Performance Modeling
of Computer Systems and Networks

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The multi-server queue

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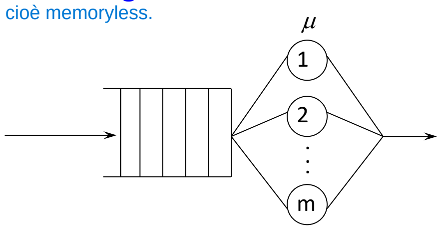
Erlang, 1917
M/M/m abstract scheduling

Arrivi e servizi esponenziali, cioè memoryless.

$$E(N_Q)_{Erlang}$$



ci sono 'n' job

$$p(n) = \begin{cases} \dfrac{1}{n!}(m\rho)^n p(0) & for\ n = 1,\ldots,m \quad \text{coda vuota, riempio solo i server} \\ \dfrac{m^m}{m!}\rho^n p(0) & for\ n > m \quad \text{oltre i server, inizio ad occupare anche la coda} \end{cases}$$

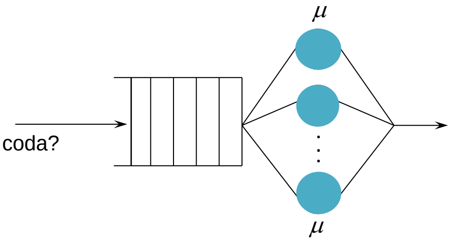
μ

$$p(0) = \left[\sum_{i=0}^{m-1}\frac{(m\rho)^i}{i!} + \frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$$

probabilità che il sistema sia vuoto

2

The *Erlang*-C formula



probabilità che quando un job arriva finisce in coda?
== probabilità che tutti i server siano pieni

$$P_Q \cong \Pr\{n \geq m\} = \sum_{n=m}^{\infty} p(n)$$

$$= \sum_{n=m}^{\infty} \frac{m^m}{m!} \rho^n p(0) = \frac{m^m}{m!} p(0) \sum_{n=m}^{\infty} \rho^n$$

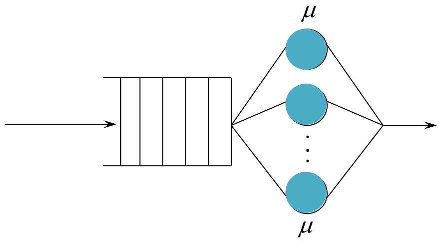
$$= \frac{m^m}{m!} p(0) \sum_{n=0}^{\infty} \rho^{n+m} = \frac{m^m}{m!} p(0) \rho^m \sum_{n=0}^{\infty} \rho^n$$

$$\frac{1}{1-\rho}$$

serie nota

Performance p.288

The *Erlang*-C formula



probabilità che siano tutti pieni,
dipende da 'm' e da 'rho'

$$P_Q = \frac{(m\rho)^m}{m!(1-\rho)} p(0)$$

$$E(N_Q)_{Erlang} = P_Q \frac{\rho}{1-\rho}$$

simile al caso servente singolo

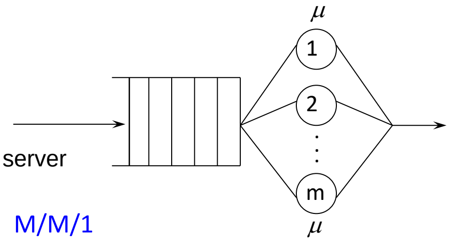
$$E(N_S) = P_Q \frac{\rho}{1-\rho} + m\rho$$

sommo quelli serviti mediamente

Little's law

$$E(T_Q) = \frac{E(N_Q)}{\lambda} \qquad E(T_Q) = P_Q \frac{\rho}{\lambda(1-\rho)} = \frac{P_Q E(S)}{1-\rho}$$

The *Erlang* formula



tempo medio per liberare uno qualsiasi dei server

M/M/m

$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho}$$

M/M/1

$$E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho} = \frac{E(S_{rem})}{1-\rho}$$

tempo per far sì che se ne liberi uno, devo metterci lei!

$$E(S) = \frac{E(S_i)}{m}$$

Infatti voglio che se ne liberi UNO qualsiasi, non uno specifico.

