DISCRETE-EVENT SIMULATION:
A FIRST COURSE

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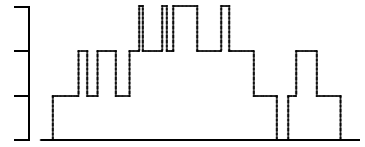*757–221–2034*
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Brief Contents

Contents

Preface

This book presents an introduction to computational and mathematical techniques for *modeling*, *simulating*, and *analyzing* the performance of various systems using simulation. For the most part the system models studied are: *stochastic* (at least some of the system state variables are random); *dynamic* (the time evolution of the system state variables is important); and *discrete-event* (significant changes in system state variables are associated with events that occur at discrete time instances only). Therefore, the book represents an introduction to what is commonly known as *discrete-event simulation*. There is also a significant, but secondary, emphasis on *Monte Carlo simulation* and its relation to *static* stochastic systems. Deterministic systems, static or dynamic, and stochastic dynamic systems that evolve continuously in time are not considered in any significant way.

Discrete-event simulation is a multi-disciplinary activity studied and applied by students of applied mathematics, computer science, industrial engineering, management science, operations research, statistics, and various hybrid versions of these disciplines found in schools of engineering, business, management, and economics. As it is presented in this book, discrete-event simulation is a computational science — a mix of theory and experimentation with a computer as the primary piece of laboratory equipment. In other words, discrete-event simulation is a form of computer-aided model building and problem solving. The goal is insight, a better understanding of how systems operate and respond to change.

Prerequisites

In terms of formal academic background, we presume the reader has taken the undergraduate equivalent of the first several courses in a conventional computer science program, two calculus courses and a course in probability or statistics. In more detail, and in decreasing order of importance, these prerequisites are as follows.

Computer Science — readers should be able to program in a contemporary high-level programming language, for example C, C++, Java, Pascal, or Ada, and have a working knowledge of algorithm complexity. Because the development of most discrete-event simulation programs necessarily involves an application of queues and event lists, some familiarity with dynamic data structures is prerequisite, as is the ability to program in a language that naturally supports such things. By design, the computer science prerequisite is strong. We firmly believe that the best way to learn about discrete-event simulation is by hands-on model building. We consistently advocate a structured approach wherein a model is constructed at three levels — conceptual, specification, and computational. At the computational level the model is built as a computer program; we believe that this construction is best done with a standard and widely available general-purpose high-level programming language, using already-familiar (editor, compiler, debugger, etc.) tools.*

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* The alternative to using a general-purpose high-level programming language is to use a (proprietary, generally unfamiliar and potentially expensive) special-purpose simulation language. In some applications this may be a superior alternative, particularly if the simulation language is already familiar (and paid for); see Chapter 1 and Appendix A for more discussion of this trade-off.

Calculus — readers should be able to do single-variable differential and integral calculus. Although still relatively strong, the calculus prerequisite is as weak as possible. That is, for example, we have generally avoided the use of multi-variate calculus; however, single-variable integration and differentiation is used as appropriate in the discussion of continuous random variables, for example. In addition, we freely use the analogous, but more computationally intuitive, discrete mathematics of summation and differencing in the discussion of discrete random variable. By design, we maintain a balance between continuous and discrete stochastic models, generally using the more easily understood, but less common, discrete (non-calculus) techniques to provide motivation for the corresponding continuous (calculus) techniques.

Probability — readers should have a working knowledge of probability including random variables, expected values, and conditioning. Some knowledge of statistics is also desirable, but not necessary. Those statistical tools most useful in discrete-event simulation are developed as needed. Because of the organization of the material, our classroom experience has been that students with strength in the computer science and calculus prerequisites only can use this book to develop a valid intuition about things stochastic. In this way the reader can learn about discrete-event simulation and, if necessary, also establish the basis for a later formal study of probability and statistics. That study is important for serious students of discrete-event simulation because without the appropriate background a student is unlikely to ever be proficient at modeling and analyzing the performance of stochastic systems.

Organization and Style

The book has ten chapters, organized into 41 sections. The shortest path through the text could *exclude* the 15 optional Sections 2.4, 2.5, 4.4, 5.3, 6.4, 6.5, 7.4, 7.5, 7.6, 8.5, 9.3, and 10.1–10.4. All the 26 remaining core sections are consistent with a 75-minute classroom lecture and together they define a traditional one-semester, three credit-hour course.* Generally, the optional sections in the first nine chapters are also consistent with a 75-minute presentation and so can be used in a classroom setting as supplemental lectures. Each section in the tenth chapter provides relatively detailed specifications for a variety of discrete-event simulation projects designed to integrate much of the core material. In addition, there are seven appendices that provide background or reference material.

In a traditional one-semester, three credit-hour course there may not be time to cover more than the 26 core sections. In a four credit-hour course there will be time to cover the core material, some of the optional sections (or appendices) and, if appropriate, structure the course around the projects in the tenth chapter as a culminating activity. Similarly, some optional sections can be covered in a three credit-hour course, provided student background is sufficient to warrant not devoting classroom time to some of the core sections.

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* Because of its multi-disciplinary nature, there is not universal agreement on what constitutes the academic core of discrete-event simulation. It is clear, however, that the core is large, sufficiently so that we have not attempted to achieve comprehensive coverage. Instead, the core sections in the first nine chapters provide a self-contained, although limited, first course in discrete-event simulation.

The book is organized consistent with a dual philosophy: (*i*) begin to model, simulate, and analyze simple-but-representative systems as soon as possible; (*ii*) whenever possible, encourage the experimental exploration and self-discovery of theoretical results before their formal presentation. As an example of (*i*), detailed trace-driven computational models of a single-server queue and a simple inventory system are developed in Chapter 1, then used to motivate the need for the random number generator developed in Chapter 2. The random number generator is used to convert the two trace-driven models into stochastic models that can be used to study both transient and steady-state system performance in Chapter 3. Similarly, as an example of (*ii*), an experimental investigation of sampling uncertainty and interval estimation is motivated in Chapters 2 and 3. A formal treatment of this topic is presented in Chapter 8.

We have tried to achieve a writing style that emphasizes *concepts* and *insight* without sacrificing rigor. Generally, formalism and proofs are not emphasized. When appropriate, however, definitions and theorems (most with proofs) are provided, particularly if their omission could create a sense of ambiguity that might impede a reader's ability to understand concepts and develop insights.

Software

Software is an integral part of the book. We provide this software as source code for several reasons. Because a computer program is the logical product of the three-level approach to model building we advocate, an introductory discrete-event simulation book based on this philosophy would be deficient if a representative sampling of such programs were not presented. Moreover, many important exercises in the book are based on the idea of extending a system model at the computational level; these exercises are conditioned on access to the source code. The software consists of many complete discrete-event programs and a variety of libraries for random number generation, random variate generation, statistical data analysis, priority queue access, and event list processing.

The software has been translated from it original development in Turbo Pascal to ANSI C with *units* converted to C libraries. Although experienced C programmers will no doubt recognize the Pascal heritage, the result is readable, structured, portable, and reasonably efficient ANSI C source code.*

Exercises

There are exercises associated with each chapter and some appendices — about 400 in all. They are an important part of the book, designed to reinforce and extend previous material and encourage computational experimentation. Some exercises are routine, others are more advanced; the advanced exercises are denoted with an '**a**' superscript. Some of the advanced exercises are sufficiently challenging and comprehensive to merit consideration as (out-of-class) exam questions or projects. Serious readers are encouraged to work a representative sample of the routine exercises and, time permitting, a large portion of the advanced exercises.

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* All the programs and libraries compile successfully, without warnings, using the GNU C compiler `gcc` with the `-ansi -Wall` switches set. Alternatively, the C++ compiler `g++` can be used instead.

Consistent with the computational philosophy of the book, a significant number of exercises require some computer programming. If required, the amount of programming is usually small for the routine exercises, less so for the advanced exercises. For some of the advanced exercises the amount of programming may be significant. In most cases when programming is required, the reader is aided by access to source code for the programs and related software tools the book provides.

Our purpose is to give an introductory, intuitive development of algorithms and methods used in Monte Carlo and discrete-event simulation modeling. More comprehensive treatments are given in the textbooks referenced throughout the text.

Acknowledgments

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CHAPTER 1

MODELS

The modeling approach in this book is based on the use of a general-purpose programming language for model implementation at the computational level. The alternative approach is to use a special-purpose simulation language; for a survey of several such languages, see Appendix A.

Sections

This chapter presents an introduction to discrete-event simulation, with an emphasis on model building. The focus in Section 1.1 is on the multiple steps required to construct a discrete-event simulation model. By design, the discussion in this section is at a high level of generality, with few details. In contrast, two specific discrete-event system models are presented in Sections 1.2 and 1.3, with a significant amount of detail. A single-server queue model is presented in Section 1.2 and a simple inventory system model is presented in Section 1.3. Both of these models are of fundamental importance because they serve as a basis for a significant amount of material in later chapters.

Although the material in this chapter also can be found in other modeling and simulation texts, there is a relatively novel emphasis on model building at the conceptual, specification and computational levels. Moreover, in Sections 1.2 and 1.3 there is a significant amount of notation, terminology, and computational philosophy which extends to subsequent chapters. For these reasons, this chapter is important to any reader of the book, even those already familiar with the rudiments of discrete-event simulation.

This book is the basis for a first course on *discrete-event simulation.* That is, the book provides an introduction to computational and mathematical techniques for *modeling, simulating* and *analyzing* the performance of discrete-event stochastic systems. By definition, the nature of discrete-event simulation is that one does not actually experiment with or modify an actual system. Instead, one develops and then works with a discrete-event simulation *model*. Consistent with that observation, the emphasis in this first chapter is on model building.

1.1.1 MODEL CHARACTERIZATION

Briefly, a discrete-event simulation model is both stochastic and dynamic with the special discrete-event property that the system state variables change value at discrete times only (see Definition 1.1.1). But what does that mean?

A system model is *deterministic* or *stochastic*. A deterministic system model has no stochastic (random) components. For example, provided the conveyor belt and machine never fail, a model of a constant velocity conveyor belt feeding parts to a machine with a constant service time is deterministic. At some level of detail, however, all systems have some stochastic components; machines fail, people are not robots, service requests occur at random, etc. An attractive feature of discrete-event simulation is that stochastic components can be accommodated, usually without a dramatic increase in the complexity of the system model at the computational level.

A system model is *static* or *dynamic*. A static system model is one in which time is not a significant variable. For example, if three million people play the state lottery this week, what is the probability that there will be at least one winner? A simulation program written to answer this question should be based on a static model; when during the week these three million people place their bets is not significant. If, however, we are interested in the probability of no winners in the next four weeks, then this model needs to be dynamic. That is, experience has revealed that each week there are no winners, the number of players in the following week increases (because the pot grows). When this happens, a dynamic system model must be used because the probability of at least one winner will increase as the number of players increases. The passage of time always plays a significant role in dynamic models.

A dynamic system model is *continuous* or *discrete*. Most of the traditional dynamic systems studied in classical mechanics have state variables that evolve continuously. A particle moving in a gravitational field, an oscillating pendulum, or a block sliding on an inclined plane are examples. In each of these cases the motion is characterized by one or more differential equations which model the continuous time evolution of the system. In contrast, the kinds of queuing, machine repair and inventory systems studied in this book are discrete because the state of the system is a *piecewise-constant* function of time. For example, the number of jobs in a queuing system is a natural state variable that only changes value at those discrete times when a job arrives (to be served) or departs (after being served).

The characterization of a system model can be summarized by a tree diagram that starts at the system model root and steps left or right at each of the three levels, as illustrated in Figure 1.1.1.
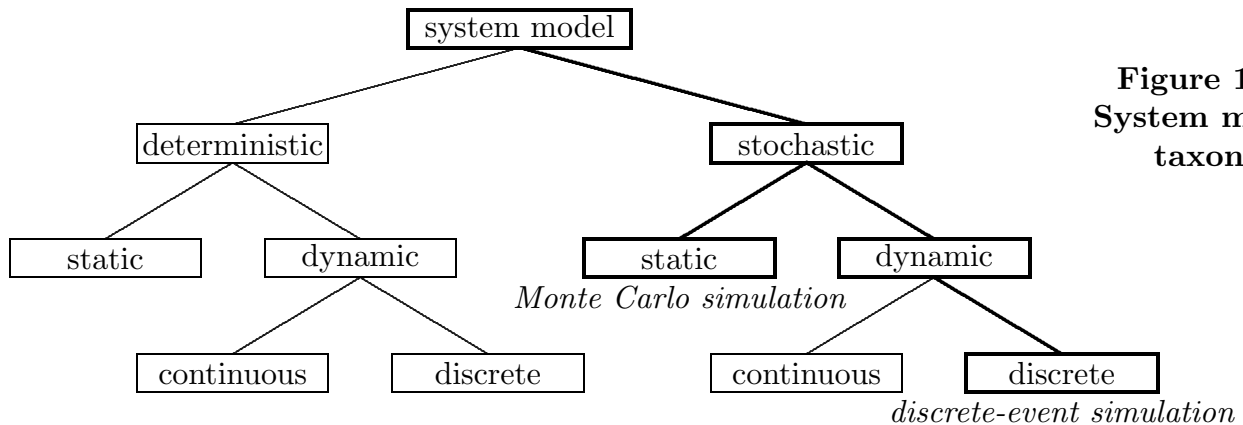
Figure 1.1.1. System model taxonomy.

As summarized by Definition 1.1.1, the system model characterized by the right-most branch of this tree is of primary interest in this book.

Definition 1.1.1   A *discrete-event simulation model* is defined by three attributes:

- *stochastic* — at least some of the system state variables are random;

- *dynamic* — the time evolution of the system state variables is important;

- *discrete-event* — significant changes in the system state variables are associated with events that occur at discrete time instances only.

One of the other five branches of the system model tree is of significant, but secondary, interest in this book. A *Monte Carlo simulation model* is stochastic and static — at least some of the system state variables are random, but the time evolution (if any) of the system state variables is not important. Accordingly, the issue of whether time flows continuously or discretely is not relevant.

Because of space constraints, the remaining four branches of the system model tree are not considered. That is, there is no material about deterministic systems, static or dynamic, or about stochastic dynamic systems that evolve continuously in time.

1.1.2   MODEL DEVELOPMENT

It is naive to think that the process of developing a discrete-event simulation model can be reduced to a simple sequential algorithm. As an instructional device, however, it is useful to consider two algorithms that outline, at a high level, how to develop a discrete-event simulation model (Algorithm 1.1.1) and then conduct a discrete-event simulation study (Algorithm 1.1.2).

Algorithm 1.1.1   If done well, a typical discrete-event simulation model will be developed consistent with the following six steps. Steps (2) through (6) are typically iterated, perhaps many times, until a (hopefully) valid computational model, a computer program, has been developed.

(1) Determine the *goals* and *objectives* of the analysis once a system of interest has been identified. These goals and objectives are often phrased as simple Boolean decisions (e.g., should an additional queuing network service node be added) or numeric decisions (e.g., how many parallel servers are necessary to provide satisfactory performance in a multi-server queuing system). Without specific goals and objectives, the remaining steps lack meaning.

(2) Build a *conceptual* model of the system based on (1). What are the *state variables*, how are they interrelated and to what extent are they dynamic? How comprehensive should the model be? Which state variables are important; which have such a negligible effect that they can be ignored? This is an intellectually challenging but rewarding activity that should not be avoided just because it is hard to do.

(3) Convert the conceptual model into a *specification* model. If this step is done well, the remaining steps are made much easier. If instead this step is done poorly (or not at all) the remaining steps are probably a waste of time. This step typically involves collecting and statistically analyzing data to provide the input models that drive the simulation. In the absence of such data, the input models must be constructed in an ad hoc manner using stochastic models believed to be representative.

(4) Turn the specification model into a *computational* model, a computer program. At this point, a fundamental choice must be made — to use a general-purpose programming language or a special-purpose simulation language. For some this is a religious issue not subject to rational debate.

(5) *Verify.* As with all computer programs, the computational model should be consistent with the specification model — did we implement the computational model correctly? This verification step is not the same as the next step.

(6) *Validate.* Is the computational model consistent with the system being analyzed — did we build the right model? Because the purpose of simulation is insight, some (including the authors) would argue that the *act* of developing the discrete-event simulation model — steps (2), (3), and (4) — is frequently as important as the tangible *product*. However, given the blind faith many people place in any computer generated output the validity of a discrete-event simulation model is always fundamentally important. One popular non-statistical, Turing-like technique for model validation is to place actual system output alongside similarly formatted output from the computational model. This output is then examined by an expert familiar with the system. Model validation is indicated if the expert is not able to determine which is the model output and which is the real thing. Interactive computer graphics (animation) can be very valuable during the verification and validation steps.

Example 1.1.1 The following *machine shop* model helps illustrate the six steps in Algorithm 1.1.1. A new machine shop has 150 identical machines; each operates continuously, 8 hours per day, 250 days per year until failure. Each machine operates independently of all the others. As machines fail they are repaired, in the order in which they fail, by a service technician. As soon as a failed machine is repaired, it is put back into operation. Each machine produces a net income of $20 per hour of operation. All service technicians are hired at once, for 2 years, at the beginning of the 2-year period with an annual salary expense of $52,000. Because of vacations, each service technician only works 230 8-hour days per year. By agreement, vacations are coordinated to maximize the number of service technicians on duty each day. How many service technicians should be hired?

(1) The objective seems clear — to find the number of service technicians for which the profit is maximized. One extreme solution is to hire one technician for each machine; this produces a huge service technician overhead but maximizes income by minimizing the amount of machine down-time. The other extreme solution is to hire just one technician; this minimizes overhead at the potential expense of large down-times and associated loss of income. In this case, neither extreme is close to optimal for typical failure and repair times.

(2) A reasonable conceptual model for this system can be expressed in terms of the state of each machine (failed or operational) and each service technician (busy or idle). These state variables provide a high-level description of the system at any time.

(3) To develop a specification model, more information is needed. Machine failures are random events; what is known (or can be assumed) about the time between failures for these machines? The time to repair a machine is also random; what, for example, is the distribution of the repair time? In addition, to develop the associated specification model some systematic method must be devised to simulate the time evolution of the system state variables.

(4) The computational model will likely include a simulation clock data structure to keep track of the current simulation time, a queue of failed machines and a queue of available service technicians. Also, to characterize the performance of the system, there will be statistics gathering data structures and associated procedures. The primary statistic of interest here is the total profit associated with the machine shop.

(5) The computational model must be verified, usually by extensive testing. Verification is a software engineering activity made easier if the model is developed in a contemporary programming environment.

(6) The validation step is used to see if the verified computational model is a reasonable approximation of the machine shop. If the machine shop is already operational, the basis for comparison is clear. If, however, the machine shop is not yet operational, validation is based primarily on consistency checks. If the number of technicians is increased, does the time-averaged number of failed machines go down; if the average service time is increased, does the time-averaged number of failed machines go up?

System Diagrams

Particularly at the conceptual level, the process of model development can be facilitated by drawing system diagrams. Indeed, when asked to explain a system, our experience is that, instinctively, many people begin by drawing a system diagram. For example, consider this system diagram of the machine shop model in Example 1.1.1.

Figure 1.1.2.
Machine shop
system diagram.

service technicians
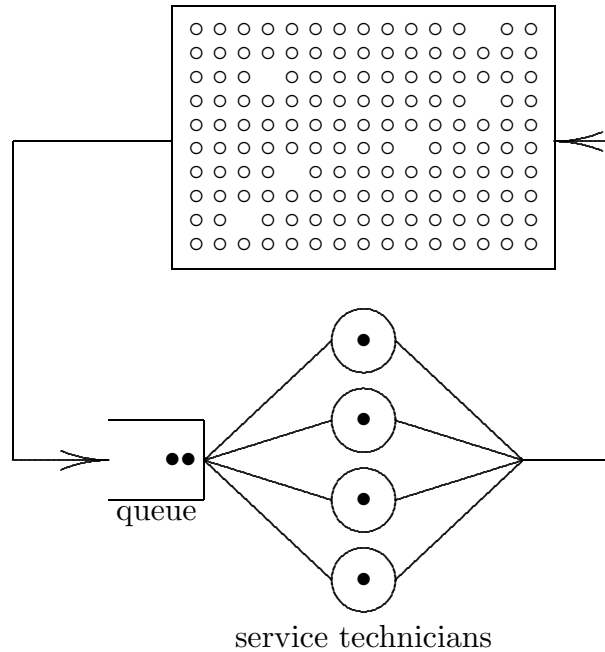
The box at the top of Figure 1.1.2 represents the pool of machines. The composite object at the bottom of the figure represents the four service technicians and an associated single queue. Operational machines are denoted with a ∘ and broken machines with a •. Conceptually, as machines break they change their state from operational (∘) to broken (•) and move along the arc on the left from the box at the top of the figure to the queue at the bottom of the figure. From the queue, a broken machine begins to be repaired as a service technician becomes available. As each broken machine is repaired, its state is changed to operational and the machine moves along the arc on the right, back to the pool of operational machines.*

As time evolves, there is a continual counter-clockwise circulation of machines from the pool at the top of Figure 1.1.2 to the service technicians at the bottom of the figure, and then back again. At the "snapshot" instant illustrated, there are six broken machines; four of these are being repaired and the other two are waiting in the queue for a service technician to become available.

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* The movement of the machines to the servers is conceptual, as is the queue. In practice, the servers would move to the machines and there would not be a physical queue of broken machines.

In general, the application of Algorithm 1.1.1 should be guided by the following observations.

- Throughout the development process, the operative principle should always be to make every discrete-event simulation model as simple as possible, but never simpler. The goal is to capture only the relevant characteristics of the system. The dual temptations of (1) ignoring relevant characteristics or (2) including characteristics that are extraneous to the goals of the model, should be avoided.

- The actual development of a complex discrete-event simulation model will not be as sequential as Algorithm 1.1.1 suggests, particularly if the development is a team activity in which case some steps will surely be worked in parallel. The different characteristics of each step should always be kept clearly in mind avoiding, for example, the natural temptation to merge steps (5) and (6).

- There is an unfortunate tendency on the part of many to largely skip over steps (1), (2), and (3), jumping rapidly to step (4). Skipping these first three steps is an approach to discrete-event simulation virtually certain to produce large, inefficient, unstructured computational models that cannot be validated. Discrete-event simulation models should *not* be developed by those who like to think a little and then program a lot.

1.1.3   SIMULATION STUDIES

Algorithm 1.1.2   Following the successful application of Algorithm 1.1.1, use of the resulting computational model (computer program) involves the following steps.

(7) Design the simulation experiments. This is not as easy as it may seem. If there are a significant number of system parameters, each with several possible values of interest, then the combinatoric possibilities to be studied make this step a real challenge.

(8) Make production runs. The runs should be made systematically, with the value of all initial conditions and input parameters recorded along with the corresponding statistical output.

(9) Analyze the simulation results. The analysis of the simulation output is statistical in nature because discrete-event simulation models have stochastic (random) components. The most common statistical analysis tools (means, standard deviations, percentiles, histograms, correlations, etc.) will be developed in later chapters.

(10) Make decisions. Hopefully the results of step (9) will lead to decisions that result in actions taken. If so, the extent to which the computational model correctly predicted the outcome of these actions is always of great interest, particularly if the model is to be further refined in the future.

(11) Document the results. If you really did gain insight, summarize it in terms of specific observations and conjectures. If not, why did you fail? Good documentation facilitates the development (or avoidance) of subsequent similar system models.

Example 1.1.2   As a continuation of Example 1.1.1, consider the application of Algorithm 1.1.2 to a verified and validated machine shop model.

(7) Since the objective of the model is to determine the optimal number of service technicians to hire to maximize profit, the number of technicians is the primary system parameter to be varied from one simulation run to the next. Other issues also contribute to the design of the simulation experiments. What are the initial conditions for the model (e.g., are all machines initially operational)? For a fixed number of service technicians, how many replications are required to reduce the natural sampling variability in the output statistics to an acceptable level?

(8) If many production runs are made, management of the output results becomes an issue. A discrete-event simulation study can produce *a lot* of output files which consume large amounts of disk space if not properly managed. Avoid the temptation to archive "raw data" (e.g., a detailed time history of simulated machine failures). If this kind of data is needed in the future, it can always be reproduced. Indeed, the ability to reproduce previous results *exactly* is an important feature which distinguishes discrete-event simulation from other, more traditional, experimental sciences.

(9) The statistical analysis of simulation output often is more difficult than classical statistical analysis, where observations are assumed to be *independent*. In particular, time-sequenced simulation-generated observations are often correlated with one another, making the analysis of such data a challenge. If the current number of failed machines is observed each hour, for example, consecutive observations will be found to be significantly positively correlated. A statistical analysis of these observations based on the (false) assumption of independence may produce erroneous conclusions.

(10) For this example, a graphical display of profit versus the number of service technicians yields both the optimal number of technicians and a measure of how sensitive the profit is to variations about this optimal number. In this way a policy decision can be made. Provided this decision does not violate any external constraints, such as labor union rules, the policy should be implemented.

(11) Documentation of the machine shop model would include a system diagram, explanations of assumptions made about machine failure rates and service repair rates, a description of the specification model, software for the computational model, tables and figures of output, and a description of the output analysis.

Insight

An important benefit of developing and using a discrete-event simulation model is that valuable insight is acquired. As conceptual models are formulated, computational models developed and output data analyzed, subtle system features and component interactions may be discovered that would not have been noticed otherwise. The systematic application of Algorithms 1.1.1 and 1.1.2 can result in better actions taken due to insight gained by an increased understanding of how the system operates.

1.1.4 PROGRAMMING LANGUAGES

There is a continuing debate in discrete-event simulation — to use a general-purpose programming language or a (special-purpose) simulation programming language. For example, two standard discrete-event simulation textbooks provide the following contradictory advice. Bratley, Fox, and Schrage (1987, page 219) state "... for any important large-scale real application we would write the programs in a standard general-purpose language, and avoid all the simulation languages we know." In contrast, Law and Kelton (2000, page 204) state "... we believe, in general, that a modeler would be prudent to give serious consideration to the use of a simulation package."

General-purpose languages are more flexible and familiar; simulation languages allow modelers to build computational models quickly. There is no easy way to resolve this debate in general. However, for the specific purpose of this book — learning the principles and techniques of discrete-event simulation — the debate is easier to resolve. Learning discrete-event simulation methodology is facilitated by using a familiar, general-purpose programming language, a philosophy that has dictated the style and content of this book.

General-Purpose Languages

Because discrete-event simulation is a specific instance of scientific computing, any *general-purpose* programming language suitable for scientific computing is similarly suitable for discrete-event simulation. Therefore, a history of the use of general-purpose programming languages in discrete-event simulation is really a history of general-purpose programming languages in scientific computing. Although this history is extensive, we will try to summarized it in a few paragraphs.

For many years FORTRAN was the primary general-purpose programming language used in discrete-event simulation. In retrospect, this was natural and appropriate because there was no well-accepted alternative. By the early 80's things began to change dramatically. Several general-purpose programming languages created in the 70's, primarily C and Pascal, were as good as or superior to FORTRAN in most respects and they began to gain acceptance in many applications, including discrete-event simulation, where FORTRAN was once dominant. Because of its structure and relative simplicity, Pascal became the de facto first programming language in many computer science departments; because of its flexibility and power, the use of C became common among professional programmers.

Personal computers became popular in the early 80's, followed soon thereafter by increasingly more powerful workstations. Concurrent with this development, it became clear that networked workstations or, to a lesser extent, stand-alone personal computers, were ideal discrete-event simulation engines. The popularity of workstation networks then helped to guarantee that C would become the general-purpose language of choice for discrete-event simulation. That is, the usual workstation network was Unix-based, an environment in which C was the natural general-purpose programming language of choice. The use of C in discrete-event simulation became wide-spread by the early 90's when C became standardized and C++, an object-oriented extension of C, gained popularity.

In addition to C, C++, FORTRAN, and Pascal, other general-purpose programming languages are occasionally used in discrete-event simulation. Of these, Ada, Java, and (modern, compiled) BASIC are probably the most common. This diversity is not surprising because every general-purpose programming language has its advocates, some quite vocal, and no matter what the language there is likely to be an advocate to argue that it is ideal for discrete-event simulation. We leave that debate for another forum, however, confident that our use of ANSI C in this book is appropriate.

Simulation Languages

Simulation languages have built-in features that provide many of the tools needed to write a discrete-event simulation program. Because of this, simulation languages support rapid prototyping and have the potential to decrease programming time significantly. Moreover, animation is a particularly important feature now built into most of these simulation languages. This is important because animation can increase the acceptance of discrete-event simulation as a legitimate problem-solving technique. By using animation, dynamic graphical images can be created that enhance verification, validation, and the development of insight. The most popular discrete-event simulation languages historically are GPSS, SIMAN, SLAM II, and SIMSCRIPT II.5. Because of our emphasis in the book on the use of general-purpose languages, any additional discussion of simulation languages is deferred to Appendix A.

Because it is not discussed in Appendix A, for historical reasons it is appropriate here to mention the simulation language Simula. This language was developed in the 60's as an object-oriented ALGOL extension. Despite its object orientation and several other novel (for the time) features, it never achieved much popularity, except in Europe. Still, like other premature-but-good ideas, the impact of Simula has proven to be profound, including serving as the inspiration for the creation of C++.

1.1.5   ORGANIZATION AND TERMINOLOGY

We conclude this first section with some brief comments about the organization of the book and the sometimes ambiguous use of the words *simulation*, *simulate*, and *model*.

Organization

The material in this book could have been organized in several ways. Perhaps the most natural sequence would be to follow, in order, the steps in Algorithms 1.1.1 and 1.1.2, devoting a chapter to each step. However, that sequence is not followed. Instead, the material is organized in a manner consistent with the experimental nature of discrete-event simulation. That is, we begin to model, simulate, and analyze simple-but-representative systems as soon as possible (indeed, in the next section). Whenever possible, new concepts are first introduced in an informal way that encourages experimental self-discovery, with a more formal treatment of the concepts deferred to later chapters. This organization has proven to be successful in the classroom.

Terminology

The words "model" and "simulation" or "simulate" are commonly used interchangeably in the discrete-event simulation literature, both as a noun and as a verb. For pedagogical reasons this word interchangeability is unfortunate because, as indicated previously, a "model" (the noun) exists at three levels of abstraction: conceptual, specification, and computational. At the computational level, a system model is a computer program; this computer program is what most people mean when they talk about a system simulation. In this context a simulation and a computational system model are equivalent. It is uncommon, however, to use the noun "simulation" as a synonym for the system model at either the conceptual or specification level. Similarly, "to model" (the verb) implies activity at three levels, but "to simulate" is usually a computational activity only.

When appropriate we will try to be careful with these words, generally using simulation or simulate in reference to a computational activity only. This is consistent with common usage of the word *simulation* to characterize not only the computational model (computer program) but also the computational process of using the discrete-event simulation model to generate output statistical data and thereby analyze system performance. In those cases when there is no real need to be fussy about terminology, we will yield to tradition and use the word simulation or simulate even though the word model may be more appropriate.

1.1.6 EXERCISES

Exercise 1.1.1 There are six leaf nodes in the system model tree in Figure 1.1.1. For each leaf node, describe a specific example of a corresponding physical system.

Exercise 1.1.2 The distinction between model *verification* and model *validation* is not always clear in practice. Generally, in the sense of Algorithm 1.1.1, the ultimate objective is a valid discrete-event simulation model. If you were told that "this discrete-event simulation model had been verified but it is not known if the model is valid" how would you interpret that statement?

Exercise 1.1.3 The *state* of a system is important, but difficult to define in a general context. (*a*) Locate at least five contemporary textbooks that discuss system modeling and, for each, research and comment on the extent to which the technical term "state" is defined. If possible, avoid example-based definitions or definitions based on a specific system. (*b*) How would you define the state of a system?

Exercise 1.1.4 (*a*) Use an Internet search engine to identify at least 10 different simulation languages that support discrete-event simulation. (Note that the '-' in discrete-event is not a universal convention.) Provide a URL, phone number, or mailing address for each and, if it is a commercial product, a price. (*b*) If you tried multiple search engines, which produced the most meaningful hits?

In this section we will construct a *trace-driven* discrete-event simulation model (i.e., a model driven by external data) of a single-server service node. We begin the construction at the conceptual level.

1.2.1   CONCEPTUAL MODEL

Definition 1.2.1   A *single-server service node* consists of a *server* plus its *queue*.*

Figure 1.2.1.
Single-server
service node
system diagram.
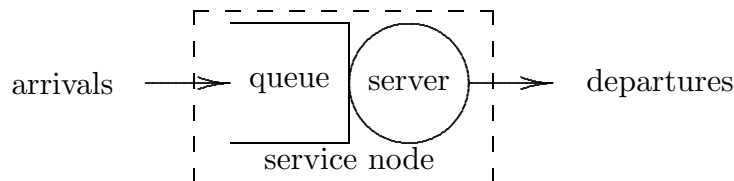
Jobs (customers) arrive at the service node at random points in time seeking service. When service is provided, the service time involved is also random. At the completion of service, jobs depart. The service node operates as follows: as each (new) job arrives, if the server is busy then the job enters the queue, else the job immediately enters service; as each (old) job departs, if the queue is empty then the server becomes idle, else a job is selected from the queue to immediately enter service. At any time, the state of the server will either be *busy* or *idle* and the state of the queue will be either *empty* or *not empty*. If the server is idle, the queue must be empty; if the queue is not empty then the server must be busy.

Example 1.2.1   If there is just one service technician, the machine shop model presented in Examples 1.1.1 and 1.1.2 is a single-server service node model. That is, the "jobs" are the machines to be repaired and the "server" is the service technician. (In this case, whether the jobs move to the server or the server moves to the jobs is not an important distinction because the repair time is the primary source of delay.)

Definition 1.2.2   Control of the queue is determined by the *queue discipline* — the algorithm used when a job is selected from the queue to enter service. The standard algorithms are:

- FIFO — first in, first out (the traditional computer science queue data structure);
- LIFO — last in, first out (the traditional computer science stack data structure);
- SIRO — service in random order;
- Priority — typically, shortest job first (SJF) or equivalently, in job-shop terminology, shortest processing time (SPT).

The maximum possible number of jobs in the service node is the *capacity*. The capacity can be either *finite* or *infinite*. If the capacity is finite then jobs that arrive and find the service node full will be *rejected* (unable to enter the service node).

---

* The term "service node" is used in anticipation of extending this model, in later chapters, to a *network* of service nodes.

Certainly the most common queue discipline is FIFO (also known as FCFS — first come, first served). If the queue discipline is FIFO, then the order of arrival to the service node and the order of departure from the service node are the same; there is no passing. In particular, upon arrival a job will enter the queue if and only if the *previous* job has not yet departed the service node. This is an important observation that can be used to simplify the simulation of a FIFO single-server service node. If the queue discipline is not FIFO then, for at least some jobs, the order of departure will differ from the order of arrival. In this book, the *default* assumptions are that the queue discipline is FIFO and the service node capacity is infinite, unless otherwise specified. Discrete-event simulation allows these assumptions to be easily altered for more realistic modeling.

There are two important additional default assumptions implicit in Definition 1.2.1. First, service is *non-preemptive* — once initiated, service on a job will be continued until completion. That is, a job in service cannot be preempted by another job arriving later. Preemption is commonly used with priority queue disciplines to prevent a job with a large service time requirement from producing excessive delays for small jobs arriving soon after service on the large job has begun. Second, service is *conservative* — the server will never remain idle if there is one or more jobs in the service node. If the queue discipline is not FIFO *and* if the next arrival time is known in advance then, even though one or more jobs are in the service node, it may be desirable for a non-conservative server to remain idle until the next job arrives. This is particularly true in non-preemptive job scheduling applications if a job in the service node has a much larger service requirement than the next job scheduled to arrive.

1.2.2   SPECIFICATION MODEL

The following variables, illustrated in Figure 1.2.2, provide the basis for moving from a conceptual model to a specification model. At their arrival to the service node, jobs are indexed by $i = 1, 2, 3, \ldots$ For each job there are six associated time variables.

- The *arrival time* of job i is a_i.

- The *delay* of job i in the queue is $d_i \geq 0$.

- The time that job i begins service is $b_i = a_i + d_i$.

- The *service time* of job i is $s_i > 0$.

- The *wait* of job i in the service node (queue and service) is $w_i = d_i + s_i$.

- The time that job i completes service (the *departure time*) is $c_i = a_i + w_i$.
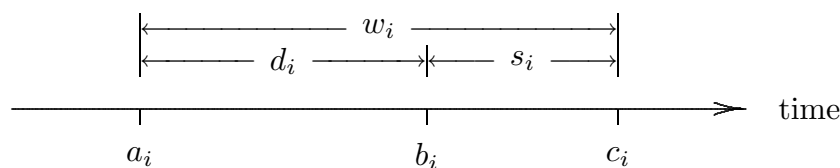
Figure 1.2.2. Six variables associated with job i.

The term "wait" can be confusing; w_i represents the total time job i spends in the service node, *not* just the time spent in the queue. The time spent in the queue (if any) is the delay d_i. In many computer science applications the term *response time* is used. To some authors this means *wait*, to others it means *delay*. Because of this ambiguity, we will generally avoid using the term "response time" choosing instead to consistently use the terminology specified previously. Similarly, we avoid the use of the common terms *sojourn time*, *flow time*, or *system time*, in place of wait.

Arrivals

As a convention, if the service node capacity is finite then rejected jobs (if any) are not indexed. That is, although rejected jobs may be counted for statistical purposes (for example, to estimate the probability of rejection), the index $i = 1, 2, 3, \ldots$ is restricted to only those jobs that actually enter the service node.

Rather than specify the *arrival* times a_1, a_2, ... explicitly, in some discrete-event simulation applications it is preferable to specify the *interarrival* times r_1, r_2, ..., thereby defining the arrival times implicitly, as shown in Figure 1.2.3 and defined in Definition 1.2.3.

Figure 1.2.3. Relationship between arrival and interarrival times.
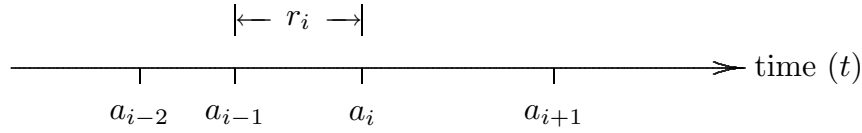
Definition 1.2.3 The *interarrival time* between jobs $i-1$ and i is $r_i = a_i - a_{i-1}$. That is, $a_i = a_{i-1} + r_i$ and so (by induction), with $a_0 = 0$ the arrival times are*

$$a_i = r_1 + r_2 + \cdots + r_i \qquad i = 1, 2, 3, \ldots$$

(We assume that $r_i > 0$ for all i, thereby eliminating the possibility of *bulk* arrivals. That is, jobs are assumed to arrive one at a time.)

Algorithmic Question

The following algorithmic question is fundamental. Given a knowledge of the arrival times a_1, a_2, ... (or, equivalently, the interarrival times r_1, r_2, ...), the associated service times s_1, s_2, ..., and the queue discipline, how can the delay times d_1, d_2, ... be computed?

As discussed in later chapters, for some queue disciplines this question is more difficult to answer than for others. If the queue discipline is FIFO, however, then the answer is particularly simple. That is, as demonstrated next, if the queue discipline is FIFO then there is a simple algorithm for computing d_i (as well as b_i, w_i, and c_i) for all i.

---

* All arrival times are referenced to the virtual arrival time a_0. Unless explicitly stated otherwise, in this chapter and elsewhere we assume that elapsed time is measured in such a way that $a_0 = 0$.

Two Cases

If the queue discipline is FIFO then the delay d_i of job $i = 1, 2, 3, \ldots$ is determined by when the job's arrival time a_i occurs relative to the departure time c_{i-1} of the previous job. There are two cases to consider.

- **Case I.** If $a_i < c_{i-1}$, i.e., if job i arrives before job $i - 1$ departs then, as illustrated, job i will experience a delay of $d_i = c_{i-1} - a_i$. Job $i - 1$'s history is displayed above the time axis and job i's history is displayed below the time axis in Figures 1.2.4 and 1.2.5.
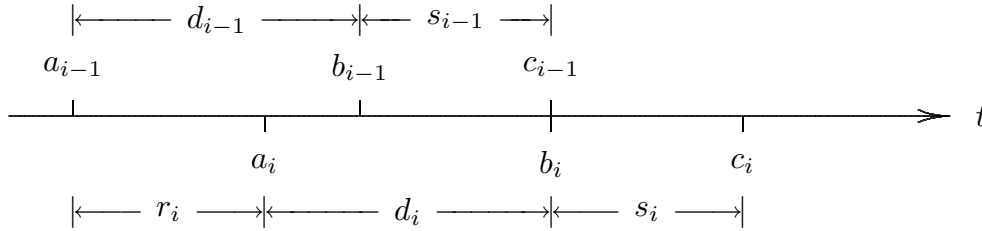
Figure 1.2.4. Job i arrives before job $i - 1$ departs.

- **Case II.** If instead $a_i \geq c_{i-1}$, i.e., if job i arrives after (or just as) job $i - 1$ departs then, as illustrated, job i will experience no delay so that $d_i = 0$.
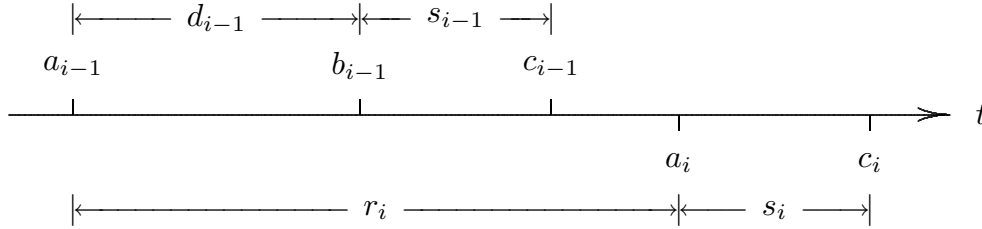
Figure 1.2.5. Job i arrives after job $i - 1$ departs.

Algorithm

The key point in algorithm development is that if the queue discipline is FIFO then the truth of the expression $a_i < c_{i-1}$ determines whether or not job i will experience a delay. Based on this logic, the computation of the delays is summarized by Algorithm 1.2.1. This algorithm, like all those presented in this book, is written in a C-like pseudo-code that is easily translated into other general-purpose programming languages.

Although it is not an explicit part of Algorithm 1.2.1, an equation can be written for the delay that depends on the interarrival and service times only. That is

$$
\begin{aligned}
c_{i-1} - a_i &= (a_{i-1} + d_{i-1} + s_{i-1}) - a_i \\
&= d_{i-1} + s_{i-1} - (a_i - a_{i-1}) \\
&= d_{i-1} + s_{i-1} - r_i.
\end{aligned}
$$

If $d_0 = s_0 = 0$ then $d_1, d_2, d_3, \ldots$ are defined by the nonlinear equation

$$
d_i = \max\{0, d_{i-1} + s_{i-1} - r_i\} \qquad i = 1, 2, 3, \ldots
$$

This equation is commonly used in theoretical studies to analyze the stochastic behavior of a FIFO service node.

Algorithm 1.2.1   If the arrival times $a_1, a_2, \ldots$ and service times $s_1, s_2, \ldots$ are known and if the server is initially idle, then this algorithm computes the delays $d_1, d_2, \ldots$ in a single-server FIFO service node with infinite capacity.

```
c0 = 0.0;                                              /* assumes that a0 = 0.0 */
i = 0;
while ( more jobs to process ) {
    i++;
    ai = GetArrival();
    if (ai < ci-1)
        di = ci-1 - ai;                                /* calculate delay for job i */
    else
        di = 0.0;                                       /* job i has no delay */
    si = GetService();
    ci = ai + di + si;                        /* calculate departure time for job i */
}
n = i;
return d1, d2, ..., dn;
```

The `GetArrival` and `GetService` procedures read the next arrival and service time from a file. (An algorithm that does not rely on the FIFO assumption is presented in Chapter 5.)

Example 1.2.2   If Algorithm 1.2.1 is used to process $n = 10$ jobs according to the input indicated below (for simplicity the a_i's and s_i's are integer time units, e.g., seconds, minutes, etc.) then the output is the sequence of delays are calculated as:

		i :	1	2	3	4	5	6	7	8	9	10
read from file	a_i :		15	47	71	111	123	152	166	226	310	320
from algorithm	d_i :		0	11	23	17	35	44	70	41	0	26
read from file	s_i :		43	36	34	30	38	40	31	29	36	30

For future reference, note that the last job arrived at time $a_n = 320$ and departed at time $c_n = a_n + d_n + s_n = 320 + 26 + 30 = 376$.

As discussed in more detail later in this section, it is a straight-forward programming exercise to produce a computational model of a single-server FIFO service node with infinite capacity using Algorithm 1.2.1. The ANSI C program `ssq1` is an example. Three features of this program are noteworthy. (*i*) Because of its reliance on previously recorded arrival and service time data read from an external file, `ssq1` is a so-called *trace-driven* discrete-event simulation program. (*ii*) Because the queue discipline is FIFO, program `ssq1` does not need to use a queue data structure. (*iii*) Rather than produce a sequence of delays as output, program `ssq1` computes four averages instead: the average interarrival time, service time, delay, and wait. These four job-averaged statistics and three corresponding time-averaged statistics are discussed next.

1.2.3 OUTPUT STATISTICS

One basic issue that must be resolved when constructing a discrete-event simulation model is the question of what statistics should be generated. The purpose of simulation is insight and we gain insight about the performance of a system by looking at meaningful statistics. Of course, a decision about what statistics are most meaningful is dependent upon your perspective. For example, from a job's (customer's) perspective the most important statistic might be the average delay or the 95th percentile of the delay — in either case, the smaller the better. On the other hand, particularly if the server is an expensive resource whose justification is based on an anticipated heavy workload, from management's perspective the server's utilization (the proportion of busy time, see Definition 1.2.7) is most important — the larger the better.

Job-Averaged Statistics

Definition 1.2.4 For the first n jobs, the *average interarrival time* and the *average service time* are, respectively*

$$\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i = \frac{a_n}{n} \qquad \text{and} \qquad \bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i.$$

The reciprocal of the average interarrival time, $1/\bar{r}$, is the *arrival rate*; the reciprocal of the average service time, $1/\bar{s}$, is the *service rate*.

Example 1.2.3 For the $n = 10$ jobs in Example 1.2.2, $\bar{r} = a_n/n = 320/10 = 32.0$ and $\bar{s} = 34.7$. If time in this example is measured in seconds, then the average interarrival time is 32.0 seconds per job and the average service time is 34.7 seconds per job. The corresponding arrival rate is $1/\bar{r} = 1/32.0 \cong 0.031$ jobs per second; the service rate is $1/\bar{s} = 1/34.7 \cong 0.029$ jobs per second. In this particular example, the server is not quite able to process jobs at the rate they arrive on average.

Definition 1.2.5 For the first n jobs, the *average delay* in the queue and the *average wait* in the service node are, respectively

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i \qquad \text{and} \qquad \bar{w} = \frac{1}{n} \sum_{i=1}^{n} w_i.$$

Recall that $w_i = d_i + s_i$ for all i. Therefore, the average time spent in the service node will be the sum of the average times spent in the queue and in service. That is

$$\bar{w} = \frac{1}{n} \sum_{i=1}^{n} w_i = \frac{1}{n} \sum_{i=1}^{n} (d_i + s_i) = \frac{1}{n} \sum_{i=1}^{n} d_i + \frac{1}{n} \sum_{i=1}^{n} s_i = \bar{d} + \bar{s}.$$

The point here is that it is sufficient to compute any two of the statistics $\bar{w}$, $\bar{d}$, $\bar{s}$. The third statistic can then be computed from the other two, if appropriate.

---

* The equation $\bar{r} = a_n/n$ follows from Definition 1.2.3 and the assumption that $a_0 = 0$.

Example 1.2.4 For the data in Example 1.2.2, $\bar{d} = 26.7$ and $\bar{s} = 34.7$. Therefore, the average wait in the node is $\bar{w} = 26.7 + 34.7 = 61.4$. (See also Example 1.2.6.)

In subsequent chapters we will construct increasingly more complex discrete-event simulation models. Because it is never easy to verify and validate a complex model, it is desirable to be able to apply as many *consistency checks* to the output data as possible. For example, although program ssq1 is certainly not a complex discrete-event simulation model, it is desirable in this program to accumulate $\bar{w}$, $\bar{d}$, and $\bar{s}$ *independently*. Then, from the program output the equation $\bar{w} = \bar{d} + \bar{s}$ can be used as a consistency check.

Time-Averaged Statistics

The three statistics $\bar{w}$, $\bar{d}$ and $\bar{s}$ are *job-averaged* statistics — the data is averaged over all jobs. Job averages are easy to understand; they are just traditional arithmetic averages. We now turn to another type of statistic that is equally meaningful, *time-averaged*. Time-averaged statistics may be less familiar, however, because they are defined by an area under a curve, i.e., by integration instead of summation.

Time-averaged statistics for a single-server service node are defined in terms of three additional variables. At any time $t > 0$:

- $l(t) = 0, 1, 2, \ldots$ is the number of jobs in the service node at time t;

- $q(t) = 0, 1, 2, \ldots$ is the number of jobs in the queue at time t;

- $x(t) = 0, 1$ is the number of jobs in service at time t.

By definition, $l(t) = q(t) + x(t)$ for any $t > 0$.

The three functions $l(\cdot)$, $q(\cdot)$, and $x(\cdot)$ are *piecewise constant*. That is, for example, a display of $l(t)$ versus t will consist of a sequence of constant segments with unit height step discontinuities as illustrated in Figure 1.2.6. (This figure corresponds to the data in Example 1.2.2. The dashed line represents the time-averaged number in the node — see Example 1.2.6.)

Figure 1.2.6. Number of jobs in the service node.
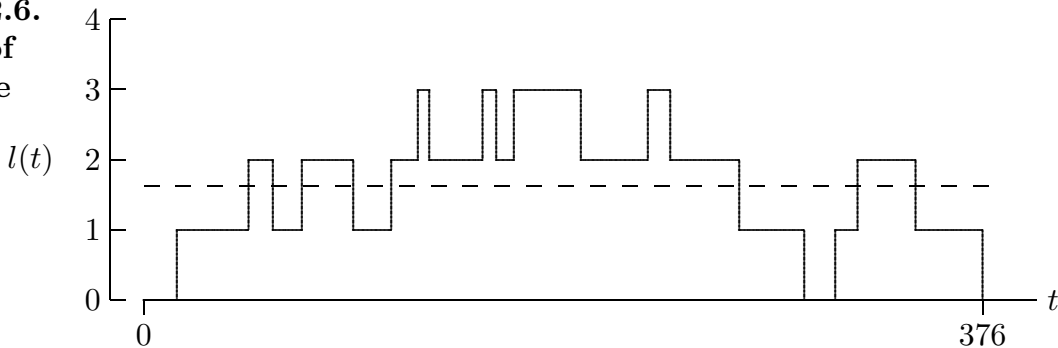
The step discontinuities are positive at the arrival times and negative at the departure times. The corresponding figures for $q(\cdot)$ and $x(\cdot)$ can be deduced from the fact that $q(t) = 0$ and $x(t) = 0$ if and only if $l(t) = 0$, otherwise $q(t) = l(t) - 1$ and $x(t) = 1$.

Definition 1.2.6 Over the time interval $(0, \tau)$ the *time-averaged number in the node* is

$$\bar{l} = \frac{1}{\tau} \int_0^\tau l(t)\, dt.$$

Similarly, the *time-averaged number in the queue* and the *time-averaged number in service* are

$$\bar{q} = \frac{1}{\tau} \int_0^\tau q(t)\, dt \qquad \text{and} \qquad \bar{x} = \frac{1}{\tau} \int_0^\tau x(t)\, dt.$$

Because $l(t) = q(t) + x(t)$ for all $t > 0$ it follows that

$$\bar{l} = \bar{q} + \bar{x}.$$

Example 1.2.5 For the data in Example 1.2.2 (with $\tau = c_{10} = 376$) the three time-averaged statistics are $\bar{l} = 1.633$, $\bar{q} = 0.710$, and $\bar{x} = 0.923$. These values can be determined by calculating the areas associated with the integrals given in Definition 1.2.6 or by exploiting a mathematical relationship between the job-averaged statistics $\bar{w}$, $\bar{d}$, and $\bar{s}$ and the time-averaged statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$, as illustrated subsequently in Example 1.2.6.

The equation $\bar{l} = \bar{q} + \bar{x}$ is the time-averaged analog of the job-averaged equation $\bar{w} = \bar{d} + \bar{s}$. As we will see in later chapters, time-averaged statistics have the following important characterizations.

- If we were to observe (sample) the number in the service node, for example, at many different times chosen *at random* between 0 and τ and then calculate the arithmetic average of all these observations, the result should be close to $\bar{l}$.

- Similarly, the arithmetic average of many random observations of the number in the queue should be close to $\bar{q}$ and the arithmetic average of many random observations of the number in service (0 or 1) should be close to $\bar{x}$.

- $\bar{x}$ must lie in the closed interval $[0, 1]$.

Definition 1.2.7 The time-averaged number in service $\bar{x}$ is also known as the server *utilization*. The reason for this terminology is that $\bar{x}$ represents the proportion of time that the server is busy.

Equivalently, if one particular time is picked *at random* between 0 and τ then $\bar{x}$ is the probability that the server is busy at that time. If $\bar{x}$ is close to 1.0 then the server is busy most of the time and correspondingly large values of $\bar{l}$ and $\bar{q}$ will be produced. On the other hand, if the utilization is close to 0.0 then the server is idle most of the time and the values of $\bar{l}$ and $\bar{q}$ will be small. The case study, presented later, is an illustration.

Little's Equations

One important issue remains — how are job averages and time averages related? Specifically, how are $\bar{w}$, $\bar{d}$, and $\bar{s}$ related to $\bar{l}$, $\bar{q}$, and $\bar{x}$?

In the particular case of an infinite capacity FIFO service node that begins and ends in an idle state, the following theorem provides an answer to this question. See Exercise 1.2.7 for a generalization of this theorem to any queue discipline.

Theorem 1.2.1   (Little, 1961) If the queue discipline is FIFO, the service node capacity is infinite, and the server is idle both initially (at $t = 0$) and immediately after the departure of the nth job (at $t = c_n$) then

$$\int_0^{c_n} l(t)\, dt = \sum_{i=1}^{n} w_i \qquad \text{and} \qquad \int_0^{c_n} q(t)\, dt = \sum_{i=1}^{n} d_i \qquad \text{and} \qquad \int_0^{c_n} x(t)\, dt = \sum_{i=1}^{n} s_i.$$

Proof   For each job $i = 1, 2, \ldots$, define an *indicator function* $\psi_i(t)$ that is 1 when the ith job is in the service node and is 0 otherwise

$$\psi_i(t) = \begin{cases} 1 & a_i < t < c_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$l(t) = \sum_{i=1}^{n} \psi_i(t) \qquad 0 < t < c_n$$

and so

$$\int_0^{c_n} l(t)\, dt = \int_0^{c_n} \sum_{i=1}^{n} \psi_i(t)\, dt = \sum_{i=1}^{n} \int_0^{c_n} \psi_i(t)\, dt = \sum_{i=1}^{n}(c_i - a_i) = \sum_{i=1}^{n} w_i.$$

The other two equations can be derived in a similar way.

Figure 1.2.7. Little's first equation.
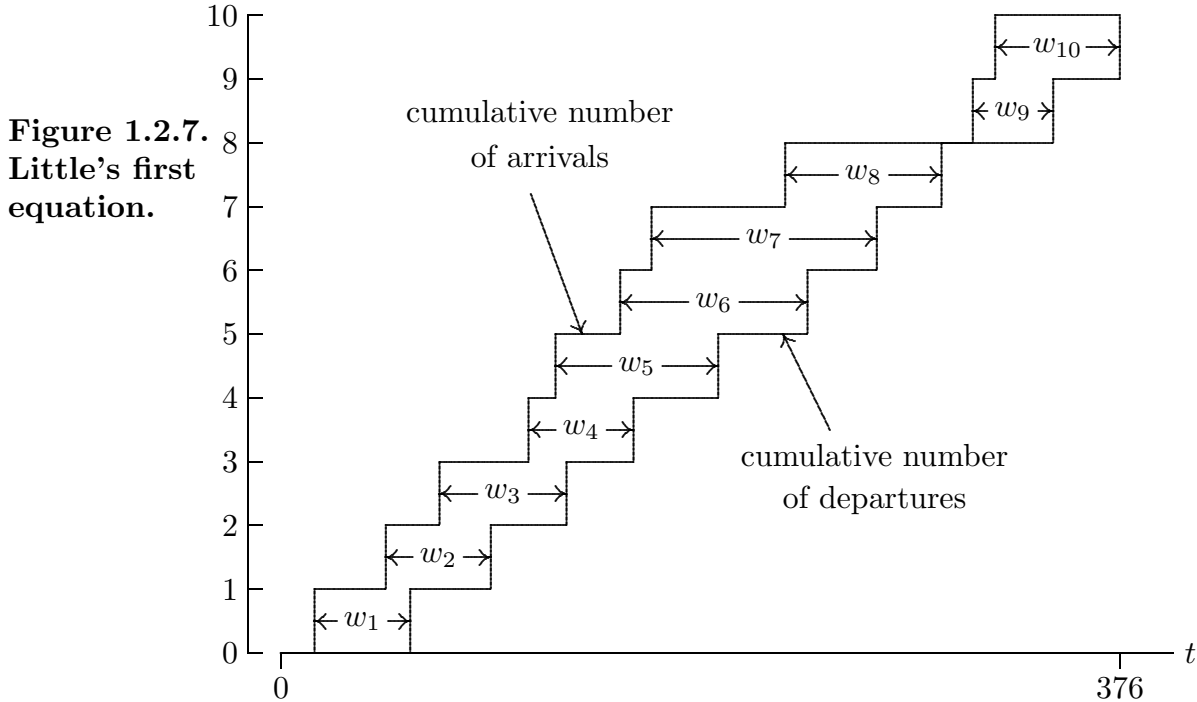
Example 1.2.6 Figure 1.2.7 illustrates Little's first equation for the data in Example 1.2.2. The top step function denotes the cumulative number of arrivals to the service node and the bottom step function denotes the cumulative number of departures from the service node. The vertical distance between the two step-functions at any time t is $l(t)$, which was plotted in Figure 1.2.6. The wait times are indicated as the horizontal distances between the risers. In this figure, it is easy to see that

$$\int_0^{376} l(t)\, dt = \sum_{i=1}^{10} w_i = 614.$$

Little's equations provide a valuable link between the job-averaged statistics $\bar{w}$, $\bar{d}$, and $\bar{s}$ and the time-averaged statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$. In Definition 1.2.6 let $\tau = c_n$. Then from Theorem 1.2.1 we have

$$c_n \bar{l} = \int_0^{c_n} l(t)\, dt = \sum_{i=1}^{n} w_i = n\bar{w}$$

so that $\bar{l} = (n/c_n)\bar{w}$. Similarly, $c_n \bar{q} = n\bar{d}$ and $c_n \bar{x} = n\bar{s}$. Therefore

$$\bar{l} = \left(\frac{n}{c_n}\right)\bar{w} \qquad \text{and} \qquad \bar{q} = \left(\frac{n}{c_n}\right)\bar{d} \qquad \text{and} \qquad \bar{x} = \left(\frac{n}{c_n}\right)\bar{s}$$

which explains how $\bar{w}$, $\bar{d}$, and $\bar{s}$ are related to $\bar{l}$, $\bar{q}$, and $\bar{x}$. These important equations relate to *steady-state* statistics and *Little's equations* — for more detail, see Chapter 8.

Example 1.2.7 For the data in Example 1.2.2 the last ($n = 10$) job departs at $c_n = 376$. From Example 1.2.4, $\bar{w} = 61.4$ and therefore $\bar{l} = (10/376)\,61.4 \cong 1.633$. Similarly, the time-averaged number in the queue and in service are $\bar{q} = (10/376)\,26.7 \cong 0.710$ and $\bar{x} = (10/376)\,34.7 \cong 0.923$.

1.2.4 COMPUTATIONAL MODEL

As discussed previously, by using Algorithm 1.2.1 in conjunction with some statistics gathering logic it is a straight-forward programming exercise to produce a computational model of a single-server FIFO service node with infinite capacity. The ANSI C program `ssq1` is an example. Like all of the software presented in this book, this program is designed with readability and extendibility considerations.

Program ssq1

Program `ssq1` reads arrival and service time data from the disk file `ssq1.dat`. This is a text file that consists of arrival times $a_1, a_2, \ldots, a_n$ and service times $s_1, s_2, \ldots, s_n$ for $n = 1000$ jobs in the format

$$
\begin{array}{cc}
a_1 & s_1 \\
a_2 & s_2 \\
\vdots & \vdots \\
a_n & s_n
\end{array}
$$

In Chapter 3 we will free this trace-driven program from its reliance on external data by using randomly generated arrival and service times instead.

Because the queue discipline is FIFO, program `ssq1` does not need to use a queue data structure. In Chapter 5 we will consider non-FIFO queue disciplines and some corresponding priority queue data structures that can be used at the computational model level.

Program `ssq1` computes the average interarrival time $\bar{r}$, the average service time $\bar{s}$, the average delay $\bar{d}$, and the average wait $\bar{w}$. In Exercise 1.2.2 you are asked to modify this program so that it will also compute the time-averaged statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$.

Example 1.2.8   For the datafile `ssq1.dat` the observed arrival rate $1/\bar{r} \cong 0.10$ is significantly less than the observed service rate $1/\bar{s} \cong 0.14$. If you modify `ssq1` to compute $\bar{l}$, $\bar{q}$, and $\bar{x}$ you will find that $1 - \bar{x} \cong 0.28$, and so the server is idle 28% of the time. Despite this significant idle time, enough jobs are delayed so that the average number in the queue is nearly 2.0.

Traffic Intensity

The ratio of the arrival rate to the service rate is commonly called the *traffic intensity*. From the equations in Definition 1.2.4 and Theorem 1.2.1 it follows that the observed traffic intensity is the ratio of the observed arrival rate to the observed service rate

$$\frac{1/\bar{r}}{1/\bar{s}} = \frac{\bar{s}}{\bar{r}} = \frac{\bar{s}}{a_n/n} = \left(\frac{c_n}{a_n}\right)\bar{x}.$$

Therefore, provided the ratio c_n/a_n is close to 1.0, the traffic intensity and utilization will be nearly equal. In particular, if the traffic intensity is less than 1.0 and n is large, then it is reasonable to expect that the ratio $c_n/a_n = 1 + w_n/a_n$ will be close to 1.0. We will return to this question in later chapters. For now, we close with an example illustrating how relatively sensitive the service node statistics $\bar{l}$, $\bar{q}$, $\bar{w}$, and $\bar{d}$ are to changes in the utilization and how nonlinear this dependence can be.

Case Study

Sven & Larry's Ice Cream Shoppe is a thriving business that can be modeled as a single-server queue. The owners are considering adding additional flavors and cone options, but are concerned about the resultant increase in service times on queue length. They decide to use a trace-driven simulation to assess the impact of the longer service times associated with the additional flavors and cone options.

The file `ssq1.dat` represents 1000 customer interactions at Sven & Larry's. The file consists of arrival times of groups of customers and the group's corresponding service times. The service times vary significantly because of the number of customers in each group.

The dependence of the average queue length $\bar{q}$ on the utilization $\bar{x}$ is illustrated in Figure 1.2.8. This figure was created by systematically increasing or decreasing each service time in the datafile `ssq1.dat` by a common multiplicative factor, thereby causing both $\bar{x}$ and $\bar{q}$ to change correspondingly. The $(\bar{x}, \bar{q}) \cong (0.72, 1.88)$ point circled corresponds to the data in `ssq1.dat` while the point immediately to its right, for example, corresponds to the same data with each service time multiplied by 1.05. The next point to the right corresponds to each service time multiplied by 1.10. From this figure, we see that even a modest increase in service times will produce a significant increase in the average queue length. A nonlinear relationship between $\bar{x}$ and $\bar{q}$ is particularly pronounced for utilizations near $\bar{x} = 1$. A 15% increase in the service times from their current values will result in a 109% increase in the time-averaged number of customers in queue, whereas a 30% increase in the service times from their current values will result in a 518% increase in the time-averaged number of customers in queue.
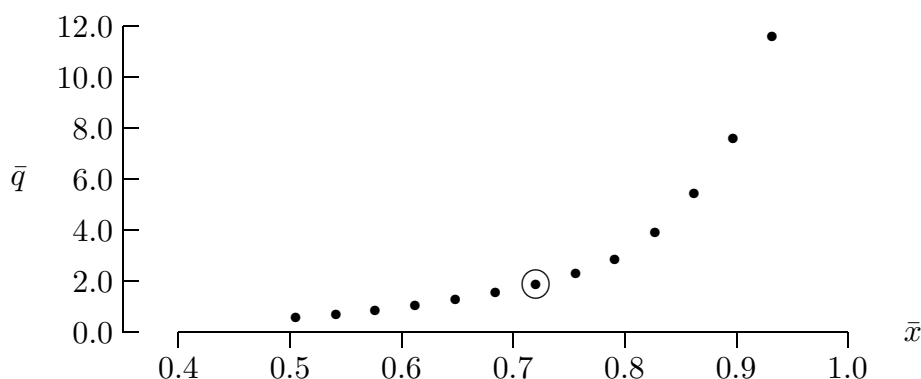
Figure 1.2.8. Average queue length vs. utilization.

