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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica
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Problem 1 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(X_n)_{n\geq 1}$ be a sequence of independent identically distributed Bernoulli random variables with success probability p. Set*

$$Z_n \overset{def}{=} \textstyle\sum_{k=1}^{n} X_n, \qquad \bar{X}_n = \frac{1}{n}\textstyle\sum_{n=1}^{n} X_k.$$

Assume that n is large. What you can say about the distributions, expectation, and variance of Z_n and $\bar{X}_n$? Consider the case $n = 100,000$ and $p = 1/2$. Use both the Central Limit Theorem and the Tchebychev inequality to estimate the probability that Z_n lies between $49,500$ and $50,500$. What you can say about the distributions, expectation, and variance of Z_n and $\bar{X}_n$ if $(X_n)_{n\geq 1}$ is a sequence of independent and Poisson distributed random variables with the same rate parameter λ?

Solution. Under the assumption that $(X_n)_{n\geq 1}$ is a sequence of independent and identically distributed Bernoulli random variables with success probability p, the random variable Z_n has the binomial distribution with parameters n and p, for every $n \in \mathbb{N}$. In symbols

$$Z_n \sim Bin\,(n, p)\,, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\mathbf{E}\left[Z_n\right] = np \quad \text{and} \quad \mathbf{D}^2\left[Z_n\right] = np\,(1-p)\,,$$

for every $n \in \mathbb{N}$. Therefore, the random variable

$$Z_n^* \equiv \frac{Z_n - np}{\sqrt{np\,(1-p)}}$$

is standardized, that is

$$\mathbf{E}\left[Z_n^*\right] = 0 \quad \text{and} \quad \mathbf{D}^2\left[Z_n^*\right] = 1.$$

for every $n \in \mathbb{N}$. By virtue of the Central Limit Theorem, we know that

$$Z_n^* \xrightarrow{\mathbf{w}} N\,(0,1)\,,$$

as $n \to \infty$. As a consequence, as n is large, the random variable Z_n^* is approximately standard Gaussian distributed. On the other hand, we can write

$$\sqrt{\frac{p\,(1-p)}{n}} Z_n^* = \sqrt{\frac{p\,(1-p)}{n}} \frac{Z_n - np}{\sqrt{np\,(1-p)}} = \frac{Z_n}{n} - p = \bar{X}_n - p,$$

that is

$$\bar{X}_n = \sqrt{\frac{p\,(1-p)}{n}} Z_n^* + p$$

This implies that, as n is large, the distribution of $\bar{X}_n$ is approximately Gaussian with

$$\mathbf{E}\left[\bar{X}_n\right] = p \quad \text{and} \quad \mathbf{D}^2\left[\bar{X}_n\right] = \frac{p\,(1-p)}{n}.$$

In the case $n = 100,000$ and $p = 1/2$, thanks to the Central Limit Theorem, we can then write

$$\begin{aligned}
\mathbf{P}\left(49,500 \leq Z_n \leq 50,500\right) &= \mathbf{P}\left(49,500 - np \leq Z_n - np \leq 50,500 - np\right) \\
&= \mathbf{P}\left(\frac{49,500 - np}{\sqrt{np\left(1-p\right)}} \leq \frac{Z_n - np}{\sqrt{np\left(1-p\right)}} \leq \frac{50,500 - np}{\sqrt{np\left(1-p\right)}}\right) \\
&= \mathbf{P}\left(\frac{49,500 - np}{\sqrt{np\left(1-p\right)}} \leq Z_n^* \leq \frac{50,500 - np}{\sqrt{np\left(1-p\right)}}\right) \\
&\simeq \Phi\left(\frac{50,500 - np}{\sqrt{np\left(1-p\right)}}\right) - \Phi\left(\frac{49,500 - np}{\sqrt{np\left(1-p\right)}}\right) \\
&= \Phi\left(\frac{500}{\sqrt{25,000}}\right) - \Phi\left(\frac{-500}{\sqrt{25,000}}\right) \\
&= 2\Phi\left(\sqrt{10}\right) - 1 \\
&= 2 \cdot 0.9992 - 1 \\
&= 0.9984.
\end{aligned}$$

where Φ is the distribution function of the standard normal. Instead, with the goal of applying the Tchebychev inequality, we can write

$$\begin{aligned}
\mathbf{P}\left(49,500 \leq Z_n \leq 50,500\right) &= \mathbf{P}\left(\frac{49,500 - np}{\sqrt{np\left(1-p\right)}} \leq \frac{Z_n - np}{\sqrt{np\left(1-p\right)}} \leq \frac{50,500 - np}{\sqrt{np\left(1-p\right)}}\right) \\
&= \mathbf{P}\left(-\sqrt{10} \leq Z_n^* \leq \sqrt{10}\right) \\
&= \mathbf{P}\left(\left|Z_n^*\right| \leq \sqrt{10}\right) \\
&= 1 - \mathbf{P}\left(\left|Z_n^*\right| > \sqrt{10}\right) \\
&= 1 - \mathbf{P}\left(\left|Z_n^*\right| \geq \sqrt{10}\right).
\end{aligned}$$

On the other hand, by the Tchebychev inequality we have

$$\mathbf{P}\left(\left|Z_n^*\right| \geq \sqrt{10}\right) \leq \frac{\mathbf{D}^2\left[Z_n^*\right]}{10} = \frac{1}{10}.$$

Therefore,

$$\mathbf{P}\left(49,500 \leq Z_n \leq 50,500\right) \geq 1 - \frac{1}{10} = \frac{9}{10} = 0.9.$$

This shows that the central limit aproach provides a sharper bound for the desired probability than the Tchebychev inequality approach. Now, if $(X_n)_{n\geq 1}$ is a sequence of independent and Poisson distributed random variables with the same rate parameter λ, the random variable Z_n has the exponential distribution with rate parameter $n\lambda$. In symbols

$$Z_n \sim Poiss\left(n\lambda\right), \quad \forall n \in \mathbb{N}.$$

Hence,

$$\mathbf{E}\left[Z_n\right] = n\lambda \quad \text{and} \quad \mathbf{D}^2\left[Z_n\right] = n\lambda.$$

for every $n \in \mathbb{N}$. Therefore, the random variable

$$Z_n^* \equiv \frac{Z_n - n\lambda}{\sqrt{n\lambda}}$$

is standardized, that is

$$\mathbf{E}\left[Z_n^*\right]=0 \quad \text{and} \quad \mathbf{D}^2\left[Z_n^*\right]=1,$$

for every $n \in \mathbb{N}$. Again, by virtue of the Central Limit Theorem, we know that

$$Z_n^* \xrightarrow{\mathbf{w}} N(0,1),$$

as $n \to \infty$. As a consequence, as n is large, the random variable Z_n^* is approximately Gaussian distributed. On the other hand, we can write

$$\sqrt{\frac{\lambda}{n}} Z_n^* = \sqrt{\frac{\lambda}{n}} \frac{Z_n - n\lambda}{\sqrt{n\lambda}} = \frac{Z_n}{n} - \lambda = \bar{X}_n - \lambda,$$

that is

$$\bar{X}_n = \sqrt{\frac{\lambda}{n}} Z_n^* + \lambda.$$

This implies that, as n is large, the distribution of $\bar{X}_n$ is approximately Gaussian with

$$\mathbf{E}\left[\bar{X}_n\right]=\lambda \quad \text{and} \quad \mathbf{D}^2\left[\bar{X}_n\right]=\frac{\lambda}{n}.$$

The solution is complete.

Problem 2 *Suppose that a random variable X, which represents the reaction time at some stimulus, has a uniform distribution on an interval $[0,\theta]$, where the parameter $\theta>0$ is unknown. An investigator wants to estimate θ on the basis of a simple random sample $X_1,\dots,X_n$ of reaction times. Since θ is the largest possible time in the entire population of reaction times, the investigator consider as a first estimator for the parameter θ the largest sample reaction time. That is to say, the investigator consider as a first estimator the statistic*

$$\hat{\theta}_1 \equiv \check{X}_n \equiv \max\left(X_1,\dots,X_n\right).$$

1. *Is $\check{X}_n$ unbiased? In case $\check{X}_n$ is not unbiased, is it possible to derive from $\check{X}_n$ an unbiased estimator of θ?*

2. *As a second estimator, the investigator consider the statistic*

$$\hat{\theta}_2 \equiv \bar{X}_n \equiv \frac{1}{n}\sum_{k=1}^{n} X_k.$$

   Is $\bar{X}_n$ unbiased? In case $\bar{X}_n$ is not unbiased, is it possible to derive from $\bar{X}_n$ an unbiased estimator of θ?

3. *In the investigator's shoes, what estimator would you prefer among those considered?*

Solution.

