogo

8) $t_1 = 0,77$ $\quad t_2 = 0,75$ $\quad t_3 = 0,70$

$t_4 = 0,72$ $\quad t_5 = 0,70$ $\quad t_6 = 0,69$ $\quad t_7 = 0,67$

$t_8 = 0,79$ $\quad t_9 = 0,64$ $\quad t_{10} = 0,72$ $\quad T \sim N(\mu, \sigma^2)$ IGNOTI

$\rightarrow \overline{x}_n = \frac{1}{10}(0,77+0,75+0,70+0,72+0,70+0,69+0,67+0,79+0,64+0,72) =$

$= 0,715$

$S^2_n = \frac{1}{9}\left[(0,77-0,715)^2+(0,75-0,715)^2+(0,70-0,715)^2+(0,72-0,715)^2+\right.$

$+(0,70-0,715)^2+(0,69-0,715)^2+(0,67-0,715)^2+(0,79-0,715)^2+$

$\left.+(0,64-0,715)^2+(0,72-0,715)^2\right] =$

$= \frac{1}{9}(0,003025+0,001225+0,000225+0,000025+0,000225+0,000625+$

$+0,002025+0,005625+0,005625+0,000025) \cong 0,00207$

$\Rightarrow S_n = \sqrt{0,00207} \approx 0,0455$

$\rightarrow$ INTERVALLO DI CONFIDENZA PER μ AL LIVELLO DI CONFIDENZA $1-\alpha = 0,95$:

$$\left(\overline{X}_n - t^+_{\frac{\alpha}{2},n-1}\frac{S_n}{\sqrt{n}} \; ; \; \overline{X}_n + t^+_{\frac{\alpha}{2},n-1}\frac{S_n}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(0,715 - t^+_{\alpha/2,n-1}\cdot\frac{0,0455}{\sqrt{10}} \; ; \; 0,715 + t^+_{\alpha/2,n-1}\cdot\frac{0,0455}{\sqrt{10}}\right)$$

$t^+_{\alpha/2,n-1} = t^-_{1-\alpha/2,n-1} = t^-_{0,975,9} = 2,262 \Rightarrow$ L'intervallo diventa:

$$\left(0,715 - 2,262\cdot\frac{0,0455}{\sqrt{10}} \; ; \; 0,715+2,262\cdot\frac{0,0455}{\sqrt{10}}\right) \equiv (0,682 \; ; \; 0,748)$$

INTERVALLO DI CONFIDENZA PER σ^2 AL LIVELLO DI CONFIDENZA $1-\alpha=0,95$:

$$\left(\frac{(n-1)S^2_{X,n}}{\chi^{2+}_{n-1,\alpha/2}} ; \frac{(n-1)S^2_{X,n}}{\chi^{2-}_{n-1,\alpha/2}}\right)$$

Realizzazione di quest'intervallo:

$$\left(\frac{9\cdot 0,00207}{\chi^{2+}_{n-1,\alpha/2}} ; \frac{9\cdot 0,00207}{\chi^{2-}_{n-1,\alpha/2}}\right)$$

$$\chi^{2+}_{n-1,\alpha/2} = \chi^{2-}_{n-1,1-\alpha/2} = \chi^{2-}_{9;0,975} = 19,023$$

$$\chi^{2-}_{n-1,\alpha/2} = \chi^{2-}_{9;0,025} = 2,700$$ ⇒ L'intervallo diventa:

$$\left(\frac{9\cdot 0,00207}{19,023} ; \frac{9\cdot 0,00207}{2,700}\right) \equiv (0,000979 ; 0,0069)$$

→ E SE AVESSIMO $\sigma^2 = 0,0025$ (NOTA)? → $\sigma = 0,05$

L'intervallo di confidenza per μ sarebbe:

$$\left(\bar{X}_n - z^+_{\alpha/2}\frac{\sigma}{\sqrt{n}} ; \bar{X}_n + z^+_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(0,715 - z^+_{\alpha/2}\frac{0,05}{\sqrt{10}} ; 0,715 + z^+_{\alpha/2}\frac{0,05}{\sqrt{10}}\right)$$

$$z^+_{\alpha/2} = z^-_{1-\alpha/2} = z^-_{0,975} = 1,96$$ ⇒ L'intervallo è:

$$\left(0,715 - 1,96\cdot\frac{0,05}{\sqrt{10}} ; 0,715 + 1,96\cdot\frac{0,05}{\sqrt{10}}\right) \equiv (0,684 ; 0,746)$$

AMPIEZZA: $2\cdot 1,96\cdot\frac{0,05}{\sqrt{n}} \leq 10 = W$

$$\Rightarrow \sqrt{n} \geq \frac{1,96}{5}\cdot 0,05$$

$$\Rightarrow n \geq 0,0196 \Rightarrow n \geq 1$$

↳ No OK, mi so non ho capito

9) $n_1 = 15$ $n_2 = 20$ $1-\alpha = 0,95$

$\bar{x}_{n_1} = 24,0$ $\bar{x}_{n_2} = 26,0$ → Stessa varianza σ^2_x NON nota

$S^2_{n_1} = 4,5$ $S^2_{n_2} = 5,0$

→ INTERVALLO DI CONFIDENZA PER $\mu_X - \mu_Y$ CON LIVELLO DI CONFIDENZA $1-\alpha = 0,95$:

$$\left(\bar{X}_{n_1} - \bar{Y}_{n_2} - t^+_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} ; \bar{X}_{n_1} - \bar{Y}_{n_2} + t^+_{\frac{\alpha}{2}, n_1+n_2-2}\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}\right)$$

$$S_p^2 = \frac{(n_1-1)S^2_{X,n_1} + (n_2-1)S^2_{Y,n_2}}{n_1+n_2-2}$$

$$\Rightarrow S_p^2 = \frac{14\cdot 4,5 + 19\cdot 5}{33} \simeq 4,79$$

La realizzazione dell'intervallo è:

$$\left(24 - 26 - t^+_{\frac{\alpha}{2},33}\sqrt{4,79\left(\frac{1}{15}+\frac{1}{20}\right)} ; 24 - 26 + t^+_{\alpha/2,33}\sqrt{4,79\left(\frac{1}{15}+\frac{1}{20}\right)}\right)$$

$$t^+_{\frac{\alpha}{2},33} = t^-_{1-\alpha/2,33} = t^-_{0,975,33} = 2,035$$ ⇒ l'intervallo è:

$$(-2 - 2,035\cdot 0,748 ; -2 + 2,035\cdot 0,748) \equiv (-3,522 ; -0,478)$$

27/01/2020 | 1) $X \sim N(\mu_x, \sigma_x^2)$ μ_x, σ_x^2 NOTE

data

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$X_1, \dots, X_n$ gaussiane e indipendenti $\Rightarrow \bar{X}_n \sim N(\mu_x, \frac{\sigma_x^2}{n})$

$$P(\bar{X}_n > 2\mu_x) = 1 - P(\bar{X}_n \le 2\mu_x) = 1 - F_{\bar{X}_n}(2\mu_x) =$$

$$= 1 - \frac{1}{2}\left[1 + erf\left(\frac{2\mu_x - \mu_x}{\sigma_x \sqrt{2}} \cdot \sqrt{n}\right)\right] = \frac{1}{2} - \frac{1}{2} erf\left(\frac{\mu_x \sqrt{n}}{\sigma_x \sqrt{2}}\right)$$

Sappiamo che: X gaussiana $\Rightarrow \frac{(n-1) S_n^2(X)}{\sigma_x^2} \sim \chi^2_{n-1}$

$$\Rightarrow S_n^2(X) \sim \frac{\sigma_x^2}{n-1} \chi^2_{n-1}$$

$$P(S_n^2(X) > 2\sigma_x^2) = P\left(\frac{\sigma_x^2}{n-1} \chi^2_{n-1} > 2\sigma_x^2\right) = P(\chi^2_{n-1} > 2n-2) =$$

$$= 1 - F_{n-1}(2n-2) = 1 - \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{x/2} t^{\frac{n-1}{2}-1} e^{-t} \, dt$$