Sven & Larry need to assess the impact of the increased service time associated with new flavors and cones on their operation. If the service times increase by only a modest amount, say 5% or 10% above the current times, the average queue length will grow modestly. The new flavors and cone options may, however, also increase the arrival rate — potentially exacerbating the problem with long lines. If queues grow to the point where the owners believe that customers are taking their ice cream business elsewhere, they should consider hiring a second server. A separate analysis would be necessary to determine the probability that an arriving group of customers will balk (never enter the queue) or renege (depart from the queue after entering) as a function of the queue's length.

Graphics Considerations

Figure 1.2.8 presents "raw" simulation output data. That is, each • represents a computed $(\bar{x}, \bar{q})$ point. Because there is nothing inherently discrete about either $\bar{x}$ or $\bar{q}$, many additional points could have been computed and displayed to produce an (essentially) continuous $\bar{q}$ versus $\bar{x}$ curve. In this case, however, additional computations seem redundant; few would question the validity of the smooth curve produced by connecting the •'s with lines, as illustrated in Figure 1.2.9. The nonlinear dependence of $\bar{q}$ on $\bar{x}$ is evident, particularly as $\bar{x}$ approaches 1.0 and the corresponding increase in $\bar{q}$ becomes dramatic.

Figure 1.2.9.
Average queue
length vs.
utilization
with linear
interpolation.
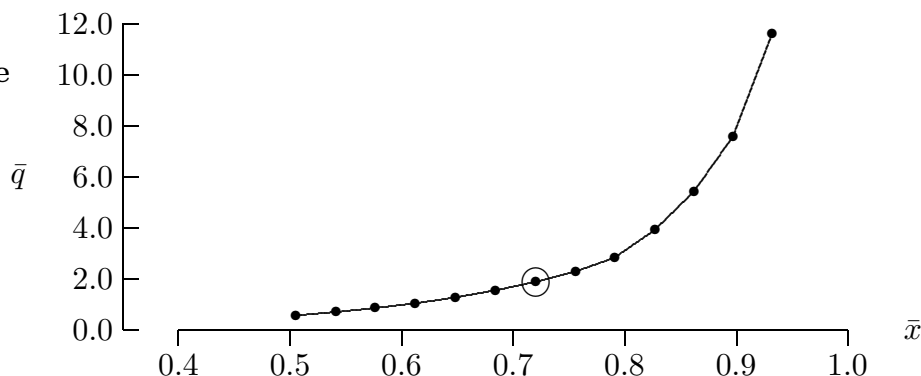
Perhaps because we were taught to do this as children, there is a natural tendency to *always* "connect the dots" (interpolate) when presenting a discrete set of experimental data. (The three most common interpolating functions are linear, quadratic, and spline functions.) Before taking such artistic liberties, however, consider the following guidelines.

- If the data has essentially no uncertainty and if the resulting interpolating curve is smooth, then there is little danger in connecting the dots — provided the original dots are left in the figure to remind the reader that some artistic license was used.

- If the data has essentially no uncertainty but the resulting interpolating curve is not smooth then more dots need to be generated to achieve a graphics scale at which smooth interpolation is reasonable.

- If the dots correspond to uncertain (noisy) data then interpolation is not justified; instead, either approximation should be used in place of interpolation, or the temptation to superimpose a continuous curve should be resisted completely.

These guidelines presume that the data is not inherently discrete. If the data is inherently discrete then it is illogical and potentially confusing to superimpose a continuous (interpolating or approximating) curve. Example 1.3.7 in the next section is an illustration of data that is inherently discrete.*

1.2.5   EXERCISES

Exercise 1.2.1[a]   How would you use the table in Example 1.2.2 to construct the associated $l(t)$ versus t figure? That is, construct an algorithm that will compute *in order* the interlaced arrival and departure times that define the points at which $l(t)$ changes. (Avoid storing $a_1, a_2, \ldots, a_n$ and $c_1, c_2, \ldots, c_n$ as two arrays, linked lists or external disk files and then merging the two into one due to memory and CPU considerations for large n.)

---

* Those interested in an excellent discussion and illustration of graphics considerations are encouraged to read the classic *The Visual Display of Quantitative Information* (Tufte, 2001). The author discusses clarity of presentation through uncluttered graphics that maximize information transmission with minimal ink. The accurate display of simulation output will be stressed throughout this text.

Exercise 1.2.2   (a) Modify program `ssq1` to output the additional statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$. (b) Similar to the case study, use this program to compute a table of $\bar{l}$, $\bar{q}$, and $\bar{x}$ for traffic intensities of 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, and 1.2. (c) Comment on how $\bar{l}$, $\bar{q}$, and $\bar{x}$ depend on the traffic intensity. (d) Relative to the case study, if it is decided that $\bar{q}$ greater than 5.0 is not acceptable, what systematic increase in service times would be acceptable? Use $d.dd$ precision.

Exercise 1.2.3   (a) Modify program `ssq1` by adding the capability to compute the maximum delay, the number of jobs in the service node at a specified time (known at compile time) and the proportion of jobs delayed. (b) What was the maximum delay experienced? (c) How many jobs were in the service node at $t = 400$ and how does the computation of this number relate to the proof of Theorem 1.2.1? (d) What proportion of jobs were delayed and how does this proportion relate to the utilization?

Exercise 1.2.4   Complete the proof of Theorem 1.2.1.

Exercise 1.2.5   If the traffic intensity is less than 1.0, use Theorem 1.2.1 to argue why for large n you would expect to find that $\bar{l} \cong \lambda\,\bar{w}$, $\bar{q} \cong \lambda\,\bar{d}$, and $\bar{x} \cong \lambda\,\bar{s}$, where the observed arrival rate is $\lambda = 1/\bar{r}$.

Exercise 1.2.6   The text file `ac.dat` consists of the arrival times $a_1, a_2, \ldots, a_n$ and the departure times $c_1, c_2, \ldots, c_n$ for $n = 500$ jobs in the format

$$
\begin{array}{cc}
a_1 & c_1 \\
a_2 & c_2 \\
\vdots & \vdots \\
a_n & c_n
\end{array}
$$

(a) If these times are for an initially idle single-server FIFO service node with infinite capacity, calculate the average service time, the server's utilization and the traffic intensity. (b) Be explicit: for $i = 1, 2, \ldots, n$ how does s_i relate to a_{i-1}, a_i, c_{i-1}, and c_i?

Exercise 1.2.7[a]   State and prove a theorem analogous to Theorem 1.2.1 but valid for *any* queue discipline. *Hint*: in place of c_n use $\tau_n = \max\{c_1, c_2, \ldots, c_n\}$. For a conservative server prove that τ_n is *independent* of the queue discipline.

Exercise 1.2.8   (a) Similar to Exercise 1.2.2, modify program `ssq1` to output the additional statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$. (b) By using the arrival times in the file `ssq1.dat` and an appropriate *constant* service time in place of the service times in the file `ssq1.dat`, use the modified program to compute a table of $\bar{l}$, $\bar{q}$, and $\bar{x}$ for traffic intensities of 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, and 1.2. (c) Comment on how $\bar{l}$, $\bar{q}$, and $\bar{x}$ depend on the traffic intensity.

Exercise 1.2.9[a]   (a) Work Exercises 1.2.2 and 1.2.8. (b) Compare the two tables produced and explain (or conjecture) why the two tables are different. Be specific.

The inputs to program `ssq1`, the arrival times and the service times, can have any positive real value — they are *continuous* variables. In some models, however, the input variables are inherently *discrete*. That is the case with the (trace-driven) discrete-event simulation model of a simple inventory system constructed in this section. As in the previous section, we begin with a conceptual model then move to a specification model and, finally, to a computational model.

1.3.1   CONCEPTUAL MODEL

Definition 1.3.1   An *inventory system* consists of a facility that distributes items from its current inventory to its customers in response to a customer demand that is typically random, as illustrated in Figure 1.3.1. Moreover, the demand is integer-valued (discrete) because customers do not want a portion of an item.* Because there is a *holding cost* associated with items in inventory, it is undesirable for the inventory level to be too high. On the other hand, if the inventory level is too low, the facility is in danger of incurring a *shortage cost* whenever a demand occurs that cannot be met.

Figure 1.3.1.
Simple inventory
system diagram.
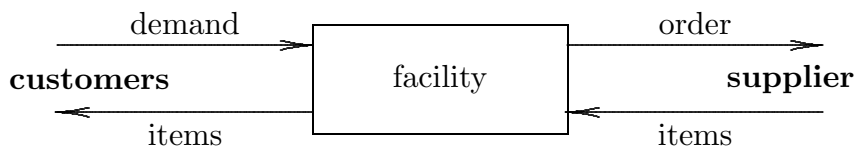
As a policy, the inventory level is reviewed periodically and new items are then (and only then) ordered from a supplier, if necessary.** When items are ordered, the facility incurs an *ordering cost* that is the sum of a fixed *setup cost* independent of the amount ordered plus an *item cost* proportional to the number of items ordered. This *periodic inventory review policy* is defined by two parameters, conventionally denoted s and S.

- s is the *minimum* inventory level — if at the time of review the current inventory level is below the threshold s then an order will be placed with the supplier to replenish the inventory. If the current inventory level is at or above s then no order will be placed.

- S is the *maximum* inventory level — when an order is placed, the amount ordered is the number of items required to bring the inventory back up to the level S.

- The (s, S) parameters are constant in time with $0 \leq s < S$.

---

  * Some inventory systems distribute "items" that are not inherently discrete, for example, a service station that sells gasoline. With minor modifications, the model developed in this section is applicable to these inventory systems as well.

  ** An alternate to the periodic inventory review policy is a *transaction reporting inventory policy*. With this policy, inventory review occurs after *each* demand instance. Because inventory review occurs more frequently, significantly more labor may be required to implement a transaction reporting inventory policy. (The scanners at a grocery store, however, require no extra labor). The transaction reporting policy has the desirable property that, for the same value of s, it is less likely for the inventory system to experience a shortage.

A discrete-event simulation model can be used to compute the cost of operating the facility. In some cases, the values of s and S are fixed; if so, the cost of operating the facility is also fixed. In other cases, if at least one of the (s, S) values (usually s) is not fixed, the cost of operating the facility can be modified and it is natural to search for values of (s, S) for which the cost of operating the facility is minimized.

To complete the conceptual model of this simple (one type of item) inventory system we make three additional assumptions. (a) *Back ordering* (backlogging) is possible — the inventory level can become negative in order to model customer demands not immediately satisfied. (b) There is no *delivery lag* — an order placed with the supplier will be delivered immediately. Usually this is an unrealistic assumption; it will be removed in Chapter 3. (c) The *initial* inventory level is S.

Example 1.3.5, presented later in this section, describes an automobile dealership as an example of an inventory system with back ordering and no delivery lag. In this example the periodic inventory review occurs each week. The value of S is fixed, the value of s is not.

1.3.2   SPECIFICATION MODEL

The following variables provide the basis for a specification model of a simple inventory system. Time begins at $t = 0$ and is measured in a coordinate system in which the inventory review times are $t = 0, 1, 2, 3, \ldots$ with the convention that the ith time interval begins at time $t = i - 1$ and ends at $t = i$.

- The inventory level at the *beginning* of the ith time interval is an integer l_{i-1}.

- The amount ordered (if any) at time $t = i - 1$ is an integer $o_{i-1} \geq 0$.

- The demand quantity *during* the ith time interval is an integer $d_i \geq 0$.

Because we have assumed that back ordering is possible, if the demand during the ith time interval is greater than the inventory level at the beginning of the interval (plus the amount ordered, if any) then the inventory level at the end of the interval will be negative.

The inventory level is reviewed at $t = i - 1$. If l_{i-1} is greater than or equal to s then no items are ordered so that $o_{i-1} = 0$. If instead l_{i-1} is less than s then $o_{i-1} = S - l_{i-1}$ items are ordered to replenish inventory. In this case, because we have assumed there is no delivery lag, ordered items are delivered immediately (at $t = i - 1$) thereby restoring inventory to the level S. In either case, the inventory level at the end of the ith time interval is diminished by d_i. Therefore, as summarized by Algorithm 1.3.1, with $l_0 = S$ the inventory orders $o_0, o_1, o_2, \ldots$ and corresponding inventory levels $l_1, l_2, \ldots$ are defined by

$$
o_{i-1} = \begin{cases} 0 & l_{i-1} \geq s \\ S - l_{i-1} & l_{i-1} < s \end{cases} \qquad \text{and} \qquad l_i = l_{i-1} + o_{i-1} - d_i.
$$

Note that $l_0 = S > s$ and so o_0 must be zero; accordingly, only $o_1, o_2, \ldots$ are of interest.

Algorithm 1.3.1   If the demands $d_1, d_2, \ldots$ are known then this algorithm computes the discrete time evolution of the inventory level for a simple (s, S) inventory system with back ordering and no delivery lag.

```
l₀ = S;                          /* the initial inventory level is S */
i = 0;
while ( more demand to process ) {
   i++;
   if (lᵢ₋₁ < s)
     oᵢ₋₁ = S - lᵢ₋₁;
   else
     oᵢ₋₁ = 0;
   dᵢ = GetDemand();
   lᵢ = lᵢ₋₁ + oᵢ₋₁ - dᵢ;
}
n = i;
oₙ = S - lₙ;
lₙ = S;                          /* the terminal inventory level is S */
return l₁, l₂, ..., lₙ and o₁, o₂, ..., oₙ;
```

Example 1.3.1   Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 to process $n = 12$ time intervals of operation with the input demand schedule:

i :	1	2	3	4	5	6	7	8	9	10	11	12
input d_i :	30	15	25	15	45	30	25	15	20	35	20	30

As illustrated in Figure 1.3.2, the time evolution of the inventory level typically features several intervals of decline, followed by an increase when an order is placed (indicated by the vertical dotted line) and, because there is no delivery lag, is immediately delivered.

Figure 1.3.2. Inventory levels.
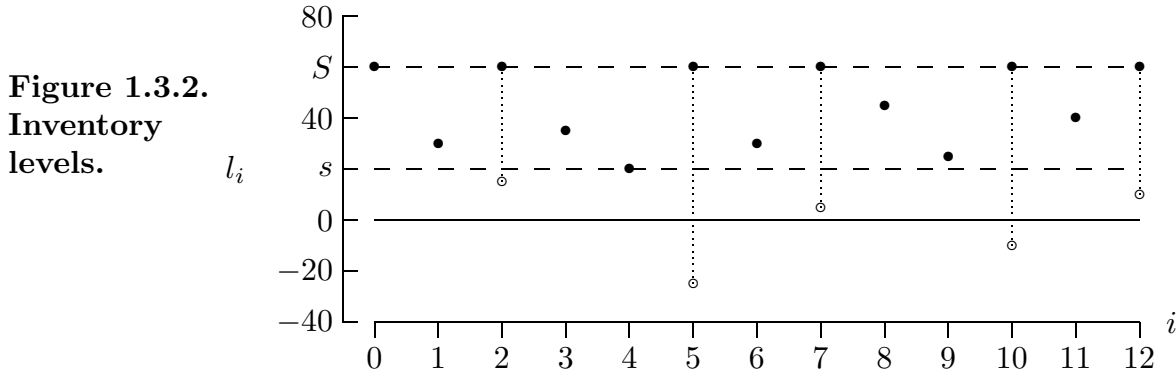
At the end of the last interval (at $t = n = 12$) an order for $o_n = 50$ inventory units was placed. The immediate delivery of this order restores the inventory level at the end of the simulation to the initial inventory level S, as shown in Figure 1.3.2.

1.3.3 OUTPUT STATISTICS

As with the development of program `ssq1` in the previous section, we must address the issue of what statistics should be computed to measure the performance of a simple inventory system. As always, our objective is to analyze these statistics and, by so doing, better understand how the system operates.

Definition 1.3.2   The *average demand* and *average order* are, respectively

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i \qquad \text{and} \qquad \bar{o} = \frac{1}{n} \sum_{i=1}^{n} o_i.$$

Example 1.3.2   For the data in Example 1.3.1, $\bar{d} = \bar{o} = 305/12 \cong 25.42$ items per time interval. As explained next, these two averages *must* be equal.

The terminal condition in Algorithm 1.3.1 is that at the end of the nth time interval an order is placed to return the inventory to its initial level. Because of this terminal condition, independent of the value of s and S, the average demand $\bar{d}$ and the average order $\bar{o}$ must be equal. That is, over the course of the simulated period of operation, all demand is satisfied (although not immediately when back ordering occurs). Therefore, if the inventory level is the same at the beginning and end of the simulation then the average "flow" of items into the facility from the supplier, $\bar{o}$, must have been equal to the average "flow" of items out of the facility to the customers, $\bar{d}$. With respect to the flow of items into and out of the facility, the inventory system is said to be *flow balanced*.
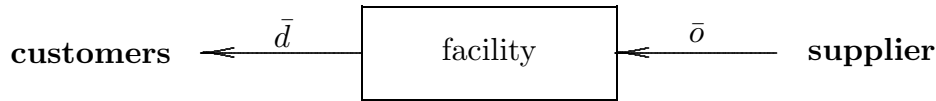
Figure 1.3.3.
Flow balance.

Average Inventory Level

The holding cost and shortage cost are proportional to time-averaged inventory levels. To compute these averages it is necessary to know the inventory level for *all t*, not just at the inventory review times. Therefore, we *assume* that the *demand rate* is constant between review times so that the continuous time evolution of the inventory level is *piecewise linear* as illustrated in Figure 1.3.4.
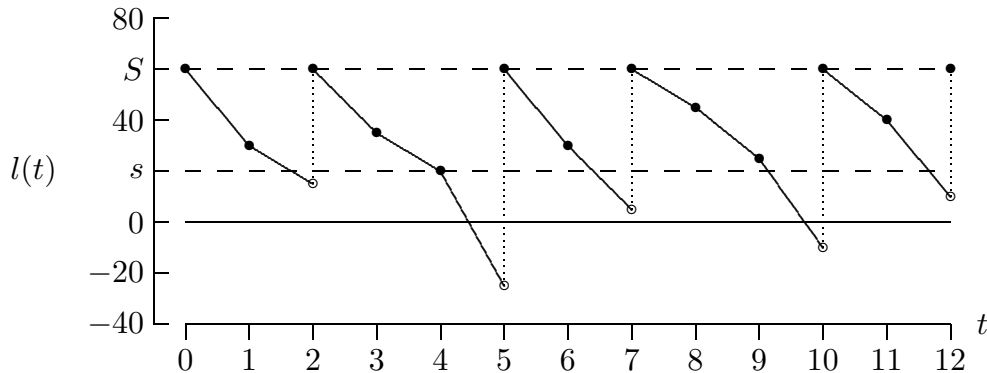
Figure 1.3.4.
Piecewise linear inventory levels.

Definition 1.3.3   If the demand rate is constant between review times, then at any time t in the ith time interval the inventory level is $l(t) = l'_{i-1} - (t - i + 1)d_i$, as illustrated in Figure 1.3.5.

**Figure 1.3.5.
Linear inventory
level in time
interval i.**
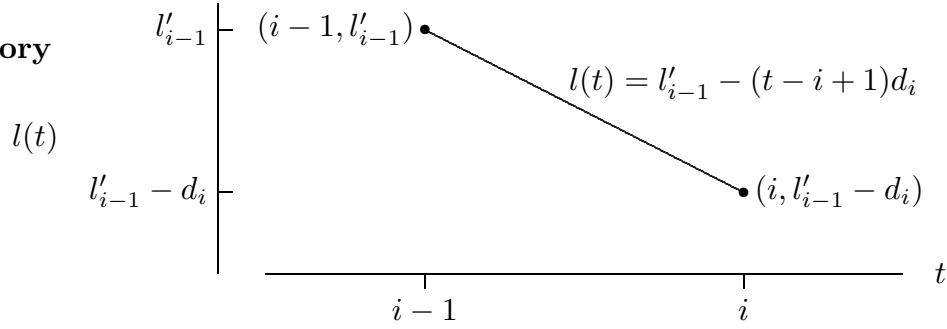
In this figure and related figures and equations elsewhere in this section, $l'_{i-1} = l_{i-1} + o_{i-1}$ represents the inventory level *after* inventory review. Accordingly, $l'_{i-1} \geq s$ for all i. (For the figure in Example 1.3.1, the ∘'s and •'s represent l_{i-1} and l'_{i-1} respectively).

The equation for $l(t)$ is the basis for calculating the time-averaged inventory level for the ith time interval.* There are two cases to consider. If $l(t)$ remains non-negative over the ith time interval then there is only a time-averaged *holding level* integral to evaluate:

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)\, dt.$$

If instead $l(t)$ becomes negative at some time τ interior to the ith interval then, in addition to a time-averaged holding level integral, there is also a time-averaged *shortage level* integral to evaluate. In this case the two integrals are

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)\, dt \qquad \text{and} \qquad \bar{l}_i^- = -\int_{\tau}^{i} l(t)\, dt.$$

---

* Because the inventory level at any time is an *integer*, the figure in Definition 1.3.3 is technically incorrect. Instead, rounding to an integer value should be used to produce the inventory level time history illustrated in Figure 1.3.6. ($\lfloor z \rfloor$ is the floor function; $\lfloor z + 0.5 \rfloor$ is z rounded to the nearest integer.)

**Figure 1.3.6.
Piecewise
constant
inventory
level in
time
interval i.**
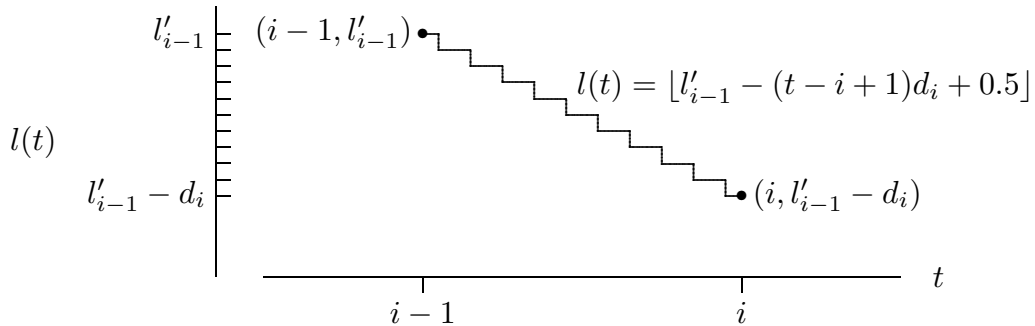
It can be shown, however, that rounding has *no* effect on the value of $\bar{l}_i^+$ and $\bar{l}_i^-$.

No Back Ordering

The inventory level $l(t)$ remains non-negative throughout the ith time interval if and only if the inventory level at the end of this interval is non-negative, as in Figure 1.3.7.
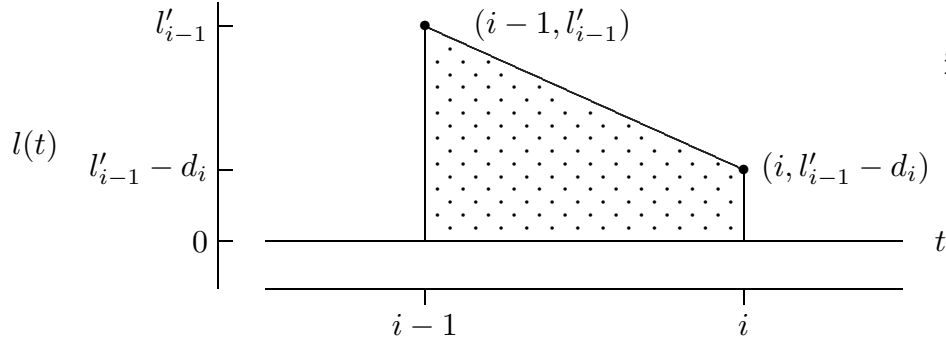
Figure 1.3.7. **Inventory level** **in time interval** i **with no** **backordering.**

Therefore, there is no shortage during the ith time interval if and only if $d_i \leq l'_{i-1}$. In this case the time-averaged holding level integral for the ith time interval can be evaluated as the area of a trapezoid so that

$$\bar{l}_i^+ = \int_{i-1}^{i} l(t)\, dt = \frac{l'_{i-1} + (l'_{i-1} - d_i)}{2} = l'_{i-1} - \frac{1}{2}d_i \qquad \text{and} \qquad \bar{l}_i^- = 0.$$

With Back Ordering

The inventory level becomes negative at some point τ in the ith time interval if and only if $d_i > l'_{i-1}$, as illustrated in Figure 1.3.8.
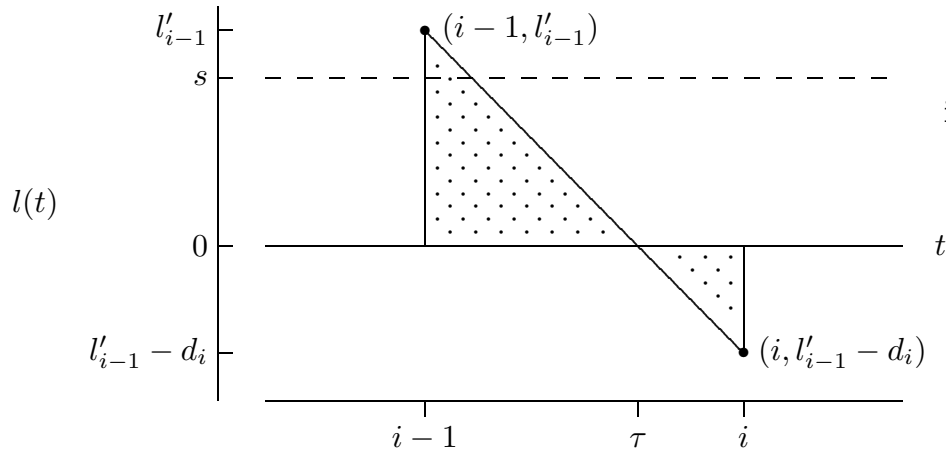
Figure 1.3.8. **Inventory level** **in time interval** i **with** **backordering.**

By using similar triangles, it can be shown that $\tau = i - 1 + (l'_{i-1}/d_i)$. In this case, the time-averaged holding level integral and shortage level integral for the ith time interval can be evaluated as the area of a triangle so that

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t)\, dt = \cdots = \frac{(l'_{i-1})^2}{2d_i} \qquad \text{and} \qquad \bar{l}_i^- = -\int_{\tau}^{i} l(t)\, dt = \cdots = \frac{(d_i - l'_{i-1})^2}{2d_i}.$$

The time-averaged holding level and shortage level for each time interval can be summed over all intervals with the resulting sums divided by the number of intervals. Consistent with Definition 1.3.4, the result represents the average number of items "held" and "short" respectively, with the average taken over all time intervals.

Definition 1.3.4   The *time-averaged holding level* and the *time-averaged shortage level* are, respectively

$$\bar{l}^+ = \frac{1}{n}\sum_{i=1}^{n} \bar{l}_i^+ \qquad \text{and} \qquad \bar{l}^- = \frac{1}{n}\sum_{i=1}^{n} \bar{l}_i^-.$$

It is potentially confusing to define the time-averaged shortage level as a *positive* number, as we have done in Definition 1.3.3. In particular, it would be a mistake to compute the time-averaged inventory level as the sum of $\bar{l}^+$ and $\bar{l}^-$. Instead, the *time-averaged inventory level* is the difference

$$\bar{l} = \frac{1}{n}\int_0^n l(t)\,dt = \bar{l}^+ - \bar{l}^-.$$

The proof of this result is left as an exercise.

Example 1.3.3   For the data in Example 1.3.1, $\bar{l}^+ = 31.74$ and $\bar{l}^- = 0.70$. Therefore, over the 12 time intervals, the average number of items held was 31.74, the average number of items short was 0.70, and the average inventory level was 31.04.

1.3.4   COMPUTATIONAL MODEL

Algorithm 1.3.1 is the basis for program `sis1` presented at the end of this section — a trace-driven computational model of a simple inventory system.

Program sis1

Program `sis1` computes five statistics: $\bar{d}$, $\bar{o}$, $\bar{l}^+$, $\bar{l}^-$ and the order frequency $\bar{u}$, which is

$$\bar{u} = \frac{\text{number of orders}}{n}.$$

Because the simulated system is flow balanced, $\bar{o} = \bar{d}$ and so it would be sufficient for program `sis1` to compute just one of these two statistics. The independent computation of both $\bar{o}$ and $\bar{d}$ is desirable, however, because it provides an important consistency check for a (flow balanced) simple inventory system.

Example 1.3.4   Program `sis1` reads input data corresponding to $n = 100$ time intervals from the file `sis1.dat`. With the inventory policy parameter values $(s, S) = (20, 80)$ the results (with *dd.dd* precision) are

$$\bar{o} = \bar{d} = 29.29 \qquad \bar{u} = 0.39 \qquad \bar{l}^+ = 42.40 \qquad \bar{l}^- = 0.25.$$

As with program `ssq1`, in Chapter 3 we will free program `sis1` from its reliance on external data by using randomly generated demand data instead.

1.3.5 OPERATING COST

Definition 1.3.5 In conjunction with the four statistics $\bar{o}$, $\bar{u}$, $\bar{l}^+$ and $\bar{l}^-$, a facility's cost of operation is determined by four constants:

- c_{item} — the (unit) cost of a new item;

- c_{setup} — the setup cost associated with placing an order;

- c_{hold} — the cost to hold one item for one time interval;

- c_{short} — the cost of being short one item for one time interval.

Case Study

Consider a hypothetical automobile dealership that uses a weekly periodic inventory review policy. The facility is the dealer's showroom, service area and surrounding storage lot and the items that flow into and out of the facility are new cars. The supplier is the manufacturer of the cars and the customers are people convinced by clever advertising that their lives will be improved significantly if they purchase a new car from this dealer.

Example 1.3.5 Suppose space in the facility is limited to a maximum of, say $S = 80$, cars. (This is a small dealership.) Every Monday morning the dealer's inventory of cars is reviewed and if the inventory level at that time falls below a threshold, say $s = 20$, then enough new cars are ordered from the supplier to restock the inventory to level S.*

- The (unit) cost to the dealer for each new car ordered is $c_{\text{item}} = \$8000$.

- The setup cost associated with deciding what cars to order (color, model, options, etc.) and arranging for additional bank financing (this is not a rich automobile dealer) is $c_{\text{setup}} = \$1000$, independent of the number ordered.

- The holding cost (interest charges primarily) to the dealer, per week, to have a car sit unsold in his facility is $c_{\text{hold}} = \$25$.

- The shortage cost to the dealer, per week, to not have a car in inventory is hard to determine because, in our model, we have assumed that *all* demand will ultimately be satisfied. Therefore, any customer who wants to buy a new car, even if none are available, will agree to wait until next Monday when new cars arrive. Thus the shortage cost to the dealer is primarily in goodwill. Our dealer realizes, however, that in this situation customers may buy from another dealer and so, when a shortage occurs, he sweetens his deals by agreeing to pay "shorted" customers $100 cash *per day* when they come back on Monday to pick up their new car. This means that the cost of being short one car for one week is $c_{\text{short}} = \$700$.

---

* There will be some, perhaps quite significant, delivery lag but that is ignored, for now, in our model. In effect, we are assuming that this dealer is located adjacent to the supplier and that the supplier responds immediately to each order.

Definition 1.3.6   A simple inventory system's average operating costs *per time interval* are defined as follows:

- item cost: $c_{\text{item}} \cdot \bar{o}$;

- setup cost: $c_{\text{setup}} \cdot \bar{u}$;

- holding cost: $c_{\text{hold}} \cdot \bar{l}^+$;

- shortage cost: $c_{\text{short}} \cdot \bar{l}^-$.

The average total cost of operation per time interval is the sum of these four costs. This sum multiplied by the number of time intervals is the total cost of operation.

Example 1.3.6   From the statistics in Example 1.3.4 and the constants in Example 1.3.5, for our auto dealership the average costs are:

- the item cost is $\$8000 \cdot 29.29 = \$234,320$;

- the setup cost is $\$1000 \cdot 0.39 = \$390$;

- the holding cost is $\$25 \cdot 42.40 = \$1060$;

- the shortage cost is $\$700 \cdot 0.25 = \$175$.

Each of these costs is a per week average.

Cost Minimization

Although the inventory system statistic of primary interest is the average total cost per time interval, it is important to know the four components of this total cost. By varying the value of s (and possibly S) it seems reasonable to expect that an optimal (minimal average cost) periodic inventory review policy can be determined for which these components are properly balanced.

In a search for optimal (s, S) values, because $\bar{o} = \bar{d}$ and $\bar{d}$ depends only on the demand sequence, it is important to note that the item cost is *independent* of (s, S). Therefore, the only cost that can be controlled by adjusting the inventory policy parameters is the sum of the average setup, holding, and shortage costs. In Example 1.3.7, this sum is called the average *dependent cost*. For reference, in Example 1.3.6 the average dependent cost is $\$390 + \$1060 + \$175 = \$1625$ per week.

If S and the demand sequence is fixed, and if s is systematically increased, say from 0 to some large value less than S, then we expect to see the following.

- Generally, the average setup cost and holding cost will increase with increasing s.

- Generally, the average shortage cost will decrease with increasing s.

- Generally, the average total cost will have a 'U' shape indicating the presence of one (or more) optimal value(s) of s.

Example 1.3.7 is an illustration.

Example 1.3.7 With S fixed at 80, a modified version of program `sis1` was used to study how the total cost relates to the value of s. That is, the cost constants in Example 1.3.5 were used to compute the average setup, holding, shortage, and dependent cost for a range of s values from 1 to 40. As illustrated in Figure 1.3.9, the minimum average dependent cost is approximately \$1550 at $s = 22$.
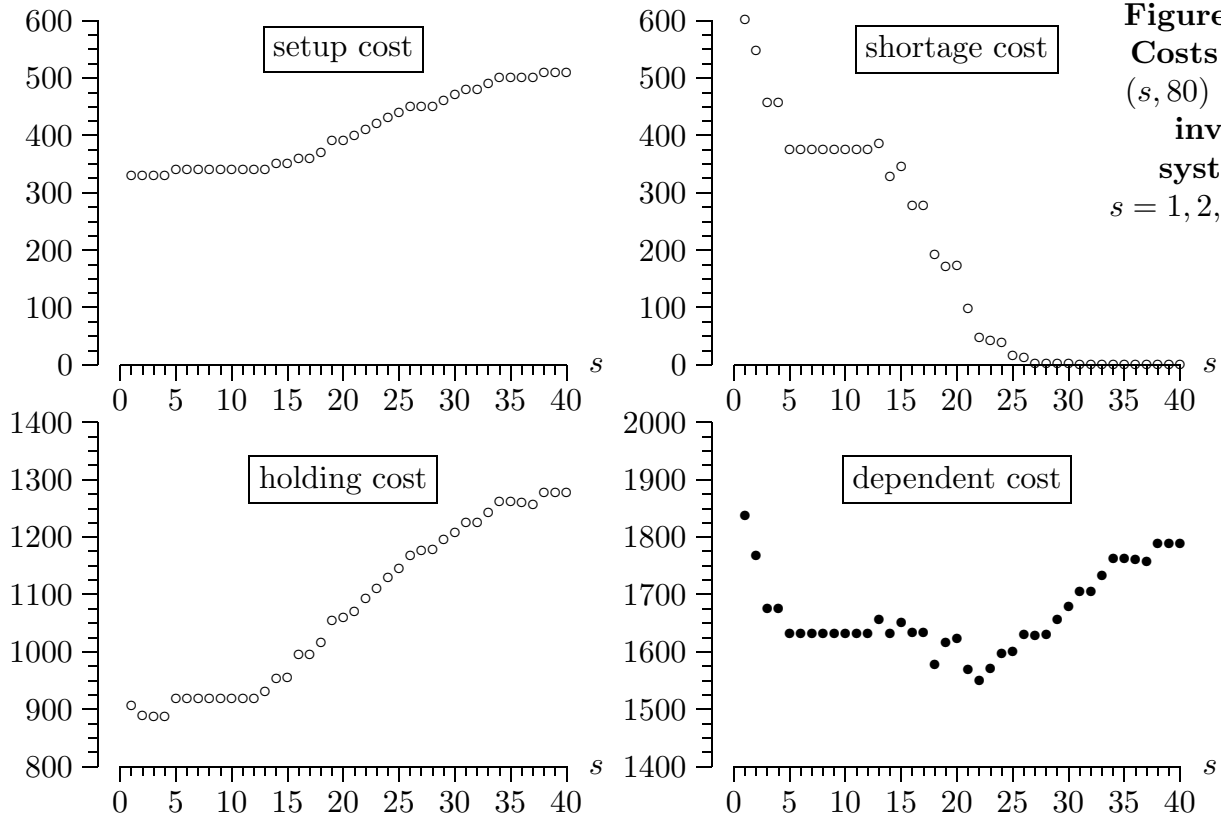
Figure 1.3.9. Costs for an $(s, 80)$ simple inventory system for $s = 1, 2, \ldots, 40$.

As in the case study concerning the ice cream shop, the "raw" simulation output data is presented. In this case, however, because the parameter s is inherently integer-valued (and so there is no "missing" output data at, say, $s = 22.5$) no interpolating or approximating curve is superimposed. [For a more general treatment of issues surrounding simulation optimization, see Andradóttir (1998).]

1.3.6 EXERCISES

Exercise 1.3.1 Verify that the results in Example 1.3.1 and the averages in Examples 1.3.2 and 1.3.3 are correct.

Exercise 1.3.2 (*a*) Using the cost constants in Example 1.3.5, modify program `sis1` to compute all four components of the total average cost per week. (*b*) These four costs may differ somewhat from the numbers in Example 1.3.6. Why? (*c*) By constructing a graph like that in Example 1.3.7, explain the trade-offs involved in concluding that $s = 22$ is the optimum value (when $S = 80$). (*d*) Comment on how well-defined this optimum is.

Exercise 1.3.3 Suppose that the inventory level $l(t)$ has a constant rate of change over the time interval $a \le t \le b$ and both $l(a)$ and $l(b)$ are integers. (a) Prove that

$$\int_a^b l(t)\, dt = \int_a^b \lfloor l(t) + 0.5 \rfloor\, dt = \frac{1}{2}(b-a)\big(l(a) + l(b)\big).$$

(b) What is the value of this integral if $l(t)$ is truncated rather than rounded (i.e., if the 0.5 is omitted in the second integral)?

Exercise 1.3.4 (a) Construct a table or figure similar to Figure 1.3.7 but for $S = 100$ and $S = 60$. (b) How does the minimum cost value of s seem to depend on S? (See Exercise 1.3.2.)

Exercise 1.3.5 Provided there is no delivery lag, prove that if $d_i \le s$ for $i = 1, 2, \ldots, n$, then $\bar{l}^- = 0$.

Exercise 1.3.6 (a) Provided there is no delivery lag, prove that if $S - s < d_i \le S$ for $i = 1, 2, \ldots, n$ then $\bar{l}^+ = S - \bar{d}/2$. (b) What is the value of $\bar{l}^-$ and $\bar{u}$ in this case?

Exercise 1.3.7 Use Definitions 1.3.3 and 1.3.4 to prove that the average inventory level equation

$$\bar{l} = \frac{1}{n} \int_0^n l(t)\, dt = \bar{l}^+ - \bar{l}^-$$

is correct. *Hint*: use the $(\cdot)^+$ and $(\cdot)^-$ functions defined for any integer (or real number) x as

$$x^+ = \frac{|x| + x}{2} \qquad \text{and} \qquad x^- = \frac{|x| - x}{2}$$

and recognize that $x = x^+ - x^-$.

Exercise 1.3.8 (a) Modify program `sis1` so that the demands are first read into a circular array, then read out of that array, as needed, during program execution. (b) By experimenting with different starting locations for reading the demands from the circular array, explore how sensitive the program's statistical output is to the order in which the demands occur.

Exercise 1.3.9[a] (a) Consider a variant of Exercise 1.3.8, where you use a conventional (non-circular) array and randomly *shuffle* the demands within this array before the demands are then read out of the array, as needed, during program execution. (b) Repeat for at least 10 different random shuffles and explore how sensitive the program's statistical output is to the order in which the demands occur.

CHAPTER 2

RANDOM NUMBER GENERATION

The material in this chapter presumes some knowledge of integer arithmetic and elementary number theory; for background material see Appendix B.

Sections

Discrete-event and Monte Carlo simulation cannot be done correctly without access to a good random number generator. Ideally, computer users should be able to assume the existence of a good system-supplied generator that can be used as a "black box" like other standard, reliable mathematical functions (`sqrt`, `sin`, `exp`, etc.). Unfortunately, history suggests this is frequently a false assumption. For that reason, the first two sections of this chapter provide a comprehensive discussion of an easily understood random number generation algorithm that can be used with confidence.

Section 2.1 is introductory, beginning with a conceptual model of the two-parameter algorithm as equivalent to drawing, at random, from an urn whose contents are determined by the choice of parameters. Given an appropriate choice of parameter values, Section 2.2 is primarily concerned with the important issue of correct algorithm implementation. That is, a software implementation of the random number generation algorithm is developed that is correct, efficient, and portable to essentially any contemporary computing system. This software implementation is then used in Section 2.3 as the basis for introducing Monte Carlo simulation as it relates to the estimation of probabilities.

Section 2.4 provides Monte Carlo examples that are more involved and interesting than those in the third section. Unlike the material in the first two sections, which is specific to a particular random number generation algorithm, Section 2.5 provides a more general discussion of random number generation algorithms.

Programs `ssq1` and `sis1` both require input data from an external source. Because of this, the usefulness of these programs is limited by the amount of available input data. What if more input data is needed? Or, what if the model is changed; can the input data be modified accordingly? Or, what can be done if only a small amount of input data is available, or perhaps, none at all?

In each case the answer is to use a *random number generator*. By convention, each call to the random number generator will produce a real-valued result between 0.0 and 1.0 that becomes the inherent source of randomness for a discrete-event simulation model. As illustrated later in this chapter, consistent with user-defined stochastic models of arrival times, service times, demand amounts, etc., the random number generator's output can be converted to a *random variate* via an appropriate mathematical transformation. The random variates are used to approximate some probabilistic element (e.g., service times) of a real-world system for a discrete-event simulation model.

2.1.1   RANDOM NUMBER GENERATION

Historically three types of random number generators have been advocated for computational applications: (*a*) 1950's-style table look-up generators like, for example, the RAND corporation table of a million random digits; (*b*) hardware generators like, for example, thermal "white noise" devices; and (*c*) algorithmic (software) generators. Of these three types, only algorithmic generators have achieved widespread acceptance. The reason for this is that only algorithmic generators have the potential to satisfy *all* of the following generally well-accepted random number generation criteria. A generator should be:

- *random* — able to produce output that passes all reasonable statistical tests of randomness;

- *controllable* — able to reproduce its output, if desired;

- *portable* — able to produce the same output on a wide variety of computer systems;

- *efficient* — fast, with minimal computer resource requirements;

- *documented* — theoretically analyzed and extensively tested.

In addition, as discussed and illustrated in Chapter 3, it should be easy to partition the generator's output into multiple "streams".

Definition 2.1.1   An *ideal* random number generator is a function, say `Random`, with the property that *each* assignment

u = Random();

will produce a real-valued (floating-point) number u between 0.0 and 1.0 in such a way that any value in the interval $0.0 < u < 1.0$ is *equally likely* to occur. A *good* random number generator produces results that are statistically indistinguishable, for all practical purposes, from those produced by an ideal generator.

What we will do in this chapter is construct, from first principles, a good random number generator that will satisfy all the criteria listed previously. We begin with the following conceptual model.

- Choose a *large* positive integer m. This defines the set $\mathcal{X}_m = \{1, 2, \ldots, m-1\}$.

- Fill a (conceptual) urn with the elements of $\mathcal{X}_m$.

- Each time a random number u is needed, draw an integer x "at random" from the urn and let $u = x/m$.

Each draw *simulates* a realization (observation, sample) of an independent identically distributed (*iid*) sequence of so-called *Uniform*$(0, 1)$ random variables. Because the possible values of u are $1/m$, $2/m$, ..., $1 - 1/m$, it is important for m to be so large that the possible values of u will be densely distributed between 0.0 and 1.0. Note that the values $u = 0.0$ and $u = 1.0$ are impossible. As discussed in later chapters, excluding these two extreme values is important for avoiding problems associated with certain random variate generation algorithms.

Ideally we would like to draw from the urn independently and *with* replacement. If so then each of the $m - 1$ possible values of u would be equally likely to be selected on each draw. For practical reasons, however, we will use a random number generation algorithm that simulates drawing from the urn *without* replacement. Fortunately, if m is large and the number of draws is small relative to m then the distinction between drawing with and without replacement is largely irrelevant. To turn the conceptual urn model into an specification model we will use a time-tested algorithm suggested by Lehmer (1951).

2.1.2 LEHMER'S ALGORITHM

Definition 2.1.2 *Lehmer's algorithm* for random number generation is defined in terms of two fixed parameters

- *modulus* m, a fixed large *prime* integer

- *multiplier* a, a fixed integer in $\mathcal{X}_m$

and the subsequent generation of the integer sequence x_0, x_1, x_2, ... via the iterative equation

- $x_{i+1} = g(x_i)$ $\qquad i = 0, 1, 2, \ldots$

where the function $g(\cdot)$ is defined for all $x \in \mathcal{X}_m = \{1, 2, \ldots, m-1\}$ as

- $g(x) = ax \bmod m$

and the *initial seed* x_0 is chosen from the set $\mathcal{X}_m$. The *modulus function* mod gives the remainder when the first argument (ax in this case) is divided by the second argument (the modulus m in this case). It is defined more carefully in Appendix B. A random number generator based on Lehmer's algorithm is called a *Lehmer generator*.

Because of the mod (remainder) operator, the value of $g(x)$ is always an integer between 0 and $m - 1$; however, it is important to exclude 0 as a possible value. That is, $g(0) = 0$ and so if 0 ever occurs in the sequence $x_0, x_1, x_2, \ldots$ then *all* the terms in the sequence will be 0 from that point on. Fortunately, if (a, m) is chosen consistent with Definition 2.1.2, it can be shown that $g(x) \neq 0$ for all $x \in \mathcal{X}_m$ so that $g : \mathcal{X}_m \to \mathcal{X}_m$. This guaranteed avoidance of 0 is part of the reason for choosing m to be prime. Because $g : \mathcal{X}_m \to \mathcal{X}_m$, it follows (by induction) that if $x_0 \in \mathcal{X}_m$ then $x_i \in \mathcal{X}_m$ for all $i = 0, 1, 2, \ldots$

Lehmer's algorithm represents a good example of the elegance of simplicity. The genius of Lehmer's algorithm is that *if* the multiplier a and prime modulus m are properly chosen, the resulting sequence $x_0, x_1, x_2, \ldots, x_{m-2}$ will be statistically indistinguishable from a sequence drawn at random, albeit without replacement, from $\mathcal{X}_m$. Indeed, it is *only* in the sense of *simulating* this random draw that the algorithm is random; there is actually *nothing* random about Lehmer's algorithm, except possibly the choice of the initial seed. For this reason Lehmer generators are sometimes called *pseudo*-random.*

An intuitive explanation of why Lehmer's algorithm simulates randomness is based on the observation that if two large integers, a and x, are multiplied and the product divided by another large integer, m, then the remainder, $g(x) = ax \bmod m$, is "likely" to take on any value between 0 and $m - 1$, as illustrated in Figure 2.1.1.

Figure 2.1.1.
Lehmer
generator
geometry.
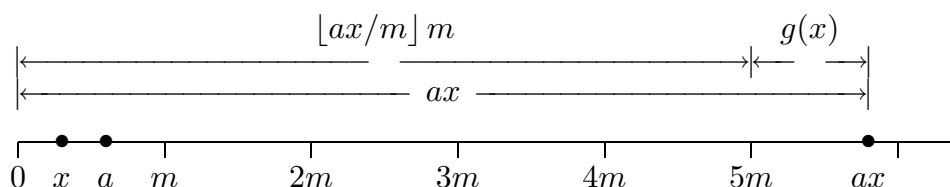
This is somewhat like going to the grocery store with no change, only dollars, and buying many identical items. After you pay for the items, the change you will receive is likely to be any value between 0¢ and 99¢. [That is, a is the price of the item in cents, x is the number of items purchased, and $m = 100$. The analogy is not exact, however, because the change is $m - g(x)$, not $g(x)$].

Modulus and Multiplier Considerations

To construct a Lehmer generator, standard practice is to choose the large prime modulus m first, then choose the multiplier a. The choice of m is dictated, in part, by system considerations. For example, on a computer system that supports 32-bit, 2's complement integer arithmetic, $m = 2^{31} - 1$ is a natural choice because it is the largest possible positive integer and it happens to be prime. (For 16-bit integer arithmetic we are not so fortunate, however, because $2^{15} - 1$ is not prime. See Exercise 2.1.6. Similarly, for 64-bit integer arithmetic $2^{63} - 1$ is not prime. See Exercise 2.1.10.) Given m, the subsequent choice of a must be made with great care. The following example is an illustration.

---

* Lehmer generators are also known as (take a deep breath) prime modulus multiplicative linear congruential pseudo-random number generators, abbreviated PMMLCG.

Example 2.1.1 As a tiny example, consider the prime modulus $m = 13$.*

* If the multiplier is $a = 6$ and the initial seed is $x_0 = 1$ then the resulting sequence of x's is

$$1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11, 1, \ldots$$

where, as the ellipses indicate, the sequence is actually *periodic* because it begins to cycle (with a *full* period of length $m - 1 = 12$) when the initial seed reappears. The point is that any 12 consecutive terms in this sequence appear to have been drawn at random, without replacement, from the set $\mathcal{X}_{13} = \{1, 2, \ldots, 12\}$.

* If the multiplier is $a = 7$ and the initial seed is $x_0 = 1$ then the resulting full-period sequence of x's is

$$1, 7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1, \ldots$$

Randomness is, like beauty, only in the eye of the beholder. Because of the $12, 6, 3$ and $8, 4, 2, 1$ patterns, however, most people would consider this second sequence to be "less random" than the first.

* If the multiplier is $a = 5$ then either

$$1, 5, 12, 8, 1, \ldots \qquad \text{or} \qquad 2, 10, 11, 3, 2, \ldots \qquad \text{or} \qquad 4, 7, 9, 6, 4, \ldots$$

will be produced, depending on the initial seed $x_0 = 1, 2$ or 4. This type of less-than-full-period behavior is clearly undesirable because, in terms of our conceptual model of random number generation, this behavior corresponds to first partitioning the set $\mathcal{X}_m$ into several disjoint subsets (urns), then selecting one subset and thereafter drawing exclusively from it.

Example 2.1.1 illustrates two of the three central issues that must be resolved when choosing (a, m).

* Does the function $g(\cdot)$ generate a full-period sequence?

* If a full-period sequence is generated, how random does the sequence appear to be?

The third central issue is implementation.

* Can the $ax \bmod m$ operation be evaluated efficiently and correctly for all $x \in \mathcal{X}_m$?

For Example 2.1.1 the issue of implementation is trivial. However, for realistically large values of a and m, a portable, efficient implementation in a high-level language is a substantive issue because of potential integer overflow associated with the ax product. We will return to this implementation issue in the next section. In the remainder of this section we will concentrate on the full-period issue.

---

* For a hypothetical computer system with 5-bit 2's complement integer arithmetic, the largest positive integer would be $2^4 - 1 = 15$. In this case $m = 13$ would be the largest possible prime and, in that sense, a natural choice for the modulus.

Full Period Considerations

From the definition of the mod function (see Appendix B), if $x_{i+1} = g(x_i)$ then there exists a non-negative integer $c_i = \lfloor ax_i/m \rfloor$ such that

$$x_{i+1} = g(x_i) = ax_i \bmod m = ax_i - mc_i \qquad i = 0, 1, 2, \ldots$$

Therefore (by induction)

$$
\begin{aligned}
x_1 &= ax_0 - mc_0 \\
x_2 &= ax_1 - mc_1 = a^2 x_0 - m(ac_0 + c_1) \\
x_3 &= ax_2 - mc_2 = a^3 x_0 - m(a^2 c_0 + ac_1 + c_2) \\
&\;\;\vdots \\
x_i &= ax_{i-1} - mc_{i-1} = a^i x_0 - m(a^{i-1} c_0 + a^{i-2} c_1 + \cdots + c_{i-1}).
\end{aligned}
$$

Because $x_i \in \mathcal{X}_m$ we have $x_i = x_i \bmod m$. Moreover, $(a^i x_0 - mc) \bmod m = a^i x_0 \bmod m$ independent of the value of the integer $c = a^{i-1} c_0 + a^{i-2} c_1 + \cdots + c_{i-1}$. Therefore, we have proven the following theorem.

Theorem 2.1.1   If the sequence x_0, x_1, x_2, ... is produced by a Lehmer generator with multiplier a and modulus m then

$$x_i = a^i x_0 \bmod m \qquad i = 0, 1, 2, \ldots$$

Although it would be an eminently bad idea to compute x_i by first computing a^i, particularly if i is large, Theorem 2.1.1 has significant theoretical importance due, in part, to the fact that x_i can be written equivalently as

$$x_i = a^i x_0 \bmod m = [(a^i \bmod m)\,(x_0 \bmod m)] \bmod m = [(a^i \bmod m)\,x_0] \bmod m$$

for $i = 0, 1, 2, \ldots$ via Theorem B.2 in Appendix B. In particular, this is true for $i = m - 1$. Therefore, because m is prime and $a \bmod m \neq 0$, from Fermat's little theorem (see Appendix B) it follows that $a^{m-1} \bmod m = 1$ and so $x_{m-1} = x_0$. This observation is the key to proving the following theorem. The details of the proof are left as an exercise.

Theorem 2.1.2   If $x_0 \in \mathcal{X}_m$ and the sequence x_0, x_1, x_2, ... is produced by a Lehmer generator with multiplier a and (prime) modulus m then there is a positive integer p with $p \leq m - 1$ such that $x_0, x_1, x_2, \ldots, x_{p-1}$ are all different and

$$x_{i+p} = x_i \qquad i = 0, 1, 2, \ldots$$

That is, the sequence $x_0, x_1, x_2, \ldots$ is *periodic* with *fundamental period* p. In addition, $(m - 1) \bmod p = 0$.

The significance of Theorem 2.1.2 is profound. If we pick *any* initial seed $x_0 \in \mathcal{X}_m$ and start to generate the sequence $x_0, x_1, x_2, \ldots$ then we are guaranteed that the initial seed will reappear. In addition, the first instance of reappearance is guaranteed to occur at some index p that is either $m - 1$ or an integer divisor of $m - 1$ and from that index on the sequence will cycle through the same p distinct values as illustrated

$$\underbrace{x_0, x_1, \ldots, x_{p-1}}_{\text{period}}, \underbrace{x_0, x_1, \ldots, x_{p-1}}_{\text{period}}, x_0, \ldots$$

Consistent with Definition 2.1.3, we are interested in choosing multipliers for which the fundamental period is $p = m - 1$.

Full Period Multipliers

Definition 2.1.3   The sequence x_0, x_1, x_2, ... produced by a Lehmer generator with modulus m and multiplier a has a *full period* if and only if the fundamental period p is $m - 1$. If the sequence has a full period then a is said to be a *full-period multiplier* relative to m.*

If the sequence x_0, x_1, x_2, ... has a full period then any $m - 1$ consecutive terms in the sequence have been drawn without replacement from the set $\mathcal{X}_m$. In effect a full-period Lehmer generator creates a virtual *circular list* with $m - 1$ distinct elements. The initial seed provides a starting list element; subsequent calls to the generator traverse the list.

Example 2.1.2   From Example 2.1.1, if $m = 13$ then $a = 6$ and $a = 7$ are full-period multipliers ($p = 12$) and $a = 5$ is not ($p = 4$). The virtual circular lists (with traversal in a clockwise direction) corresponding to these two full-period multipliers are shown in Figure 2.1.2.
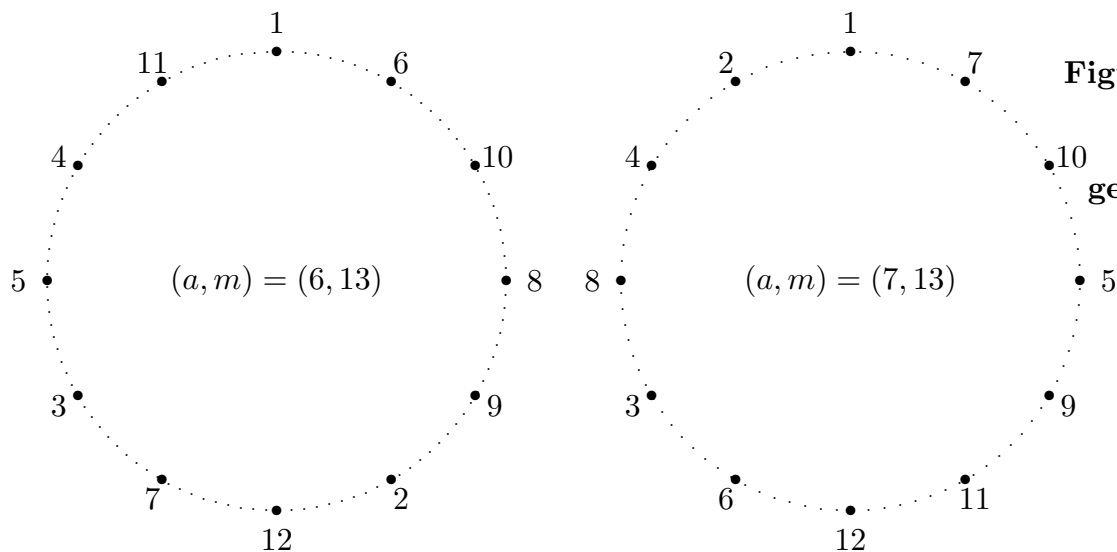
Figure 2.1.2. Two full period generators.

---

* A full-period multiplier a relative to m is also said to be a *primitive root* of m.

Algorithm 2.1.1   This algorithm, based upon Theorem 2.1.2, can be used to determine if a is a full-period multiplier relative to the prime modulus m.

```
p = 1;
x = a;
while (x != 1) {
    p++;
    x = (a * x) % m;                      /* beware of a * x overflow */
}
if (p == m - 1)
    /* a is a full-period multiplier */
else
    /* a is not a full-period multiplier */
```

If m is not prime Algorithm 2.1.1 may not halt. It provides a slow-but-sure $O(m)$* test for a full-period multiplier. The algorithm uses $x_0 = 1$ as an arbitrary initial seed (note that $x_1 = a$) and the recursive generation of new values of x until the initial seed reappears. Before starting to search for full-period multipliers, however, it would be good to know that they exist and with what frequency. That is, given a prime modulus m, how many corresponding full-period multipliers are there? The following theorem provides the answer to this question.

Theorem 2.1.3   If m is prime and $p_1, p_2, \ldots, p_r$ are the (unique) *prime factors* of $m - 1$ then the number of full-period multipliers in $\mathcal{X}_m$ is

$$\frac{(p_1 - 1)(p_2 - 1) \ldots (p_r - 1)}{p_1 p_2 \ldots p_r} (m - 1).$$

Example 2.1.3   If $m = 13$ then $m - 1 = 2^2 \cdot 3$. From the equation in Theorem 2.1.3 this prime modulus has $\frac{(2-1)(3-1)}{2 \cdot 3}(13 - 1) = 4$ full-period multipliers: $a = 2$, 6, 7, and 11.

Example 2.1.4   The Lehmer generator used in this book has the (Mersenne, i.e., of the form $2^k - 1$, where k is a positive integer) prime modulus $m = 2^{31} - 1 = 2\,147\,483\,647$. Because the prime decomposition of $m - 1$ is

$$m - 1 = 2^{31} - 2 = 2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$$

from the equation in Theorem 2.1.3 the number of full-period multipliers is

$$\left(\frac{1 \cdot 2 \cdot 6 \cdot 10 \cdot 30 \cdot 150 \cdot 330}{2 \cdot 3 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331} \right) (2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331) = 534\,600\,000.$$

Therefore, for this prime modulus approximately 25% of the multipliers between 1 and $m - 1$ are full-period multipliers.

---

  * A function f is called "order m", for example, written $O(m)$, if there exist real positive constants c_1 and c_2 independent of m such that $c_1 m \leq f(m) \leq c_2 m$.

Using Algorithm 2.1.1 (and *a lot* of computer time) it can be shown that if $m = 2^{31} - 1$ then $a = 2, 3, 4, 5, 6$ are not full-period multipliers but that $a = 7$ is. Remarkably, once *one* full-period multiplier has been found, in this case $a = 7$, then *all* the others can be found using the following $O(m)$ algorithm. This algorithm presumes the availability of a function `gcd` that returns the greatest common divisor of two positive integers (see Appendix B).

Algorithm 2.1.2 Given the prime modulus m and any full-period multiplier a, the following algorithm generates all the full-period multipliers relative to m.*

```
i = 1;
x = a;
while (x != 1) {
   if (gcd(i, m − 1) == 1)
     /* x is a full-period multiplier equal to a^i mod m */
   i++;
   x = (a * x) % m;                       /* beware of a * x overflow */
}
```

Algorithm 2.1.2 is based on Theorem 2.1.4 which establishes a one-to-one correspondence between integers $i \in \mathcal{X}_m$ that are relatively prime to $m-1$ and full-period multipliers $a^i \bmod m \in \mathcal{X}_m$. That is, the equation in Theorem 2.1.3 counts both the number of full-period multipliers *and* the number of integers in $\mathcal{X}_m$ that are relatively prime to $m - 1$. The proof of Theorem 2.1.4 is left as an (advanced) exercise.

Theorem 2.1.4 If a is any full-period multiplier relative to the prime modulus m then each of the integers

$$a^i \bmod m \in \mathcal{X}_m \qquad i = 1, 2, 3, \ldots, m - 1$$

is also a full-period multiplier relative to m if and only if the integer i has no prime factors in common with the prime factors of $m - 1$, i.e., i and $m - 1$ are relatively prime (see Appendix B).

Example 2.1.5 If $m = 13$ then from Example 2.1.3 there are 4 integers between 1 and 12 that are relatively prime to 12. They are $i = 1, 5, 7, 11$. From Example 2.1.1, $a = 6$ is a full-period multiplier relative to 13; from Theorem 2.1.4 the 4 full-period multipliers relative to 13 are therefore

$$6^1 \bmod 13 = 6, \qquad 6^5 \bmod 13 = 2, \qquad 6^7 \bmod 13 = 7, \qquad 6^{11} \bmod 13 = 11.$$

Equivalently, if we had known that $a = 2$ is a full-period multiplier relative to 13 we could have used Algorithm 2.1.2 with $a = 2$ to determine the full-period multipliers as

$$2^1 \bmod 13 = 2, \qquad 2^5 \bmod 13 = 6, \qquad 2^7 \bmod 13 = 11, \qquad 2^{11} \bmod 13 = 7.$$

---

* See Exercises 2.1.4 and 2.1.5.

Example 2.1.6 If $m = 2^{31} - 1$ then from Example 2.1.4 there are 534600000 integers between 1 and $m - 1$ that are relatively prime to $m - 1$. The first few of these are $i = 1, 5, 13, 17, 19$. As discussed previously, $a = 7$ is a full-period multiplier relative to this modulus and so from Algorithm 2.1.2

$$7^1 \bmod 2147483647 = 7$$
$$7^5 \bmod 2147483647 = 16807$$
$$7^{13} \bmod 2147483647 = 252246292$$
$$7^{17} \bmod 2147483647 = 52958638$$
$$7^{19} \bmod 2147483647 = 447489615$$

are full-period multipliers relative to $2^{31} - 1 = 2147483647$. Note that the full-period multiplier 16807 is a *logical* choice; 7 is the smallest full-period multiplier and 5 is the smallest integer (other than 1) that is relatively prime to $m - 1$. However, that the multiplier 16807 is a logical choice does not mean that the resulting sequence it generates is necessarily "random". We will have more to say about this in the next section.

Standard Algorithmic Generators

Most of the standard algorithmic generators in use today are one of the following three *linear congruential* types (Law and Kelton, 2000, pages 406–412):

- *mixed*: $g(x) = (ax + c) \bmod m$ with $m = 2^b$ ($b = 31$ typically), $a \bmod 4 = 1$ and $c \bmod 2 = 1$. All integer values of $x \in \{0\} \cup \mathcal{X}_m$ are possible and the period is m.