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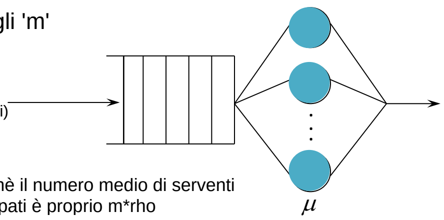
The Multi Server Queue

numero server occupati tra gli 'm'

c ≅ busy servers



(quando non sono tutti pieni) + (quando sono tutti pieni)

la media dei server occupati è: $E(c) = \sum_{n=0}^{m-1} np(n) + \sum_{n=m}^{\infty} mp(n) = m\rho$

perchè il numero medio di serventi occupati è proprio m*rho

definizione di Pq della C-Erlang (prob tutti occupati)

$$\rho = \sum_{n=0}^{m-1} \frac{n}{m} p(n) + \sum_{n=m}^{\infty} p(n) = \sum_{n=0}^{m-1} \frac{n}{m} p(n) + P_Q \Rightarrow \rho \geq P_Q$$

$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho} \leq E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho}$$

dato un certo carico, lambda e mu, la probabilità che siano tutti pieni è più piccola della probabilità che sia pieno solo uno.

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Nel multiserver ho più "sedie" su cui far sedere i job, se devo ottimizzare l'attesa, conviene distribuire la capacità, avere ad esempio 10 server meno potenti che uno 10 volte più potente, perchè dal punto di vista dell'attesa rho>Pq
Se devo minimizzare tempi di attesa è meglio la soluzione distribuita!

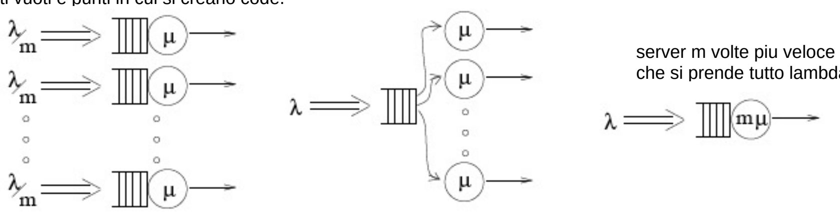
dipende sempre dal contesto!

Server Organizations

m server disgiunti che si dividono il traffico, potrei avere punti vuoti e punti in cui si creano code.

m server divisi ma con traffico convogliato tutto insieme, quindi non posso avere punti vuoti e punti con code

server m volte piu veloce che si prende tutto lambda.



$$\rho = \frac{\lambda}{m\mu}$$

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Communication systems

1
2
3
m

communication line

independent Poisson packet streams

each with an arrival rate of λ/m packets per second

the transmission time for each packet Exponential(1/μ)

Frequency-division Multiplexing

Statistical Multiplexing

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Communication systems

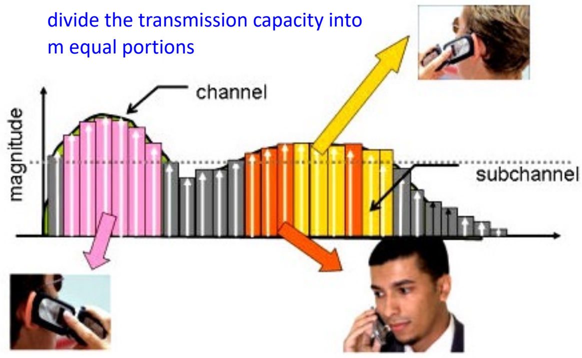
Frequency-division Multiplexing

divide the transmission capacity into
m equal portions

1
2
3

m

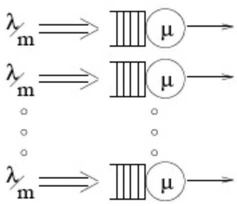
separated streams



tengo gli 'm' flussi separati quando si divide la capacità trasmissiva in 'm' porzioni uguali.

Communication systems

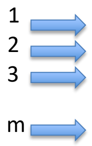
Frequency-division Multiplexing



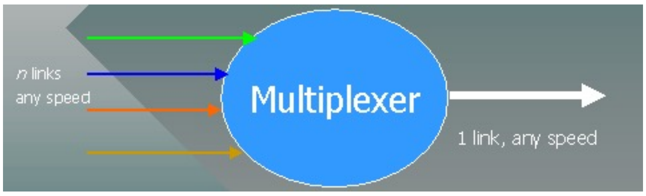
Communication systems

Statistical Multiplexing

keep the transmission capacity as a whole



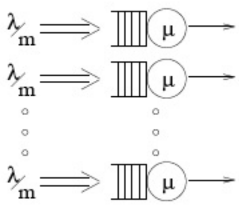
merge into a single stream



Qui gli 'm' flussi vengono convogliati in un unico flusso, fa un 'merge' e li manda su un unico link.

