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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica
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Problem 1 *Let $(X_n)_{n\geq 1}$ a sequence of independent real random variables on a probability space Ω such that*

$$\mathbf{P}(X_n = x) = \begin{cases} 1-\frac{1}{n} & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \sqrt{n} \\ 0 & \text{otherwise} \end{cases}$$

In the assigned order, check whether the sequence $(X_n)_{n\geq 1}$ converges in distribution, converges in probability, converges almost surely, converges in mean, and converges in square mean.

Solution. According to the definition of the probability distribution, X_n is a Bernoulli random variable on Ω with states 0, $\sqrt{n}$ and success probability $\frac{1}{n}$, for every $n \geq 1$. In symbols,

$$X_n = \begin{cases} 0 & \mathbf{P}(X_n = 0) = 1-\frac{1}{n} \\ \sqrt{n} & \mathbf{P}(X_n = \sqrt{n}) = \frac{1}{n} \end{cases}.$$

Then, considering the distribution function $F_{X_n} : \mathbb{R} \to \mathbb{R}_+$ of X_n, we have

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-\frac{1}{n} & \text{if } 0 \leq x < \sqrt{n} \\ 1 & \text{if } \sqrt{n} \leq x \end{cases}.$$

As a consequence,

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

In fact, for any $x < 0$ we have $F_{X_n}(x) = 0$, for every $n \geq 1$, which clearly implies $\lim_{n\to\infty} F_{X_n}(x) = 0$. On the other hand, for any $x \geq 0$ there clearly exist $n_x \geq 1$ such that $x \leq \sqrt{n}$ for every $n \geq n_x$. It follows, $F_{X_n}(x) = 1 - \frac{1}{n}$ for any $n \geq n_x$ and $\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$.

Therefore, the sequence $(F_{X_n})_{n\geq 0}$ of distribution functions converges to the Heaviside function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

for every $x \in \mathbb{R}$. In particular for all $x \in \mathbb{R}$ where F is continuous. This means that the sequence $(X_n)_{n\geq 1}$ converges in distribution to the Dirac random variable concentrated at 0. Write $X \equiv Dir(0)$. Now, for any $0 < \varepsilon < 1$ consider the event $\{|X_n - X| \geq \varepsilon\}$ on varying of $n \geq 1$. We have

$$|X_n - X| = \begin{cases} 0 & \text{if } X_n = 0 \\ \sqrt{n} & \text{if } X_n = \sqrt{n} \end{cases}.$$

Hence,

$$|X_n - X| = X_n.$$

The latter implies

$$\{|X_n - X| \geq \varepsilon\} = \{X_n \geq \varepsilon\} = \{X_n = \sqrt{n}\}.$$

It follows

$$\mathbf{P}(|X_n - X| \geq \varepsilon) = \mathbf{P}\left(X_n = \sqrt{n}\right) = \frac{1}{n},$$

for every $n \geq 1$. As a consequence,

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - X| \geq \varepsilon\right) = \lim_{n\to\infty} \frac{1}{n} = 0,$$

which yields the convergence in probability of the sequence $(X_n)_{n\geq 1}$ to X. Note that this result is implied by the theorem stating that any sequence $(X_n)_{n\geq 1}$ which converges in distribution to a Dirac random variable converges also in probability to this Dirac random variable.

To check the almost sure convergence of the sequence $(X_n)_{n\geq 1}$ to X, choosing any $\varepsilon < 1$, on account of the independence of the random variables of the sequence $(X_n)_{n\geq 1}$, we estimate

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{n\geq m}\{|X_n| \leq \varepsilon\}\right) &\leq \mathbf{P}\left(\bigcap_{n=m}^{2m}\{|X_n| \leq \varepsilon\}\right) = \prod_{n=m}^{2m}\mathbf{P}\left(|X_n| \leq \varepsilon\right) \\
&= \prod_{n=m}^{2m}\mathbf{P}\left(X_n = 0\right) = \prod_{n=m}^{2m}\left(1 - \frac{1}{n}\right) \\
&\leq \prod_{n=m}^{2m}\left(1 - \frac{1}{2m}\right) = \left(1 - \frac{1}{2m}\right)^m.
\end{aligned}$$

As a consequence,

$$\lim_{m\to\infty} \mathbf{P}\left(\bigcap_{n\geq m}\{|X_n| \leq \varepsilon\}\right) \leq \lim_{m\to\infty}\left(1 - \frac{1}{2m}\right)^m = e^{-1/2} < 1.$$

This prevents that

$$\lim_{m\to\infty} \mathbf{P}\left(\bigcap_{n\geq m}\{|X_n| \leq \varepsilon\}\right) = 1,$$

so that $X_n \overset{\textbf{a.s.}}{\nrightarrow} 0$.