→ Se invece $X \sim Ber(p)$, con p NOTO, n "grande", allora sappiamo che APPROSSIMATIVAMENTE $Z_n \sim N(np, np(1-p))$

$$\Rightarrow \bar{X}_n \sim \frac{1}{n} N(np, np(1-p)) \sim N\left(p, \frac{p(1-p)}{n}\right)$$

$$P(\bar{X}_n > 2p) = 1 - P(\bar{X}_n \le 2p) = 1 - F_{\bar{X}_n}(2p) =$$

$$= 1 - \frac{1}{2}\left[1 + erf\left(\frac{2p - p}{\sqrt{p(1-p)2}} \sqrt{n}\right)\right] = \frac{1}{2} - \frac{1}{2} erf\left(\frac{\sqrt{p}\sqrt{n}}{\sqrt{2}\sqrt{1-p}}\right)$$

Chiedere conferma

2) $f_X(x) := \frac{3x^2}{\theta^3} \mathbb{1}_{[0,\theta]}(x)$ $\forall x \in \mathbb{R}$ $\theta \ge 0$

data

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METODO DEI MOMENTI:

$$E[X] = \int_{\mathbb{R}} x f_X(x) \, d\mu_L(x) = \int_{\mathbb{R}} \frac{3}{\theta^3} x^3 \mathbb{1}_{[0,\theta]}(x) \, d\mu_L(x) = \frac{3}{\theta^3} \int_0^\theta x^3 dx =$$

$$= \frac{3}{\theta^3} \left[\frac{x^4}{4}\right]_0^\theta = \frac{3}{\theta^3} \cdot \frac{\theta^4}{4} = \frac{3}{4}\theta$$

$$\Rightarrow \bar{X}_n = \frac{3}{4} \hat{\theta}_n^M \Rightarrow \hat{\theta}_n^M = \frac{4}{3} \bar{X}_n$$

$$\rightarrow E[\hat{\theta}_n^M] = E\left[\frac{4}{3}\bar{X}_n\right] = \frac{4}{3} E[\bar{X}_n] = \frac{4}{3} E[X] = \frac{4}{3} \cdot \frac{3}{4}\theta = \theta$$

$\Rightarrow \hat{\theta}_n^M$ è uno stimatore non distorto.

$$\rightarrow E[(\hat{\theta}_n^M - \theta)^2] = E[(\hat{\theta}_n^M - E[\hat{\theta}_n^M])^2] = D^2[\hat{\theta}_n^M] = D^2\left[\frac{4}{3}\bar{X}_n\right] =$$

$$= \frac{16}{9} D^2[\bar{X}_n] = \frac{16}{9} \cdot \frac{D^2[X]}{n}$$

$$\cdot E[X^2] = \int_{\mathbb{R}} x^2 f_X(x) \, d\mu_L(x) = \int_{\mathbb{R}} \frac{3}{\theta^3} x^4 \mathbb{1}_{[0,\theta]}(x) \, d\mu_L(x) =$$

$$= \frac{3}{\theta^3} \int_0^\theta x^4 dx = \frac{3}{\theta^3} \left[\frac{x^5}{5}\right]_0^\theta = \frac{3}{5}\theta^2$$

$$\Rightarrow D^2[X] = E[X^2] - E^2[X] = \frac{3}{5}\theta^2 - \frac{9}{16}\theta^2 = \frac{48-45}{80}\theta^2 = \frac{3}{80}\theta^2$$

$$\Rightarrow E[(\hat{\theta}_n^M - \theta)^2] = \frac{16}{9n} \cdot \frac{3}{80}\theta^2 \xrightarrow{n \to +\infty} 0$$

$\Rightarrow \hat{\theta}_n^M$ è consistente in media quadratica.

Poiché $\hat{\theta}_n^M$ è non distorto, possiamo dire che è consistente anche in probabilità.

METODO DELLA MASSIMA VEROSIMIGLIANZA:

$$\mathcal{L}_{X_1,\dots,X_n}(\theta;x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;\theta) =$$

$$= \prod_{k=1}^{n} \frac{3x_k^2}{\theta^3} \mathbb{1}_{[0,\theta]}(x_k) = \left(\frac{3}{\theta^3}\right)^n \prod_{k=1}^{n} x_k^2 \mathbb{1}_{[0,+\infty)}(x_k) \mathbb{1}_{[x_k,+\infty)}(\theta)$$

$$\to \mathbb{1}_{[0,\theta]}(x_k) = \mathbb{1}_{[0,+\infty)}(x_k) \mathbb{1}_{[x_k,+\infty)}(\theta)$$

Dobbiamo assumere:

- $x_k \geq 0 \quad \forall k = 1,\dots,n$
- $\theta \geq \max\{x_1,\dots,x_n\}$

Tuttavia, affinché la funzione sia massimizzata, θ deve essere il più piccolo possibile in modulo (compare al denominatore elevato alla $3n$).

In definitiva, otteniamo $\theta = \max\{x_1,\dots,x_n\}$

$$\Rightarrow \hat{\theta}_n^{ML} = \check{X}_n$$

$$\to E[\hat{\theta}_n^{ML}] = E[\check{X}_n]$$

$X_1,\dots,X_n$ indip.

$$F_{\check{X}_n}(x) = P(\check{X}_n \leq x) = P(X_1 \leq x,\dots,X_n \leq x) =$$

$$= \prod_{k=1}^{n} P(X_k \leq x) = \prod_{k=1}^{n} P(X \leq x) = [P(X \leq x)]^n = F_X^n(x)$$

$X_1,\dots,X_n$ identicam. distribuite

$$F_X(x) = \int_{(-\infty,x]} f_X(u)\, d\mu_L(u) = \int_{(-\infty,x]} \frac{3u^2}{\theta^3} \mathbb{1}_{[0,\theta]}(u)\, d\mu_L(u) =$$

$\to x < 0 \Rightarrow 0$

$$\to 0 \leq x \leq \theta \Rightarrow \int_{[0,x]} \frac{3u^2}{\theta^3}\, d\mu_L(u) = \frac{3}{\theta^3}\int_0^x u^2\, du = \frac{3}{\theta^3}\left[\frac{u^3}{3}\right]_0^x =$$

$$= \frac{3}{\theta^3}\cdot\frac{x^3}{3} = \left(\frac{x}{\theta}\right)^3$$

$$\to x > \theta \Rightarrow \int_{[0,\theta]} \frac{3u^2}{\theta^3}\, d\mu_L(u) = \frac{3}{\theta^3}\left[\frac{u^3}{3}\right]_0^\theta = 1$$

$$\Rightarrow F_X(x) = \left(\frac{x}{\theta}\right)^3 \mathbb{1}_{[0,\theta]}(x) + \mathbb{1}_{(\theta,+\infty)}(x)$$

$$\Rightarrow F_X^n(x) = \left(\frac{x}{\theta}\right)^{3n} \mathbb{1}_{[0,\theta]}(x) + \mathbb{1}_{(\theta,+\infty)}(x) = F_{\check{X}_n}(x)$$

$$\Rightarrow F'_{\check{X}_n}(x) = \frac{1}{\theta^{3n}}\cdot 3n x^{3n-1} \mathbb{1}_{(0,\theta)}(x)$$

$$\rightsquigarrow \int_{(-\infty,x]} F'_{\check{X}_n}(u)\, d\mu_L(u) = \int_{(-\infty,x]} \frac{3n}{\theta^{3n}} u^{3n-1} \mathbb{1}_{[0,\theta]}(u)\, d\mu_L(u) =$$

$\to x < 0 \Rightarrow 0$

$$\to 0 \leq x \leq \theta \Rightarrow \int_{[0,x]} \frac{3n}{\theta^{3n}} u^{3n-1}\, d\mu_L(u) = \frac{3n}{\theta^{3n}}\int_0^x u^{3n-1}\, du =$$

$$= \frac{3n}{\theta^{3n}}\left[\frac{u^{3n}}{3n}\right]_0^x = \left(\frac{x}{\theta}\right)^{3n}$$

$$\to x > \theta \Rightarrow \int_{[0,\theta]} \frac{3n}{\theta} u^{3n-1}\, d\mu_L(u) = \frac{3n}{\theta^{3n}}\left[\frac{u^{3n}}{3n}\right]_0^\theta = 1$$