Communication systems

Frequency-division Multiplexing



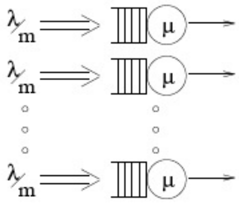
Statistical multiplexing

How do the two approaches compare with respect to mean response time?

p.289 performance

Communication systems

Frequency-division Multiplexing



Statistical multiplexing

$$E(T_S) = \frac{\rho E(S)}{1-\rho} + E(S) = \frac{E(S)}{1-\rho}$$

$$E(T_S) = \frac{1}{\mu\left(1 - \frac{\lambda}{\mu}\right)} = \frac{1}{\mu - \lambda}$$

M/M/1

$\lambda \quad \mu$

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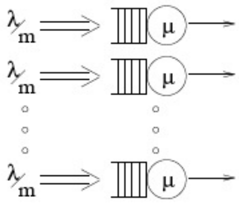
Communication systems

Frequency-division Multiplexing



Statistical multiplexing

$$E(T_S)^{FDM} = \frac{1}{\mu - \frac{\lambda}{m}} = \frac{m}{m\mu - \lambda}$$

$$E(T_S)^{SM} = \frac{1}{m\mu - \lambda}$$

FDM shows a response time *m* times greater then for *SM* !

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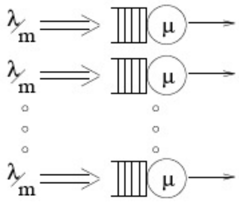
FDM però garantisce a ciascun flusso una specifica frequenza di servizio!

Communication systems

Frequency-division Multiplexing



Statistical multiplexing

1 QoS guarantee for each stream: a specific service rate to each stream

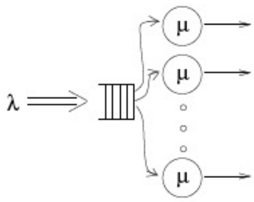
No QoS guarantee

se avessi Poisson, riotterrei Poisson

If the original m streams were very regular (not Poisson), i.e., they were much less variable than Poisson, by merging them, we introduce lots of variability into the arrival stream.

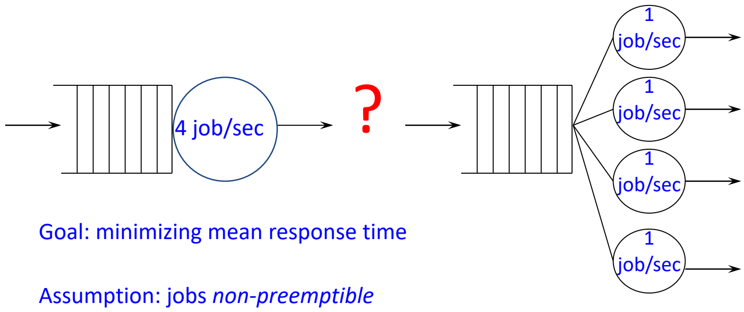
2 This leads to problems if the application requires a low variability in delay, e.g., voice or video.

Server Organizations



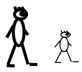
Modelling power: Design tool

Design Example 4: One machine or many?



Goal: minimizing mean response time

Assumption: jobs *non-preemptible*
each job must be run to completion

high variability 

Modelling power: Design tool

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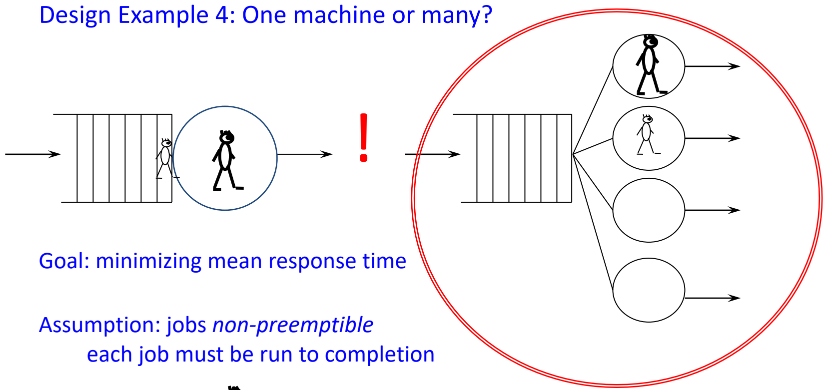