- *multiplicative* with $m = 2^b$: $g(x) = ax \bmod m$ with $m = 2^b$ ($b = 31$ typically) and $a \bmod 8 = 3$ or $a \bmod 8 = 5$. To achieve the maximum period, which is only $m/4$, x must be restricted to the odd integers in $\mathcal{X}_m$.

- *multiplicative* with m prime: $g(x) = ax \bmod m$ with m prime and a a full-period multiplier. All integer values of $x \in \mathcal{X}_m = \{1, 2, \ldots, m - 1\}$ are possible and the period is $m - 1$.

Of these three, specialists generally consider the third generator, which has been presented in this chapter, to be best. Random number generation remains an area of active research, however, and reasonable people will probably always disagree about the *best* algorithm for random number generation. Section 2.5 contains a more general discussion of random number generators.

For now we will avoid the temptation to be drawn further into the "what is the best possible random number generator" debate. Instead, as discussed in the next section, we will use a Lehmer generator with modulus $m = 2^{31} - 1$ and a corresponding full-period multiplier $a = 48271$ carefully selected to provide generally acceptable randomness *and* facilitate efficient software implementation. With this generator in hand, we can then turn our attention to the main topic of this book — the modeling, simulation and analysis of discrete-event stochastic systems.

2.1.3 EXERCISES

Exercise 2.1.1   For the tiny Lehmer generator defined by $g(x) = ax \bmod 127$, find all the full-period multipliers. (*a*) How many are there? (*b*) What is the smallest multiplier?

Exercise 2.1.2   Prove Theorem 2.1.2.

Exercise 2.1.3   Prove that if (a, m) are chosen consistent with Definition 2.1.2, then $g : \mathcal{X}_m \to \mathcal{X}_m$ is a bijection (one-to-one and onto).

Exercise 2.1.4ª   (*a*) Prove that if (a, m) are chosen consistent with Definition 2.1.2, then $g(x) \neq 0$ for all $x \in \mathcal{X}_m$ and so $g : \mathcal{X}_m \to \mathcal{X}_m$. (*b*) In addition, relative to this definition, prove that there is nothing to be gained by considering integer multipliers outside of $\mathcal{X}_m$. Consider the two cases $a \geq m$ and $a \leq 0$.

Exercise 2.1.5   (*a*) Except for the special case $m = 2$, prove that $a = 1$ cannot be a full-period multiplier. (*b*) What about $a = m - 1$?

Exercise 2.1.6   In ANSI C an `int` is guaranteed to hold all integer values between $-(2^{15} - 1)$ and $2^{15} - 1$ inclusive. (*a*) What is the largest prime modulus in this range? (*b*) How many corresponding full-period multipliers are there and what is the smallest one?

Exercise 2.1.7ª   Prove Theorem 2.1.4.

Exercise 2.1.8   (*a*) Evaluate $7^i \bmod 13$ and $11^i \bmod 13$ for $i = 1, 5, 7, 11$. (*b*) How does this relate to Example 2.1.5?

Exercise 2.1.9   (*a*) Verify that the list of five full-period multipliers in Example 2.1.6 is correct. (*b*) What are the next five elements in this list?

Exercise 2.1.10ª   (*a*) What is the largest prime modulus less than or equal to $2^{63} - 1$? (*b*) How many corresponding full-period multipliers are there? (*c*) How does this relate to the use of Lehmer random number generators on computer systems that support 64-bit integer arithmetic?

Exercise 2.1.11   For the first few prime moduli, this table lists the number of full-period multipliers and the smallest full-period multiplier. Add the next 10 rows to this table.

prime modulus m	number of full-period multipliers	smallest full-period multiplier a
2	1	1
3	1	2
5	2	2
7	2	3
11	4	2
13	4	2

Recall that one good reason to choose $m = 2^{31} - 1$ as the modulus for a Lehmer random number generator is that virtually all contemporary computer systems support 32-bit 2's complement integer arithmetic and on such systems $2^{31} - 1$ is the largest possible prime. Consistent with that observation, a portable and efficient algorithmic implementation of an $m = 2^{31} - 1$ Lehmer generator that is valid (provided all integers between $-m$ and m can be represented exactly) is developed in this section.

We can use ANSI C for the implementation because 32-bit signed integer arithmetic is supported in a natural way. That is, the ANSI C type **long** is required to be valid for all integers between **LONG_MIN** and **LONG_MAX** inclusive and, although the values are implementation dependent, **LONG_MAX** and **LONG_MIN** are required to be at least $2^{31} - 1$ and at most $-(2^{31} - 1)$ respectively.*

2.2.1 IMPLEMENTATION

If there is no guarantee that integers larger than m can be represented exactly, then the implementation issue of *potential integer overflow* must be addressed. That is, for any full-period Lehmer generator, the product ax can be as large as $a(m-1)$. Therefore, unless values of t as large as $a(m - 1)$ can be represented exactly, it is *not* possible to evaluate $g(x) = ax \bmod m$ in the "obvious" way by first computing the intermediate product $t = ax$ and then computing $t \bmod m = t - \lfloor t/m \rfloor m$.

Example 2.2.1   If $(a, m) = (48271, 2^{31} - 1)$ then $a(m - 1) \cong 1.47 \times 2^{46}$. Therefore, it would not be possible to implement this (a, m) Lehmer generator in the obvious way without access to a register that is at least 47 bits wide to store the intermediate product t. This is true even though $t \bmod m$ is no more than 31 bits wide. (If $a = 16807$ then a 46 bit register would be required to hold the intermediate product.)

Type Considerations

If we wish to implement a Lehmer generator with $m = 2^{31} - 1$ in ANSI C, and do it in the obvious way, then the type declaration of t will dictate the number of bits available to store the intermediate $t = ax$ product. Correspondingly, if t is (naturally) an *integer* type then the integer division t/m can be used to evaluate $\lfloor t/m \rfloor$; or if t is a *floating point* type then a floating point division t/m followed by a floating-point-to-integer cast can be used to evaluate $\lfloor t/m \rfloor$.

Example 2.2.2   If the variable t is declared to be a **long** and $m = 2^{31} - 1$ then the obvious implementation will be correct only if **LONG_MAX** is $a(m - 1)$ or larger. Most contemporary computer systems do not support integers this large and thus for a Lehmer generator with $m = 2^{31} - 1$ the obvious implementation is *not* a viable algorithm option.

---

* The macros **LONG_MIN** and **LONG_MAX** are defined in the ANSI C library **<limits.h>**. The type **long** is a shorthand representation for the type **long int**. **LONG_LONG** is supported in C99, the new (maybe latest) ANSI/ISO Standard for C for 64-bit integers. This option is only appropriate if the hardware supports 64-bit arithmetic (e.g., Apple, AMD).

Example 2.2.3   If the variable t is declared to be the ANSI C floating point type `double` then the obvious implementation may be correct provided the multiplier a is not too large. That is, `double` is generally consistent with the IEEE 754 64-bit floating point standard which specifies a 53-bit mantissa (including the sign bit) and if $m = 2^{31} - 1$ a mantissa this large allows for t to be much larger than m. So it *may* be possible to implement a $m = 2^{31} - 1$ Lehmer generator with a sufficiently small multiplier in the obvious way by doing the *integer* calculations in *floating point* arithmetic. However, when portability is required and an efficient integer-based implementation is possible, only the unwise would use a floating point implementation instead.

Algorithm Development

Consistent with the previous examples, it is desirable to have an integer-based implementation of Lehmer's algorithm that will port to any system which supports the ANSI C type `long`. This can be done provided no integer calculation produces an intermediate or final result larger than $m = 2^{31} - 1$ in magnitude. With this constraint in mind we must be prepared to do some algorithm development.

If it were possible to factor the modulus as $m = aq$ for some integer q then $g(x)$ could be written as $g(x) = ax \bmod m = a(x \bmod q)$, enabling us to do the mod *before* the multiply and thereby avoid the potential overflow problem. That is, in this case the largest possible value of $ax \bmod m$ would be $a(q-1) = m - a$ and this is less than m. Of course if m is prime no such factorization is possible. It is always possible, however, to "approximately factor" m as $m = aq + r$ with $q = \lfloor m/a \rfloor$ and $r = m \bmod a$. As demonstrated in this section, if the *remainder r* is small relative to the *quotient q*, specifically if $r < q$, then this (q, r) decomposition of m provides the basis for an algorithm to evaluate $g(x)$ in such a way that integer overflow is eliminated.

Example 2.2.4   If $(a, m) = (48271, 2^{31} - 1)$ then the quotient is $q = \lfloor m/a \rfloor = 44488$ and the remainder is $r = m \bmod a = 3399$. Similarly, if $a = 16807$ then $q = 127773$ and $r = 2836$. In both cases $r < q$.

For all $x \in \mathcal{X}_m = \{1, 2, \ldots, m - 1\}$ define the two functions

$$\gamma(x) = a(x \bmod q) - r\lfloor x/q \rfloor \qquad \text{and} \qquad \delta(x) = \lfloor x/q \rfloor - \lfloor ax/m \rfloor.$$

Then, for any $x \in \mathcal{X}_m$

$$
\begin{aligned}
g(x) = ax \bmod m &= ax - m\lfloor ax/m \rfloor \\
&= ax - m\lfloor x/q \rfloor + m\lfloor x/q \rfloor - m\lfloor ax/m \rfloor \\
&= ax - (aq + r)\lfloor x/q \rfloor + m\delta(x) \\
&= a\big(x - q\lfloor x/q \rfloor\big) - r\lfloor x/q \rfloor + m\delta(x) \\
&= a(x \bmod q) - r\lfloor x/q \rfloor + m\delta(x) = \gamma(x) + m\delta(x).
\end{aligned}
$$

An efficient, portable implementation of a Lehmer random number generator is based upon this alternate representation of $g(x)$ and the following theorem.

Theorem 2.2.1   If $m = aq + r$ is prime, $r < q$, and $x \in \mathcal{X}_m$, then $\delta(x) = \lfloor x/q \rfloor - \lfloor ax/m \rfloor$ is either 0 or 1. Moreover, with $\gamma(x) = a(x \bmod q) - r\lfloor x/q \rfloor$

- $\delta(x) = 0$ if and only if $\gamma(x) \in \mathcal{X}_m$;

- $\delta(x) = 1$ if and only if $-\gamma(x) \in \mathcal{X}_m$.

Proof   First observe that if u and v are real numbers with $0 < u - v < 1$ then the integer difference $\lfloor u \rfloor - \lfloor v \rfloor$ is either 0 or 1. Therefore, because $\delta(x) = \lfloor x/q \rfloor - \lfloor ax/m \rfloor$, the first part of the theorem is true if we can show that

$$0 < \frac{x}{q} - \frac{ax}{m} < 1.$$

Rewriting the center of the inequality as

$$\frac{x}{q} - \frac{ax}{m} = x\left(\frac{1}{q} - \frac{a}{m}\right) = x\left(\frac{m - aq}{mq}\right) = \frac{xr}{mq}$$

and because $r < q$

$$0 < \frac{xr}{mq} < \frac{x}{m} \leq \frac{m-1}{m} < 1$$

which establishes that if $x \in \mathcal{X}_m$ then $\delta(x)$ is either 0 or 1. To prove the second part of this theorem recall that if $x \in \mathcal{X}_m$ then $g(x) = \gamma(x) + m\delta(x)$ is in $\mathcal{X}_m$. Therefore, if $\delta(x) = 0$ then $\gamma(x) = g(x)$ is in $\mathcal{X}_m$; conversely if $\gamma(x) \in \mathcal{X}_m$ then $\delta(x)$ cannot be 1 for if it were then $g(x) = \gamma(x) + m$ would be $m + 1$ or larger which contradicts $g(x) \in \mathcal{X}_m$. Similarly, if $\delta(x) = 1$ then $-\gamma(x) = m - g(x)$ is in $\mathcal{X}_m$; conversely if $-\gamma(x) \in \mathcal{X}_m$ then $\delta(x)$ cannot be 0 for if it were then $g(x) = \gamma(x)$ would not be in $\mathcal{X}_m$.

Algorithm

The key to avoiding overflow in the evaluation of $g(x)$ is that the calculation with the potential to cause overflow, the ax product, is "trapped" in $\delta(x)$. This is important because, from Theorem 2.2.1, the value of $\delta(x)$ can be deduced from the value of $\gamma(x)$ and it can be shown that $\gamma(x)$ can be computed without overflow — see Exercise 2.2.5. This comment and Theorem 2.2.1 can be summarized with the following algorithm.

Algorithm 2.2.1   If $m = aq + r$ is prime, $r < q$, and $x \in \mathcal{X}_m$, then $g(x) = ax \bmod m$ can be evaluated as follows without producing any intermediate or final values larger than $m - 1$ in magnitude.

```
t = a * (x % q) - r * (x / q);                                    /* t = γ(x) */
if (t > 0)
    return (t);                                                   /* δ(x) = 0 */
else
    return (t + m);                                               /* δ(x) = 1 */
```

Modulus Compatibility

Definition 2.2.1 The multiplier a is *modulus-compatible* with the prime modulus m if and only if the remainder $r = m \bmod a$ is less than the quotient $q = \lfloor m/a \rfloor$.

If a is modulus-compatible with m then Algorithm 2.2.1 can be used as the basis for an implementation of a Lehmer random number generator. In particular, if the multiplier is modulus-compatible with $m = 2^{31} - 1$ then Algorithm 2.2.1 can be used to implement the corresponding Lehmer random number generator in such a way that it will port to any system that supports 32-bit integer arithmetic. From Example 2.2.4, for example, the full-period multiplier $a = 48271$ is modulus-compatible with $m = 2^{31} - 1$.

In general, there are no modulus-compatible multipliers beyond $(m-1)/2$. Moreover, as the following example illustrates, modulus-compatible multipliers are much more densely distributed on the low end of the $1 \le a \le (m-1)/2$ scale.

Example 2.2.5 The (tiny) modulus $m = 401$ is prime. Figure 2.2.1 illustrates, on the first line, the 38 associated modulus-compatible multipliers. On the second line are the 160 full-period multipliers and the third line illustrates the ten multipliers (3, 6, 12, 13, 15, 17, 19, 21, 23, and 66) that are *both* modulus-compatible and full-period.
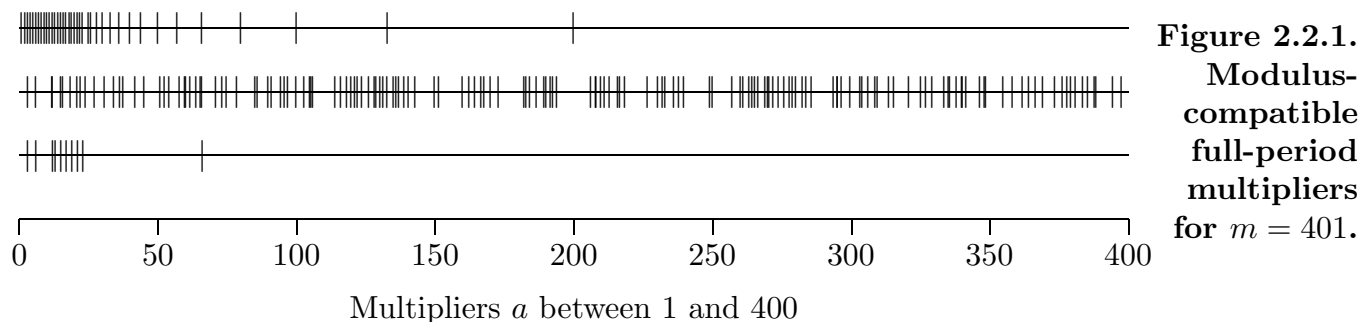
Figure 2.2.1.
Modulus-
compatible
full-period
multipliers
for $m = 401$.

Multipliers a between 1 and 400

If you have a very high resolution graphics device, a very long piece of paper, a magnifying glass, and a few hours of CPU time, you are encouraged to make the corresponding figure for $m = 2^{31} - 1$. (For this modulus there are 92679 modulus-compatible multipliers. Of these, 23093 are also full-period — see Example 2.2.6.)

Modulus-compatibility is closely related to the smallness of the multiplier. In particular, a is defined to be "small" if and only if $a^2 < m$. If a is small then a is modulus-compatible with m — see Exercise 2.2.4. From this result it follows, for example, that all the multipliers between 1 and 46340 inclusive are modulus-compatible with $m = 2^{31} - 1$. Thus smallness is sufficient to guarantee modulus-compatibility. Smallness is not a necessary condition, however. In particular, $a = 48271$ is modulus-compatible with $m = 2^{31} - 1$, but it is not small. Similarly, of the 38 modulus-compatible multipliers for $m = 401$ in Example 2.2.5, only about half (those between 1 and 20) satisfy the $a^2 < m$ small multiplier test.

As illustrated in Example 2.2.5, for a given prime modulus there are relatively few associated full-period, modulus-compatible multipliers. A mechanical way to find one of these is to start with the small (and thus modulus-compatible) multipliers $a = 2$, $a = 3$, etc.; until the first full-period multiplier is found — see Algorithm 2.1.1. Given that we have found one full-period, modulus-compatible multiplier, the following $O(m)$ algorithm can be used to generate *all* the others. This algorithm, an extension of Algorithm 2.1.2, presumes the availability of a function `gcd` that returns the greatest common divisor of two positive integers (see Appendix B).

Algorithm 2.2.2   Given the prime modulus m and any associated full-period, modulus-compatible multiplier a the following algorithm generates all the full-period, modulus-compatible multipliers relative to m.

```
i = 1;
x = a;
while (x != 1) {
   if ((m % x < m / x) and (gcd(i, m - 1) == 1))
      /* x is a full-period modulus-compatible multiplier */
   i++;
   x = g(x);        /* use Algorithm 2.2.1 to evaluate g(x) = ax mod m */
}
```

Example 2.2.6   Using Algorithm 2.2.2 with $m = 2^{31} - 1 = 2147483647$, $a = 7$ and a lot of CPU cycles we find that there are a total of 23093 associated full-period, modulus-compatible multipliers, the first few of which are

$$7^1 \bmod 2147483647 = 7$$
$$7^5 \bmod 2147483647 = 16807$$
$$7^{113039} \bmod 2147483647 = 41214$$
$$7^{188509} \bmod 2147483647 = 25697$$
$$7^{536035} \bmod 2147483647 = 63295.$$

Of these, the multiplier $a = 16807$ deserves special mention. It was first suggested by Lewis, Goodman, and Miller (1969), largely because it was easy to prove (as we have done) that it is a full-period multiplier. Since then it has become something of a "minimal" standard (Park and Miller, 1988).

In retrospect $a = 16807$ was such an obvious choice that it seems unlikely to be the best possible full-period multiplier relative to $m = 2^{31} - 1$ and, indeed, subsequent testing for randomness has verified that, at least in theory, other full-period multipliers generate (slightly) more random sequences. Although several decades of generally favorable user experience with $a = 16807$ is not easily ignored, we will use $a = 48271$ instead.

Randomness

For a given (prime) modulus m, from among all the full-period, modulus-compatible multipliers relative to m we would like to choose the one that generates the "most random" sequence. As suggested in Example 2.1.1, however, there is no simple and universal definition of randomness and so it should come as no surprise to find that there is less than complete agreement on what this metric should be.

Example 2.2.7   To the extent that there is agreement on a randomness metric, it is based on a fundamental geometric characteristic of all Lehmer generators, first established by Marsaglia (1968), that "random numbers fall mainly in the planes." That is, in 2-space for example, the points

$$(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots$$

all fall on a finite, and possibly small, number of parallel lines to form a *lattice* structure as illustrated in Figure 2.2.2 for $(a, m) = (23, 401)$ and $(66, 401)$. If a simulation model of a nuclear power plant, for example, required two consecutive small random numbers (e.g., both smaller than 0.08) for a particular rare event (e.g., melt down), this would *never* occur for the $(66, 401)$ generator due to the void in the southwest corner of the graph. [Using the axiomatic approach to probability, the exact probability is $(0.08)(0.08) = 0.0064$.]

Figure 2.2.2. Random numbers falling in planes.

$$(a, m) = (23, 401)$$

$$(a, m) = (66, 401)$$
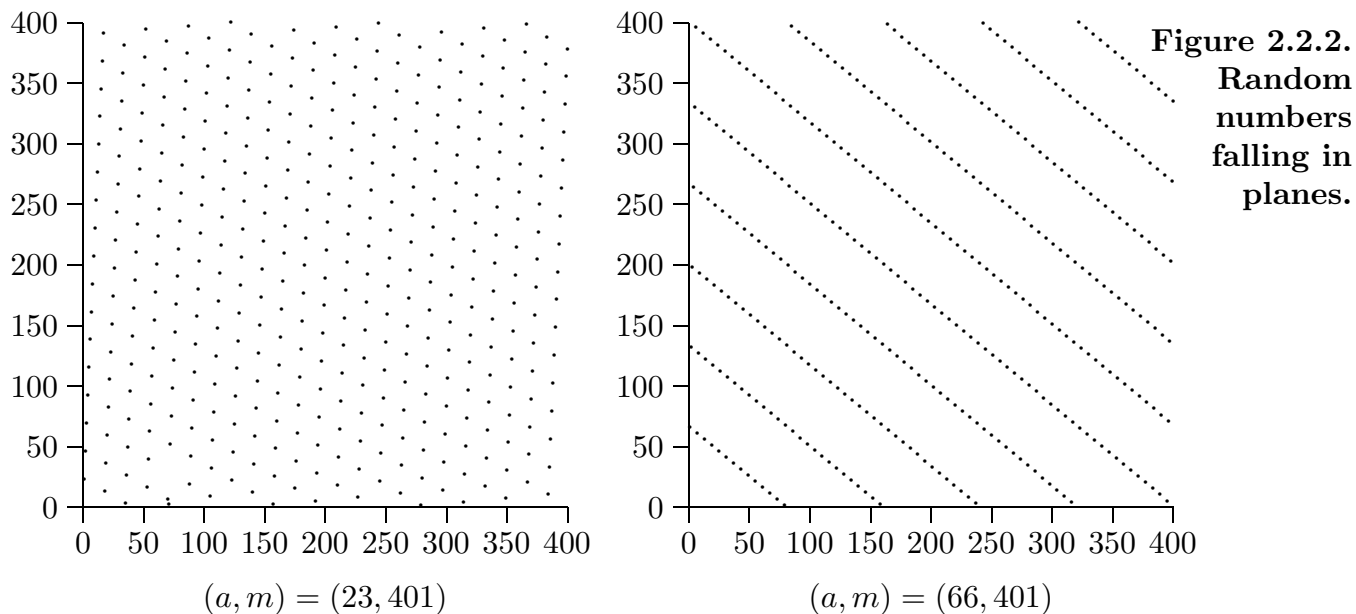
In general, for any integer $k \geq 2$ the points

$$(x_0, x_1, \ldots, x_{k-1}), (x_1, x_2, \ldots, x_k), (x_2, x_3, \ldots, x_{k+1}), \ldots$$

form a lattice structure. The best Lehmer generator randomness metrics are based on analyzing (numerically, not visually) the *uniformity* of this lattice structure in k-space for small values of k. Knuth (1998) gives one popular randomness metric of this type known as the *spectral test*. Consistent with these metrics, for $m = 401$ the multiplier $a = 23$ would be a much better full-period, modulus-compatible multiplier than $a = 66$.

ANSI C Implementation

The kind of theoretical testing for randomness illustrated in Example 2.2.7 has been done for all of the 23093 full-period, modulus-compatible multipliers relative to $m = 2^{31} - 1$. Of these, the winner is $a = 48271$ with $q = \lfloor m/a \rfloor = 44488$ and $r = m \bmod a = 3399$.*

Example 2.2.8   A Lehmer random number generator with $(a, m) = (48271, 2^{31} - 1)$ can be implemented correctly, efficiently and portably in ANSI C as follows

```
double Random(void)
{
   const long A     = 48271;                          /* multiplier */
   const long M     = 2147483647;                      /* modulus    */
   const long Q     = M / A;                           /* quotient   */
   const long R     = M % A;                           /* remainder  */
   static long state = 1;
         long t     = A * (state % Q) - R * (state / Q);
   if (t > 0)
      state = t;
   else
      state = t + M;
   return ((double) state / M);
}
```

With minor implementation-dependent modifications, the random number generator in Example 2.2.8 is the basis for all the simulated stochastic results presented in this book. There are three important points relative to this particular implementation.

- The static variable `state` is used to hold the current *state* of the random number generator, initialized to 1. There is nothing magic about 1 as an initial state (seed); any value between 1 and 2 147 483 646 could have been used.

- Because `state` is a static variable, the state of the generator will be retained between successive calls to `Random`, as must be the case for the generator to operate properly. Moreover, because the scope of `state` is local to the function, the state of the generator is protected — it cannot be changed in any way other than by a call to `Random`.

- If the implementation is correct, tests for randomness are redundant; if the implementation is incorrect, it should be discarded. A standard way of testing for a correct implementation is based on the fact that if the initial value of `state` is 1, then after 10 000 calls to `Random` the value of `state` should be 399 268 537.

---

* In addition to "in theory" testing, the randomness associated with this (a, m) pair has also been subjected to a significant amount of "in practice" empirical testing — see Section 10.1.

Example 2.2.9   As a potential alternative to the generator in Example 2.2.8, the random number generator in the ANSI C library `<stdlib.h>` is the function `rand`. The intent of this function is to simulate drawing at random from the set $\{0, 1, 2, \ldots, m - 1\}$ with m required to be at least 2^{15}. That is, `rand` returns an `int` between 0 and `RAND_MAX` inclusive where the macro constant `RAND_MAX` (defined in the same library) is required to be at least $2^{15} - 1 = 32767$. To convert the integer value returned by `rand` to a floating point number between 0.0 and 1.0 (consistent with Definition 2.1.1) it is conventional to use an assignment like

   `u = (double) rand() / RAND_MAX;`

Note, however, that the ANSI C standard does *not* specify the details of the algorithm on which this generator is based. Indeed, the standard does not even require the output to be random! For scientific applications it is generally a good idea to avoid using `rand`, as indicated in Section 13.15 of Summit (1995).

Random Number Generation Library

   The random number generation library used in this course is based upon the implementation considerations developed in this section. This library is defined by the header file `"rng.h"` and is recommended as a replacement for the standard ANSI C library functions `rand` and `srand`, particularly in simulation applications where the statistical goodness of the random number generator is important. The library provides the following capabilities.

- `double Random(void)` — This is the Lehmer random number generator in Example 2.2.8. We recommended it as a replacement for the standard ANSI C library function `rand`.

- `void PutSeed(long seed)` — This function can be used to initialize or reset the current state of the random number generator. We recommended it as a replacement for the standard ANSI C library function `srand`. If `seed` is positive then that value becomes the current state of the generator. If `seed` is 0 then the user is prompted to set the state of the generator interactively via keyboard input. If `seed` is negative then the state of the generator is set by the system clock.*

- `void GetSeed(long *seed)` — This function can be used to get the current state of the random number generator.*

- `void TestRandom(void)` — This function can be used to test for a correct implementation of the library.

Although we recommend the use of the multiplier 48271, and that is the value used in the library `rng`, as discussed in Example 2.2.6 the 16807 multiplier is something of a minimal standard. Accordingly, the library is designed so that it is easy to use 16807 as an alternative to 48271.

---

   * See Section 2.3 for more discussion about the use of `PutSeed` and `GetSeed`.

Example 2.2.10   As an example of the use of `Random` and `PutSeed`, this algorithm was used to create the two scatterplots illustrated in Figure 2.2.3 for two different values of the initial seed.

```
seed = 123456789;                                   /* or 987654321 */
PutSeed(seed);
x_0 = Random();
for (i = 0;  i < 400;  i++) {
   x_{i+1} = Random();
   Plot(x_i,  x_{i+1});                    /* a generic graphics function */
}
```

Unlike the lattice structure so obvious in Example 2.2.7, the (x_i, x_{i+1}) pairs in this case appear to be random with no lattice structure evident, as desired. (For a more direct comparison all the integer-valued coordinates in Example 2.2.7 would need to be normalized to 1.0 via division by $m = 401$. That would be a purely cosmetic change, however.)

Figure 2.2.3. Scatterplots of 400 random number pairs.

Initial seed $x_0 = 123456789$ Initial seed $x_0 = 987654321$
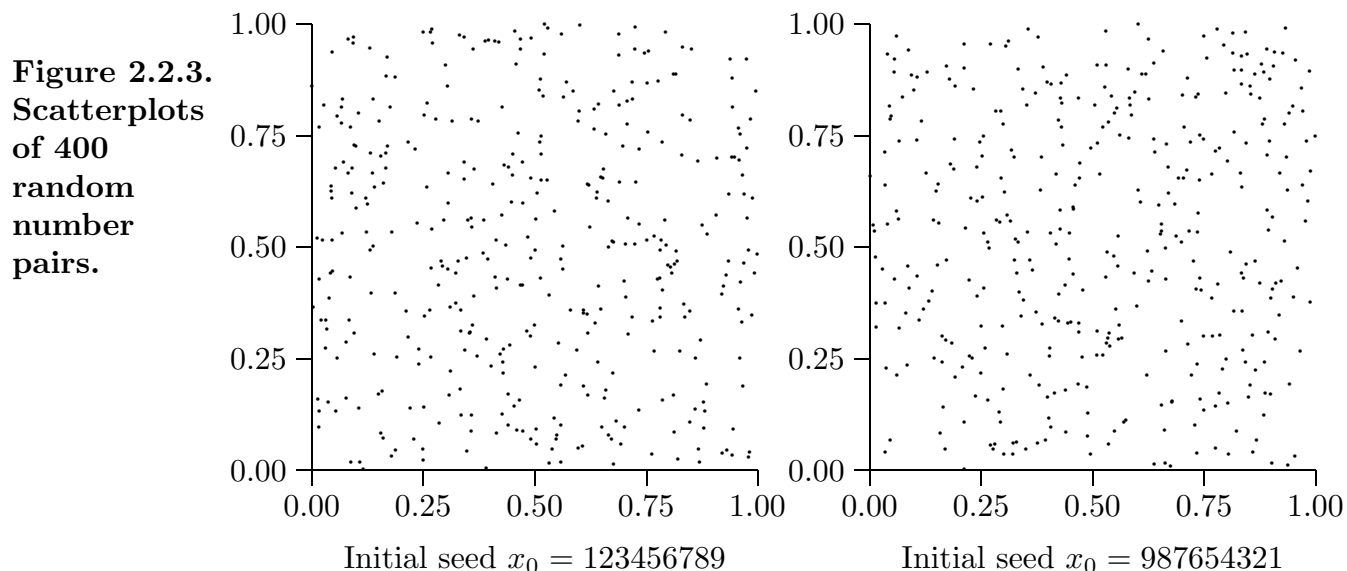
Consistent with the discussion in Example 2.2.7, it should be mentioned that in a sense the appearance of randomness in Example 2.2.10 is an illusion. That is, if *all* of the possible pairs of (x_i, x_{i+1}) points were generated (there are $m - 1 = 2^{31} - 2$ of these) and if it were somehow possible to plot all of these pairs of points at a fine enough scale to avoid blackening the page, then like the figures in Example 2.2.7 a micro-scale lattice structure would be evident.* This observation is one of the reasons why we distinguish between *ideal* and *good* random number generators (see Definition 2.1.1).

---

* Contemplate what size graphics device and associated dpi (dots per inch) resolution would be required to actually do this.

Example 2.2.11   Plotting consecutive, overlapping random number pairs (x_i, x_{i+1}) from a full-period Lehmer generator with $m = 2^{31} - 1$ would blacken the unit square, obscuring the lattice structure. The fact that any tiny square contained in the unit square will exhibit approximately the same appearance is exploited in the algorithm below, where *all* of the random numbers are generated by `Random()`, but only those that fall in the square with opposite corners $(0, 0)$ and $(0.001, 0.001)$ are plotted. One would expect that approximately $(0.001)(0.001)(2^{31} - 2) \cong 2147$ of the points would fall in the tiny square.

```
seed = 123456789;
PutSeed(seed);
x₀ = Random();
for (i = 0; i < 2147483646; i++) {
    x_{i+1} = Random();
    if ((x_i < 0.001) and (x_{i+1} < 0.001)) Plot(x_i, x_{i+1});
}
```

The results of the implementation of the algorithm are displayed in Figure 2.2.4 for the multipliers $a = 16807$ (on the left) and $a = 48271$ (on the right). The random numbers produced by $a = 16807$ fall in just 17 nearly-vertical parallel lines, whereas the random numbers produced by $a = 48271$ fall in 47 parallel lines. These scatterplots provide further evidence for our choice of $a = 48271$ over $a = 16807$ in the random number generator provided in the library `rng`.

Figure 2.2.4. Scatterplots for $a = 16807$ and $a = 48271$.

Multiplier $a = 16807$ Multiplier $a = 48271$
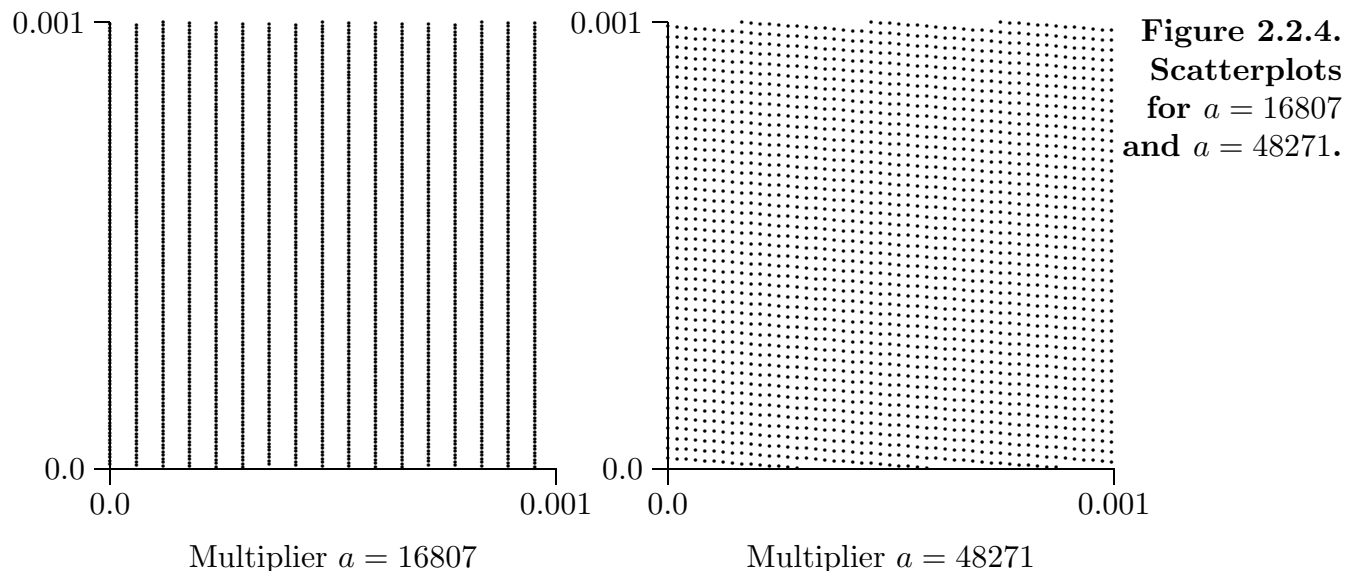
We have shown that as we zoom in on the unit square the (unwanted) lattice structure of the (x_i, x_{i+1}) pairs produced by our *good* random number generator becomes more apparent. It is appropriate at this juncture to contemplate what would happen with an *ideal* random number generator. An ideal generator exhibits no lattice behavior as you zoom in on increasingly tiny squares. Like a fractal or a scale-free network, all scatterplots will look like those in Figure 2.2.3, assuming more points are generated as you zoom in.

Other Multipliers

Recall that for the modulus $m = 2^{31} - 1 = 2\,147\,483\,647$, there are $534\,600\,000$ multipliers a that result in a full-period random number generator. Of these, $23\,093$ are modulus-compatible. Although the modulus-compatible, full-period multiplier $a = 48\,271$ is used in library **rng**, are there other multipliers that are recommended? One rather tedious way of approaching this question is to execute the statistical tests for random number generators outlined in Section 10.1 on all of the generators. A second, more efficient approach is to consult the experts who perform research in this area. Fishman (2001, pages 428–445) notes that $a = 16\,807$, $a = 630\,360\,016$, and $a = 742\,938\,285$ are employed by the discrete-event simulation languages SIMAN, SIMSCRIPT II.5, and GPSS/H, respectively, ordered here by improving lattice structure. In addition to $a = 742\,938\,285$, he also suggests four other multipliers with similar performance in terms of parallel hyperplanes in 2, 3, 4, 5, and 6 dimensions: $a = 950\,706\,376$, $a = 1\,226\,874\,159$, $a = 62\,089\,911$, and $a = 1\,343\,714\,438$. Determining whether these multipliers are modulus-compatible is left as an exercise.

Good Generator — Non-representative Subsequences

In any simulation run, only a subset of the random numbers produced by a generator are typically used. What if, for example, only 20 random numbers were needed for a simulation. If you had the *extreme* misfortune of choosing the seed $x_0 = 109\,869\,724$ with the generator in the library **rng**, the resulting 20 random numbers (to only $0.xx$ precision) are:

<div align="center">

0.64   0.72   0.77   0.93   0.82   0.88   0.67   0.76   0.84   0.84

0.74   0.76   0.80   0.75   0.63   0.94   0.86   0.63   0.78   0.67.

</div>

Is there something wrong with this generator to have produced 20 consecutive random numbers exceeding 0.62? Certainly not. It will only occur once in a blue moon, but this particular seed resulted in this rather unique sequence of random numbers.* This sequence of 20 consecutive random numbers may initially seem analogous to an *outlier* from statistics. Statisticians sometimes discard outliers as being unrepresentative, but it is *never* appropriate to do so in simulation. The random number generator will have a few sequences of unusually high random numbers as well as a few sequences of unusually low random numbers, as it should. Analogously, if we were to perform the experiment of tossing a fair coin 20 times for a large number of replications, there should be a few rare cases when all 20 tosses come up heads (or tails).

This string of 20 large random numbers highlights the importance of *replicating* a simulation many times so as to average out these unusual cases. Replication will be considered in detail in Chapter 8.

---

* The probability of 20 consecutive random numbers exceeding 0.62 using the axiomatic approach to probability is $(0.38)^{20} \cong 4 \cdot 10^{-9}$.

The Curse of Fast CPUs

When discrete-event simulation was in its infancy, the time required to cycle through a full-period Lehmer generator with modulus $m = 2^{31} - 1 = 2\,147\,483\,647$ was measured in days. This quickly shrank to hours, and now only takes a few minutes on a desktop machine. Before long, the time to complete a cycle of such a generator will be measured in seconds. The problem of *cycling*, that is, reusing the same random numbers in a single simulation, must be avoided because of the resultant dependency in the simulation output.

Extending the period length of Lehmer generators has been an active research topic within the simulation community for many years. One simple solution is to run all simulations on 64-bit machines. Since the largest prime number less than $2^{63} - 1$ is $2^{63} - 25 = 9\,223\,372\,036\,854\,775\,783$, the period is lengthened from about $2.14 \cdot 10^9$ to $9.22 \cdot 10^{18}$. In this case, cycle times are measured in years. Obviously, more machines supporting 64-bit integer arithmetic is beneficial to the simulation community. Since it is too early to know if 64-bit machines will become the norm, other portable techniques on 32-bit machines for achieving longer periods have been developed.

One technique for achieving a longer period is to use a *multiple recursive generator*, where our usual $x_{i+1} = ax_i \bmod m$ is replaced by

$$x_{i+1} = (a_1 x_i + a_2 x_{i-1} + \cdots + a_q x_{i-q}) \bmod m.$$

These generators can produce periods as long as $m^q - 1$ if the parameters are chosen properly. A second technique for extending the period is to use a *composite generator*, where several Lehmer generators can be combined in a manner to extend the period and improve statistical behavior. A third technique is to use a *Tausworthe*, or *shift-register* generator, where the modulo function is applied to bits, rather than large integers. The interested reader should consult Chapter 6 of Bratley, Fox, and Schrage (1987), Chapter 5 of Lewis and Orav (1989), Chapter 7 of Law and Kelton (2000), Chapter 9 of Fishman (2001), Gentle (2003), and L'Ecuyer, Simard, Chen, and Kelton (2002) for more details.

Any of these generators that have been well-tested by multiple authors yield two benefits: longer periods and better statistical behavior (e.g., fewer hyperplanes). The cost is always the same: increased CPU time.

2.2.2 EXERCISES

Exercise 2.2.1 Prove that if u and v are real numbers with $0 < u - v < 1$ then the integer difference $\lfloor u \rfloor - \lfloor v \rfloor$ is either 0 or 1.

Exercise 2.2.2[a] If $g(\cdot)$ is a Lehmer generator (full period or not) then there must exist an integer $x \in \mathcal{X}_m$ such that $g(x) = 1$. (a) Why? (b) Use the $m = aq + r$ decomposition of m to derive a $O(a)$ algorithm that will solve for this x. (c) If $a = 48271$ and $m = 2^{31} - 1$ then what is x? (d) Same question if $a = 16807$.

Exercise 2.2.3[a]   Derive a $O(\log(a))$ algorithm to solve Exercise 2.2.2. *Hint*: a and m are relatively prime.

Exercise 2.2.4   Prove that if a, m are positive integers and if a is "small" in the sense that $a^2 < m$, then $r < q$ where $r = m \bmod a$ and $q = \lfloor m/a \rfloor$.

Exercise 2.2.5   Write $\gamma(x) = \alpha(x) - \beta(x)$ where $\alpha(x) = a(x \bmod q)$ and $\beta(x) = r\lfloor x/q \rfloor$ with $m = aq + r$ and $r = m \bmod a$. Prove that if $r < q$ then for all $x \in \mathcal{X}_m$ both $\alpha(x)$ and $\beta(x)$ are in $\{0, 1, 2, \ldots, m - 1\}$.

Exercise 2.2.6   Is Algorithm 2.2.1 valid if m is not prime? If not, how should it be modified?

Exercise 2.2.7   (a) Implement a correct version of `Random` using floating point arithmetic and do a timing study. (b) Comment.

Exercise 2.2.8   Prove that if a, x, q are positive integers then $ax \bmod aq = a(x \bmod q)$.

Exercise 2.2.9   You have been hired as a consultant by *XYZ Inc* to assess the market potential of a relatively inexpensive *hardware* random number generator they may develop for high-speed scientific computing applications. List all the technical reasons you can think of to convince them this is a bad idea.

Exercise 2.2.10   There are exactly 400 points in each of the figures in Example 2.2.7. (a) Why? (b) How many points would there be if a were not a full-period multiplier?

Exercise 2.2.11   Let m be the largest prime modulus less than or equal to $2^{15} - 1$ (see Exercise 2.1.6). (a) Compute all the corresponding modulus-compatible full-period multipliers. (b) Comment on how this result relates to random number generation on systems that support 16-bit integer arithmetic only.

Exercise 2.2.12[a]   (a) Prove that if m is prime with $m \bmod 4 = 1$ then a is a full-period multiplier if and only if $m - a$ is also a full-period multiplier. (b) What if $m \bmod 4 = 3$?

Exercise 2.2.13[a]   If $m = 2^{31} - 1$ compute the $x \in \mathcal{X}_m$ for which $7^x \bmod m = 48271$.

Exercise 2.2.14   The lines on the scatterplot in Figure 2.2.4 associated with the multiplier $a = 16807$ appear to be vertical. Argue that the lines must *not* be vertical based on the fact that $(a, m) = (16807, 2^{31} - 1)$ is a full-period generator.

Exercise 2.2.15   Determine whether the multipliers associated with $m = 2^{31} - 1$ given by Fishman (2001): $a = 630\,360\,016$, $a = 742\,938\,285$ $a = 950\,706\,376$, $a = 1\,226\,874\,159$, $a = 62\,089\,911$, and $a = 1\,343\,714\,438$ are modulus-compatible.

This section presents four applications of Monte Carlo simulation that are designed to augment the elementary examples considered in Section 2.3. The applications have been chosen to highlight the diversity of problems that can be addressed by Monte Carlo simulation, and have been arranged in increasing order of the complexity of their implementation. The problems are (1) estimating the probability that the determinant of a 3×3 matrix of random numbers having a particular sign pattern is positive (program `det`), (2) estimating the probability of winning in Craps (program `craps`), (3) estimating the probability that a hatcheck girl will return all of the hats to the wrong owners when she returns n hats at random (program `hat`) and (4) estimating the mean time to complete a stochastic activity network (program `san`). Although the axiomatic approach to probability can be used to solve some of these problems, a minor twist in the assumptions associated with the problems often sinks an elegant axiomatic solution. A minor twist in the assumptions typically does not cause serious difficulties with the Monte Carlo simulation approach.

2.4.1 RANDOM MATRICES

Although the elementary definitions associated with matrices may be familiar to readers who have taken a course in linear algebra, we begin by reviewing some elementary definitions associated with matrices.

Matrices and Determinants

Definition 2.4.1   A *matrix* is a set of real or complex numbers arranged in a rectangular array.

Upper-case letters are typically used to denote matrices, with subscripted lower-case letters used for their elements. The first subscript denotes the row where the element resides and the second subscript denotes the column. Thus a generic $m \times n$ array A containing m rows and n columns is written as:

$$
A = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}.
$$

There are many operations (e.g., addition, multiplication, inversion), quantities (e.g., eigenvalues, trace, rank), and properties (e.g., positive definiteness) associated with matrices. We consider just one particular quantity associated with a matrix: its determinant, a single number associated with a square ($n \times n$) matrix A, typically denoted* as $|A|$ or det A.

---

* The bars around A associated with the notation $|A|$ have nothing to do with absolute value.

Definition 2.4.2   The *determinant* of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The determinant of a 3×3 matrix can be defined as a function of the determinants of 2×2 sub-matrices as follows.

Definition 2.4.3   The *determinant* of a 3×3 matrix A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Random Matrices

Matrix theory traditionally emphasizes matrices that consist of real or complex constants. But what if the elements of a matrix are random variables? Such matrices are referred to as "stochastic" or "random" matrices.

Although a myriad of questions can be asked concerning random matrices, our emphasis here will be limited to the following question: if the elements of a 3×3 matrix are independent random numbers with positive diagonal elements and negative off-diagonal elements, what is the probability that the matrix has a positive determinant? That is, find the probability that

$$\begin{vmatrix} +u_{11} & -u_{12} & -u_{13} \\ -u_{21} & +u_{22} & -u_{23} \\ -u_{31} & -u_{32} & +u_{33} \end{vmatrix} > 0,$$

where the u_{ij}'s are independent random numbers. This question is rather vexing using the axiomatic approach to probability due to the appearance of some of the random numbers multiple times on the right hand side of Definition 2.4.3.*

---

* This question is of interest to matrix theorists as it is the first example of a probability that cannot be calculated easily. A positive determinant in this case is equivalent to the matrix being of a special type called an *M-matrix* (Horn and Johnson, 1990). A positive determinant in a matrix with this particular sign pattern (positive diagonal elements and negative off-diagonal elements) corresponds to having all three 2×2 principal minors (determinants of sub-matrices determined by deleting the same numbered row and column) being positive.

Having completed the conceptual formulation of this problem, we can go directly to the implementation step since the algorithm is straightforward. The ANSI C program `det` generates random matrices in a loop, counts the number of these matrices that have a positive determinant and prints the ratio of this count to the number of replications.

In order to estimate the probability with some precision, it is reasonable to make one long run. We should ensure, however, that we do not recycle random numbers. Since nine random numbers are used to generate the 3×3 random matrix in each replication, we should use fewer than $(2^{31} - 1)/9 \cong 239\,000\,000$ replications. For an initial random number generator seed of 987654321 and 200\,000\,000 replications, the program returns an estimated probability of a positive determinant as 0.05017347.

The point estimate for the probability has been reported to seven significant digits in the preceding paragraph. How many of the leading digits are significant, and how many of the trailing digits are sampling variability (noise) that should not be reported? One way to address this question is to re-run the simulation with the same number of replications and seed, but a different multiplier a. (Simply using a new seed will not be effective because the random numbers will be recycled due to the large number of replications.) Using $a = 16807$ and $a = 41214$, the estimates of the probability of a positive determinant are 0.050168935 and 0.05021236, respectively. Making a minimum of three runs in Monte Carlo and discrete-event simulation is good practice because it allows you to see the spread of the point estimates. Using common sense or some statistical techniques that will be developed in subsequent chapters, it is reasonable to report the point estimate to just three significant digits: 0.0502. Obtaining a fourth digit of accuracy pushes the limits of independence and uniformity of the random number generator.

As illustrated in the exercises, this Monte Carlo simulation solution can be extended to cover other random variables as entries in the matrix or larger matrices.

2.4.2 CRAPS

The gambling game known as "Craps" provides a slightly more complicated Monte Carlo simulation application. The game involves tossing a pair of fair dice one or more times and observing the total number of spots showing on the up faces. If a 7 or 11 is tossed on the first roll, the player wins immediately. If a 2, 3, or 12 is tossed on the first roll, the player loses immediately. If any other number is tossed on the first roll, this number is called the "point". The dice are rolled repeatedly until the point is tossed (and the player wins) or a 7 is tossed (and the player loses).

The problem is conceptually straightforward. The algorithm development is slightly more complicated than in the previous example involving the 3×3 random matrix. An *Equilikely*$(1, 6)$ random variate is used to model the roll of a single fair die. The algorithm that follows uses N for the number of replications of the game of Craps. The variable *wins* counts the number of wins and the `do .. while` loop is used to simulate the player attempting to make the point when more than one roll is necessary to complete the game.

Algorithm 2.4.1   This algorithm estimates the winning probability using N replications.

```
wins = 0;
for (i = 1; i <= N; i++) {
    roll = Equilikely(1, 6) + Equilikely(1, 6);
    if (roll == 7 or roll == 11)
        wins++;
    else if (roll != 2 and roll != 3 and roll != 12) {
        point = roll;
        do {
            roll = Equilikely(1, 6) + Equilikely(1, 6);
            if (roll == point) wins++;
        } while (roll != point and roll != 7)
    }
}
return (wins / N);
```

The algorithm has been implemented in program `craps`. The function `Roll` returns the outcome when a pair of dice are tossed by adding the results of two calls to *Equilikely*$(1,6)$. The `switch` statement with several `case` prefixes (for `2 .. 12`) has been used to determine the result of the game based on the outcome of a roll. The program was executed for $10\,000$ replications with seeds 987654321, 123456789, and 555555555, yielding estimates of winning a game 0.497, 0.485, and 0.502, respectively. Unlike the previous example, the solution using the axiomatic approach is known in this case — it is $244/495 \cong 0.4929$, which seems to be consistent with the Monte Carlo simulation results. The fact that the probability of winning is slightly less than 0.5 lures gamblers to the game, yet assures that the house will extract money from gamblers in the long run.

2.4.3   HATCHECK GIRL PROBLEM

A hatcheck girl at a fancy restaurant collects n hats and returns them at random. What is the probability that all of the hats will be returned to the wrong owner?

Monte Carlo simulation can be used to solve this classic probability problem. We begin by numbering the hats $1, 2, \ldots, n$, which is suitable for algorithm development and computer implementation. The approach will be to generate one vector containing the hats in the order that they are checked and another for the order that they are returned. Since the order that they are checked is arbitrary, we can assume that the order that the hats are checked is $1, 2, \ldots, n$. Next, it is necessary to determine the meaning of returning the hats "at random". One reasonable interpretation is that as each customer finishes dining and requests their hat from the hatcheck girl, she chooses one of the remaining hats in an equally likely manner. If this is the case then all of the $n!$ permutations are equally likely. When $n = 3$, for example, the $3! = 6$ permutations:

$$1, 2, 3 \qquad 1, 3, 2 \qquad 2, 1, 3 \qquad 2, 3, 1 \qquad 3, 1, 2 \qquad 3, 2, 1$$

are equally likely. Since only the permutations $2, 3, 1$ and $3, 1, 2$ correspond to everyone leaving the restaurant with the wrong hat (since the hats are checked in as $1, 2, 3$), the probability that all three hats went to the wrong owner is $2/6 = 1/3$.

Thus the algorithmic portion of solving this problem via Monte Carlo simulation reduces to devising an algorithm for generating a random permutation which represents the order that the hats are returned.

One way to generate an element in a random permutation is to pick one of the n hats randomly, and return it to the customer if it has not already been returned. Generating a random permutation by this "obvious" algorithm is inefficient because: (1) it spends a substantial amount of time checking to see if the hat has already been selected, and (2) it may require several randomly generated hats in order to produce one that has not yet been returned. Both of these detractors are particularly pronounced for the last few customers that retrieve their hats. Fortunately, there is a very clever algorithm for generating a random permutation that avoids these difficulties. Although it will be presented in detail in Section 6.5, the reader is encouraged to think carefully about the $O(n)$ algorithm below, which is used to generate a random permutation of the array `a` with elements `a[0]`, `a[1]`, ..., `a[n - 1]` from the array `a` with elements input *in any order*.

```
for (i = 0; i < n - 1; i++) {
    j = Equilikely(i, n - 1);
    hold = a[j];
    a[j] = a[i];                              /* swap a[i] and a[j] */
    a[i] = hold;
}
```

The Monte Carlo simulation to estimate the probability that all n customers will receive the wrong hat is implemented in program `hat`. The function `Shuffle` is used to generate a random permutation using the algorithm given above and the function `Check` is used to check whether any customers left the restaurant with their own hat. When run for $10\,000$ replications with $n = 10$ hats and initial seeds 987654321, 123456789, and 555555555, the estimates of the probability that all customers leave with the wrong hats are 0.369, 0.369, and 0.368, respectively.

A question of interest here is the behavior of this probability as n increases. Intuition might suggest that the probability goes to one in the limit as $n \to \infty$, but this is not the case. One way to approach this problem is to simulate for increasing values of n and fashion a guess based on the results of multiple long simulations. One problem with this approach is that you can never be sure that n is large enough. For this problem, the axiomatic approach is a better and more elegant way to determine the behavior of this probability as $n \to \infty$.

The hatcheck girl problem can be solved using the axiomatic approach to probability. The probability of no matches is:

$$1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}\right)$$

(Ross, 2002a, pages 42–44), which is $16481/44800 \cong 0.3679$ when $n = 10$. This result helps to validate the implementation of the program `hat` for the case of $n = 10$ hats.*

Since the Taylor expansion of e^{-x} around $x = 0$ is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

the limiting probability (as the number of hats $n \to \infty$) of no matches is $1/e \cong 0.36787944$. This axiomatic approach is clearly superior to making long Monte Carlo simulation runs for various increasing values of n in order to determine this limiting probability.

2.4.4   STOCHASTIC ACTIVITY NETWORKS

The fourth and final example of a Monte Carlo simulation study is more comprehensive than the first three. We consider the estimation of the mean time to complete a *stochastic activity network*, where arcs in the network represent activities that must be completed according to prescribed precedences.

Project Management and Networks

Project management is a field of study devoted to the analysis and management of projects. Project management analyzes a project that typically occurs once (such as building a stadium, landing a man on the moon, building a dam, laying out a job shop) whereas *production management* involves overseeing the manufacture of many items (such as automobiles on an assembly line). Production management is more forgiving than project management in that there are multiple opportunities to correct faulty design decisions midstream. Production management is repetitive in nature whereas project management considers only a single undertaking.

Projects consist of activities. One aspect of project management is the sequencing of these activities. When constructing a stadium, for example, the foundation must be poured before the seats can be installed. *Precedence relationships* can be established between the various activities. Certain activities cannot be started until others have been completed. Project managers often use *networks* to graphically illustrate the precedence relationships.

---

* The fact that all three simulations yielded probability estimates slightly *above* the theoretical value, however, would provide some impetus to make a few more runs. In an analogous fashion, flipping three heads in a row is not sufficient evidence to conclude that a coin is biased, but a prudent person would certainly make a few more flips to be on the safe side.

Generally speaking, a network is a collection of nodes and arcs. In our setting, the arcs represent activities and the nodes are used to delay the beginning of outgoing activities until all of the incoming activities have been completed. The network shown in Figure 2.4.1, adopted from Pritsker (1995, pages 216–221), has six nodes and nine activities. The arc connecting nodes i and j is denoted by a_{ij}. Activity start times are constrained in that no activity emanating from a given node can start until all activities which enter that node have been completed. In the network in Figure 2.4.1, activity a_{46}, for example, may not start until activities a_{14} *and* a_{34} have both been completed.

**Figure 2.4.1.
Six node,
nine activity
network.**
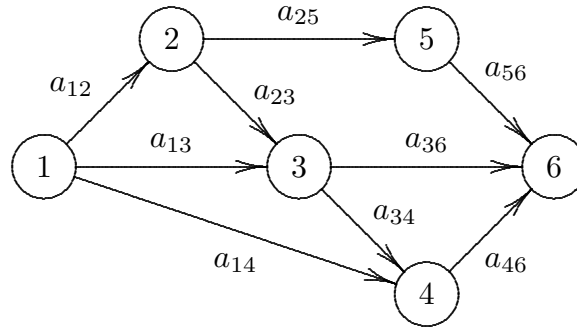
Activity Durations and Critical Paths

The nodes and arcs in an activity network can be used to mark certain points in the time evolution of the project:

- the nodes can mark the maximum completion times (e.g., the end time of the latest activity completed for entering activities and the beginning time of all exiting activities);

- the arcs in an activity network represent activity durations.

Figure 2.4.2 shows an example of a deterministic activity network, where the integers shown near each arc are the time to complete each activity. If these activity durations are fixed constants known with certainty, how long will it take to complete this network?

**Figure 2.4.2.
Deterministic
activity
network.**
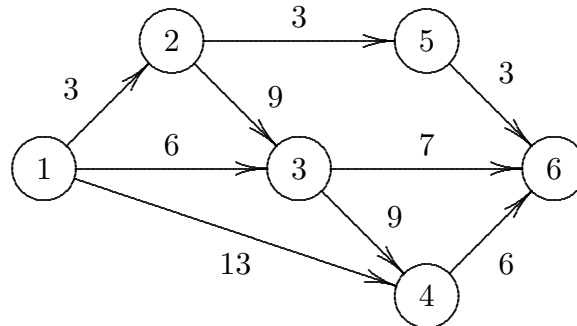
To answer this question, we must introduce the concept of *paths*. A path is an ordered set (sequence) of arcs leading in succession from node 1 to node n. Label the paths $\pi_1, \pi_2, ..., \pi_r$. Thus arc $a_{ij} \in \pi_k$ means that arc a_{ij} is along the path π_k. The *length* of path π_k, denoted L_k, is the sum of all of the activity durations corresponding to the arcs $a_{ij} \in \pi_k$.

Returning to the network in Figure 2.4.2, there are $r = 6$ paths in the network:

k	Node sequence	π_k	L_k
1	$1 \to 3 \to 6$	$\{a_{13}, a_{36}\}$	13
2	$1 \to 2 \to 3 \to 6$	$\{a_{12}, a_{23}, a_{36}\}$	19
3	$1 \to 2 \to 5 \to 6$	$\{a_{12}, a_{25}, a_{56}\}$	9
4	$1 \to 4 \to 6$	$\{a_{14}, a_{46}\}$	19
5	$1 \to 3 \to 4 \to 6$	$\{a_{13}, a_{34}, a_{46}\}$	21
6	$1 \to 2 \to 3 \to 4 \to 6$	$\{a_{12}, a_{23}, a_{34}, a_{46}\}$	27

The ordering of the six paths is arbitrary. Assume that time is measured in days. The longest of these paths, namely π_6, which has a total duration of $L_6 = 27$ days, is known as the *critical path*. An algorithm known as the *critical path method* (CPM) can be used to identify and determine the length of the critical path for large networks. In general we denote the critical path as π_c which is the path with the longest length, $L_c = \max\{L_1, L_2, ..., L_r\}$. The length of the critical path determines the time to complete the entire network. Any path i whose length is shorter than the critical path (that is, $L_i < L_c$) *can* be delayed without lengthening the duration of the project. Activity a_{56}, for example, from the network in Figure 2.4.2 is on only one path, π_3. Its duration could be changed from 3 days to as long as $27 - 9 = 18$ days without extending the duration of the project.

The analysis of a project with deterministic activity durations, such as the one in Figure 2.4.2, is straightforward. In most practical applications, however, activity durations are not deterministic. We consider the analysis of a *stochastic activity network*, where the activity durations are positive random variables. In practice, contracts often have penalties for projects that exceed a deadline and bonuses for projects that are completed early. Hence the accurate analysis of the time to complete a project can be quite important.* We begin with the conceptual development of the problem, where the notation necessary to develop an algorithm is defined. We will alternate between general notation and our sample network.

Conceptual Development

Activity networks have a single source node labeled 1 and a single terminal node labeled n, where n is the number of nodes in the network. Node 1 has one or more arcs leaving it and node n only has arcs entering it. All other nodes must have at least one arc entering and at least one arc leaving. There are m arcs (activities) in the network.

---

* The analysis of stochastic activity networks is often performed by a method known as the *project evaluation and review technique* (PERT), which yields the approximate distribution of the time to complete the network based on a result from probability theory known as the *central limit theorem*. Our development here estimates the exact distribution of the time to complete the network via simulation.

Each arc a_{ij} has a positive random activity duration Y_{ij}. Node j has a random time value T_j which denotes the time of completion of all activities entering node j, $j = 1, 2, \ldots, m$. The random time T_n is therefore the completion time of the entire network.

In the deterministic network in Figure 2.4.2, there is always one or more critical paths. In a stochastic activity network, a path is critical with a certain probability. For each realization of a stochastic network, a path is the critical path with some probability $p(\pi_k) = \Pr(\pi_k \equiv \pi_c)$, $k = 1, 2, \ldots, r$.

We need to transform the network given in Figure 2.4.1 to a mathematical entity appropriate for algorithm development. A matrix is well-suited for this task because two subscripts can be used to define arc a_{ij} and the ease of computer implementation. We use a well-known tool known as a node-arc incidence matrix. A node-arc incidence matrix is an $n \times m$ matrix N, where each row represents a node and each column represents an arc. Let

$$N[i, j] = \begin{cases} 1 & \text{arc } j \text{ leaves node } i \\ -1 & \text{arc } j \text{ enters node } i \\ 0 & \text{otherwise.} \end{cases}$$

We now apply the general notation developed so far to specific questions associated with a specific network. For the network in Figure 2.4.3, use Monte Carlo simulation to estimate:

- the mean time to complete the network;

- the probability that each path is the critical path.

We assume that each activity duration is a uniform random variate with a lower limit of zero and an upper limit given in Figure 2.4.2 [e.g., Y_{12} has a *Uniform*$(0, 3)$ distribution].

Figure 2.4.3. Stochastic activity network.
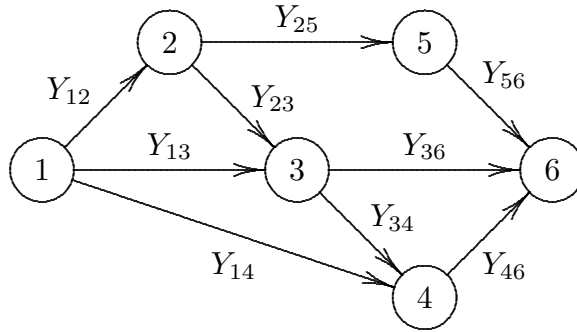
It may initially seem that this problem is best addressed by discrete-event simulation, rather than Monte Carlo simulation, complete with an event calendar to keep track of the timing of the activities. The following mathematical relationship, however, can be used to analyze any network of this type by looking at just one activity at a time, and exploiting the use of recursion in a programming language. If T_1 is defined to be 0.0, then

$$T_j = \max_{i \in \mathcal{B}(j)} \{T_i + Y_{ij}\},$$

for $j = 2, 3, \ldots, n$, where the set $\mathcal{B}(j)$ is the set of all nodes immediately *before* node j. In the six-node network with $j = 6$, for example,

$$T_6 = \max\{T_3 + Y_{36}, T_4 + Y_{46}, T_5 + Y_{56}\}.$$

This observation allows us to write a function that returns a T_j value by recursively calling for the completion times of all nodes before node j.

The 6×9 matrix

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

is a node-arc incidence matrix of the network with $n = 6$ nodes and $m = 9$ arcs in Figure 2.4.3. The number of 1's in each row indicates the number of arcs exiting a particular node and the number of -1's indicates the number of arcs entering a particular node. There is exactly one 1 and one -1 in each column and their positions indicates the beginning and ending node of one particular activity.

Finally, we discuss the appropriate point estimators for the two measures of performance prior to presenting the algorithm. The point estimator for the mean time to complete the network, for example, is the sample mean of the T_n's generated using the algorithm to follow (which is generated by the recursive expression for T_j given earlier). The point estimator for the probability $p(\pi_k)$ that path π_k is a critical path is the fraction of the networks generated that have π_k as the critical path, $k = 1, 2, \ldots, r$.