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$$\Rightarrow \int_{(-\infty,x]} F'_{\check{X}_n}(u)\, d\mu_L(u) = F_{\check{X}_n}(x)$$

$$\Rightarrow F'_{\check{X}_n}(x) = f_{\check{X}_n}(x) = \frac{3n}{\theta^{3n}} x^{3n-1} \mathbb{1}_{[0,\theta]}(x)$$

$$\Rightarrow E[\hat{\theta}_n^{ML}] = E[\check{X}_n] = \int_{\mathbb{R}} x f_{\check{X}_n}(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{3n}{\theta^{3n}} x^{3n} \mathbb{1}_{[0,\theta]}(x)\, d\mu_L(x) =$$

$$= \frac{3n}{\theta^{3n}} \int_0^\theta x^{3n}\, dx = \frac{3n}{\theta^{3n}} \left[\frac{x^{3n+1}}{3n+1}\right]_0^\theta = \frac{3n}{\theta^{3n}} \cdot \frac{\theta^{3n+1}}{3n+1} = \frac{3n}{3n+1}\theta$$

$\Rightarrow \hat{\theta}_n^{ML}$ è uno stimatore distorto.

$$\rightsquigarrow P\left(\left|\frac{3n+1}{3n}\hat{\theta}_n^{ML} - \theta\right| \geq \varepsilon\right) = P\left(\left|\frac{3n+1}{3n}\hat{\theta}_n^{ML} - E\left[\frac{3n+1}{3n}\hat{\theta}_n^{ML}\right]\right| \geq \varepsilon\right)$$

$$\leq \frac{D^2\left[\frac{3n+1}{3n}\hat{\theta}_n^{ML}\right]}{\varepsilon^2}$$

$$\bullet\ E[\check{X}_n^2] = \int_{\mathbb{R}} x^2 f_{\check{X}_n}(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{3n}{\theta^{3n}} x^{3n+1} \mathbb{1}_{[0,\theta]}(x)\, d\mu_L(x) =$$

$$= \frac{3n}{\theta^{3n}} \int_0^\theta x^{3n+1}\, dx = \frac{3n}{\theta^{3n}} \left[\frac{x^{3n+2}}{3n+2}\right]_0^\theta = \frac{3n}{3n+2}\theta^2$$

$$\bullet\ D^2[\check{X}_n] = E[\check{X}_n^2] - E^2[\check{X}_n] = \frac{3n}{3n+2}\theta^2 - \frac{9n^2}{(3n+1)^2}\theta^2 =$$

$$= \frac{3n(9n^2+6n+1) - 9n^2(3n+2)}{(3n+2)(3n+1)^2}\theta^2 = \frac{27n^3+18n^2+3n-27n^3-18n^2}{(3n+2)(3n+1)^2}\theta^2 =$$

$$= \frac{3n\theta^2}{(3n+2)(3n+1)^2}$$

$$\Rightarrow P\left(\left|\frac{3n+1}{3n}\hat{\theta}_n^{ML} - \theta\right| \geq \varepsilon\right) \leq \frac{(3n+1)^2}{\varepsilon^2 \cdot 9n^2} \cdot \frac{3n\theta^2}{(3n+2)(3n+1)^2} =$$

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$$= \frac{\theta^2}{\varepsilon^2 \cdot 3n(3n+2)} \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow \frac{3n+1}{3n}\hat{\theta}_n^{ML} \xrightarrow{P} \theta$$

Inoltre, $\frac{3n}{3n+1} \xrightarrow{P} 1$

$$\Rightarrow \frac{3n+1}{3n} \cdot \hat{\theta}_n^{ML} \cdot \frac{3n}{3n+1} = \hat{\theta}_n^{ML} \xrightarrow{P} \theta \cdot 1 = \theta$$

$\Rightarrow$ Lo stimatore $\hat{\theta}_n^{ML}$ è consistente in probabilità.

$$\rightsquigarrow E[(\hat{\theta}_n^{ML} - \theta)^2] = E[(\hat{\theta}_n^{ML})^2 - 2\theta\hat{\theta}_n^{ML} + \theta^2] =$$

$$= E[(\hat{\theta}_n^{ML})^2] - 2\theta E[\hat{\theta}_n^{ML}] + \theta^2 =$$

$$= E[\check{X}_n^2] - 2\theta E[\check{X}_n] + \theta^2 =$$

$$= \frac{3n}{3n+2}\theta^2 - 2\theta \cdot \frac{3n}{3n+1}\theta + \theta^2 = \frac{3n(3n+1) - 6n(3n+2) + (3n+2)(3n+1)}{(3n+2)(3n+1)}\theta^2$$

$$= \frac{9n^2+3n-18n^2-12n+9n^2+3n+6n+2}{(3n+2)(3n+1)}\theta^2 = \frac{2\theta^2}{(3n+2)(3n+1)} \xrightarrow{n\to+\infty} 0$$

$\Rightarrow$ Lo stimatore $\hat{\theta}_n^{ML}$ è consistente anche in media quadratica.

→ Costruiamo gli stimatori per la media μ_X e la varianza σ_X^2.

 $\hat{\mu}^{M}_{X_n} = \bar{X}_n = \frac{3}{4}\hat{\theta}^{M}_n$

$$\hat{\mu}^{ML}_{X,n} = \frac{3}{4}\hat{\theta}^{LM}_n = \frac{3}{4}\check{X}_n$$

$$\hat{\sigma}^{2,M}_{X,n} = \frac{3}{80}\left(\hat{\theta}^{M}_n\right)^2 = \frac{3}{80}\cdot\frac{16}{9}\bar{X}_n^2 = \frac{1}{15}\bar{X}_n^2$$

$$\hat{\sigma}^{2,M}_{X,n} = \frac{3}{80}\left(\hat{\theta}^{LM}_n\right)^2 = \frac{3}{80}\check{X}_n^2$$

3) $X \sim N(\mu_X, \sigma_X^2)$ $\quad \mu_X, \sigma_X^2$ IGNOTE

$n=10$ → ho le varie realizzazioni $x_1, \dots, x_{10}$

→ INTERVALLO DI CONFIDENZA PER μ_X AL LIVELLO DI CONFIDENZA $1-\alpha=0{,}99$:

$$\left(\bar{X}_n - t^{+}_{\frac{\alpha}{2},n-1}\frac{S_n}{\sqrt{n}} \; ; \; \bar{X}_n + t^{+}_{\frac{\alpha}{2},n-1}\frac{S_n}{\sqrt{n}}\right)$$

Realizzazione di quest'intervallo:

$$\left(\bar{x}_n - t^{+}_{\frac{\alpha}{2},n-1}\frac{s_n}{\sqrt{n}} \; ; \; \bar{x}_n + t^{+}_{\frac{\alpha}{2},n-1}\frac{s_n}{\sqrt{n}}\right)$$

$$\to \bar{x}_n = \frac{1}{10}(-2{,}5-1.5-1.0-0.5+0.5+1.0+1.5+1.5+2.5+3.0) = 0.45$$

$$\to s_n^2 = \frac{1}{9}\left[(-2.5-0.45)^2+(-1.5-0.45)^2+(-1.0-0.45)^2+(-0.5-0.45)^2+(0.5-0.45)^2+(1.0-0.45)^2+(1.5-0.45)^2+(1.5-0.45)^2+(2.5-0.45)^2+(3.0-0.45)^2\right] =$$

$$= \frac{1}{9}(8{,}7025+3{,}8025+2{,}1025+0{,}9025+0{,}0025+0{,}3025+1{,}1025+1{,}1025+4{,}2025+6{,}5025) \simeq 3{,}19$$

$\Rightarrow s_n \approx 1{,}79$

$$\to t^{+}_{\frac{\alpha}{2},n-1} = t^{-}_{1-\frac{\alpha}{2},n-1} = t^{-}_{0{,}995,9} = 3{,}250$$

⇒ L'intervallo diventa:

$$\left(0.45 - 3{,}250\cdot\frac{1{,}79}{\sqrt{10}} \; ; \; 0.45 + 3{,}250\cdot\frac{1{,}79}{\sqrt{10}}\right) = (-1{,}39 \; ; \; 2{,}29)$$

$$\to W = 0{,}5 \Rightarrow 2t^{+}_{\frac{\alpha}{2},n-1}\frac{S_n}{\sqrt{n}} = 0{,}5 \Rightarrow t^{+}_{\frac{\alpha}{2},n-1} = \frac{0{,}5}{2}\cdot\frac{\sqrt{10}}{1{,}79} \approx 0{,}44.$$