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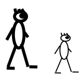
high variability 

Modelling power: Design tool

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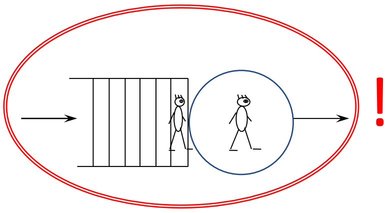
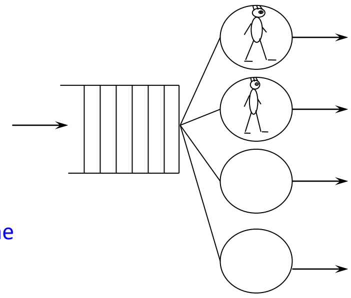
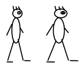




Goal: minimizing mean response time

Assumption: jobs *non-preemptible*
each job must be run to completion

low variability 

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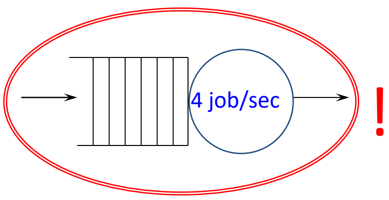
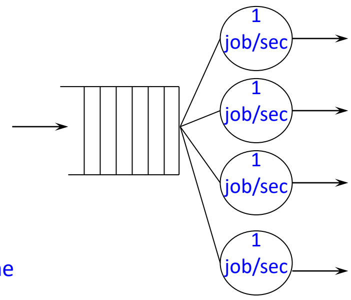
Modelling power: Design tool

Design Example 4: One machine or many?





Goal: minimizing mean response time

Assumption: jobs *preemptible*
each job can be stopped and restarted where they left off

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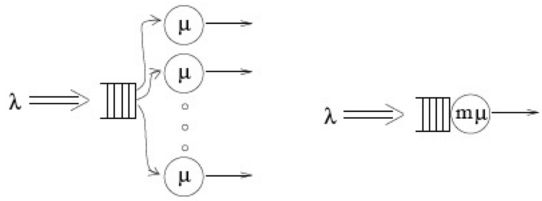
Server Organizations

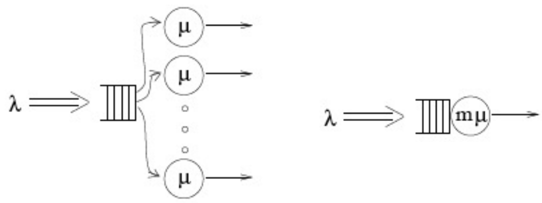


$$E(T_Q)_{Erlang} = \frac{P_Q E(S)}{1-\rho} \qquad E(T_Q)_{KP} = \frac{\rho E(S)}{1-\rho}$$

$\rho \ge P_Q$

from the waiting time perspective the distributed capacity solution produces an improvement in the user perceived QoS

Server Organizations

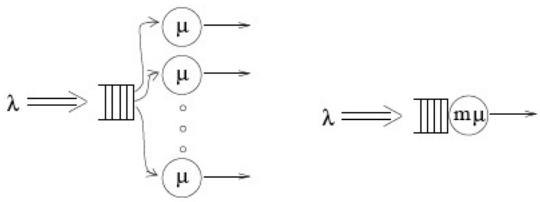


What about the response time perspective??

$$E(T_S)_{Erlang} = \frac{P_Q E(S)}{1-\rho} + E(S_i) \qquad E(T_S)_{KP} = \frac{\rho E(S)}{1-\rho} + E(S)$$

$$E(S_i) = \frac{1}{\mu} = m\frac{1}{m\mu} = mE(S)$$

Server Organizations



What about the response time perspective??