To check the convergence in mean of the sequence $(X_n)_{n\geq 1}$, recall first that if a sequence $(X_n)_{n\geq 1}$ of real random variables converges in probability to a real random variable X and $(X_n)_{n\geq 1}$ converges also in mean, then $(X_n)_{n\geq 1}$ converges in mean to X. In light of this, we check the convergence in mean of the sequence $(X_n)_{n\geq 1}$ to X. By virtue of (**??**), we have

$$\mathbf{E}\left[|X_n - X|\right] = \mathbf{E}\left[X_n\right].$$

On the other hand,

$$\mathbf{E}\left[X_n\right] = 0 \cdot \mathbf{P}\left(X_n = 0\right) + \sqrt{n} \cdot \mathbf{P}\left(X_n = \sqrt{n}\right) = 0\left(1 - \frac{1}{n}\right) + \sqrt{n}\frac{1}{n} = \frac{1}{\sqrt{n}}.$$

Therefore,

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X|\right] = \lim_{n\to\infty} \mathbf{E}\left[X_n\right] = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

which yields the convergence in mean of the sequence $(X_n)_{n\geq 1}$ to X.

To check the convergence in square mean of the sequence $(X_n)_{n\geq 1}$, recall first that if a sequence $(X_n)_{n\geq 1}$ of real random variables converges in probability to a real random variable X and $(X_n)_{n\geq 1}$ converges also in square mean, then $(X_n)_{n\geq 1}$ converges in square mean to X. In light of this, we check convergence in square mean of the sequence $(X_n)_{n\geq 1}$ to X. By virtue of (**??**), we have

$$\mathbf{E}\left[|X_n - X|^2\right] = \mathbf{E}\left[X_n^2\right].$$

On the other hand,

$$\mathbf{E}\left[X_n^2\right] = 0 \cdot \mathbf{P}\left(X_n = 0\right) + n \cdot \mathbf{P}\left(X_n = \sqrt{n}\right) = 0\left(1 - \frac{1}{n}\right) + n\frac{1}{n} = 1.$$

Therefore,

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X|^2\right] = \lim_{n\to\infty} \mathbf{E}\left[X_n^2\right] = \lim_{n\to\infty} 1 = 1.$$

This prevents the convergence in square mean of the sequence $(X_n)_{n\geq 1}$ to X.

Problem 2 *Let $X \sim U(0,1)$ and let $(Y_n)_{n\geq 1}$ be the sequence of real random variables given by*

$$Y_n \stackrel{def}{=} \begin{cases} n & \text{if } 0 \leq X < \frac{1}{n}, \\ 0 & \text{if } 1/n \leq X \leq 1. \end{cases}, \quad \forall n \geq 1$$

Check whether the sequence $(Y_n)_{n\geq 1}$ converges in probability, converges in mean, converges almost surely, in the assigned order.

Exercise 3 *Hint: to deal with the almost sure convergence consider the event $E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}$ and the complement E_0^c.*

Solution. Note that, according to the definition

$$\mathbf{P}(Y_n = n) = \mathbf{P}\left(0 \leq X < \frac{1}{n}\right) = \frac{1}{n} \quad \text{and} \quad \mathbf{P}(Y_n = 0) = \mathbf{P}\left(\frac{1}{n} \leq X \leq 1\right) = 1 - \frac{1}{n}.$$

Therefore,

$$\mathbf{P}(|Y_n| \leq \varepsilon) \geq \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n}.$$

It follows

$$\lim_{n\to\infty} \mathbf{P}(|Y_n| \leq \varepsilon) \geq \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1,$$

which implies the convergence in probability to 0. Now, to check the convergence in mean, we consider

$$\mathbf{E}[Y_n] = n\mathbf{P}(Y_n = n) + 0\mathbf{P}(Y_n = 0) = 1$$

It follows that

$$\lim_{n\to\infty} \mathbf{E}[|Y_n - 0|] = \lim_{n\to\infty} \mathbf{E}[Y_n] = 1 \neq 0.$$

Hence, $(Y_n)_{n\geq 1}$ does not converge in mean. In the end, consider the event

$$E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}.$$

Since $X \sim U(0,1)$ we have $\mathbf{P}(E_0) = 0$. In addition, for every $\omega \in E_0^c$ we have $X(\omega) > 0$ and it is possible to find n_ω such that for every $n > n_\omega$

$$\frac{1}{n} < X(\omega).$$

It then follows that

$$Y_n(\omega) = 0$$

for every $n > n_\omega$. This implies

$$\lim_{n\to\infty} Y_n(\omega) = 0, \quad \forall \omega \in E_0^c$$

which yields the almost sure convergence to 0 of the sequence $(Y_n)_{n\geq 1}$.