~ Continuo a non capire questa richiesta :C

→ INTERVALLO DI CONFIDENZA PER σ_X^2 AL LIVELLO DI CONFIDENZA $1-\alpha=0{,}9$:

$$\left(\frac{(n-1)S_n^2}{\chi^{2+}_{n-1,\alpha/2}} \; ; \; \frac{(n-1)S_n^2}{\chi^{2-}_{n-1,\alpha/2}}\right)$$

Realizzazione di quest'intervallo:

$$\left(\frac{9\cdot 3{,}19}{\chi^{2+}_{n-1,\alpha/2}} \; ; \; \frac{9\cdot 3{,}19}{\chi^{2-}_{n-1,\alpha/2}}\right).$$

$$\to \chi^{2+}_{n-1,\alpha/2} = \chi^{2-}_{n-1,1-\alpha/2} = \chi^{2-}_{9,0{,}950} = 16{,}919$$

$$\to \chi^{2-}_{n-1,\alpha/2} = \chi^{2-}_{9,0{,}050} = 3{,}325$$

⇒ L'intervallo diventa:

$$\left(\frac{9\cdot 3{,}19}{16{,}919} \; ; \; \frac{9\cdot 3{,}19}{3{,}325}\right) = (1{,}70 \; ; \; 8{,}63) \xrightarrow{\text{PASSANDO ALLA DEV. STANDARD}} (1{,}30 \; ; \; 2{,}94)$$

TO DO: ▸ 12/02/2020 ⇒ 1 ▸ 25/02/2020 ⇒ 1, 4

data

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12/02/2020

1) $X \sim N(\mu_x, \sigma_x^2)$ $Y \sim N(\mu_Y, \sigma_Y^2)$ ← TUTTI I PARAMETRI SONO NOTI

$\rightsquigarrow X, Y$ gaussiane $\wedge$ $X_1, \dots, X_m$ indip. $\wedge$ $Y_1, \dots, Y_n$ indip. $\Rightarrow$

▸ $S_m^2(X) \sim \frac{\sigma_x^2}{m-1} \chi^2_{m-1}$; $S_n^2(Y) \sim \frac{\sigma_Y^2}{n-1} \chi^2_{n-1}$

▸ $\overline{X}_m \sim N(\mu_x, \frac{\sigma_x^2}{m})$; $\overline{Y}_n \sim N(\mu_Y, \frac{\sigma_Y^2}{n})$

$$P(x \le S_m^2(Y) \le y) = P\left(\frac{(m-1)x}{\sigma_Y^2} \le \chi^2_{n-1} \le \frac{(m-1)y}{\sigma_Y^2}\right) =$$

$$= \int_{\left[\frac{(m-1)x}{\sigma_Y^2} ; \frac{(m-1)y}{\sigma_Y^2}\right]} \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} u^{\frac{n-1}{2}-1} e^{-u/2} \, d\mu_L(u)$$

$$P(\overline{X}_m > x) = 1 - P(\overline{X}_m \le x) = 1 - F_{\overline{X}_m}(x) = 0{,}25$$

$$\Rightarrow F_{\overline{X}_m}(x) = 0{,}75$$

dove $Z \sim N(0,1)$

$$P(\overline{X}_m \le x) = P\left(\frac{Z + \mu_x}{\sigma_x^2} \le x\right) = P(Z \le x\sigma_x^2 - \mu_x) = F_Z(x\sigma_x^2 - \mu_x)$$

$$P(\overline{X}_m - x > \overline{Y}_n + y) = P(\overline{X}_m > \overline{Y}_n + y + x) =$$

$$= \int_{\{\overline{x} \in (\overline{y}+y+x; +\infty)\} \cap \{\overline{y} \in \mathbb{R}\}} \frac{\sqrt{m}}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(u-\mu_x)^2}{\sigma_x^2} m} \cdot \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{(v-\mu_Y)^2}{\sigma_Y^2} n} \, d\mu_L^2(u,v).$$

↪ Pure qua chiedere conferma

25/02/2020

1) $X \sim Geom_{mod}(p)$ p IGNOTO

data

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Art-ware bellissime arti

METODO DEI MOMENTI:

$E[X] = 1/p$ Infatti:

~~$E[X] = \sum_{k=1}^{n} k\, P(X = x_k) = \sum_{k=1}^{n} kp(1-p)$~~

$$E[X] = \sum_{n=1}^{\infty} n\, P(X = n) = \sum_{n=1}^{\infty} n p (1-p)^{n-1} = \frac{p}{1-p} \sum_{n=1}^{\infty} n (1-p)^n =$$

$$= \frac{p}{1-p} \cdot \frac{1-p}{(1-1+p)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\Rightarrow \overline{X}_n = \frac{1}{\hat{p}_m^M} \Rightarrow \hat{p}_m^M = \frac{1}{\overline{X}_n}$$

$E[\hat{p}_m^M] = E[\frac{1}{\overline{X}_n}]$ ~~→ Qui applichiamo il metodo Delta~~

~~Prendo $f(x) = \frac{1}{x}$~~

↘ Non è detto che sia uguale a p per via del fatto che $E[\frac{1}{\overline{X}_n}] \neq \frac{1}{E[\overline{X}_n]}$

Questo aggi' ess' chiest'.

Notiamo che $P(\overline{X}_n = 0) = 0$ poiché $P(X_k \le 0) = 0 \quad \forall k$

$$\rightsquigarrow P(|\overline{X}_n - 1/p| \ge \varepsilon) = P(|\overline{X}_n - E[\overline{X}_n]| \ge \varepsilon) \le \frac{D^2[\overline{X}_n]}{\varepsilon^2} =$$

$$= \frac{D^2[X]}{n\varepsilon^2} = \frac{1-p}{p^2} \cdot \frac{1}{n\varepsilon^2} \xrightarrow{n \to +\infty} 0$$

$$\Rightarrow \overline{X}_n \xrightarrow{P} 1/p$$

$\Rightarrow \frac{1}{\overline{X}_n} \xrightarrow{P} \frac{1}{1/p} = p \Rightarrow \hat{p}_m^M$ è consistente in probabilità.

METODO DELLA MASSIMA VEROSIMIGLIANZA:

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$$\mathcal{L}_{x_1,\dots,x_n}(p;x_1,\dots,x_n) = \prod_{k=1}^{n} f_X(x_k;p) =$$

$$= \prod_{k=1}^{n} \sum_{m=1}^{\infty} p(1-p)^{m-1} 1_{\{m\}}(x_k) = \prod_{k=1}^{n} p(1-p)^{x_k-1} = p^n \prod_{k=1}^{n}(1-p)^{x_k-1}$$

↳ Proviamo a passare al logaritmo:

$$n\log(p) + \sum_{k=1}^{n}(x_k-1)\log(1-p)$$

Ora deriviamo rispetto a p:

$$\frac{n}{p} + \sum_{k=1}^{n}(x_k-1)\cdot\frac{1}{1-p} = \frac{n}{p} + \frac{1}{p-1}\sum_{k=1}^{n}(x_k-1) =$$

$$= \frac{n}{p} + \frac{1}{p-1}\left(\sum_{k=1}^{n} x_k - n\right) = \frac{n}{p} + \frac{n}{1-p} + \frac{1}{p-1}\sum_{k=1}^{n} x_k =$$

$$= \frac{n-np+np}{p(1-p)} + \frac{1}{p-1}\sum_{k=1}^{n} x_k = \frac{n}{p(1-p)} + \frac{1}{p-1}\sum_{k=1}^{n} x_k = 0$$

$$\Rightarrow \frac{1}{1-p}\sum_{k=1}^{n} x_k = \frac{n}{p(1-p)} \Rightarrow \sum_{k=1}^{n} x_k = \frac{n}{p} \Rightarrow p = \frac{n}{\sum_{k=1}^{n} x_k} = \frac{1}{\bar{x}_n}$$

$$\Rightarrow \hat{p}_n^{LH} = \frac{1}{\bar{X}_n} \qquad (= \hat{p}_n^{M})$$

$n = 50$; $\sum_{k=1}^{50} x_k = 150 \Rightarrow$

$$\Rightarrow \frac{1}{50}\sum_{k=1}^{50} x_k = 3 = \bar{x}_n$$

$$\Rightarrow \hat{p}_n^{M}(\omega) = \hat{p}_n^{LH}(\omega) = \frac{1}{\bar{x}_n} = \frac{1}{3}$$