$$E(T_S)_{Erlang} = \frac{P_Q E(S)}{1-\rho} + mE(S)$$

Decreasing less than linear $\wedge$ $\vee$ linear growth

$$E(T_S)_{KP} = \frac{\rho E(S)}{1-\rho} + E(S)$$

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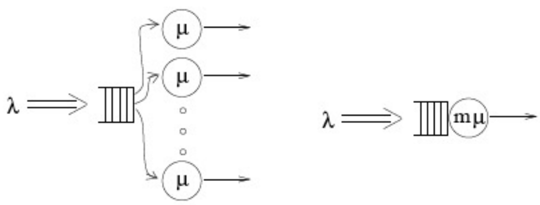
Server Organizations



Performance goal:

Waiting time perspective

Distributed capacity

Response time perspective

$\rho \to 0$ distrib. capac. gives an *m* times slower organization

$\rho \to 1$ approximately the same response time

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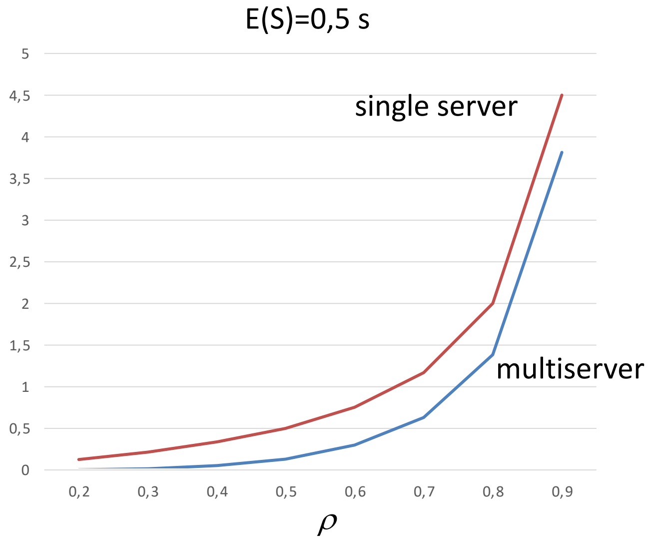
Waiting time perspective



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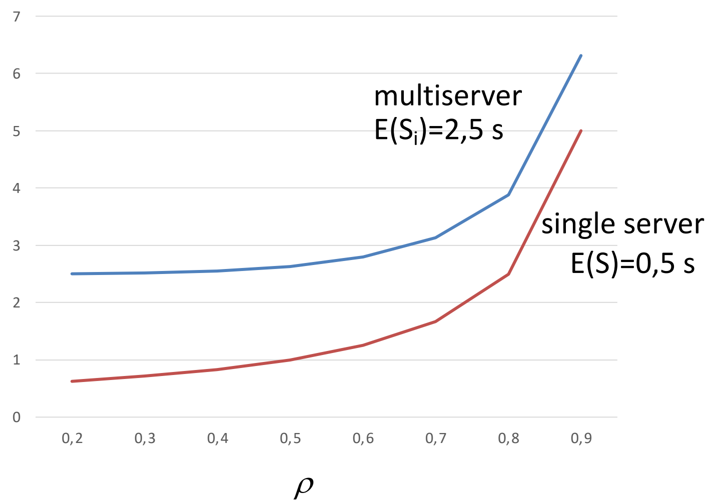
Response time perspective



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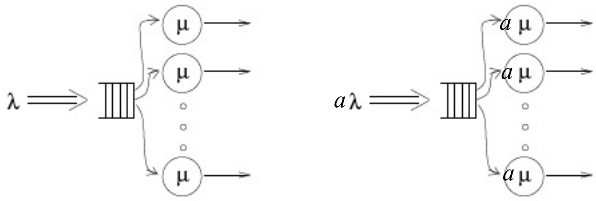
Scaling factor

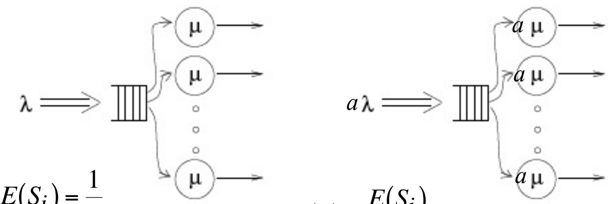


What about waiting and response time?

$$\rho = \frac{\lambda}{m\mu} \qquad\qquad \rho = \frac{a\lambda}{ma\mu} = \frac{\lambda}{m\mu}$$

$$E(S_i) = \frac{1}{\mu} \qquad\qquad E(S_i) = \frac{1}{a\mu} \qquad E(S) = \frac{E(S_i)}{m}$$