1. Writing $F_{\check{X}_n}:\mathbb{R}\to\mathbb{R}$ for the distribution function of the statistic $\check{X}_n$, we have

$$\begin{aligned} F_{\check{X}_n}(x) &= \mathbf{P}\left(\check{X}_n \le x\right) = \mathbf{P}\left(X_1 \le x,\dots,X_n \le x\right) = \prod_{k=1}^{n}\mathbf{P}\left(X_k \le x\right) \\ &= \prod_{k=1}^{n}\mathbf{P}\left(X \le x\right) = \mathbf{P}\left(X \le x\right)^n = F_X(x)^n. \end{aligned}$$

for every $x \in \mathbb{R}$, where $F_X : \mathbb{R} \to \mathbb{R}$ is the distribution function of the random variable X. On the other hand, since X is uniformly distributed on $[0, \theta]$, X is absolutely continuous with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) \overset{\text{def}}{=} \frac{1}{\theta} 1_{[0,\theta]}(x), \quad \forall x \in \mathbb{R}.$$

Hence,

$$\begin{aligned} F_X(x) &= \int_{(-\infty,x]} f_X(u)\ d\mu_L(u) = \int_{(-\infty,x]} \frac{1}{\theta} 1_{[0,\theta]}(u)\ d\mu_L(u) = \frac{1}{\theta} \int_{(-\infty,x]\cap[0,\theta]} d\mu_L(u) \\ &= \begin{cases} \frac{1}{\theta} \int_{\varnothing}\ d\mu_L(u) = 0, & \text{if } x < 0, \\ \frac{1}{\theta} \int_{[0,x]}\ d\mu_L(u) = \frac{x}{\theta}, & \text{if } 0 \leq x \leq \theta, \\ \frac{1}{\theta} \int_{[0,\theta]}\ d\mu_L(u) = 1, & \text{if } \theta < x. \end{cases} \end{aligned}$$

More briefly

$$F_X(x) = \frac{x}{\theta} 1_{[0,\theta]}(x) + 1_{(\theta,+\infty)}(x),$$

for every $x \in \mathbb{R}$. It then follows,

$$F_{\check{X}_n}(x) = F_X(x)^n = \frac{x^n}{\theta^n} 1_{[0,\theta]}(x) + 1_{(\theta,+\infty)}(x),$$

for every $x \in \mathbb{R}$. Now, we have

$$F'_{\check{X}_n}(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{nx^{n-1}}{\theta^n}, & \text{if } 0 < x < \theta, \\ 0, & \text{if } \theta < x, \end{cases}$$

but $F_{\check{X}_n}$ is not everywhere differentable. Eventually, is not differentiable at the point $x = \theta$. However, considering the function $f_{\check{X}_n} : \mathbb{R} \to \mathbb{R}$ given by

$$f_{\check{X}_n}(x) \overset{\text{def}}{=} \frac{nx^{n-1}}{\theta^n} 1_{(0,\theta)}(x), \quad \forall x \in \mathbb{R},$$

a straightforward computation shows that

$$F_{\check{X}_n}(x) = \int_{(-\infty,x]} f_{\check{X}_n}(u)\ d\mu_L(u),$$

for every $x \in \mathbb{R}$. This implies that $\check{X}_n$ is absolutely continuous with density $f_{\check{X}_n}$. As a consequence,

$$\begin{aligned} \mathbf{E}\left[\check{X}_n\right] &= \int_{\mathbb{R}} x f_{\check{X}_n}(x)\ d\mu_L(x) = \int_{\mathbb{R}} x \frac{nx^{n-1}}{\theta^n} 1_{(0,\theta)}(x)\ d\mu_L(x) = \frac{n}{\theta^n} \int_{(0,\theta)} x^n\ d\mu_L(x) \\ &= \frac{n}{\theta^n} \int_0^\theta x^n\ dx = \frac{n}{\theta^n} \left. \frac{x^{n+1}}{n+1} \right|_0^\theta = \frac{n}{n+1}\theta. \end{aligned}$$

We can conclude that $\check{X}_n$ is not a unbiased estimator of θ but $\frac{n+1}{n}\check{X}_n$ is an unbiased estimator of θ.

2. We have

$$\begin{aligned} \mathbf{E}\left[\bar{X}_n\right] = \mathbf{E}[X] &= \int_{\mathbb{R}} x f_X(x)\ d\mu_L(x) = \int_{\mathbb{R}} \frac{x}{\theta} 1_{[0,\theta]}(x)\ d\mu_L(x) \\ &= \frac{1}{\theta} \int_{[0,\theta]} x\ d\mu_L(x) = \frac{1}{\theta} \int_0^\theta x\ dx = \frac{1}{\theta} \left. \frac{x^2}{2} \right|_0^\theta = \frac{\theta}{2}. \end{aligned}$$

Hence, $\bar{X}_n$ is not a unbiased estimator of θ but $2\bar{X}_n$ is an unbiased estimator of θ.

3. From 1. and 2. we know that

$$\mathbf{E}\left[\frac{n+1}{n}\check{X}_n\right]=\theta \qquad \text{and} \qquad \mathbf{E}\left[2\bar{X}_n\right]=\theta.$$

Hence, both $\frac{n+1}{n}\check{X}_n$ and $2\bar{X}_n$ are unbiased estimators of the parameter θ. To choose which is preferable between them, we consider

$$\mathbf{D}^2\left[\frac{n+1}{n}\check{X}_n\right] \qquad \text{and} \qquad \mathbf{D}^2\left[2\bar{X}_n\right].$$

We have

$$\begin{aligned}\mathbf{E}\left[\check{X}_n^2\right] &= \int_{\mathbb{R}} x^2 f_{\check{X}_n}(x)\; d\mu_L(x) = \int_{\mathbb{R}} x^2 \frac{nx^{n-1}}{\theta^n} 1_{(0,\theta)}(x)\; d\mu_L(x) = \frac{n}{\theta^n}\int_{(0,\theta)} x^{n+1}\; d\mu_L(x)\\ &= \frac{n}{\theta^n}\int_0^\theta x^{n+1}\; dx = \frac{n}{\theta^n}\left.\frac{x^{n+2}}{n+2}\right|_0^\theta = \frac{n}{n+2}\theta^2.\end{aligned}$$

Therefore,

$$\mathbf{D}^2\left[\check{X}_n\right] = \mathbf{E}\left[\check{X}_n^2\right]-\mathbf{E}\left[\check{X}_n\right]^2 = \frac{n}{n+2}\theta^2-\frac{n^2}{(n+1)^2}\theta^2 = \frac{n}{(n+1)^2(n+2)}\theta^2.$$

As a consequence,

$$\mathbf{D}^2\left[\frac{n+1}{n}\check{X}_n\right] = \left(\frac{n+1}{n}\right)^2\mathbf{D}^2\left[\check{X}_n\right] = \left(\frac{n+1}{n}\right)^2\frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{\theta^2}{n(n+2)}.$$

On the other hand,

$$\mathbf{D}^2\left[2\bar{X}_n\right] = 4\mathbf{D}^2\left[\bar{X}_n\right] = \frac{4}{n}\mathbf{D}^2[X] = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}.$$

Now, for any $n>1$ we clearly have

$$\mathbf{D}^2\left[\frac{n+1}{n}\check{X}_n\right] < \mathbf{D}^2\left[2\bar{X}_n\right].$$

It follows that the estimator $\frac{n+1}{n}\check{X}_n$ is preferable to $2\bar{X}_n$.

Problem 3 *Let X be a binomially distributed real random variable with known number of trials parameter m and unknown success parameter p. An investigator wants to estimate p on the basis of a simple random sample $X_1,\dots,X_n$ of size n drawn from X.*

1. *Assume the investigator applies the method of moments. What is the estimator $\hat{p}_n^M$?*
2. *Assume the investigator applies the likelihood method. What is the estimator $\hat{p}_n^{ML}$?*
3. *Given that $m=10$ and we observe a realization $4,4,3,5,6$ of a sample $X_1,\dots,X_5$ of size 5 drawn from X what is the estimate of p by the estimators $\hat{p}_n^M$?*
4. *Can you give an estimate of $\mathbf{E}[X]$ and $\mathbf{D}^2[X]$ by means of the estimator $\hat{p}_n^M$ and the information provided at 3.?*

Solution.