Algorithm

The recursive algorithm named T (for Time) that follows generates a single time to completion T_j for some node j given that the network is represented by the node-arc incidence matrix N. The realization of the stochastic activity durations Y_{ij} associated with each arc a_{ij} are generated *prior* to the call to T. All that is needed to complete the algorithm is to embed this code in a loop for a particular number of replications and compute point estimates for the measures of performance listed in the previous paragraph.

The two (global) parameters for this algorithm are the $n \times m$ node-arc incidence matrix N and one realization of the activity durations Y_{ij} generated as random variates. The procedure T has a single argument j, the node whose completion time is of interest (typically n to determine the time to complete the entire network). When T is passed the argument 1, it will return a completion time of 0.0, since 0.0 is assumed to be the beginning time of the network.

Algorithm 2.4.2   This algorithm returns a random time to complete all activities prior to node j for a single stochastic activity network with node-arc incidence matrix N.

```
k = 1;                             /* initialize index for columns of N */
l = 0;                       /* initialize index for predecessors to node j */
t_max = 0.0;           /* initialize longest time of all paths to node j */
while (l < |B(j)|) {          /* loop through predecessor nodes to node j */
   if (N[j,k] == -1) {   /* if column k of N has arc entering node j */
      i  = 1;                       /* begin search for predecessor node */
      while (N[i, k] != 1) {      /* while i not a predecessor index */
         i++;                                         /* increment i */
      }
      t = T_i + Y_ij      /* recursive call:  t is completion time of a_ij */
      if (t >= t_max) t_max = t;       /* choose largest completion time */
      l++;                               /* increment predecessor index */
   }
   k++;                                        /* increment column index */
}
return (t_max);                          /* return completion time T_j */
```

In most cases, this algorithm is called with argument n so that a realization of the time to complete the entire network T_n is generated. In order to estimate $p(\pi_k)$, for $k = 1, 2, \ldots, r$, each path's length must be calculated for each realization, then the fraction of times that the length is the longest is calculated.

Implementation

The Monte Carlo simulation program `san` implements the recursive algorithm for generating `nrep` replications of the stochastic activity network completion times and estimates the mean time to complete the network and the probability that each of the six paths is the critical path.

When T is called with the argument $j = 6$, one realization of the nine uniform Y_{ij}'s are generated, then the algorithm given above is called to find T_6. The recursive calls associated with the predecessors (for node 6, the predecessor nodes are 3, 4, and 5), and subsequent recursive calls will be made to their predecessors. When the algorithm is called to generate the T_6's, the order of the recursive calls associated with the node-arc incidence matrix given above is T_6, T_3, T_1, T_2, T_1, etc.

The simulation was run for $10\,000$ network realizations with initial seeds of 987654321, 123456789, and 555555555. The mean times to complete the network were 14.64, 14.59, and 14.57, respectively. The point estimates for $p(\pi_k)$ for each of the three runs are given in the table that follows, where $\hat{p}_1(\pi_k)$, $\hat{p}_2(\pi_k)$, and $\hat{p}_3(\pi_k)$ denote the estimates of $p(\pi_k)$ for the three initial seeds and $\hat{p}_a(\pi_k)$ denotes the average of the three estimates.

k	π_k	$\hat{p}_1(\pi_k)$	$\hat{p}_2(\pi_k)$	$\hat{p}_3(\pi_k)$	$\hat{p}_a(\pi_k)$
1	$\{a_{13}, a_{36}\}$	0.0168	0.0181	0.0193	0.0181
2	$\{a_{12}, a_{23}, a_{36}\}$	0.0962	0.0970	0.0904	0.0945
3	$\{a_{12}, a_{25}, a_{56}\}$	0.0013	0.0020	0.0013	0.0015
4	$\{a_{14}, a_{46}\}$	0.1952	0.1974	0.1907	0.1944
5	$\{a_{13}, a_{34}, a_{46}\}$	0.1161	0.1223	0.1182	0.1189
6	$\{a_{12}, a_{23}, a_{34}, a_{46}\}$	0.5744	0.5632	0.5801	0.5726

The path through the nodes $1 \to 2 \to 3 \to 4 \to 6$ is the most likely of the paths to be the critical path. It is critical for 57.26% of the 30 000 realizations of the network generated.

2.4.5 EXERCISES

Exercise 2.4.1 Modify program det so that all $2^{31} - 1$ possible matrices associated with the random number generator with $(a, m) = (48271, 2^{31} - 1)$ are generated.

Exercise 2.4.2 Modify program det so as to test the other random variates as entries: (a) Uniform(0, 2), (b) Uniform(1, 2), (c) Exponential(1) (see Section 3.1 for the algorithm to generate an Exponential(1) random variate).

Exercise 2.4.3 Modify program det so that it estimates the probability that a $n \times n$ matrix of random numbers with positive diagonal and negative off-diagonal signs has a positive determinant for $n = 2, 3, 4, 5$. Use 1 000 000 replications and an initial seed 987654321.

Exercise 2.4.4[a] Show that the exact probability of winning at Craps is $\frac{244}{495}$.

Exercise 2.4.5 Dice can be *loaded* with weights so that one particular outcome is more likely (e.g., rolling a six), although the outcome on the opposite side of the die (e.g., rolling a one) becomes less likely. Assuming that one and six are on opposite faces, two and five are on opposite faces, and three and four are on opposite faces, is there a way to load a die so that the probability of winning at Craps exceeds 0.5?

Exercise 2.4.6 You make one $1000 bet on a game of Craps. I make 1000 $1 bets. Find the probability that each of us comes out ahead. (The result should show you why a casino would prefer many small bets to few large bets).

Exercise 2.4.7 Modify program Craps to estimate the mean number of rolls required per game.

Exercise 2.4.8   Modify program `hat` to estimate, for $i = 1, 2, \ldots, 20$, (a) the average number of hats returned correctly, (b) the average number of hats returned prior to the first hat returned correctly conditioned on the correct return of one or more hats.

Exercise 2.4.9[a]   Show that the exact probability that the hatcheck girl will return all of the n hats to the improper person is

$$1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!} \right).$$

Exercise 2.4.10   Modify program `san` to estimate the mean time to complete the network of $Uniform(0, 1)$ activity durations given below.
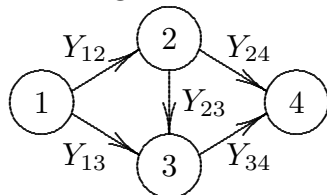
Exercise 2.4.11   Modify program `san` to store intermediate node completion times so that no redundant recursive calls are made.

Exercise 2.4.12   Write a program `bubble` that estimates the mean number of comparisons and the mean number of swaps required to bubble sort a list of n numbers in random order. Test your program for $n = 2, 3, \ldots, 20$.

Exercise 2.4.13   A *traveling salesman* has his home in a certain city represented by a random position on the unit square. (Any other shaped city could be modeled by using the acceptance/rejection technique illustrated in Algorithm 2.3.1.) The salesman wishes to travel to each of the $n - 1$ cities, also uniformly distributed on the unit square, where he does business and returns home. Develop an algorithm and write a Monte Carlo simulation program `tsp` that estimates the average distance for the salesman to take the shortest route that visits each city exactly once and returns home. Code `tsp` to measure distance between cities: (a) as the crow flies (Euclidean), and (b) only on North-South and East-West roads (rectilinear, as in a large city). This *traveling salesman problem* (Hillier and Lieberman, 2005) plays a central role in combinatorics and in a branch of operations research known as "discrete optimization."

Exercise 2.4.14[a]   *Bootstrapping* is a statistical technique that is used to analyze a data set without making any assumptions about the population distribution (for example, that the population is bell-shaped). Efron and Tibshirani (1993, page 11) provide the rat survival data given below. Seven rats are given a treatment and their survival times given in days, are 16, 23, 38, 94, 99, 141, 197. Nine other rats constitute a control group, and their survival times are 10, 27, 30, 40, 46, 51, 52, 104, 146.

The median of the treatment group is 94 days and the median of the control group is 46 days. Do the medians differ by enough to conclude that there is a statistically significant difference between them? The *bootstrapping* approach to estimating the variability of the median for the treatment group is to generate B bootstrap samples, each of which consists of seven samples drawn with replacement from 16, 23, 38, 94, 99, 141, and 197. Draw $B = 1000$ such samples using an *Equilikely*$(1, 7)$ random variate to determine the index of the value sampled, and perform a similar sampling of $B = 1000$ values from the control group. Perform whatever analysis you deem appropriate to determine whether there is a statistically significant difference between the median survival times.

Exercise 2.4.15[a] During World War II, Axis Forces unwisely numbered their tanks sequentially from 1. As the Allied Forces captured tanks, they recorded the serial numbers in an effort to determine enemy strength by estimating the total number of tanks. Assume that four tanks, numbered 952, 1923, 1927, and 2000, have been captured. If an *Equilikely*$(1, N)$ random variate is used to model the number of tanks, where the total number of tanks N is unknown, here are four reasonable ways to estimate the number of tanks:

- **Maximum likelihood.** Choose the estimate that maximizes the likelihood of observing 2000 as the largest tank number. Obviously any estimate less that 2000 is impossible (since 2000 was observed), so using $\hat{N}_1 = 2000$ as an estimate would yield the probability of observing 2000 as the largest tank number as $1/2000$. Any larger value, such as 2007, will have a lower probability (that is, $1/2007$).

- **Mean matching.** Since the theoretical mean of the *Equilikely*$(1, N)$ distribution is $(N + 1)/2$, the theoretical mean can be equated to the sample mean:
$$\frac{N + 1}{2} = \frac{952 + 1923 + 1927 + 2000}{4}.$$
Solving for N yields $\hat{N}_2 = 3400$.

- **Gaps.** Since four tanks were captured, the average gap between the data values is
$$\frac{2000 - 4}{4} = \frac{1996}{4} = 499$$
tanks. Thus if consistent gaps were maintained, a guess for the highest numbered tank would be $\hat{N}_3 = 2000 + 499 + 1 = 2500$.

- **Minimum Variance Unbiased Estimator.** For sampling with replacement (not a brilliant military strategy) from an *Equilikely*$(1, N)$ distribution, Hogg, McKean, and Craig (2005, page 388) suggest the estimator
$$\hat{N}_4 = \frac{Y^{n+1} - (Y - 1)^{n+1}}{Y^n - (Y - 1)^n},$$
where n is the sample size and Y is the largest data value. For our data, $\hat{N}_4 = 2499.38$.

Devise a Monte Carlo simulation program that evaluates these four very different estimates for the number of tanks using criteria of your choice.

At a high level of abstraction, a common deterministic system model consists of a *state space* that defines the set of possible system states and a corresponding *state transition function* that determines how the system will evolve from state to state. Because the system is deterministic, there is no uncertainty about state transitions: when the state of the system changes, the new state is uniquely defined by the prior state of the system.

As illustrated in Section 2.1, this simple deterministic model is directly applicable to the computational process associated with random number generation. In this kind of application the size (cardinality) of the state space is typically large but *finite*. Because the state space is finite, as the system transitions from state to state it is inevitable that eventually some previously occurring state will occur again. At that point a *cycle* has occurred and, because each state determines its successor uniquely, the cycle will then repeat endlessly. The primary focus of this section is the development and implementation of three algorithms to determine when this cycle begins and, when it does, the length of the cycle so produced.

2.5.1 BACKGROUND

Definition 2.5.1   A deterministic finite-state system is defined in terms of:

- a *finite* state space (set) $\mathcal{X}$ with $|\mathcal{X}|$ elements (typically $|\mathcal{X}|$ is very large);

- a state transition function $g \colon \mathcal{X} \to \mathcal{X}$;

- finite-state sequences $x_0, x_1, x_2, x_3, \ldots$ generated by selecting x_0 (perhaps at random) from $\mathcal{X}$ and then defining $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, 3, \ldots$

Example 2.5.1   As a simple example, suppose that $\mathcal{X}$ is the set of all integers between 0 and 9999 ($|\mathcal{X}| = 10000$). Thus the states are the 4-digit non-negative integers with the understanding that if a state x is between 0 and 999 it will be made into a 4-digit integer by appending 0's on the left as necessary. Define the state transition function $g(\cdot)$ using the now infamous *midsquares* method first suggested by von Neumann and Metropolis in the 1940's as a possible way to generate random numbers. That is, define the transition from state to state as follows. Take the current state, a 4-digit number, and square it. Write the result as an 8-digit number by appending 0's on the left if necessary. The next state is then the 4-digit number defined by extracting the *middle* four digits. For example, if the current state is $x_i = 1726$ then $x_i^2 = 02\underline{9790}76$ and so $x_{i+1} = 9790$. You should convince yourself that the 4-digit midsquares method is equivalent to applying the following state transition function $g(x) = \lfloor x^2/100 \rfloor \bmod 10000$. To illustrate, the finite-state sequences corresponding to the initial states $x_0 = 1726, 6283, 5600$ are

> 1726, 9790, 8441, 2504, 2700, 2900, 4100, 8100, 6100, 2100, 4100, ...
>
> 6283, 4760, 6576, 2437, 9389, 1533, 3500, 2500, 2500, ...
>
> 5600, 3600, 9600, 1600, 5600, ...

Definition 2.5.2 The sequence $x_0, x_1, x_2, \ldots$ is *ultimately periodic* if and only if there is a *period* $p \geq 1$ and *starting index* $s \geq 0$ such that $x_{i+p} = x_i$ for all $i \geq s$.

Consistent with the following theorem, all three finite-state sequence in Example 2.5.1 are ultimately periodic. Indeed, because of Theorem 2.5.1, when dealing with any finite-state sequences the existence of (s, p) is guaranteed.

Theorem 2.5.1 For any finite state space $\mathcal{X}$, state transition function $g \colon \mathcal{X} \to \mathcal{X}$, and initial state $x_0 \in \mathcal{X}$, the corresponding infinite sequence of states $x_0, x_1, x_2, \ldots$ defined by $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, \ldots$ is ultimately periodic. Moreover, there is a unique *fundamental* (smallest) period $p \geq 1$ and unique associated starting index $s \geq 0$ such that:

- $s + p \leq |\mathcal{X}|$;

- if $s > 0$ the first s (*transient*) states $x_0, x_1, \ldots, x_{s-1}$ are all different and never reappear in the infinite sequence;

- the next p (*periodic*) states $x_s, x_{s+1}, \ldots, x_{s+p-1}$ are all different but each reappears in the infinite sequence consistent with the periodic pattern $x_{i+p} = x_i$ for $i \geq s$.

Proof Because $\mathcal{X}$ is finite and $g \colon \mathcal{X} \to \mathcal{X}$, from the pigeonhole principle at least two of the first $|\mathcal{X}| + 1$ states in the sequence $x_0, x_1, x_2, \ldots$ must be the same. That is, there are indices s and t with $0 \leq s < t \leq |\mathcal{X}|$ such that $x_s = x_t$. Because any set of non-negative integers has a least element, there is no loss of generality if it is assumed that s and t are the *smallest* such integers. This defines s. Moreover, if $p = t - s$ then $x_{p+s} = x_t = x_s$ which establishes that p is the fundamental (smallest) period. The remaining details, including the inductive proof that $x_{p+i} = x_i$ for all $i \geq s$, are left as an exercise.

Example 2.5.2 In general, the (s, p) pair depends on x_0. For Example 2.5.1:

- if the initial state is $x_0 = 1726$ so that

$$1726, \ 9790, \ 8441, \ 2504, \ 2700, \ 2900, \ 4100, \ 8100, \ 6100, \ 2100, \ 4100, \ldots$$

then $(s, p) = (6, 4)$;

- if the initial state is $x_0 = 6283$ so that

$$6283, \ 4760, \ 6576, \ 2437, \ 9389, \ 1533, \ 3500, \ 2500, \ 2500, \ \ldots$$

then $(s, p) = (7, 1)$;

- if the initial state is $x_0 = 5600$ so that

$$5600, \ 3600, \ 9600, \ 1600, \ 5600, \ \ldots$$

then $(s, p) = (0, 4)$.

Three algorithms for computing (s, p) are presented in this section. The first two of these algorithms are "obvious", the third (and best) is not.

2.5.2 ALGORITHMS

The first two algorithms are based on the observation that (s, p) can be determined as follows.

- Pick x_0 and then generate $x_1 = g(x_0)$. If $x_0 = x_1$ then $(s, p) = (0, 1)$.

- If x_0 and x_1 are different, then generate $x_2 = g(x_1)$. If $x_0 = x_2$ then $(s, p) = (0, 2)$, else if $x_1 = x_2$ then $(s, p) = (1, 1)$.

- If x_0, x_1, and x_2 are all different, then generate $x_3 = g(x_2)$. If $x_0 = x_3$ then $(s, p) = (0, 3)$, else if $x_1 = x_3$ then $(s, p) = (1, 2)$, else if $x_2 = x_3$ then $(s, p) = (2, 1)$.

- If x_0, x_1, x_2, x_3 are all different, then generate $x_4 = g(x_3)$ and compare, etc.

Eventually this algorithm will terminate with an index $1 \leq t \leq |\mathcal{X}|$ and corresponding state x_t such that the t states $x_0, x_1, \ldots x_{t-1}$ are all different but exactly one of these states, say x_s, is equal to x_t:

$$\underbrace{x_0, x_1, x_2, x_3, \ldots, x_s, \ldots, x_{t-1}}_{\text{all are different, but } x_s = x_t}, x_t, \ldots$$

At this point s and $p = t - s$ are both determined. Indeed, this algorithm is a constructive proof of Theorem 2.5.1.

Algorithm 2.5.1

To implement the algorithm outlined above, note that if $|\mathcal{X}|$ is small enough then all the intermediate states $x_0, x_1, x_2, \ldots, x_t$ can be stored in a linked list as they are generated. This is summarized by the following algorithm which has $(s+p)$ storage requirements and makes $(s + p)$ calls to $g(\cdot)$.

Algorithm 2.5.1   Given the state transition function $g(\cdot)$ and initial state x_0, this algorithm determines the fundamental pair (s, p).

```
x₀ =  initial state;                        /* initialize the list */
t = 0;
s = 0;
while (s == t) {                            /* while no match found */
    x_{t+1} = g(x_t);                        /* add a state to the list */
    t++;
    s = 0;
    while (x_s != x_t)                      /* traverse the list */
        s++;
}
p = t - s;                     /* period is the distance between matches */
return s, p;
```

There is an important time-space trade-off in Algorithm 2.5.1. When the algorithm terminates $(s + p)$ may be as large as $|\mathcal{X}|$. In many important applications $|\mathcal{X}|$ is large, or unknown, relative to the amount of available storage for the intermediate states. In this case storing all the intermediate states may not be possible; if so, as an alternative it will be necessary to trade time for space by generating these states *repeatedly* as necessary, once for each comparison.

Algorithm 2.5.2

Algorithm 2.5.1 can be modified to eliminate the need to store all the intermediate states by observing that only *three* states need to be stored — the initial state x_0 and the current value of x_s and x_t. In the following algorithm these three states are denoted $x.o$, $x.s$, and $x.t$, respectively. This algorithm has minimal storage requirements and is easy to program; however, the number of calls to $g(\cdot)$ is $s + (p + s)(p + s + 1)/2$.

Algorithm 2.5.2   Given the state transition function $g(\cdot)$ and initial state $x.o$, this algorithm determines the corresponding fundamental pair (s, p).

```
x.t = x.o;
t = 0;
s = 0;
while (s == t) {
   x.t = g(x.t);
   t++;
   x.s = x.o;
   s = 0;
   while (x.s != x.t) {
      x.s = g(x.s);
      s++;
   }
}
p = t - s;
return s, p;
```

Example 2.5.3   Algorithm 2.5.1 or 2.5.2 applied to the 6-digit midsquares state transition function

$$g(x) = \lfloor x^2/1000 \rfloor \bmod 1000000$$

yields the following results (albeit slowly):

initial state x_0	:	141138	119448	586593	735812	613282
fundamental pair (s, p)	:	$(296, 29)$	$(428, 210)$	$(48, 13)$	$(225, 1)$	$(469, 20)$

The time complexity of Algorithm 2.5.2 will be dominated by calls to the state transition function $g(\cdot)$. For many practical applications, for example analyzing the (fundamental) period of a (good) random number generator, $s + (s+p)(s+p+1)/2$ is so large that the complexity (speed) of Algorithm 2.5.2 is unacceptable. For that reason, we are motivated to look for a better algorithm to determine (s, p) — an algorithm with the best features of Algorithms 2.5.1 and 2.5.2.

Algorithm 2.5.3

The development of this better algorithm is based on the following theorem which provides two important characterizations of periods, fundamental or not. One characterization is that the starting index s has the property that $x_{2q} = x_q$ if and only if q is a period at least as large as s; the other is that the fundamental period p has the property that the positive integer q is a period if and only if q is an integer multiple of p. The proof of this theorem is left as an exercise.

Theorem 2.5.2   If the finite-state sequence $x_0, x_1, x_2, \ldots$ has the fundamental pair (s, p) and if $q > 0$ then:

- q is a period and $q \geq s$ if and only if $x_{2q} = x_q$;

- q is a period if and only if $q \bmod p = 0$.

Given the fundamental pair (s, p), the index

$$q = \begin{cases} p & s = 0 \\ \lceil s/p \rceil p & s > 0 \end{cases}$$

plays an important role in the development of Algorithm 2.5.3. That is, since q is either p or an integer multiple of p, it follows from Theorem 2.5.2 that q is a period of the infinite sequence. In addition, if $s > 0$ then $q = \lceil s/p \rceil p$ so that

$$s = (s/p)p \leq \lceil s/p \rceil p = q \qquad \text{and} \qquad q = \lceil s/p \rceil p < (1 + s/p)p = p + s.$$

Similarly, if $s = 0$ then $q = p$. Therefore, the period q satisfies the inequality $s \leq q \leq p + s$. Because $q \geq s$, it follows from Theorem 2.5.2 that $x_{2q} = x_q$.

Now, suppose that $x_{2i} = x_i$ for *any* integer $i > 0$. It follows from Theorem 2.5.2 that i is a period with $i \geq s$ and that there exists an integer $\alpha \geq 1$ such that $i = \alpha p \geq s$. If $s > 0$ then from the condition $\alpha p \geq s$, it follows that $\alpha \geq \lceil s/p \rceil$. If we multiply both sides of this inequality by p, it follows that $i = \alpha p \geq \lceil s/p \rceil p = q$. Similarly, if $s = 0$ then $q = p$ and because $i = \alpha p \geq p$, it follows that $i \geq q$. Therefore, independent of the value of s, if $x_i = x_{2i}$ for some $i > 0$ then $i \geq q \geq s$.

From the previous discussion we see that although q may or may not be the smallest (fundamental) period, it is at least the smallest period that is not smaller than s. This is summarized by the following theorem.

Theorem 2.5.3 Given the finite-state sequence $x_0, x_1, x_2, \ldots$ with fundamental pair (s, p), the smallest index $i > 0$ such that $x_{2i} = x_i$ is $i = q$ where

$$q = \begin{cases} p & s = 0 \\ \lceil s/p \rceil p & s > 0. \end{cases}$$

Moreover, $i = q$ is a period that satisfies the inequality $s \leq q \leq s + p$.

Theorem 2.5.3 is the basis for a 3-step algorithm with at most $(5s + 4p)$ calls to $g(\cdot)$ and minimal storage requirements that will find (s, p). This algorithm is based on the "racetrack analogy" in step (1).

(1) *To determine q and x_q* — think of the two sequences

$$\begin{array}{ccccccc} x_0, & x_1, & x_2, & x_3, & \cdots, & x_i, & \cdots \\ x_0, & x_2, & x_4, & x_6, & \cdots, & x_{2i}, & \cdots \end{array}$$

as points (states) that start together and then move around a race track with the second sequence moving twice as fast as the first. From Theorem 2.5.3 the second sequence will catch the first sequence for the first time at $i = q$. Since $q \leq s + p$ the number of steps in this race is no more than $s + p$. Thus there is an algorithm for finding q and x_q with no more than $(3s + 3p)$ calls to $g(\cdot)$. Moreover, since the only states that are needed in this race are the current values of x_i and x_{2i}, the storage requirements for this step are minimal.

(2) *To determine s and x_s* — because q is a period, $x_s = x_{s+q}$. Therefore, the starting index s and corresponding state x_s can be determined from a knowledge of x_q by beginning with the states x_0, x_q and generating the successor states

$$\begin{array}{ccccccc} x_0, & x_1, & x_2, & x_3, & \cdots, & x_i, & \cdots \\ x_q, & x_{1+q}, & x_{2+q}, & x_{3+q}, & \cdots, & x_{i+q}, & \cdots \end{array}$$

The starting index s is the first value of $i \geq 0$ for which $x_i = x_{i+q}$. The number of calls to $g(\cdot)$ for this step is $2s$ and the storage requirements are minimal.

(3) *To determine p* — once s, q, and x_s have been determined, the fundamental period p can be determined as follows.

(a) If $2s \leq q$ (this is the usual case) then $p = q$. That is, because $q \leq s + p$, it follows that $s + q \leq 2s + p$. Therefore, if $2s \leq q$ then $s + q \leq q + p$ which is equivalent to $s \leq p$. If $s \leq p$, however, then $q = p$.

(b) If $2s > q$ then because $x_s = x_{s+p}$, a linear search can be used to determine p. That is, p can be determined by starting with x_s and generating the successor states x_{s+i}. The fundamental period p is then the first value of $i > 0$ for which $x_s = x_{s+i}$. The number of calls to $g(\cdot)$ for this step is p with minimal storage requirements.

Algorithm 2.5.3 is an implementation of this 3-step approach.

Algorithm 2.5.3 Given the state transition function $g(\cdot)$ and initial state $x.o$, this algorithm determines the corresponding fundamental pair (s, p).

```
q   = 1;
x.q = g(x.o);
z   = g(x.q);
while (x.q != z) {                      /* step 1:   determine q and x.q */
   q++;
   x.q = g(x.q);
   z   = g(g(z));
}
s   = 0;
x.s = x.o;
z   = x.q;
while (x.s != z) {                      /* step 2:   determine s and x.s */
   s++;
   x.s = g(x.s);
   z   = g(z);
}
if (2 * s <= q)                              /* step 3:   determine p */
   p = q;
else {
   p = 1;
   z = g(x.s);
   while (x.s != z) {
     p++;
     z = g(z);
   }
}
return s, p;
```

Example 2.5.4 As a continuation of Example 2.5.3, Algorithm 2.5.3 applied to the 6-digit midsquares state transition function

$$g(x) = \lfloor x^2/1000 \rfloor \bmod 1000000$$

yields the following results:

initial state	:	141138	119448	586593	735812	613282
speed up	:	34	72	7	21	48

The "speed up" row indicates the efficiency of Algorithm 2.5.3 relative to Algorithm 2.5.2, e.g., for $x_0 = 141138$ Algorithm 2.5.3 is approximately 34 times faster than Algorithm 2.5.2.

Full-Period Sequences

As suggested previously, a primary application for Algorithm 2.5.3 is to analyze the period of a random number generator. Algorithm 2.5.3 is attractive in this application because it is efficient and (s, p) can be determined without any knowledge of the "details" of $g(\cdot)$. All that is needed to apply Algorithm 2.5.3 is a function that implements $g(\cdot)$ and, perhaps, lots of CPU cycles.

Consistent with the discussion in Sections 2.1 and 2.2, when analyzing the period of a random number generator, we are primarily concerned with *full-period* sequences. In the generalized context of this section, that idea is expressed in terms of (s, p) by the following definition and associated theorem.

Definition 2.5.3 The function $g \colon \mathcal{X} \to \mathcal{X}$ is a *full-period* state transition function and the corresponding sequence $x_0, x_1, x_2, \ldots$ has a *full-period* if and only if $p = |\mathcal{X}|$.

Note that $p = |\mathcal{X}|$ is possible only if $s = 0$. That is, $s = 0$ is a necessary (but not sufficient) condition for $g(\cdot)$ to be a full-period state transition function. Of course, if (somehow) you *know* that $s = 0$ then there is no need to use Algorithm 2.5.3 to test for a full period. Instead, it is sufficient to use the following variant of Algorithm 2.1.1

```
p = 1;
x = g(x₀);
while (x != x₀) {
    p++;
    x = g(x);
}
return p;
```

If $p = |\mathcal{X}|$ then $g(\cdot)$ is a full-period state transition function.

Example 2.5.5 If $g(\cdot)$ is a Lehmer generator (full period or not) then, from Theorem 2.1.2, for any initial state we know that $s = 0$. Therefore, (with $x_0 = 1$) the use of Algorithm 2.1.1 is justified as a test for full period.

The following theorem is an immediate consequence of Definition 2.5.3 and Theorem 2.5.1. The proof is left as an exercise.

Theorem 2.5.4 The infinite finite-state sequence $x_0, x_1, x_2, \ldots$ has a full period if and only if each possible state appears exactly once in *any* $|\mathcal{X}|$ consecutive terms of the sequence.

As an exercise you are encouraged to construct your own many-digit state transition function using a variety of obscure bit-level and byte-level operations to "randomize" the output. You will be impressed with how difficult it is to construct a full-period state transition function. In that regard, see Algorithm K in Knuth (1998).

2.5.3   STATE SPACE PARTITION

Provided (a, m) are properly chosen, Lehmer generators: (1) have full period, (2) have been extensively studied, (3) provide multiple streams, and (4) produce output that is generally considered to be acceptably random for discrete-event simulation purposes. Given all of that, one could ask why anyone would be interested in other types of generators, particularly if they are not full-period generators. But, the discussion in this section is not focused on random number generators exclusively, and certainly not on Lehmer generators. Instead, the focus is on general state transition functions and corresponding finite-state sequences that may not be (and generally will not be) full period. Correspondingly, in the remainder of this section we will study a state space partition that is naturally associated with state transition functions that are *not* full period. The study of this state space partition is facilitated by introducing the idea of "jump" functions, defined as follows.

Jump Functions

If $g\colon \mathcal{X} \to \mathcal{X}$ then for all $x \in \mathcal{X}$ inductively define the *jump function* $g^j(x)$ as follows

$$g^j(x) = \begin{cases} x & j = 0 \\ g\big(g^{j-1}(x)\big) & j = 1, 2, 3, \ldots \end{cases}$$

Because $g : \mathcal{X} \to \mathcal{X}$ it follows that $g^j : \mathcal{X} \to \mathcal{X}$ for any $j \geq 0$. That is, the jump function $g^j(\cdot)$ is also a state transition function. Moreover, if $g(\cdot)$ generates the sequence $x_0, x_1, x_2, x_3, \ldots$ then $g^j(\cdot)$ generates the sequence $x_0, x_j, x_{2j}, x_{3j}, \ldots$ In terms of the race-track analogy, if $j \geq 2$ then the state transition function $g^j(\cdot)$ runs the race j times faster than $g(\cdot)$ by jumping ahead j states at a time. This important idea is used in Chapter 3 to construct a Lehmer random number generator that can create and maintain multiple *streams* of random numbers.

Periodic and Transient States

The following definition is the key to studying the state space partition that is naturally associated with a state transition function that is not full period.

Definition 2.5.4   Given a finite state space $\mathcal{X}$ and state transition function $g\colon \mathcal{X} \to \mathcal{X}$, then $x \in \mathcal{X}$ is a *periodic* (or recurrent) state if and only if there exists a positive integer j with $0 < j \leq |\mathcal{X}|$ such that $g^j(x) = x$. If $x \in \mathcal{X}$ is not a periodic state, then it is said to be a *transient* state.

Theorem 2.5.1 is the basis for this definition. If $x = x_0$ is the initial state in a finite-state sequence, then either $s = 0$ in which case x is a periodic state because $g^p(x) = x$, or $s > 0$ in which case x is a transient state because $g^j(x) \neq x$ for all $j = 1, 2, \ldots$

Definition 2.5.5   If $x \in \mathcal{X}$ is a periodic state, then the *recurrent class* corresponding to x is the set

$$\mathcal{R} = \{g^i(x) : i = 0, 1, 2, \ldots, j - 1\}$$

where j is the smallest positive integer such that $g^j(x) = x$. Note that $|\mathcal{R}| = j$.

From Theorem 2.5.1, there must be at least one periodic state. Therefore, there must be at least one recurrent class. Moreover, if $x \in \mathcal{X}$ is a periodic state with corresponding recurrent class $\mathcal{R}$ and if $x' \in \mathcal{X}$ is a periodic state with corresponding recurrent class $\mathcal{R}'$, then either $\mathcal{R} = \mathcal{R}'$ or $\mathcal{R} \cap \mathcal{R}' = \emptyset$. That is, the set of periodic states is *partitioned* into recurrent classes. Equivalently, as Examples 2.5.6 and 2.6.7 illustrate, if there are multiple recurrent classes then the classes are disjoint — each periodic state belongs to exactly one recurrent class.

For reference, note that if $g(\cdot)$ is a full-period state transition function then all states are periodic and there is only one recurrent class, which is $\mathcal{X}$. If $g(\cdot)$ is not a full-period state transition function then, generally, there will be transient states and more than one recurrent class, but not necessarily.

Example 2.5.6   The 1-digit midsquares state transition function
$$g(x) = \lfloor x^2 \rfloor \bmod 10 = x^2 \bmod 10$$
has four periodic states (0, 1, 5, 6) and four transient states (2, 3, 4, 7, 8, 9). The periodic states are partitioned into four recurrent classes: $\{0\}$, $\{1\}$, $\{5\}$, and $\{6\}$, as illustrated in Figure 2.5.1.
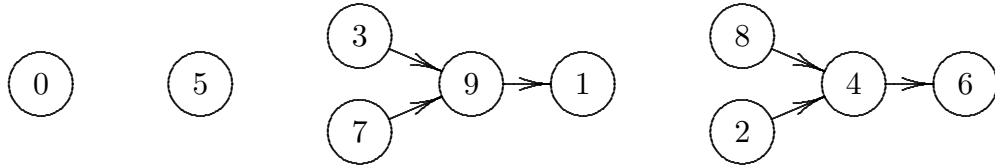
Figure 2.5.1. Periodic and transient. states.

Example 2.5.7   For the 4-digit midsquares state transition function, it can be shown that there are 17 periodic states and 9983 transient states. The periodic states are partitioned into eight recurrent classes: $\{2100, 4100, 8100, 6100\}$, $\{0\}$, $\{100\}$, $\{2500\}$, $\{7600\}$, $\{1600, 5600, 3600, 9600\}$, $\{540, 2916, 5030, 3009\}$, and $\{3792\}$.

Although the recurrent classes contain periodic states only, from Theorem 2.5.1 any transient state $x \in \mathcal{X}$ is "connected to" a recurrent class in the sense that there is a positive integer j such that $g^j(x)$ is a periodic state. This observation is summarized by the following theorem.

Theorem 2.5.5   If $x \in \mathcal{X}$ is a transient state then there then there exists a unique recurrent class $\mathcal{R}$ such that $g^j(x) \in \mathcal{R}$ for some j with $0 < j < |\mathcal{X}|$. In this case x is said to be *connected to* $\mathcal{R}$.

It follows from Theorem 2.5.5 that each recurrent class has associated with it a (perhaps empty) set of transient states that are connected to it. In this way, each recurrent class can be naturally enlarged to an associated "cluster".

Definition 2.5.6   For each recurrent class $\mathcal{R}$, the associated recurrent class *cluster* is
$$\mathcal{C} = \mathcal{R} \cup \{\text{all the transient states that are connected to } \mathcal{R}\}.$$

There is one cluster for each recurrent class, the clusters are disjoint, and each state (transient or periodic) is in exactly one cluster. Equivalently, the clusters partition the state space. This is an important conceptual result, particularly as it relates to understanding the behavior of bad random number generators.

Relative to the conceptual model that random number generation corresponds to drawing at random from an urn, the clusters correspond to urns, the initial state of the generator determines the urn (cluster) from which *all* subsequent draws will be made, and the recurrent class in the cluster defines the periodic cycle into which the output of the generator must ultimately collapse. If the initial state of the generator is selected at random, the cardinality of each cluster determines the probability that cluster (urn) will be selected.

Example 2.5.8   As a continuation of Example 2.5.7, for the 4-digit midsquares state transition function, there are eight clusters. The recurrent class associated with each cluster is illustrated in the table below, along with the size of each cluster. The table is sorted by decreasing cluster size

cluster size	recurrent class
6291	$\{2100, 4100, 8100, 6100\}$
1968	$\{0\}$
1360	$\{1600, 5600, 3600, 9600\}$
130	$\{2500\}$
104	$\{100\}$
86	$\{540, 2916, 5030, 3009\}$
60	$\{7600\}$
1	$\{3792\}$

As a source of random numbers, the 4-digit midsquares method is a non-starter.

It is an interesting exercise to construct a time efficient algorithm that will generate the state space decomposition into recurrent class clusters. In that regard, see Exercise 2.5.4.

2.5.4   EXERCISES

Exercise 2.5.1   The BASICA random number generator for the original "true blue" IBM PC was defined by the state transition function

$$g(x) = (a(x \bmod 2^{16}) + c) \bmod 2^{24}$$

with $a = 214\,013$ and $c = 13\,523\,655$. (*a*) If $g \colon \mathcal{X} \to \mathcal{X}$ with $\mathcal{X} = \{0, 1, \ldots, m - 1\}$ what is m? (*b*) Determine (s, p) for initial seeds of 0, 1, 12, 123, 1234, and the last four digits of your social security number. (*c*) For each initial seed, how does $|\mathcal{X}| = m$ relate to p? (*d*) Comment.

Exercise 2.5.2 Determine (s, p) for both of the following state transition functions

```
long g(long x)                              /* start with x = 777 */
{
x = (123 * x + 1) % 123456;
return x;
}

long g(long x)                    /* start with x = 37911 and x = 1 */
{
x = (9806 * x + 1) % 131071;
return x;
}
```

Exercise 2.5.3 Prove or disprove the following two conjectures. Given that $|\mathcal{X}|$ is finite, $g \colon \mathcal{X} \to \mathcal{X}$, $x_0 \in \mathcal{X}$, and (s, p) are the values returned by Algorithm 2.5.3: (a) if $s = 0$ and $p = |\mathcal{X}|$ then the function g is a bijection (one-to-one and onto); (b) if the function g is a bijection then $s = 0$ and $p = |\mathcal{X}|$.

Exercise 2.5.4ª Given a finite state space $\mathcal{X}$ and state transition function $g \colon \mathcal{X} \to \mathcal{X}$, construct an efficient algorithm that partitions the state space into clusters. In particular, the algorithm should output the size (cardinality) of each cluster, the smallest periodic state in each cluster, and the size of each associated recurrent class. (a) What is the output of this algorithm when applied to the 6-digit midsquares state transition function? (b) Same question for the state transition function in Exercise 2.5.1? (c) Comment on both of these state transition functions as a possible source of random numbers. (d) What is the time complexity (relative to $g(\cdot)$ evaluations) and space complexity of your algorithm?

Exercise 2.5.5ª Consider a large round 12-hour clock with an hour hand, a minute hand, and 3600 marks (seconds) around the circumference of the clock. Both hands move in discrete steps from one mark to the next. The marks define the set $\mathcal{X} = \{0, 1, \ldots, 3599\}$. Assume that both hands are "geared" so that a manual movement of the hour hand by one mark forces the minute hand to move by 12 marks. Generate a sequence of integers $x_0, x_1, x_2, \ldots$ in $\mathcal{X}$ as follows. Start with the hands in an arbitrary location (time) and let $x_0 \in \mathcal{X}$ be the location (mark) of the minute hand. To generate x_1, move the *hour* hand clockwise to the *minute* hand mark (x_0), allowing the minutes hand to spin accordingly to a new mark; this new minute hand mark is $x_1 \in \mathcal{X}$. To generate x_2, move the hour hand to x_1 and let $x_2 \in \mathcal{X}$ be the new location of the minute hand, etc. What do you think of this as a source of random numbers? ("Not much" is not a sufficient answer.)

Exercise 2.5.6ª Same as Exercise 2.5.5 except for a clock with 360 000 marks around the circumference.

As discussed in Chapter 1, `ssq1` and `sis1` are examples of trace-driven discrete-event simulation programs. By definition, a trace-driven simulation relies on input data from an external source to supply recorded realizations of naturally occurring stochastic processes. Total reliance on such external data limits the applicability of a discrete-event simulation program, naturally inhibiting the user's ability to do "what if" studies. Given this limitation, a general discrete-event simulation objective is to develop methods for using a random number generator to convert a trace-driven discrete-event simulation program to a discrete-event simulation program that is not dependent on external data. This chapter provides several examples.

3.1.1   SINGLE-SERVER SERVICE NODE

Relative to the single-server service node model from Chapter 1, two stochastic assumptions are needed to free program `ssq1` from its reliance on external data. One assumption relates to the arrival times, the other assumption relates to the service times. We consider the service times first, using a $Uniform(a, b)$ random variate model.

Example 3.1.1   Suppose that time is measured in minutes in a single-server service node model and all that is known about the service time is that it is random with possible values between 1.0 and 2.0. That is, although we know the range of possible values, we are otherwise in such a state of ignorance about the stochastic behavior of this server that we are unwilling to say some service times (between 1.0 and 2.0) are more likely than others. In this case we have modeled service time as a $Uniform(1.0, 2.0)$ random variable. Accordingly, random variate service times, say s, can be generated via the assignment

```
s = Uniform(1.0, 2.0);
```

Exponential Random Variates

A $Uniform(a, b)$ random variable has the property that all values between a and b are equally likely. In most applications this is an unrealistic assumption; instead, some values will be more likely than others. Specifically, there are many discrete-event simulation applications that require a continuous random variate, say x, that can take on any positive value but in such a way that small values of x are more likely than large values.

To generate such a random variate we need a *nonlinear* transformation that maps values of the random number u between 0.0 and 1.0 to values of x between 0.0 and ∞ and does so by "stretching" large values of u much more so than small values. Although a variety of such nonlinear transformation are possible, for example $x = u/(1 - u)$, perhaps the most common is $x = -\mu \ln(1 - u)$ where $\mu > 0$ is a parameter that controls the rate of stretching and $\ln(\cdot)$ is the natural logarithm (base e).* As explained in Chapter 7, this transformation generates what is known as an $Exponential(\mu)$ random variate.

---

* The function `log(x)` in the ANSI C library `<math.h>` represents the mathematical natural log function $\ln(x)$. In the equation $x = -\mu \ln(1 - u)$ do not confuse the positive, real-valued parameter μ with the $Uniform(0, 1)$ random variate u.

It is often appropriate to generate interarrivals as *Exponential*(μ) random variates, as we will do later in Example 3.1.2. In some important theoretical situations, service times are also modeled this way (see Section 8.5). The geometry associated with the *Exponential*(μ) random variate transformation $x = -\mu \ln(1 - u)$ is illustrated in Figure 3.1.1.

**Figure 3.1.1.
Exponential variate
generation geometry.**
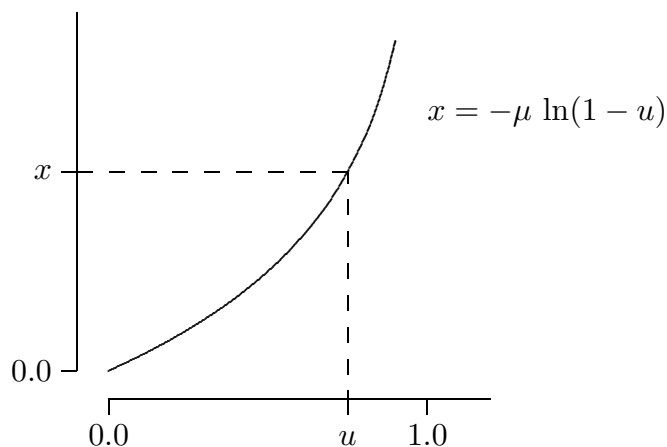
By inspection we see that the transformation is monotone increasing and, for any value of the parameter $\mu > 0$, the interval $0 < u < 1$ is mapped one-to-one and onto the interval $0 < x < \infty$. That is,

$$
\begin{aligned}
0 < u < 1 &\iff 0 < (1 - u) < 1 \\
&\iff -\infty < \ln(1 - u) < 0 \\
&\iff 0 < -\mu \ln(1 - u) < \infty \\
&\iff 0 < x < \infty.
\end{aligned}
$$

Definition 3.1.1   This ANSI C function generates an *Exponential*(μ) random variate*

```
double Exponential(double μ)                          /* use μ > 0.0 */
{
   return (-μ * log(1.0 - Random()));
}
```

The statistical significance of the parameter μ is that if repeated calls to the function `Exponential`(μ) are used to generate a random variate sample x_1, x_2, ..., x_n then, in the limit as $n \to \infty$, the sample mean (average) of this sample will converge to μ. In the same sense, repeated calls to the function `Uniform`(a, b) will produce a sample whose mean converges to $(a + b)/2$ and repeated calls to the function `Equilikely`(a, b) will also produce a sample whose mean converges to $(a + b)/2$.

---

* The ANSI C standard says that `log(0.0)` may produce "a range error". This possibility is naturally avoided in the function `Exponential` because the largest possible value of `Random` is less than 1.0. See Exercise 3.1.3.

Example 3.1.2   In a single-server service node simulation, to generate a sequence of random variate *arrival* times $a_1, a_2, a_3, \ldots, a_n$ with an average interarrival time that will converge to μ as $n \to \infty$ it is common to generate *Exponential*(μ) *interarrival* times (see Definition 1.2.3) and then (with $a_0 = 0$) create the arrival times by the assignment

$\quad$ `a`$_i$ `= a`$_{i-1}$ `+ Exponential(`μ`);` $\qquad\qquad$ $i = 1, 2, 3, \ldots, n$

As discussed in Chapter 7, this use of an *Exponential*(μ) random variate corresponds naturally to the idea of jobs arriving *at random* with an arrival rate that will converge to $1/\mu$ as $n \to \infty$. Similarly, as in Example 3.1.1, to generate a random variate sequence of service times $s_1, s_2, s_3, \ldots, s_n$ equally likely to lie anywhere between a and b (with $0 \le a < b$) and with an average service time that converges to $(a+b)/2$, the assignment

$\quad$ `s`$_i$ `= Uniform(`a`, `b`);` $\qquad\qquad$ $i = 1, 2, 3, \ldots, n$

can be used. The average service time $(a+b)/2$ corresponds to a service rate of $2/(a+b)$.

Program ssq2

$\quad$ Program `ssq2` is based on the two stochastic modeling assumptions in Example 3.1.2.* Program `ssq2` is an extension of program `ssq1` in that the arrival times and service times are generated randomly (rather than relying on a trace-driven input) and a complete set of first-order statistics $\bar{r}$, $\bar{w}$, $\bar{d}$, $\bar{s}$, $\bar{l}$, $\bar{q}$, $\bar{x}$ is generated. Note that each time the function `GetArrival()` is called the static variable `arrival`, which represents an *arrival* time, is incremented by a call to the function `Exponential(2.0)`, which generates an *interarrival* time.

$\quad$ Because program `ssq2` generates stochastic data as needed there is essentially no restriction on the number of jobs that can be processed. Therefore, the program can be used to study the *steady-state* behavior of a single-server service node. That is, by experimenting with an increasing number of jobs processed, one can investigate whether or not the service node statistics will converge to constant values, *independent* of the choice of the `rng` initial seed and the initial state of the service node. Steady-state behavior — can it be achieved and, if so, how many jobs does it take to do so — is an important issue that will be explored briefly in this chapter and in more detail in Chapter 8.

$\quad$ Program `ssq2` can also be used to study the *transient* behavior of a single-server service node. The idea in this case is to fix the number of jobs processed at some finite value and run (replicate) the program repeatedly with the initial state of the service node fixed, changing *only* the `rng` initial seed from run to run. In this case replication will produce a natural variation in the service node statistics consistent with the fact that for a fixed number of jobs, the service node statistics are *not* independent of the initial seed or the initial state of the service node. Transient behavior, and its relation to steady-state behavior, will be considered in Chapter 8.

$\quad$ * In the jargon of queuing theory (see Section 8.5), program `ssq2` simulates what is known as an $M/G/1$ queue (see Kleinrock, 1975, 1976, or Gross and Harris, 1985).

Steady-State Statistics

Example 3.1.3   If the *Exponential*(μ) interarrival time parameter is set to $\mu = 2.0$ so that the steady-state arrival rate is $1/\mu = 0.5$ and if the *Uniform*(a, b) service time parameters are set to $a = 1.0$ and $b = 2.0$ respectively so that the steady-state service rate is $2/(a + b) \cong 0.67$, then to *d.dd* precision the theoretical steady-state statistics generated from an analytic model (exact, not estimates by simulation) program `ssq2` will produce are (see Gross and Harris, 1985)

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	3.83	2.33	1.50	1.92	1.17	0.75

Therefore, although the server is only busy 75% of the time ($\bar{x} = 0.75$), "on average" there are approximately two jobs ($\bar{l} = 23/12 \cong 1.92$) in the service node and a job can expect to spend more time ($\bar{d} = 23/6 - 3/2 \cong 2.33$ time units) in the queue than in service ($\bar{s} = 1.50$ time units). As illustrated in Figure 3.1.2 for the average wait, the number of jobs that must be pushed through the service node to achieve these steady-state statistics is large. To produce this figure, program `ssq2` was modified to print the accumulated average wait every 20 jobs. The results are presented for three choices of the `rng` initial seed.* The solid, horizontal line at height $23/6 \cong 3.83$ represents the steady-state value of $\bar{w}$.

Figure 3.1.2. Average wait times.
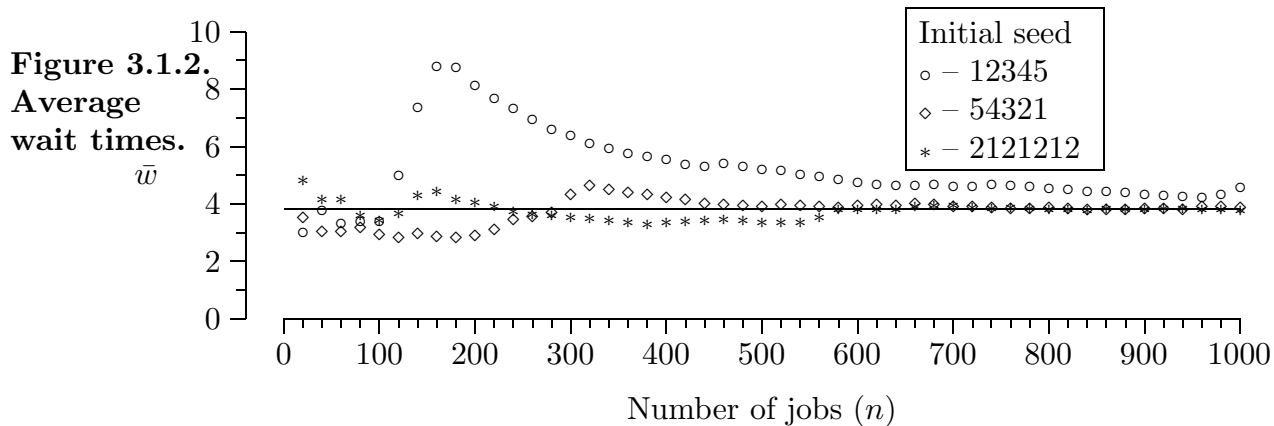
In Example 3.1.3 convergence of $\bar{w}$ to the steady-state value 3.83 is slow, erratic, and very dependent on the random variate sequence of stochastic arrival and service times, as manifested by the choice of initial seed. Note, for example, the dramatic rise in average wait beginning at about job 100 associated with the `rng` initial seed 12345. You are encouraged to add diagnostic printing and additional statistics gathering to program `ssq2` to better understand what combination of chance events occurred to produce this rise.

The *Uniform*(a, b) service time model in Example 3.1.3 may not be realistic. Service times seldom "cut off" beyond a minimum and maximum value. More detail will be given in subsequent chapters on how to select realistic distributions for input models.

---

* There is nothing special about these three initial seeds. Do not fall into the common trap of thinking that some `rng` initial seeds are necessarily better than others.

Example 3.1.3 shows that the stochastic character of the arrival times and service times, as manifested by the choice of `rng` initial seed, has a significant effect on the transition-to-steady-state behavior of a single-server service node. This example also illustrates the use of the library `rng` to conduct *controlled* "what if" experiments. Studies like this are the stuff of discrete-event simulation. Additional examples of this kind of experimentation, and the selection of values of μ, a, and b, are presented in later chapters.

Geometric Random Variates

As discussed in Chapter 2, an *Equilikely*(a, b) random variate is the discrete analog of a continuous *Uniform*(a, b) random variate. Consistent with this characterization, one way to generate an *Equilikely*(a, b) random variate is to generate a *Uniform*$(a, b + 1)$ random variate instead (note that the upper limit is $b + 1$, not b) and then convert (cast) the resulting floating-point result to an integer. That is, if a and b are integers with $a < b$ and if x is a *Uniform*$(a, b + 1)$ random variate then $\lfloor x \rfloor$ is an *Equilikely*(a, b) random variate.

Given the analogy between *Uniform*(a, b) and *Equilikely*(a, b) random variates, it is reasonable to expect that there is a discrete analog to a continuous *Exponential*(μ) random variate; and there is. Specifically, if x is an *Exponential*(μ) random variate, let y be the *discrete* random variate defined by $y = \lfloor x \rfloor$. For a better understanding of this discrete random variate, let $p = \Pr(y \neq 0)$ denote the probability that y is not zero. Since x is generated as $x = -\mu \ln(1 - u)$ with u a *Uniform*$(0, 1)$ random variate, $y = \lfloor x \rfloor$ will not be zero if and only if $x \geq 1$. Equivalently

$$x \geq 1 \iff -\mu \ln(1 - u) \geq 1$$
$$\iff \ln(1 - u) \leq -1/\mu$$
$$\iff 1 - u \leq \exp(-1/\mu)$$

where $\exp(-1/\mu)$ is $e^{-1/\mu}$, and so $y \neq 0$ if and only if $1 - u \leq \exp(-1/\mu)$. Like u, $1 - u$ is also a *Uniform*$(0, 1)$ random variate. Moreover, for any $0 < p < 1$, the condition $1 - u \leq p$ is true with probability p (see Section 7.1). Therefore, $p = \Pr(y \neq 0) = \exp(-1/\mu)$.

If x is an *Exponential*(μ) random variate and if $y = \lfloor x \rfloor$ with $p = \exp(-1/\mu)$, then it is conventional to call y a *Geometric*(p) random variable (see Chapter 6). Moreover, it is conventional to use p rather than $\mu = -1/\ln(p)$ to define y directly by the equation

$$y = \lfloor \ln(1 - u)/\ln(p) \rfloor.$$

Definition 3.1.2 This ANSI C function generates a *Geometric*(p) random variate*

```
long Geometric(double p)                              /* use 0.0 < p < 1.0 */
{
   return ((long) (log(1.0 - Random()) / log(p)));
}
```

---

* Note that `log(0.0)` is avoided in this function because the largest possible value returned by `Random` is less than 1.0.

In addition to its significance as $\Pr(y \neq 0)$, the parameter p is also related to the mean of a *Geometric*(p) sample. Specifically, if repeated calls to the function `Geometric(p)` are used to generate a random variate sample $y_1, y_2, \ldots, y_n$ then, in the limit as $n \to \infty$, the mean of this sample will converge to $p/(1-p)$. Note that if p is close to 0.0 then the mean will be close to 0.0. At the other extreme, if p is close to 1.0 then the mean will be large.

In the following example, a *Geometric*(p) random variate is used as part of a *composite* service time model. In this example the parameter p has been adjusted to make the average service time match that of the *Uniform*$(1.0, 2.0)$ server in program `ssq2`.

Example 3.1.4 Usually one has sufficient information to argue that a *Uniform*(a, b) random variate service time model is not appropriate; instead, a more sophisticated model is justified. Consider a hypothetical server that, as in program `ssq2`, processes a stream of jobs arriving, at random, with a steady-state arrival rate of 0.5 jobs per minute. The service requirement associated with each arriving job has two stochastic components:

* the *number* of service tasks is one plus a *Geometric*(0.9) random variate;

* the *time* (in minutes) per task is, independently for each task, a *Uniform*$(0.1, 0.2)$ random variate.

In this case, program `ssq2` would need to be modified by including the function `Geometric` from Definition 3.1.2 and changing the function `GetService` to the following.

```
double GetService(void)
{
   long   k;
   double sum   = 0.0;
   long   tasks = 1 + Geometric(0.9);

   for (k = 0; k < tasks; k++)
     sum += Uniform(0.1, 0.2);
   return (sum);
}
```

With this modification, the population steady-state statistics from the analytic model to $d.dd$ precision that will be produced are:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.77	4.27	1.50	2.89	2.14	0.75

Program `ssq2` will produce results that converge to these values for a very long run. When compared with the steady-state results in Example 3.1.3, note that although the arrival rate $1/\bar{r} = 0.50$, the service rate $1/\bar{s} = 0.67$, and the utilization $\bar{x} = 0.75$ are the same, the other four statistics are significantly larger than in Example 3.1.3. This difference illustrates the sensitivity of the performance measures to the service time distribution. This highlights the importance of using an accurate service time model. See Exercise 3.1.5.

3.1.2 SIMPLE INVENTORY SYSTEM

Example 3.1.5 In a simple inventory system simulation, to generate a random variate sequence of demands d_1, d_2, d_3, ... equally likely to have any integer value between a and b inclusive and with an average of $(a + b)/2$ use

   d_i = Equilikely(a, b) $i = 1, 2, 3, \ldots$

Recognize, however, that the previous discussion about $Uniform(a, b)$ service times as an unrealistic model also applies here in the discrete case. That is, in this application the modeling assumption that all the demands between a and b are equally likely is probably unrealistic; some demands should be more likely than others. As an alternative model we could consider generating the demands as $Geometric(p)$ random variates. In this particular case, however, a $Geometric(p)$ demand model is probably not very realistic either. In Chapter 6 we will consider alternative models that are more appropriate.

Program sis2

   Program sis2 is based on the $Equilikely(a, b)$ stochastic modeling assumption in Example 3.1.5. This program is an extension of program sis1 in that the demands are generated randomly (rather than relying on a trace-driven input). Consequently, in a manner analogous to that illustrated in Example 3.1.3, program sis2 can be used to study the transition-to-steady-state behavior of a simple inventory system.

Example 3.1.6 If the $Equilikely(a, b)$ demand parameters are set to $(a, b) = (10, 50)$ so that the average demand per time interval is $(a + b)/2 = 30$, then with $(s, S) = (20, 80)$ the (approximate) steady-state statistics program sis2 will produce are

$\bar{d}$	$\bar{o}$	$\bar{u}$	$\bar{l}^+$	$\bar{l}^-$
30.00	30.00	0.39	42.86	0.26

As illustrated in Figure 3.1.3 (using the same three rng initial seeds used in Example 3.1.3) for the average inventory level $\bar{l} = \bar{l}^+ - \bar{l}^-$, at least several hundred time intervals must be simulated to approximate these steady-state statistics. To produce Figure 3.1.3, program sis2 was modified to print the accumulated value of $\bar{l}$ every 5 time intervals.
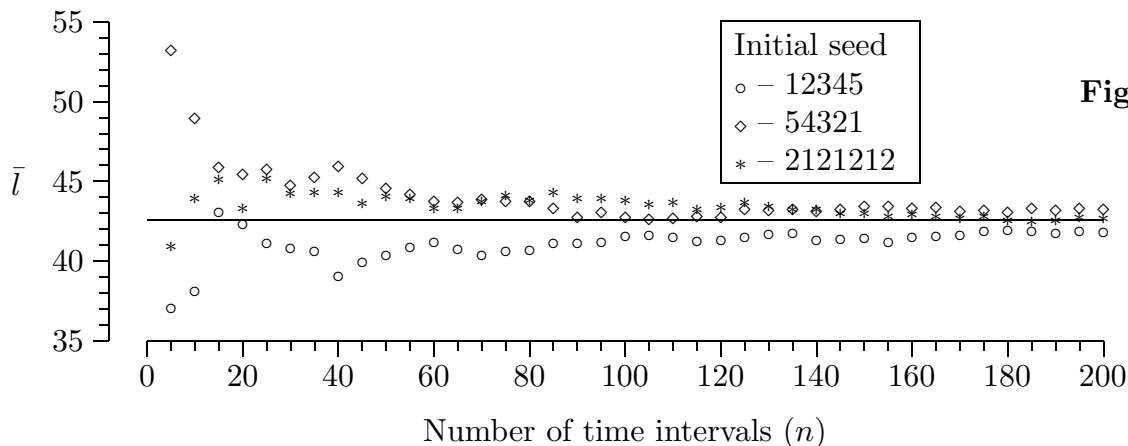
Figure 3.1.3. Average inventory level.

Number of time intervals (n)

Optimal Steady-State System Performance

Example 3.1.7   As an extension of Example 1.3.7, a modified version of program `sis2` was used to simulate $n = 100$ weeks (about two years) of automobile dealership operation with $S = 80$ and values of s varied from 0 to 60. The cost parameters from Example 1.3.5 were used to determine the variable part of the dealership's average weekly cost of operation. That is, as discussed in Section 1.3, the (large) part of the average weekly cost that is proportional to the average weekly order $\bar{o}$, and therefore is independent of (s, S), was ignored so that only the dependent cost was computed. The results are presented with ∘'s representing the $n = 100$ averages. As illustrated in Figure 3.1.4, there is a relatively well-defined minimum with an optimal value of s somewhere between 20 and 30.

Figure 3.1.4. Dependent cost for (s, S) inventory system.
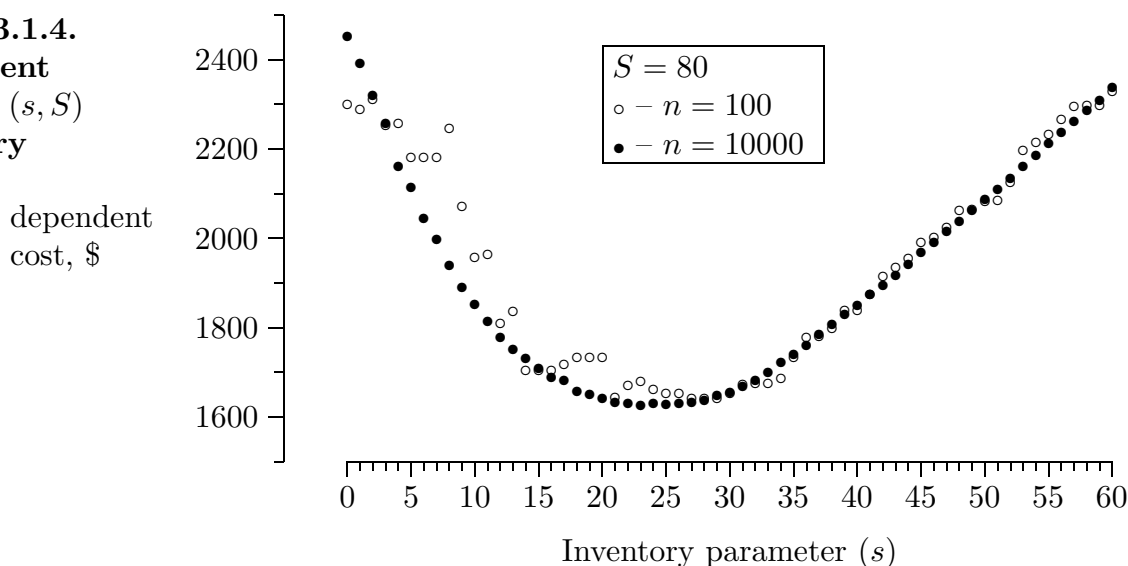
The same initial seed (12345) was used for each value of s. Fixing the initial seed guarantees that *exactly* the same sequence of demands will be processed; therefore, any changes in the resulting system statistics are due to changes in s only. As in Example 3.1.6, the demands were generated as *Equilikely*(10, 50) random variates. To compute steady-state statistics, all the modeling assumptions upon which this simulation is based would have to remain valid for many years. For that reason, steady-state statistics are not very meaningful in this example. Indeed, it is questionable to assume that the demand distribution, cost parameters, and inventory policy will remain constant for even two years. Steady-state performance statistics are fashionable, however, and represent an interesting limiting case. For that reason averages based on $n = 10000$ weeks of operation (approximately 192 years) are presented as •'s for comparison. These estimated steady-state averages have the attractive feature that they vary smoothly with s and, because of that, the optimum steady-state value of s is relatively well defined. In a world that (on average) never changes, if $S = 80$ the auto dealer should use an inventory threshold of $s = 23$. This example illustrates the more general problem of optimizing a function that is not known with certainty known as *stochastic optimization.*

3.1.3 STATISTICAL CONSIDERATIONS

The statistical analysis of simulation-generated data is discussed in Chapters 4 and 8. In anticipation of that discussion, we note that Example 3.1.7 illustrates two important ideas: *variance reduction* and *robust estimation*. We consider variance reduction first, particularly as it relates to transient (small n) output statistics.

