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Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Sequences of Random Variables with Solutions 2021-11-23

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* X *be a uniformly distributed real random variable on the interval* $[0, 1]$*. In symbols,* $X \sim U(0, 1)$*. Consider the sequence* $(Y_n)_{n\geq 1}$ *of real random variables given by*

$$Y_n \stackrel{def}{=} \begin{cases} n, & \text{if } 0 \leq X < \frac{1}{n}, \\ 0, & \text{if } 1/n \leq X \leq 1, \end{cases} \qquad \forall n \geq 1.$$

Check whether the sequence $(Y_n)_{n\geq 1}$ *converges in distribution, converges in probability, converges in mean, converges almost surely, in the assigned order.*

Exercise 2 *Hint: to deal with the almost sure convergence consider the event* $E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}$ *and the complement* E_0^c*.*

Solution. Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}$ for the distribution function of Y_n. We have

$$F_{Y_n}(y) = \mathbf{P}(Y_n \leq y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(1/n \leq X \leq 1) = 1 - 1/n, & \text{if } 0 \leq y < n, \\ 1, & \text{if } n \leq y. \end{cases}$$

On the other hand, for every $y \geq 0$ there exists $n_y \in \mathbb{N}$, (e.g. $n_y = \lceil y \rceil$, where $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{R}$, is the ceiling function), such that $y < n$ for every $n > n_y$. Therefore, definitively,

$$\mathbf{P}(Y_n \leq y) = 1 - 1/n.$$

It then follows

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y < 0, \\ \lim_{n\to\infty} 1 - 1/n = 1, & \text{if } 0 \leq y. \end{cases}$$

Considering the Heavside function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } 0 \leq y, \end{cases}$$

we clearly have

$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R}$. Hence, the sequence $(Y_n)_{n\geq 1}$ converges in distribution to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0 \in \mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. However, according to the definition, we have

$$\mathbf{P}(Y_n = n) = \mathbf{P}\left(0 \leq X < \frac{1}{n}\right) = \frac{1}{n} \quad \text{and} \quad \mathbf{P}(Y_n = 0) = \mathbf{P}\left(\frac{1}{n} \leq X \leq 1\right) = 1 - \frac{1}{n}.$$

Therefore, definitively,

$$\mathbf{P}(|Y_n| \leq \varepsilon) \geq \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n},$$

for every $\varepsilon > 0$. It follows

$$\lim_{n\to\infty} \mathbf{P}\left(|Y_n| \le \varepsilon\right) \ge \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1,$$

for every $\varepsilon > 0$, which is the convergence in probability to $Dir\,(0)$. Now, to check the convergence in mean, we consider

$$\mathbf{E}\left[Y_n\right] = n\mathbf{P}\left(Y_n = n\right) + 0\mathbf{P}\left(Y_n = 0\right) = 1.$$

It follows that

$$\lim_{n\to\infty} \mathbf{E}\left[|Y_n - 0|\right] = \lim_{n\to\infty} \mathbf{E}\left[Y_n\right] = 1 \ne 0.$$

Hence, $(Y_n)_{n\ge1}$ does not converge in mean. In the end, consider the event

$$E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}.$$

Since $X \sim U(0,1)$ we have $\mathbf{P}(E_0) = 0$. In addition, for every $\omega \in E_0^c$ we have $X(\omega) > 0$ and it is possible to find n_ω such that

$$\frac{1}{n} < X(\omega),$$

for every $n > n_\omega$. It then follows that

$$Y_n(\omega) = 0,$$

for every $n > n_\omega$. This implies

$$\lim_{n\to\infty} Y_n(\omega) = 0,$$

for every $\omega \in E_0^c$, which is the almost sure convergence of the sequence $(Y_n)_{n\ge1}$ to $Dir\,(0)$. □

