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**Dipartimento d'Ingegneria Civile e Ingegneria Informatica**
**LM in Ingegneria dell'Informazione e dell'Automazione**
**Complementi di Probabilità e Statistica - Advanced Statistics**
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**Problems on Sequences of Random Variables with Solutions 2021-12-23**

**Problem 1** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X$ be a uniformly distributed real random variable on the interval $[0,1]$. In symbols, $X \sim U(0,1)$. Consider the sequence $(Y_n)_{n\geq 1}$ of real random variables given by*

$$Y_n \overset{def}{=} \begin{cases} n, & \text{if } 0 \leq X < \frac{1}{n}, \\ 0, & \text{if } 1/n \leq X \leq 1, \end{cases} \qquad \forall n \geq 1.$$

*Check whether the sequence $(Y_n)_{n\geq 1}$ converges in distribution, converges in probability, converges almost surely, converges in mean, and converges in mean square in the indicated order.*

*Hint: to deal with the almost sure convergence consider the event $E_0 \equiv \{\omega \in \Omega : X(\omega) \leq 0\}$ and the complement $E_0^c$.*

**Solution.** Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}$ for the distribution function of $Y_n$. We have

$$F_{Y_n}(y) = \mathbf{P}(Y_n \leq y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(1/n \leq X \leq 1) = 1 - 1/n, & \text{if } 0 \leq y < n, \\ 1, & \text{if } n \leq y. \end{cases}$$

Note that for every $y \geq 0$ there exists $n(y) \in \mathbb{N}$, (e.g. $n(y) \equiv \lceil y \rceil$, where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{R}$, is the ceiling function), such that $y < n$ for every $n > n(y)$. Therefore, we have definitively,

$$\mathbf{P}(Y_n \leq y) = 1 - 1/n.$$

It then follows

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \lim_{n\to\infty} 1/n = 1, & \text{if } 0 \leq y. \end{cases}$$

Considering the Heavside function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } 0 \leq y, \end{cases}$$

we clearly have

$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R}$. Hence, the sequence $(Y_n)_{n\geq 1}$ converges in distribution to the standard Dirac real random variable $Dir(0)$.

With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0 \in \mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. However, by a direct approach, according to the definition of $Y_n$, we have

$$\mathbf{P}(Y_n = n) = \mathbf{P}\left(0 \leq X < \frac{1}{n}\right) = \frac{1}{n} \quad \text{and} \quad \mathbf{P}(Y_n = 0) = \mathbf{P}\left(\frac{1}{n} \leq X \leq 1\right) = 1 - \frac{1}{n},$$

for every $n \geq 1$. Therefore, guessing that $Y_n \to Dir(0)$, we have definitively,

$$\mathbf{P}(|Y_n - Dir(0)| \leq \varepsilon) = \mathbf{P}(|Y_n| \leq \varepsilon) \geq \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n},$$

for every $\varepsilon > 0$. It follows

$$\lim_{n\to\infty} \mathbf{P}\left(|Y_n| \leq \varepsilon\right) \geq 1 - \lim_{n\to\infty} \frac{1}{n} = 1,$$

for every $\varepsilon > 0$, which eventually shows the convergence in probability of $(Y_n)_{n\geq 1}$ to $Dir\,(0)$. Now, to check the almost sure convergence to $Dir\,(0)$ (recall that the almost sure convergence at some random variable implies convergence in probability at the same random variable) consider the event

$$E_0 \equiv \{\omega \in \Omega : X(\omega) \leq 0\}.$$

Since $X \sim U(0,1)$, we have $\mathbf{P}(E_0) = \mathbf{P}(X \leq 0) = 0$. In addition, for every $\omega \in E_0^c$ we have $X(\omega) > 0$ and it is possible to find $n(\omega)$ such that

$$\frac{1}{n} < X(\omega),$$

for every $n > n(\omega)$. It then follows that

$$Y_n(\omega) = 0,$$

for every $n > n(\omega)$. This implies

$$\lim_{n\to\infty} Y_n(\omega) = 0,$$

for every $\omega \in E_0^c$, which is the almost sure convergence of the sequence $(Y_n)_{n\geq 1}$ to $Dir\,(0)$. In the end, to check the convergence in mean to $Dir\,(0)$ (recall that the convergence in mean at some random variable implies convergence in probability at the same random variable), we consider

$$\mathbf{E}\left[|Y_n - Dir\,(0)|\right] = \mathbf{E}\left[Y_n\right] = 0 \cdot \mathbf{P}\left(Y_n = 0\right) + n \cdot \mathbf{P}\left(Y_n = n\right) = n \cdot \frac{1}{n} = 1.$$