Variance Reduction

Because the transient statistical output produced by any discrete-event simulation program depends on the sequence of random variates generated as the program executes, the output will always have some inherent uncertainty. That is, as illustrated in Examples 3.1.3 and 3.1.6, the transient statistical output depends on the value of the `rng` initial seed, particularly if the number of jobs or the number of simulated time intervals is small. Therefore, if a variety of initial seeds are used (a policy we certainly advocate) there will be a natural *variance* in any computed transient statistic.

The statistical tools used to quantify this variance/uncertainty will be developed in Chapter 8. For now, the key point to be made is that using *common random numbers*, as in Example 3.1.7 when $n = 100$, is an intuitive approach to reducing the variance in computed system statistics. This example of variance reduction (using common random numbers) is consistent with the time-honored approach to experimental scientific "what if" studies where, if possible, all variables except one are fixed. Fixing the initial seed at 12345 in Example 3.1.7 isolates the variability of the performance measure (dependent cost). This technique is more generally known as *blocking* in statistics.

Robust Estimation

The optimal (minimal cost) estimated steady-state threshold value $s = 23$ in Example 3.1.7 is a robust estimate because other values of s close to 23 yield essentially the same cost. Therefore, the impact of using one of these alternate values of s in place of $s = 23$ would be slight. In a more general sense, it is desirable for an optimal value to be robust to *all* the assumptions upon which the discrete-event simulation model is based. Relative to Example 3.1.7, this means that we would be interested in determining what happens to the optimal value of s when, for example S is varied about 80, or the average demand (per week) is varied about 30, or there are delivery lags. The estimate $s = 23$ is robust only if it remains close to the minimal cost operating policy level when all these assumptions are altered. In general, robust estimators are insensitive to model assumptions.

3.1.4 EXERCISES

Exercise 3.1.1 (*a*) Modify program `ssq2` to use *Exponential*(1.5) service times. (*b*) Process a relatively large number of jobs, say 100 000, and determine what changes this produces relative to the statistics in Example 3.1.3? (*c*) Explain (or conjecture) why some statistics change and others do not.

Exercise 3.1.2   (a) Relative to the steady-state statistics in Example 3.1.3 and the statistical equations in Section 1.2, list all of the consistency checks that should be applicable. (b) Verify that all of these consistency checks are valid.

Exercise 3.1.3   (a) Given that the Lehmer random number generator used in the library `rng` has a modulus of $2^{31} - 1$, what are the largest and smallest possible numerical values (as a function of μ) that the function `Exponential`(μ) can return? (b) Comment on this relative to the theoretical expectation that an $Exponential(\mu)$ random variate can have an arbitrarily large value and a value arbitrarily close to zero.

Exercise 3.1.4   (a) Conduct a transition-to-steady-state study like that in Example 3.1.3 except for a service time model that is $Uniform(1.3, 2.3)$. Be specific about the number of jobs that seem to be required to produce steady-state statistics. (b) Comment.

Exercise 3.1.5   (a) Verify that the mean service time in Example 3.1.4 is 1.5. (b) Verify that the steady-state statistics in Example 3.1.4 seem to be correct. (c) Note that the arrival rate, service rate, and utilization are the same as those in Example 3.1.3, yet all the other statistics are larger than those in Example 3.1.3. Explain (or conjecture) why this is so. Be specific.

Exercise 3.1.6   (a) Modify program `sis2` to compute data like that in Example 3.1.7. Use the functions `PutSeed` and `GetSeed` from the library `rng` in such a way that *one* initial seed is supplied by the system clock, printed as part of the program's output and used automatically to generate the same demand sequence for all values of s. (b) For $s = 15, 16, \ldots, 35$ create a figure (or table) similar to the one in Example 3.1.7. (c) Comment.

Exercise 3.1.7   (a) Relative to Example 3.1.5, if instead the random variate sequence of demands are generated as

$$d_i \texttt{ = Equilikely(5, 25) + Equilikely(5, 25)} \qquad i = 1, 2, 3, \ldots$$

then, when compared with those in Example 3.1.6, demonstrate that some of the steady-state statistics will be the same and others will not. (b) Explain why this is so.

Exercise 3.1.8[a]   Modify program `sis2` to simulate the operation of a simple inventory system *with a delivery lag*. (a) Specifically, assume that if an order is placed at time $t = i - 1$ then the order will arrive at the later time $t = i - 1 + \delta_i$ where the delivery lag δ_i is a $Uniform(0, 1)$ random variate, independent of the size of the order. (b) What are the equations for $\bar{l}_i^+$ and $\bar{l}_i^-$? (c) Using the same parameter values as in Example 3.1.7, determine that value of s for which the average dependent cost is least. Compare this result with that obtained in Example 3.1.7. (d) It is important to have the *same* sequence of demands for all values of s, with and without a lag. Why? How did you accomplish this? (e) Discuss what you did to convince yourself that the modification is correct. (f) For both $n = 100$ and $n = 10000$ produce a table of dependent costs corresponding to $s = 10, 20, 30, 40, 50, 60$.

A typical discrete-event simulation model will have many stochastic components. When this model is implemented at the computational level, the statistical analysis of system performance is often facilitated by having a unique source of randomness for each stochastic component. Although it may seem that the best way to meet this need for multiple sources of randomness is to create multiple random number generators, there is a simpler and better approach — use *one* random number generator to generate multiple "streams" of random numbers using multiple initial seeds as entry points, one for each stochastic system component. Consistent with this approach, in this section we extend the Lehmer random number generation algorithm from Chapter 2 by adding the ability to partition the generator's output sequence into multiple subsequences (streams).

3.2.1   STREAMS

The library `rng` provides a way to partition the random number generator's output into multiple streams by establishing multiple states for the generator, one for each stream. As illustrated by the following example, the function `PutSeed` can be used to set the state of the generator with the current state of the stream before generating a random variate appropriate to the corresponding stochastic component and the function `GetSeed` can be used to retrieve the revised state of the stream after the random variate has been generated.

Example 3.2.1   The program `ssq2` has two stochastic components, the arrival process and the service process, represented by the functions `GetArrival` and `GetService` respectively. To create a different stream of random numbers for each component, it is sufficient to allocate a different Lehmer generator state variable to each function. This is illustrated by modifying `GetService` from its original form in `ssq2`, which is

```
double GetService(void)                              /* original form */
{
   return (Uniform(1.0, 2.0));
}
```

to the multi-stream form indicated which uses the static variable `x` to represent the current state of the service process stream, initialized to 123456789.

```
double GetService(void)                           /* multi-stream form */
{
   double s;
   static long x = 123456789;      /* use your favorite initial seed */
   PutSeed(x);                      /* set the state of the generator */
   s = Uniform(1.0, 2.0);
   GetSeed(&x);                        /* save the new generator state */
   return (s);
}
```

Example 3.2.2   As in the previous example, the function `GetArrival` should be modified similarly, with a corresponding static variable to represent the current state of the arrival process stream, but initialized to a *different* value. That is, the original form of `GetArrival` in program `ssq2`

```
double GetArrival(void)                                /* original form */
{
   static double arrival = START;
   arrival += Exponential(2.0);
   return (arrival);
}
```

should be modified to something like

```
double GetArrival(void)                                /* multi-stream form */
{
   static double arrival = START;
   static long x = 987654321;    /* use an appropriate initial seed */
   PutSeed(x);                       /* set the state of the generator */
   arrival += Exponential(2.0);
   GetSeed(&x);                          /* save the new generator state */
   return (arrival);
}
```

As in Example 3.2.1, in the multi-stream form the static variable x represents the current state of the arrival process stream, initialized in this case to 987654321. Note that there is nothing magic about this initial state (relative to 123456789) and, indeed, it may not even be a particularly good choice — more about that point later in this section.

  If `GetService` and `GetArrival` are modified as in Examples 3.2.1 and 3.2.2, then the arrival times will be drawn from one stream of random numbers and the service times will be drawn from another stream. Provided the two streams don't overlap, in this way the arrival process and service process will be uncoupled.* As the following example illustrates, the cost of this uncoupling in terms of execution time is modest.

Example 3.2.3   The parameter `LAST` in program `ssq2` was changed to process 1 000 000 jobs and the execution time to process this many jobs was recorded. (A large number of jobs was used to get an accurate time comparison.) Program `ssq2` was then modified as in Examples 3.2.1 and 3.2.2 and used to process 1 000 000 jobs, with the execution time recorded. The increase in execution time was 20%.

---

  * Also, because the scope of the two stream state variables (both are called x) is local to their corresponding functions, the use of `PutSeed` in program `ssq2` to initialize the generator can, and should, be eliminated from `main`.

Jump Multipliers

As illustrated in the previous examples, the library `rng` can be used to support the allocation of a unique stream of random numbers to each stochastic component in a discrete-event simulation program. There is, however, a potential problem with this approach — the assignment of initial seeds. That is, each stream requires a unique initial state that should be chosen to produce *disjoint* streams. But, if multiple initial states are picked at whim there is no convenient way to guarantee that the streams are disjoint; some of the initial states may be just a few calls to `Random` away from one another. With this limitation of the library `rng` in mind, we now turn to the issue of constructing a random number generation library called `rngs` which is a multi-stream version of the library `rng`. We begin by recalling two key points from Section 2.1.

- A Lehmer random number generator is defined by the function

$$g(x) = ax \bmod m,$$

  where the modulus m is a large prime integer, the full-period multiplier a is modulus compatible with m, and $x \in \mathcal{X}_m = \{1, 2, \ldots, m - 1\}$.

- If $x_0, x_1, x_2, \ldots$ is an infinite sequence in $\mathcal{X}_m$ generated by $g(x) = ax \bmod m$ then each x_i is related to x_0 by the equation

$$x_i = a^i x_0 \bmod m \qquad i = 1, 2, \ldots$$

The following theorem is the key to creating the library `rngs`. The proof is left as an exercise.

Theorem 3.2.1   Given a Lehmer random number generator defined by $g(x) = ax \bmod m$ and any integer j with $1 < j < m - 1$, the associated *jump function* is

$$g^j(x) = (a^j \bmod m) x \bmod m$$

with the *jump multiplier* $a^j \bmod m$. For any $x_0 \in \mathcal{X}_m$, if the function $g(\cdot)$ generates the sequence $x_0, x_1, x_2, \ldots$ then the jump function $g^j(\cdot)$ generates the sequence $x_0, x_j, x_{2j}, \ldots$

Example 3.2.4   If $m = 31$, $a = 3$, and $j = 6$ then the jump multiplier is

$$a^j \bmod m = 3^6 \bmod 31 = 16.$$

Starting with $x_0 = 1$ the function $g(x) = 3x \bmod 31$ generates the sequence

$$\underline{1}, 3, 9, 27, 19, 26, \underline{16}, 17, 20, 29, 25, 13, \underline{8}, 24, 10, 30, 28, 22, \underline{4}, 12, 5, 15, 14, 11, \underline{2}, 6, \ldots$$

while the jump function $g^6(x) = 16x \bmod 31$ generates the sequence of underlined terms

$$1, 16, 8, 4, 2, \ldots$$

That is, the first sequence is x_0, x_1, x_2, ... and the second sequence is x_0, x_6, x_{12}, ...

The previous example illustrates that once the jump multiplier $a^j \bmod m$ has been computed — this is a one-time cost — then the jump function $g^j(\cdot)$ provides a mechanism to jump from x_0 to x_j to x_{2j}, etc. If j is properly chosen then the jump function can be used in conjunction with a user supplied initial seed to "plant" additional initial seeds, each separated one from the next by j calls to `Random`. In this way disjoint streams can be automatically created with the initial state of each stream dictated by the choice of just *one* initial state.

Example 3.2.5   There are approximately 2^{31} possible values in the full period of our standard $(a, m) = (48271, 2^{31} - 1)$ Lehmer random number generator. Therefore, if we wish to maintain $256 = 2^8$ streams of random numbers (the choice of 256 is largely arbitrary) it is natural to partition the periodic sequence of possible values into 256 disjoint subsequences, each of equal length. This is accomplished by finding the largest value of j less than $2^{31}/2^8 = 2^{23} = 8388608$ such that the associated jump multiplier $48271^j \bmod m$ is modulus-compatible with m. Because this jump multiplier is modulus-compatible, the jump function

$$g^j(x) = (48271^j \bmod m)\, x \bmod m$$

can be implemented using Algorithm 2.2.1. This jump function can then be used in conjunction with one user supplied initial seed to efficiently plant the other 255 additional initial seeds, each separated one from the next by $j \cong 2^{23}$ steps.* By planting the additional seeds this way, the possibility of stream overlap is minimized.

Maximal Modulus-Compatible Jump Multipliers

Definition 3.2.1   Given a Lehmer random number generator with (prime) modulus m, full-period modulus-compatible multiplier a, and a requirement for s disjoint streams as widely separated as possible, the *maximal* jump multiplier is $a^j \bmod m$ where j is the largest integer less than $\lfloor m/s \rfloor$ such that $a^j \bmod m$ is modulus compatible with m.

Example 3.2.6   Consistent with Definition 3.2.1 and with $(a, m) = (48271, 2^{31} - 1)$ a table of maximal modulus-compatible jump multipliers can be constructed for 1024, 512, 256, and 128 streams, as illustrated.

# of streams s	$\lfloor m/s \rfloor$	jump size j	jump multiplier $a^j \bmod m$
1024	2097151	2082675	97070
512	4194303	4170283	44857
256	8388607	8367782	22925
128	16777215	16775552	40509

Computation of the corresponding table for $a = 16807$ (the minimal standard multiplier) is left as an exercise.

---

* Because j is less than $2^{31}/2^8$, the last planted initial seed will be more than j steps from the first.

Library rngs

The library `rngs` is an upward-compatible multi-stream replacement for the library `rng`. The library `rngs` can be used as an alternative to `rng` in any of the programs presented earlier by replacing

    #include "rng.h"

with

    #include "rngs.h"

As configured `rngs` provides for 256 streams, indexed from 0 to 255, with 0 as the default stream. Although the library is designed so that all streams will be initialized to default values if necessary, the recommended way to initialize all streams is by using the function `PlantSeeds`. Only one stream is *active* at any time; the other 255 are *passive*. The function `SelectStream` is used to define the active stream. If the default stream is used exclusively, so that 0 is *always* the active stream, then the library `rngs` is functionally equivalent to the library `rng` in the sense that `rngs` will produce *exactly* the same `Random` output as `rng` (for the same initial seed, of course).

The library `rngs` provides six functions, the first four of which correspond to analogous functions in the library `rng`.

- `double Random(void)` — This is the Lehmer random number generator used throughout this book.

- `void PutSeed(long x)` — This function can be used to set the state of the active stream.

- `void GetSeed(long *x)` — This function can be used to get the state of the active stream.

- `void TestRandom(void)` — This function can be used to test for a correct implementation of the library.

- `void SelectStream(int s)` — This function can be used to define the active stream, i.e., the stream from which the next random number will come. The active stream will remain as the source of future random numbers until another active stream is selected by calling `SelectStream` with a different stream index `s`.

- `void PlantSeeds(long x)` — This function can be used to set the state of all the streams by "planting" a sequence of states (seeds), one per stream, with all states dictated by the state of the default stream. The following convention is used to set the state of the default stream:

    if `x` is positive then `x` is the state;

    if `x` is negative then the state is obtained from the system clock;

    if `x` is 0 then the state is to be supplied interactively.

3.2.2 EXAMPLES

The following examples illustrate how to use the library **rngs** to allocate a separate stream of random numbers to each stochastic component of a discrete-event simulation model. We will see additional illustrations of how to use **rngs** in this and later chapters. From this point on **rngs** will be the basic random number generation library used for *all* the discrete-event simulation programs in this book.

Example 3.2.7 As a superior alternative to the multi-stream generator approach in Examples 3.2.1 and 3.2.2, the functions **GetArrival** and **GetService** in program **ssq2** can be modified to use the library **rngs**, as illustrated

```
double GetArrival(void)
{
   static double arrival = START;
   SelectStream(0);                              /* this line is new */
   arrival += Exponential(2.0);
   return (arrival);
}

double GetService(void)
{
   SelectStream(2);                              /* this line is new */
   return (Uniform(1.0, 2.0));
}
```

The other modification is to include **"rngs.h"** in place of **"rng.h"** and use the function **PlantSeeds(123456789)** in place of **PutSeed(123456789)** to initialize the streams.*

If program **ssq2** is modified consistent with Example 3.2.7, then the arrival process will be *uncoupled* from the service process. That is important because we may want to study what happens to system performance if, for example, the **return** in the function **GetService** is replaced with

```
return (Uniform(0.0, 1.5) + Uniform(0.0, 1.5));
```

Although two calls to **Random** are now required to generate each service time, this new service process "sees" *exactly* the same job arrival sequence as did the old service process. This kind of uncoupling provides a desirable variance reduction technique when discrete-event simulation is used to compare the performance of different systems.

---

* Note that there is nothing magic about the use of **rngs** stream 0 for the arrival process and stream 2 for the service process — any two different streams can be used. In particular, if even more separation between streams is required then, for example, streams 0 and 10 can be used.

A Single-Server Service Node With Multiple Job Types

A meaningful extension to the single-server service node model is *multiple job types*, each with its own arrival and service process. This model extension is easily accommodated at the conceptual level; each arriving job carries a job type that determines the kind of service provided when the job enters service. Similarly, provided the queue discipline is FIFO the model extension is straightforward at the specification level. Therefore, using program `ssq2` as a starting point, we can focus on the model extension at the implementation level. Moreover, we recognize that to facilitate the use of common random numbers, the library `rngs` can be used with a different stream allocated to each of the stochastic arrival and service processes in the model. The following example is an illustration.

Example 3.2.8 Suppose that there are two job types arriving independently, one with *Exponential*(4.0) interarrivals and *Uniform*(1.0, 3.0) service times and the other with *Exponential*(6.0) interarrivals and *Uniform*(0.0, 4.0) service times. In this case, the arrival process generator in program `ssq2` can be modified as

```
double GetArrival(int *j)                          /* j denotes job type */
{
  const  double mean[2]    = {4.0, 6.0};
  static double arrival[2] = {START, START};
  static int    init       = 1;
        double temp;
  if (init) {                          /* initialize the arrival array */
    SelectStream(0);
    arrival[0] += Exponential(mean[0]);
    SelectStream(1);
    arrival[1] += Exponential(mean[1]);
    init       = 0;
  }
  if (arrival[0] <= arrival[1])
    *j = 0;                                    /* next arrival is job type 0 */
  else
    *j = 1;                                    /* next arrival is job type 1 */
  temp = arrival[*j];                                  /* next arrival time */
  SelectStream(*j);                            /* use stream j for job type j */
  arrival[*j] += Exponential(mean[*j]);
  return (temp);
}
```

Note that `GetArrival` returns the next arrival time *and* the job type as an index with value 0 or 1, as appropriate.

Example 3.2.9   As a continuation of Example 3.2.8, the corresponding service process generator in program `ssq2` can be modified as

```
double GetService(int j)
{
   const double min[2] = {1.0, 0.0};
   const double max[2] = {3.0, 4.0};
   SelectStream(j + 2);              /* use stream j + 2 for job type j */
   return (Uniform(min[j], max[j]));
}
```

Relative to Example 3.2.9, note that the job type index `j` is used in `GetService` to insure that the service time corresponds to the appropriate job type. Also, `rngs` streams 2 and 3 are allocated to job types 0 and 1 respectively. In this way all four simulated stochastic processes are uncoupled. Thus, the random variate model corresponding to any one of these four processes could be changed without altering the generated sequence of random variates corresponding to the other three processes.

Consistency Checks

Beyond the modifications in Examples 3.2.8 and 3.2.9, some job-type-specific statistics gathering needs to be added in `main` to complete the modification of program `ssq2` to accommodate multiple job types. If these modifications are made correctly, with $d.dd$ precision the steady-state statistics that will be produced are

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.40	7.92	5.92	2.00	3.30	2.47	0.83

The details are left in Exercise 3.2.4. How do we know these values are correct?

In addition to $\bar{w} = \bar{d} + \bar{s}$ and $\bar{l} = \bar{q} + \bar{x}$, the three following intuitive consistency checks give us increased confidence in these (estimated) steady-state results:

- Both job types have an average service time of 2.0, so that $\bar{s}$ should be 2.00. The corresponding service rate is 0.5.

- The arrival rate of job types 0 and 1 are 1/4 and 1/6 respectively. Intuitively, the net arrival rate should then be $1/4 + 1/6 = 5/12$ which corresponds to $\bar{r} = 12/5 = 2.40$.

- The steady-state utilization should be the ratio of the arrival rate to the service rate, which is $(5/12)/(1/2) = 5/6 \cong 0.83$.

3.2.3   EXERCISES

Exercise 3.2.1   (*a*) Construct the $a = 16807$ version of the table in Example 3.2.6. (*b*) What is the $O(\cdot)$ time complexity of the algorithm you used?

Exercise 3.2.2[a]   (*a*) Prove that if m is prime, $1 \leq a \leq m - 1$, and $a^* = a^{m-2} \bmod m$ then

$$a^* a \bmod m = 1.$$

Now define

$$g(x) = ax \bmod m \qquad \text{and} \qquad g^*(x) = a^* x \bmod m$$

for all $x \in \mathcal{X}_m = \{1, 2, \dots, m - 1\}$. (*b*) Prove that the functions $g(\cdot)$ and $g^*(\cdot)$ generate the same sequence of states, except in *opposite* orders. (*c*) Comment on the implication of this relative to full period multipliers. (*d*) If $m = 2^{31} - 1$ and $a = 48271$ what is a^*?

Exercise 3.2.3   Modify program `ssq2` as suggested in Example 3.2.7 to create two programs that differ only in the function `GetService`. For one of these programs, use the function as implemented in Example 3.2.7; for the other program, use

```
double GetService(void)
{
   SelectStream(2);                              /* this line is new */
   return (Uniform(0.0, 1.5) + Uniform(0.0, 1.5));
}
```

(*a*) For both programs verify that *exactly* the same average interarrival time is produced (print the average with *d.dddddd* precision). Note that the average service time is approximately the same in both cases, as is the utilization, yet the service nodes statistics $\bar{w}$, $\bar{d}$, $\bar{l}$, and $\bar{q}$ are different. (*b*) Why?

Exercise 3.2.4   Modify program `ssq2` as suggested in Examples 3.2.8 and 3.2.9. (*a*) What proportion of processed jobs are type 0? (*b*) What are $\bar{w}$, $\bar{d}$, $\bar{s}$, $\bar{l}$, $\bar{q}$, and $\bar{x}$ for each job type? (*c*) What did you do to convince yourself that your results are valid? (*d*) Why are $\bar{w}$, $\bar{d}$, and $\bar{s}$ the same for both job types but $\bar{l}$, $\bar{q}$, and $\bar{x}$ are different?

Exercise 3.2.5   Prove Theorem 3.2.1.

Exercise 3.2.6   Same as Exercise 3.2.3, but using the `GetService` function in Example 3.1.4 instead of the `GetService` function in Exercise 3.2.3.

Exercise 3.2.7   Suppose there are three job types arriving independently to a single-server service node. The interarrival times and service times have the following characterization

job type	interarrival times	service times
0	$Exponential(4.0)$	$Uniform(0.0, 2.0)$
1	$Exponential(6.0)$	$Uniform(1.0, 2.0)$
2	$Exponential(8.0)$	$Uniform(1.0, 5.0)$

(*a*) What is the proportion of processed jobs for each type? (*b*) What are $\bar{w}$, $\bar{d}$, $\bar{s}$, $\bar{l}$, $\bar{q}$, and $\bar{x}$ for each job type? (*c*) What did you do to convince yourself that your results are valid? (Simulate at least 100 000 processed jobs.)

In this section we will consider three discrete-event system models, each of which is an extension of a model considered previously. The three models are (*i*) a single-server service node *with immediate feedback*, (*ii*) a simple inventory system *with delivery lag*, and (*iii*) a single-server machine shop.

3.3.1   SINGLE-SERVER SERVICE NODE WITH IMMEDIATE FEEDBACK

We begin by considering an important extension of the single-server service node model first introduced in Section 1.2. Consistent with the following definition (based on Definition 1.2.1) the extension is *immediate feedback* — the possibility that the service a job just received was incomplete or otherwise unsatisfactory and, if so, the job feeds back to once again request service.

Definition 3.3.1   A single-server service node *with immediate feedback* consists of a server plus its queue with a feedback mechanism, as illustrated in Figure 3.3.1.

**Figure 3.3.1.
Single-server
service node
with immediate
feedback.**
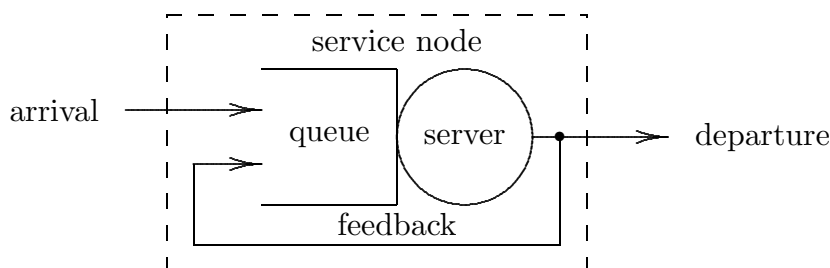
Jobs arrive at the service node, generally at random, seeking service. When service is provided, the time involved is generally also random. At the completion of service, jobs either depart the service node (forever) or immediately feed back and once again seek service. The service node operates as follows: as each job arrives, if the server is busy then the job enters the queue, else the job immediately enters service; as each job completes service, either a departure or a feedback occurs, generally at random. When a *departure* occurs, if the queue is empty then the server becomes idle, else another job is selected from the queue to immediately enter service. When a *feedback* occurs, if the queue is empty then the job immediately re-enters service, else the job enters the queue, after which one of the jobs in the queue is selected to immediately enter service. At any instant in time, the state of the server will either be busy or idle and the state of the queue will be either empty or not empty. If the server is idle then the queue must be empty; if the queue is not empty then the server must be busy.

Note the distinction between the two events "completion of service" and "departure". If there is no feedback these two events are equivalent; if feedback is possible then it is important to make a distinction. When the distinction is important, the completion-of-service event is more fundamental because at the completion of service, either a departure event or a feedback event then occurs. This kind of "which event comes first" causal reasoning is important at the conceptual model-building level.

Model Considerations

When feedback occurs we assume that the job joins the queue (if any) consistent with the queue discipline. For example, if the queue discipline is FIFO then a fed-back job would receive no priority; it would join the queue at the end, in effect becoming indistinguishable from an arriving job. Of course, other feedback queue disciplines are possible, the most common of which involves assigning a priority to jobs that are fed back. If feedback is possible the default assumption in this book is that a fed-back job will join the queue consistent with the queue discipline and a new service time will be required, independent of any prior service provided. Similarly, the default assumption is that the decision to depart or feed back is random with *feedback probability* β, as illustrated in Figure 3.3.2.
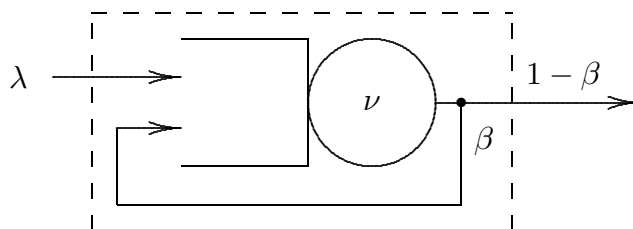
Figure 3.3.2. Single-server service node with feedback probability β.

In addition to β, the other two parameters that characterize the stochastic behavior of a single-server service node with immediate feedback are the *arrival rate* λ and the *service rate* ν. Consistent with Definition 1.2.5, $1/\lambda$ is the average interarrival time and $1/\nu$ is the average service time.*

As each job completes service, it departs the service node with probability $1 - \beta$ or feeds back with probability β. Consistent with this model, feedback is independent of past history and so a job may feed back more than once. Indeed, in theory, a job may feed back arbitrarily many times — see Exercise 3.3.1. Typically β is close to 0.0 indicating that feedback is a rare event. This is not a universal assumption, however, and so a well written discrete-event simulation program should accommodate any probability of feedback in the range $0.0 \leq \beta < 1.0$. At the computational model-building level, feedback can be modeled with a boolean-valued function as illustrated.

```
int GetFeedback(double beta)                    /* use 0.0 <= beta < 1.0 */
{
   SelectStream(2);                  /* use rngs stream 2 for feedback */
   if (Random() < beta)
     return (1);                                   /* feedback occurs */
   else
     return (0);                                   /* no feedback     */
}
```

---

* The use of the symbol ν to denote the service rate is non-standard. Instead, the usual convention is to use the symbol μ. See Section 8.5 for more discussion of arrival rates, service rates, and our justification for the use of ν in place of μ.

Statistical Considerations

If properly interpreted, the mathematical variables and associated definitions in Section 1.2 remain valid if immediate feedback is possible. The interpretation required is that the index $i = 1, 2, 3, \ldots$ counts jobs that enter the service node; once indexed in this way, a fed-back job is not counted again. Because of this indexing all the job-averaged statistics defined in Section 1.2 remain valid *provided* delay times, wait times, and service times are incremented each time a job is fed back. For example, the average wait is the sum of the waits experienced by all the jobs that enter the service node divided by the number of such jobs; each time a job is fed back it contributes an additional wait to the sum of waits, but it does *not* cause the number of jobs to be increased. Similarly, the time-averaged statistics defined in Section 1.2 also remain valid if feedback is possible.

The key feature of immediate feedback is that jobs from outside the system are merged with jobs from the feedback process. In this way, the (steady-state) request-for-service rate is larger than λ by the positive additive factor $\beta \bar{x} \nu$. As illustrated later in Example 3.3.2, if there is no corresponding increase in service rate this increase in the request-for-service rate will cause job-averaged and time-averaged statistics to increase from their non-feedback values. This increase is intuitive — if you are entering a grocery store and the check-out queues are already long, you certainly do not want to see customers re-entering these queues because they just realized they were short-changed at check-out or forgot to buy a gallon of milk.

Note that indexing by arriving jobs will cause the average service time $\bar{s}$ to increase as the feedback probability increases. In this case do not confuse $\bar{s}$ with the reciprocal of the service rate; $1/\nu$ is the (theoretical) average service time *per service request*, irrespective of whether that request is by an arriving job or by a fed back job.

Algorithm and Data Structure Considerations

Example 3.3.1   Consider the following arrival times, service times, and completion times for the first 9 jobs entering a single-server FIFO service node with immediate feedback. (For simplicity all times are integers.)

job index	:	1	2	3	4	5	·	6	·	7	8	·	9	$\cdots$
arrival/feedback	:	1	3	4	7	10	**13**	14	**15**	19	24	**26**	30	$\cdots$
service	:	9	3	2	4	7	5	6	3	4	6	3	7	$\cdots$
completion	:	10	**13**	**15**	19	**26**	31	37	**40**	44	50	53	**60**	$\cdots$

The bold-face times correspond to jobs that were fed back. For example, the third job completed service at time 15 and immediately fed back. At the computational level, note that some algorithm and data structure is necessary to insert fed back jobs into the arrival stream. That is, an inspection of the yet-to-be-inserted feedback times $(15, 26)$ reveals that the job fed back at time 15 must be inserted in the arrival stream after job 6 (which arrived at time 14) and before job 7 (which arrived at time 19).

The reader is encouraged to extend the specification model of a single-server service node in Section 1.2 to account for immediate feedback. Then, extend this model at the computational level by starting with program `ssq2` and using a different `rngs` stream for each stochastic process. Example 3.3.1 provides insight into an algorithmic extension and associated data structure that can be used to accomplish this.

Example 3.3.2 Program `ssq2` was modified to account for immediate feedback. Consistent with the stochastic modeling assumptions in Example 3.1.3, the arrival process has *Exponential*(2.0) random variate interarrivals corresponding to a fixed arrival rate of $\lambda = 0.50$, the service process has *Uniform*(1.0, 2.0) random variate service times corresponding to a fixed service rate of $\nu \cong 0.67$ and the feedback probability is $0.0 \leq \beta \leq 1.0$. To illustrate the effect of feedback, the modified program was used to simulate the operation of a single-server service node with nine different values of levels of feedback varied from $\beta = 0.0$ (no feedback) to $\beta = 0.20$. In each case 100 000 arrivals were simulated. Utilization $\bar{x}$ as a function of β is illustrated on the left-hand-side of Figure 3.3.3; the average number in the queue $\bar{q}$ as a function of β is illustrated on the right-hand-side.

Figure 3.3.3. Utilization and average number in queue as a function of β.
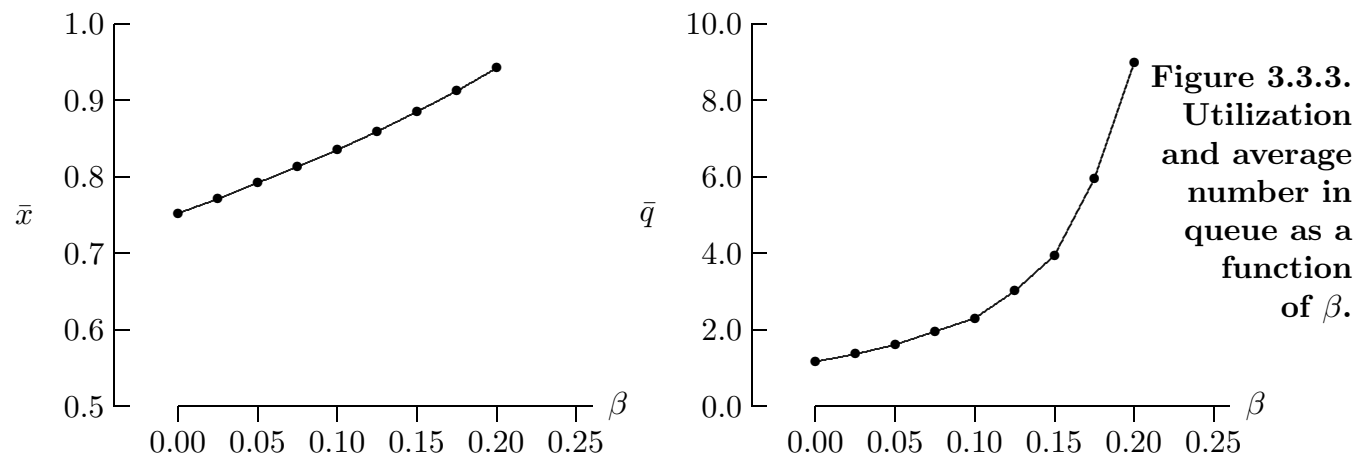
As the probability of feedback increases, the utilization increases from the steady-state value of $\bar{x} = 0.75$ when there is no feedback toward the maximum possible value $\bar{x} = 1.0$. If this $\bar{x}$ versus β figure is extrapolated, it appears that saturation ($\bar{x} = 1.0$) is achieved as $\beta \to 0.25$.

Flow Balance and Saturation

The observation that saturation occurs as β approaches 0.25 is an important consistency check based on steady-state *flow balance* considerations. That is, jobs flow *into* the service node at the average rate of λ. To remain flow balanced jobs must flow *out* of the service node at the same average rate. Because the average rate at which jobs flow out of the service node is $\bar{x}(1 - \beta)\nu$, flow balance is achieved when $\lambda = \bar{x}(1 - \beta)\nu$. Saturation is achieved when $\bar{x} = 1$; this happens as $\beta \to 1 - \lambda/\nu = 0.25$. Consistent with saturation, in Example 3.3.2 we see that the average number in the queue increases dramatically as β increases, becoming effectively infinite as $\beta \to 0.25$.

3.3.2   SIMPLE INVENTORY SYSTEM WITH DELIVERY LAG

The second discrete-event system model we will consider in this section represents an important extension of the periodic review simple inventory system model first introduced in Section 1.3. The extension is *delivery lag* (or *lead time*) — an inventory replacement order placed with the supplier will not be delivered immediately; instead, there will be a lag between the time an order is placed and the time the order is delivered. Unless stated otherwise, this lag is assumed to be random and independent of the amount ordered.

If there are no delivery lags then a typical inventory time history looks like the one in Section 1.3, reproduced in Figure 3.3.4 for convenience, with jump discontinuities possible only at the (integer-valued) times of inventory review.

Figure 3.3.4.
Inventory
levels with
no delivery
lags.
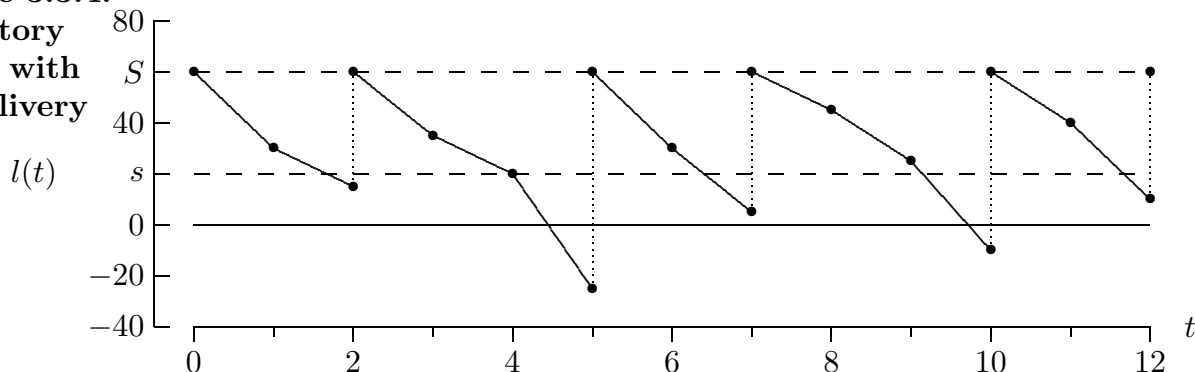
If delivery lags are possible, a typical inventory time history would have jump discontinuities at arbitrary times, as illustrated in Figure 3.3.5. (The special-case order at the terminal time $t = 12$ is assumed to be delivered with zero lag.)

Figure 3.3.5.
Inventory
levels with
delivery
lags.
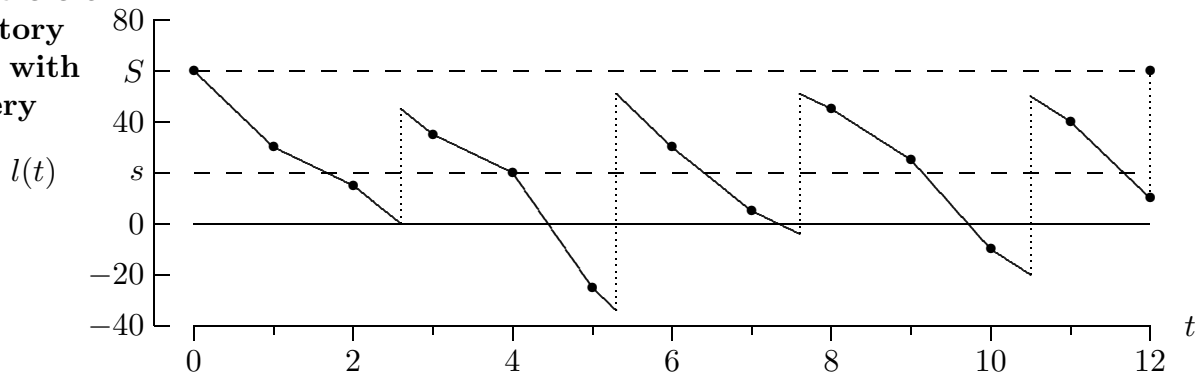
Unless stated otherwise, we assume that any order placed at the beginning of a time interval (at times $t = 2, 5, 7$, and 10 in this case) will be delivered before the end of the time interval. With this assumption there is *no* change in the simple inventory system model at the specification level (see Algorithm 1.3.1). There is, however, a significant change in how the system statistics are computed. For those time intervals in which a delivery lag occurs, the time-averaged holding and shortage integrals in Section 1.3 must be modified.

Statistical Considerations

As in Section 1.3, l_{i-1} denotes the inventory level at the beginning of the ith time interval (at $t = i-1$) and d_i denotes the amount of demand during this interval. Consistent with the model in Section 1.3, the demand rate is assumed to be constant between review times. Given d_i, l_{i-1}, and this assumption, there are two cases to consider.

If $l_{i-1} \geq s$ then, because no order is placed at $t = i - 1$, the inventory decreases at a constant rate throughout the interval with no "jump" in level. The inventory level at the end of the interval is $l_{i-1} - d_i$. In this case, the equations for $\bar{l}_i^+$ and $\bar{l}_i^-$ in Section 1.3 remain valid (with l_{i-1} in place of l'_{i-1}.)

If $l_{i-1} < s$ then an order is placed at $t = i - 1$ which later causes a jump in the inventory level when the order is delivered. In this case the equations for $\bar{l}_i^+$ and $\bar{l}_i^-$ in Section 1.3 must be modified. That is, if $l_{i-1} < s$ then an order for $S - l_{i-1}$ items is placed at $t = i - 1$ and a *delivery lag* $0 < \delta_i < 1$ occurs during which time the inventory level drops at a constant rate to $l_{i-1} - \delta_i d_i$. When the order is delivered at $t = i - 1 + \delta_i$ the inventory level jumps to $S - \delta_i d_i$. During the remainder of the interval the inventory level drops at the same constant rate to its final level $S - d_i$ at $t = i$. All of this is summarized with the observation that, in this case, the inventory-level time history during the ith time interval is defined by one vertical and two parallel lines, as illustrated in Figure 3.3.6.*
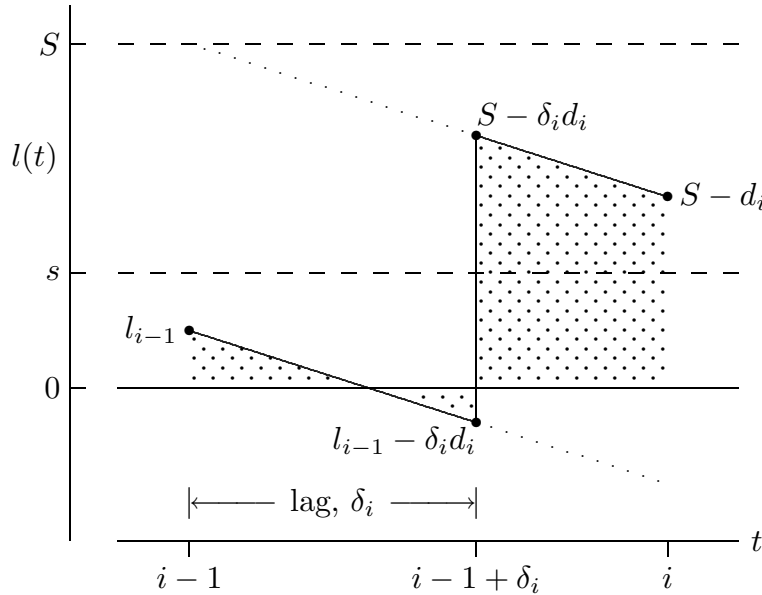
Figure 3.3.6. Inventory level during time interval i when an order is placed.

Depending on the location of the four line-segment endpoints indicated by •'s, with each location measured relative to the line $l(t) = 0$, either triangular or trapezoidal figures will be generated. To determine the time-averaged holding level $\bar{l}_i^+$ and time-averaged shortage level $\bar{l}_i^-$ (see Definition 1.3.3), it is necessary to determine the area of each figure. The details are left as an exercise.

---

* Note that $\delta_i d_i$ must be integer-valued.

Consistency Checks

When system models are extended, it is fundamentally important to verify that the extended model is consistent with the parent model (the model before extension). This is usually accomplished by setting system parameters to special values. For example, if the feedback probability is set to zero an extended computational model that simulates a single-server service node *with* feedback reduces to a parent computational model of a single-server service node *without* feedback. At the computational level the usual way to make this kind of consistency check is to compare output system statistics and verify that, with the extension removed, the output statistics produced by the extended model agree with the output statistics produced by the parent model. Use of the library `rngs` facilitates this kind of comparison. In addition to these "extension removal" consistency checks, it is also good practice to check for intuitive "small-perturbation" consistency. For example, if the feedback probability is small, but non-zero, the average number in the queue should be slightly larger than its feedback-free value. The following example applies this idea to a simple inventory system model with delivery lag.

Example 3.3.3   For a simple inventory system with delivery lag we adopt the convention that δ_i is defined for all $i = 1, 2, \ldots, n$ with $\delta_i = 0.0$ if and only if there is no order placed at the beginning of the ith time interval (that is, if $l_{i-1} \geq s$). If an order is placed then $0.0 < \delta_i < 1.0$. With this convention the stochastic time evolution of a simple inventory system with delivery lag is driven by the two n-point stochastic sequences $d_1, d_2, \ldots, d_n$ and $\delta_1, \delta_2, \ldots, \delta_n$. The simple inventory system is *lag-free* if and only if $\delta_i = 0.0$ for all $i = 1, 2, \ldots, n$; if $\delta_i > 0.0$ for at least one i then the system is not lag-free. Relative to the five system statistics in Section 1.3, if the inventory parameters (S, s) are fixed then, even if the delivery lags are small, the following points are valid.

- The average order $\bar{o}$, average demand $\bar{d}$, and relative frequency of setups $\bar{u}$ are exactly the same whether the system is lag-free or not.

- Compared to the lag-free value, if the system is not lag-free then the time-averaged holding level $\bar{l}^+$ will decrease.

- Compared to the lag-free value, if the system is not lag-free then the time-averaged shortage level $\bar{l}^-$ will either remain unchanged or it will increase.

At the computational level these three points provide valuable consistency checks for a simple inventory system discrete-event simulation program.

Delivery Lag

If the statistics-gathering logic in program `sis2` is modified to be consistent with the previous discussion, then the resulting program will provide a computational model of a simple inventory system with delivery lag. To complete this modification, a stochastic model of delivery lag is needed. In the absence of information to the contrary, we assume that each delivery lag is an independent *Uniform*$(0, 1)$ random variate.

Example 3.3.4   Program `sis2` was modified to account for $Uniform(0, 1)$ random variate delivery lags, independent of the size of the order. As an extension of the automobile dealership example (Example 3.1.7), this modified program was used to study the effect of delivery lag. That is, with $S = 80$ the average weekly cost was computed for a range of inventory threshold values s between 20 and 60. To avoid clutter only steady-state cost estimates (based on $n = 10\,000$ time intervals) are illustrated. For comparison, the corresponding lag-free cost values from Example 3.1.7 are also illustrated in Figure 3.3.7.
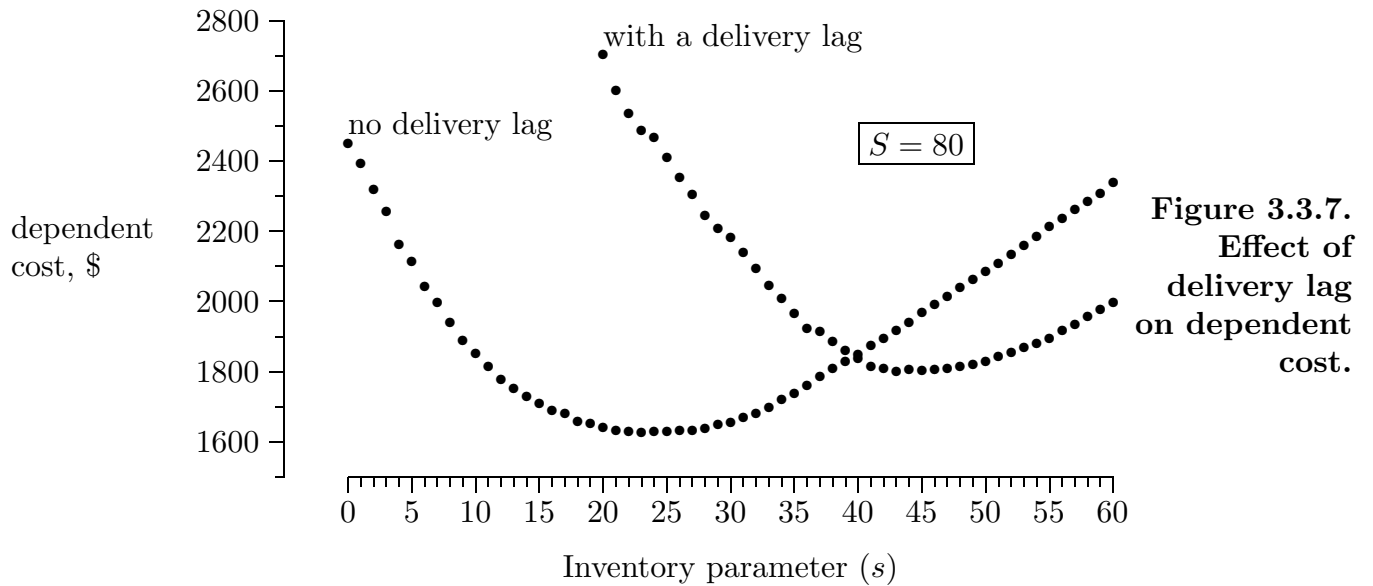
Figure 3.3.7.
Effect of
delivery lag
on dependent
cost.

Figure 3.3.7 shows that the effect of delivery lag is profound; the U-shaped cost-versus-s curve is shifted up and to the right. Because of this shift the optimum (minimum cost) value of s is increased by approximately 20 automobiles and the corresponding minimum weekly cost is increased by almost $200.*

The shift in the U-shaped curve in Example 3.3.4 is consistent with the second and third points in Example 3.3.3. That is, delivery lags cause $\bar{l}^+$ to decrease and $\bar{l}^-$ to increase (or remain the same). Because the holding cost coefficient is $C_{\text{hold}} = \$25$ and the shortage cost coefficient is $C_{\text{short}} = \$700$ (see Example 1.3.5), delivery lags will cause holding costs to decrease a little for all values of s and will cause shortage costs to increase a lot, but only for small values of s. The shift in the U-shaped curve is the result.

Examples 3.3.2 and 3.3.4 present results corresponding to significant extensions of the two canonical system models used throughout this book. The reader is strongly encouraged to work through the details of these extensions at the computational level and reproduce the results in Examples 3.3.2 and 3.3.4.

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* Because of the dramatic shift in the optimum value of s from the lag-free value of $s \cong 23$ to the with-lag value of $s \cong 43$, we see that the optimal value of s is *not* robust with respect to model assumptions about the delivery lag.

3.3.3   SINGLE-SERVER MACHINE SHOP

The third discrete-event simulation model considered in this section is a single-server machine shop. This is a simplified version of the multi-server machine shop model used in Section 1.1 to illustrate model building at the conceptual, specification, and computational level (see Example 1.1.1).

A single-server machine shop model is essentially identical to the single-server service node model first introduced in Section 1.2, except for one important difference. The service node model is *open* in the sense that an effectively infinite number of jobs are available to arrive from "outside" the system and, after service is complete, return to the "outside". In contrast, the machine shop model is *closed* because there are a finite number of machines (jobs) that are part of the system — as illustrated in Figure 3.3.8, there is no "outside".

Figure 3.3.8. Single-server machine shop system diagram.
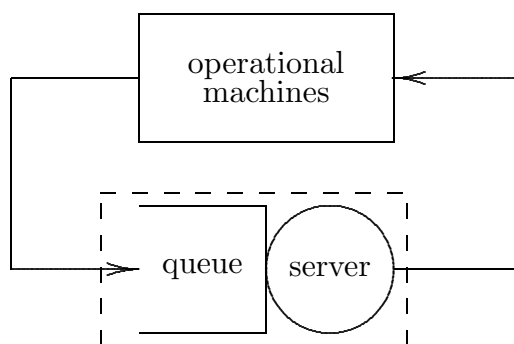
In more detail, there is a finite population of statistically identical machines, all of which are initially in an *operational* state (so the server is initially idle and the queue is empty). Over time these machines fail, independently, at which time they enter a *broken* state and request repair service at the single-server service node.* Once repaired, a machine immediately re-enters an operational state and remains in this state until it fails again. Machines are repaired in the order in which they fail, without interruption. Correspondingly, the queue discipline is FIFO, non-preemptive and conservative. There is no feedback.

To make the single-server machine shop model specific, we assume that the service (repair) time is a *Uniform*(1.0, 2.0) random variate, that there are M machines, and that the time a machine spends in the operational state is an *Exponential*(100.0) random variate. All times are in hours. Based on 100 000 simulated machine failures, we want to estimate the steady-state time-averaged number of operational machines and the server utilization as a function of M.

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* Conceptually the machines move along the network arcs indicated, from the operational pool into and out of service and then back to the operational pool. In practice, the machines are usually stationary and the server moves to the machines. The time, if any, for the server to move from one machine to another is part of the service time.

Program ssms

Program `ssms` simulates the single-server machine shop described in this section. This program is similar to program `ssq2`, but with two important differences.

- The library `rngs` is used to provide an independent source of random numbers to both the simulated machine failure process and the machine repair process.

- The failure process is defined by the array `failure` which represents the time of next failure for each of the M machines.

The time-of-next-failure list (array) is not maintained in sorted order and so it must be searched completely each time a machine failure is simulated. The efficiency of this $O(M)$ search could be a problem for large M. Exercise 3.3.7 investigates computational efficiency improvements associated with an alternative algorithm and associated data structure.

Example 3.3.5   Because the time-averaged number in the service node $\bar{l}$ represents the time-averaged number of broken machines, $M - \bar{l}$ represents the time-averaged number of operational machines. Program `ssms` was used to estimate $M - \bar{l}$ for values of M between 20 and 100, as illustrated in Figure 3.3.9.
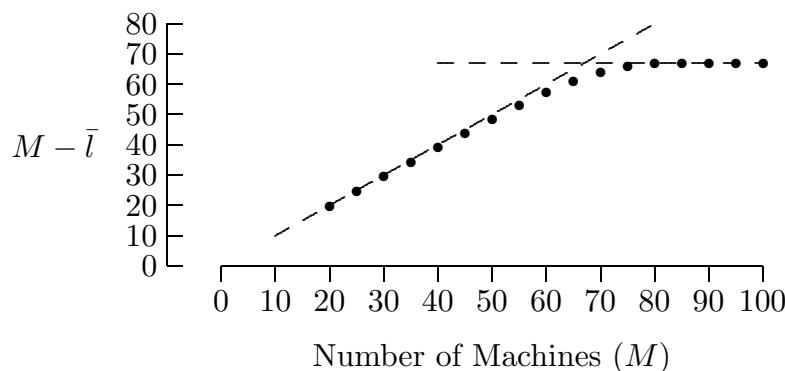
Figure 3.3.9. Time-averaged number of operational machines as a function of the number of machines.

As expected, for small values of M the time-averaged number of operational machines is essentially M. This is consistent with low server utilization and a correspondingly small value of $\bar{l}$. The angled dashed line indicates the ideal situation where all machines are continuously operational (i.e., $\bar{l} = 0$). Also, as expected, for large values of M the time-averaged number of operational machines is essentially constant, independent of M. This is consistent with a saturated server (utilization of 1.0) and a correspondingly large value of $\bar{l}$. The horizontal dashed line indicates that this saturated-server constant value is approximately 67.

Distribution Parameters

The parameters used in the distributions in the models presented in this section, e.g., $\mu = 2.0$ for the average interarrival time and $\beta = 0.20$ for the feedback probability in the single-server service node with feedback, have been drawn from thin air. This has been done in order to focus on the simulation modeling and associated algorithms. Chapter 9 on "Input Modeling" focuses on techniques for estimating realistic parameters from data.

3.3.4   EXERCISES

Exercise 3.3.1   Let β be the probability of feedback and let the integer-valued random variable X be the number of times a job feeds back. (a) For $x = 0, 1, 2, \ldots$ what is $\Pr(X = x)$? (b) How does this relate to the discussion of acceptance/rejection in Section 2.3?

Exercise 3.3.2[a]   (a) Relative to Example 3.3.2, based on $1\,000\,000$ arrivals, generate a table of $\bar{x}$ and $\bar{q}$ values for β from 0.00 to 0.24 in steps of 0.02. (b) What data structure did you use and why? (c) Discuss how external arrivals are merged with fed back jobs.

Exercise 3.3.3   For the model of a single-server service node with feedback presented in this section, there is nothing to prevent a fed-back job from colliding with an arriving job. Is this a model deficiency that needs to be fixed and, if so, how would you do it?

Exercise 3.3.4[a]   Modify program `ssq2` to account for a *finite* service node capacity. (a) For capacities of 1, 2, 3, 4, 5, and 6 construct a table of the estimated steady-state probability of rejection. (b) Also, construct a similar table if the service time distribution is changed to be $Uniform(1.0, 3.0)$. (c) Comment on how the probability of rejection depends on the service process. (d) How did you convince yourself these tables are correct?

Exercise 3.3.5[a]   Verify that the results in Example 3.3.4 are correct. Provide a table of values corresponding to the figure in this example.

Exercise 3.3.6   (a) Relative to Example 3.3.5, construct a figure or table illustrating how $\bar{x}$ (utilization) depends on M. (b) If you extrapolate linearly from small values of M, at what value of M will saturation ($\bar{x} = 1$) occur? (c) Can you provide an empirical argument or equation to justify this value?

Exercise 3.3.7[a]   In program `ssms` the time-of-next-failure list (array) is not maintained in sorted order and so the list must be searched completely each time another machine failure is simulated. As an alternative, implement an algorithm and associated sorted data structure to determine if a significant improvement in computational efficiency can be obtained. (You may need to simulate a huge number of machine failures to get an accurate estimate of computational efficiency improvement.)

Exercise 3.3.8   (a) Relative to Example 3.3.2, compare a FIFO queue discipline with a priority queue discipline where fed-back jobs go the head of the queue (i.e., re-enter service immediately). (b) Is the following conjecture true or false: although statistics for the fed-back jobs change, system statistics do not change?

Exercise 3.3.9[a]   (a) Repeat Exercise 3.3.7 using $M = 120$ machines, with the time a machine spends in the operational state increased to an $Exponential(200.0)$ random variate. (b) Use $M = 180$ machines with the time spent in the operational increased accordingly. (c) What does the $O(\cdot)$ computational complexity of your algorithm seem to be?

CHAPTER 4

STATISTICS

Sections

The first three sections in this chapter consider the computation and interpretation of univariate-data sample statistics. If the size of the sample is small then essentially all that can be done is compute the sample mean and standard deviation. Welford's algorithm for computing these quantities is presented in Section 4.1. In addition, Chebyshev's inequality is derived and used to illustrate how the sample mean and standard deviation are related to the distribution of the data in the sample.

If the size of the sample is not small, then a sample data histogram can be computed and used to analyze the distribution of the data in the sample. Two algorithms are presented in Section 4.2 for computing a discrete-data histogram. A variety of examples are presented as well. Similarly, a continuous-data histogram algorithm and a variety of examples are presented in Section 4.3. In both of these sections concerning histograms, the empirical cumulative distribution function is presented as an alternative way to graphically present a data set. The fourth section deals with the computation and interpretation of paired-data sample statistics, in both paired-correlation and auto-correlation applications. Correlation is apparent in many discrete-event simulations. The wait times of adjacent jobs in a single-server node, for example, tend to be above their means and below their means together, that is, they are "positively" correlated.

4.1 Sample Statistics

Discrete-event simulations generate experimental data — frequently *a lot* of experimental data. To facilitate the analysis of all this data, it is conventional to compress the data into a handful of meaningful statistics. We have already seen examples of this in Sections 1.2 and 3.1, where job averages and time averages were used to characterize the performance of a single-server service node. Similarly, in Sections 1.3 and 3.1, averages were used to characterize the performance of a simple inventory system. This kind of "within-the-run" data generation and statistical computation (see programs `ssq2` and `sis2`, for example) is common in discrete-event simulation. Although less common, there is also a "between-the-runs" kind of statistical analysis commonly used in discrete-event simulation. That is, a discrete-event simulation program can be used to simulate the same stochastic system repeatedly — all you need to do is change the initial seed for the random number generator from run to run. This process is known as *replication*.

In either case, each time a discrete-event simulation program is used to generate data it is important to appreciate that this data is only a *sample* from that much larger *population* that would be produced, in the first case, if the program were run for a much longer time or, in the second case, if more replications were used. Analyzing a sample and then inferring something about the population from which the sample was drawn is the essence of statistics. We begin by defining the sample mean and standard deviation.

4.1.1 SAMPLE MEAN AND STANDARD DEVIATION

Definition 4.1.1   Given the sample x_1, x_2, ..., x_n (either continuous or discrete data) the *sample mean* is

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

The *sample variance* is the average of the squared differences about the sample mean

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

The *sample standard deviation* is the positive square root of the sample variance, $s = \sqrt{s^2}$.

The sample mean is a measure of *central tendency* of the data values. The sample variance and standard deviation are measures of *dispersion* — the spread of the data about the sample mean. The sample standard deviation has the same "units" as the data and the sample mean. For example, if the data has units of *sec* then so also does the sample mean and standard deviation. The sample variance would have units of sec^2. Although the sample variance is more amenable to mathematical manipulation (because it is free of the square root), the sample standard deviation is typically the preferred measure of dispersion since it has the same units as the data. (Because $\bar{x}$ and s have the same units, the ratio $s/\bar{x}$, known as the *coefficient of variation*, has no units. This statistic is commonly used only if the data is inherently non-negative, although it is inferior to s as a measure of dispersion because a common shift in the data results in a change in $s/\bar{x}$.)

For statistical reasons that will be explained further in Chapter 8, a common alternative definition of the sample variance (and thus the sample standard deviation) is

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad \text{rather than} \qquad \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Provided one of these definitions is used consistently for *all* computations, the choice of which equation to use for s^2 is largely a matter of taste, based on statistical considerations listed below. There are three reasons that the $1/(n-1)$ form of the sample variance appears almost universally in introductory statistics books:

- The sample variance is undefined when $n = 1$ when using the $1/(n-1)$ form. This is intuitive in the sense that a single observed data value indicates nothing about the spread of the distribution from which the data value is drawn.

- The $1/(n-1)$ form of the sample variance is an *unbiased estimate* of the population variance when the data values are drawn independently from a population with a finite mean and variance. The designation of an estimator as "unbiased" in statistical terminology implies that the sample average of many such sample variances converges to the population variance.

- The statistic

$$\sum_{i=1}^{n}(x_i - \bar{x})^2$$

  has $n-1$ "degrees of freedom". It is common practice in statistics (particularly in a sub-field known as *analysis of variance*) to divide a statistic by its degrees of freedom.

Despite these compelling reasons, why do we still use the $1/n$ form of the sample variance as given in Definition 4.1.1 consistently throughout the book? Here are five reasons:

- The sample size n is typically *large* in discrete-event simulation, making the difference between the results obtained by using the two definitions small.

- The unbiased property associated with the $1/(n-1)$ form of the sample variance applies only when observations are independent. Observations within a simulation run are typically not independent in discrete-event simulation.

- In the unlikely case that only $n = 1$ observation is collected, the algorithms presented in this text using the $1/n$ form are able to compute a numeric value (zero) for the sample variance. The reader should understand that a sample variance of zero in the case of $n = 1$ does *not* imply that there is no variability in the population.

- The $1/n$ form of the sample variance enjoys the status as a "plug-in" (details given in Chapter 6) estimate of the population variance. It also is the "maximum likelihood estimate" (details given in Chapter 9) when sampling from bell-shaped populations.

- Authors of many higher-level books on mathematical statistics prefer the $1/n$ form of the sample variance.

The relationship between the sample mean and sample standard deviation is summarized by Theorem 4.1.1. The proof is left as an exercise.

Theorem 4.1.1   In a *root-mean-square* (rms) sense, the sample mean is that unique value that best fits the sample $x_1, x_2, \ldots, x_n$ and the sample standard deviation is the corresponding smallest possible rms value. In other words, if the rms value associated with any value of x is

$$d(x) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - x)^2}$$

then the smallest possible rms value of $d(x)$ is $d(\bar{x}) = s$ and this value is achieved if and only if $x = \bar{x}$.

Example 4.1.1   Theorem 4.1.1 is illustrated in Figure 4.1.1 for a random variate sample of size $n = 50$ that was generated using a modified version of program `buffon`. The data corresponds to 50 observations of the x-coordinate of the righthand endpoint of a unit-length needle dropped at random (see Example 2.3.10). The 50 points in the sample are indicated with hash marks.