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(X_n)_{\geq n}$ be a sequence of real random variables on Ω. Assume that $(X_n)_{\geq n}$ are identically distributed and let $f_X : \mathbb{R} \to \mathbb{R}_+$ their common density function given by*

$$f_X(x) \overset{def}{=} \frac{2}{x^3} 1_{(1,+\infty)}(x), \quad \forall x \in \mathbb{R}.$$

Set

$$Y_n \equiv \frac{X_n}{n^\alpha}, \quad \forall n \geq 1,$$

where $\alpha > 0$.

1. *Study the convergence in distribution, probability, and L^p of the sequence $(Y_n)_{n\geq 1}$ on varying of $\alpha > 0$.*

2. *Under the additional assumption of independence of the random variables of the sequence $(X_n)_{\geq n}$, does the sequence $(Y_n)_{n\geq 1}$ converge almost surely?*

Solution. Note that the random variables of the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are almost surely positive. Hence, we have

$$\begin{aligned}\mathbf{E}[|Y_n|^p] = \mathbf{E}[Y_n^p] &= \int_\Omega Y_n^p \, d\mathbf{P} = \frac{1}{n^{\alpha p}} \int_\Omega X_n^p \, d\mathbf{P} = \frac{1}{n^{\alpha p}} \int_\mathbb{R} x^p f_X(x) \, dx \\ &= \frac{1}{n^{\alpha p}} \int_\mathbb{R} 2x^{p-3} 1_{(1,+\infty)}(x) \, dx = \frac{2}{n^{\alpha p}} \int_1^{+\infty} x^{p-3} \, dx,\end{aligned}$$

for every $\alpha > 0$, $p \geq 1$. On the other hand,

$$\int_1^{+\infty} x^{p-3} \, dx = \begin{cases} \frac{1}{p-2} x^{p-2} \Big|_1^{+\infty} = \frac{1}{2-p} & \text{if } 1 \leq p < 2 \\ +\infty & \text{if } p \geq 2 \end{cases}.$$

It then follows

$$\mathbf{E}[|Y_n|^p] = \begin{cases} \frac{1}{2-p} \frac{2}{n^{\alpha p}} & \text{if } 1 \leq p < 2 \\ +\infty & \text{if } p \geq 2 \end{cases}.$$

for every $\alpha > 0$, As a consequence, $Y_n \overset{\mathbf{L}^p}{\to} 0$ for every $\alpha > 0$ if and only if $1 \leq p < 2$. In particular, $Y_n \overset{\mathbf{P}}{\to} 0$ and $Y_n \overset{\mathbf{w}}{\to} 0$ for every $\alpha > 0$. Now, we can write

$$\mathbf{P}(|Y_n| > \varepsilon) = \mathbf{P}(Y_n > \varepsilon) = \mathbf{P}(X_n > n^\alpha \varepsilon) = \int_{n^\alpha \varepsilon}^{+\infty} 2x^{-3} 1_{(1,+\infty)}(x) \, dx = -x^{-2}\Big|_{n^\alpha \varepsilon}^{+\infty} = \frac{1}{\varepsilon^2 n^{2\alpha}}$$

for every $\varepsilon > 0$ and every $n > n_\varepsilon$, where $n_\varepsilon \in \mathbb{N}$ is such that $n^\alpha \varepsilon > 1$. As a consequence,

$$\textstyle\sum_{n=1}^\infty \mathbf{P}(|Y_n| > \varepsilon) < \infty \Leftrightarrow \frac{1}{\varepsilon^2} \sum_{n=1}^\infty \frac{1}{n^{2\alpha}} < \infty \Leftrightarrow \alpha > \frac{1}{2}.$$

Therefore, we have

$$Y_n \overset{\mathbf{a.s.}}{\not\to} 0, \quad \forall \alpha \leq \frac{1}{2} \quad \text{and} \quad Y_n \overset{\mathbf{a.s.}}{\to} 0, \quad \forall \alpha > \frac{1}{2}.$$