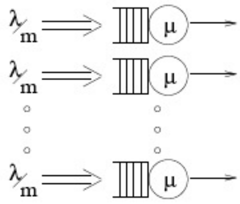
Scaling factor

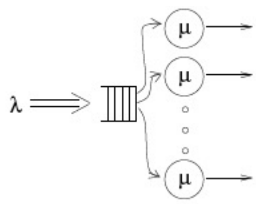


$$E(S_i) = \frac{1}{\mu} \qquad E(S) = \frac{E(S_i)}{m} \qquad E(S_i) = \frac{1}{a\mu}$$

Mean waiting time

$$E(T_Q)_{m,a} = \frac{P_Q E(S)_{m,a}}{1-\rho} = \frac{P_Q}{ma\mu(1-\rho)} = \frac{1}{a}\frac{P_Q E(S)m,1}{(1-\rho)} = \frac{1}{a}E(T_Q)_{m,1}$$





$\lambda = 4$ j/s,   m=4, μ=1.5 j/s E(S)=0.666667 s

$$E(S) = \frac{E(S_i)}{4} = 0.1667$$

$$\rho = 0.666667$$

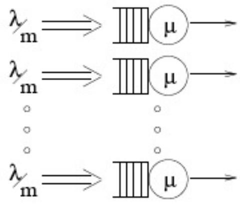
$$E(T_S) = \frac{1}{\mu - \lambda} = 2\ s$$

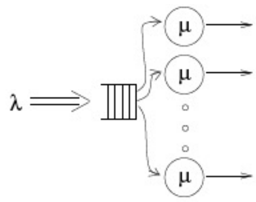
$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336$$

$$\rho = 0.666667$$

$$p(0) = \left[\sum_{i=0}^{3} \frac{(4\rho)^i}{i!} + \frac{(4\rho)^4}{4!(1-\rho)}\right]^{-1}$$

$$= \left[1 + 4\rho + \frac{(4\rho)^2}{2} + \frac{(4\rho)^3}{6} + \frac{(4\rho)^4}{24(1-\rho)}\right]^{-1} = 0.059857$$





$\lambda = 4$ j/s,   m=4, μ=1.5 j/s E(S)=0.666667 s

$$E(S) = \frac{E(S_i)}{4} = 0.1667$$

$E(S_i)$=0.666667 s

$$\rho = 0.666667$$

$$E(T_S) = \frac{1}{\mu - \lambda} = 2\ s$$

$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336$$

$$\rho = 0.666667$$

$$P_Q = \frac{(4\rho)^4}{4!(1-\rho)} p(0) = 0.37847$$

$$E(T_S) = \frac{P_Q E(S)}{1-\rho} + E(S_i) = 0.855992$$

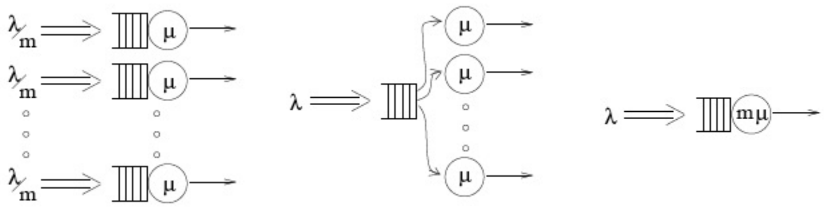
$$E(T_Q) = 0.189292$$

$\lambda = 4$ j/s, $m\mu = 4\text{x}1.5 = 6$ j/s E(S)=0.166667 s

$\rho = 0.666667$

$$E(T_S) = \frac{1}{m\mu - \lambda} = 0.5$$

$$E(T_Q) = 0.3334$$



$\rho = 0.666667$

$$E(T_S) = \frac{1}{\mu - \lambda} = 2 \qquad E(T_S) = 0.855992 \qquad \boxed{E(T_S) = \frac{1}{\mu - \lambda} = 0.5\ s}$$

$$E(T_Q) = \frac{\rho E(S)}{1-\rho} = 1.3336 \qquad \boxed{E(T_Q) = 0.189292} \qquad E(T_Q) = 0.3334$$

Esercizio proposto: $\rho = 0.533334 \quad \rho = 0.8$