Figure 4.1.1.
Sample mean
minimizes.
rms.
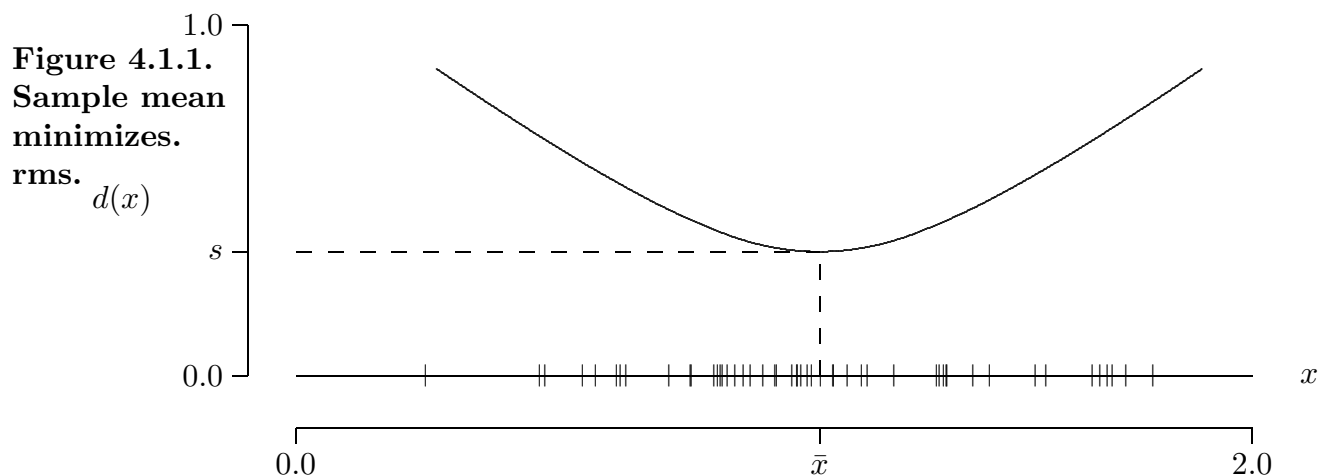
For this sample $\bar{x} \cong 1.095$ and $s \cong 0.354$. Consistent with Theorem 4.1.1 the smallest value of $d(x)$ is $d(\bar{x}) = s$ as illustrated.

The portion of the figure in Figure 4.1.1 that looks like

is known as a univariate *scatter diagram* — a convenient way to visualize a sample, provided the sample size is not too large. Later in this chapter we will discuss *histograms* — a superior way to visualize a univariate sample if the sample size is sufficiently large.

Chebyshev's Inequality

To better understand how the sample mean and standard deviation are related, appreciate that the number of points in a sample that lie within k standard deviations of the mean can be bounded by Chebyshev's inequality. The derivation of this fundamental result is based on a simple observation — the points that make the largest contribution to the sample standard deviation are those that are most distant from the sample mean.

Given the sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ with mean $\bar{x}$ and standard deviation s, and given a parameter $k > 1$, define the set*

$$\mathcal{S}_k = \{x_i \mid \bar{x} - ks < x_i < \bar{x} + ks\}$$

consisting of all those $x_i \in \mathcal{S}$ for which $|x_i - \bar{x}| < ks$. For $k = 2$ and the sample in Example 4.1.1, the set $\mathcal{S}_2$ is defined by the hash marks that lie within the $2ks = 4s$ wide interval centered on $\bar{x}$ as shown by the portion of Figure 4.1.1 given below.
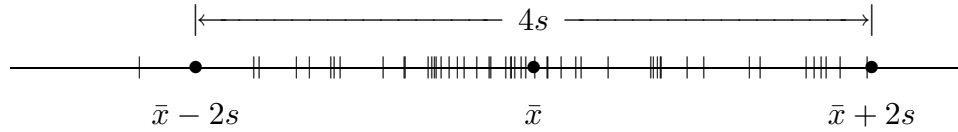
Let $p_k = |\mathcal{S}_k|/n$ be the *proportion* of x_i that lie within $\pm ks$ of $\bar{x}$. This proportion is the probability that an x_i selected at random from the sample will be in $\mathcal{S}_k$. From the definition of s^2

$$ns^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{x_i \in \mathcal{S}_k}(x_i - \bar{x})^2 + \sum_{x_i \in \overline{\mathcal{S}}_k}(x_i - \bar{x})^2$$

where the set $\overline{\mathcal{S}}_k$ is the complement of $\mathcal{S}_k$. If the contribution to s^2 from all the points close to the sample mean (the points in $\mathcal{S}_k$) is ignored then, because the contribution to s^2 from each point in $\overline{\mathcal{S}}_k$ (the points far from the sample mean) is at least $(ks)^2$, the previous equation becomes the following inequality

$$ns^2 \geq \sum_{x_i \in \overline{\mathcal{S}}_k}(x_i - \bar{x})^2 \geq \sum_{x_i \in \overline{\mathcal{S}}_k}(ks)^2 = |\overline{\mathcal{S}}_k|(ks)^2 = n(1 - p_k)k^2 s^2.$$

If ns^2 is eliminated the result is *Chebyshev's inequality*

$$p_k \geq 1 - \frac{1}{k^2} \qquad (k > 1).$$

(This inequality says *nothing* for $k \leq 1$.) From this inequality it follows that, in particular, $p_2 \geq 0.75$. That is, *for any sample* at least 75% of the points in the sample must lie within $\pm 2s$ of the sample mean.

---

* Some values in the sample may occur more than once. Therefore, $\mathcal{S}$ is actually a *multiset*. For the purposes of counting, however, the elements of $\mathcal{S}$ are treated as distinguishable so that the cardinality (size) of $\mathcal{S}$ is $|\mathcal{S}| = n$.

For the data illustrated by the scatter diagram the true percentage of points within $\pm 2s$ of the sample mean is 98%. That is, for the sample in Example 4.1.1, as for most samples, the 75% in the $k = 2$ form of Chebyshev's inequality is very conservative. Indeed, it is common to find approximately 95% of the points in a sample within $\pm 2s$ of the sample mean. (See Exercise 4.1.7.)

The primary issue here is not the accuracy (or lack thereof) of Chebyshev's inequality. Instead, Chebyshev's inequality and practical experience with actual data suggest that the $\bar{x} \pm 2s$ interval defines the "effective width" of a sample.* As a rule of thumb, most but not all of the points in a sample will fall within this interval. With this in mind, when analyzing experimental data it is common to look for *outliers* — values so far from the sample mean, for example $\pm 3s$ or more — that they must be viewed with suspicion.

Linear Data Transformations

It is sometimes the case that the output data generated by a discrete-event simulation, for example the wait times in a single-server service node, are statistically analyzed in one system of units (say seconds) and later there is a need to convert to a different system of units (say minutes). Usually this conversion of units is a *linear* data transformation and, if so, the change in system statistics can be determined directly, without any need to re-process the converted data.

That is, if $x_i' = ax_i + b$ for $i = 1, 2, \ldots, n$ then how do the sample mean $\bar{x}$ and standard deviation s of the x-data relate to the sample mean $\bar{x}'$ and standard deviation s' of the x'-data? The answer is that

$$\bar{x}' = \frac{1}{n} \sum_{i=1}^{n} x_i' = \frac{1}{n} \sum_{i=1}^{n} (ax_i + b) = \frac{a}{n} \left(\sum_{i=1}^{n} x_i \right) + b = a\bar{x} + b$$

and

$$(s')^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i' - \bar{x}')^2 = \frac{1}{n} \sum_{i=1}^{n} (ax_i + b - a\bar{x} - b)^2 = \frac{a^2}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = a^2 s^2.$$

Therefore

$$\bar{x}' = a\bar{x} + b \qquad \text{and} \qquad s' = |a|s.$$

Example 4.1.2   Suppose the sample $x_1, x_2, \ldots, x_n$ is measured in seconds. To convert to minutes, the transformation is $x_i' = x_i/60$ so that if, say $\bar{x} = 45$ (seconds) and $s = 15$ (seconds), then

$$\bar{x}' = \frac{45}{60} = 0.75 \text{ (minutes)} \qquad \text{and} \qquad s' = \frac{15}{60} = 0.25 \text{ (minutes)}.$$

---

   * There is nothing magic about the value $k = 2$. Other values like $k = 2.5$ and $k = 3$ are sometimes offered as alternatives to define the $\bar{x} \pm ks$ interval. To the extent that there is a standard, however, $k = 2$ is it. This issue will be revisited in Chapter 8.

Example 4.1.3 Some people, particularly those used to analyzing so-called "normally distributed" data, like to *standardize* the data by subtracting the sample mean and dividing the result by the sample standard deviation. That is, given a sample $x_1, x_2, \ldots, x_n$ with sample mean $\bar{x}$ and standard deviation s, the corresponding standardized sample is computed as

$$x_i' = \frac{x_i - \bar{x}}{s} \qquad i = 1, 2, \ldots, n.$$

It follows that the sample mean of the standardized sample is $\bar{x}' = 0$ and the sample standard deviation is $s' = 1$. Standardization is commonly used to avoid potential numerical problems associated with samples containing very large or very small values.

Nonlinear Data Transformations

Although nonlinear data transformations are difficult to analyze in general, there are times when data is used to create a Boolean (two-state) outcome. That is, the *value* of x_i is not as important as the *effect* — does x_i cause something to occur or not? In particular, if $\mathcal{A}$ is a fixed set and if for $i = 1, 2, \ldots, n$ we define the nonlinear data transformation

$$x_i' = \begin{cases} 1 & x_i \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

then what are $\bar{x}'$ and s'? To answer this question let the proportion of x_i that fall in $\mathcal{A}$ be

$$p = \frac{\text{the number of } x_i \text{ in } \mathcal{A}}{n}$$

then

$$\bar{x}' = \frac{1}{n} \sum_{i=1}^{n} x_i' = p$$

and

$$(s')^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i' - p)^2 = \cdots = (1-p)(-p)^2 + p(1-p)^2 = p(1-p).$$

Therefore

$$\bar{x}' = p \qquad \text{and} \qquad s' = \sqrt{p(1-p)}.$$

Example 4.1.4 For a single-server service node, let $x_i = d_i$ be the delay experienced by the ith job and let $\mathcal{A}$ be the set of positive real numbers. Then x_i' is 1 if and only if $d_i > 0$ and p is the proportion of jobs (out of n) that experience a non-zero delay. If, as in Exercise 1.2.3, $p = 0.723$ then $\bar{x}' = 0.723$ and $s' = \sqrt{(0.723)(0.277)} = 0.448$.

Example 4.1.5 As illustrated in Section 2.3, a Monte Carlo simulation used to estimate a probability ultimately involves the generation of a sequence of 0's and 1's with the probability estimate p being the ratio of the number of 1's to the number of trials. Thus p is the sample mean of this sequence of 0's and 1's and $\sqrt{p(1-p)}$ is the sample standard deviation.

4.1.2   COMPUTATIONAL CONSIDERATIONS

One significant computational drawback to calculating the sample standard deviation using the equation

$$s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

is that it requires a *two-pass* algorithm. That is, the sample must be scanned once to calculate the sample mean by accumulating the $\sum x_i$ partial sums, and then scanned a second time to calculate the sample standard deviation by accumulating the $\sum (x_i - \bar{x})^2$ partial sums. This two-pass approach is usually undesirable in discrete-event simulation because it creates a need to temporarily store or re-create the *entire* sample, which is undesirable when n is large.

Conventional One-Pass Algorithm

There is an alternate, mathematically equivalent, equation for s^2 (and thus s) that can be implemented as a *one-pass* algorithm. The derivation of this equation is

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2\bar{x}x_i + \bar{x}^2)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right) - \left(\frac{2}{n}\bar{x}\sum_{i=1}^{n}x_i\right) + \left(\frac{1}{n}\sum_{i=1}^{n}\bar{x}^2\right)$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right) - 2\bar{x}^2 + \bar{x}^2$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right) - \bar{x}^2.$$

If this alternate equation is used, the sample mean and standard deviation can be calculated in one pass through the sample by accumulating the $\sum x_i$ and $\sum x_i^2$ partial sums, thereby eliminating the need to store the sample. As each new observation arises in a Monte Carlo or discrete-event simulation, only three memory locations are needed in order to store $\sum x_i$, $\sum x_i^2$, and the number of observations to date. There is, however, a potential problem with this one-pass approach — floating-point round-off error. To avoid overflow, the sums are typically accumulated with floating-point arithmetic, even if the data is integer-valued. Floating-point arithmetic is problematic in this case because the sample variance is ultimately calculated by subtracting two quantities that may be *very* large relative to their difference. That is, if the true value of s is tiny relative to $|\bar{x}|$ then accumulated floating-point round-off error may yield an incorrect value for s.

Fortunately there is an alternate one-pass algorithm (due to B. P. Welford in 1962) that can be used to calculate the sample mean and standard deviation. This algorithm is superior to the conventional one-pass algorithm in that it is much less prone to significant floating-point round-off error. The original algorithm is given in Welford (1962). Chan, Golub, and LeVeque (1983) survey other alternative algorithms. Welford's algorithm is based upon the following definition and associated theorem.

Welford's One-Pass Algorithm

Definition 4.1.2   Given the sample $x_1, x_2, x_3, \ldots$, for $i = 1, 2, 3, \ldots$, define the running sample mean and the running sample sum of squared deviations as

$$\bar{x}_i = \frac{1}{i}(x_1 + x_2 + \cdots + x_i)$$
$$v_i = (x_1 - \bar{x}_i)^2 + (x_2 - \bar{x}_i)^2 + \cdots + (x_i - \bar{x}_i)^2$$

where $\bar{x}_i$ and v_i/i are the sample mean and variance, respectively, of $x_1, x_2, \ldots, x_i$.

Theorem 4.1.2   For $i = 1, 2, 3, \ldots$, the variables $\bar{x}_i$ and v_i in Definition 4.1.2 can be computed recursively by

$$\bar{x}_i = \bar{x}_{i-1} + \frac{1}{i}(x_i - \bar{x}_{i-1})$$
$$v_i = v_{i-1} + \left(\frac{i-1}{i}\right)(x_i - \bar{x}_{i-1})^2$$

with the initial conditions $\bar{x}_0 = 0$ and $v_0 = 0$.

Proof   Both equations in this theorem are valid by inspection if $i = 1$. If $i > 1$ then we can write
$$i\bar{x}_i = x_1 + x_2 + \cdots + x_i$$
$$= (x_1 + x_2 + \cdots + x_{i-1}) + x_i$$
$$= (i-1)\bar{x}_{i-1} + x_i$$
$$= i\bar{x}_{i-1} + (x_i - \bar{x}_{i-1})$$

from which the first equation follows via division by i. This establishes the first equation in the theorem. As in the prior discussion of the conventional one-pass algorithm, an alternate equation for v_i is

$$v_i = x_1^2 + x_2^2 + \cdots + x_i^2 - i\bar{x}_i^2 \qquad i = 1, 2, 3, \ldots$$

From this alternate equation and the first equation in the theorem, if $i > 1$ we can write

$$v_i = (x_1^2 + x_2^2 + \cdots + x_{i-1}^2) + x_i^2 - i\bar{x}_i^2$$
$$= v_{i-1} + (i-1)\bar{x}_{i-1}^2 + x_i^2 - i\bar{x}_i^2$$
$$= v_{i-1} - i(\bar{x}_i^2 - \bar{x}_{i-1}^2) + (x_i^2 - \bar{x}_{i-1}^2)$$
$$= v_{i-1} - i(\bar{x}_i - \bar{x}_{i-1})(\bar{x}_i + \bar{x}_{i-1}) + (x_i - \bar{x}_{i-1})(x_i + \bar{x}_{i-1})$$
$$= v_{i-1} - (x_i - \bar{x}_{i-1})(\bar{x}_i + \bar{x}_{i-1}) + (x_i - \bar{x}_{i-1})(x_i + \bar{x}_{i-1})$$
$$= v_{i-1} + (x_i - \bar{x}_{i-1})(x_i - \bar{x}_i)$$
$$= v_{i-1} + (x_i - \bar{x}_{i-1})\left(x_i - \bar{x}_{i-1} - \frac{1}{i}(x_i - \bar{x}_{i-1})\right)$$
$$= v_{i-1} + (x_i - \bar{x}_{i-1})^2 - \frac{1}{i}(x_i - \bar{x}_{i-1})^2$$

which establishes the second equation in the theorem.

Algorithm 4.1.1   Given the sample $x_1, x_2, x_3, \ldots$, Welford's algorithm for calculating the sample mean $\bar{x}$ and standard deviation s is

```
n = 0;
x̄ = 0.0;
v = 0.0;
while ( more data ) {
    x = GetData();
    n++;
    d = x - x̄;                                    /* temporary variable */
    v = v + d * d * (n - 1) / n;
    x̄ = x̄ + d / n;
}
s = sqrt(v / n);
return n, x̄, s;
```

Program uvs

Algorithm 4.1.1 is a one-pass $O(n)$ algorithm that does not require prior knowledge of the sample size n. The algorithm is not as prone to accumulated floating-point round-off error as is the conventional one-pass algorithm, yet it is essentially as efficient. Program uvs at the end of this section is based on a robust version of Algorithm 4.1.1 designed to not fail if applied, by mistake, to an empty sample. This program computes the sample mean and standard deviation as well as the sample minimum and maximum.

Input to program uvs can be integer-valued or real-valued. The input is assumed to be a *text* data file, in a one-value-per-line format with no blank lines in the file. (The file uvs.dat is an example.) To use the program, compile uvs.c to produce the executable file uvs. Then at a command line prompt, uvs can be used in three ways:

- To have `uvs` read a disk data file, say `uvs.dat`, at a command line prompt use '`<`' redirection as:

      uvs < uvs.dat

- To have `uvs` filter the numerical output of a program, say `test`, at a command line prompt use a '`|`' pipe as:

      test | uvs

- To use `uvs` with keyboard input, at a command line prompt enter:

      uvs

  then enter the data, one value per line, being sure to signify an end-of-file as the last line of input [`^d` (Ctrl-d) in Unix or `^z` (Ctrl-z) in Microsoft Windows].

4.1.3   EXAMPLES

The two examples given here illustrate the effect of independent and dependent sampling on the sample statistics $\bar{x}$ and s.

Example 4.1.6   If a *Uniform*(a, b) random variate sample x_1, x_2, $\ldots$, x_n is generated then, in the limit as $n \to \infty$, the sample mean and sample standard deviation will converge to the limits indicated

$$\bar{x} \to \frac{a + b}{2} \qquad \text{and} \qquad s \to \frac{b - a}{\sqrt{12}}.$$

(see Chapter 7 for more details concerning *Uniform*(a, b) random variates.)

Welford's algorithm was used in conjunction with a random variate sample generated by calls to `Random` (drawn from stream 0 with an `rngs` initial seed of 12345). Since the random variates so generated are independent *Uniform*$(0, 1)$, as the sample size increases the sample mean and standard deviation should converge to

$$\frac{0 + 1}{2} = 0.5 \qquad \text{and} \qquad \frac{1 - 0}{\sqrt{12}} \cong 0.2887.$$

In the upper display in Figure 4.1.2, the sample values are indicated with ∘'s. The •'s show the running values of $\bar{x}$ and indicate the convergence of the corresponding sample mean $\bar{x}_n$ to 0.5 (the solid, horizontal line). As expected, the ∘ and the • overlap when $n = 1$. The plot of $\bar{x}$ may remind students of grade point average (GPA) calculations which can be significantly influenced during the freshman and sophomore years, but remains fairly constant during the junior and senior years. Similarly, in the lower display in Figure 4.1.2, the •'s show the running values of s and indicate the convergence of the sample standard deviation $s_n = \sqrt{v_n/n}$ to 0.2887 (the solid, horizontal line). As discussed earlier in this section, the sample standard deviation is 0 when $n = 1$. Consistent with the discussion in Section 2.3, the convergence of both $\bar{x}$ and s to their theoretical values is not necessarily monotone with increasing n. The final values plotted on each display corresponding to $n = 100$ are $\bar{x} = 5.15$ and $s = 0.292$.

**Figure 4.1.2.
Running
$\bar{x}$ and s
of random
numbers.**

Number of samples (n)
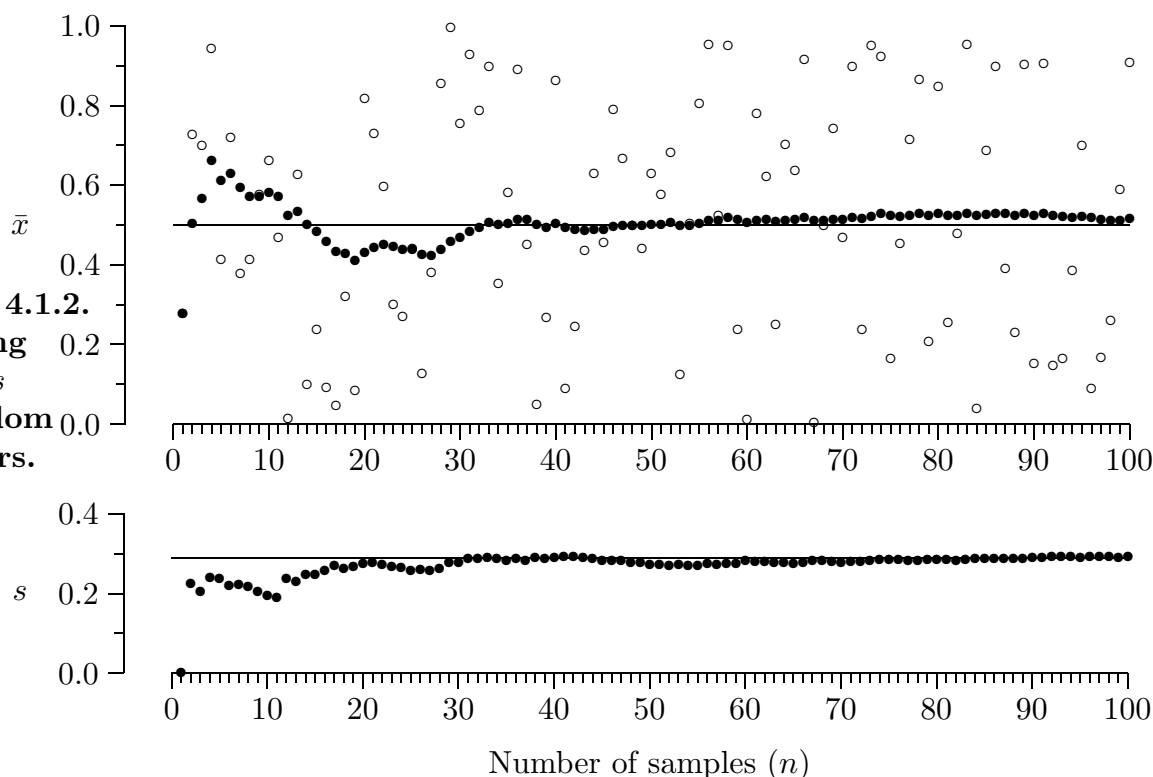
Analogous to Example 4.1.6, if an *Equilikely*(a, b) random variate sample is generated then, in the limit as $n \to \infty$, the mean and standard deviation of the sample will converge to the limits indicated

$$\bar{x} \to \frac{a + b}{2} \qquad \text{and} \qquad s \to \sqrt{\frac{(b - a + 1)^2 - 1}{12}}.$$

Similarly, if an *Exponential*(μ) random variate sample is generated then

$$\bar{x} \to \mu \qquad \text{and} \qquad s \to \mu$$

as $n \to \infty$ and for a *Geometric*(p) random variate sample

$$\bar{x} \to \frac{p}{1 - p} \qquad \text{and} \qquad s \to \frac{\sqrt{p}}{1 - p}.$$

(Chapters 6 and 7 contain derivations of many of these results.)

Serial Correlation

The random variates generated in Example 4.1.6 were *independent* samples. Informally, independence means that each x_i in the sample $x_1, x_2, \ldots, x_n$ has a value that does not depend in any way on any other point in the sample. When an independent sample is displayed, as in Example 4.1.6, there will not be any deterministic "pattern" or "trend" to the data.

Because repeated calls to `Random` produce a simulated independent sample, there is no deterministic pattern or trend to the sample in Example 4.1.6. If, however, we were to display the time-sequenced output from a discrete-event simulation then that sample typically will not be independent. For example, if we display a sample of random waits experienced by consecutive jobs passing through a single-server service node then a pattern will be evident because the sample is *not* independent, particularly if the utilization is large. That is, if the ith job experiences a large wait then the $(i+1)^{\text{th}}$ job is likely to also experience a large wait with an analogous statement true if the ith job experiences a small wait. In statistical terms, the wait times of consecutive jobs have positive *serial correlation* (see Section 4.4 for more details) and because of this the wait times in the sample will be dependent (i.e., not independent).

Hence independence is typically an appropriate assumption for Monte Carlo simulation, as illustrated in Example 4.1.6. It is typically *not* an appropriate assumption for discrete-event simulation, however, as illustrated in the following example.

Example 4.1.7 Program `ssq2`, with *Exponential*(2) interarrival times and *Uniform*(1, 2) service times, was modified to output the waits $w_1, w_2, \ldots, w_n$ experienced by the first $n = 100$ jobs. To simulate starting the service node in (approximate) steady-state the initial value of the program variable `departure` was set to 3.0. The `rng` initial seed was 12345. For this single-server service node it can be shown (see Section 8.5) that as $n \to \infty$ the sample mean and standard deviation of the wait times should converge to 3.83 and 2.83 respectively, as indicated by the solid horizontal lines in Figure 4.1.3.
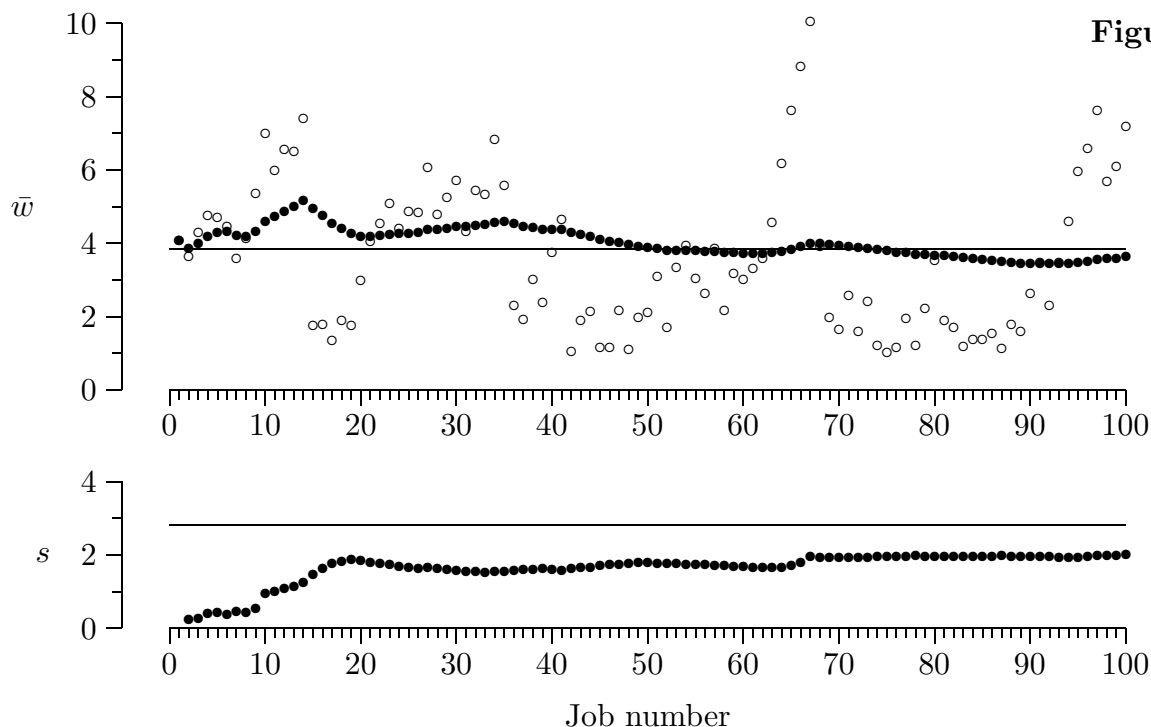
Figure 4.1.3. Running $\bar{x}$ and s of wait times.

Job number

As in Example 4.1.6, the sample values are indicated in the upper figure with ∘'s and the •'s indicate the (unbiased) convergence of the corresponding sample mean to its expected steady-state value.* Unlike the upper figure in Example 4.1.6, this sample reveals a lack-of-independence pattern caused by high positive serial correlation in the sample wait times $w_1, w_2, \ldots, w_n$. A sequence of longer-than-average waits is followed by a sequence of smaller-than-average waits. This pattern is quite different from that in Figure 4.1.2, where each observation is above or below the mean $(1/2)$ as in a sequence of tosses of a fair coin. The running average wait does, however, converge to the theoretical value. Moreover, as illustrated in the lower figure, the high positive serial correlation produces a pronounced *bias* in the sample standard deviation. That is, the sample standard deviation consistently underestimates the expected value. In this case a profound modification to the sample variance equation (and thus the sample standard deviation equation) is required to remove the bias; just replacing $1/n$ with $1/(n-1)$ will not do it. (See Appendix F.)

4.1.4   TIME-AVERAGED SAMPLE STATISTICS

As discussed in previous chapters, time-averaged statistics play an important role in discrete-event simulation. The following definition is the time-averaged analog of Definition 4.1.1.

Definition 4.1.3   Given a function $x(t)$ defined for all $0 < t < \tau$ as a realization (sample path) of a stochastic process the *sample-path mean* is

$$\bar{x} = \frac{1}{\tau} \int_0^\tau x(t)\, dt.$$

The associated *sample-path variance* is

$$s^2 = \frac{1}{\tau} \int_0^\tau \big(x(t) - \bar{x}\big)^2 dt$$

and the *sample-path standard deviation* is $s = \sqrt{s^2}$.

The equation for s^2 in Definition 4.1.3 is the two-pass variance equation; the corresponding one-pass equation is

$$s^2 = \left(\frac{1}{\tau} \int_0^\tau x^2(t)\, dt\right) - \bar{x}^2.$$

As the time-averaged statistic changes at the event times in a discrete-event simulation, only three memory locations are needed in order to store the running statistics $\int x(t)\, dt$, $\int x^2(t)\, dt$, and the length of the observation period to date.

---

* If the initial state of the service node had been idle, rather than adjusted to simulate starting the service node in (approximate) steady-state, then there would have been some *initial state bias* in the sample means.

For a discrete-event simulation a sample path is naturally *piecewise constant.* That is, as illustrated in Figure 4.1.4, there are *event times* $0 = t_0 < t_1 < t_2 < \cdots < t_n = \tau$ and associated state values $x_1, x_2, \ldots, x_n$ on the time intervals $(t_0, t_1], (t_1, t_2], \ldots, (t_{n-1}, t_n]$. The choice of making the left endpoint of these intervals open and the right endpoint of these intervals closed is arbitrary.
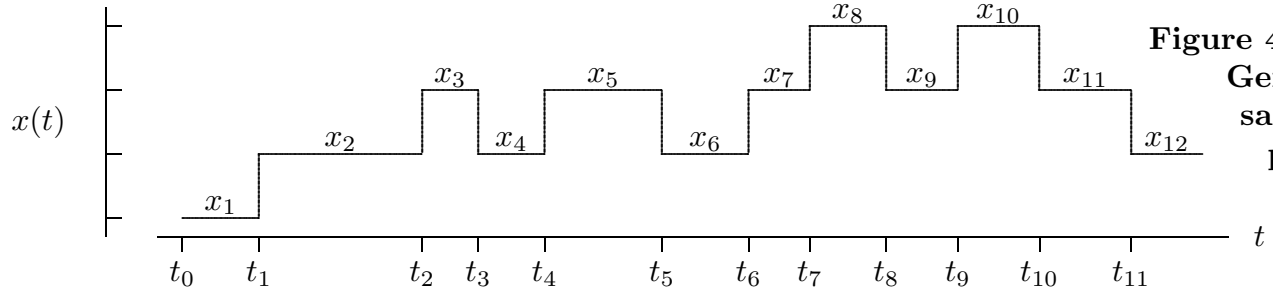
Figure 4.1.4. Generic sample path.

If a sample path is piecewise constant then the sample-path *integral* equations for $\bar{x}$ and s^2 in Definition 4.1.3 can be reduced to *summation* equations, as summarized by Theorem 4.1.3 that follows. The *defining formulas* given in Definition 4.1.3 are useful conceptually, but Theorem 4.1.3 gives the *computational formulas* that are used to compute $\bar{x}$ and s^2 in practice.

Theorem 4.1.3   Given a sample path represented by the piecewise constant function

$$
x(t) = \begin{cases} x_1 & t_0 < t \leq t_1 \\ x_2 & t_1 < t \leq t_2 \\ \vdots & \vdots \\ x_n & t_{n-1} < t \leq t_n. \end{cases}
$$

and the inter-event times $\delta_i = t_i - t_{i-1}$ for $i = 1, 2, \ldots, n$ then with $t_0 = 0$ and $\tau = t_n$ the sample-path mean is

$$
\bar{x} = \frac{1}{\tau} \int_0^\tau x(t)\, dt = \frac{1}{t_n} \sum_{i=1}^n x_i\, \delta_i
$$

and the sample-path variance is

$$
s^2 = \frac{1}{\tau} \int_0^\tau \left(x(t) - \bar{x} \right)^2 dt = \frac{1}{t_n} \sum_{i=1}^n \left(x_i - \bar{x} \right)^2 \delta_i = \left(\frac{1}{t_n} \sum_{i=1}^n x_i^2\, \delta_i \right) - \bar{x}^2.
$$

Welford's Sample-Path Algorithm

The two-pass and one-pass algorithms are susceptible to floating-point round-off error as described earlier. Welford's algorithm can be extended to the time-averaged case by defining running sample means and running sample sum of squared deviations analogous to the earlier case.

Definition 4.1.4   With reference to Theorem 4.1.3, for $i = 1, 2, \ldots, n$ the sample-path variant of Welford's algorithm is based on the definitions

$$\bar{x}_i = \frac{1}{t_i}(x_1\delta_1 + x_2\delta_2 + \cdots + x_i\delta_i)$$

$$v_i = (x_1 - \bar{x}_i)^2\delta_1 + (x_2 - \bar{x}_i)^2\delta_2 + \cdots + (x_i - \bar{x}_i)^2\delta_i$$

where $\bar{x}_i$ and v_i/t_i are the sample-path mean and variance, respectively, of the sample path $x(t)$ for $t_0 \le t \le t_i$.

Theorem 4.1.4   For $i = 1, 2, \ldots, n$ the variables $\bar{x}_i$ and v_i in Definition 4.1.4 can be computed recursively by

$$\bar{x}_i = \bar{x}_{i-1} + \frac{\delta_i}{t_i}(x_i - \bar{x}_{i-1})$$

$$v_i = v_{i-1} + \frac{\delta_i t_{i-1}}{t_i}(x_i - \bar{x}_{i-1})^2$$

with the initial conditions $\bar{x}_0 = 0$ and $v_0 = 0$.

4.1.5   EXERCISES

Exercise 4.1.1   Prove Theorem 4.1.1 by first proving that $d^2(x) = (x - \bar{x})^2 + s^2$.

Exercise 4.1.2   Prove Theorem 4.1.1 for the $1/(n-1)$ form of the sample variance.

Exercise 4.1.3   Relative to Theorem 4.1.2, prove that the sequence $v_1, v_2, \ldots, v_n$ satisfies the inequality $0 = v_1 \le v_2 \le \cdots \le v_{n-1} \le v_n = ns^2$.

Exercise 4.1.4[a]   What common sample statistic best fits the sample $x_1, x_2, \ldots, x_n$ in the sense of minimizing

$$d(x) = \frac{1}{n}\sum_{i=1}^{n}|x_i - x|?$$

Exercise 4.1.5   The statistic q^3 is the so-called sample *skewness* defined by

$$q^3 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s}\right)^3.$$

(a) What is the one-pass version of the equation for q^3? (b) Extend the conventional (non-Welford) one-pass algorithm to also compute the sample statistic $q = (q^3)^{1/3}$. (c) What is the value of q for the data in the file `uvs.dat`?

Exercise 4.1.6   Look up the article by Chan, Golub, and LeVeque (1983) in the bibliography. Give a survey of existing algorithms for computing the sample variance listing the pros and cons of each method.

Exercise 4.1.7   (*a*) Generate an *Exponential*(9) random variate sample of size $n = 100$ and compute the proportion of points in the sample that fall within the intervals $\bar{x} \pm 2s$ and $\bar{x} \pm 3s$. Do this for 10 different `rngs` streams. (*b*) In each case, compare the results with Chebyshev's inequality. (*c*) Comment.

Exercise 4.1.8   Generate a plot similar to that in Figure 4.1.2 with calls to *Exponential*(17), rather than `Random` to generate the variates. Indicate the values to which the sample mean and sample standard deviation will converge.

Exercise 4.1.9[a]   Prove Theorems 4.1.3 and 4.1.4.

Exercise 4.1.10   (*a*) Given x_1, x_2, ..., x_n with $a \leq x_i \leq b$ for $i = 1, 2, \ldots, n$, what is the largest and smallest possible value of $\bar{x}$? (*b*) Same question for s.

Exercise 4.1.11   Calculate $\bar{x}$ and s by hand using the 2-pass algorithm, the 1-pass algorithm, and Welford's algorithm in the following two cases. (*a*) The data based on $n = 3$ observations: $x_1 = 1$, $x_2 = 6$, and $x_3 = 2$. (*b*) The sample path $x(t) = 3$ for $0 < t \leq 2$, and $x(t) = 8$ for $2 < t \leq 5$, over the time interval $0 < t \leq 5$.

Exercise 4.1.12[a]   The extent to which accumulated round-off error is a potential problem in any calculation involving floating-point arithmetic is determined, in part, by what is known as the computational system's "floating-point precision." One way to determine this precision is by executing the code fragment below.

```
    typedef float FP;                          /* float or double */
    const FP  ONE   = 1.0;
    const FP  TWO   = 2.0;
          FP  tiny  = ONE;
          int count = 0;
    while (ONE + tiny != ONE) {
       tiny  /= TWO;
       count++;
    }
```

(*a*) Experiment with this code fragment on at least three systems being sure to use both `float` and `double` data types for the generic floating-point type `FP`. (*b*) Explain what this code fragment does, why the `while` loop terminates, and the significance of the variables `tiny` and `count`. Be explicit.

4.2 Discrete-Data Histograms

Given a univariate sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, if the sample size is sufficiently large (say $n > 50$ or so) then generally it is meaningful to do more than just calculate the sample mean and standard deviation. In particular, the sample can be processed to form a *histogram* and thereby gain insight into the distribution of the data. There are two cases to consider, discrete data and continuous data. In this section we deal with the easier case, discrete data; continuous data is considered in the next section.

4.2.1 DISCRETE-DATA HISTOGRAMS

Definition 4.2.1 Given the discrete-data sample (multiset) $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, let $\mathcal{X}$ be the set of possible distinct values in $\mathcal{S}$. For each $x \in \mathcal{X}$ the *relative frequency* (*proportion*) is

$$\hat{f}(x) = \frac{\text{the number of } x_i \in \mathcal{S} \text{ for which } x_i = x}{n}$$

A *discrete-data histogram* is a graphical display of $\hat{f}(x)$ versus x.*

The point here is that if the sample size $n = |\mathcal{S}|$ is large and the data is discrete then it is reasonable to expect that the number of distinct values in the sample will be much smaller than the sample size — values will appear multiple times. A discrete-data histogram is nothing more than a graphical display of the relative frequency with which each distinct value in the sample appears.

Example 4.2.1 As an example of a discrete-data histogram, a modified version of program `galileo` was used to replicate $n = 1000$ rolls of three fair dice (with an `rng` initial seed of 12345) and thereby create a discrete-data sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ with each x_i an integer between 3 and 18 inclusive. (Each x_i is the sum of the three up faces.) Therefore, $\mathcal{X} = \{3, 4, \ldots, 18\}$. The resulting relative frequencies (probability estimates) are plotted as the discrete-data histogram on the left side of Figure 4.1.2. Illustrated on the right is the same histogram with the theoretical probabilities associated with the sum-of-three-dice experiment superimposed as $\bullet$'s. Each theoretical probability represents the limit to which the corresponding relative frequency will converge as $n \to \infty$.

Figure 4.2.1. Histograms for program `galileo`.
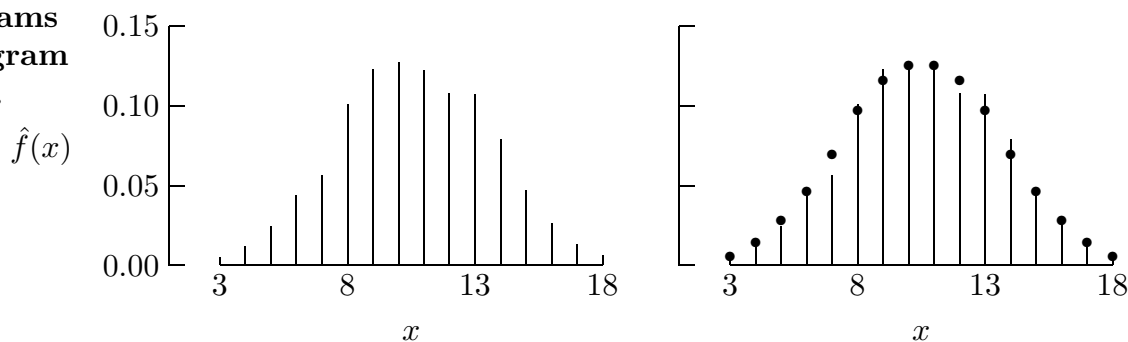
* The reason for the $\hat{f}(x)$ "hat" notation will be explained in Chapter 8.

No matter how many replications are used, in Example 4.2.1 we know *a priori* that $\mathcal{X} = \{3, 4, \ldots, 18\}$. Because of that it is natural to use an array to accumulate the relative frequencies, as in program `galileo`. An array can also be used to accumulate the relative frequencies in the next example. In this case, however, the use of an array is less natural because, although we know that $\mathcal{X} = \{0, 1, 2, \ldots, b\}$ for some value of b, a reasonable value for b, valid for *any* value of n, is not easily established a priori. If an array is not appropriate, a more flexible data structure based on dynamic memory allocation is required to tally relative frequencies. As an illustration, a linked-list implementation of a discrete-data histogram algorithm is presented later in this section.

Example 4.2.2 Suppose that $2n = 2000$ balls are placed *at random* into $n = 1000$ boxes. That is, for each ball a box is selected at random and the ball is placed in it. The Monte Carlo simulation algorithm given below can be used to generate a random sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ where, for each i, x_i is the number of balls placed in box i.

```
n = 1000;
for (i = 1; i <= n; i++)                      /* i counts boxes */
    x_i = 0;
for (j = 0; j < 2 * n; j++) {                 /* j counts balls */
    i = Equilikely(1, n);              /* pick a box at random  */
    x_i++;                            /* then put a ball in it */
}
return x_1, x_2, ..., x_n;
```

When 2000 balls are placed into 1000 boxes in this way then each box will have exactly two balls in it on average. Some boxes will be empty, however, some will have just one ball, some will have two balls, some will have three balls, etc., as illustrated in Figure 4.2.2.
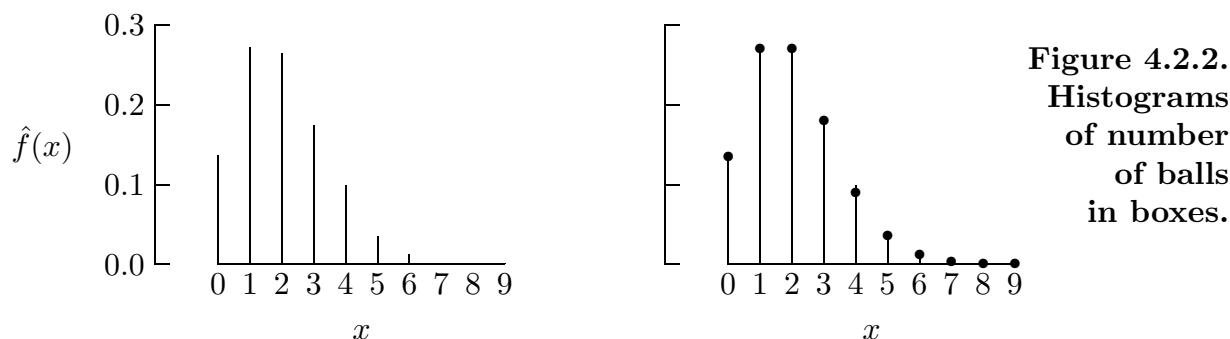
Figure 4.2.2. Histograms of number of balls in boxes.

In this case, for an `rng` initial seed of 12345, $\mathcal{X} = \{0, 1, 2, \ldots, 9\}$. As in Example 4.2.1, the discrete-data histogram is on the left and the figure on the right is the same histogram with theoretical probabilities superimposed as •'s. Each theoretical probability represents the limit to which the corresponding relative frequency will converge as $n \to \infty$. (The relative frequencies $\hat{f}(7) = 0.002$ and $\hat{f}(9) = 0.001$ are too small to be visible in the histogram and, because no boxes ended up with 8 balls, $\hat{f}(8) = 0.000$.)

Note the asymmetric shape of the histogram in Example 4.2.2. This asymmetry is frequently associated with non-negative data when the mean is small. As an exercise you should repeat this example in the situation where there are 1000 boxes and 10 000 balls. In that case the shape of the histogram will change to a distribution symmetric about the expected number of balls in each box, which is 10. The reason for this shift to a symmetric distribution will be discussed in Chapter 6.

Histogram Mean and Standard Deviation

Definition 4.2.2   Given the relative frequencies from Definition 4.2.1, the *discrete-data histogram mean* is

$$\bar{x} = \sum_x x \hat{f}(x)$$

and the associated *discrete-data histogram standard deviation* is

$$s = \sqrt{\sum_x (x - \bar{x})^2 \hat{f}(x)}$$

where the sum is over all $x \in \mathcal{X}$. The discrete-data histogram variance is s^2.

Consistent with their interpretation as probability estimates, the relative frequencies from Definition 4.2.1 are defined so that $\hat{f}(x) \geq 0$ for all $x \in \mathcal{X}$ and

$$\sum_x \hat{f}(x) = 1.$$

Moreover, it follows from the definition of $\mathcal{S}$ and $\mathcal{X}$ that

$$\sum_{i=1}^n x_i = \sum_x x n \hat{f}(x) \qquad \text{and} \qquad \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_x (x - \bar{x})^2 \, n \hat{f}(x).$$

(In both equations, the two summations compute the same thing, but in a different order.) Therefore, from the equations in Definition 4.1.1 and Definition 4.2.2 it follows that the *sample* mean and standard deviation are mathematically equivalent to the *discrete-data histogram* mean and standard deviation, respectively.

Provided the relative frequencies have already been computed, it is generally preferable to compute $\bar{x}$ and s using the discrete-data histogram equations. Moreover, the histogram-based equations have great theoretical significance in that they provide the motivation for defining the mean and standard deviation of discrete random variables — see Chapter 6.

The equation for s in Definition 4.2.2 is the two-pass version of the standard deviation equation. The mathematically equivalent one-pass version of this equation is

$$s = \sqrt{\left(\sum_x x^2 \hat{f}(x) \right) - \bar{x}^2}$$

where the sum is over all $x \in \mathcal{X}$.

Example 4.2.3   For the data in Example 4.2.1

$$\bar{x} = \sum_{x=3}^{18} x\,\hat{f}(x) \cong 10.609 \qquad \text{and} \qquad s = \sqrt{\sum_{x=3}^{18}(x-\bar{x})^2\hat{f}(x)} \cong 2.925$$

Similarly, for the data in Example 4.2.2

$$\bar{x} = \sum_{x=0}^{9} x\,\hat{f}(x) = 2.0 \qquad \text{and} \qquad s = \sqrt{\sum_{x=0}^{9}(x-\bar{x})^2\hat{f}(x)} \cong 1.419$$

4.2.2   COMPUTATIONAL CONSIDERATIONS

As illustrated in Examples 4.2.1 and 4.2.2, in many simulation applications the discrete data will, in fact, be *integer*-valued. For integer-valued data, the usual way to tally the discrete-data histogram is to use an array. Recognize, however, that the use of an array data structure for the histogram involves the allocation of memory with an associated requirement to know the range of data values. That is, when memory is allocated it is necessary to assume that $\mathcal{X} = \{a, a+1, a+2, \ldots, b\}$ for reasonable integer values of $a < b$. Given a knowledge of a, b the following one-pass $O(n)$ algorithm is the preferred way to compute a discrete-data histogram for integer-valued data.

Algorithm 4.2.1   Given integers a, b with $a < b$ and integer-valued data $x_1, x_2, \ldots$ the following algorithm computes a discrete-data histogram.

```
long count[b - a + 1];
n = 0;
for (x = a; x <= b; x++)
   count[x - a] = 0;
outliers.lo = 0;
outliers.hi = 0;
while ( more data ) {
   x = GetData();
   n++;
   if ((a <= x) and (x <= b))
     count[x - a]++;
   else if (a > x)
     outliers.lo++;
   else
     outliers.hi++;
}
return n, count[], outliers                /* f̂(x) is (count[x − a] / n) */
```

Outliers

By necessity Algorithm 4.2.1 allows for the possibility of *outliers* — occasional x_i that fall outside the range $a \leq x_i \leq b$. Generally with simulation-generated data there should not be any outliers. That is, an outlier is a data point that some omniscient being considers to be so different from the rest that it should be excluded from any statistical analysis. Although outliers are common with some kinds of experimentally measured data, it is difficult to argue that *any* data generated by a *valid* discrete-event simulation program is an outlier, no matter how unlikely the value may appear to be.

General-Purpose Discrete-Data Histogram Algorithm

Algorithm 4.2.1 is not appropriate as a *general-purpose* discrete-data histogram algorithm for two reasons.

- If the discrete-data sample is integer-valued and the a, b parameters are not chosen properly then either outliers will be produced, perhaps without justification, or excessive computational memory (to store the `count` array) may be required needlessly.

- If the data is not integer-valued then Algorithm 4.2.1 is not applicable.

In either case we must find an alternative, more general, algorithm that uses dynamic memory allocation and has the ability to handle both real-valued and integer-valued data. The construction of this algorithm is one of several occasions in this book where we will use dynamic memory allocation, in this case in the form of a linked-list, as the basis for an algorithm that is well suited to accommodate the uncertainty and variability naturally associated with discrete-event simulation data.

Definition 4.2.3   A linked-list discrete-data histogram algorithm with general applicability can be constructed using a *list pointer* `head` and multiple *list nodes* (structures) each consisting of three fields `value`, `count`, and `next`, as illustrated in Figure 4.2.3.

Figure 4.2.3.
Linked list data
structure for
discrete data.
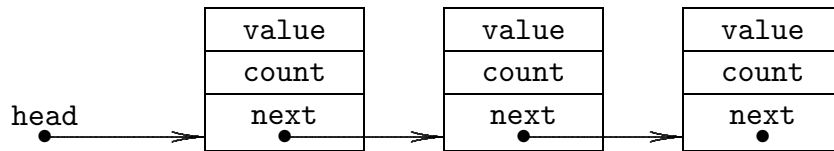
The association between the first two fields in each list node and the corresponding terms in Definition 4.2.1 is $x \sim$ `value` and, after all data is processed, $n\hat{f}(x) \sim$ `count`. The `next` field is the link (pointer) from one list node to the next.

Algorithm 4.2.2   Given the linked-list data structure in Definition 4.2.3 and the discrete data $x_1, x_2, \ldots, x_n$ a discrete-data histogram is computed by using x_1 to initialize the first list node. Then, for $i = 2, 3, \ldots, n$, as each x_i is read the list is searched (linearly from the head by following the links) to see if a list node with `value` equal to x_i is already present in the list. If so the corresponding list node `count` is increased by 1; otherwise a new list node is added to the end of the list with `value` equal to x_i and `count` equal to 1.

Example 4.2.4   For the discrete data $3.2, 3.7, 3.7, 2.9, 3.7, 3.2, 3.7, 3.2$, Algorithm 4.2.2 generates the corresponding linked-list illustrated in Figure 4.2.4.
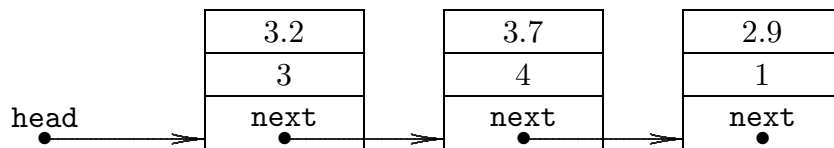
Figure 4.2.4. Sample linked list.

Note that the order of the linked-list nodes is dictated by the order in which the data appears in the sample. As an alternative, it may be better to use the `count` field and maintain the list in *sorted* order by decreasing relative frequency, or to use the `value` field and sort by data value — see Exercise 4.2.6.

Program ddh

The discrete-data histogram program `ddh` is based on Algorithm 4.2.2 and the linked-list data structure in Definition 4.2.3. The program has been designed so that the linked list is sorted by value prior to output. This program is valid for both integer-valued and real-valued input with no artificial outlier check imposed and no restriction on the sample size. Like program `uvs`, program `ddh` supports file redirection, as illustrated by Example 4.2.5 (to follow).

If program `ddh` is used improperly to tally a large sample that is *not* discrete the program's execution time may be excessive. That is, the design of program `ddh` is based on the assumption that, even though the sample size $|\mathcal{S}|$ is essentially arbitrary, the number of distinct values in the sample $|\mathcal{X}|$ is not too large, say a few hundred or less. If the number of distinct values is large then the $O(|\mathcal{X}|)$ complexity (per sample value) of the function `Insert` and the one-time $O(|\mathcal{X}|^2)$ complexity of the function `Sort` will become the source of excessive execution time. Note, however, that a discrete-data sample with more than a few hundred distinct values would be *very* unusual.

Example 4.2.5   Program `sis2` computes the statistics $\bar{d}$, $\bar{o}$, $\bar{u}$, $\bar{l}^+$, and $\bar{l}^-$. If we were interested in a more detailed look at this simple inventory system we could, for example, construct a histogram of the inventory level prior to inventory review. This is easily accomplished by eliminating the printing of summary statistics in program `sis2` and modifying the `while` loop in `main` by inserting the one line indicated

```
index++;
printf("%ld\n", inventory);                      /* this line is new */
if (inventory < MINIMUM) {
```

If the new program `sis2` is compiled to disk (with $STOP = 10\,000$), the command

```
sis2 | ddh > sis2.out
```

will then produce a discrete-data histogram file `sis2.out` from which an inventory level histogram can be constructed.

Example 4.2.6   As a continuation of Example 4.2.5 the inventory level histogram is illustrated (x denotes the inventory level prior to review) in Figure 4.2.5.

Figure 4.2.5.
Inventory
level
with
Equilikely
demands.
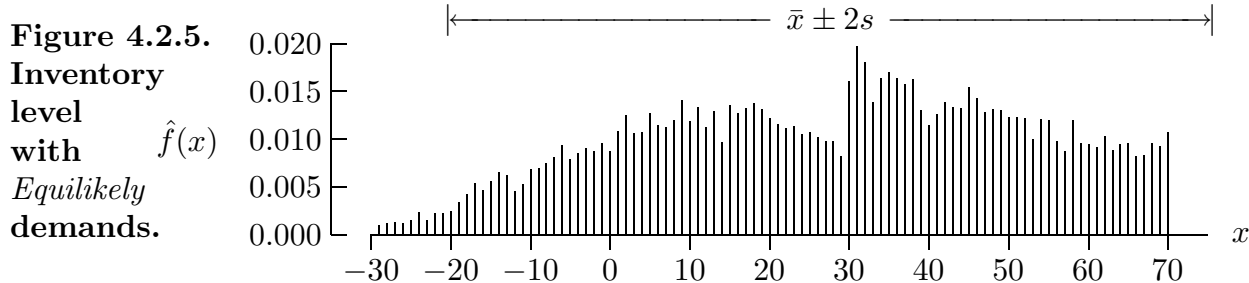
The sample mean and standard deviation for this data are $\bar{x} = 27.63$ and $s = 23.98$. The corresponding $\bar{x} \pm 2s$ interval is indicated. In this case approximately 98.5% of the data falls in this interval. Therefore, consistent with the discussion in the previous section, the "at least 75%" estimate guaranteed by Chebyshev's inequality is very conservative. That is, the $\bar{x} \pm 2s$ interval contains almost all the sample data.

As discussed in Example 3.1.5, the assumption of *Equilikely*(10, 50) random variate demands in program `sis2` is questionable. Generally one would expect the extreme values of the demand, in this case 10 and 50, to be much less likely than an intermediate value like 30. One way to accomplish this is use the assignment

```
return(Equilikely(5, 25) + Equilikely(5, 25));
```

in the function `GetDemand`. It can be verified by the axiomatic approach to probability that the theoretical mean and standard deviation of the demand for this assignment is 30 and $\sqrt{220/3} \cong 8.56$ respectively. Moreover, the distribution and associated histogram will have a triangular shape with a peak at 30. Of course, other stochastic demand models are possible as well. We will investigate this issue in more detail in Chapter 6.

Example 4.2.7   Compared to the histogram in Example 4.2.6, the use of the more realistic discrete triangular random variate demand model produces a corresponding change in the inventory level histogram, as illustrated in Figure 4.2.6.

Figure 4.2.6.
Inventory
level with
discrete
triangular
demands.
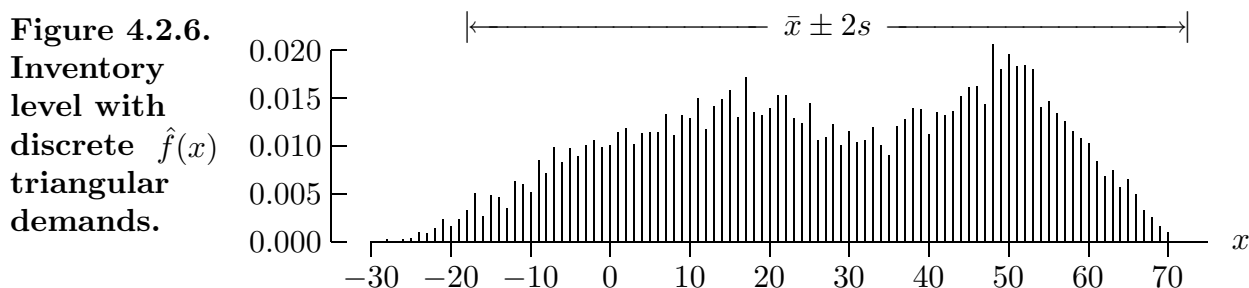
This inventory level histogram is much more tapered at the extreme values, particularly at 70. The sample mean and standard deviation are 27.29 and 22.59 respectively. The corresponding $\bar{x} \pm 2s$ interval is slightly smaller, as illustrated. As in Example 4.2.6, approximately 98.5% of the data lie in this interval.

Accuracy of Point Estimates

We now consider the accuracy of probability estimates derived by Monte Carlo simulation. This issue was first raised in Chapter 2 and, at that point, largely dismissed. Now we can use program `ddh` as an experimental tool to study the inherent uncertainty in such probability estimates. The following example uses the Monte Carlo program `craps` from Section 2.4 to generate 1000 point estimates of the probability of winning in the dice game craps. Because of the inherent uncertainty in any one of these estimates, when many estimates are generated a natural distribution of values will be produced. This distribution of values can be characterized with a discrete-data histogram and in that way we can gain significant insight into just how uncertain any one probability estimate may be.

Example 4.2.8   A Monte Carlo simulation of the dice game craps was used to generate 1000 estimates of the probability of winning. In the top figure each estimate is based on just $N = 25$ plays of the game; in the bottom figure $N = 100$ plays per estimate were used. For either figure, let p_1, p_2, p_3, ..., p_n denote the $n = 1000$ estimates where $p_i = x_i/N$ and x_i is the number of wins in N plays of the game. Since only $N + 1$ values of p_i are possible, it is natural to use a discrete-data histogram in this application, as illustrated in Figure 4.2.7.
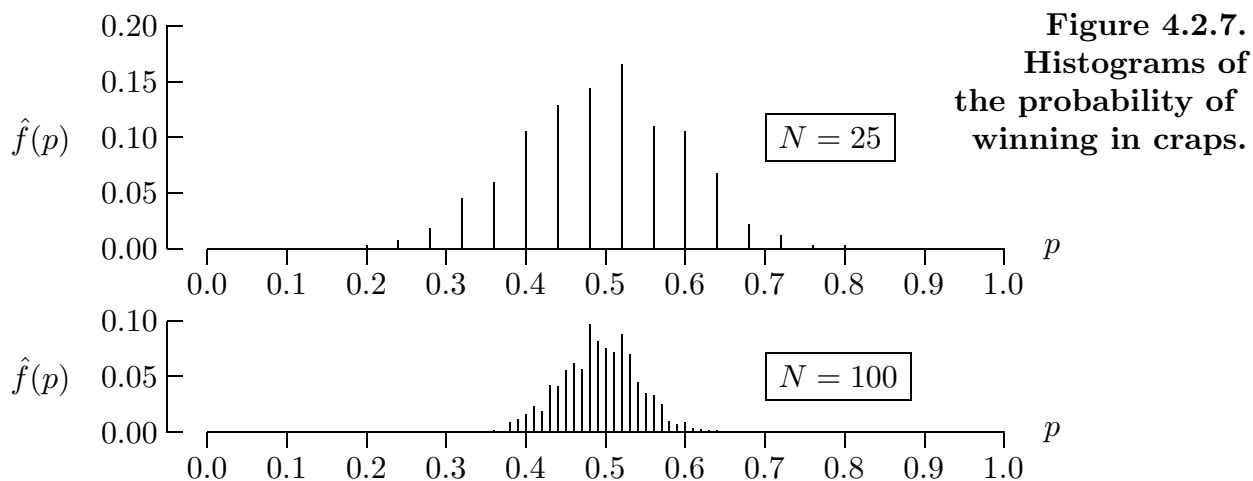
Figure 4.2.7. Histograms of the probability of winning in craps.

In the top histogram, the sample mean is 0.494, the sample standard deviation is 0.102 and there is significant variation about the true probability of winning at craps, which is known to be $244/495 \cong 0.4929$ using the analytic approach to probability. This variation corresponds to an inherently large uncertainty in any one estimate. That is, if just *one* probability estimate were generated based on 25 plays of the game, then our experiment with 1000 replications shows that probability estimates as far away from the true probability as $5/25 = 0.2$ and $20/25 = 0.8$ could be generated. In the bottom histogram, the associated sample mean and standard deviation are 0.492 and 0.048 respectively. When compared with the top histogram there is still significant variability, but it is reduced. In particular, the standard deviation is reduced by a factor of about two (from 0.102 to 0.048) resulting in a corresponding two-fold reduction in the uncertainty of any one estimate.

The reduction in the sample standard deviation from 0.102 to 0.048 in Example 4.2.8 is good. That a *four*-fold increase in the number of replications (games) is required to produce a *two*-fold reduction in uncertainty, however, is not so good. We will have much more to say about this in Chapter 8.

In Example 4.2.8 the "bell-shape" of the discrete-data histogram is important. As will be discussed in Chapter 8, this bell (or Gaussian) shape, which shows up in a wide variety of applications, is accurately described by a well-known mathematical equation that provides the basis for establishing a probability estimate *interval* within which the true (theoretical) probability lies with high confidence.

4.2.3 EMPIRICAL CUMULATIVE DISTRIBUTION FUNCTIONS

The histogram is an effective tool for estimating the shape of a distribution. There are occasions, however, when a cumulative version of the histogram is preferred. Two such occasions are when quantiles (e.g., the 90th quantile of a distribution) are of interest or when two or more distributions are to be compared. An *empirical cumulative distribution function*, which simply takes an upward step of $1/n$ at each of the n data values $x_1, x_2, \ldots, x_n$, is easily computed. In the case of discrete data, of course, there will typically be many ties, so the step function will have varying heights to the risers.

Example 4.2.9 The $n = 1000$ estimates of the probability of winning in craps for $N = 100$ plays from Example 4.2.8 range from 0.33 to 0.64. Four *empirical cumulative distribution functions* for this data set using different formats are plotted in Figure 4.2.8 for $0.3 < p < 0.7$. The top two graphs plot the cumulative probability only at the observed values, while the bottom two graphs are step functions plotted for all values of p. The choice between these four styles is largely a matter of taste.