It follows that

$$\lim_{n\to\infty} \mathbf{E}\left[|Y_n - Dir\,(0)|\right] = 1 \neq 0.$$

Hence, $(Y_n)_{n\geq 1}$ does not converge in mean to $Dir\,(0)$. This also implies that $(Y_n)_{n\geq 1}$ does not converge in mean at all (recall that convergence in mean at some random variable implies convergence in probability at the same random variable). As a consequence, $(Y_n)_{n\geq 1}$ does not converge in mean square. $\square$

**Problem 2** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_L) \equiv \mathbb{R}$ be the real Borel-Lebesgue state space. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) \overset{def}{=} \frac{\alpha - 1}{x^\alpha} 1_{[1,+\infty)}, \quad \forall x \in \mathbb{R},$$

*where $\alpha > 1$, is a density. Thereafter, consider a real random variable $X$ on $\Omega$ with density $f_X = f$ and the sequence $(Y_n)_{n\geq 1}$ of real random variables on $\Omega$ given by*

$$Y_n \overset{def}{=} \frac{X}{n}, \quad \forall n \in \mathbb{N}.$$

1. *Study the convergence in distribution, in probability, almost sure, and in p-th mean of the sequence $(Y_n)_{n\geq 1}$, on varying of $\alpha > 1$, in the indicated order.*
2. *Assume that $(X_n)_{n\geq 1}$ is a sequence of (totally) independent real random variables on $\Omega$ with density $f_{X_n} = f$, for every $n \in \mathbb{N}$, and consider the sequence $(Z_n)_{n\geq 1}$ of real random variables on $\Omega$ given by*

$$Z_n \overset{def}{=} \frac{X_n}{n}, \quad \forall n \in \mathbb{N}.$$

*Study the convergence in distribution, in probability, almost sure, and in p-th mean of the sequence $(Z_n)_{n\geq 1}$, on varying of $\alpha > 1$, in the indicated order.*

**Solution.**

1. Since $\alpha > 1$, we have
$$f(x) \geq 0,$$
for every $x \in \mathbb{R}$, and, since $1 - \alpha < 0$, we have
$$\begin{aligned}\int_{\mathbb{R}} f(x)\, d\mu_L(x) &= \int_{\mathbb{R}} \frac{\alpha-1}{x^\alpha} 1_{[1,+\infty)}(x)\, d\mu_L(x) = \int_{[1,+\infty)} \frac{\alpha-1}{x^\alpha} d\mu_L(x) \\ &= \int_1^{+\infty} \frac{\alpha-1}{x^\alpha} dx = \lim_{x\to+\infty} \int_1^x \frac{\alpha-1}{u^\alpha} du = -\lim_{x\to+\infty} \int_1^x du^{1-\alpha} \\ &= -\lim_{x\to+\infty} u^{1-\alpha}\Big|_1^x = 1 - \lim_{x\to+\infty} x^{1-\alpha} = 1.\end{aligned}$$
This shows that $f : \mathbb{R} \to \mathbb{R}$ is a density.
Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}$ for the distribution function of $Y_n$, for every $n \geq 1$. We have
$$F_{Y_n}(y) = \mathbf{P}(Y_n \leq y) = \mathbf{P}(X/n \leq y) = \mathbf{P}(X \leq ny) = \int_{(-\infty,ny]} f(x)\, d\mu_L(x).$$
for every $y \in \mathbb{R}$. On the other hand,
$$\begin{aligned}\int_{(-\infty,ny]} f(x)\, d\mu_L(x) &= \int_{(-\infty,ny]} \frac{\alpha-1}{x^\alpha} 1_{[1,+\infty)}(x)\, d\mu_L(x) \\ &= \int_{(-\infty,ny]\cap[1,+\infty)} \frac{\alpha-1}{x^\alpha} d\mu_L(x) \\ &= \begin{cases} \int_{\varnothing} \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } ny < 1, \\ \int_{\{ny\}} \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } ny = 1, \\ \int_{[1,ny]} \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } 1 < ny, \end{cases}\end{aligned}$$
where
$$\int_{\varnothing} \frac{\alpha-1}{x^\alpha} d\mu_L(x) = \int_{\{ny\}} \frac{\alpha-1}{x^\alpha} d\mu_L(x) = 0$$
and
$$\int_{[1,ny]} \frac{\alpha-1}{x^\alpha} d\mu_L(x) = \int_1^{ny} \frac{\alpha-1}{x^\alpha} dx = -\int_1^{ny} dx^{1-\alpha} = -\, x^{1-\alpha}\Big|_1^{ny} = 1 - \frac{1}{n^{\alpha-1} y^{\alpha-1}}.$$
Therefore,
$$F_{Y_n}(y) = \begin{cases} 0, & \text{if } y \leq \frac{1}{n}, \\ 1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}, & \text{if } \frac{1}{n} < y, \end{cases} = \left(1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}\right) 1_{(1/n,+\infty)}(y),$$
for every $y \in \mathbb{R}$. As a consequence,
$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ 1 - \lim_{n\to\infty} \frac{1}{n^{\alpha-1}y^{\alpha-1}} = 1, & \text{if } 0 < y. \end{cases}$$
Considering the Heavside function $H : \mathbb{R} \to \mathbb{R}$ given by
$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } 0 \leq y, \end{cases}$$
we then have
$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R}- \{0\}$, where the Heavside function is continuous. Hence, the sequence $(Y_n)_{n\geq 1}$ converges in distribution to the standard Dirac real random variable $Dir\,(0)$ on $\Omega$. Regarding to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir\,(y_0)$, concentrated at some $y_0 \in \mathbb{R}$, implies also the convergence in probability to $Dir\,(y_0)$. Thus, we can state that the sequence $(Y_n)_{n\geq 1}$ converges in distribution to $Dir\,(0)$. However, by a direct approach, observe that