1. Under the assumption considered, we know that the first population moment is given by

$$\mathbf{E}[X] = m \cdot p$$

Equating the first population moment to the first sample moment $\bar{X}_n$ and replacing $\hat{p}_n^M$ to p we obtain

$$m \cdot \hat{p}_n^M = \bar{X}_n.$$

Therefore,

$$\hat{p}_n^M = \frac{\bar{X}_n}{m}.$$

2. The density function $f_X : \mathbb{N}_0 \times (0,1) \to \mathbb{R}_+$ of a binomial random variable with fixed number of trials parameter m and variable success parameter p can be written as

$$f_X(x;p) = \frac{m!}{(m-x)!x!} p^x (1-p)^{m-x} \cdot 1_{\{0,1,\ldots,m\}}(x),$$

for every $x \in \mathbb{N}_0$ and every $p \in (0,1)$. Let $X_1, \ldots, X_n$ be a simple random sample of size n drawn from X. Then the likelihood function $\mathcal{L}_{X_1,\ldots,X_n} : (0,1) \times \mathbb{N}_0^n \to \mathbb{R}$ of the sample $X_1, \ldots, X_n$ is given by

$$\begin{aligned}
&\mathcal{L}_{X_1,\ldots,X_n}(p; x_1, \ldots, x_n) \\
&= \textstyle\prod_{k=1}^n \dfrac{m!}{(m-x_k)!x_k!} p^{x_k} (1-p)^{m-x_k} \cdot 1_{\{0,1,\ldots,m\}}(x_k) \\
&= \left(\textstyle\prod_{k=1}^n \dfrac{m!}{(m-x_k)!x_k!} \right) p^{\sum_{k=1}^n x_k} (1-p)^{n \cdot m - \sum_{k=1}^n x_k} 1_{\{0,1,\ldots,m\}^n}(x_1, \ldots, x_n)
\end{aligned}$$

for every $p \in (0,1)$ and every realizations $x_1, \ldots, x_n$ of the sample $X_1, \ldots, X_n$. Note that

$$\begin{aligned}
&\mathcal{L}_{X_1,\ldots,X_n}(p; x_1, \ldots, x_n) \\
&= \begin{cases} \left(\prod_{k=1}^n \frac{m!}{(m-x_k)!x_k!} \right) p^{\sum_{k=1}^n x_k} (1-p)^{n \cdot m - \sum_{k=1}^n x_k} > 0, & \text{if } (x_1, \ldots, x_n) \in \{0,1,\ldots,m\}^n, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore,

$$\underset{\theta \in \mathbb{R}_{++}}{\arg\max} \mathcal{L}_{X_1,\ldots,X_n}(p; x_1, \ldots, x_n) = \underset{\theta \in \mathbb{R}_{++}}{\arg\max} \left(\textstyle\prod_{k=1}^n \dfrac{m!}{(m-x_k)!x_k!} \right) p^{\sum_{k=1}^n x_k} (1-p)^{n \cdot m - \sum_{k=1}^n x_k}$$

Hence, we can consider as the log-likelihood function of the sample $X_1, \ldots, X_n$ the function $\log \mathcal{L}_{X_1,\ldots,X_n} : (0,1) \times \{0,1,\ldots,m\}^n \to \mathbb{R}$ given by

$$\begin{aligned}
&\log \mathcal{L}_{X_1,\ldots,X_n}(p; x_1, \ldots, x_n) \stackrel{\text{def}}{=} \ln \left(\left(\textstyle\prod_{k=1}^n \dfrac{m!}{(m-x_k)!x_k!} \right) p^{\sum_{k=1}^n x_k} (1-p)^{n \cdot m - \sum_{k=1}^n x_k} \right) \\
&= \sum_{k=1}^n \ln \left(\frac{m!}{(m-x_k)!x_k!} \right) + \left(\textstyle\sum_{k=1}^n x_k \right) \ln(p) + \left(n \cdot m - \textstyle\sum_{k=1}^n x_k \right) \ln(1-p).
\end{aligned}$$

To determine $\hat{p}_n^{ML}$ we consider the first order condition

$$\frac{d}{dp} \log \mathcal{L}_{X_1,\ldots,X_n}(p; x_1, \ldots, x_n) = 0,$$

which yields

$$\left(\sum_{k=1}^{n} x_k\right) \frac{1}{p} - \left(n \cdot m - \sum_{k=1}^{n} x_k\right) \frac{1}{1-p} = 0.$$

On account that $p \in (0,1)$, the latter becomes

$$\left(\sum_{k=1}^{n} x_k\right)(1-p) - \left(n \cdot m - \sum_{k=1}^{n} x_k\right) p = 0.$$

That is

$$\sum_{k=1}^{n} x_k - n \cdot m \cdot p = 0,$$

which implies

$$p = \frac{\sum_{k=1}^{n} x_k}{n \cdot m} = \frac{\bar{x}_n}{m}.$$

In addition,

$$\begin{aligned}
\frac{d^2}{dp^2} \log \mathcal{L}_{X_1,\dots,X_n}(p; x_1, \dots, x_n) &= -\left(\sum_{k=1}^{n} x_k\right) \frac{1}{p^2} - \left(n \cdot m - \sum_{k=1}^{n} x_k\right) \frac{1}{(1-p)^2} \\
&= \frac{-\left(\sum_{k=1}^{n} x_k\right)(1-p)^2 - \left(n \cdot m - \sum_{k=1}^{n} x_k\right) p^2}{p^2 (1-p)^2} \\
&= \frac{-\sum_{k=1}^{n} x_k + 2\left(\sum_{k=1}^{n} x_k\right) p - n \cdot m \cdot p^2}{p^2 (1-p)^2} \\
&= \frac{-n\bar{x}_n + 2n\bar{x}_n p - n \cdot m \cdot p^2}{p^2 (1-p)^2} \\
&= -\frac{n}{p^2 (1-p)^2} \left(\bar{x}_n - 2\bar{x}_n p + m \cdot p^2\right).
\end{aligned}$$

Now, we have

$$\left(\bar{x}_n - 2\bar{x}_n p + m \cdot p^2\right)_{p=\frac{\bar{x}_n}{m}} = \left(\bar{x}_n - \frac{2}{m}\bar{x}_n^2 + m \cdot \frac{\bar{x}_n^2}{m^2}\right) = \bar{x}_n \left(1 - \frac{1}{m}\bar{x}_n\right).$$

On the other hand, we clearly have

$$\bar{x}_n \leq m,$$

for every $(x_1, \dots, x_n) \in \{0, 1, \dots, m\}^n$. It follows

$$\frac{d^2}{dp^2} \log \mathcal{L}_{X_1,\dots,X_n}(p; x_1, \dots, x_n) \leq 0$$

which implies that

$$p = \frac{\bar{x}_n}{m}$$

is a maximum for $\log \mathcal{L}_{X_1,\dots,X_n}(p; x_1, \dots, x_n)$. As a consequence, we obtain that the maximum likelihood estimator fo p is given by

$$\hat{p}_n^{ML} = \frac{\bar{X}_n}{m}.$$

3. Given that $m = 10$ and we observe a realization $4, 4, 3, 5, 6$ of a sample $X_1, \dots, X_5$ of size 5 drawn from X, we obtain

$$\hat{p}_n^{M}(\omega) = \frac{\bar{X}_5(\omega)}{10} = \frac{\frac{1}{5}(4+4+3+5+6)}{10} = 0.44.$$

4. We know that
$$\mathbf{E}[X] = m \cdot p \qquad \text{and} \qquad \mathbf{D}^2[X] = m \cdot p(1-p),$$
where p is the true value of the success parameter, As a consequence given the estimators $\hat{p}_n^M$ an estimator $\hat{\mu}_X$ [resp. $\hat{\sigma}_X^2$] of the expectation [resp. variance] of X is given by
$$\hat{\mu}_X = m \cdot \hat{p}_n^M \quad \text{and} \quad \hat{\sigma}_X^2 = m \cdot \hat{p}_n^M \left(1 - \hat{p}_n^M\right).$$
An estimate of $\mathbf{E}[X]$ and $\mathbf{D}^2[X]$ by means of the estimator $\hat{p}_n^M$ and the information provided at 3. is the given by
$$\hat{\mu}_X(\omega) = m \cdot \hat{p}_n^M(\omega) = 10 \cdot 0.44 = 4.4$$
and
$$\hat{\sigma}_X^2(\omega) = m \cdot \hat{p}_n^M(\omega)\left(1 - \hat{p}_n^M(\omega)\right) = 10 \cdot 0.44 \cdot (1 - 0.44) = 2.464.$$
This completes the solution.

Problem 4 *Let X be a normally distributed random variable with unknown mean μ_X and variance σ_X^2. An investigator wants to estimate μ and σ^2 on the basis of a simple random sample $X_1, \ldots, X_n$ of size n drawn from X.*

1. *Assume the investigator applies the likelihood methods. What are the estimator $\hat{\mu}_n^{LM}$ and $\hat{\sigma}_n^{2^{LM}}$?*
2. *Assume the investigator applies the method of moments. What are the estimators $\hat{\mu}_n^M$ and $\hat{\sigma}_n^{2^M}$? Hint: guess what $\hat{\sigma}_n^{2^M}$ could be and get it!*

Solution.