Figure 4.2.8. Empirical cumulative distribution functions.
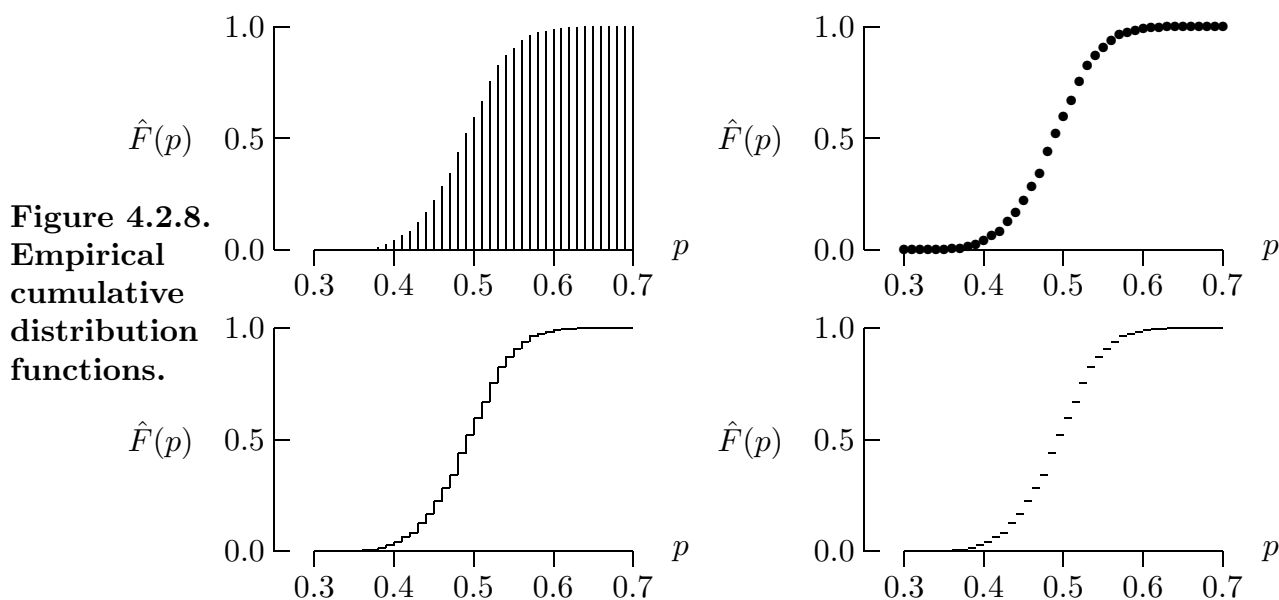
4.2.4   EXERCISES

Exercise 4.2.1[a]   Find the theoretical values of the probabilities, mean, and variance for program `galileo` in Example 4.2.1.

Exercise 4.2.2   (*a*) Generate the 2000-ball histogram in Example 4.2.2. (*b*) Verify that the resulting relative frequencies $\hat{f}(x)$ satisfy the equation

$$\hat{f}(x) \cong \frac{2^x \exp(-2)}{x!} \qquad x = 0, 1, 2, \ldots$$

(*c*) Then generate the corresponding histogram if 10 000 balls are placed, at random, in 1000 boxes. (*d*) Find an equation that seems to fit the resulting relative frequencies well and illustrate the quality of the fit.

Exercise 4.2.3   The gap in the histogram in Figure 4.2.2 in Example 4.2.2 corresponding to no boxes having exactly eight balls brings up an important topic in Monte Carlo and discrete-event simulation: the generation of *rare events*. To use the probabilist's terminology, there is poorer precision for estimating the probability of events out in the fringes (or "tails") of a distribution. (*a*) In theory, could all of the 2000 balls be in just one of the boxes? If so, give the probability of this event. If not, explain why not. (*b*) In practice, with our random number generator `rng`, could all of the 2000 balls be in one of the boxes?

Exercise 4.2.4[a]   Find the theoretical probabilities required to generate the right-hand plot in Figure 4.2.2.

Exercise 4.2.5   Program `ddh` computes the *histogram* mean and standard deviation. If you were to modify this program so that it also computes the *sample* mean and standard deviation using Welford's algorithm you would observe that (except perhaps for floating-point round-off errors) these two ways of computing the mean and standard deviation produce identical results. (*a*) If, instead, Algorithm 4.2.1 were used to compute the histogram mean and standard deviation, would they necessarily agree exactly with the sample mean and standard deviation? (*b*) Why or why not?

Exercise 4.2.6   Although the output of program `ddh` is sorted by `value`, this is only a convention. As an alternative, modify program `ddh` so that the output is sorted by `count` (in decreasing order).

Exercise 4.2.7[a]   Use simulation to explore the "efficiency increasing" modifications suggested in Example 4.2.4. That is, either of these modifications will make the algorithm more complicated for what may be, in return, only a marginal increase in efficiency. Because of its quadratic complexity beware of simply using a modified version of the function `Sort` in this application.

Exercise 4.2.8   How do the inventory level histograms in Examples 4.2.6 and 4.2.7 relate to the relative frequency of setups $\bar{u}$?

Exercise 4.2.9[a]   Generate a random variate demand sample of size $n = 10\,000$ as

$$d_i \text{ = Equilikely(5, 25) + Equilikely(5, 25);} \qquad i = 1, 2, \dots, n$$

(*a*) Why is the sample mean 30? (*b*) Why is the the sample standard deviation $\sqrt{220/3}$? (*c*) Why is the shape of the histogram triangular?

Exercise 4.2.10   A discrete-data histogram of orders is illustrated (corresponding to the demand distribution in Example 4.2.6). Based on this histogram, *by inspection* estimate the histogram mean and standard deviation.
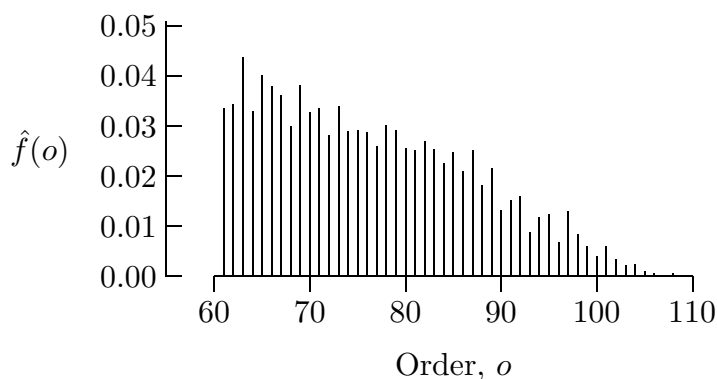
Exercise 4.2.11[a]   A test is compiled by selecting 12 different questions, at random and without replacement, from a well-publicized list of 120 questions. After studying this list you are able to classify all 120 questions into two classes, I and II. Class I questions are those about which you feel confident; the remaining questions define class II. Assume that your grade probability, conditioned on the class of the problems, is

	A	B	C	D	F
class I	0.6	0.3	0.1	0.0	0.0
class II	0.0	0.1	0.4	0.4	0.1

Each test question is graded on an $A = 4$, $B = 3$, $C = 2$, $D = 1$, $F = 0$ scale and a score of 36 or better is required to pass the test. (*a*) If there are 90 class I questions in the list, use Monte Carlo simulation and 100\,000 replications to generate a discrete-data histogram of scores. (*b*) Based on this histogram what is the probability that you will pass the test?

Exercise 4.2.12[a]   Modify program `ssq2` so that each time a new job arrives the program outputs the number of jobs in the service node *prior* to the job's arrival. (*a*) Generate a discrete-data histogram for 10\,000 jobs. (*b*) Comment on the shape of the histogram and compare the histogram mean with $\bar{l}$. (*c*) How does $\hat{f}(0)$ relate to $\bar{x}$?

As in the previous section, we assume a sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ with n sufficiently large that it is reasonable to do more than just calculate the sample mean and standard deviation. In sharp contrast to the discrete-data situation in the previous section, however, we now consider continuous-data histograms where the data values $x_1, x_2, \ldots, x_n$ are assumed to be real-valued and generally distinct.

4.3.1 CONTINUOUS-DATA HISTOGRAMS

Given a real-valued sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, without loss of generality we can assume the existence of real-valued lower and upper bounds a, b with the property that

$$a \leq x_i < b \qquad i = 1, 2, \ldots, n.$$

This defines an interval of possible values for some random variable X as $\mathcal{X} = [a, b) = \{x \mid a \leq x < b\}$, that can be partitioned into k equal-width *bins* (k is a positive integer) as

$$[a, b) = \bigcup_{j=0}^{k-1} \mathcal{B}_j = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{k-1},$$

where the bins are $\mathcal{B}_0 = [a, a + \delta)$, $\mathcal{B}_1 = [a + \delta, a + 2\delta)$, ... and the width of each bin is

$$\delta = \frac{b - a}{k}$$

as illustrated on the axis below.
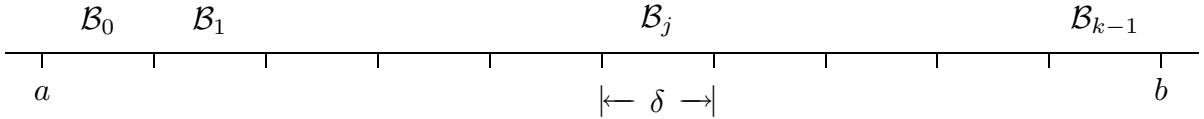
Definition 4.3.1   Given the sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ and the related parameters a, b, and either k or δ, for each $x \in [a, b)$ there is a unique bin $\mathcal{B}_j$ with $x \in \mathcal{B}_j$. The estimated *density* of the random variable X is then

$$\hat{f}(x) = \frac{\text{the number of } x_i \in \mathcal{S} \text{ for which } x_i \in \mathcal{B}_j}{n\,\delta} \qquad a \leq x < b.$$

A *continuous-data histogram* is a "bar" plot of $\hat{f}(x)$ versus x.*

As the following example illustrates, $\hat{f}(\cdot)$ is a piecewise constant function (constant over each bin) with discontinuities at the histogram bin boundaries. Since simulations tend to produce large data sets, we adopt the graphics convention illustrated in Example 4.3.1 of drawing $\hat{f}(\cdot)$ as a sequence of piecewise constant horizontal segments connected by vertical lines. This decision is consistent with maximizing the "data-to-ink" ratio (see Tufte, 2001).

---

* Compare Definition 4.3.1 with Definition 4.2.1. The bin index is $j = \lfloor (x - a)/\delta \rfloor$. The *density* is the relative frequency of the data in bin $\mathcal{B}_j$ *normalized* via a division by δ.

Example 4.3.1   As an extension of Example 4.1.1, a modified version of program `buf-fon` was used to generate a random variate sample of $n = 1000$ observations of the x-coordinate of the righthand endpoint of a unit-length needle dropped at random. To form a continuous-data histogram of the sample, the values $a = 0.0$ and $b = 2.0$ are, in this case, obvious choices for lower and upper bounds.* The number of histogram bins was selected, somewhat arbitrarily, as $k = 20$ so that $\delta = (b - a)/k = 0.1$. The resulting histogram is illustrated on the left side of Figure 4.3.1.

Figure 4.3.1.
Histograms
for $k = 20$.
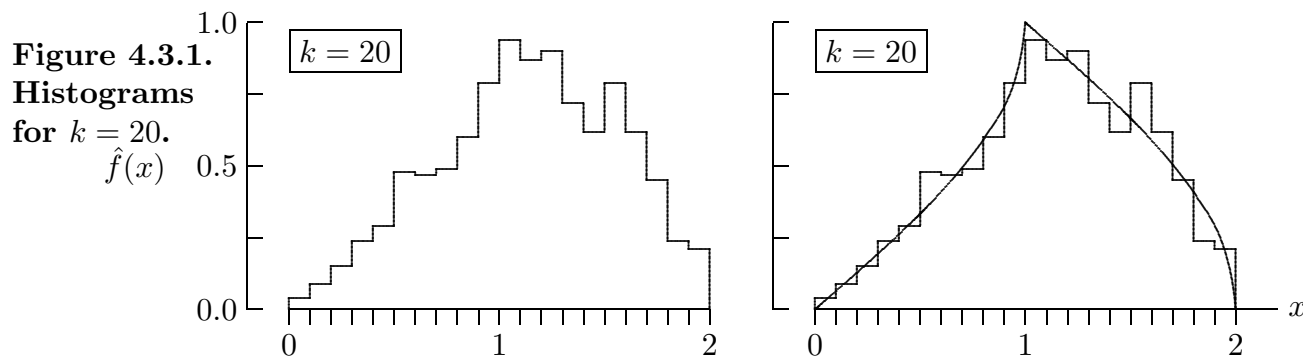
Illustrated on the right is the same histogram with a continuous curve superimposed. As discussed later in Chapter 7, this curve represents the *probability density function* of the righthand endpoint, which is the limit to which the histogram will converge as the sample size (n) approaches infinity *and* simultaneously k approaches infinity or, equivalently, δ approaches zero.

Histogram Parameter Guidelines

There is an experimental component to choosing the continuous-data histogram parameters a, b, and either k or δ. These guidelines are certainly not rigid rules.

- For data produced by a (valid) simulation there should be few, if any, outliers. That is, the bounds a, b should be chosen so that few, if any, data points in the sample are excluded. Of course, as in the discrete-data case, prior knowledge of reasonable values for a, b may not always be easily obtained.

- If k is too large (δ is too small) then the histogram will be too "noisy" with the potential of exhibiting false features caused by natural sampling variability; if k is too small (δ is too large) then the histogram will be too "smooth" with the potential of masking significant features — see Example 4.3.2.

- The histogram parameters should always be chosen with the aesthetics of the resulting figure in mind, e.g., for $a = 0$ and $b = 2$, if n is sufficiently large a choice of $k = 20$ ($\delta = 0.1$) would be a better choice than $k = 19$ ($\delta \cong 0.10526$).

- Typically $\lfloor \log_2(n) \rfloor \le k \le \lfloor \sqrt{n} \rfloor$ with a bias toward $k \cong \lfloor (5/3)\sqrt[3]{n} \rfloor$ (Wand, 1997).

- Sturges's rule (Law and Kelton, 2000, page 336) suggests $k \cong \lfloor 1 + \log_2 n \rfloor$.

---

* Because the needle has unit length, $0.0 < x_i < 2.0$ for $i = 1, 2, \ldots, n$.

Example 4.3.2   As a continuation of Example 4.3.1, two additional histograms are illustrated in Figure 4.3.2 corresponding to $k = 10$ ($\delta = 0.2$) on the left and to $k = 40$ ($\delta = 0.05$) on the right. The histogram on the right is clearly too noisy consistent with a choice of k that violates the second histogram parameter guideline (i.e., k is too large relative to n). Although this characterization is less clear, the histogram on the left may be too smooth because k is too small. For this sample, the best choice of k seems to be somewhere between 10 and 20 (using the last of the histogram guidelines, $9 \leq k \leq 31$ and $k \cong \lfloor (5/3)\sqrt[3]{1000} \rfloor = 16$). Since simulations typically produce large numbers of observations, and therefore many histogram bins are required, we avoid the common practice of dropping the vertical lines to the horizontal axis which visually partitions the bins. Including these lines unnecessarily clutters the histogram and obscures the shape of the histogram, particularly for large k.

Figure 4.3.2. Histograms for $k = 10$ and $k = 40$.
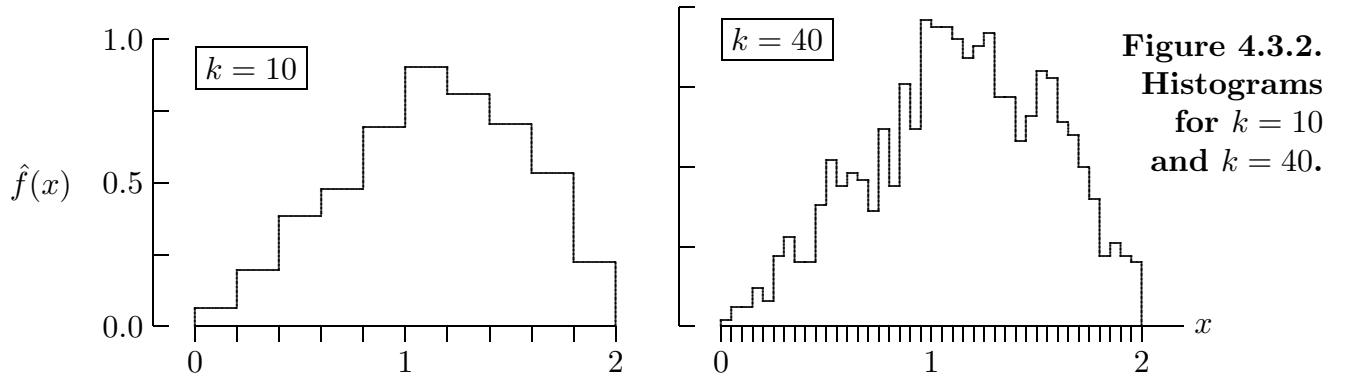
Histogram Integrals

Definition 4.3.2   As an extension of Definition 4.3.1, for each $j = 0, 1, \ldots, k - 1$ define p_j as the *relative frequency* of points in $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ that fall into bin $\mathcal{B}_j$. Since the bins form a partition of $[a, b)$ each point in $\mathcal{S}$ is counted exactly once (assuming no outliers) and so $p_0 + p_1 + \cdots + p_{k-1} = 1$. In addition, define the *bin midpoints*

$$m_j = a + \left(j + \frac{1}{2} \right) \delta \qquad j = 0, 1, \ldots, k - 1$$

as illustrated on the axis below.
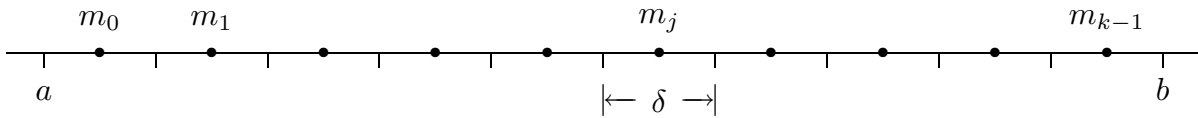
It follows from Definitions 4.3.1 and 4.3.2 that

$$p_j = \delta \hat{f}(m_j) \qquad j = 0, 1, \ldots, k - 1$$

and that $\hat{f}(\cdot)$ is a non-negative function with unit area. That is

$$\int_a^b \hat{f}(x)\, dx = \sum_{j=0}^{k-1} \int_{\mathcal{B}_j} \hat{f}(x)\, dx = \sum_{j=0}^{k-1} \hat{f}(m_j) \int_{\mathcal{B}_j} dx = \sum_{j=0}^{k-1} \left(\frac{p_j}{\delta} \right) \delta = \sum_{j=0}^{k-1} p_j = 1.$$

In addition to proving that $\int_a^b \hat{f}(x)\,dx = 1$, the previous derivation can be extended to the two integrals

$$\int_a^b x\hat{f}(x)\,dx \qquad \text{and} \qquad \int_a^b x^2\hat{f}(x)\,dx.$$

That is, for the first of these two integrals,

$$\int_a^b x\hat{f}(x)\,dx = \sum_{j=0}^{k-1} \int_{\mathcal{B}_j} x\hat{f}(x)\,dx = \sum_{j=0}^{k-1} \hat{f}(m_j) \int_{\mathcal{B}_j} x\,dx = \sum_{j=0}^{k-1} \left(\frac{p_j}{\delta}\right) \int_{\mathcal{B}_j} x\,dx$$

and in analogous fashion, for the second integral

$$\int_a^b x^2\hat{f}(x)\,dx = \cdots = \sum_{j=0}^{k-1} \left(\frac{p_j}{\delta}\right) \int_{\mathcal{B}_j} x^2\,dx.$$

In this way, because $\hat{f}(\cdot)$ is piecewise constant, the two integrals over $[a,b]$ are reduced to simple polynomial integration over each histogram bin. In particular, for $j = 0, 1, \ldots, k-1$,

$$\int_{\mathcal{B}_j} x\,dx = \int_{m_j - \delta/2}^{m_j + \delta/2} x\,dx = \frac{(m_j + \delta/2)^2 - (m_j - \delta/2)^2}{2} = \cdots = m_j\delta,$$

so that the first integral reduces to

$$\int_a^b x\hat{f}(x)\,dx = \sum_{j=0}^{k-1} \left(\frac{p_j}{\delta}\right) \int_{\mathcal{B}_j} x\,dx = \sum_{j=0}^{k-1} m_j\,p_j.$$

Similarly

$$\int_{\mathcal{B}_j} x^2\,dx = \frac{(m_j + \delta/2)^3 - (m_j - \delta/2)^3}{3} = \cdots = m_j^2\delta + \frac{\delta^3}{12}$$

so that the second integral reduces to

$$\int_a^b x^2\hat{f}(x)\,dx = \sum_{j=0}^{k-1} \left(\frac{p_j}{\delta}\right) \int_{\mathcal{B}_j} x^2\,dx = \left(\sum_{j=0}^{k-1} m_j^2\,p_j\right) + \frac{\delta^2}{12}$$

Therefore, the two integrals $\int_a^b x\hat{f}(x)\,dx$ and $\int_a^b x^2\hat{f}(x)\,dx$ can be evaluated *exactly* by finite summation. This is significant because the continuous-data histogram sample mean and sample standard deviation are defined in terms of these two integrals.

Histogram Mean and Standard Deviation

Definition 4.3.3   Analogous to the discrete-data equations in Definition 4.2.2 (replacing $\sum$'s with $\int$'s), the *continuous-data histogram mean* and *standard deviation* are defined as

$$\bar{x} = \int_a^b x \hat{f}(x)\, dx \qquad \text{and} \qquad s = \sqrt{\int_a^b (x - \bar{x})^2 \hat{f}(x)\, dx}.$$

The continuous-data histogram variance is s^2.

The integral equations in Definition 4.3.3 provide the motivation for defining the population mean and standard deviation of continuous random variables in Chapter 7. From the integral equations derived previously it follows that a continuous-data histogram mean can be evaluated exactly by finite summation with the equation

$$\bar{x} = \sum_{j=0}^{k-1} m_j\, p_j.$$

Moreover,

$$s^2 = \int_a^b (x - \bar{x})^2 \hat{f}(x)\, dx = \cdots = \left(\int_a^b x^2 \hat{f}(x)\, dx \right) - \bar{x}^2$$

and, similarly,

$$\sum_{j=0}^{k-1} (m_j - \bar{x})^2\, p_j = \cdots = \left(\sum_{j=0}^{k-1} m_j^2\, p_j \right) - \bar{x}^2.$$

From these last two equations it follows that a continuous-data histogram standard deviation can be evaluated exactly by finite summation with either of the two equations*

$$s = \sqrt{\left(\sum_{j=0}^{k-1} (m_j - \bar{x})^2\, p_j \right) + \frac{\delta^2}{12}} \qquad \text{or} \qquad s = \sqrt{\left(\sum_{j=0}^{k-1} m_j^2\, p_j \right) - \bar{x}^2 + \frac{\delta^2}{12}}.$$

In general the continuous-data *histogram* mean and standard deviation will differ slightly from the *sample* mean and standard deviation, even if there are no outliers. This difference is caused by the *quantization error* associated with the arbitrary binning of continuous data. In any case this difference should be slight — if the difference is not slight then the histogram parameters a, b, and either k or δ should be adjusted. Although the histogram mean and standard deviation are inferior to the sample mean and standard deviation, there are circumstances where a data analyst is presented with binned data and does not have access to the associated raw data.

---

* There is some disagreement in the literature relative to these two equations. Many authors use these equations, with the $\delta^2/12$ term ignored, to *define* the continuous-data histogram standard deviation.

Example 4.3.3   For the 1000-point sample in Example 4.3.1, using $a = 0.0$, $b = 2.0$, $k = 20$, the difference between the sample and histogram statistics is slight:

	raw data	histogram	histogram with $\delta = 0$
$\bar{x}$	1.135	1.134	1.134
s	0.424	0.426	0.425

Moreover, by comparing the last two columns in this table we see that, in this case, there is essentially no impact of the $\delta^2/12 = (0.1)^2/12 = 1/1200$ term in the computation of the histogram standard deviation.

4.3.2   COMPUTATIONAL CONSIDERATIONS

Algorithm 4.3.1   Given the parameters a, b, k, and the real-valued data $x_1, x_2, \ldots$, this algorithm computes a continuous-data histogram.

```
long count[k];
δ = (b - a) / k;
n = 0;
for (j = 0; j < k; j++)
   count[j] = 0;                               /* initialize bin counters */
outliers.lo = 0;              /* initialize outlier counter on (-∞,a) */
outliers.hi = 0;               /* initialize outlier counter on [b,∞) */
while ( more data ) {
   x = GetData();
   n++;
   if ((a <= x) and (x < b)) {
      j = (long) (x - a) / δ;
      count[j]++;                    /* increment appropriate bin counter */
   }
   else if (a > x)
      outliers.lo++;
   else
      outliers.hi++;
}
return n, count[], outliers                 /* p_j is (count[j] / n) */
```

The previously derived summation equations for $\bar{x}$ and s can then be used to compute the histogram mean and standard deviation.

If the sample is written to a disk file (using sufficient floating-point precision), then one can experiment with different values for the continuous-data histogram parameters in an *interactive* graphics environment. Consistent with this, program cdh illustrates the construction of a continuous-data histogram for data read from a text file. For an alternative approach see Exercise 4.3.7.

Example 4.3.4   As in Example 4.1.7, a modified version of program `ssq2` was used to generate a sample consisting of the waits $w_1, w_2, \ldots, w_n$ experienced by the first $n = 1000$ jobs. The simulation was initialized to simulate steady-state and the `rng` initial seed was 12345. Program `cdh` was used to process this sample with the continuous-data histogram parameters set to $(a, b, k) = (0.0, 30.0, 30)$, as illustrated in Figure 4.3.3.
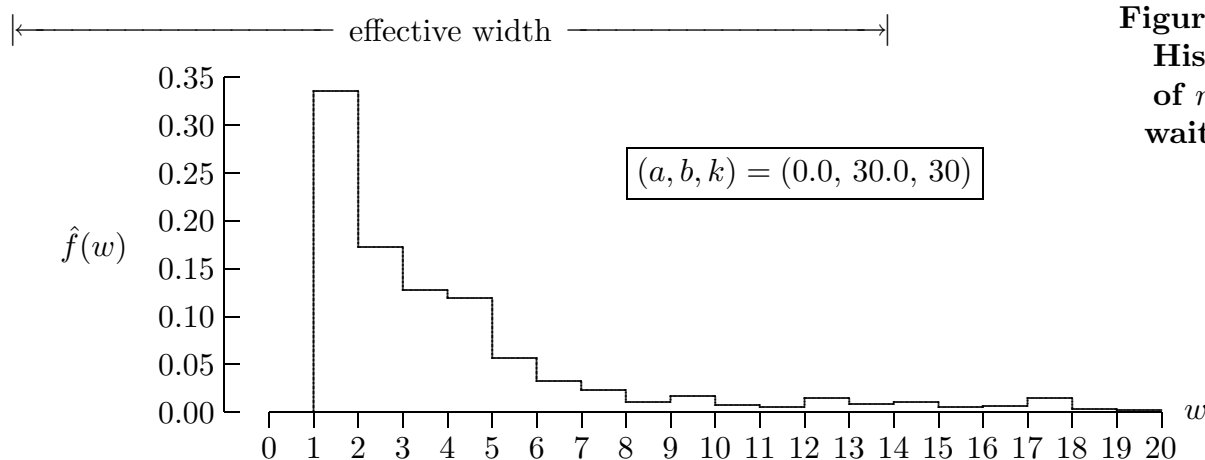
Figure 4.3.3. Histogram of $n = 1000$ wait times.

The histogram mean is 4.57 and the histogram standard deviation is 4.65. The associated two-standard deviation "effective width" of the sample is 4.57 ± 9.30, as illustrated. Consistent with previous examples this interval includes most of the points in the sample.*

Point Estimation

The issue of sampling variability and how it relates to the uncertainty of any probability estimate derived from a Monte Carlo simulation was considered in the previous section. This issue is sufficiently important to warrant reconsideration here in the context of continuous-data histograms. In particular, recall that for Example 4.2.8 there was a sample $\mathcal{S} = \{p_1, p_2, \ldots, p_n\}$ of $n = 1000$ point estimates of the probability of winning at the game of craps. One figure in Example 4.2.8 corresponds to probability estimates based on 25 plays of the game per estimate; the other figure is based on 100 plays per estimate. For both figures, the samples were displayed as discrete-data histograms and we were primarily interested in studying how the width of the histograms decreased with an increase in the number of games per estimate.

If we were to study the issue of how the histogram width depends on the number of games in more detail, it would be natural to quadruple the number of games per estimate to 400, 1600, 6400, etc. As the number of games increases, the associated discrete-data histograms will look more continuous as the histogram spikes get closer together. Given that, it is natural to ignore the inherently discrete nature of the probability estimates and, instead, treat the sample as though it were continuous data. That is what is done in the following example.

---

* The interval 4.57 ± 9.30 includes approximately 93% of the sample. There are 21 points in the sample with a value larger than 20.0, the largest of which is 28.2.

Example 4.3.5   The sample of $n = 1000$ probability estimates based on $N = 25$ plays of the game of craps from Example 4.2.8 was processed as a continuous-data histogram with parameters $(a, b, k) = (0.18, 0.82, 16)$. This choice of parameters is matched to the "resolution" of the estimates (which is $\delta = 0.04$) with the center of each histogram bin corresponding to exactly one possible value of an estimate.*

Figure 4.3.4. Histogram of $n = 1000$ estimates of winning at craps from $N = 25$ plays.
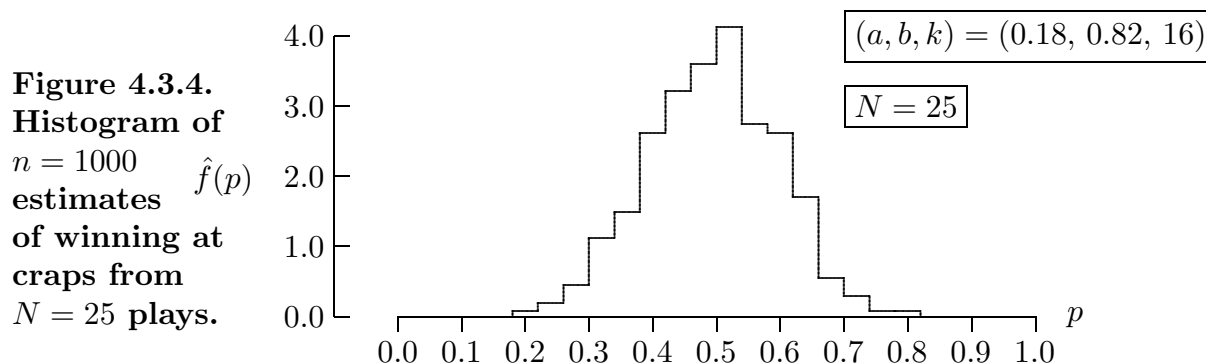
In a similar way the sample of $n = 1000$ probability estimates based on $N = 100$ plays of the game was processed as a continuous-data histogram with parameters $(a, b, k) = (0.325, 0.645, 16)$. This choice of parameters is matched to half the resolution of the estimates ($\delta = 0.02$) with the center of each histogram bin corresponding to the midpoint of exactly two possible values of an estimate, as illustrated in Figure 4.3.5.

Figure 4.3.5. Histogram of $n = 1000$ estimates of winning at craps from $N = 100$ plays.

----
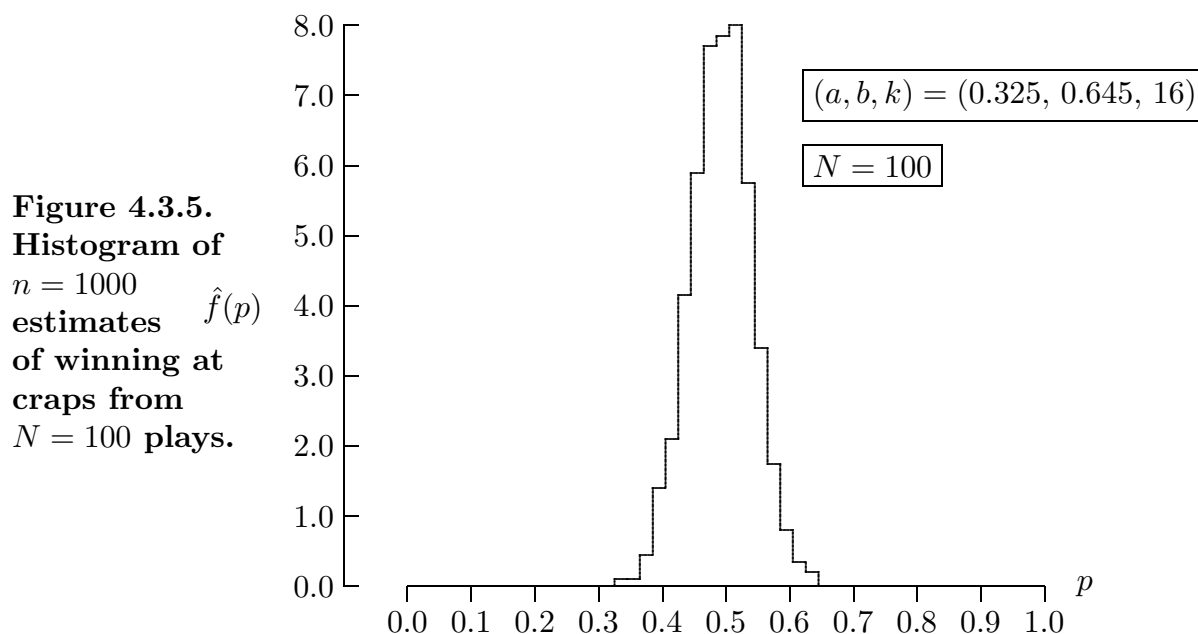
* Continuous-data histograms are *density* estimates, not *probability* estimates. Thus values of $\hat{f}(p)$ can be greater than 1.0 are possible, as in this example, but the histogram value can't stay at that height too long since $\int_0^1 \hat{f}(p)\, dp = 1$.

As in Example 4.2.8, we see that increasing the number of replications per estimate by a factor of *four* causes the uncertainty in any one probability estimate to decrease by a factor of *two*. The advantage to using a continuous-data histogram representation in this example is that experimentation with more games per estimate can be naturally accommodated. As the number of games per estimate is increased the histogram will become taller and narrower, always centered near the true value of $244/495 \cong 0.4929$, and always consistent with the invariant unit-area requirement $\int_0^1 \hat{f}(p)\, dp = 1$.

Random Events Yield Exponentially Distributed Inter-Events

Example 4.3.6 As another continuous-data histogram example, pick a positive parameter $t > 0$ and suppose that n calls to the function $\texttt{Uniform}(0, t)$ are used to generate a random variate sample of n events occurring *at random* in the interval $(0, t)$. If these n event times are then sorted into increasing order, the result is a sequence of events times $u_1, u_2, \ldots, u_n$ ordered so that $0 < u_1 < u_2 < \cdots < u_n < t$. With $u_0 = 0$ define

$$x_i = u_i - u_{i-1} > 0 \qquad\qquad i = 1, 2, \ldots, n,$$

as the inter-event times. Let $\mu = t/n$ and recognize that

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{u_n - u_0}{n} \cong \frac{t}{n} = \mu$$

so that the sample mean of the inter-event times is approximately μ. One might expect that a histogram of the inter-event times will be approximately "bell-shaped" and centered at μ. As illustrated in Figure 4.3.6, however, that is *not* the case — the histogram has an *exponential* shape.* In particular, the smallest inter-event times are the most likely.

$$t = 2000$$
$$n = 1000$$
$$\mu = 2.0$$

$$(a, b, k) = (0.0, 17.0, 34)$$
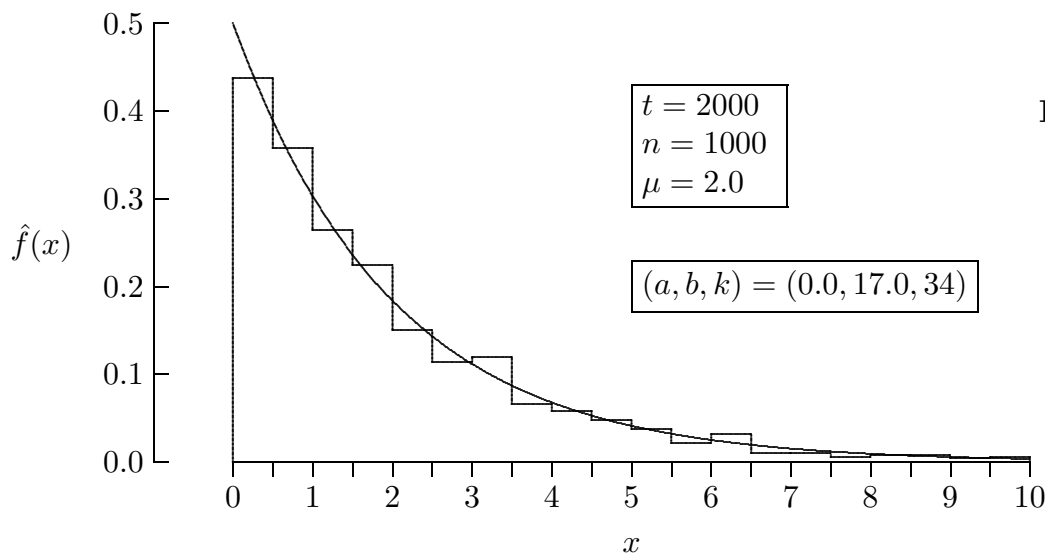
**Figure 4.3.6.
Histogram
of $n = 1000$
inter-event
times.**

---

* The histogram actually has an even longer tail than that illustrated in Figure 4.3.6. There are eight data points in the sample (out of 1000) with a value larger than 10.

The continuous curve superimposed illustrates that in this case $\hat{f}(x) \cong f(x)$ where

$$f(x) = \frac{1}{\mu} \exp(-x/\mu) \qquad x > 0.$$

Indeed, in the limit as $n \to \infty$ and $\delta \to 0$ (with μ held constant) $\hat{f}(x) \to f(x)$ for all $x > 0$.* We will return to this important example in Chapter 7.

4.3.3 EMPIRICAL CUMULATIVE DISTRIBUTION FUNCTIONS

The fact that parameters, such as the number of bins k, must be chosen by the modeler is a distinct drawback for continuous-data histograms. Two different data analysts could have the extreme misfortune of choosing different binning schemes for the same data set and produce histograms with somewhat different shapes. This is particularly true if the sampling variability inherent in the data set conspires with the two different binning schemes to accentuate the difference between the two histograms.

An alternative approach to plotting continuous data that avoids arbitrary parameters from the modeler is known as the *empirical cumulative distribution function*.

Definition 4.3.4   Given the sample $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, the estimated *cumulative distribution function* of the random variable X is

$$\hat{F}(x) = \frac{\text{the number of } x_i \in \mathcal{S} \text{ for which } x_i \leq x}{n}.$$

The *empirical cumulative distribution function* is a plot of $\hat{F}(x)$ versus x.

When $x_1, x_2, \ldots, x_n$ are distinct, the plot of $\hat{F}(x)$ versus x is a step function with an upward step of $1/n$ at each data value. On the other hand, if d values are tied at a particular x-value, the height of the riser on that particular step is d/n.

The empirical cumulative distribution function requires no parameters from the modeler, which means that one data set always produces the same empirical cumulative distribution function.

We now compare the computational complexity and memory requirements of the two graphical procedures. The continuous-data histogram algorithm performs a single-pass through the data values, with time complexity $O(n)$ and requires k memory locations. Plotting the empirical cumulative distribution function requires a sort, with time complexity $O(n \log n)$ at best, and all data values must be stored simultaneously, requiring n memory locations.

---

* See, for example, Rigdon and Basu (2000, pages 50–52) for the details concerning the relationship between the uniform and exponential distributions in this case.

Example 4.3.7 Consider again the modified version of program `buffon` that was used to generate a random variate sample of $n = 50$ observations of the x-coordinate of the righthand endpoint of a unit-length needle dropped at random using an initial seed of 123456789. The empirical cumulative distribution function is plotted in Figure 4.3.6 using our choice among the four plotting formats displayed for discrete data in Figure 4.2.8. There is an upward step of 1/50 at each of the values generated. The minimum and maximum values generated by the program are 0.45688 and 1.79410, respectively, which correspond to the horizontal position of the first and last step of the plot of the empirical cumulative distribution function. In this example it is possible to compute the theoretical cumulative distribution function using the axiomatic approach to probability. This is the smooth curve superimposed in Figure 4.3.7. The difference between the step function and the smooth curve is due to random sampling variability.
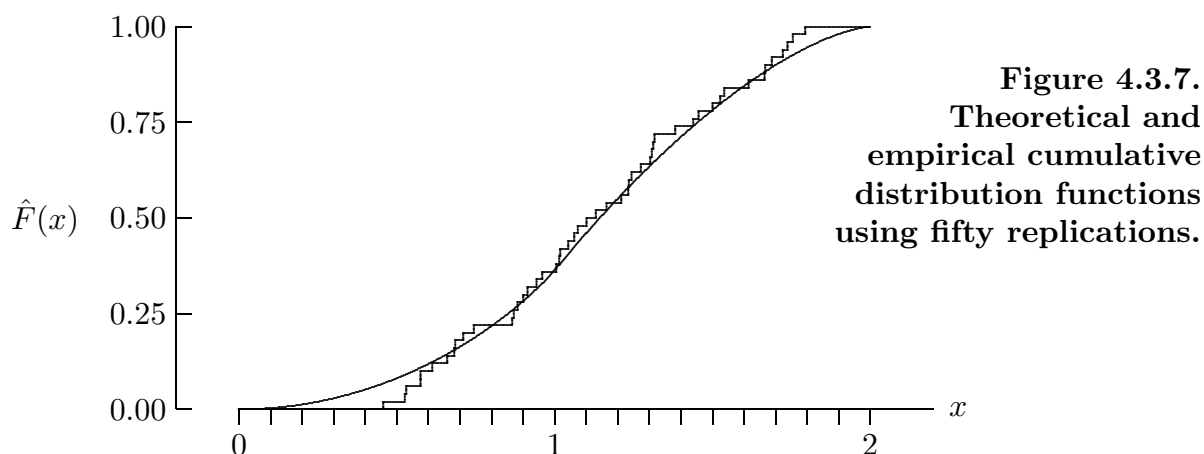
Figure 4.3.7. Theoretical and empirical cumulative distribution functions using fifty replications.

How does one compare the advantages and disadvantages of the continuous-data histogram and empirical cumulative distribution function? The histogram is clearly superior at detecting the *shape* of the distribution of the random quantity of interest. The arbitrary parameters associated with binning are its only downside. Selecting the continuous-data histogram parameters is more of an art than a science, which drives us to an alternative. The empirical cumulative distribution function is nonparametric, and thus less susceptible to the effects of sampling variability since there is no binning. Unfortunately, its shape is less distinct than the continuous-data histogram. It is often used to compare a hypothesized or fitted distribution to a data set using a statistical "goodness-of-fit" test.*

Increased CPU speeds makes generating large data sets possible in simulation, putting a strain on the memory and speed associated with plotting an empirical cumulative distribution function. Fortunately this is typically only done once during a simulation run.

We end this section with an example which combines some of the best features of continuous-data histograms and empirical cumulative distribution functions.

---

* The Kolmogorov–Smirnov, Anderson–Darling, and Cramer–von Mises are three well-known statistical goodness-of-fit tests for continuous data.

Example 4.3.8 If the sample size from the previous example ($n = 50$) were dramatically increased to $n = 1\,000\,000\,000$, one would expect the empirical cumulative distribution function to become very smooth and approximate the theoretical curve. We plot an empirical cumulative distribution function in order to take advantage of its nonparametric nature. Unfortunately, plotting the empirical cumulative distribution function requires that we store and sort the one billion righthand needle endpoints. Since each data value lies on $0 \le x \le 2$, we can create a close approximation to the empirical cumulative distribution function by defining, for instance, 200 equal-width cells on $0 \le x \le 2$, i.e., $[0, 0.01), [0.01, 0.02), \ldots, [1.99, 2.00)$.* Counts associated with these cells are accumulated, and plot of the cumulative proportions associated with these cells should be virtually identical to a plot from the raw data. The cells take advantage of the fact that the counts can be updated as the data is generated, eliminating the need for storage and sorting. The plot of the cumulative proportions shown in Figure 4.3.8 is much smoother than the plot for $n = 50$ in Figure 4.3.7 because of the huge number of replications. The difference between the plot in Figure 4.3.8 and the true cumulative distribution function could only be apparent to someone with a powerful microscope!

**Figure 4.3.8.
Approximate
empirical
cumulative
distribution
function
with one
billion
replications.**
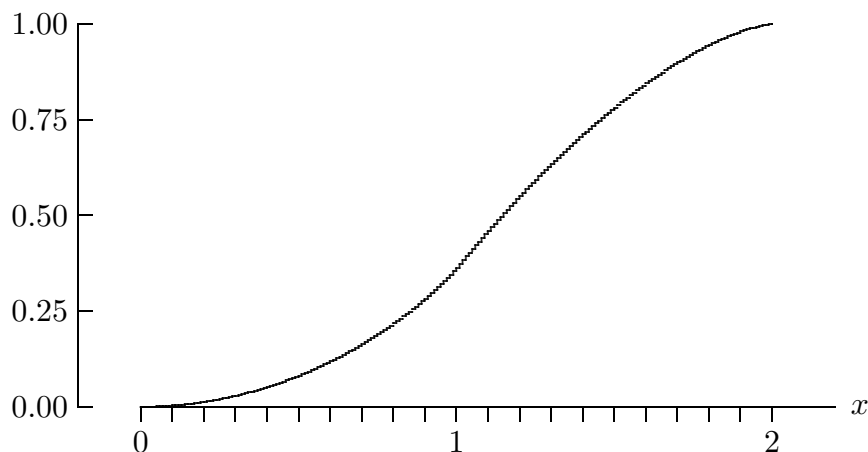
4.3.4   EXERCISES

Exercise 4.3.1   (*a*) Use program `cdh` to construct a continuous-data histogram like the one on the left in Example 4.3.1, but corresponding to a needle of length $r = 2$. (*b*) Based on this histogram what is the probability that the needle will cross at least one line. (*c*) What is the corresponding axiomatic probability that a needle of length $r = 2$ will cross at least one line?

---

* The number of cells chosen in this example (200) is arbitrary. The formulas given in Section 4.3.1 for choosing the number of histogram cells do not apply here. The choice depends on the physical size of the plot, the dsired smoothness, and the sample size.

Exercise 4.3.2   Repeat the experiment in Example 4.3.6 with $t = 5000$ and $n = 2000$. Do not use a bubble sort.

Exercise 4.3.3   Fill in the $= \cdots =$'s in the derivation of the two equations

$$\int_a^b x \hat{f}(x)\, dx = \cdots = \sum_{j=0}^{k-1} m_j\, p_j \qquad \text{and} \qquad \int_a^b x^2 \hat{f}(x)\, dx = \cdots = \left(\sum_{j=0}^{k-1} m_j^2\, p_j \right) + \frac{\delta^2}{12}.$$

Exercise 4.3.4   Generate a random variate sample $x_1, x_2, \ldots, x_n$ of size $n = 10\,000$ as follows:

```
for (i = 1; i <= n; i++)
    x_i = Random() + Random();
```

(*a*) Use program cdh to construct a 20-bin continuous-data histogram. (*b*) Can you find an equation that seems to fit the histogram density well?

Exercise 4.3.5   (*a*) As a continuation of Exercise 1.2.6, construct a continuous-data histogram of the service times. (*b*) Compare the *histogram* mean and standard deviation with the corresponding *sample* mean and standard deviation and justify your choice of the histogram parameters a, b, and either k or δ.

Exercise 4.3.6   As an extension of Definition 4.3.1, the *cumulative* histogram density is defined as

$$\hat{F}(x) = \int_a^x \hat{f}(t)\, dt \qquad a \le x < b.$$

Derive a finite summation equation for $\hat{F}(x)$.

Exercise 4.3.7[a]   To have more general applicability, program cdh needs to be restructured to support file redirection, like programs uvs and ddh. That part is easy. The ultimate objective here, however, should be a large-sample "auto-cdh" program that buffers, say, the first 1000 sample values and then automatically computes good values for the histogram parameters a, b, and either k or δ with a dynamic data structure allowance for additional bins to be added at the tails of the histogram and thereby avoid outliers if extreme values occur. (*a*) Construct such a program. (*b*) Discuss the logic you used for computing good values for the histogram parameters.

Exercise 4.3.8[a]   Show that the theoretical cumulative distribution function superimposed over the empirical cumulative distribution function in Figure 4.3.7 is

$$F(x) = \begin{cases} \dfrac{2(x\,\arcsin(x) + \sqrt{1 - x^2} - 1)}{\pi} & 0 < x < 1 \\[3mm] \dfrac{2(1 - x)\,\arcsin(x - 1) - 2\sqrt{x\,(2 - x)} + \pi x}{\pi} & 1 < x < 2. \end{cases}$$

4.4 Correlation

In a discrete-event simulation model it is often the case that two random variables are "co-related". That is, the natural variation in these two variables is somehow coupled. The statistical term for this is *correlation.* For example, in a FIFO single-server service node there is (positive) correlation between a job's wait in the service node and the wait time of the preceding job. This section examines two types of correlation: *paired* and *serial.*

4.4.1 PAIRED CORRELATION

Given a *paired sample* (u_i, v_i) for $i = 1, 2, \ldots, n$, we begin by considering how to compute a statistic that measures the extent to which this sample exhibits correlation.

Definition 4.4.1   A display of the paired data (u_i, v_i) for $i = 1, 2, \ldots, n$ as illustrated in Figure 4.1.1 is called a *bivariate scatterplot.*

**Figure 4.4.1.
Bivariate
scatterplot.**
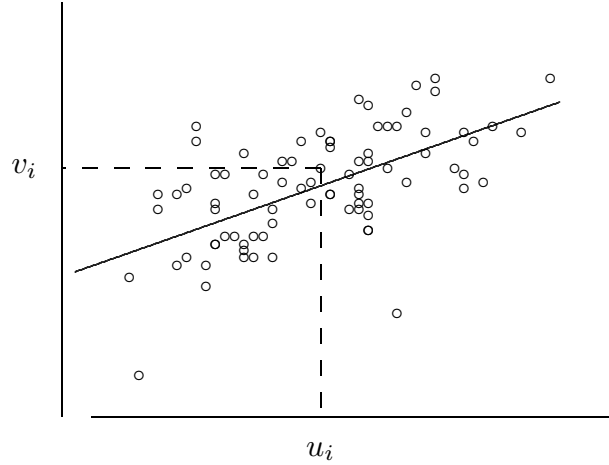
When a bivariate scatterplot is created, it is sometimes the case that the (u_i, v_i) points lie primarily clustered around a line, as is the case in Figure 4.4.1. If this is the case then it is natural to ask what line "best fits" the scatterplot data. It is in this sense that what we are discussing is called *linear* correlation. Consider the line in the (u, v) plane defined by the equation $au + bv + c = 0$ and for each point (u_i, v_i), let d_i be the *orthogonal* distance from this point to the line, as illustrated below.

$$d_i = \frac{|au_i + bv_i + c|}{\sqrt{a^2 + b^2}}$$

From calculus recall that the equation for d_i is as indicated. Thus we choose the (a, b, c) line parameters that *minimize* the mean-square orthogonal distance

$$D = \frac{1}{n} \sum_{i=1}^{n} d_i^2 = \frac{1}{n(a^2 + b^2)} \sum_{i=1}^{n} (au_i + bv_i + c)^2.$$

Three-Parameter Minimization

To determine the line that best fits the data, or equivalently to determine the choice of the (a, b, c) line parameters that will minimize D, we begin with the observation that D can be written as*

$$D = \frac{1}{n(a^2 + b^2)} \sum_{i=1}^{n} \Big(a(u_i - \bar{u}) + b(v_i - \bar{v}) \Big)^2 + \frac{1}{n(a^2 + b^2)} \sum_{i=1}^{n} (a\,\bar{u} + b\,\bar{v} + c)^2$$

where

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i \qquad \text{and} \qquad \bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i$$

are the sample means of the u and v data respectively. Both terms in the equation for D are non-negative since they involve sums of squares. The first term is independent of c; the second term is not. Therefore, to minimize D the line parameter c should be chosen so as to minimize the second term. By inspection, the minimum value of the second term is zero and this is achieved by choosing c so that

$$a\,\bar{u} + b\,\bar{v} + c = 0,$$

which eliminates the second term. For any choice of the (a, b) line parameters, if $a\,\bar{u} + b\,\bar{v} + c = 0$ it follows that the line must pass through the point $(u, v) = (\bar{u}, \bar{v})$. In particular, this geometric property is true for the choice of the (a, b) line parameters that will minimize D. The three-parameter minimization problem has been reduced to a two-parameter minimization problem.

Two-Parameter Minimization

Because $c = -a\,\bar{u} - b\,\bar{v}$ the equation $au + bv + c = 0$ can be written equivalently as $a(u - \bar{u}) + b(v - \bar{v}) = 0$. Moreover, since the (a, b) line parameters cannot both be zero, we can simplify the equation for D by assuming that (a, b) are normalized so that $a^2 + b^2 = 1$. The following theorem summarizes this discussion.

Theorem 4.4.1 The line that best fits the data (u_i, v_i) for $i = 1, 2, \ldots, n$ in a mean-squared orthogonal distance sense is given by the equation

$$a(u - \bar{u}) + b(v - \bar{v}) = 0$$

where the (a, b) line parameters are chosen to minimize

$$D = \frac{1}{n} \sum_{i=1}^{n} \Big(a(u_i - \bar{u}) + b(v_i - \bar{v}) \Big)^2$$

subject to the constraint $a^2 + b^2 = 1$.

---

* The details of this derivation are left as an exercise.

Covariance and Correlation

We now re-write the algebraic expression for D in Theorem 4.4.1. We do this by introducing two important (related) statistical measures of how the u's and v's are "co-related".

Definition 4.4.2   Given the bivariate sample (u_i, v_i) for $i = 1, 2, \ldots, n$ the (linear) *sample covariance* is

$$c_{uv} = \frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v})$$

and, provided both s_u and s_v are not zero, the (linear) *sample correlation coefficient* is

$$r = \frac{c_{uv}}{s_u s_v}$$

where $\bar{u}$, $\bar{v}$,

$$s_u^2 = \frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})^2, \qquad \text{and} \qquad s_v^2 = \frac{1}{n} \sum_{i=1}^{n} (v_i - \bar{v})^2$$

are the sample means and sample variances of the u and v data respectively.*

As we will see, the correlation coefficient r measures the "spread" (dispersion) of the u, v data about the line that best fits the data. By using Definition 4.4.2 the expression for D can be written in terms of s_u^2, s_v^2, and r as

$$
\begin{aligned}
D &= \frac{1}{n} \sum_{i=1}^{n} \Big(a(u_i - \bar{u}) + b(v_i - \bar{v}) \Big)^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} \Big(a^2(u_i - \bar{u})^2 + 2ab(u_i - \bar{u})(v_i - \bar{v}) + b^2(v_i - \bar{v})^2 \Big) \\
&= \frac{a^2}{n} \left(\sum_{i=1}^{n} (u_i - \bar{u})^2 \right) + \frac{2ab}{n} \left(\sum_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v}) \right) + \frac{b^2}{n} \left(\sum_{i=1}^{n} (v_i - \bar{v})^2 \right) \\
&= a^2 s_u^2 + 2abr s_u s_v + b^2 s_v^2.
\end{aligned}
$$

Note that $r = 1$ if and only if $D = (as_u + bs_v)^2$ and that $r = -1$ if and only if $D = (as_u - bs_v)^2$. Therefore, if $|r| = 1$ then it is possible to choose (a, b) in such a way that $D = 0$. Indeed, as we will see, the following three conditions are equivalent

$$|r| = 1 \quad \Longleftrightarrow \quad D = 0 \quad \Longleftrightarrow \quad \text{\textit{all} the points } (u_i, v_i) \text{ lie on a line.}$$

---

* The covariance is a generalization of the variance in the sense that $c_{uu} = s_u^2$ and $c_{vv} = s_v^2$. Note that the covariance derives its "dimensions" from u and v. The correlation coefficient is dimensionless.

The covariance equation in Definition 4.4.2 is a two-pass expression. As in Section 4.1 it can be shown that an equivalent one-pass expression for the covariance is

$$c_{uv} = \frac{1}{n} \left(\sum_{i=1}^{n} u_i v_i \right) - \bar{u}\bar{v}.$$

The derivation is left as an exercise. We will consider the computational significance of this result later in the section.

One-Parameter Minimization

Let θ be the angle of the line $a(u - \bar{u}) + b(v - \bar{v}) = 0$ measured counterclockwise relative to the u-axis, as illustrated in Figure 4.4.2.
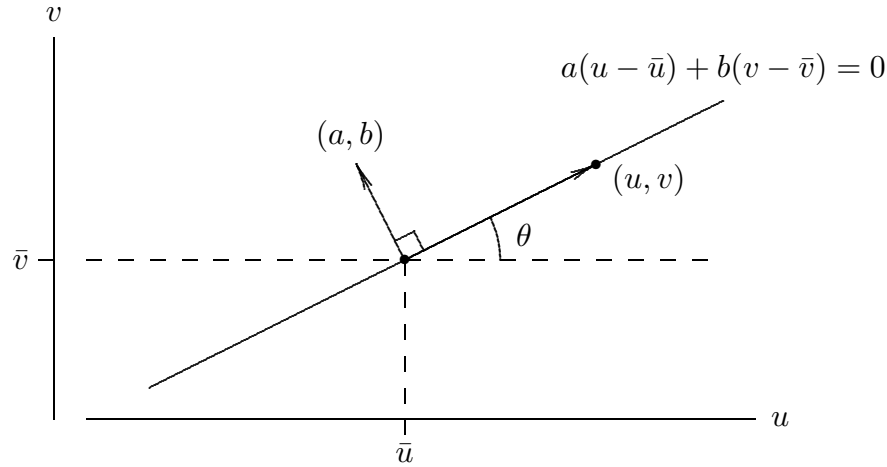
Figure 4.4.2. Defining the angle θ.

Because $a(u - \bar{u}) + b(v - \bar{v}) = 0$ and $a^2 + b^2 = 1$, the (a, b) line parameters form a unit vector that is *orthogonal* to the line. Therefore the relation between (a, b) and θ using elementary trigonometry is

$$a = -\sin\theta \qquad \text{and} \qquad b = \cos\theta.$$

Consistent with Theorem 4.4.1, minimizing D is accomplished by rotating the line about the point $(\bar{u}, \bar{v})$ to find the angle θ for which D is smallest. To find this angle, we can write D in terms of θ as

$$\begin{aligned}
D &= a^2 s_u^2 + 2ab c_{uv} + b^2 s_v^2 \\
&= s_u^2 \sin^2\theta - 2c_{uv} \sin\theta \cos\theta + s_v^2 \cos^2\theta \\
&= s_u^2(1 - \cos^2\theta) - c_{uv} \sin 2\theta + s_v^2 \cos^2\theta \\
&\ \ \vdots \\
&= \frac{1}{2}(s_u^2 + s_v^2) - c_{uv} \sin 2\theta - \frac{1}{2}(s_u^2 - s_v^2) \cos 2\theta.
\end{aligned}$$

The value of θ that minimizes

$$D = \frac{1}{2}(s_u^2 + s_v^2) - c_{uv} \sin 2\theta - \frac{1}{2}(s_u^2 - s_v^2) \cos 2\theta$$

solves the equation

$$\frac{dD}{d\theta} = -2c_{uv} \cos 2\theta + (s_u^2 - s_v^2) \sin 2\theta = 0.$$

This equation can be solved for $\tan 2\theta$ to yield

$$\tan 2\theta = \frac{2c_{uv}}{s_u^2 - s_v^2} \qquad (s_u \neq s_v)$$

as illustrated in Figure 4.4.3.

Figure 4.4.3. Finding the value of θ that minimizes D.
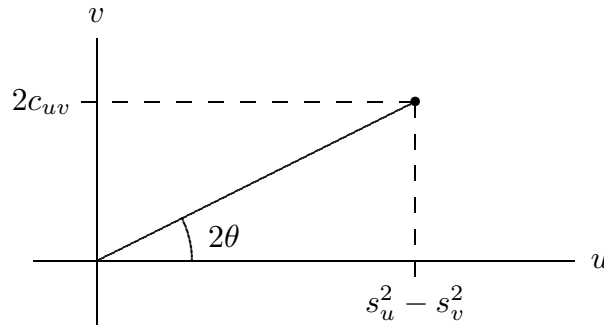
Therefore, the angle which minimizes D is

$$\theta = \frac{1}{2} \tan^{-1}(s_u^2 - s_v^2, \, 2c_{uv}),$$

where $\tan^{-1}(u, v)$ is the usual 4-quadrant inverse tangent function, measured counterclockwise from the positive u-axis.* This discussion is summarized by the following theorem.

Theorem 4.4.2   The line that best fits the data (u_i, v_i) for $i = 1, 2, \ldots, n$ in a mean-squared orthogonal distance sense passes through the point $(\bar{u}, \bar{v})$ at the angle

$$\theta = \frac{1}{2} \tan^{-1}(s_u^2 - s_v^2, \, 2c_{uv})$$

measured counterclockwise relative to the positive u-axis. The equation of the line is

$$v = (u - \bar{u}) \tan(\theta) + \bar{v}$$

provided $\theta \neq \pi/2$. (By convention $-\pi < \tan^{-1}(u, v) \leq \pi$ so that $-\pi/2 < \theta \leq \pi/2$.)

---

   * The function `atan2(v, u)` in the ANSI C library `<math.h>` represents the mathematical function $\tan^{-1}(u, v)$. Note the (u, v) switch.

Definition 4.4.3   The line that best fits the data (u_i, v_i) for $i = 1, 2, \ldots, n$ is known as the (mean-square orthogonal distance) *linear regression* line.*

As a corollary to Theorem 4.4.2, it can be shown that the smallest possible value of D is

$$2D_{\min} = (s_u^2 + s_v^2) - \sqrt{(s_u^2 - s_v^2)^2 + 4r^2 s_u^2 s_v^2},$$

which can be written equivalently as

$$D_{\min} = \frac{2(1 - r^2)s_u^2 s_v^2}{s_u^2 + s_v^2 + \sqrt{(s_u^2 - s_v^2)^2 + 4r^2 s_u^2 s_v^2}}.$$

The details of this derivation are left as an exercise. There are three important observations that follow immediately from this equation and the fact that $D_{\min}$ cannot be negative.

- The correlation coefficient satisfies the inequality $-1 \le r \le 1$.

- The closer $|r|$ is to 1, the smaller the dispersion of the (u, v) data about the regression line, and the better the (linear) fit.

- All the (u_i, v_i) points lie on the regression line if and only if $D_{\min} = 0$ or equivalently if and only if $|r| = 1$.

Example 4.4.1   The scatterplot in Definition 4.4.1, reproduced in Figure 4.4.4, corresponds to 82 student scores on two standardized tests of English verbal skills: the Test of English as a Foreign Language (TOEFL) and Graduate Record Examination (GRE). Although one might hope that the two test scores would be highly correlated with r close to 1, in this case $r = 0.59$. That is, consistent with the considerable scatter that is evident about the linear regression line, the correlation is not particularly high. The consistency between the two tests is certainly less than desirable.
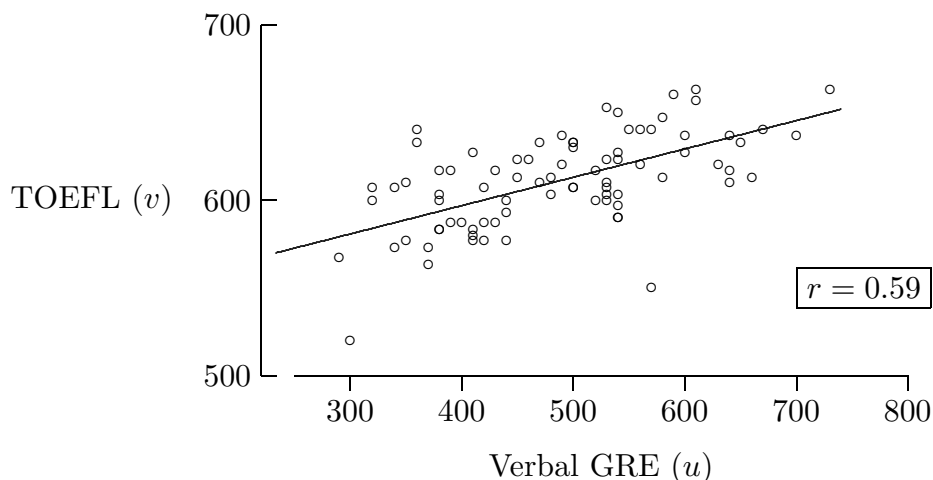
Figure 4.4.4. English verbal measures scatterplot.

* The derivation of an alternate mean-square *non-orthogonal* distance linear regression line is outlined in the exercises.

Significance

The magnitude of $|r|$ is a measure of the extent to which there is a *linear* relation between the u and v data. Associated terminology is provided by the following definition.

Definition 4.4.4   If $r \neq 0$ then the slope of the regression line is positive ($\theta > 0$) if and only if $r > 0$ and the slope of the regression line is negative ($\theta < 0$) if and only if $r < 0$. If r is close to $+1$ the data is said to be *positively correlated*. If r is close to -1 the data is said to be *negatively correlated*. If r is close to 0 then the data is said to be *uncorrelated*.