$$F'_{Y_n}(y) = \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y),$$

for every $y \in \mathbb{R}- \{1/n\}$, and

$$\int_{(-\infty,y]} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} 1_{(1/n,+\infty)}(u)\, d\mu_L(u) = \int_{(-\infty,y]\cap(1/n,+\infty)} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} d\mu_L(u) = \begin{cases} \int_{\varnothing} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} d\mu_L(u), & \text{if } y \leq 1 \\ \int_{[1/n,y]} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} d\mu_L(u), & \text{if } y > 1 \end{cases}$$

where

$$\int_{\varnothing} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} d\mu_L(u) = 0$$

and

$$\begin{aligned}\int_{[1/n,y]} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} d\mu_L(u) &= \frac{1}{n^{\alpha-1}} \int_{1/n}^{y} \frac{\alpha-1}{u^{\alpha}} du = -\frac{1}{n^{\alpha-1}} \int_{1/n}^{y} du^{1-\alpha} = -\frac{1}{n^{\alpha-1}}\ u^{1-\alpha}\Big|_{1/n}^{y} \\ &= \frac{1}{n^{\alpha-1}} \left( \frac{1}{n^{1-\alpha}} - \frac{1}{y^{\alpha-1}} \right) = 1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}.\end{aligned}$$

Therefore, we can write

$$\int_{(-\infty,y]} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} 1_{(1/n,+\infty)}(u)\, d\mu_L(u) = \left(1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}\right) 1_{(1/n,+\infty)}(y).$$

This shows that $Y_n$ is absolutely continuous with density $f_{Y_n} : \mathbb{R} \to \mathbb{R}$, given by

$$f_{Y_n}(y) \overset{\text{def}}{=} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y), \quad \forall y \in \mathbb{R}.$$

As a consequence, provided $n$ is sufficiently large,

$$\begin{aligned}
\mathbf{P}\left(Y_n \geq \varepsilon\right) &= \int_{[\varepsilon,+\infty)} f_{Y_n}(y)\, d\mu_L(y) \\
&= \int_{[\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y)\, d\mu_L(y) \\
&= \int_{[\varepsilon,+\infty)\cap(1/n,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{[\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{\varepsilon}^{+\infty} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} dy \\
&= \frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} \frac{\alpha-1}{u^{\alpha}} du \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} du^{\alpha-1} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{1-\alpha}\Big|_{\varepsilon}^{y} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \left(\frac{1}{y^{\alpha-1}} - \frac{1}{\varepsilon^{\alpha-1}}\right) \\
&= \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}}.
\end{aligned}$$

It follows

$$\lim_{n\to\infty} \mathbf{P}\left(\left|Y_n - Dir(0)\right| \geq \varepsilon\right) = \lim_{n\to\infty} \mathbf{P}\left(Y_n \geq \varepsilon\right) = \lim_{n\to\infty} \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}} = 0,$$

for every $\varepsilon > 0$. This proves directly that $(Y_n)_{n\geq1}$ converges in probability to $Dir(0)$. In addition, we have

$$\lim_{n\to\infty} Y_n(\omega) = \lim_{n\to\infty} \frac{X(\omega)}{n} = 0,$$

for every $\omega \in \Omega$, which proves that $(Y_n)_{n\geq1}$ converges almost surely to $Dir(0)$.