→ STIMA MEDIA DI X: $\hat{\mu}_n(\omega) = \bar{x}_n = \frac{1}{\hat{p}_n^{M}(\omega)} = \frac{1}{\hat{p}_n^{LH}(\omega)} = 3$

data

luogo

→ STIMA VARIANZA DI X: $\hat{\sigma}_n^2(\omega) = \frac{1-\hat{p}_n^{M}(\omega)}{(\hat{p}_n^{M}(\omega))^2} =$

$$= \frac{1-\hat{p}_n^{LH}(\omega)}{(\hat{p}_n^{LH}(\omega))^2} = \frac{1-1/3}{1/9} = 9\cdot\frac{2}{3} = 6$$

---

4) $X \sim N(\mu_X, \sigma_X^2)$ $\qquad Y \sim N(\mu_Y, \sigma_X^2)$

Penso che possiamo assumere X, Y indipendenti.

$$\bar{x}_{13} = \frac{1}{13}(1.60+1.70+1.55+1.70+1.75+1.35+1.65+1.75+1.70+1.65+1.56+1.80+1.80) \simeq 1{,}64$$

$$\bar{y}_{12} = \frac{1}{12}(1.70+1.35+1.50+1.65+1.50+1.60+1.70+1.75+1.60+1.65+1.55+1.45) \approx 1{,}58$$

$$S^2_{X,13} = \frac{1}{12}\Big[(1{,}60-1{,}64)^2+(1{,}70-1{,}64)^2+(1{,}55-1{,}64)^2+(1{,}70-1{,}64)^2+(1{,}75-1{,}64)^2+(1{,}35-1{,}64)^2+(1{,}65-1{,}64)^2+(1{,}75-1{,}64)^2+(1{,}70-1{,}64)^2+(1{,}65-1{,}64)^2+(1{,}56-1{,}64)^2+(1{,}60-1{,}64)^2+(1{,}80-1{,}6$$

$$= \frac{1}{12}(0{,}0016+0{,}0036+0{,}0081+0{,}0036+0{,}0121+0{,}0841+0{,}0001+0{,}0121+0{,}0036+0{,}0001+0{,}0064+0{,}0016+0{,}0256) = 0{,}01355$$

$$\Rightarrow S_{X,13} \simeq 0{,}1164$$

$$S^2_{Y,12} = \frac{1}{11}\Big[(1{,}70-1{,}58)^2+(1{,}35-1{,}58)^2+(1{,}50-1{,}58)^2+(1{,}65-1{,}58)^2+(1{,}50-1{,}58)^2+(1{,}60-1{,}58)^2+(1{,}70-1{,}58)^2+(1{,}75-1{,}58)^2+$$
$$+(1{,}60-1{,}58)^2+(1{,}65-1{,}58)^2+(1{,}55-1{,}58)^2+(1{,}45-1{,}58)^2\Big] =$$

$$= \frac{1}{11}\big(0{,}0144+0{,}0529+0{,}0064+0{,}0049+0{,}0064+0{,}0004+0{,}0144+$$
$$+0{,}0289+0{,}0004+0{,}0049+0{,}0009+0{,}0169\big) = 0{,}0138$$

$$\Rightarrow S_{Y,12} \simeq 0{,}1175$$

$$\rightarrow S_p^2 = \frac{(m-1)S^2_{X,m}+(n-1)S^2_{Y,n}}{m+n-2}$$

$$\Rightarrow S_p^2 = \frac{12\cdot 0{,}01355+11\cdot 0{,}0138}{23} \approx 0{,}01367$$

$\rightarrow$ INTERVALLO DI CONFIDENZA PER $\mu_X-\mu_Y$ AL LIVELLO DI CONFIDENZA $1-\alpha = 0{,}95$:

$$\left(\overline{X}_m-\overline{Y}_n-t^+_{\frac{\alpha}{2},m+n-2}\sqrt{S_p^2\left(\frac{1}{m}+\frac{1}{n}\right)}\ ;\ \overline{X}_m-\overline{Y}_n+t^+_{\frac{\alpha}{2},m+n-2}\sqrt{S_p^2\left(\frac{1}{m}+\frac{1}{n}\right)}\right)$$

Realizzazione di quest'intervallo:

$$\left(1{,}64-1{,}58-t^+_{\frac{\alpha}{2},m+n-2}\sqrt{0{,}01367\left(\frac{1}{13}+\frac{1}{12}\right)}\ ;\ 1{,}64-1{,}58+t^+_{\alpha/2,m+n-2}\sqrt{0{,}01367\left(\frac{1}{13}+\frac{1}{12}\right)}\right)$$

$$\rightarrow t^+_{\frac{\alpha}{2},m+n-2} = t^-_{1-\frac{\alpha}{2},m+n-2} = t^-_{0{,}975,23} = 2{,}069 \Rightarrow \text{l'intervallo è:}$$

$$\left(0{,}06-2{,}069\sqrt{0{,}00219}\ ;\ 0{,}06+2{,}069\sqrt{0{,}00219}\right) = (-0{,}03684; 0{,}15684)$$

Torniamo a fare qualcosa sulle successioni di v.a.:

$X \sim Unif(0,1)$

$$Y_n := \begin{cases} n & \text{se } 0\le X\le 1/n \\ 0 & \text{se } 1/n\le X\le 1 \end{cases} \longrightarrow Y_n = \begin{cases} n & \text{con probabilità } 1/n \\ 0 & \text{con probabilità } 1-1/n \end{cases}$$

$$\Rightarrow Y_n \sim Ber(1/n)\cdot n$$

CONVERGENZA IN DISTRIBUZIONE

$$F_{Y_n}(y) = \begin{cases} 0 & \text{se } y<0 \\ 1-1/n & \text{se } 0\le y<n \\ 1 & \text{se } y\ge n \end{cases}$$

$$F_Y(y) = \lim_{n\to+\infty} F_{Y_n}(y) = \begin{cases} 0 & \text{se } y<0 \\ 1 & \text{se } y\ge 0 \end{cases}$$

$$\Rightarrow Y_n \xrightarrow{W} Dirac(0)$$

CONVERGENZA IN PROBABILITÀ

$$P(|Y_n - Dirac(0)|<\varepsilon) = P(Y_n<\varepsilon) \underset{\varepsilon \text{ arbitrariamente piccolo}}{=} P(Y_n=0) = 1-\frac{1}{n} \xrightarrow{n\to+\infty} 1$$

$$\Rightarrow Y_n \xrightarrow{P} Dirac(0)$$

Qui potevo anche usare il teorema

CONVERGENZA QUASI CERTA:

$$\lim_{n\to+\infty} Y_n = \begin{cases} \lim_{n\to+\infty} n & \text{con probabilità } \lim_{n\to+\infty}\frac{1}{n}=0 \\ 0 & \text{con probabilità } \lim_{n\to+\infty} 1-\frac{1}{n} = 1 \end{cases}$$

$$\Rightarrow \lim_{n\to+\infty} Y_n = 0 \quad (Dirac(0))$$

$$\Rightarrow Y_n \xrightarrow{a.s.} Dirac(0)$$

CONVERGENZA IN MEDIA

data

$E[|Y_n - Dirac(0)|] = E[Y_n] = n \cdot \frac{1}{n} = 1$

luogo $\Rightarrow Y_n$ NON converge in media a Dirac(0)

$\Rightarrow$ Non può convergere neanche in media quadratica a Dirac(0).

$$f_X(x) = \frac{\alpha-1}{x^\alpha} \mathbb{1}_{[1,+\infty)}(x) \qquad \alpha>1$$

$$Y_n = \frac{X}{n}$$

CONVERGENZA IN DISTRIBUZIONE:

$$F_X(x) = \int_{(-\infty,x]} f_X(u)\, d\mu_L(u) = \int_{(-\infty,x]} \frac{\alpha-1}{u^\alpha} \mathbb{1}_{[1,+\infty)}(u)\, d\mu_L(u) =$$