1. We know that the joint density function $f_{X_1,\ldots,X_n} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ of the sample $X_1, \ldots, X_n$ is given by
$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\mu,\sigma) \overset{\text{def}}{=} \prod_{k=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_k-\mu)^2}{2\sigma^2}}, \quad \forall (x_1,\ldots,x_n) \in \mathbb{R}^n, \quad \forall (\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_{++},$$
that is
$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n;\mu,\sigma) = \frac{1}{\sqrt{2^n\pi^n}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n (x_k-\mu)^2}.$$
Hence, the likelihood function $\mathcal{L}_{X_1,\ldots,X_n} : \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}$ of the sample is given by
$$\mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n) = \frac{1}{\sqrt{2^n\pi^n}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n (x_k-\mu)^2}, \quad \forall (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_{++}, \quad \forall (x_1,\ldots,x_n) \in \mathbb{R}^n.$$
Thanks to the stucture of $\mathcal{L}_{X_1,\ldots,X_n}$ it is convenient to consider the log-likelihood function $\log \mathcal{L}_{X_1,\ldots,X_n} : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ of the sample which is given by
$$\log \mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n) \overset{\text{def}}{=} (\log \circ \mathcal{L}_{X_1,\ldots,X_n})(\mu,\sigma;x_1,\ldots,x_n), \quad \forall (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_{++}, \quad \forall (x_1,\ldots,x_n) \in \mathbb{R}^n$$
that is
$$\log \mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n) = -n\left(\frac{1}{2}\ln(2\pi) + \ln(\sigma)\right) - \frac{1}{2\sigma^2}\sum_{k=1}^n (x_k-\mu)^2.$$
Now, to determine $\underset{(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}_{++}}{\arg\max} \log \mathcal{L}_{X_1,\ldots,X_n}$ we can consider the first order conditions
$$\frac{\partial}{\partial\mu} \log \mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n) = 0 \quad \text{and} \quad \frac{\partial}{\partial\sigma} \log \mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n) = 0.$$

We have

$$\frac{\partial}{\partial\mu}\log\mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n)=\frac{1}{\sigma^2}\textstyle\sum_{k=1}^n(x_k-\mu)$$

and

$$\frac{\partial}{\partial\sigma}\log\mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n)=\frac{1}{\sigma}\left(\frac{1}{\sigma^2}\textstyle\sum_{k=1}^n(x_k-\mu)^2-n\right).$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial\mu}\log\mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n)=0&\Rightarrow\textstyle\sum_{k=1}^n(x_k-\mu)=0,\\&\Rightarrow\mu=\frac{1}{n}\textstyle\sum_{k=1}^n x_k,\\&\Rightarrow\mu=\bar{x}_n\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\sigma}\log\mathcal{L}_{X_1,\ldots,X_n}(\mu,\sigma;x_1,\ldots,x_n)=0&\Rightarrow\frac{1}{\sigma^2}\textstyle\sum_{k=1}^n(x_k-\mu)^2-n,\\&\Rightarrow\sigma^2=\frac{1}{n}\textstyle\sum_{k=1}^n(x_k-\mu)^2,\\&\Rightarrow\sigma^2=\frac{1}{n}\textstyle\sum_{k=1}^n(x_k-\bar{x}_n)^2.\end{aligned}$$

The above equations imply that we have

$$\hat{\mu}_n^{LM}=\bar{X}_n\quad\text{and}\quad\hat{\sigma}_n^{2^{LM}}=\tilde{S}_n^2(X),$$

where $\bar{X}_n$ [resp. $\tilde{S}_n^2(X)$] is the sample mean [resp. unbiased sample variance] of $X_1,\ldots,X_n$.

2. We know that

$$\mathbf{E}[X]=\mu\quad\text{and}\quad\mathbf{E}\left[X^2\right]=\mu^2+\sigma^2.$$

Hence, applying the method of moments, the investigator writes

$$\frac{1}{n}\textstyle\sum_{k=1}^n X_k=\hat{\mu}_n^M\quad\text{and}\quad\frac{1}{n}\textstyle\sum_{k=1}^n X_k^2=\left(\hat{\mu}_n^M\right)^2+\hat{\sigma}_n^{2^M}.$$

The first of the two equations clearly yields

$$\hat{\mu}_n^M=\bar{X}_n.$$

The second equation, on account of the first, yields

$$\hat{\sigma}_n^{2^M}=\frac{1}{n}\textstyle\sum_{k=1}^n X_k^2-\bar{X}_n^2.$$

On the other hand,

$$\begin{aligned}\tilde{S}_n^2(X)&=\frac{1}{n}\textstyle\sum_{k=1}^n\left(X_k-\bar{X}_n\right)^2\\&=\frac{1}{n}\textstyle\sum_{k=1}^n\left(X_k^2-2X_k\bar{X}_n+\bar{X}_n^2\right)\\&=\frac{1}{n}\left(\textstyle\sum_{k=1}^n X_k^2-2\bar{X}_n\textstyle\sum_{k=1}^n X_k+\textstyle\sum_{k=1}^n\bar{X}_n^2\right)\\&=\frac{1}{n}\left(\textstyle\sum_{k=1}^n X_k^2-2n\bar{X}_n^2+n\bar{X}_n^2\right)\\&=\frac{1}{n}\left(\textstyle\sum_{k=1}^n X_k^2-n\bar{X}_n^2\right)\\&=\frac{1}{n}\textstyle\sum_{k=1}^n X_k^2-\bar{X}_n^2\end{aligned}$$

It follows that
$$\hat{\sigma}_n^{2^M} = \tilde{S}_n^2(X).$$
This completes the solution.

Problem 5 *Let X a random variable representing a characteristic of a certain population. Assume that X has a density $f_X : \mathbb{R} \to \mathbb{R}$ given by*
$$f_X(x) \overset{def}{=} \frac{1}{\theta} e^{-\frac{x-3}{\theta}} 1_{[3,+\infty)}(x), \quad \forall x \in \mathbb{R},$$
where θ is a positive parameter.

1. *Apply the method of moments to find the estimator $\hat{\theta}_M$ of the parameter θ.*
2. *Apply the maximum likelihood method to find the estimator $\hat{\theta}_{ML}$ of the parameter θ.*
3. *Use the estimators $\hat{\theta}_M$ and $\hat{\theta}_{ML}$ to build estimators for $\mathbf{E}[X]$ and $\mathbf{D}^2[X]$.*

Solution.

1. We have
$$\begin{aligned}
\mathbf{E}[X] &= \int_{\mathbb{R}} x f_X(x)\ d\mu_L(x) = \int_{\mathbb{R}} x \frac{1}{\theta} e^{-\frac{x-3}{\theta}} 1_{[3,+\infty)}(x)\ d\mu_L(x) = \int_{[3,+\infty)} x \frac{1}{\theta} e^{-\frac{x-3}{\theta}}\ d\mu_L(x) \\
&= \int_3^{+\infty} x \frac{1}{\theta} e^{-\frac{x-3}{\theta}}\ dx = \lim_{x\to+\infty} \int_3^x u \frac{1}{\theta} e^{-\frac{u-3}{\theta}}\ du = \lim_{x\to+\infty} \int_0^{\frac{x-3}{\theta}} (\theta v + 3) e^{-v}\ dv \\
&= \lim_{x\to+\infty} \left(\theta \int_0^{\frac{x-3}{\theta}} v e^{-v}\ dv + 3 \int_0^{\frac{x-3}{\theta}} e^{-v}\ dv \right) = \theta \lim_{x\to+\infty} \int_0^{\frac{x-3}{\theta}} v e^{-v}\ dv + 3 \lim_{x\to+\infty} \int_0^{\frac{x-3}{\theta}} e^{-v}\ dv.
\end{aligned}$$
Now,
$$\int_0^{\frac{x-3}{\theta}} e^{-v}\ dv = -\int_0^{-\frac{x-3}{\theta}} e^{v}\ dv = \int_{-\frac{x-3}{\theta}}^{0} e^{v}\ dv = e^v \Big|_{-\frac{x-3}{\theta}}^{0} = 1 - e^{-\frac{x-3}{\theta}}$$
and
$$\begin{aligned}
\int_0^{\frac{x-3}{\theta}} v e^{-v}\ dv &= -\int_0^{\frac{x-3}{\theta}} v\ de^{-v} = -\left(v e^{-v} \Big|_0^{\frac{x-3}{\theta}} - \int_0^{\frac{x-3}{\theta}} e^{-v}\ dv \right) \\
&= -\left(v e^{-v} \Big|_0^{\frac{x-3}{\theta}} - \left(1 - e^{-\frac{x-3}{\theta}}\right)\right) = -\left(\frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} + e^{-\frac{x-3}{\theta}} - 1 \right) \\
&= 1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}}.
\end{aligned}$$
It follows
$$\mathbf{E}[X] = \theta \lim_{x\to+\infty} \left(1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}} \right) + 3 \lim_{x\to+\infty} \left(1 - e^{-\frac{x-3}{\theta}} \right) = \theta + 3.$$
As a consequence, setting
$$\frac{1}{n} \textstyle\sum_{k=1}^n X_k = \hat{\theta}_n^M + 3$$
we obtain
$$\hat{\theta}_n^M = \bar{X}_n - 3.$$

2. We know that the joint density function $f_{X_1,\ldots,X_n} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$ of a simple random sample $X_1, \ldots, X_n$ drawn from X is given by

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n; \theta) \overset{\text{def}}{=} \prod_{k=1}^n \frac{1}{\theta} e^{-\frac{x_k-3}{\theta}} 1_{[3,+\infty)}(x_k), \quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \forall \theta \in \mathbb{R}_{++},$$

that is

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n; \theta) &= \left(\prod_{k=1}^n \frac{1}{\theta}\right)\left(\prod_{k=1}^n e^{-\frac{x_k-3}{\theta}}\right) \prod_{k=1}^n 1_{[3,+\infty)}(x_k) \\ &= \frac{1}{\theta^n} e^{-\frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)} \prod_{k=1}^n 1_{[3,+\infty)}(x_k). \end{aligned}$$