Problem 3 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ be the Euclidean real line endowed with the Borel σ-algebra. Prove that the function $f : \mathbb{R} \to \mathbb{R}_+$ given by*

$$f(x) \stackrel{def}{=} \frac{\alpha - 1}{x^\alpha} 1_{[1,+\infty)}, \quad \forall x \in \mathbb{R},$$

where $\alpha > 1$, is a density. Then, consider a random variable X with density $f_X = f$ and the sequence $(Y_n)_{n\ge1}$ of random variables given by

$$Y_n \stackrel{def}{=} \frac{X}{n}, \quad \forall n \in \mathbb{N}.$$

Exercise 4 *Study the convergence in distribution, in probability and in p-th mean of the sequence $(Y_n)_{n\ge1}$ on varying of $\alpha > 1$.*

Solution. Since $\alpha > 1$, that is $\alpha - 1 > 0$ and $1 - \alpha < 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\, d\mu_L(x) &= \int_{\mathbb{R}} \frac{\alpha-1}{x^\alpha} 1_{[1,+\infty)}(x)\, d\mu_L(x) = \int_{[1,+\infty)} \frac{\alpha-1}{x^\alpha} d\mu_L(x) \\ &= \int_1^{+\infty} \frac{\alpha-1}{x^\alpha} dx = \lim_{x\to+\infty} \int_1^x \frac{\alpha-1}{u^\alpha} du = \lim_{x\to+\infty} -\int_1^x du^{1-\alpha} \\ &= \lim_{x\to+\infty} - u^{1-\alpha}\Big|_1^x = 1 - \lim_{x\to+\infty} x^{1-\alpha} = 1.\end{aligned}$$

This proves that $f : \mathbb{R} \to \mathbb{R}_+$ is a density.

Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}_+$ for the distribution function of Y_n, for every $n \ge 1$. We have

$$F_{Y_n}(y) = \mathbf{P}(Y_n \le y) = \mathbf{P}(X/n \le y) = \mathbf{P}(X \le ny) = \int_{(-\infty, ny]} f(x)\, d\mu_L(x).$$

On the other hand,

$$\begin{aligned}\int_{(-\infty,ny]} f(x)\,d\mu_L(x) &= \int_{(-\infty,ny]} \frac{\alpha-1}{x^\alpha} 1_{[1,+\infty)}(x)\,d\mu_L(x)\\ &= \int_{(-\infty,ny]\cap[1,+\infty)} \frac{\alpha-1}{x^\alpha} d\mu_L(x)\\ &= \begin{cases} \int_\varnothing \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } ny<1,\\ \int_{\{ny\}} \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } ny=1,\\ \int_{[1,ny]} \frac{\alpha-1}{x^\alpha} d\mu_L(x), & \text{if } 1<ny,\end{cases}\end{aligned}$$

where

$$\int_\varnothing \frac{\alpha-1}{x^\alpha} d\mu_L(x) = \int_{\{ny\}} \frac{\alpha-1}{x^\alpha} d\mu_L(x) = 0$$

and

$$\int_{[1,ny]} \frac{\alpha-1}{x^\alpha} d\mu_L(x) = \int_1^{ny} \frac{\alpha-1}{x^\alpha} dx = -\int_1^{ny} dx^{1-\alpha} = -\left. x^{1-\alpha}\right|_1^{ny} = 1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}.$$

Therefore,

$$F_{Y_n}(y) = \begin{cases} 0, & \text{if } y \le \frac{1}{n},\\ 1-\frac{1}{n^{\alpha-1}y^{\alpha-1}}, & \text{if } \frac{1}{n}<y.\end{cases}$$

As a consequence,

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y\le 0,\\ \lim_{n\to\infty} 1-\frac{1}{n^{\alpha-1}y^{\alpha-1}} = 1, & \text{if } 0<y.\end{cases}$$

Considering the Heavside function $H:\mathbb{R}\to\mathbb{R}_+$ given by

$$H(y) = \begin{cases} 0, & \text{if } y<0,\\ 1, & \text{if } 0\le y,\end{cases}$$

we clearly have

$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R}-\{0\}$, where the Heavside function is continuous. Hence, the sequence $(Y_n)_{n\ge1}$ converges in distribution to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0\in\mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. However, since

$$F_{Y_n}(y) = \int_{(-\infty,y]} \frac{1-\alpha}{n^{\alpha-1}u^\alpha} 1_{(1/n,+\infty)}(u)\,d\mu_L(u)$$