By virtue of what shown above, to study the convergence in $p$-th mean of the sequence $(Y_n)_{n\geq1}$, we have to study the convergence in $p$-th mean of the sequence $(Y_n)_{n\geq1}$ to $Dir(0)$. Hence, it is sufficient to show whether

$$\lim_{n\to\infty} \mathbf{E}\left[Y_n^p\right] = 0.$$

We have

$$\begin{aligned}
\mathbf{E}\left[Y_n^p\right] &= \int_{\mathbb{R}} y^p f_{Y_n}(y)\, d\mu_L(u) = \int_{\mathbb{R}} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha-p}} 1_{(1/n,+\infty)}(y)\, d\mu_L(u) = \frac{1-\alpha}{n^{\alpha-1}} \int_{(1/n,+\infty)} \frac{1}{y^{\alpha-p}} d\mu_L(u) \\
&= \frac{1-\alpha}{n^{\alpha-1}} \int_{1/n}^{+\infty} \frac{1}{y^{\alpha-p}} dy = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} du^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{p-\alpha+1}\Big|_{1/n}^{y}, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} d\ln(u) = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \ln(u)\Big|_{1/n}^{y}, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mathbf{E}\left[Y_n^p\right] &= \mathbf{E}\left[\left(\frac{X}{n}\right)^p\right] = \int_{\mathbb{R}} \frac{x^p}{n^p} f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{x^p}{n^p}\frac{\alpha-1}{x^\alpha} 1_{[1,+\infty)}(x)\, d\mu_L(x) \\
&= \frac{\alpha-1}{n^p}\int_{[1,+\infty)} \frac{1}{x^{\alpha-p}} d\mu_L(x) = \frac{\alpha-1}{n^p}\int_1^{+\infty} \frac{1}{x^{\alpha-p}} dx = \frac{\alpha-1}{n^p}\lim_{x\to+\infty}\int_1^x \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}\lim_{x\to+\infty}\int_1^x du^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}\lim_{x\to+\infty} u^{p-\alpha+1}\big|_1^x, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^p}\lim_{x\to+\infty}\int_1^x d\ln(u) = \frac{\alpha-1}{n^p}\lim_{x\to+\infty}\ln(u)\big|_1^x, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}$$

Now, if $p \geq \alpha-1$ we have that $\mathbf{E}\left[Y_n^p\right]$ is not finite. Therfore, the sequence $(Y_n)_{n\geq1}$ cannot converge in $p$-th mean. If $1 \leq p < \alpha-1$, we have

$$\mathbf{E}\left[Y_n^p\right] = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}}\frac{1}{n^{p-\alpha+1}} = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}.$$

Hence,

$$\lim_{n\to\infty}\mathbf{E}\left[Y_n^p\right] = -\lim_{n\to\infty}\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} = 0.$$

In this case, the sequence $(Y_n)_{n\geq1}$ converges in $p$-th mean to $Dir(0)$.

2. Clearly, the random variable $Z_n$ has the same distribution function of the random variable $Y_n$ for every $n \geq 1$. It follows that

$$Z_n \xrightarrow{\mathbf{w}} Dir(0) \quad \text{and} \quad Z_n \xrightarrow{\mathbf{P}} Dir(0).$$

Regarding to the almost sure convergence, observe that, with the same computations as above, we obtain

$$\mathbf{P}(|Z_n| \geq \varepsilon) = \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}}.$$

Therefore,

$$\sum_{n=1}^{\infty}\mathbf{P}(|Z_n| \geq \varepsilon) = \frac{1}{\varepsilon^{\alpha-1}}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha-1}}.$$