Exercise 5 *Consider the interval $[0,1]$ of the real Euclidean line. Let $\mathcal{B}([0,1])$ the Borel σ-algebra on $[0,1]$ and let $\mu_L : \mathcal{B}([0,1]) \to \mathbb{R}_+$ be the Lebesge measure on $[0,1]$. Hence, consider the probability space $(\Omega, \mathcal{E}, \mathbf{P})$ where $\Omega \equiv [0,1]$, $\mathcal{E} = \mathcal{B}([0,1])$, and $\mathbf{P} \equiv \mu_L$. Prove,* ***following the prescribed order whitout independence assumption****, that the sequence $(X_n)_{n\geq 1}$ of random variables given by*

$$X_n \overset{def}{=} \sqrt{n} 1_{[0,1/n]}, \quad \forall n \geq 1 \tag{1}$$

converges in distribution, in probability, almost surely, and in mean to the Dirac random variable concentrated at 0. Prove also that $(X_n)_{n\geq 1}$ does not converge in square mean.

Solution. According to Definition 1 we have

$$X_n(\omega) = \begin{cases} \sqrt{n} & \text{if } \omega \in [0, 1/n] \\ 0 & \text{if } \omega \in (1/n, 1] \end{cases},$$

for every $n \geq 1$. Hence,

$$\mathbf{P}\left(X_n = \sqrt{n}\right) = \frac{1}{n} \quad \text{and} \quad \mathbf{P}\left(X_n = 0\right) = \frac{n-1}{n}.$$

That is X_n is a Bernoulli random variable with states $0, \sqrt{n}$ and success probability $\frac{1}{n}$. As a consequence, considering the distribution function $F_{X_n} : \mathbb{R} \to \mathbb{R}_+$ of X_n, we have

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{n-1}{n} & \text{if } 0 \leq x < \sqrt{n} \\ 1 & \text{if } \sqrt{n} \leq x \end{cases},$$

for every $n \geq 1$. It follows

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \end{cases}.$$

Therefore,

$$\lim_{n\to\infty} F_{X_n}(x) = H(x),$$

for every $x \in \mathbb{R}$, where $H(x)$ is the Heaviside function which is the distribution function of the Dirac random variable concentrated at 0. This shows that

$$X_n \xrightarrow{\mathbf{d}} X_0,$$

where X_0 is the Dirac random variable concentrated at 0

$$X_0(\omega) \overset{\text{def}}{=} 0, \quad \mathbf{P}(X_0 = 0) = 1.$$

To prove the convergence in probability of X_n to X_0, we have to show that

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - X_0| < \varepsilon\right) = 1,$$

for every $\varepsilon > 0$. We have

$$|X_n - X_0| = X_n \quad \text{a.s. on } \Omega.$$

Hence,

$$\mathbf{P}\left(|X_n - X_0| < \varepsilon\right) = \mathbf{P}\left(X_n < \varepsilon\right).$$

Now, for any $\varepsilon \leq 1$ we have

$$\mathbf{P}\left(X_n < \varepsilon\right) = \mathbf{P}\left(X_n = 0\right) = \mu_L\left((1/n, 1]\right) = 1 - \frac{1}{n}.$$

It follows,

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - X_0| < \varepsilon\right) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1.$$

this is sufficient to prove that

$$X_n \xrightarrow{\mathbf{P}} X_0.$$

To prove the almost sure convergence of X_n to X_0, we need to show that there exists an event $E \in \mathcal{E}$ such that $\mathbf{P}(E) = 1$ and

$$\lim_{n\to\infty} X_n(\omega) = X_0(\omega),$$

for every $\omega \in E$. To this goal consider the event $E = (0, 1]$. We have

$$\mathbf{P}(E) = \mu_L(E) = 1.$$

In addition for every $\omega \in E$ there exists n_ω such that

$$\frac{1}{n} < \omega$$

for every $n > n_\omega$. This implies that

$$X_n(\omega) = 0$$

for every $n > n_\omega$ and it follows

$$\lim_{n\to\infty} X_n(\omega) = 0 = X_0(\omega).$$

That is

$$X_n \overset{\mathbf{a.s.}}{\to} X_0.$$

To prove the convergence in mean of X_n to X_0, we have to show that

$$\lim_{n\to\infty} \mathbf{E}[|X_n - X_0|] = 0.$$

We have

$$\mathbf{E}[|X_n - X_0|] = \mathbf{E}[X_n] = \sqrt{n}\mathbf{P}\left(X_n = \sqrt{n}\right) = \sqrt{n}\mathbf{P}([0, 1/n]) = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}.$$