for every $y\in\mathbb{R}$, we have that the random variables of the sequence $(Y_n)_{n\ge1}$ are absolutely continuous with density $f_{Y_n}:\mathbb{R}\to\mathbb{R}_+$ given by

$$f_{Y_n}(y) = \frac{1-\alpha}{n^{\alpha-1}y^\alpha} 1_{(1/n,+\infty)}(y).$$

Note that

$$F'_{Y_n}(y) = f_{Y_n}(y),$$

for every $y \neq 1/n$. As a consequence, provided n is sufficiently large,

$$
\begin{aligned}
\mathbf{P}\left(|Y_n| > \varepsilon\right) = \mathbf{P}\left(Y_n > \varepsilon\right) &= \int_{(\varepsilon,+\infty)} f_{Y_n}(y)\, d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y)\, d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)\cap(1/n,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{\varepsilon}^{+\infty} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} dy \\
&= \frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} \frac{\alpha-1}{u^{\alpha}} du \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} du^{\alpha-1} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{1-\alpha}\Big|_{\varepsilon}^{y} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \left(\frac{1}{y^{\alpha-1}} - \frac{1}{\varepsilon^{\alpha-1}}\right) \\
&= \frac{1}{n^{\alpha-1}\varepsilon^{\alpha-1}}.
\end{aligned}
$$

It follows

$$
\lim_{n\to\infty} \mathbf{P}\left(|Y_n| > \varepsilon\right) = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}\varepsilon^{\alpha-1}} = 0,
$$

for every $\varepsilon > 0$. This proves directly that $(Y_n)_{n\geq 1}$ converges in probability to $Dir(0)$.

By virtue of what shown above, to study the convergence in p-th mean of the sequence $(Y_n)_{n\geq 1}$ it is sufficient to consider

$$
\begin{aligned}
\mathbf{E}\left[Y_n^p\right] &= \int_{\mathbb{R}} y^p f_{Y_n}(y)\, d\mu_L(u) = \int_{\mathbb{R}} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha-p}} 1_{(1/n,+\infty)}(y)\, d\mu_L(u) = \frac{1-\alpha}{n^{\alpha-1}} \int_{(1/n,+\infty)} \frac{1}{y^{\alpha-p}} d\mu_L(u) \\
&= \frac{1-\alpha}{n^{\alpha-1}} \int_{1/n}^{+\infty} \frac{1}{y^{\alpha-p}} dy = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} du^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{p-\alpha+1}\Big|_{1/n}^{y}, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} d\ln(u) = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \ln(u)\Big|_{1/n}^{y}, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\mathbf{E}\left[Y_n^p\right] = \mathbf{E}\left[\left(\frac{X}{n}\right)^p\right] &= \int_{\mathbb{R}} \frac{x^p}{n^p} f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{x^p}{n^p} \frac{\alpha-1}{x^{\alpha}} 1_{[1,+\infty)}(x)\, d\mu_L(x) \\
&= \frac{\alpha-1}{n^p} \int_{[1,+\infty)} \frac{1}{x^{\alpha-p}} d\mu_L(x) = \frac{\alpha-1}{n^p} \int_{1}^{+\infty} \frac{1}{x^{\alpha-p}} dx = \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} \frac{1}{u^{\alpha-p}} u^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} \lim_{x\to+\infty} u^{p-\alpha+1}\Big|_{1}^{x}, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} d\ln(u) = \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \ln(u)\Big|_{1}^{x}, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}
$$

Now, if $p \geq \alpha - 1$ we have that $\mathbf{E}\left[Y_n^p\right]$ is not finite. The sequence $(Y_n)_{n\geq 1}$ cannot converge in p-th mean. If $1 \leq p < \alpha - 1$, we have

$$
\mathbf{E}\left[Y_n^p\right] = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}}\frac{1}{n^{p-\alpha+1}} = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}.
$$

Hence,

$$\lim_{n\to\infty}\mathbf{E}\left[Y_n^p\right]=\lim_{n\to\infty}-\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}=0.$$

The sequence converges in p-th mean to the standard Dirac random variable.