Hence, the likelihood function $\mathcal{L}_{X_1,\ldots,X_n} : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}$ of the sample is given by

$$\mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)} \prod_{k=1}^n 1_{[3,+\infty)}(x_k), \quad \forall \theta \in \mathbb{R}_{++}, \quad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Note that

$$\mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = \begin{cases} \frac{1}{\theta^n} e^{-\frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)} > 0, & \text{if } (x_1, \ldots, x_n) \in \times_{k=1}^n [3, +\infty), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\underset{\theta \in \mathbb{R}_{++}}{\arg\max} \mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = \underset{\theta \in \mathbb{R}_{++}}{\arg\max} \frac{1}{\theta^n} e^{-\frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)}$$

Hence, we can consider as the log-likelihood function of the sample $X_1, \ldots, X_n$ the function $\log \mathcal{L}_{X_1,\ldots,X_n} : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\log \mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) \overset{\text{def}}{=} \log\left(\frac{1}{\theta^n} e^{-\frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)}\right) = -n\ln(\theta) - \frac{1}{\theta}\left(\sum_{k=1}^n x_k - 3n\right)$$

Now, to determine $\underset{\theta \in \mathbb{R}_{++}}{\arg\max} \log \mathcal{L}_{X_1,\ldots,X_n}$ we can consider the first order condition

$$\frac{\partial}{\partial \theta} \log \mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = 0.$$

We have

$$\frac{\partial}{\partial \theta} \log \mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = -\frac{n}{\theta} + \left(\sum_{k=1}^n x_k - 3n\right)\frac{1}{\theta^2}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial \theta} \log \mathcal{L}_{X_1,\ldots,X_n}(\theta; x_1, \ldots, x_n) = 0 &\Rightarrow \frac{1}{\theta}\left(\left(\sum_{k=1}^n x_k - 3n\right)\frac{1}{\theta} - n\right) = 0 \\ &\Rightarrow \theta = \frac{1}{n}\left(\sum_{k=1}^n x_k - 3n\right) \\ &\Rightarrow \theta = \frac{1}{n}\sum_{k=1}^n x_k - 3. \end{aligned}$$

The above equation imply that we have

$$\hat{\theta}_n^{LM} = \bar{X}_n - 3.$$

3. Note that
$$\hat{\theta}_n^M = \hat{\theta}_n^{LM} \equiv \bar{X}_n - 3.$$

Since
$$\mathbf{E}[X] = \theta + 3,$$
it clearly follows that the desired estimator for $\mathbf{E}[X]$ is $\bar{X}_n$.

To build an estimator for $\mathbf{D}^2[X]$ we have first to determine the second order moment $\mathbf{E}\left[X^2\right]$ of X. We have
$$\begin{aligned}
\mathbf{E}\left[X^2\right] &= \int_{\mathbb{R}} x^2 f_X(x)\ d\mu_L(x) = \int_{\mathbb{R}} x^2 \frac{1}{\theta} e^{-\frac{x-3}{\theta}} 1_{[3,+\infty)}(x)\ d\mu_L(x) = \int_{[3,+\infty)} x^2 \frac{1}{\theta} e^{-\frac{x-3}{\theta}}\ d\mu_L(x) \\
&= \int_3^{+\infty} x^2 \frac{1}{\theta} e^{-\frac{x-3}{\theta}}\ dx = \lim_{x\to+\infty} \int_3^x u^2 \frac{1}{\theta} e^{-\frac{u-3}{\theta}}\ dx = \lim_{x\to+\infty} \int_0^{\frac{x-3}{\theta}} (\theta v + 3)^2 e^{-v}\ dv \\
&= \lim_{x\to+\infty} \int_0^{\frac{x-3}{\theta}} \left(\theta^2 v^2 + 6\theta v + 9\right) e^{-v}\ dv \\
&= \lim_{x\to+\infty} \left(\theta^2 \int_0^{\frac{x-3}{\theta}} v^2 e^{-v}\ dv + 6\theta \int_0^{\frac{x-3}{\theta}} v e^{-v}\ du + 9 \int_0^{\frac{x-3}{\theta}} e^{-v}\ dv\right)
\end{aligned}$$

We already know that
$$\int_0^{\frac{x-3}{\theta}} e^{-v}\ dv = 1 - e^{-\frac{x-3}{\theta}} \quad \text{and} \quad \int_0^{\frac{x-3}{\theta}} v e^{-v}\ dv = 1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}}.$$

Hence, we compute
$$\begin{aligned}
\int_0^{\frac{x-3}{\theta}} v^2 e^{-v}\ dv &= -\int_0^{\frac{x-3}{\theta}} v^2\ de^{-v} = -\left(v^2 e^{-v}\Big|_0^{\frac{x-3}{\theta}} - 2\int_0^{\frac{x-3}{\theta}} v e^{-v}\ dv\right) \\
&= 2\left(1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}}\right) - \left(\frac{x-3}{\theta}\right)^2 e^{-\frac{x-3}{\theta}}.
\end{aligned}$$

We then have
$$\begin{aligned}
\mathbf{E}\left[X^2\right] &= \lim_{x\to+\infty} \theta^2 \left(2\left(1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}}\right) - \left(\frac{x-3}{\theta}\right)^2 e^{-\frac{x-3}{\theta}}\right) \\
&\quad + \lim_{x\to+\infty} 6\theta \left(1 - \frac{x-3}{\theta} e^{-\frac{x-3}{\theta}} - e^{-\frac{x-3}{\theta}}\right) + 9 \lim_{x\to+\infty} \left(1 - e^{-\frac{x-3}{\theta}}\right) \\
&\quad = 2\theta^2 + 6\theta + 9.
\end{aligned}$$

It follows
$$\mathbf{D}^2[X] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X]^2 = 2\theta^2 + 6\theta + 9 - (\theta+3)^2 = \theta^2.$$

Therefore, the desired estimators for $\mathbf{D}^2[X]$ is
$$\left(\hat{\theta}_n^M\right)^2 = \left(\bar{X}_n - 3\right)^2 = \bar{X}_n^2 - 6\bar{X}_n + 9.$$

This completes the solution.

Problem 6 *Assume that the returns of a stock in a financial market are Gaussian distributed with unknown mean μ and variance σ^2. Let X be the normal random variable representing the realization of the returns and let $X_1, \dots, X_n$ be a simple random sample of size n drawn from X. Assume that $n = 5$ and the realizations of the sample are*

$$x_1 \equiv -1.5, \quad x_2 \equiv -0.5, \quad x_3 \equiv 1.5, \quad x_4 \equiv 2.0, \quad x_5 \equiv 2.5$$

1. *Determine a 99% confidence interval for the mean μ.*
2. *Find the confidence for an interval of width 0.1.*
3. *Determine a 90% confidence interval for the standard deviation σ.*

Solution.

1. From data we obtain

$$\bar{x}_5 \equiv \frac{1}{5}\sum_{k=1}^{5} x_k = 0.8$$

and

$$s_5^2(X) = \frac{1}{4}\sum_{k=1}^{5}(x_k - \bar{x}_5)^2 = 2.95 \Rightarrow s_5(X) = 1.72$$

Now, since X is Gaussian distributed with unknown variance and the size of the sample is small, to determine a $100(1-\alpha)\%$ confidence interval for the mean μ the statistic to be considered is

$$\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} \sim t_{n-1}.$$

The achievement of a $100(1-\alpha)\%$ confidence interval requires to use the $\alpha/2$ critical value $t_{n-1,\alpha/2}$ of t_{n-1} for $\alpha = 0.01$. In fact, we have

$$\begin{aligned}
1-\alpha &= \mathbf{P}\left(-t_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} < t_{n-1,\alpha/2}\right) \\
&= \mathbf{P}\left(-\left(\bar{X}_n + t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right) < -\mu < -\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right)\right) \\
&= \mathbf{P}\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}} < \mu < \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right)
\end{aligned}$$

It follows that the desired confidence interval for μ is given by the random interval

$$\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}},\ \bar{X}_n + t_{n-1,\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right)$$

A realization of such a confidence interval is then given by

$$\left(\bar{x}_n - t_{n-1,\alpha/2}\frac{s_n(X)}{\sqrt{n}},\ \bar{x}_n + t_{n-1,\alpha/2}\frac{s_n(X)}{\sqrt{n}}\right).$$