It follows

$$\lim_{n\to\infty} \mathbf{E}[|X_n - X_0|] = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0,$$

which proves

$$X_n \overset{\mathbf{L}^1}{\to} X_0.$$

To prove that X_n does not converge in square mean to X_0, we compute

$$\mathbf{E}\left[|X_n - X_0|^2\right] = \mathbf{E}\left[X_n^2\right] = n\mathbf{P}\left(X_n = \sqrt{n}\right) = n\mathbf{P}([0, 1/n]) = \frac{n}{n} = 1.$$

It follows

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X_0|^2\right] = 1,$$

which prevents

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X_0|^2\right] = 0,$$

as it would be necessary to have

$$X_n \overset{\mathbf{L}^2}{\to} X_0.$$

Problem 6 *Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) \overset{def}{=} \begin{cases} \frac{y-x}{2} & \textit{if } (x, y) \in [-1, 0] \times [0, 1] \\ \frac{x-y}{2} & \textit{if } (x, y) \in [0, 1] \times [-1, 0] \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density. Hence, consider the random vector $(X, Y)^\intercal$ with density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_{X,Y}(x, y) \overset{def}{=} f(x, y).$$

Determine the marginal densities of entries X and Y of $(X, Y)^\intercal$. Are X and Y correlated? Are X and Y independent? Compute

$$\mathbf{P}(X + Y \geq 0).$$

Solution. To prove that $f : \mathbb{R}^2 \to \mathbb{R}$ is a probability density, we need to show that

$$f(x,y) \geq 0,$$

for almost every $(x,y) \in \mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} f(x,y)\ d\mu_L(x,y) = 1.$$

Since in $[-1,0]\times[0,1]$ [resp. $[0,1]\times[-1,0]$] we have $x \leq 0$ and $y \geq 0$ [resp. $x \geq 0$ and $y \leq 0$] it follows $y - x \geq 0$ [resp. $x - y \geq 0$]. This proves the positivity of $f(x,y)$ for every $(x,y) \in \mathbb{R}^2$. Since we can write

$$f(x,y) = \frac{y-x}{2} 1_{[-1,0]\times[0,1]}(x,y) + \frac{x-y}{2} 1_{[0,1]\times[-1,0]}(x,y),$$

for every $(x,y) \in \mathbb{R}^2$, thanks to the properties of the Lebesgue integral for positive functions, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x,y)\ d\mu_L(x,y) &= \int_{\mathbb{R}^2} \left(\frac{y-x}{2} 1_{[-1,0]\times[0,1]}(x,y) + \frac{x-y}{2} 1_{[0,1]\times[-1,0]}(x,y) \right)\ d\mu_L(x,y) \\
&= \int_{\mathbb{R}^2} \frac{y-x}{2} 1_{[-1,0]\times[0,1]}(x,y)\ d\mu_L(x,y) + \int_{\mathbb{R}^2} \frac{x-y}{2} 1_{[0,1]\times[-1,0]}(x,y)\ d\mu_L(x,y) \\
&= \int_{[-1,0]\times[0,1]} \frac{y-x}{2}\ d\mu_L(x,y) + \int_{[0,1]\times[-1,0]} \frac{x-y}{2}\ d\mu_L(x,y).
\end{aligned}$$

On the other hand, the function $y - x$ [resp. $x - y$] is continuous on $[-1,0]\times[0,1]$ [resp. $[0,1]\times[-1,0]$]. It follows

$$\int_{[-1,0]\times[0,1]} \frac{y-x}{2}\ d\mu_L(x,y) = \int_{x=-1}^{0} \int_{y=0}^{1} \frac{y-x}{2} dxdy \quad \text{[resp. } \int_{[0,1]\times[-1,0]} \frac{x-y}{2}\ d\mu_L(x,y) = \int_{x=1}^{0} \int_{y=-1}^{0} \frac{y-x}{2} dxdy].$$