- $x<1 \Rightarrow 0$
- $x \geq 1 \Rightarrow \int_{[1,x]} (\alpha-1) u^{-\alpha}\, d\mu_L(u) = (\alpha-1)\int_1^x u^{-\alpha}\, du = (\alpha-1)\left[\frac{u^{-\alpha+1}}{-\alpha+1}\right]_1^x =$

$$= (\alpha-1)\left(\frac{-1}{x^{+(\alpha-1)}(\alpha-1)}\right) + (\alpha-1)\left(\frac{+1}{\alpha-1}\right) = 1 - \frac{1}{x^{\alpha-1}}$$

$$\Rightarrow F_X(x) = \left(1 - \frac{1}{x^{\alpha-1}}\right)\mathbb{1}_{[1,+\infty)}(x)$$

$$F_{Y_n}(x) = P(Y_n \leq x) = P\left(\frac{X}{n} \leq x\right) = P(X \leq nx) = F_X(nx) =$$

$$= \left(1 - \frac{1}{n^{\alpha-1}x^{\alpha-1}}\right)\mathbb{1}_{[1,+\infty)}(nx) = \left(1 - \frac{1}{(nx)^{\alpha-1}}\right)\mathbb{1}_{[1/n,+\infty)}(x)$$

$$\Rightarrow F_Y(x) = \lim_{n\to+\infty} F_{Y_n}(x) = \lim_{n\to+\infty} \mathbb{1}_{[1,+\infty)}(nx) = \begin{cases} 0 & \text{se } x < 1/n \xrightarrow{n\to+\infty} 0 \\ 1 & \text{se } x \geq 1/n \xrightarrow{n\to+\infty} 0 \end{cases}$$

$$\Rightarrow Y_n \xrightarrow{W} Dirac(0)$$

Poiché Y_n converge debolmente a una v.a. di Dirac, allora vi converge anche in probabilità:

data

luogo

$$Y_n \xrightarrow{P} Dirac(0)$$

CONVERGENZA IN MEDIA:

$$E[|Y_n - 0|] = E[Y_n]$$

$$\rightarrow F'_{Y_n}(x) = \frac{d}{dx}\left(-\frac{1}{n^{\alpha-1}} x^{1-\alpha}\right)\mathbb{1}_{[1/n,+\infty)}(x) = +\frac{1}{n^{\alpha-1}}(\alpha-1)x^{-\alpha}\mathbb{1}_{[1/n,+\infty)}(x) =$$

$$= (\alpha-1)\frac{1}{n^{\alpha-1}x^\alpha}\mathbb{1}_{[1/n,+\infty)}(x)$$

$$\rightarrow \int_{(-\infty,x]} (\alpha-1)\frac{u^{-\alpha}}{n^{\alpha-1}}\mathbb{1}_{[1/n,+\infty)}(u)\, d\mu_L(u) =$$

- $x < 1/n \Rightarrow 0$
- $x \geq 1/n \Rightarrow \int_{[1/n,x]} (\alpha-1)\frac{u^{-\alpha}}{n^{\alpha-1}}\, d\mu_L(u) = \int_{1/n}^x \frac{(\alpha-1)}{n^{\alpha-1}} u^{-\alpha}\, du =$

$$= \frac{\alpha-1}{n^{\alpha-1}}\left[\frac{u^{-\alpha+1}}{1-\alpha}\right]_{1/n}^x = \frac{\alpha-1}{n^{\alpha-1}}\cdot\frac{1}{x^{\alpha-1}(1-\alpha)} + \frac{\alpha-1}{n^{\alpha-1}}\cdot\frac{n^{\alpha-1}}{\alpha-1} =$$

$$= 1 - \frac{1}{n^{\alpha-1}x^{\alpha-1}}$$

$$\Rightarrow \int_{(-\infty,x]} F'_{Y_n}(u)\, d\mu_L(u) = \left(1 - \frac{1}{n^{\alpha-1}x^{\alpha-1}}\right)\mathbb{1}_{[1/n,+\infty)}(x) = F_{Y_n}(x)$$

$$\Rightarrow f_{Y_n}(x) = F'_{Y_n}(x) = \frac{\alpha-1}{n^{\alpha-1}}\cdot\frac{1}{x^\alpha}\mathbb{1}_{[1/n,+\infty)}(x)$$

$$\rightarrow E[Y_n] = \int_{\mathbb{R}} \frac{\alpha-1}{n^{\alpha-1}}\cdot x^{-\alpha+1}\mathbb{1}_{[1/n,+\infty)}(x)\, d\mu_L(x) = \frac{\alpha-1}{n^{\alpha-1}}\int_{1/n}^{+\infty} x^{-\alpha+1}\, dx =$$

data

luogo

$$= \frac{\alpha-1}{n^{\alpha-1}} \left[\frac{x^{-\alpha+2}}{-\alpha+2}\right]_{1/n}^{+\infty} = \frac{\alpha-1}{n^{\alpha-1}} \left[\frac{1}{(-\alpha+2)\,x^{\alpha-2}}\right]_{1/n}^{+\infty} =$$

$$= \frac{\alpha-1}{n^{\alpha-1}} \left(\frac{n^{\alpha-2}}{(\alpha-2)}\right) = \frac{\alpha-1}{\alpha-2}\cdot\frac{1}{n} \xrightarrow{n\to+\infty} 0 \Longrightarrow Y_n \xrightarrow{\mathcal{L}^1} \text{Dirac}(0)$$

CONVERGENZA QUASI CERTA:

X non dipende da n

$$\Longrightarrow \lim_{n\to+\infty} Y_n = \lim_{n\to+\infty} \frac{X}{n} = 0 \Longrightarrow Y_n \xrightarrow{a.s.} \text{Dirac}(0)$$

data $X \sim N(\mu_X, \sigma_X^2)$ $\quad X_1, \dots, X_n$

luogo $Y \sim N(\mu_Y, \sigma_Y^2)$ $\quad Y_1, \dots, Y_n$

NON INDIPENDENTI

$D := Y - X$ $\quad D_1, \dots, D_n$

Possiamo assumere essere gaussiana con media $\mu_D = \mu_Y - \mu_X$ e varianza σ_D^2 ignota.

→ Abbiamo le nostre misurazioni $x_1, \dots, x_7$; $y_1, \dots, y_7$ $(n=7)$

$H_0: \mu_Y = \mu_X$ $\qquad H_1: \mu_Y \neq \mu_X$ $\qquad H_2: \mu_Y > \mu_X$

$\alpha = 0{,}05$

→ Dovrebbe intervenire una statistica con distribuzione di Student:

$$\frac{\overline{D}_n - \mu_D}{S_{D_n}/\sqrt{n}} \sim t_{n-1} \xrightarrow{H_0 \text{ VERA}} \frac{\overline{D}_n}{S_{D_n}/\sqrt{n}} \sim t_{n-1}$$

$$\overline{d}_n = \frac{1}{n}\sum_{k=1}^{n} d_k = \frac{1}{7}\sum_{k=1}^{7}(y_k - x_k) = 0{,}857$$

$$S_{D_n}^2 = \frac{1}{n-1}\sum_{k=1}^{n}(d_k - \overline{d}_n)^2 \simeq \frac{1}{6}\left(1{,}088 + 0{,}002 + 0{,}1962 + 0{,}0018 + 5{,}0940 + 0{,}0660 + 0{,}8892\right) = 1{,}2228 \Rightarrow S_{D_n} \simeq 1{,}1058$$

La realizzazione della statistica è data da:

$$\frac{\overline{d}_n}{S_{D_n}/\sqrt{n}} = \frac{0{,}857}{1{,}1058/\sqrt{7}} \simeq 2{,}0505$$

→ $t_{n-1, 1-\alpha/2} = t_{6;0,975} = 2{,}447$; $t_{n-1,1-\alpha} = t_{6;0,95} = 1{,}943$

data luogo

REGIONE DI RIGETTO (con H_1):

Abbiamo il rigetto se: $2{,}0505 \in (-\infty; -t_{n-1,1-\alpha/2}) \cup (+t_{n-1,1-\alpha/2}; +\infty)$

ovvero se: $2{,}0505 \in (-\infty; -2{,}447) \cup (2{,}447, +\infty)$

↳ NON VERO

⇒ Non possiamo rigettare $H_0: \mu_Y = \mu_X$ a favore di $H_1: \mu_Y \neq \mu_X$ con una significatività di $\alpha = 0{,}05$.

REGIONE DI RIGETTO (con H_2):

Abbiamo il rigetto se: $2{,}0505 \in (t_{n-1,1-\alpha}; +\infty)$ ovvero se:

$2{,}0505 \in (1{,}943, +\infty)$ → VERO

⇒ Possiamo rigettare $H_0: \mu_Y = \mu_X$ a favore di $H_2: \mu_Y > \mu_X$ con una significatività di $\alpha = 0{,}05$

P-VALUE (con H_1):

Abbiamo il rigetto se: $2P(t_6 \geq 2{,}0505) \leq 0{,}05$

ovvero se: $P(t_6 \geq 2{,}0505) \leq 0{,}025$

QUESTA PROBABILITÀ È COMPRESA TRA 0,025 E 0,05

→ Non possiamo rigettare $H_0: \mu_Y = \mu_X$ a favore di $H_1: \mu_Y \neq \mu_X$ con una significatività di $\alpha = 0{,}05$.