In the case under scrutiny, since $t_{n-1,\alpha/2} \equiv t_{4,0.005} = 4.60$, $\bar{x}_n \equiv \bar{x}_5 = 0.80$, $s_n \equiv s_5 = 1.72$, the realization of the confidence interval becomes

$$(-3.16,\ 4.76).$$

2. From 1. it is clearly seen the width w of a $100\,(1-\alpha)\,\%$ confidence interval is given by

$$w = 2t_{n-1,\alpha/2}\frac{S_n\,(X)}{\sqrt{n}}.$$

As a consequence, the size n of the sample which gives a $100\,(1-\alpha)\,\%$ confidence interval of a given widht w is given by the solution of the equation

$$\frac{n}{t^2_{n-1,\alpha/2}} = \left[4\frac{S^2_n\,(X)}{w^2}\right] + 1.$$

To determine n we need to consider as many realizations $x_1,\ldots,x_n$ of the simple random sample $X_1,\ldots,X_n$ such that

$$\frac{n}{t^2_{n-1,\alpha/2}} = \left[4\frac{s^2_n\,(X)}{0.01}\right] + 1.$$

3. Again, since X is Gaussian distributed and the size of the sample is small, to determine a $100\,(1-\alpha)\,\%$ confidence interval for the standard deviation σ the statistic to be considered is

$$\frac{(n-1)\,S^2_n\,(X)}{\sigma^2} \sim \chi^2_{n-1}.$$

Since χ^2_{n-1} is not symmetric, the achievement of a $100\,(1-\alpha)\,\%$ confidence interval requires to exploit the $\alpha/2$ and the $1-\alpha/2$ critical value $\chi^2_{n-1,\alpha/2,-} \equiv \chi^2_{n-1,\alpha/2}$ and $\chi^2_{n-1,\alpha/2,+} = \chi^2_{n-1,1-\alpha/2}$ of χ^2_{n-1} for $\alpha = 0.1$, where $\chi^2_{n-1,\alpha/2}$ [resp. $\chi^2_{n-1,1-\alpha/2}$] is the $\alpha/2$-quantile [$1-\alpha/2$-quantile] of the χ^2_{n-1} distribution. In fact, we have

$$1-\alpha = \mathbf{P}\left(\chi^2_{n-1,\alpha/2,-} < \frac{(n-1)\,S^2_n\,(X)}{\sigma^2} < \chi^2_{n-1,\alpha/2,+}\right) = \mathbf{P}\left(\frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,\alpha/2,+}} < \sigma^2 < \frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,\alpha/2,-}}\right).$$

It follows that the desired confidence interval for the variance σ^2 is given by

$$\left(\frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,\alpha/2,+}},\ \frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,\alpha/2,-}}\right) = \left(\frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,1-\alpha/2}},\ \frac{(n-1)\,S^2_n\,(X)}{\chi^2_{n-1,\alpha/2}}\right).$$

In the case under scrutiny, since $\chi^2_{n-1,\alpha/2} \equiv \chi^2_{4,0.5} = 0.71$, $\chi^2_{n-1,1-\alpha/2} \equiv \chi^2_{4,0.95} = 9.49$, $\bar{x}_n \equiv \bar{x}_5 = 0.80$, $s^2_n\,(X) \equiv s^2_5\,(X) = 2.95$, a realization of the confidence interval is given by

$$\left(\frac{4s^2_n\,(X)}{\chi^2_{4,0.95}},\ \frac{4s^2_n\,(X)}{\chi^2_{4,0.05}}\right) = \left(\frac{4\cdot 2.95}{9.49},\ \frac{4\cdot 2.95}{0.71}\right) = (1.24,\ 16.62)\,.$$

As a consequence the $100\,(1-0.1)\,\%$ confidence interval for the standard deviation σ is

$$(1.11,\ 4.08)\,.$$

This completes the solution.

Problem 7 *Assume that a library master believes that the mean duration in days of the borrowing period is* $20d$. *However, the library master selects a simple random sample of* 100 *books in the library and discovers that the sample mean and variance of the borrowing days are* $18d$ *and* $8d^2$, *respectively. Determine a* 99% *confidence interval for the mean duration of the borrowing days to check whether library master's initial guess is correct.*

Solution. Note that the distribution of the random variable X representing the duration in days of the borrowing period is unknown. However, are known the sample mean and variance realizations referred to a simple sample of size $n = 100$, which may be considered a large sample. In this case the statistic to be considered is given by

$$\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} \to Z,$$

where $Z \sim N(0,1)$. The achievement of a $100(1-\alpha)\%$ confidence interval requires to exploit the $\alpha/2$ critical value $z_{\alpha/2}$ of Z for $\alpha = 0.01$. In fact, we have

$$\begin{aligned}
1-\alpha &\approx \mathbf{P}\left(-z_{\alpha/2} < \frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} < z_{\alpha/2}\right) \\
&= \mathbf{P}\left(-\left(\bar{X}_n + z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right) < -\mu < -\left(\bar{X}_n - z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right)\right) \\
&= \mathbf{P}\left(\bar{X}_n - z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}} < \mu < \bar{X}_n + z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right)
\end{aligned}$$

It follows that the desired confidence interval for μ is given by

$$\left(\bar{X}_n - z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}},\ \bar{X}_n + z_{\alpha/2}\frac{S_n(X)}{\sqrt{n}}\right).$$

In the case under scrutiny,

$$n = 100, \quad \bar{x}_n \equiv \bar{x}_{100} = 18, \quad s_n(X) \equiv s_{100}(X) = \sqrt{8}, \quad z_{\alpha/2} \equiv 2.58.$$

Therefore, a realization of the confidence interval is given by

$$\left(\bar{x}_n - z_{\alpha/2}\frac{s_n(X)}{\sqrt{n}},\ \bar{x}_n + z_{\alpha/2}\frac{s_n(X)}{\sqrt{n}}\right) = \left(18 - 2.58\cdot\frac{\sqrt{8}}{\sqrt{100}}, 18 + 2.58\cdot\frac{\sqrt{8}}{\sqrt{100}}\right) = (17.27,\ 18.73).$$

It follows that library master's initial guess is not supported by data. Note that this problem can be tackled also exploiting the hypothesis test method. In fact, assume as the null hypothesis that library master's assumption is correct, that is $H_0 : \mu = \mu_0$, and as the alternative hypothesis that library master's assumption is wrong, that is $H_0 : \mu \neq \mu_0$. The same consideration as above on the available information on the random variable X led to consider the rejection region

$$R = \left\{\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} < -z_{\alpha/2}\right\} \cup \left\{\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} > z_{\alpha/2}\right\} = \left\{\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} < -2.58\right\} \cup \left\{\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} > 2.58\right\},$$

where $\mu = 20$. Computing the statistic $\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}}$ for the available realization, we obtain

$$\frac{\bar{x}_n - \mu}{s_n(X)/\sqrt{n}} = \frac{18 - 20}{\sqrt{8}/10} = -7.07 \in R.$$

Hence, the library master's assumption has to be rejected.

Problem 8 *The mark of a infamous exam of Probability and Statistics are normally distributed with standard deviation $\sigma = 2$. A simple random sample of nine students is selected end the following evaluations are computed*

$$\textstyle\sum_{k=1}^{9} x_k = 237 \quad \textit{and} \quad \sum_{k=1}^{9} x_k^2 = 6295.$$

1. *Find a* 90% *confidence interval for the mean mark.*
2. *Discuss, without computation, whether the lenght of a* 95% *confidence interval would be smaller, greater or equal than the lenght of the interval previously determined.*
3. *How large the minimum sample size should be to obtain a* 90% *confidence interval for the mean mark with width equal to* 3*? Besides the confidence interval method is it possible to apply the Tchebychev inequality?*

Solution.

1. We have

$$\bar{x}_n = \frac{1}{n}\sum_{\ell=1}^{n} x_\ell$$

and

$$\begin{aligned} s_n^2(X) &= \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x}_n)^2 = \frac{1}{n-1}\left(\sum_{k=1}^{n}\left(x_k^2 - 2x_k\bar{x}_n + \bar{x}_n^2\right)\right) \\ &= \frac{1}{n-1}\left(\sum_{k=1}^{n}x_k^2 - 2\bar{x}_n\sum_{k=1}^{n}x_k + \sum_{k=1}^{n}\bar{x}_n^2\right) = \frac{1}{n-1}\left(\sum_{k=1}^{n}x_k^2 - 2n\bar{x}_n^2 + n\bar{x}_n^2\right) \\ &= \frac{1}{n-1}\left(\sum_{k=1}^{n}x_k^2 - n\bar{x}_n^2\right). \end{aligned}$$

Hence, in our case

$$n = 9, \quad \bar{x}_n \equiv \bar{x}_9 = \frac{1}{9}237 = 26.33, \quad s_n^2(X) \equiv s_9^2(X) = \frac{1}{8}\left(6295 - \frac{1}{9}237^2\right) = 6.75.$$

Now, the random variable X representing the mark of the exam is normally distributed and the size of the sample considered is small. On the other hand, we know the variance of X and we also know the realization of the sample variance. Therefore, we can considered two different approaches.

(i) Exploit the statistic

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim Z$$

where $Z \sim N(0,1)$.

(ii) Exploit the statistic

$$\frac{\bar{X}_n - \mu}{S_n(X)/\sqrt{n}} \sim t_{n-1}.$$

where t_{n-1} is Student distributed with $n-1$ degrees of freedom.