Now,

$$\begin{aligned}
\int_{x=-1}^{0} \int_{y=0}^{1} \frac{y-x}{2} dxdy &= \frac{1}{2} \int_{x=-1}^{0} \left(\int_{y=0}^{1} (y-x)\, dy \right) dx \\
&= \frac{1}{2} \int_{x=-1}^{0} \left(\left. \frac{(y-x)^2}{2} \right|_{y=0}^{1} \right) dx \\
&= \frac{1}{2} \int_{x=-1}^{0} \left(\frac{(1-x)^2}{2} - \frac{x^2}{2} \right) dx \\
&= \frac{1}{2} \int_{x=-1}^{0} \left(\frac{1-2x+x^2}{2} - \frac{x^2}{2} \right) dx \\
&= \frac{1}{2} \int_{x=-1}^{0} \left(\frac{1}{2} - x \right) dx \\
&= \frac{1}{2} \left. \frac{1}{2} x - \frac{x^2}{2} \right|_{x=-1}^{0} \\
&= \frac{1}{4}\ x - x^2 \Big|_{x=-1}^{0} \\
&= \frac{1}{2}.
\end{aligned}$$

Moreover, we clearly have

$$\int_{x=1}^{0} \int_{y=-1}^{0} \frac{y-x}{2} dxdy = \int_{x=-1}^{0} \int_{y=0}^{1} \frac{y-x}{2} dxdy.$$

It then follows that $f:\mathbb{R}^2\to\mathbb{R}$ is a probability density. The marginal densities $f_X:\mathbb{R}\to\mathbb{R}$ and $f_Y:\mathbb{R}\to\mathbb{R}$ are given by

$$f_X(x)\overset{\text{def}}{=}\int_{\mathbb{R}} f_{X,Y}(x,y)\ d\mu_L(y)\quad\text{and}\quad f_Y(y)\overset{\text{def}}{=}\int_{\mathbb{R}} f_{X,Y}(x,y)\ d\mu_L(x).$$

Since we can write

$$f(x,y)=\frac{y-x}{2}1_{[-1,0]}(x)\,1_{[0,1]}(y)+\frac{x-y}{2}1_{[0,1]}(x)\,1_{[-1,0]}(y),$$

we have

$$\begin{aligned}
f_X(x)&=\int_{\mathbb{R}}\frac{y-x}{2}1_{[-1,0]}(x)\,1_{[0,1]}(y)\ d\mu_L(y)+\int_{\mathbb{R}}\frac{x-y}{2}1_{[0,1]}(x)\,1_{[-1,0]}(y)\ d\mu_L(y)\\
&=1_{[-1,0]}(x)\int_{\mathbb{R}}\frac{y-x}{2}1_{[0,1]}(y)\ d\mu_L(y)+1_{[0,1]}(x)\int_{\mathbb{R}}\frac{x-y}{2}1_{[-1,0]}(y)\ d\mu_L(y)\\
&=1_{[-1,0]}(x)\int_{[0,1]}\frac{y-x}{2}\ d\mu_L(y)+1_{[0,1]}(x)\int_{[-1,0]}\frac{x-y}{2}\ d\mu_L(y)\\
&=\frac{1}{2}1_{[-1,0]}(x)\int_0^1(y-x)\ dy+\frac{1}{2}1_{[0,1]}(x)\int_{-1}^0(x-y)\ dy\\
&=\frac{1}{2}1_{[-1,0]}(x)\left.\frac{(y-x)^2}{2}\right|_{y=0}^1-\frac{1}{2}1_{[0,1]}(x)\left.\frac{(x-y)^2}{2}\right|_{y=-1}^0\\
&=\frac{1}{2}1_{[-1,0]}(x)\left(\frac{(1-x)^2}{2}-\frac{x^2}{2}\right)-\frac{1}{2}1_{[0,1]}(x)\left(\frac{x^2}{2}-\frac{(x+1)^2}{2}\right)\\
&=\frac{1}{2}\left(\frac{1-2x}{2}\right)1_{[-1,0]}(x)+\frac{1}{2}\left(\frac{1+2x}{2}\right)1_{[0,1]}(x)\\
&=\frac{1}{4}\left((1-2x)\,1_{[-1,0]}(x)+(1+2x)\,1_{[0,1]}(x)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_Y(y)&=\int_{\mathbb{R}}\frac{y-x}{2}1_{[-1,0]}(x)\,1_{[0,1]}(y)\ d\mu_L(x)+\int_{\mathbb{R}}\frac{x-y}{2}1_{[0,1]}(x)\,1_{[-1,0]}(y)\ d\mu_L(x)\\
&=1_{[0,1]}(y)\int_{\mathbb{R}}\frac{y-x}{2}1_{[-1,0]}(x)\ d\mu_L(x)+1_{[-1,0]}(y)\int_{\mathbb{R}}\frac{x-y}{2}1_{[0,1]}(x)\ d\mu_L(x)\\
&=1_{[0,1]}(y)\int_{[-1,0]}\frac{y-x}{2}\ d\mu_L(x)+1_{[-1,0]}(y)\int_{[0,1]}\frac{x-y}{2}\ d\mu_L(x)\\
&=\frac{1}{2}1_{[0,1]}(y)\int_{-1}^0(y-x)\ dx+\frac{1}{2}1_{[-1,0]}(y)\int_0^1(x-y)\ dx\\
&=-\frac{1}{2}1_{[0,1]}(y)\left.\frac{(y-x)^2}{2}\right|_{x=-1}^0+\frac{1}{2}1_{[-1,0]}(y)\left.\frac{(x-y)^2}{2}\right|_{x=0}^1\\
&=-\frac{1}{2}1_{[0,1]}(y)\left(\frac{y^2}{2}-\frac{(y+1)^2}{2}\right)+\frac{1}{2}1_{[-1,0]}(y)\left(\frac{(1-y)^2}{2}-\frac{y^2}{2}\right)\\
&=\frac{1}{2}\left(\frac{1+2y}{2}\right)1_{[0,1]}(y)+\frac{1}{2}\left(\frac{1-2y}{2}\right)1_{[-1,0]}(y)\\
&=\frac{1}{4}\left((1-2y)\,1_{[-1,0]}(y)+(1+2y)\,1_{[0,1]}(y)\right).
\end{aligned}$$