P-VALUE (con H_2):

Abbiamo il rigetto se $P(t_6 \geq 2{,}0505) \leq 0{,}05$ ← VERO

→ Possiamo rigettare $H_0: \mu_Y = \mu_X$ a favore di $H_2: \mu_Y > \mu_X$ con una significatività di $\alpha = 0{,}05$.

$\sigma_X = 5{,}51$ $\quad \rho_{X,Y} = 0{,}98$ $\quad \alpha = 0{,}05$

$H_0: \sigma_Y^2 = \sigma_X^2$ $\quad H_1: \sigma_Y^2 \neq \sigma_X^2$

$H_2: \sigma_Y^2 < \sigma_X^2$

$$\sigma_D^2 = \sigma_X^2 + \sigma_Y^2 - 2Cov(X,Y) = \sigma_X^2 + \sigma_Y^2 - 2\sigma_X \sigma_Y \rho_{X,Y} =$$

$$\simeq 30{,}36 + \sigma_Y^2 - 10{,}80\,\sigma_Y =$$ ← H_0 vera

$$= 30{,}36 + 30{,}36 - 59{,}51 = 1{,}21$$

$\Rightarrow H_0: \sigma_D^2 = 1{,}21 \quad [H_1: \sigma_D^2 \neq 1{,}21 \;,\; H_2: \sigma_D^2 > 1{,}21]$

La statistica che interviene stavolta ha una distribuzione chi-quadro:

$$(n-1)\frac{S_{D,n}^2}{\sigma_D^2} \sim \chi^2_{n-1}$$

↳ La realizzazione di tale statistica è:

$$\frac{6 \cdot 1{,}2228}{1{,}21} \simeq 6{,}0635$$

- $\chi^{2-}_{n-1,\alpha/2} = \chi^{2-}_{6,\,0{,}025} = 1{,}237$
- $\chi^{2+}_{n-1,\alpha/2} = \chi^{2+}_{6,\,0{,}025} = 14{,}449$
- $\chi^{2-}_{n-1,\alpha} = \chi^{2-}_{6,\,0{,}05} = 1{,}635$
- $\chi^{2+}_{n-1,\alpha} = \chi^{2+}_{6,\,0.05} = 12{,}592$

REGIONE DI RIGETTO (con H_1):

Abbiamo il rigetto se: $6{,}0635 \in [0; \chi^{2-}_{n-1,\alpha/2}) \cup (\chi^{2+}_{n-1,\alpha/2}; +\infty)$

Ovvero se: $6{,}06356 \in [0; 1.237) \cup (14{,}449; +\infty)$ ← NON VERO

$\Rightarrow$ Non possiamo rigettare $H_0: \sigma_Y^2 = \sigma_X^2$ a favore di $H_1: \sigma_Y^2 \neq \sigma_X^2$ con una significatività di $\alpha = 0{,}05$

REGIONE DI RIGETTO (con H_2):

Abbiamo il rigetto se: $6{,}0635 \in (\chi^{+}_{n-1,\alpha}; +\infty)$ ovvero se:

$6{,}0635 \in (12{,}592; +\infty)$ ← NON VERO

$\Rightarrow$ Non possiamo rigettare $H_0: \sigma_Y^2 = \sigma_X^2$ a favore di $H_2: \sigma_Y^2 < \sigma_X^2$ con una significatività di $\alpha = 0.05$.

P-VALUE (con H_1):

Abbiamo il rigetto se: $\min\{p^+, p^-\} \leq 0{,}05$

dove
$$p^+ = P(\chi_6^2 \geq 6{,}0635)$$
$$p^- = P(\chi_6^2 \leq 6{,}0635)$$
} ENTRAMBI COMPRESI TRA 0,1 E 0,9

$\Rightarrow$ Non possiamo rigettare $H_0: \sigma_Y^2 = \sigma_X^2$ a favore di $H_1: \sigma_Y^2 \neq \sigma_X^2$ con una significatività di $\alpha = 0{,}05$.

P-VALUE (con H_2):

Abbiamo il rigetto se: $P(\chi_6^2 \geq 6{,}0635) \leq 0{,}05$ ← NON VERO

$\Rightarrow$ Non possiamo rigettare $H_0: \sigma_Y^2 = \sigma_X^2$ a favore di $H_2: \sigma_Y^2 < \sigma_X^2$ con una significatività di $\alpha = 0{,}05$.

$\sim 10/2/2020$ 5) $X \sim N(\mu_X, \sigma_X^2)$ INDIPENDENTI $X_1, \dots, X_m$

data $Y \sim N(\mu_Y, \sigma_Y^2)$ $Y_1, \dots, Y_n$

luogo

Abbiamo le realizzazioni $x_1, \dots, x_7$; $y_1, \dots, y_9$ ($m=7$, $n=9$)

$H_0 : \mu_Y = \mu_X$ $\quad H_1 : \mu_Y \neq \mu_X$ $\quad H_2 : \mu_Y > \mu_X$

$\alpha = 0{,}05$

→ Considero la variabile aleatoria $\overline{D}_{m,n} = \overline{X}_m - \overline{Y}_n$

dove $\overline{X}_m \sim N(\mu_X, \frac{\sigma_X^2}{m})$, $\overline{Y}_n \sim N(\mu_Y, \frac{\sigma_Y^2}{n})$ (indip. tra loro)

$\Rightarrow \overline{D}_{m,n} \simeq N(\mu_X - \mu_Y ; \frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n})$

→ Interviene la statistica di Student:

$$\frac{\overline{D}_{m,n} - \mu_D}{\sqrt{\frac{S_{X_m}^2}{m} + \frac{S_{Y_n}^2}{n}}} = \frac{\overline{X}_m - \overline{Y}_n - \mu_X + \mu_Y}{\sqrt{\frac{S_{X_m}^2}{m} + \frac{S_{Y_n}^2}{n}}} \sim t_{\hat{\nu}}$$

→ $\overline{x}_m = \frac{1}{m} \sum_{k=1}^{m} x_k = 5{,}80$

→ $\overline{y}_n = \frac{1}{n} \sum_{k=1}^{n} y_k = 3{,}92$

→ $S_{x_m}^2 = \frac{1}{m-1} \sum_{k=1}^{m} (x_k - \overline{x}_m)^2 = \frac{1}{6}(0 + 0{,}01 + 0{,}25 + 0{,}04 + 0{,}01 + 0{,}16$

$+ 0{,}09) \simeq 0{,}093$

→ $S_{Y_n}^2 = \frac{1}{n-1} \sum_{k=1}^{n} (y_k - \overline{y}_n)^2 = \frac{1}{8}(0{,}5184 + 0{,}2704 + 0{,}6084 + 0{,}38\dots$

$+ 0{,}3364 + 0{,}1024 + 0{,}1444 + 0{,}0784 + 0{,}0324) \simeq 0{,}309$

$$\hat{\nu} = \left[\frac{\left(\frac{S^2_{X_m}}{m} + \frac{S^2_{Y_n}}{n}\right)^2}{\frac{S^4_{X_m}}{(m-1)m^2} + \frac{S^4_{Y_n}}{(n-1)n^2}} \right] = \left[\frac{\left(\frac{0,093}{7} + \frac{0,309}{9}\right)^2}{\frac{0,093^2}{6 \cdot 49} + \frac{0,309^2}{8 \cdot 81}} \right]$$

$$= \left[\frac{0,00227}{0,00018} \right] = 12$$ (non so se serviva ma vbb)

→ Assumendo che l'ipotesi nulla sia vera, abbiamo che:

$$\frac{\overline{X}_m - \overline{Y}_n}{\sqrt{\frac{S^2_{X_m}}{m} + \frac{S^2_{Y_n}}{n}}} \sim t_{\hat{\nu}}$$

La realizzazione di tale statistica è:

$$\frac{\overline{x}_m - \overline{y}_n}{\sqrt{\frac{s^2_{x_m}}{m} + \frac{s^2_{y_n}}{n}}} = \frac{5,80 - 3,92}{\sqrt{\frac{0,093}{7} + \frac{0,309}{9}}} \approx \frac{1,88}{\sqrt{0,0476}} \approx 8,617$$

$\rightarrow t_{12,\,0,95} = 1,782$ $\rightarrow t_{12,\,0,975} = 2,179$

REGIONE DI RIGETTO (con H_1):

Abbiamo il rigetto se: $8,617 \in (-\infty, -2.179) \cup (2,179, +\infty)$

$\Rightarrow$ Possiamo rigettare $H_0 : \mu_Y = \mu_X$ a favore di $H_1 : \mu_Y \neq \mu_X$ con significatività $\alpha = 0,05$.