In the first case, a $100(1-\alpha)\%$ confidence interval requires to consider the $\alpha/2$ critical value $z_{\alpha/2}$ of Z for $\alpha = 0.01$. It follows that the desired confidence interval for μ is given by

$$\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{X}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

On account that $z_{\alpha/2} \equiv z_{0.05} \equiv 1.64$ and $\sigma = 2$, a realization of the confidence interval, is then given by

$$I_\alpha = \left(\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \left(26.33 - 1.64\frac{2}{3},\ 26.33 + 1.64\frac{2}{3}\right) = (25.24,\ 27.42)$$

In the second case, a $100\,(1-\alpha)\,\%$ confidence interval requires to consider the $\alpha/2$ critical value $t_{n-1,\alpha/2}$ of T for $\alpha = 0.1$. Hence, the desired confidence interval for μ is given by

$$\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{S_n\,(X)}{\sqrt{n}},\ \bar{X}_n - t_{n-1,\alpha/2}\frac{S_n\,(X)}{\sqrt{n}}\right),$$

On account that $t_{n-1,\alpha/2} \equiv t_{8,0.05} = 1.86$, a realization of the confidence interval, is then given by

$$J_\alpha = \left(\bar{x}_n - t_{n-1,\alpha/2}\frac{s_n\,(X)}{\sqrt{n}},\ \bar{x}_n + t_{n-1,\alpha/2}\frac{s_n\,(X)}{\sqrt{n}}\right) = \left(26.33 - 1.86\frac{\sqrt{6.75}}{3},\ 26.33 + 1.86\frac{\sqrt{6.75}}{3}\right) = (24.72,\ 2$$

Note that

$$I_\alpha \subset J_\alpha.$$

2. The lenght of a 95% confidence interval will be greater then the lenght of the interval previously determined. There is a theorem stating that the higher the confidence is, the larger the confidence interval is.

3. Considering Case (i), the width w of a 90% confidence interval for the mean mark is given by

$$w = 2\ z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Solving for n we obtain

$$n = \frac{4\ z_{\alpha/2}^2\sigma^2}{w^2}$$

Setting $w = 3$, and keeping all other parameters unchanged, it follows

$$n = 12.$$

With this value of n, keeping all other parameters unchanged, the confidence interval becomes

$$(24.84,\ 27.82)\,,$$

which is narrower than I_α.

FROM NOW ON THE SOLUTIONS TO THE PROBLEMS HAVE NOT BEEN REVIEWED YET. HOWEVER, THEY SHOULD BE ESSENTIALLY CORRECT. PLEASE, LET ME KNOW WHETHER YOU FIND INCONGRUENCES.

Problem 9 *Let $X_1, \dots, X_n, X_{n+1}$ be a simple random sample of size $n+1$ drawn from a Gaussian distributed random variable X with unknown mean μ and variance σ^2. Assume that we have observed $X_1, \dots, X_n$ and we want use the observed values $x_1, \dots, x_n$ to determine a confidence interval for the prediction of X_{n+1}. To this goal give detailed answers to the following questions:*

1. *what is the distribution of the statistic $\bar{X}_n$?*
2. *what is the distribution of the statistic $\left(X_{n+1} - \bar{X}_n\right)/\sigma\sqrt{1+1/n}$?*

3. *are the statistics* $X_{n+1} - \bar{X}_n$ *and* $S_n^2 \equiv \frac{1}{n-1} \sum_{k=1}^{n} (X_k - \bar{X}_n)^2$ *independent?*

4. *what is the distribution of the statistic* $(X_{n+1} - \bar{X}_n) / S_n \sqrt{1 + 1/n}$*?*

Exercise 10 *After answering the above questions, build an interval in which the random variable* X_{n+1} *takes its values with probability* α *and determine the corresponding confidence interval for the prediction of* X_{n+1}*. In the end, assume that* $n = 7$ *and we have*

$$x_1 = 7005, \quad x_2 = 7432, \quad x_3 = 7420, \quad x_4 = 6822, \quad x_5 = 6752, \quad x_6 = 5333, \quad x_7 = 6552.$$

compute the 95% *confidence interval for the prediction of* X_8*.*

Problem 11 *Let* X *be a Gaussian random variable with unknown mean* μ_X *and variance* σ_X^2 *representing a certain characteristic of a population. Assume that testing the sample mean* $\bar{X}_n$ *and the sample standard deviation* S_n *of a simple random sample* $X_1, \ldots, X_n$ *of size* $n \equiv 9$ *drawn from* X *we obtain the value* $\bar{X}_n(\omega) \equiv \bar{x}_n = 251.50cm$ *and* $S_n(\omega) \equiv s_n = 2.30cm$*.*

1. *Considering both the rejection region method and the p-value method, should the null hypothesis* $H_0 : \mu_X = 250cm$ *be rejected against the alternative* $H_a : \mu_X \neq 250cm$ *at the significance level* $\alpha = 0.1$*?*

2. *Considering both the rejection region method and the p-value method, should the null hypothesis* $H_0 : \sigma_X^2 = 4$ *be rejected against of the alternative* $H_a : \sigma_X^2 > 4$ *at the significance level* $\alpha = 0.05$*? Calculate the probability* $\beta(5)$ *of a II type error.*

Solution.

1. Since X is Gaussian distributed with unknown mean and variance and the size of the sample is small, the statistic to be used is
$$\frac{\bar{X}_n - \mu_X}{S_n/\sqrt{n}}. \tag{1}$$
Consider testing the null hypothesis $H_0 : \mu_X = \mu_0$, where $\mu_0 \equiv 250cm$, against the alternative $H_1 : \mu_X \neq \mu_0$ at the significance level $\alpha = 0.1$. Under the assumption that the null hypothesis is true the statistic (1) with $\mu_X = \mu_0$ has the Student distribution with $n-1$ degree of freedom, that is
$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim T_{n-1}.$$
Moreover, the stucture of the alternative hypothesis calls for a rejection region of the form
$$R = \left\{T_{n-1} < t_{n-1,1-\alpha/2}\right\} \cup \left\{T_{n-1} > t_{n-1,\alpha/2}\right\}.$$
where,
$$t_{n-1,\alpha/2} = t_{8,0.05} = 1.860 \quad \text{and} \quad t_{n-1,1-\alpha/2} = -t_{n-1,\alpha/2} = -t_{8,0.05} = -1.860.$$
Hence,
$$R = (-\infty, -1.860) \cup (1.860, +\infty)$$
Computing the realization of the statistic, we have
$$\frac{\bar{X}_n(\omega) - \mu_X}{S_n(\omega)/\sqrt{n}} = \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{9}} = \frac{251.50 - 250}{2.30/3} = 1.96 \in R.$$

This implies a rejection of the null hypothesis in favor of the alternative. Adopting the p-value method, we recall that, on account of the alternative hypothesis, the p-value is the probability that the absolute value of the test statistic under the null assumption yields a value not less than the realization of the statistic. We will reject the null hypothesis when the computed p-value is smaller that the given significance level α. In symbols,

$$p = \mathbf{P}\left(|T_{n-1}| \geq \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{9}} \mid H_0 = T\right) = \mathbf{P}\left(T_8 \leq -1.96\right) + \mathbf{P}\left(T_8 \geq 1.96\right) = 0.087 < 0.1,$$

which confirms the rejection of the null hypothesis.

2. Since ve are interested in testing a hypothesis on the variance of X, which is normally distributed with unknown variance and the size of the sample is small, the statistic to be used is

$$\frac{(n-1)\,S_n^2}{\sigma_X^2}. \tag{2}$$

Consider testing the null hypothesis $H_0 : \sigma_X^2 = \sigma_0^2$, where $\sigma_0^2 \equiv 4$ against the alternative $H_1 : \sigma_X^2 > \sigma_0^2$ at the significance level $\alpha \equiv 0.05$. Under the assumption that the null hypothesis is true the statistic (2) with $\sigma_X^2 = \sigma_0^2$ has the chi-square distribution with $n-1$ degrees of freedom, that is

$$\frac{(n-1)\,S_n^2}{\sigma_0^2} \sim \chi_{n-1}^2.$$

Hence, the upper tail rejection region is given by

$$R = \left\{\chi_{n-1}^2 > \chi_{n-1,\alpha}^2\right\}$$

where, the upper $\alpha = 0.05$ critical value $\chi_{n-1,\alpha}^2$ of the chi-square distribution with $n-1=8$ degrees of freedom is given by

$$\chi_{8,0.05}^2 \simeq 15.51.$$

Hence,

$$R = (15.51, +\infty)$$

Computing the realization of the statistic, we have

$$\frac{(n-1)\,S_n^2(\omega)}{\sigma_X^2} = \frac{(n-1)\,s_n^2}{\sigma_0^2} = \frac{8 \cdot 2.30^2}{4} = 10.58 \notin R$$

Hence, the realization of the statistic does not belong to the rejection region. This implies that H_0 cannot be rejected.