This converges for $\alpha > 2$, which implies the almost sure convergence of the sequence $(Z_n)_{n\geq1}$ to $Dir(0)$, for $\alpha > 2$. In case $1 < \alpha \leq 2$, we have

$$\mathbf{P}\left(\bigcap_{n\geq m}\{|Z_n| < \varepsilon\}\right) \leq \mathbf{P}\left(\bigcap_{n\geq m}^{s}\{|Z_n| < \varepsilon\}\right)$$

for every $s \geq m$. On the other hand, thanks to the independence of the random variables in $(X_n)_{n\geq1}$, we can write

$$\mathbf{P}\left(\bigcap_{n\geq m}^{s}\{|Z_n| < \varepsilon\}\right) = \prod_{n=m}^{s}\mathbf{P}(\{|Z_n| < \varepsilon\}) = \prod_{n=m}^{s}\left(1 - \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}}\right).$$

Now, the function $g : \mathbb{R}_+ \to \mathbb{R}$, given by

$$g(x) \stackrel{\text{def}}{=} x^{\alpha-1}, \quad \forall x \in \mathbb{R}_+$$

is increasing. Therefore

$$\varepsilon^{\alpha-1}n^{\alpha-1} \leq \varepsilon^{\alpha-1}s^{\alpha-1}$$

for every $n \leq s$. In addition, we have

$$\varepsilon^{\alpha-1}s^{\alpha-1} \leq \varepsilon^{\alpha-1}s$$

for every $s \in \mathbb{N}$ and every $\alpha$ such that $1 < \alpha \leq 2$. We can then write

$$1 - \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}} \leq 1 - \frac{1}{\varepsilon^{\alpha-1}s^{\alpha-1}} \leq 1 - \frac{1}{\varepsilon^{\alpha-1}s},$$

for every $n \leq s$. As a consequence, in case $1 < \alpha \leq 2$, we can write

$$\prod_{n=m}^{s}\left(1 - \frac{1}{\varepsilon^{\alpha-1}n^{\alpha-1}}\right) \leq \prod_{n=m}^{s}\left(1 - \frac{1}{\varepsilon^{\alpha-1}s}\right).$$

In particular, setting $s = 2m$, we obtain

$$\mathbf{P}\left(\bigcap_{n\geq m}\{|Z_n| < \varepsilon\}\right) \leq \mathbf{P}\left(\bigcap_{n\geq m}^{2m}\{|Z_n| < \varepsilon\}\right) \leq \prod_{n=m}^{2m}\left(1 - \frac{1}{2\varepsilon^{\alpha-1}m}\right) = \left(1 - \frac{1}{2\varepsilon^{\alpha-1}m}\right)^{m}.$$

In the end, since

$$\lim_{m\to\infty}\left(1 - \frac{1}{2\varepsilon^{\alpha-1}m}\right)^{m} = \frac{1}{e^{\frac{1}{2}\varepsilon^{1-\alpha}}} < 1,$$

for every $\varepsilon > 0$, we cannot have the almost sure convergence of the sequence $(Z_n)_{n\geq 1}$ to $Dir\,(0)$. Alternatively, we obtain the same result by observing that, since the random variables of the sequence $(Z_n)_{n\geq 1}$ are independent, we have

$$Z_n \overset{\mathbf{a.s.}}{\to} Dir\,(0) \Leftrightarrow \sum_{n=1}^{\infty}\mathbf{P}\,(|Y_n| \geq \varepsilon) < \infty, \quad \forall\varepsilon > 0$$

(see Rohatgi, 1976, p. 265) and the series

$$\sum_{n=1}^{\infty}\mathbf{P}\,(|Z_n| \geq \varepsilon) = \frac{1}{\varepsilon^{\alpha-1}}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha-1}}$$

does not converge in case $1 < \alpha \leq 2$.

**Problem 3** *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(X_n)_{\geq n}$ be a sequence of real random variables on $\Omega$. Assume that $(X_n)_{\geq n}$ are identically distributed and let $f_X : \mathbb{R} \to \mathbb{R}$ their common density function given by*

$$f_X\,(x) \overset{def}{=} \frac{2}{x^3}1_{[1,+\infty)}\,(x)\,, \quad \forall x \in \mathbb{R}.$$

*Consider the random variable $X$ with density $f_X : \mathbb{R} \to \mathbb{R}$ and set*

$$Y_n \overset{def}{=} \frac{X}{n^{\alpha}}, \quad \forall n \geq 1,$$

*where $\alpha > 0$.*

1. *Study the convergence in distribution, probability, almost surely, and $L^p$ of the sequence $(Y_n)_{n\geq 1}$, on varying of $\alpha > 0$, in the indicated order.*