REGIONE DI RIGETTO (con H_2):

Abbiamo il rigetto se: $8,617 \in (-\infty, -1,782)$

$\Rightarrow$ Non possiamo rigettare $H_0 : \mu_Y = \mu_X$ a favore di $H_2 : \mu_Y > \mu_X$ con significatività $\alpha = 0,05$.

data
luogo

P-VALUE (con H_1):

Abbiamo il rigetto se: $2P(t_{12} \geq 8{,}617) \leq 0{,}05$

$\Rightarrow P(t_{12} \geq 8{,}617) \leq 0{,}025$ AVOJA

(già è vero $P(t_{12} \geq 2.3) \leq 0{,}025$)

$\Rightarrow$ Possiamo rigettare $H_0: \mu_Y = \mu_X$ in favore di $H_1: \mu_Y \neq \mu_X$ con significatività $\alpha = 0{,}05$.

P-VALUE (con H_2):

Abbiamo il rigetto se: $P(t_{12} \leq +8{,}617) \leq 0{,}05$

MA MAI NELLA VITA: GIÀ ABBIAMO $P(t_{12} \leq 1{,}356) = 0{,}9$ $\boxed{!}$

$\Rightarrow$ Non possiamo rigettare $H_0: \mu_Y = \mu_X$ in favore di $H_2: \mu_Y > \mu_X$ con significatività $\alpha = 0{,}05$.

U, V indipendenti $\quad U \sim \chi^2_m \quad V \sim \chi^2_n$

$\Rightarrow \dfrac{U/m}{V/n} \sim F(m,n)$

$\triangleright f^-_{m\,n\,\alpha}, \; f^+_{m\,n\,\alpha}$ NOTI

Aggi introduco una statistica per testare l'ipotesi nulla $H_0: \sigma_X^2 = \sigma_Y^2$ contro $H_1: \sigma_X^2 > \sigma_Y^2$ e $H_2: \sigma_X^2 < \sigma_Y^2$

~~Riprendiamo $D = X - Y \sim N(\mu_X - \mu_Y, \frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n})$~~

Consideriamo: $\frac{(m-1) S_{Xm}^2}{\sigma_X^2} \sim \chi^2_{m-1}$

data

luogo

$$\frac{(n-1) S_{Xn}^2}{\sigma_Y^2} \sim \chi^2_{n-1}$$

Allora: $\frac{(m-1) S_{Xm}^2}{(m-1)\sigma_X^2} \cdot \frac{(n-1)\sigma_Y^2}{(n-1) S_{Xn}^2} = \frac{S_{Xm}^2/\sigma_X^2}{S_{Xn}^2/\sigma_Y^2} \sim F(m-1, n-1)$

Se l'ipotesi nulla è vera, allora:

$$\frac{S_{Xm}^2}{S_{Yn}^2} \sim F(m-1, n-1)$$

Mi sa proprio che è questa la statistica che prendiamo

La sua realizzazione è: $\frac{S_{xm}^2}{S_{xn}^2}$

DUNQUE:

REGIONE DI RIGETTO:

Ho il rigetto a favore di $H_1: \sigma_X^2 > \sigma_Y^2$ se

$$\frac{S_{xm}^2}{S_{xn}^2} \in \left(f^+_{m-1,n-1,\alpha}\,;+\infty\right)$$

Ho il rigetto a favore di $H_2: \sigma_X^2 < \sigma_Y^2$ se

$$\frac{S_{xm}^2}{S_{xn}^2} \in \left[0, f^-_{m-1,n-1,\alpha}\right)$$

P-VALUE:

Ho il rigetto a favore di $H_1: \sigma_X^2 > \sigma_Y^2$ se $P\left(f_{m-1,n-1} > \frac{S_{xm}^2}{S_{xn}^2}\right) \le \alpha$

Ho il rigetto a favore ... $P(f ... \frac{S_{xm}^2}{...}) \le \alpha$

3) $\mu_0 = 20$ $\quad n=100$ $\quad \bar{x}_n = 18$ $\quad s_n = 4$ $\quad 1-\alpha = 0{,}95$

data
luogo

ASSUMO $X \sim N(\mu_x, \sigma_x^2)$ $\quad \mu_x, \sigma_x^2$ IGNOTE

Interviene la statistica di Student:

$$\frac{\bar{X}_n - \mu_x}{S_n/\sqrt{n}} \sim t_{n-1} \Rightarrow \mu_x = \bar{X}_n - t_{n-1}\frac{S_n}{\sqrt{n}}$$

$\Rightarrow$ L'intervallo di confidenza al livello di confidenza $1-\alpha = 0{,}95$ per μ_x è:

$$\left(\bar{X}_n - t_{n-1;1-\alpha/2}\frac{S_n}{\sqrt{n}} \; ; \; \bar{X}_n + t_{n-1;1-\alpha/2}\frac{S_n}{\sqrt{n}}\right)$$

$\rightarrow t_{n-1;1-\alpha/2} = t_{99;0{,}975} \approx 1{,}98$

$\Rightarrow$ La realizzazione dell'intervallo è:

$$\left(18 - 1{,}98 \cdot \frac{4}{10} \; ; \; 18 + 1{,}98 \cdot \frac{4}{10}\right) \equiv (17{,}21 \; ; 18{,}79)$$

$\mu_0 = 20$ non vi appartiene.

$H_0: \mu = 20$ $\quad H_1: \mu > 20$ $\quad \alpha = 0{,}05$

Al solito, qui interviene la statistica

$$\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1} \quad \leftarrow \text{Assumendo } H_0 \text{ vera}$$

La realizzazione di tale statistica è data da:

$$\frac{18-20}{4/10} = -2 \cdot \frac{10}{4} = -5$$

$\rightarrow t_{n-1,1-\alpha} = t_{99;0.95} = 1{,}66$

REGIONE DI RIGETTO:

Abbiamo il rigetto se $-5 \in (1{,}66 \; ; +\infty)$ $\quad \leftarrow$ NON VERO

data
luogo

$\Rightarrow$ Non possiamo rigettare $H_0: \mu = 20$ in favore di $H_1: \mu > 20$ con significatività $\alpha = 0{,}05$.

P-VALUE:

Abbiamo il rigetto se $P(t_{99} > -5) \leq 0{,}05$ $\quad$ MA MANCO PE' NIENTE!!

$\hookrightarrow$ Non possiamo rigettare $H_0: \mu = 20$ in favore di $H_1: \mu > 20$ con significatività $\alpha = 0{,}05$.

~~$\beta(23) = P(\text{Accetto } H_0 \mid \mu = 23) = P\left(\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \leq 1{,}66 \mid \mu = 23\right)$~~ SBAGLIATO

~~$= P\left(\frac{\bar{X}_{100} - 23}{S_{100}/10} \leq 1{,}66\right) = P(t_{99} \leq 1{,}66) = 0{,}95$~~

Noi abbiamo appena lavorato con la statistica $\frac{\bar{X}_n - 20}{S_n/\sqrt{n}} =$

$$= \frac{\bar{X}_n - \mu + \mu - 20}{S_n/\sqrt{n}} = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} + \frac{\mu - 20}{S_n/\sqrt{n}}$$

$$\Rightarrow \beta(23) = P(\text{Accetto } H_0 \mid \mu = 23) = P\left(\frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} + \frac{\mu - 20}{S_n/\sqrt{n}} \leq 1{,}66 \mid \mu = 23\right) =$$

$$= P\left(\frac{\bar{X}_n - 23}{S_n/\sqrt{n}} \leq 1{,}66 - \frac{3}{S_n/\sqrt{n}}\right)$$

Vabbè ce semo capiti