Note that we have[1]

$$
\begin{aligned}
f_X(-x) &= \frac{1}{4}\big((1+2x)\,1_{[-1,0]}(-x) + (1-2x)\,1_{[0,1]}(-x)\big) \\
&= \frac{1}{4}\big((1+2x)\,1_{[0,1]}(x) + (1-2x)\,1_{[-1,0]}(x)\big) \\
&= f_X(x).
\end{aligned}
$$

That is $f_X : \mathbb{R} \to \mathbb{R}_+$ is an even function. Moreover,

$$f_Y(y) = f_X(x)\,|_{x=y}.$$

As a consequence,

$$\mathbf{E}[X] = \int_{\mathbb{R}} x f_X(x)\ d\mu_L(x) = 0 = \mathbf{E}[Y].$$

Otherwise, in terms of explicit computations, we can write,

$$
\begin{aligned}
\mathbf{E}[X] &= \int_{\mathbb{R}} x f_X(x)\ d\mu_L(x) \\
&= \int_{\mathbb{R}} \frac{1}{4}\big(x(1-2x)\,1_{[-1,0]}(x) + x(1+2x)\,1_{[0,1]}(x)\big)\ d\mu_L(x) \\
&= \frac{1}{4}\left(\int_{[-1,0]} x(1-2x)\ d\mu_L(x) + \int_{[0,1]} x(1+2x)\ d\mu_L(x)\right) \\
&= \frac{1}{4}\left(\int_{-1}^{0} \left(x - 2x^2\right)\ dx + \int_{0}^{1} \left(x + 2x^2\right)\ dx\right) \\
&= \frac{1}{4}\left(\left.\frac{x^2}{2} - \frac{2}{3}x^3\right|_{x=-1}^{0} + \left.\frac{x^2}{2} + \frac{2}{3}x^3\right|_{x=0}^{1}\right) \\
&= \frac{1}{4}\left(-\frac{1}{2} - \frac{2}{3} + \frac{1}{2} + \frac{2}{3}\right) \\
&= 0.
\end{aligned}
$$

The same computation yields

$$
\begin{aligned}
\mathbf{E}[Y] &= \int_{\mathbb{R}} y f_X(y)\ d\mu_L(y) \\
&= \int_{\mathbb{R}} \frac{1}{4}\big(1_{[-1,0]}(y)\,y(1-2y) + 1_{[0,1]}(y)\,y(1+2y)\big)\ d\mu_L(y) \\
&= \frac{1}{0}.
\end{aligned}
$$