In terms of p-value we have to compute

$$\mathbf{P}\left(\chi_{n-1}^2 \geq \frac{(n-1)\,s_n^2}{\sigma_0^2} \mid H_0 \text{ true}\right) = \mathbf{P}\left(\chi_8^2 \geq 10.58\right) = 1 - \mathbf{P}\left(\chi_8^2 \leq 10.58\right) = 0.227 > 0.05.$$

The p-value method confirms that H_0 cannot be rejected.

With regard to the evaluation of $\beta(5)$, setting $\sigma_1^2 \equiv 5$, we have

$$\begin{aligned}
\beta(5) &= \mathbf{P}\left(\text{accept } H_0 \mid \sigma_X^2 = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \frac{\sigma_1^2}{\sigma_1^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_1^2} \frac{\sigma_1^2}{\sigma_0^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_X^2} \frac{\sigma_1^2}{\sigma_0^2} \notin R\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2 \frac{\sigma_1^2}{\sigma_0^2} \notin R\right) \\
&= \mathbf{P}\left(\chi_8^2 \frac{\sigma_1^2}{\sigma_0^2} \leq 15.51\right) \\
&= \mathbf{P}\left(\chi_8^2 \leq 15.51 \cdot \frac{\sigma_0^2}{\sigma_1^2}\right) \\
&= \mathbf{P}\left(\chi_8^2 \leq 8.46\right) \\
&= 0.61.
\end{aligned}$$

.

Problem 12 *Let X be a Gaussian random variable with unknown mean μ and variance σ^2 representing a certain characteristic of a population and let $X_1, \ldots, X_n$ be a simple random sample of size n drawn from X. Assume that $n = 25$ and that the realizations $x_1, \ldots, x_{25}$ of the sample give an information summarized by*

$$\textstyle\sum_{k=1}^{25} x_k = 100 \quad \textit{and} \quad \sum_{k=1}^{25} x_k^2 = 560$$

1. *Considering both the rejection region method and the p-value method, should the null hypothesis $H_0 : \sigma^2 = 4$ be rejected against of the alternative $H_1 : \sigma^2 > 4$ with a significance level $\alpha = 0.05$? Calculate the probability $\beta(5)$ of a II type error.*

2. *Considering both the rejection region method and the p-value method, should the null hypothesis $H_0 : \sigma^2 = 4$ be rejected against of the alternative $H_1 : \sigma^2 \neq 4$ with a significance level $\alpha = 0.05$? Calculate the probability $\beta(5)$ of a II type error.*

Solution. Note that the given information allows the knowledge of the realization of sample variance, which is given by

$$
\begin{aligned}
s_n^2 &= \frac{1}{n-1}\sum_{k=1}^n \left(x_k - \frac{1}{n}\sum_{\ell=1}^n x_\ell\right)^2 \\
&= \frac{1}{n-1}\sum_{k=1}^n \left(x_k^2 - \frac{2}{n}x_k\sum_{\ell=1}^n x_\ell + \frac{1}{n^2}\left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{2}{n}\left(\sum_{k=1}^n x_k\right)\left(\sum_{\ell=1}^n x_\ell\right) + \frac{1}{n^2}\sum_{k=1}^n \left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{2}{n}\left(\sum_{k=1}^n x_k\right)^2 + \frac{1}{n^2}n\left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{1}{n}\left(\sum_{k=1}^n x_k\right)^2\right).
\end{aligned}
$$

Hence, in our case

$$
s_n^2 = \frac{1}{24}\left(560 - \frac{1}{25}100^2\right) = \frac{20}{3} = 6.67.
$$

Now, since ve are interested in testing a hypothesis on the variance of X, which is normally distributed, the statistic to be used is

$$
\frac{(n-1)\,S_n^2}{\sigma_X^2}. \tag{3}
$$

Consider testing the null hypothesis $H_0 : \sigma_X^2 = \sigma_0^2$, where $\sigma_0^2 \equiv 4$ against the alternative $H_1 : \sigma_X^2 > \sigma_0^2$ at the significance level $\alpha \equiv 0.05$. Under the assumption that the null hypothesis is true the statistic (3) with $\sigma_X^2 = \sigma_0^2$ has the chi-square distribution with $n-1$ degrees of freedom, that is

$$
\frac{(n-1)\,S_n^2}{\sigma_0^2} \sim \chi_{n-1}^2.
$$

1. By virtue of the above considerations, the upper tail rejection region is given by

   $$
   R = \left\{\chi_{n-1}^2 > \chi_{n-1,\alpha}^2\right\}
   $$

   where, the upper $\alpha = 0.05$ critical value of the chi-square distribution with $n-1 = 24$ degrees of freedom is given by

   $$
   \chi_{24,0.05}^2 = 36.415.
   $$

   Hence,

   $$
   R = (36.415, +\infty)
   $$

   Now, we have

   $$
   \frac{(n-1)\,S_n^2(\omega)}{\sigma_X^2} = \frac{(n-1)\,s_n^2}{\sigma_0^2} = \frac{24}{4}\cdot\frac{20}{3} = 40.00 \in R.
   $$

   Thus. the realization of the statistic belongs to the rejection region. This implies that H_0 is rejected.

   In terms of p-value we have to compute

   $$
   \mathbf{P}\left(\chi_{n-1}^2 \geq \frac{(n-1)\,s_n^2}{\sigma_0^2} \mid H_0 \text{ true}\right) = \mathbf{P}\left(\chi_{24}^2 \geq 40.00\right) = 1 - \mathbf{P}\left(\chi_{24}^2 \leq 40.00\right) = 0.0213 < 0.05.
   $$

   The p-value method confirms the rejection of H_0.

With regard to the evaluation of $\beta(5)$, $\sigma_0^2 \equiv 4$, $\sigma_1^2 \equiv 5$, and $n = 25$, we have that

$$\begin{aligned}
\beta(5) &= \mathbf{P}(\text{II type error}) = \mathbf{P}(\text{accept } H_0 \mid H_0 \text{ false}) = \mathbf{P}\left(\text{accept } H_0 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2}\frac{\sigma_1^2}{\sigma_1^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_1^2}\frac{\sigma_1^2}{\sigma_0^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2 \frac{\sigma_1^2}{\sigma_0^2} \notin R\right) \\
&= \mathbf{P}\left(\chi_{24}^2 \leq 36.415 \frac{\sigma_0^2}{\sigma_1^2}\right) \\
&= \mathbf{P}\left(\chi_{24}^2 \leq 36.415 \frac{4}{5}\right) \\
&= \mathbf{P}\left(\chi_{24}^2 \leq 29.132\right) \\
&= 0.7848.
\end{aligned}$$

2. In this case the rejection region is given by

$$R = \left\{\frac{(n-1)\, S_n^2}{\sigma^2} < \chi^2_{n-1,1-\alpha/2}\right\} \cup \left\{\frac{(n-1)\, S_n^2}{\sigma^2} > \chi^2_{n-1,\alpha/2}\right\}$$

where, lower $1-\alpha/2 = 0.975$ and the upper $\alpha/2 = 0.025$ critical values of the chi-square distribution with $n-1 = 24$ degrees of freedom are given by

$$\chi^2_{24,0.975} = 12.40 \quad \text{and} \quad \chi^2_{24,0.025} = 39.36.$$

In this case

$$\frac{(n-1)\, s_n^2}{\sigma^2} = \frac{24}{4} \cdot \frac{20}{3} = 40.00 \in R$$

belongs to the rejection region. Therefore, H_0 can be rejected against H_1. In terms of p-value we have to compute

$$2\mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma^2} \geq \frac{(n-1)\, s_n^2}{\sigma^2} \mid H_0 \text{ true}\right) = 2\mathbf{P}\left(\chi_{24}^2 \geq 40.00\right) = 2\left(1 - \mathbf{P}\left(\chi_{24}^2 \leq 40.00\right)\right) = 0.0428 < 0.050.$$

Hence, the p-value method confirms that H_0 should not be rejected.

With regard to the evaluation of $\beta(5)$, setting $\sigma_0^2 \equiv 4$, $\sigma_1^2 \equiv 5$, and $n = 25$, we have that

$$\begin{aligned}
\beta(5) &= \mathbf{P}(\text{II type error}) = \mathbf{P}(\text{accept } H_0 \mid H_0 \text{ false}) = \mathbf{P}\left(\text{accept } H_0 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)S_n^2}{\sigma_0^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.42 \leq \frac{(n-1)S_n^2}{\sigma_0^2}\frac{\sigma_1^2}{\sigma_1^2} \leq 39.36 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.42\frac{\sigma_0^2}{\sigma_1^2} \leq \frac{(n-1)S_n^2}{\sigma_1^2} \leq 39.36\frac{\sigma_0^2}{\sigma_1^2} \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.42\frac{4}{5} \leq \frac{(n-1)S_n^2}{\sigma^2} \leq 39.36\frac{4}{5}\right) \\
&= \mathbf{P}\left(9.94 \leq \chi_{n-1}^2 \leq 31.49\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2 \leq 31.49\right) - \mathbf{P}\left(\chi_{n-1}^2 \leq 9.94\right) \\
&= 0.860 - 0.005 \\
&= 0.855.
\end{aligned}$$