Problem 5 *Let $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ be a probability space and let $(X_n)_{\geq n}$ be a sequence of real random variables on Ω. Assume that $(X_n)_{\geq n}$ are identically distributed and let $f_X:\mathbb{R}\to\mathbb{R}_+$ their common density function given by*

$$f_X(x)\overset{def}{=}\frac{2}{x^3}1_{(1,+\infty)}(x),\quad\forall x\in\mathbb{R}.$$

Set

$$Y_n\equiv\frac{X_n}{n^\alpha},\quad\forall n\geq 1,$$

where $\alpha>0$.

1. *Study the convergence in distribution, probability, and L^p of the sequence $(Y_n)_{n\geq1}$ on varying of $\alpha>0$.*

2. *Under the additional assumption of independence of the random variables of the sequence $(X_n)_{\geq n}$, compute $\limsup_{n\to\infty}Y_n$ and $\liminf_{n\to\infty}Y_n$ on varying of $\alpha>0$. Does the sequence $(Y_n)_{n\geq1}$ converge almost surely?*

Solution. . □

Problem 6 *Let $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ be a complete probability space and let $(X_n)_{n\geq1}$ be a sequence of independent real random variables such that $X_n\sim Ber(1/n^\alpha)$ for some $\alpha>0$. Consider the sequence $(Y_n)_{n\geq1}$ of real random variables on Ω given by*

$$Y_n\overset{def}{=}\min\{X_1,\dots,X_n\}.$$

1. *study the convergence in distribution, in probability and in $L^p(\Omega;\mathbb{R})$ of $(X_n)_{n\geq1}$ and $(Y_n)_{n\geq1}$ on varying of $\alpha>0$;*

2. *study the almost sure convergence of $(X_n)_{n\geq1}$ and $(Y_n)_{n\geq1}$ on varying of $\alpha>0$.*

Solution.

1. We clearly have

$$Y_n(\omega)=\begin{cases}1 & \Leftrightarrow X_1(\omega)=\dots=X_n(\omega)=1\\ 0 & \text{otherwise}\end{cases}$$

Hence, by virtue of the independence of the random variables of the sequence $(X_n)_{n\geq1}$, we have

$$\mathbf{P}(Y_n=1)=\mathbf{P}(X_1=1,\dots,X_n=1)=\mathbf{P}(X_1=1)\cdots\mathbf{P}(X_n=1)=\textstyle\prod_{k=1}^n\frac{1}{k^\alpha}=\frac{1}{n!^\alpha}$$

and

$$\mathbf{P}(Y_n=0)=1-\mathbf{P}(Y_n=1)=1-\frac{1}{n!^\alpha}.$$

In other words, $(Y_n)_{n\geq 1}$ is a sequence of standard Bernoulli random variables with succes probability $\frac{1}{n!^\alpha}$. Considering the distribution functions $F_{X_n}:\mathbb{R}\to\mathbb{R}_+$ and $F_{Y_n}:\mathbb{R}\to\mathbb{R}_+$ of X_n and Y_n, respectively, we have

$$F_{X_n}(x)\overset{\text{def}}{=}\mathbf{P}(X_n\leq x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n^\alpha}, & \text{if } 0\leq x<1,\\ 1, & \text{if } 1\leq x,\end{cases}$$

and

$$F_{Y_n}(x)\overset{\text{def}}{=}\mathbf{P}(Y_n\leq x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n!^\alpha}, & \text{if } 0\leq x<1,\\ 1, & \text{if } 1\leq x.\end{cases}$$

Therefore, considering the Heaviside function $H:\mathbb{R}\to\mathbb{R}_+$ given by

$$H(x)\overset{\text{def}}{=}\begin{cases}0, & \text{if } x<0,\\ 1, & \text{if } 0\leq x,\end{cases}$$

we have

$$\lim_{n\to\infty}F_{X_n}(x)=\lim_{n\to\infty}F_{Y_n}(x)=H(x),$$

for every $x\in\mathbb{R}$. Thus, both the sequences $(F_{X_n})_{n\geq 1}$ and $(F_{Y_n})_{n\geq 1}$ converge pointwise to H. It follows that both the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ converge to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0\in\mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. However, according to the definition, we have definitively