[1] Thanks to Tiziana Mannucci

On the other hand,

$$
\begin{aligned}
\mathbf{E}[XY] &= \int_{\mathbb{R}^2} xy f_{X,Y}(x,y)\ d\mu_L(x,y) \\
&= \int_{\mathbb{R}^2} \left(\frac{xy(y-x)}{2} 1_{[-1,0]\times[0,1]}(x,y) + \frac{xy(x-y)}{2} 1_{[0,1]\times[-1,0]}(x,y) \right)\ d\mu_L(x,y) \\
&= \int_{\mathbb{R}^2} \frac{xy(y-x)}{2} 1_{[-1,0]\times[0,1]}(x,y)\ d\mu_L(x,y) + \int_{\mathbb{R}^2} \frac{xy(x-y)}{2} 1_{[0,1]\times[-1,0]}(x,y)\ d\mu_L(x,y) \\
&= \int_{[-1,0]\times[0,1]} \frac{xy(y-x)}{2}\ d\mu_L(x,y) + \int_{[0,1]\times[-1,0]} \frac{xy(x-y)}{2}\ d\mu_L(x,y) \\
&= \frac{1}{2} \left(\int_{y=0}^{1} \left(\int_{x=-1}^{0} \left(xy^2 - x^2 y\right)\ dx \right) dy + \int_{y=-1}^{0} \left(\int_{x=0}^{1} \left(x^2 y - xy^2\right)\ dx \right) dy \right) \\
&= \frac{1}{2} \left(\int_{y=0}^{1} \frac{1}{2}x^2y^2 - \frac{1}{3}x^3 y \Big|_{x=-1}^{0} dy + \int_{y=-1}^{0} \frac{1}{3}x^3 y - \frac{1}{2}x^2y^2 \Big|_{x=0}^{1} dy \right) \\
&= \frac{1}{2} \left(-\int_{y=0}^{1} \left(\frac{1}{2}y^2 + \frac{1}{3}y \right) dy + \int_{y=-1}^{0} \frac{1}{3}y - \frac{1}{2}y^2 dy \right) \\
&= \frac{1}{2} \left(-\frac{1}{6}y^3 + \frac{1}{6}y^2 \Big|_{y=0}^{1} + \frac{1}{6}y^2 - \frac{1}{6}y^3 \Big|_{y=-1}^{0} \right) \\
&= \frac{1}{12}(-2-2) \\
&= -\frac{1}{3}.
\end{aligned}
$$

It follows

$$
\mathbf{E}[XY] \neq \mathbf{E}[X]\,\mathbf{E}[Y],
$$

which shows that X and Y are correlated. Therefore, X and Y are not independent. In the end, setting

$$
H = \left\{ (x,y) \in \mathbb{R}^2 : x + y \geq 0 \right\},
$$

which is the half-plane on the right of the line of equation

$$
x + y = 0,
$$

we have

$$
\mathbf{P}(X + Y \geq 0) = \int_H f_{X,Y}(x,y)\, d\mu_L^2(x,y).
$$

Now, since ve can write

$$
\begin{aligned}
f(x,y) &= \frac{y-x}{2} 1_{[-1,0]\times[0,1]}(x,y) + \frac{x-y}{2} 1_{[0,1]\times[-1,0]}(x,y) \\
&= \frac{y-x}{2} 1_{[-1,0]}(x)\, 1_{[0,1]}(y) + \frac{x-y}{2} 1_{[0,1]}(x)\, 1_{[-1,0]}(y)
\end{aligned}
$$

we have

$$
\begin{aligned}
f(-x,-y) &= \frac{-y+x}{2} 1_{[-1,0]}(-x)\, 1_{[0,1]}(-y) + \frac{-x+y}{2} 1_{[0,1]}(-x)\, 1_{[-1,0]}(-y) \\
&= \frac{x-y}{2} 1_{[0,1]}(x)\, 1_{[-1,0]}(y) + \frac{y-x}{2} 1_{[-1,0]}(x)\, 1_{[0,1]}(y) \\
&= f(x,y).
\end{aligned}
$$

and

$$\begin{aligned} f(y,x) &= \frac{x-y}{2} 1_{[-1,0]}(y) 1_{[0,1]}(x) + \frac{y-x}{2} 1_{[0,1]}(y) 1_{[-1,0]}(x) \\ &= \frac{y-x}{2} 1_{[-1,0]}(x) 1_{[0,1]}(y) + \frac{x-y}{2} 1_{[0,1]}(x) 1_{[-1,0]}(y) \\ &= f(x,y) \end{aligned}$$

This means that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is symmetric with respect to the point $(0,0)$ and the line $x+y=0$. As a consequence,

$$\int_H f_{X,Y}(x,y)\, d\mu_L^2(x,y) = \frac{1}{2}.$$