2. *Under the additional assumption of independence of the random variables of the sequence* $(X_n)_{\geq n}$, *consider the sequence*

$$Y_n \equiv \frac{X_n}{n^\alpha}, \quad \forall n \geq 1,$$

*Does the sequence* $(Y_n)_{n\geq 1}$ *converge almost surely?*

**Solution.** .

**Problem 4** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a complete probability space and let* $(X_n)_{n\geq 1}$ *be a sequence of independent real random variables such that* $X_n \sim Ber(1/n^\alpha)$ *for some* $\alpha > 0$. *Consider the sequence* $(Y_n)_{n\geq 1}$ *of real random variables on* $\Omega$ *given by*

$$Y_n \stackrel{def}{=} \min\{X_1, \dots, X_n\}.$$

1. *study the convergence in distribution, in probability and in* $L^p(\Omega; \mathbb{R})$ *of* $(X_n)_{n\geq 1}$ *and* $(Y_n)_{n\geq 1}$ *on varying of* $\alpha > 0$;

2. *study the almost sure convergence of* $(X_n)_{n\geq 1}$ *and* $(Y_n)_{n\geq 1}$ *on varying of* $\alpha > 0$.

**Solution.** .

**Problem 5** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(X_n)_{\geq n}$ *be a sequence of real random variables on* $\Omega$. *Prove that*

$$X_n \xrightarrow{\mathbf{P}} 0 \Leftrightarrow \lim_{n\to\infty} \mathbf{E}\left[\frac{|X_n|}{1+|X_n|}\right] = 0.$$

**Solution.** .

**Problem 6** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the Euclidean real line endowed with the Borel* $\sigma$*-algebra. Assume that* $\Omega$ *is countable and* $\mathcal{E} = \mathcal{P}(\Omega)$. *Let* $(X_n)_{n\geq 1}$ *be a sequence of real random variables on* $\Omega$. *Show that if there exists a real random variable* $X$ *on* $\Omega$ *such that* $X_n \xrightarrow{\mathbf{P}} X$, *then* $X_n \xrightarrow{a.s.} X$.

**Solution.** .

**Problem 7** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the Euclidean real line endowed with the Borel* $\sigma$*-algebra. Let* $(X_n)_{n\geq 1}$ *be a sequence of independent standard Bernoulli random variables on* $\Omega$ *each* $X_n$ *of which has success probability* $p_n$. *Prove that*

**Exercise 8**

1. $X_n \xrightarrow{\mathbf{P}} 0$ *if and only if* $\lim_{n\to\infty} p_n = 0$.

2. $X_n \xrightarrow{a.s.} 0$ *if and only if* $\sum_{n=1}^{\infty} p_n < \infty$.

**Solution.** .

**Exercise 9 (Sheldon M. Ross - 4.2 - 4.3)** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(X_n)_{n\geq 1}$ *be a sequence of independent Bernoulli random variables. Recall that for a Bernoulli random variable* $X$ *we have*

$$X = \begin{cases} 1 & \mathbf{P}(X=1) = p \\ 0 & \mathbf{P}(X=0) = q \end{cases},$$

*where* $p \in (0,1)$ *and* $q \equiv 1-p$*. Consider the sequence* $(Z_n)_{n\geq 1}$ *of random variables given by*

$$Z_n \overset{def}{=} \textstyle\sum_{k=1}^{n} X_k, \quad \forall n \geq 1$$

*and let* $(H_n)_{n\geq 1}$ *be the sequence of random variables given by*

$$H_n \overset{def}{=} 2Z_n - n, \quad \forall n \geq 1.$$

1. *Assume that* $X_n$ *is the random variable which represents the toss of a coin with the convention that "success" [resp. "failure"] is for the outcome "head" [resp. "tail"] represented, in turn, by the outcome* $1$ *[resp.* $0$*]. Give an interpretation of the random variables* $Z_n$ *and* $H_n$ *and compute their meand and variance.*

2. *Assume that* $p = 1/2$*. Prove that we have*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{H_n}{\sqrt{n}} < z\right) = \Phi(z),$$

*for every* $z \in \mathbb{R}$*, where* $\Phi$ *is the standard notation for the distribution function of the standard normal random variable.*

**Solution.** .