$$\mathbf{P}(|X_n-Dir(0)|<\varepsilon)=\mathbf{P}(X_n<\varepsilon)=\mathbf{P}(X_n=0)=1-\frac{1}{n^\alpha}$$

and

$$\mathbf{P}(|Y_n-Dir(0)|<\varepsilon)=\mathbf{P}(Y_n<\varepsilon)=\mathbf{P}(Y_n=0)=1-\frac{1}{n!^\alpha},$$

for every $\varepsilon>0$. Therefore,

$$\lim_{n\to\infty}\mathbf{P}(|X_n-Dir(0)|<\varepsilon)=\lim_{n\to\infty}\left(1-\frac{1}{n^\alpha}\right)=1$$

and

$$\lim_{n\to\infty}\mathbf{P}(|Y_n-Dir(0)|<\varepsilon)=\lim_{n\to\infty}\left(1-\frac{1}{n!^\alpha}\right)=1,$$

which is the convergence in probabiliyty of both $(X_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ to $Dir(0)$. To check the convergence in $L^p(\Omega;\mathbb{R})$, we need to consider $\|X_n-Dir(0)\|_p$ and $\|Y_n-Dir(0)\|_p$, because in case of convergence the limit has to be $Dir(0)$. We then have

$$\|X_n-Dir(0)\|_p=\left(\int_\Omega|X_n-Dir(0)|^p\ d\mathbf{P}\right)^{1/p}=\left(\int_\Omega X_n^p\ d\mathbf{P}\right)^{1/p}=\mathbf{P}(X_n=1)^{1/p}=\frac{1}{n^{\frac{\alpha}{p}}}$$

and

$$\|Y_n-Dir(0)\|_p=\left(\int_\Omega|Y_n-Dir(0)|^p\ d\mathbf{P}\right)^{1/p}=\left(\int_\Omega Y_n^p\ d\mathbf{P}\right)^{1/p}=\mathbf{P}(Y_n=1)^{1/p}=\frac{1}{n!^{\frac{\alpha}{p}}},$$

for every $\alpha>0$. Hence, we obtain

$$\lim_{n\to\infty}\|X_n-X\|_p=\lim_{n\to\infty}\|Y_n-X\|_p=0,$$

which proves the convergence of both $(X_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ to $Dir(0)$ in $L^p(\Omega;\mathbb{R})$.

2. With regard to the almost sure convergence, note that also in this case, if the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ converge almost surely, the limit has to be $Dir\,(0)$. Hence, for any fixed $\varepsilon > 0$ consider the events $E_0 \equiv \{|X_n - Dir\,(0)| \geq \varepsilon\}$ and $F_0 \equiv \{|Y_n - Dir\,(0)| \geq \varepsilon\}$ we have

$$\{|X_n - Dir\,(0)| \geq \varepsilon\} = \{X_n \geq \varepsilon\} \quad \text{and} \quad \{|Y_n - Dir\,(0)| \geq \varepsilon\} = \{Y_n \geq \varepsilon\}.$$

Hence,

$$\mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \mathbf{P}\left(X_n = 1\right) = \frac{1}{n^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1, \end{cases}$$

and

$$\mathbf{P}\left(|Y_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \mathbf{P}\left(Y_n = 1\right) = \frac{1}{n!^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1. \end{cases}$$

As a consequence,

$$\textstyle\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1, \end{cases}$$

and

$$\textstyle\sum_{n=1}^{\infty} \mathbf{P}\left(|Y_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n!^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1. \end{cases}$$

It then follows that $\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right)$ converges for every $\alpha > 1$ and $\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right)$ converges for every $\alpha > 0$. This yields the almost sure convergence of $(X_n)_{n\geq 1}$ to $Dir\,(0)$ for every $\alpha > 1$ and the almost sure convergence of $(Y_n)_{n\geq 1}$ to $Dir\,(0)$ for every $\alpha > 0$. In fact, the convergence of the series implies that

$$\lim_{m\to\infty} \mathbf{P}\left(\bigcup_{n\geq m} \{|Z_n - Z| \geq \varepsilon\}\right) \leq \textstyle\sum_{n=m}^{\infty} \mathbf{P}\left(|Z_n - Z| \geq \varepsilon\right) = 0.$$