Example 4.4.2   A modified version of program `ssq2` was used to generate interarrival, service, delay, and wait times for a steady-state sample of 100 jobs passing through an $M/M/1$ service node with arrival rate 1.0 and service rate 1.25. An $M/M/1$ service node has *Exponential* interarrival time, *Exponential* service times, and a single server. By pairing these times, a total of six bivariate scatterplots could be formed, four of which are illustrated in Figure 4.4.5.
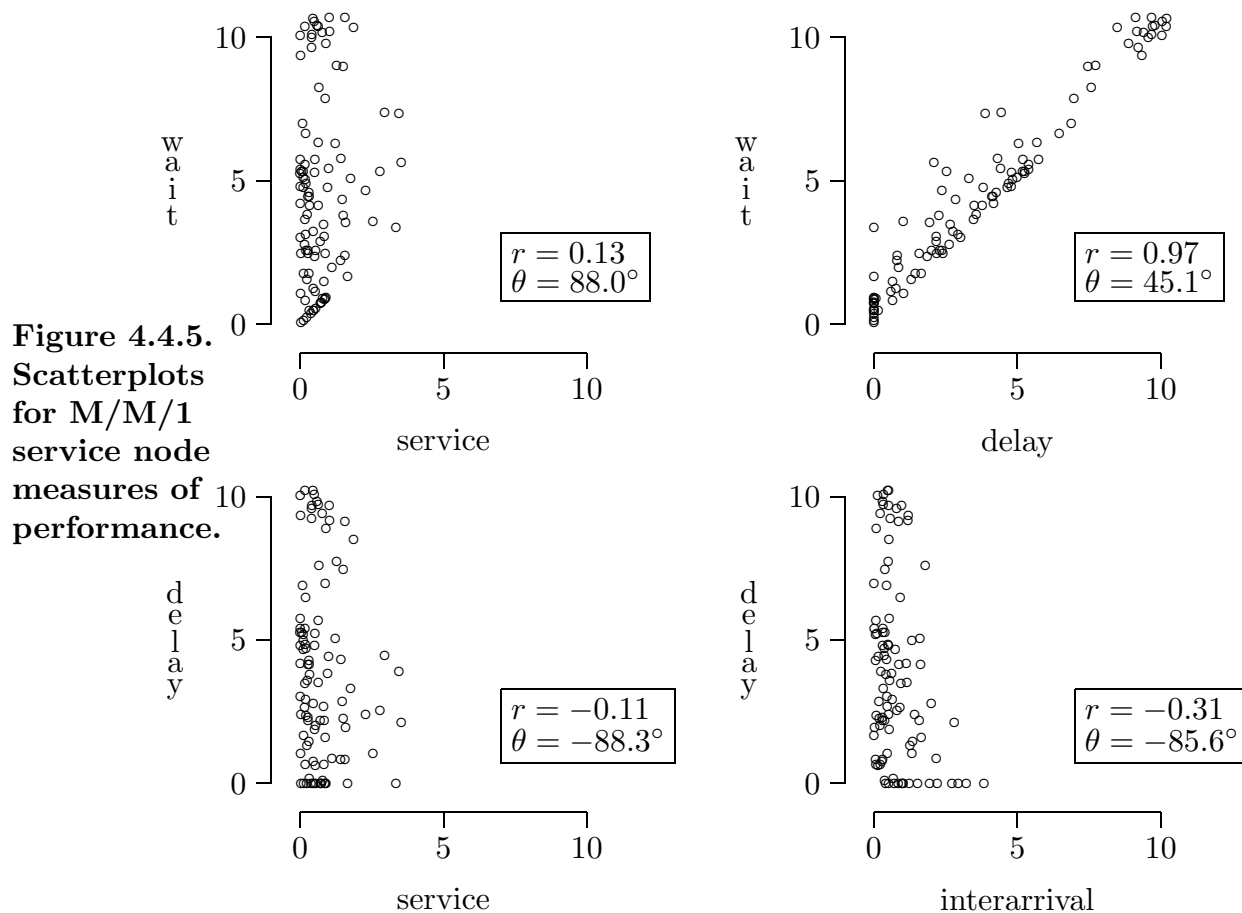
Figure 4.4.5. Scatterplots for M/M/1 service node measures of performance.

In this case we see, as expected, that the strongest positive correlation is between a job's delay and wait. For the other three pairings illustrated, the correlation is weak, if it is non-zero at all at all.

The statistical question of how far $|r|$ has to be from 0 to conclude that the bivariate data is actually correlated is a difficult one made more complicated by the fact that the decision depends on the sample size and the joint probability distribution of the two random variables. The smaller the sample size, the larger $|r|$ needs to be before one can safely conclude that the data is correlated.

If the sample size is small then the value of r is uncertain in the sense that another sample of the same size could produce a significantly different value of r. For example, for an $M/M/1$ service node, a job's delay and service time are *uncorrelated*. Why? Thus the weak $r = -0.11$ negative correlation indicated in Figure 4.4.5 is not statistically different from zero — another sample of size 100 will produce a *positive* correlation with probability $1/2$.

Computational Considerations

To return to the covariance equation in Definition 4.4.2, recall from Section 4.1 that there are two ways to calculate a variance (or standard deviation). One method involves *two* passes through the data, the first to evaluate the mean and then the second to sum the squares of the deviations about the mean. The other method involves just *one* pass through the data. An analogous result applies to the calculation of the covariance and, therefore, the correlation coefficient. Indeed, we have already observed that there are two mathematically equivalent expressions for the covariance

$$\underbrace{c_{uv} = \frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v})}_{\text{two-pass}} \qquad \text{and} \qquad \underbrace{c_{uv} = \frac{1}{n} \sum_{i=1}^{n} u_i v_i - \bar{u}\bar{v}}_{\text{one-pass}}.$$

For the same reasons discussed in Section 4.1, the one-pass algorithm is virtually always preferred in discrete-event simulation. Moreover, there is an extension to Welford's algorithm (Algorithm 4.1.2) that applies in this case, based on the following theorem.

Theorem 4.4.3 Let $\bar{u}_i$ and $\bar{v}_i$ denote the sample means of $u_1, u_2, \ldots, u_i$ and $v_1, v_2, \ldots, v_i$ respectively and define

$$w_i = (u_1 - \bar{u}_i)(v_1 - \bar{v}_i) + (u_2 - \bar{u}_i)(v_2 - \bar{v}_i) + \cdots + (u_i - \bar{u}_i)(v_i - \bar{v}_i)$$

for $i = 1, 2, \ldots, n$ where w_i/i is the covariance of the first i data pairs. Then, with the initial condition $w_0 = 0$,

$$w_i = w_{i-1} + \left(\frac{i-1}{i}\right)(u_i - \bar{u}_{i-1})(v_i - \bar{v}_{i-1}) \qquad i = 1, 2, \ldots, n$$

which provides a one-pass recursive algorithm to compute $c_{uv} = w_n/n$.

Program bvs

The program `bvs` is based upon the extended version of Welford's algorithm in Theorem 4.4.3. This program illustrates the calculation of the *bivariate* sample statistics $\bar{u}$, s_u, $\bar{v}$, s_v, r, and the linear regression line angle θ.

4.4.2   SERIAL CORRELATION

It is frequently the case that one is interested in the extent to which a set of data is *auto*-correlated (e.g., self-correlated). This is particularly true, for example, in a steady-state analysis of the waits experienced by consecutive jobs entering a service node. Intuitively, particularly if the utilization of the service node is high, there will be a high positive correlation between the wait w_i experienced by the ith job and the wait w_{i+1} experienced by the next job. Indeed, there will be a statistically significant positive correlation between w_i and w_{i+j} for some range of small, positive j values.

In general, let $x_1, x_2, \ldots, x_n$ be data which is presumed to represent n *consecutive* observations of some stochastic process whose serial correlation we wish to characterize. In the (u_i, v_i) notation used previously in this section, we pick a (small) fixed positive integer $j \ll n$ and then associate u_i with x_i and v_i with x_{i+j} as illustrated

$$
\begin{array}{ccccccccccc}
u & : & & & x_1 & x_2 & x_3 & \cdots & x_i & \cdots & x_{n-j} & x_{n-j+1} & \cdots & x_n \\
v & : & x_1 & \cdots & x_j & x_{1+j} & x_{2+j} & x_{3+j} & \cdots & x_{i+j} & \cdots & x_n
\end{array}
$$

The integer $j > 0$ is called the *autocorrelation lag* (or *shift*). Although the value $j = 1$ is generally of primary interest, it is conventional to calculate the serial correlation for a *range* of lag values $j = 1, 2, \ldots, k$ where $k \ll n$.*

Because of the lag we must resolve how to handle the "non-overlap" in the data at the beginning and end. The standard way to handle this non-overlap is to do the obvious — ignore the extreme data values. That is, define the sample autocovariance for lag j, based only on the $n - j$ overlapping values, as

$$
c_j = \frac{1}{n-j} \sum_{i=1}^{n-j} (x_i - \bar{x})(x_{i+j} - \bar{x}) \qquad\qquad j = 1, 2, \ldots, k,
$$

where the sample mean, based on all n values, is

$$
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.
$$

The associated autocorrelation is then defined as follows.

Definition 4.4.5   The *sample autocorrelation* for lag j is

$$
r_j = \frac{c_j}{c_0} \qquad\qquad j = 1, 2, \ldots, k
$$

where the sample variance is

$$
c_0 = s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2.
$$

---

* Serial statistics are commonly known as *auto*-statistics, e.g., *autocovariance* and *auto-correlation*.

Computational Considerations

The problem with the "obvious" definition of the sample autocovariance is that an implementation based on this definition would involve a two-pass algorithm. For that reason, it is common to use the following alternate definition of the sample autocovariance. Although this definition is *not* algebraically equivalent to the "obvious" definition, if $j \ll n$ then the numerical difference between these two autocovariance definitions is slight. Because it can be implemented as a one-pass algorithm, Definition 4.4.6 is preferred (in conjunction with Definition 4.4.5).

Definition 4.4.6   The *sample autocovariance* for lag j is

$$c_j = \left(\frac{1}{n-j} \sum_{i=1}^{n-j} x_i x_{i+j} \right) - \bar{x}^2 \qquad j = 1, 2, \ldots, k.$$

Example 4.4.3   A modified version of program `ssq2` was used to generate a sample of waits and services experienced by $10\,000$ consecutive jobs processed through an $M/M/1$ service node, in steady-state, with arrival rate 1.0, service rate 1.25, and utilization $1/1.25 = 0.8$. Definitions 4.4.5 and 4.4.6 were used to compute the corresponding sample autocorrelations r_j for $j = 1, 2, \ldots, 50$, in what is commonly known as an *sample autocorrelation function*, or *correlogram*, illustrated in Figure 4.4.6.
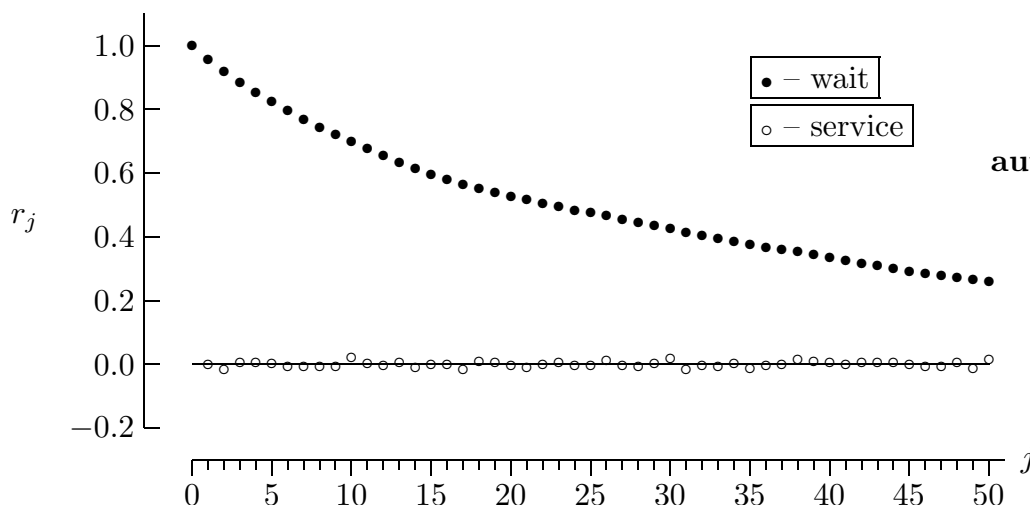
Figure 4.4.6. Sample autocorrelation functions for the M/M/1 queue.

As expected, the sample *wait* autocorrelation is positive and high for small values of j, indicating that each job's wait is strongly (auto)correlated with the wait of the next few jobs that follow. Also, as expected, the sample autocorrelation decreases monotonically toward zero as j increases. The rate of decrease may be slower than expected; if the utilization were smaller (larger), the rate of decrease would be higher (lower). It is quite surprising that the wait times of two jobs separated by 49 intervening jobs have a moderately strong positive correlation. Also, as expected, the sample *service* autocorrelation is essentially zero for all values of j, consistent with the stochastic independence of the service times.

We select the first three values of the the sample autocorrelation function for the wait times in order to interpret the magnitude of the r_j values. First, $r_0 = 1$ means that there is perfect correlation between each observation and itself, since the lag associated with r_0 is zero. The next sample autocorrelation value, $r_1 = 0.957$, indicates that adjacent (lag 1) jobs, such as job number 15 and job number 16, have a statistically significant strong positive correlation. If the 15th job has a long wait, then the 16th job is almost certain to also have a long wait. Likewise, if the 15th job has a short wait, then the 16th job is almost certain to also have a short wait. Anyone who has waited in a busy queue recognizes this notion intuitively. Finally, consider the estimated lag-two sample autocorrelation $r_2 = 0.918$. This autocorrelation is not quite a strong as the lag-one autocorrelation due to the increased temporal distance between the wait times. The positive value of r_2 indicates that wait times two jobs apart (e.g., the 29th and the 31st wait times) *tend* to be above the mean wait time together or below the mean wait time together.

Graphical Considerations

Several formats are common for displaying the sample autocorrelation function. In Figure 4.4.6, we plot the r_j values as points. Another common practice is to draw "spikes" from the horizontal axis to the $r_0, r_1, \ldots, r_k$ values. It is certainly *not* appropriate to connect the points to produce a piecewise-linear function. This would imply that r_j is defined for non-integer values of j — which it is not.

Statistical Considerations

The previous example indicated that jobs separated by 50 lags have wait times that are positively correlated. But how do we know that r_{50} differs significantly from 0. Leaving out the details, Chatfield (2004, page 56) indicates that an r_j value will fall outside the limits $\pm 2/\sqrt{n}$ with approximate probability 0.95 when the lag j values are uncorrelated. In the previous example with $n = 10\,000$, for instance, these limits are at ± 0.02, indicating the *all* of the wait time sample autocorrelation values plotted differ significantly from 0. For the service times, only $r_{10} = 0.022$ falls outside of these limits. Experience dictates that this is simply a function of random sampling variability rather than some relationship between service times separated by 10 jobs. We have set up our service time model with independent service times, so we expect a flat sample autocorrelation function for service times. The spurious value can be ignored.

The high autocorrelation that typically exists in the time-sequenced stochastic data produced by a simulation makes the statistical analysis of the data a challenge. Specifically, if we wish to make an *interval* estimate of some steady-state statistic like, for example, the average wait in a service node, we must be prepared to deal with the impact of autocorrelation on our ability to make an accurate estimates of the standard deviation. Most of so-called "classical" statistics relies on the assumption that the values sampled are drawn independently from a population. This is often not the case in discrete-event simulation and appropriate measures must be taken in order to compute appropriate interval estimates.

Program acs

To implement Definition 4.4.6 as a one-pass algorithm for a fixed lag-value j involves nothing more than storing the values x_i, x_{i+j}, accumulating the x_i sum, and accumulating the x_i^2 and $x_i x_{i+j}$ "cosums." It is a greater challenge to construct a one-pass algorithm that will compute c_j for a *range* of lags $j = 1, 2, \ldots, k$. In addition to the accumulation of the x_i sum, the simultaneous computation of $c_0, c_1, \ldots, c_k$ involves storing the $k + 1$ consecutive values $x_i, x_{i+1}, \ldots, x_{i+k}$ and accumulating the $k + 1$ (lagged) $x_i x_{i+j}$ cosums for $j = 0, 1, 2, \ldots, k$. The $k + 1$ cosums can be stored as an array of length $k + 1$. A more interesting queue data structure is required to store the values $x_i, x_{i+1}, \ldots, x_{i+k}$. This queue has been implemented as a *circular* array in the program `acs`. A circular array is a natural choice here because the queue length is fixed at $k+1$ and efficient access to *all* the elements in the queue, not just the head and tail, is required. In the following algorithm the box indicates the rotating head of the circular queue. An array index p keeps track of the current location of the rotating head; the initial value is $p = 0$.

Algorithm 4.4.1 Program `acs` is based on the following algorithm. A circular queue is initially filled with $x_1, x_2, \ldots, x_k, x_{k+1}$, as illustrated by the boxed elements below. The lagged products $x_1 x_{1+j}$ are computed for all $j = 0, 1, \ldots, k$ thereby initializing the $k + 1$ cosums. Then the next data value is read into the (old) head of the queue location, p is incremented by 1 to define a new head of the queue location, the lagged products $x_2 x_{2+j}$ are computed for all $j = 0, 1, \ldots, k$, and the cosums are updated. This process is continued until all the data has been read and processed. (The case $n \bmod (k+1) = 2$ is illustrated.)

$$
\begin{array}{llllllllll}
(i = k + 1) & \boxed{x_1} & x_2 & x_3 & \cdots & x_{k-1} & x_k & x_{k+1} & (p = 0) \\
(i = k + 2) & x_{k+2} & \boxed{x_2} & x_3 & \cdots & x_{k-1} & x_k & x_{k+1} & (p = 1) \\
(i = k + 3) & x_{k+2} & x_{k+3} & \boxed{x_3} & \cdots & x_{k-1} & x_k & x_{k+1} & (p = 2) \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
(i = 2k) & x_{k+2} & x_{k+3} & x_{k+4} & \cdots & x_{2k} & \boxed{x_k} & x_{k+1} & (p = k) \\
(i = 2k + 1) & x_{k+2} & x_{k+3} & x_{k+4} & \cdots & x_{2k} & x_{2k+1} & \boxed{x_{k+1}} & (p = k + 1) \\
(i = 2k + 2) & \boxed{x_{k+2}} & x_{k+3} & x_{k+4} & \cdots & x_{2k} & x_{2k+1} & x_{2k+2} & (p = 0) \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
(i = n) & x_{n-1} & x_n & \boxed{x_{n-k}} & \cdots & x_{n-4} & x_{n-3} & x_{n-2} & (p = 2)
\end{array}
$$

After the last data value, x_n, has been read, the associated lagged products computed, and the cosums updated, all that remains is to "empty" the queue. This can be accomplished by effectively reading k additional 0-valued data values. For more details, see program `acs`.

4.4.3   EXERCISES

Exercise 4.4.1   Prove that the orthogonal distance from the point (u_i, v_i) to the line $a\,u + b\,v + c = 0$ is in fact

$$d_i = \frac{|a\,u_i + b\,v_i + c|}{\sqrt{a^2 + b^2}}.$$

Hint: consider the squared distance $(u - u_i)^2 + (v - v_i)^2$ from (u_i, v_i) to a point (u, v) on the line and show that d_i^2 is the smallest possible value of this distance.

Exercise 4.4.2   (a) If $u'_i = \alpha_u u_i + \beta_u$ and $v'_i = \alpha_v v_i + \beta_v$ for $i = 1, 2, \ldots, n$ and constants α_u, α_v, β_u, and β_v how does the covariance of the u', v' data relate to the covariance of the u, v data? (b) Same question for the correlation coefficients? (c) Comment.

Exercise 4.4.3[a]   The orthogonal distance regression derivation presented in this section treats both variables equally — there is no presumption that one variable is "independent" and the other is "dependent." Consider the more common regression approach in which the equation of the regression line is $v = au + b$, consistent with a model that treats u as independent and v as dependent. That is, given the data (u_i, v_i) for $i = 1, 2, \ldots, n$ and the line defined by the equation $v = au + b$, the *conventional* (non-orthogonal) distance from the point (u_i, v_i) to the line is

$$\delta_i = |v_i - (au_i + b)|.$$

(a) What choice of the (a, b) parameters will minimize the conventional mean-square distance

$$\Delta = \frac{1}{n} \sum_{i=1}^{n} \delta_i^2 = \frac{1}{n} \sum_{i=1}^{n} (v_i - au_i - b)^2.$$

(b) Prove that the minimum value of Δ is $(1 - r^2)s_v^2$.

Exercise 4.4.4   Prove Theorem 4.4.3.

Exercise 4.4.5   To what extent are these two definitions of the autocovariance different?

$$c'_j = \frac{1}{n - j} \sum_{i=1}^{n-j} (x_i - \bar{x})(x_{i+j} - \bar{x}) \qquad \text{and} \qquad c_j = \left(\frac{1}{n - j} \sum_{i=1}^{n-j} x_i x_{i+j} \right) - \bar{x}^2$$

Exercise 4.4.6   (a) Generate a figure like the one in Figure 4.4.6 but corresponding to a utilization of 0.9. Do this for three different `rngs` streams. Comment. (b) Repeat for a utilization of 0.7. (You can ignore the service autocorrelations.) Comment.

Exercise 4.4.7   If Definition 4.4.6 is used in conjunction with Definition 4.4.5, there is no *guarantee* that $|r_j| \leq 1$ for all $j = 1, 2, \ldots, k$. If it is important to guarantee that this inequality is true, then how should the two definitions be modified?

CHAPTER 5

NEXT-EVENT SIMULATION

Sections

The three sections in this chapter all concern the *next-event* approach to discrete-event simulation. Section 5.1 defines the fundamental terminology used in next-event simulation such as system state, events, simulation clock, event scheduling, and event list (which is also known as the *calendar*), and provides an introduction to this fundamental approach as it applies to the simulation of a single-server service node with and without feedback. The algorithm associated with next-event simulation initializes the simulation clock (typically to time zero), event list (with an initial arrival, for example, in a queuing model), and system state to begin the simulation. The simulation model continues to (1) remove the next event from the event list, (2) update the simulation clock to the time of the next event, (3) process the event, and (4) schedule the time of occurrence of any future events spawned by the event, until some terminal condition is satisfied.

Section 5.2 provides further illustrations of this approach relative to the simulation of a simple inventory system with delivery lag and a multi-server service node. The multi-server service node provides an illustration of an event list which can have an arbitrarily large number of elements.

As the simulations in Sections 5.1 and 5.2 illustrate, an *event list* is an integral feature of the next-event approach. The data structures and algorithms that are used to manage the event list are crucial to the efficiency of a next-event simulation. Section 5.3 provides a sequence of examples associated with the management of an event list that begin with a naive and inefficient data structure and algorithm and iterate toward a more efficient scheme.

In this section we will present a general *next-event* approach to building discrete-event simulation models. From this chapter on, this next-event approach will be the basis for all the discrete-event simulation models developed in this book.

The motivation for considering the next-event approach to discrete-event simulation is provided by considering the relative complexity of the effort required to extend the discrete-event simulation models in Section 3.1 to accommodate the slightly more sophisticated corresponding models in Section 3.3. That is, at the computational level compare the simplicity of program `ssq2` in Section 3.1 with the increased complexity of the extension to `ssq2` that would be required to reproduce the results in Example 3.3.2. Yet the only increase in the complexity of the associated single-server service node model is the addition of immediate feedback. Similarly, compare the simplicity of program `sis2` in Section 3.1 with the increased complexity of the extension to `sis2` that would be required to reproduce the results in Example 3.3.4. Yet in this case the only increase in the complexity of the associated simple inventory system model is the addition of a delivery lag.

5.1.1 DEFINITIONS AND TERMINOLOGY

While programs `ssq2` and `sis2` and their corresponding extensions in Section 3.3 are valid and meaningful (albeit simple) discrete-event simulation programs, they do not adapt easily to increased model complexity and they do not generalize well to other systems. Based on these observations we see the need for a more general approach to discrete-event simulation that applies to queuing systems, inventory systems and a variety of other systems as well. This more general approach — next-event simulation — is based on some important definitions and terminology: (1) system state, (2) events, (3) simulation clock, (4) event scheduling, and (5) event list (calendar).

System State

Definition 5.1.1 The *state* of a system is a complete characterization of the system at an instance in time — a comprehensive "snapshot" in time. To the extent that the state of a system can be characterized by assigning values to variables, then *state variables* are what is used for this purpose.

To build a discrete-event simulation model using the next-event approach, the focus is on refining a description of the state of the system and its evolution in time. At the *conceptual* model level the state of a system exists only in the abstract as a collection of possible answers to the following questions: what are the state variables, how are they interrelated, and how do they evolve in time? At the *specification* level the state of the system exists as a collection of mathematical variables (the state variables) together with equations and logic describing how the state variables are interrelated and an algorithm for computing their interaction and evolution in time. At the *computational* level the state of the system exists as a collection of program variables that collectively characterize the system and are systematically updated as (simulated) time evolves.

Example 5.1.1   A natural way to describe the state of a single-server service node is to use the number of jobs in the service node as a state variable. As demonstrated later in this section, by refining this system state description we can construct a next-event simulation model for a single-server service node with or without immediate feedback.

Example 5.1.2   Similarly, a natural way to describe the state of a simple inventory system is to use the current inventory level and the amount of inventory on order (if any) as state variables. As demonstrated in the next section, by refining this system state description we can construct a next-event simulation model of a simple inventory system with or without delivery lag.

Events

Definition 5.1.2   An *event* is an occurrence that may change the state of the system. By definition, the state of the system can only change at an event time. Each event has an associated *event type.*

Example 5.1.3   For a single-server service node model with or without immediate feedback, there are two types of events: the *arrival* of a job and the *completion of service* for a job. These two types of occurrences are events because they have the potential to change the state of the system. An arrival will always increase the number in the service node by one; if there is no feedback, a completion of service will always decrease the number in the service node by one. When there is feedback, a completion *may* decrease the number in the service node by one. In this case there are two event types because the "arrival" event type and the "completion of service" event type are not the same.

Example 5.1.4   For a simple inventory system with delivery lag there are three event types: the occurrence of a *demand* instance, an *inventory review*, and the *arrival* of an inventory replenishment order. These are events because they have the potential to change the state of the system: a demand will decrease the inventory level by one, an inventory review may increase the amount of inventory on order, and the arrival of an inventory replenishment order will increase the inventory level and decrease the amount of inventory on order.

The *may* in Definition 5.1.2 is important; it is not necessary for an event to cause a change in the state of the system, as illustrated in the following four examples: (1) events can be scheduled that statistically *sample*, but do not change, the state of a system, (2) for a single-server service node with immediate feedback, a job's completion of service will only change the state of the system if the job is not fed back, (3) for a single-server service node, an event may be scheduled at a prescribed time (e.g., 5 PM) to cut off the stream of arriving jobs to the node, which will not change the state of the system, and (4) for a simple inventory system with delivery lag, an inventory review will only change the state of the system if an order is placed.

Simulation Clock

Because a discrete-event simulation model is dynamic, as the simulated system evolves it is necessary to keep track of the current value of simulated time. In the implementation phase of a next-event simulation, the natural way keep track of simulated time is with a floating point variable, which is typically named `t`, `time`, `tnow`, or `clock` in discrete-event simulation packages. The two examples that follow the definition of the simulation clock highlight the inability of the discrete-event simulation approach to easily generalize or embellish models. The next-event framework overcomes this limitation.

Definition 5.1.3   The variable that represents the current value of simulated time in a next-event simulation model is called the *simulation clock*.

Example 5.1.5   The discrete-event simulation model that program `ssq2` represents is heavily dependent on the job processing order imposed by the FIFO queue discipline. Therefore, it is difficult to extend the model to account for immediate feedback, or a finite service node capacity, or a priority queue discipline. In part, the reason for this difficulty is that there are effectively *two* simulation clocks with one coupled to the arrival events and the other coupled to the completion of service events. These two clocks are not synchronized and so it is difficult to reason about the temporal order of events if arrivals and completions of service are merged by feedback.

Example 5.1.6   The discrete-event simulation model that program `sis2` represents has only one type of event, inventory review, and events of this type occur deterministically at the beginning of each time interval. There is a simulation clock, but it is *integer-valued* and so is primitive at best. Because the simulation clock is integer-valued we are essentially forced to ignore the individual demand instances that occur within each time interval. Instead, all the demands per time interval are aggregated into one random variable. This aggregation makes for a computationally efficient discrete-event simulation program, but forces us in return to do some calculus to derive equations for the time-averaged holding and shortage levels. As outlined in Section 3.3, when there is a delivery lag the derivation of those equations is a significant task.

Event Scheduling

In a discrete-event simulation model it is necessary to use a *time-advance mechanism* to guarantee that events occur in the correct order — that is, to guarantee that the simulation clock never runs backward. The primary time-advance mechanism used in discrete-event simulation is known as *next-event* time advance; this mechanism is typically used in conjunction with *event scheduling*.

Definition 5.1.4   If event scheduling is used with a next-event time-advance mechanism as the basis for developing a discrete-event simulation model, the result is called a *next-event* simulation model.

To construct a next-event simulation model, three things must be done:

- construct a set of state variables that together provide a complete system description;

- identify the system event types;

- construct a collection of algorithms that define the state changes that will take place when each type of event occurs.

The model is constructed so as to cause the (simulated) system to evolve in (simulated) time by executing the events in increasing order of their scheduled time of occurrence. Time does not flow continuously; instead, the simulation clock is advanced discontinuously from event time to event time. At the computational level, the simulation clock is frozen during the execution of each state-change algorithm so that each change of state, no matter how computationally complex, occurs instantaneously relative to the simulation clock.

Event List

Definition 5.1.5   The data structure that represents the scheduled time of occurrence for the next possible event of each type is called the *event list* or *calendar*.

The event list is often, but not necessarily, represented as a priority queue sorted by the next scheduled time of occurrence for each event type.

5.1.2   NEXT-EVENT SIMULATION

Algorithm 5.1.1   A next-event simulation model consists of the following four steps:

- **Initialize.** The simulation clock is initialized (usually to zero) and, by looking ahead, the first time of occurrence of each possible event type is determined and scheduled, thereby initializing the event list.

- **Process current event.** The event list is scanned to determine the *most imminent* possible event, the simulation clock is then advanced to this event's scheduled time of occurrence, and the state of the system is updated to account for the occurrence of this event. This event is known as the "current" event.

- **Schedule new events.** New events (if any) that may be spawned by the current event are placed on the event list (typically in chronological order).

- **Terminate.** The process of advancing the simulation clock from one event time to the next continues until some terminal condition is satisfied. This terminal condition may be specified as a *pseudo*-event that only occurs once, at the end of the simulation, with the specification based on processing a fixed number of events, exceeding a fixed simulation clock time, or estimating an output measure to a prescribed precision.

The next-event simulation model initializes once at the beginning of a simulation replication, then alternates between the second step (processing the current event) and third step (scheduling subsequent events) until some terminate criteria is encountered.

Because event times are typically random, the simulation clock runs *asynchronously*. Moreover, because state changes *only* occur at event times, periods of system inactivity are ignored by advancing the simulation clock from event time to event time. Compared to the alternate, which is a *fixed-increment* time-advance mechanism, there is a clear computational efficiency advantage to this type of asynchronous next-event processing.*

In the remainder of this section, next-event simulation will be illustrated by constructing a next-event model of a single-server service node. Additional illustrations are provided in the next section by constructing next-event simulation models of a simple inventory system with delivery lag and a multi-server service node.

5.1.3   SINGLE-SERVER SERVICE NODE

The state variable $l(t)$ provides a *complete* characterization of the state of a single-server service node in the sense that

$$l(t) = 0 \iff q(t) = 0 \qquad \text{and} \qquad x(t) = 0$$
$$l(t) > 0 \iff q(t) = l(t) - 1 \qquad \text{and} \qquad x(t) = 1$$

where $l(t)$, $q(t)$, and $x(t)$ represent the number in the node, in the queue, and in service respectively at time $t > 0$. In words, if the number in the service node is known, then the number in the queue and the status (idle or busy) of the server is also known. Given that the state of the system is characterized by $l(t)$, we then ask what events can cause $l(t)$ to change? The answer is that there are two such events: (1) an *arrival* in which case $l(t)$ is increased by 1; and (2) a *completion of service* in which case $l(t)$ is decreased by 1. Therefore, our conceptual model of a single-server service node consists of the state variable $l(t)$ and two associated event types, arrival and completion of service.

To turn this next-event conceptual model into a specification model, three additional assumptions must be made.

- The *initial state* $l(0)$ can have any non-negative integer value. It is common, however, to assume that $l(0) = 0$, often referred to as "empty and idle" in reference to the initial queue condition and server status, respectively. Therefore, the first event must be an arrival.

---

* Note that asynchronous next-event processing cannot be used if there is a need at the computational level for the simulation program to interact synchronously with some other process. For example, because of the need to interact with a person, so called "real time" or "person-in-the-loop" simulation programs must use a fixed-increment time-advance mechanism. In this case the underlying system model is usually based on a system of ordinary differential equations. In any case, fixed-increment time-advance simulation models are outside the scope of this book, but are included in some of the languages surveyed in Appendix A.

- Although the *terminal state* can also have any non-negative integer value, it is common to assume, as we will do, that the terminal state is also idle. Rather than specifying the number of jobs processed, our stopping criteria will be specified in terms of a time τ beyond which no new jobs can arrive. This assumption effectively "closes the door" at time τ but allows the system to continue operation until all jobs have been completely served. This would be the case, for instance, at an ice cream shop that closes at a particular hour, but allows remaining customers to be served. Therefore, the last event must be a completion of service.*

- Some mechanism must be used to denote an event as *impossible*. One way to do this is to structure the event list so that it contains *possible* events only. This is particularly desirable if the number of event types is large. As an alternate, if the number of event types is not large then the event list can be structured so that it contains both possible and impossible events — but with a numeric constant "∞" used for an event time to denote the impossibility of an event. For simplicity, this alternate event list structure is used in Algorithm 5.1.2.

To complete the development of a specification model, the following notation is used. The next-event specification model is then sufficiently simple that we can write Algorithm 5.1.2 directly.

- The simulation clock (current time) is t.

- The terminal ("close the door") time is τ.

- The next scheduled arrival time is t_a.

- The next scheduled service completion time is t_c.

- The number in the node (state variable) is l.

The genius and allure of both discrete-event and next-event simulation is apparent, for example, in the generation of arrival times in Algorithm 5.1.2. The naive approach of generating and storing all arrivals prior to the execution of the simulation is not necessary. Even if this naive approach were taken, the modeler would be beset by the dual problems of memory consumption and not knowing how many arrivals to schedule. Next-event simulation simply primes the pump by scheduling the first arrival in the initialization phase, then schedules each subsequent arrival while processing the current arrival. Meanwhile, service completions weave their way into the event list at the appropriate moments in order to provide the appropriate sequencing of arrivals and service completions.

At the end of this section we will discuss how to extend Algorithm 5.1.2 to account for several model extensions: immediate feedback, alternative queue disciplines, finite capacity, and random sampling.

---

* The simulation will terminate at $t = \tau$ only if $l(\tau) = 0$. If instead $l(\tau) > 0$ then the simulation will terminate at $t > \tau$ because additional time will be required to complete service on the jobs in the service node.

Algorithm 5.1.2   This algorithm is a next-event simulation of a FIFO single-server service node with infinite capacity. The service node begins and ends in an empty and idle state. The algorithm presumes the existence of two functions `GetArrival` and `GetService` that return a random value of arrival time and service time respectively.

```
l = 0;                                    /* initialize the system state */
t = 0.0;                                  /* initialize the system clock */
t_a = GetArrival();                         /* initialize the event list */
t_c = ∞;                                    /* initialize the event list */
while ((t_a < τ) or (l > 0)) {       /* check for terminal condition */
  t = min(t_a, t_c);                             /* scan the event list */
  if (t == t_a) {                             /* process an arrival */
    l++;
    t_a = GetArrival();
    if (t_a > τ)
      t_a = ∞;
    if (l == 1)
      t_c = t + GetService();
  }
  else {                          /* process a completion of service */
    l--;
    if (l > 0)
      t_c = t + GetService();
    else
      t_c = ∞;
  }
}
```

If the service node is to be an M/M/1 queue (exponential interarrival and service times with a single server) with arrival rate 1.0 and service rate 1.25, for example, the two t_a = `GetArrival()` statements can be replaced with t_a = t + `Exponential(1.0)` and the two t_a = `GetService()` statements can be replaced with t_c = t + `Exponential(0.8)`. The `GetArrival` and `GetService` functions can draw their values from a file (a "trace-driven" approach) or generate variates to model these stochastic elements of the service node.

Because there are just two event types, arrival and completion of service, the event list in Algorithm 5.1.2 contains at most two elements, t_a and t_c. Given that the event list is small and its size is bounded (by 2), there is no need for any special data structure to represent it. If the event list were larger, an array or structure would be a natural choice. The only drawback to storing the event list as t_a and t_c is the need to specify the arbitrary numeric constant "∞" to denote the impossibility of an event. In practice, ∞ can be any number that is *much* larger than the terminal time τ (100τ is used in program `ssq3`).

If the event list is large and its size is dynamic then a dynamic data structure is required with careful attention paid to its organization. This is necessary because the event list is scanned and updated each time an event occurs. Efficient algorithms for the insertion and deletion of events on the event list can impact the computational time required to execute the next-event simulation model. Henriksen (1983) indicates that for telecommunications system models, the choice between an efficient and inefficient event list processing algorithm can produce a five-fold difference in total processing time. Further discussion of data structures and algorithms associated with event lists is postponed to Section 5.3.

Program ssq3

Program `ssq3` is based on Algorithm 5.1.2. Note, in particular, the state variable `number` which represents $l(t)$, the number in the service node at time t, and the important time management structure `t` that contains:

- the event list `t.arrival` and `t.completion` (t_a and t_c from Algorithm 5.1.2);

- the simulation clock `t.current` (t from Algorithm 5.1.2);

- the next event time `t.next` ($\min(t_a, t_c)$ from Algorithm 5.1.2);

- the last arrival time `t.last`.

Event list management is trivial. The event type (arrival or a completion of service) of the next event is determined by the statement `t.next = Min(t.arrival, t.completion)`.

Note also that a statistics gathering structure `area` is used to calculate the time-averaged number in the node, queue, and service. These statistics are calculated exactly by accumulating time integrals via summation, which is valid because $l(\cdot)$, $q(\cdot)$, and $x(\cdot)$ are piecewise constant functions and only change value at an event time (see Section 4.1). The structure `area` contains:

- $\displaystyle\int_0^t l(s)\,ds$ evaluated as `area.node`;

- $\displaystyle\int_0^t q(s)\,ds$ evaluated as `area.queue`;

- $\displaystyle\int_0^t x(s)\,ds$ evaluated as `area.service`.

Program `ssq3` does not accumulate job-averaged statistics. Instead, the job-averaged statistics $\bar{w}$, $\bar{d}$, and $\bar{s}$ are computed from the time-averaged statistics $\bar{l}$, $\bar{q}$, and $\bar{x}$ by using the equations in Theorem 1.2.1. The average interarrival time $\bar{r}$ is computed from the equation in Definition 1.2.4 by using the variable `t.last`. If it were not for the use of the assignment `t.arrival = INFINITY` to "close the door", $\bar{r}$ could be computed from the terminal value of `t.arrival`, thereby eliminating the need for `t.last`.

World Views and Synchronization

Programs `ssq2` and `ssq3` simulate exactly the same system. The programs work in different ways, however, with one clear consequence being that `ssq2` naturally produces *job*-averaged statistics and `ssq3` naturally produces *time*-averaged statistics. In the jargon of discrete-event simulation the two programs are said to be based upon different *world views.** In particular, program `ssq2` is based upon a *process-interaction* world view and program `ssq3` is based upon an *event-scheduling* world view. Although other world views are sometimes advocated, process interaction and event-scheduling are the two most common. Of these two, event-scheduling is the discrete-event simulation world view of choice in this and all the remaining chapters.

Because programs `ssq2` and `ssq3` simulate exactly the same system, these programs should be able to produce exactly the same output statistics. Getting them to do so, however, requires that both programs process exactly the same stochastic source of arriving jobs and associated service requirements. Because the arrival times a_i and service times s_i are ultimately produced by calls to `Random`, some thought is required to provide this synchronization. That is, the random variates in program `ssq2` are always generated in the alternating order a_1, s_1, a_2, s_2, ... while the order in which these random variates are generated in `ssq3` cannot be known a priori. The best way to produce this synchronization is to use the library `rngs`, as is done is program `ssq3`. In Exercise 5.1.3, you are asked to modify program `ssq2` to use the library `rngs` and, in that way, verify that the two programs can produce exactly the same output.

5.1.4 MODEL EXTENSIONS

We close this section by discussing how to modify program `ssq3` to accommodate several important model extensions. For each of the four extensions you are encouraged to consider what would be required to extend program `ssq2` correspondingly.

Immediate Feedback

Given the function `GetFeedback` from Section 3.3, we can modify program `ssq3` to account for immediate feedback by just adding an `if` statement so that `index` and `number` are not changed if a feedback occurs following a completion of service, as illustrated.

```
   else {                              /* process a completion of service */
     if (GetFeedback() == 0) {                  /* this statement is new */
       index++;
       number--;
     }
```

---

  * A world view is the collection of concepts and views that guide the development of a simulation model. World views are also known as *conceptual frameworks*, *simulation strategies*, and *formalisms*.

For a job that is not fed back, the counter for the number of departed jobs (`index`) is incremented and the counter for the current number of jobs in the service node (`number`) is decremented. The simplicity of the immediate feedback modification is a compelling example of how well next-event simulation models accommodate model extensions.

Alternate Queue Disciplines

Program `ssq3` can be modified to simulate *any* queue discipline. To do so, it is necessary to add a dynamic queue data structure such as, for example a singly-linked list where each list node contains the arrival time and service time for a job in the queue, as illustrated.*
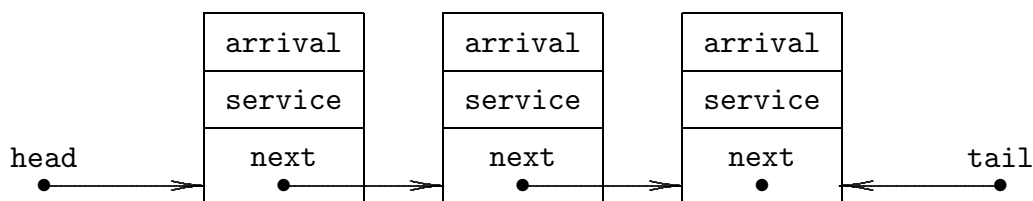
Figure 5.1.1. Queue data structure.

Two supporting queue functions, `Enqueue` and `Dequeue`, are also needed to insert and delete jobs from the queue, respectively. Then `ssq3` can be modified as follows:

- use `Enqueue` each time an arrival event occurs and the server is busy;

- use `Dequeue` each time a completion of service event occurs and the queue is not empty.

The details of this important modification are left as an exercise. This modification will result in a program that can be tested for correctness by using a FIFO queue discipline and reproducing results from program `ssq3`.

Note in particular that this modification can be combined with the immediate feedback modification illustrated previously. In this case, the `arrival` field in the linked list would hold the time of feedback for those jobs that are fed back. The resulting program would allow a priority queue discipline to be used for fed back jobs if (as is common) a priority assumption is appropriate.

Finite Service Node Capacity

Program `ssq3` can be modified to account for a finite capacity by defining a constant `CAPACITY` which represents the service node capacity (one more than the queue capacity) and declaring an integer variable `reject` which counts rejected jobs. Then all that is required is a modification to the "process an arrival" portion of the program, as illustrated.

---

* If this queue data structure is used, then service times are computed and stored at the time of arrival. In this way, each job's delay in the queue and wait in the service node can be computed at the time of entry into service, thereby eliminating the need to compute job-averaged statistics from time-averaged statistics.

```
      if (t.current == t.arrival) {                    /* process an arrival */
        if (number < CAPACITY) {
          number++;
          if (number == 1)
            t.completion = t.current + GetService();
        }
        else
          reject++;
        t.arrival     = GetArrival();
        if (t.arrival > STOP) {
          t.last       = t.current;
          t.arrival   = INFINITY;
        }
      }
```

This code replaces the code in program `ssq3` for processing an arrival. As with the immediate feedback modification, again we see the simplicity of this modification is a compelling example of how well next-event simulation models accommodate model extensions.

Random Sampling

An important feature of program `ssq3` is that its structure facilitates direct *sampling* of the current number in the service node or queue. This is easily accomplished by adding a sampling time element, say `t.sample`, to the event list and constructing an associated algorithm to process the samples as they are acquired. Sampling times can then be scheduled *deterministically*, every δ time units, or *at random* by generating sampling times with an *Exponential*(δ) random variate inter-sample time. In either case, the details of this modification are left as an exercise.

5.1.5   EXERCISES

Exercise 5.1.1   Consider a next-event simulation model of a three-server service node with a single queue and three servers. (*a*) What variable(s) are appropriate to describe the system state? (*b*) Define appropriate events for the simulation. (*c*) What is the maximum length of the event list for the simulation? (Answer with and without considering the pseudo-event for termination of the replication.)

Exercise 5.1.2   Consider a next-event simulation model of three single-server service nodes in series. (*a*) What variable(s) are appropriate to describe the system state? (*b*) Define appropriate events for the simulation. (*c*) What is the maximum length of the event list for the simulation? (Answer with and without considering the pseudo-event for termination of the replication.)

Exercise 5.1.3 (*a*) Use the library `rngs` to verify that programs `ssq2` and `ssq3` can produce *exactly* the same results. (*b*) Comment on the value of this as a consistency check for both programs.

Exercise 5.1.4 Add a sampling capability to program `ssq3`. (*a*) With deterministic inter-sample time $\delta = 1.0$, sample the number in the service node and compare the average of these samples with the value of $\bar{l}$ computed by the program. (*b*) With average inter-sample time $\delta = 1.0$, sample *at random* the number in the service node and compare the average of these samples with the value of $\bar{l}$ computed by the program. (*c*) Comment.

Exercise 5.1.5 Modify program `ssq3` by adding a FIFO queue data structure. Verify that this modified program and `ssq3` produce *exactly* the same results.

Exercise 5.1.6[a] As a continuation of Exercise 5.1.5, simulate a single-server service node for which the server uses a shortest-job-first priority queue discipline based upon a knowledge of the service time for each job in the queue. (*a*) Generate a histogram of the wait in the node for the first 10000 jobs if interarrival times are *Exponential*(1.0) and service times are *Exponential*(0.8). (*b*) How does this histogram compare with the corresponding histogram generated when the queue discipline is FIFO? (*c*) Comment.

Exercise 5.1.7 Modify program `ssq3` to reproduce the feedback results obtained in Section 3.3.

Exercise 5.1.8 Modify program `ssq3` to account for a finite service node capacity. (*a*) Determine the proportion of rejected jobs for capacities of 1, 2, 3, 4, 5, and 6. (*b*) Repeat this experiment if the service time distribution is *Uniform*(1.0, 3.0). (*c*) Comment. (Use a large value of `STOP`.)

Exercise 5.1.9 (*a*) Construct a next-event simulation model of a single-server machine shop. (*b*) Compare your program with the program `ssms` and verify that, with a proper use of the library `rngs`, the two programs can produce *exactly* the same output. (*c*) Comment on the value of this as a consistency check for both programs.

Exercise 5.1.10[a] An M/M/1 queue can be characterized by the following system state change mechanism, where the system state is $l(t)$, the number of jobs in the node:

- The transition from state j to state $j + 1$ is exponential with rate $\lambda_1 > 0$ (the arrival rate) for $j = 0, 1, \ldots$.

- The transition from state j to state $j - 1$ is exponential with rate $\lambda_2 > 0$ (the service rate) for $j = 1, 2, \ldots$.

If the current state of the system is some positive integer j, what is the probability that the next transition will be to state $j + 1$?

5.2 Next-Event Simulation Examples

As a continuation of the discussion in the previous section, in this section two next-event simulation models will be developed. The first is a next-event simulation model of a simple inventory system with delivery lag, the second is a next-event simulation model of a multi-server service node.

5.2.1   A SIMPLE INVENTORY SYSTEM WITH DELIVERY LAG

To develop a next-event simulation model of a simple inventory system with delivery lag, we make two changes relative to the model on which program sis2 is based. The first change is consistent with the discussion of delivery lag in Section 3.3. The second change is new and provides a more realistic demand model.

- There is a lag between the time of inventory review and the delivery of any inventory replenishment order that is placed at the time of review. This delivery lag is assumed to be a $Uniform(0, 1)$ random variable, independent of the size of the order. Consistent with this assumption, the delivery lag cannot be longer than a unit time interval; consequently, any order placed at the beginning of a time interval will arrive by the end of the time interval, before the next inventory review.

- The demands per time interval are no longer aggregated into one random variable and assumed to occur at a constant rate during the time interval. Instead, individual demand instances are assumed to occur *at random* throughout the simulated period of operation with an average rate of λ demand instances per time interval. That is, each demand instance produces a demand for exactly one unit of inventory and the inter-demand time is an $Exponential(1/\lambda)$ random variable.

Figure 5.2.1 shows the first six time intervals of a typical inventory level time history $l(t)$.

Figure 5.2.1.
Inventory
level
history.
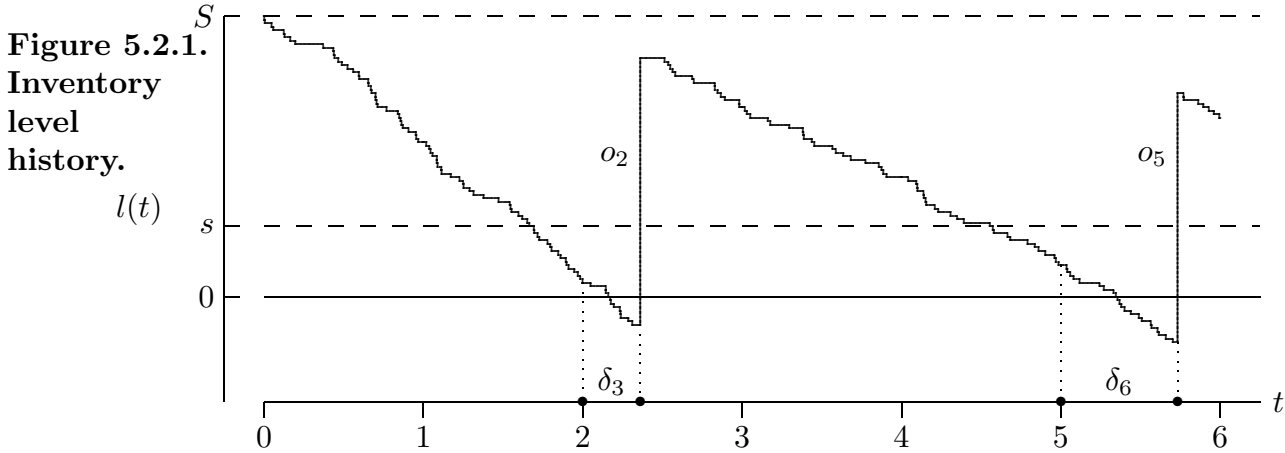
Each demand instance causes the inventory level to decrease by one. Inventory review for the ith time interval occurs at $t = i - 1 = 0, 1, 2, \ldots$ with an inventory replenishment order in the amount $o_{i-1} = S - l(i - 1)$ placed only if $l(i - 1) < s$. Following a delivery lag δ_i, the subsequent arrival of this order causes the inventory to experience an increase of o_{i-1} at time $t = i - 1 + \delta_i$, as illustrated for $i = 3$ and $i = 6$ in Figure 5.2.1.

Recall that in program `sis2` the aggregate demand in each time interval is generated as an *Equilikely*(10, 50) random variate. Although the aggregate demand in each time interval can be any value between 10 and 50, within each interval there is nothing random about the occurrence of the individual demands — the inter-demand time is constant. Thus, for example, if the random variate aggregate demand in a particular interval is 25 then the inter-demand time throughout that interval is 0.04.

In contrast to the demand model in program `sis2`, it is more realistic to generate the inter-demand time as an *Exponential*($1/\lambda$) random variate. In this way the demand is modeled as an arrival process (e.g., customers arriving at random to buy a car) with λ as the arrival rate *per time interval.* Thus, for example, if we want to generate demands with an average of 30 per time interval then we would use $\lambda = 30$.

States

To develop a next-event simulation model of this system at the specification level, the following notation is used.

- The simulation clock (current time) is t and the terminal time is τ.

- At any time $t > 0$ the current inventory level is $l(t)$.

- At any time $t > 0$ the amount of inventory *on order* (if any) is $o(t)$.

In addition to $l(t)$, the new state variable $o(t)$ is necessary to keep track of an inventory replenishment order that, because of a delivery lag, has not yet arrived. Together, $l(t)$ and $o(t)$ provide a complete state description of a simple inventory system with delivery lag. Both $l(t)$ and $o(t)$ are integer-valued. Although t is real-valued, inventory reviews occur at integer values of t only. The terminal time τ corresponds to an inventory review time and so it is integer-valued.

We assume the initial state of the inventory system is $l(0) = S$, $o(0) = 0$. That is, the initial inventory level is S and the inventory replenishment order level is 0. Similarly, the terminal state is assumed to be $l(\tau) = S$, $o(\tau) = 0$ with the understanding that the ordering cost associated with increasing $l(t)$ to S at the end of the simulation (at $t = \tau$, with no delivery lag) should be included in the accumulated system statistics.

Events

Given that the state of the system is defined by $l(t)$ and $o(t)$, there are three types of events that can change the state of the system:

- a *demand* for an item at time t, in which case $l(t)$ will decrease by 1;

- an inventory *review* at (integer-valued) time t, in which case $o(t)$ will increase from 0 to $S - l(t)$ provided $l(t) < s$, else $o(t)$ will remain 0;

- an *arrival* of an inventory replenishment order at time t, in which case $l(t)$ will increase from its current level by $o(t)$ and then $o(t)$ will decrease to 0.

To complete the development of a specification model, the time variables t_d, t_r, and t_a are used to denote the next scheduled time for the three events inventory *demand*, inventory *review*, and inventory *arrival*, respectively. As in the previous section, ∞ is used to denote (schedule) an event that is not possible.

Algorithm 5.2.1   This algorithm is a next-event simulation of a simple inventory system with delivery lag. The algorithm presumes the existence of two functions `GetLag` and `GetDemand` that return a random value of delivery lag and the next demand time respectively.

```
l = S;                                          /* initialize inventory level */
o = 0;                                          /* initialize amount on order */
t = 0.0;                                        /* initialize simulation clock */
t_d = GetDemand();                               /* initialize the event list */
t_r = t + 1.0;                                   /* initialize the event list */
t_a = ∞;                                         /* initialize the event list */
while (t < τ) {
   t = min(t_d, t_r, t_a);                           /* scan the event list */
   if (t == t_d) {                          /* process an inventory demand */
     l--;
     t_d = GetDemand();
   }
   else if (t == t_r) {                     /* process an inventory review */
     if (l < s) {
        o = S - l;
        δ = GetLag();
        t_a = t + δ;
     }
     t_r += 1.0;
   }
   else {                                   /* process an inventory arrival */
     l += o;
     o = 0;
     t_a = ∞;
   }
}
```

Program sis3

Program `sis3` is an implementation of Algorithm 5.2.1. The event list consists of three elements `t.demand`, `t.review`, and `t.arrive` corresponding to t_d, t_r, and t_a respectively. These are elements of the structure `t`. Similarly, the two state variables `inventory` and `order` correspond to $l(t)$ and $o(t)$. Also, the time-integrated holding and shortage integrals are `sum.hold` and `sum.short`.

5.2.2   A MULTI-SERVER SERVICE NODE

As another example of next-event simulation we will now consider a *multi*-server service node. The extension of this next-event simulation model to account for immediate feedback, or finite service node capacity, or a priority queue discipline is left as an exercise. This example serves three objectives.

* A multi-server service node is one natural generalization of the single-server service node.

* A multi-server service node has considerable practical and theoretical importance.

* In a next-event simulation model of a multi-server service node, the size of the event list is dictated by the number of servers and, if this number is large, the data structure used to represent the event list is important.

Definition 5.2.1   A *multi-server service node* consists of a single queue, if any, and two or more servers operating *in parallel*. At any instant in time, the state of each server will be either *busy* or *idle* and the state of the queue will be either *empty* or *not empty*. If at least one server is idle, the queue must be empty. If the queue is not empty then all the servers must be busy.
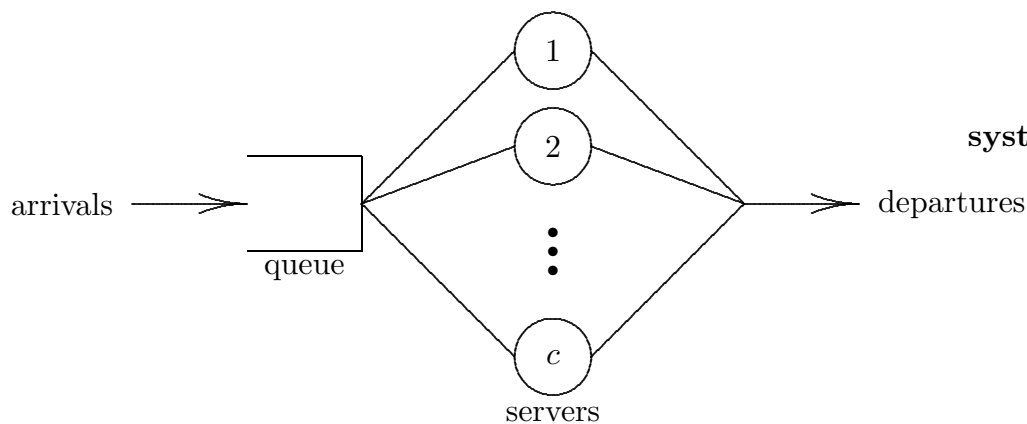
**Figure 5.2.2.
Multi-server
service node
system diagram.**

Jobs arrive at the node, generally at random, seeking service. When service is provided, generally the time involved is also random. At the completion of service, jobs depart. The service node operates as follows. As each job arrives, if all servers are busy then the job enters the queue, else an available server is selected and the job enters service. As each job departs a server, if the queue is empty then the server becomes idle, else a job is selected from the queue to enter service at this server. Servers process jobs independently — they do not "team up" to process jobs more efficiently during periods of light traffic. This system configuration is popular, for example, at airport baggage check-in, banks, and roller coasters. Felt ropes or permanent dividers are often used to herd customers into queues. One advantage to this configuration is that it is impossible to get stuck behind a customer with an unusually long service time.

As in the single-server service node model, control of the queue is determined by the *queue discipline* — the algorithm used when a job is selected from the queue to enter service (see Section 1.2). The queue discipline is typically FIFO.

Server Selection Rule

Definition 5.2.2   A job may arrive to find two or more servers idle. In this case, the algorithm used to select an idle server is called the *server selection rule.*

There are several possible server selection rules. Of those listed below, the random, cyclic, and equity server selection rules are designed to achieve an equal utilization of all servers. With the other two server selection rules, typically some servers will be more heavily utilized than others.

- Random selection — select at random from the idle servers.

- Selection in order — select server 1 if idle, else select server 2 if idle, etc.

- Cyclic selection — select the first available server beginning with the successor (a circular search, if needed) of the last server engaged.

- Equity selection — select the server that has been idle longest *or* the idle server whose utilization is lowest.*

- Priority selection — choose the "best" idle server. This will require a specification from the modeler as how "best" is determined.

For the purposes of mathematical analysis, multi-server service nodes are frequently assumed to have *statistically identical, independent* servers. In this case, the server selection rule has no effect on the average performance of the service node. That is, although the utilization of the individual servers can be affected by the server selection rule, if the servers are statistically identical and independent, then the *net* utilization of the node is not affected by the server selection rule. Statistically identical servers are a convenient mathematical fiction; in a discrete-event simulation environment, if it is not appropriate then there is no need to assume that the service times are statistically identical.

States

In the queuing theory literature, the parallel servers in a multi-server service node are commonly called *service channels.* In the discussion that follows,

- the positive integer c will denote the number of servers (channels);

- the server index will be $s = 1, 2, \ldots, c$.

---

* There is an ambiguity in this server selection rule in that idle time can be measured from the most recent departure or from the beginning of the simulation. The modeler must specify which metric is appropriate.

As for a single-server node, the state variable $l(t)$ denotes the number of jobs in the service node at time t. For a multi-server node with distinct servers this single state variable does not provide a complete state description. If $l(t) \geq c$, then all servers are busy and $q(t) = l(t) - c$ jobs are in the queue. If $l(t) < c$, however, then for a complete state description we need to know which servers are busy and which are idle. Therefore, for $s = 1, 2, \ldots, c$ define

$$x_s(t) : \text{the number of jobs in service (0 or 1) by server } s \text{ at time } t,$$

or, equivalently, $x_s(t)$ is the state of server s at time t (with 0 denoting idle and 1 denoting busy). Finally, observe that

$$q(t) = l(t) - \sum_{s=1}^{c} x_s(t),$$

that is, the number of jobs in the queue at time t is the number of jobs in the service at time t minus the number of busy servers at time t.

Events

The $c+1$ state variables $l(t), x_1(t), x_2(t), \ldots, x_c(t)$ provide a complete state description of a multi-server service node. With a complete state description in hand we then ask what types of events can cause the state variables to change. The answer is that if the servers are distinct then there are $c + 1$ event types — either an arrival to the service node or completion of service by one of the c servers. If an *arrival* occurs at time t, then $l(t)$ is incremented by 1. Then, if $l(t) \leq c$ an idle server s is selected and the job enters service at server s (and the appropriate completion of service is scheduled), else all servers are busy and the job enters the queue. If a *completion of service* by server s occurs at time t then $l(t)$ is decremented by 1. Then, if $l(t) \geq c$ a job is selected from the queue to enter service at server s, else server s becomes idle.

The additional assumptions needed to complete the development of the next-event simulation model at the specification level are consistent with those made for the single-server model in the previous section.

- The initial state of the multi-server service node is empty and idle. Therefore, the first event must be an arrival.

- There is a terminal "close the door" time τ at which point the arrival process is turned off but the system continues operation until all jobs have been completed. Therefore, the terminal state of the multi-server node is empty and idle and the last event must be a completion of service.

- For simplicity, all servers are assumed to be independent and statistically identical. Moreover, equity selection is assumed to be the server selection rule.

All of these assumptions can be relaxed.

Event List

The event list for this next-event simulation model can be organized as an array of $c + 1$ event types indexed from 0 to c as illustrated below for the case $c = 4$.

Figure 5.2.3.
Event list
data structure
for multi-server
service node.

	t	x	
0	t	x	arrival
1	t	x	completion of service by server 1
2	t	x	completion of service by server 2
3	t	x	completion of service by server 3
4	t	x	completion of service by server 4

The `t` field in each event structure is the scheduled time of next occurrence for that event; the `x` field is the current *activity status* of the event. The status field is used in this data structure as a superior alternative to the ∞ "impossibility flag" used in the model on which programs `ssq3` and `sis3` are based. For the 0th event type, `x` denotes whether the arrival process is on (1) or off (0). For the other event types, `x` denotes whether the corresponding server is busy (1) or idle (0).

An array data structure is appropriate for the event list because the size of the event list cannot exceed $c + 1$. If c is large, however, it is preferable to use a variable-length data structure like, for example, a linked-list containing events sorted by time so that the next (most imminent) event is always at the head of the list. Moreover, in this case the event list should be partitioned into busy (`event[e].x = 1`) and idle (`event[e].x = 0`) sublists. This idea is discussed in more detail in the next section.

Program msq

Program `msq` is an implementation of the next-event multi-server service node simulation model we have just developed.

- The state variable $l(t)$ is `number`.

- The state variables $x_1(t), x_2(t), \ldots, x_c(t)$ are incorporated into the event list.

- The time-integrated statistic $\int_0^t l(\theta) \, d\theta$ is `area`.

- The array named `sum` contains structures that are used to record, for each server, the sum of service times and the number served.

- The function `NextEvent` is used to search the event list to determine the index `e` of the next event.

- The function `FindOne` is used to search the event list to determine the index `s` of the available server that has been idle longest (because an equity selection server selection rule is used).

5.2.3   EXERCISES

Exercise 5.2.1   Use program `ddh` in conjunction with program `sis3` to construct a discrete-data histogram of the total demand per time interval. (Use 10 000 time intervals.) (*a*) Compare the result with the corresponding histogram for program `sis2`. (*b*) Comment on the difference.

Exercise 5.2.2[a]   (*a*) Modify program `sis3` to account for a *Uniform*(0.5, 2.0) delivery lag. Assume that if an order is placed at time t and if $o(t) > 0$ then the amount ordered will be $S - l(t) - o(t)$. (*b*) Discuss why you think your program is correct.

Exercise 5.2.3   Modify program `sis3` so that the inventory review is no longer periodic but, instead, occurs after each demand instance. (This is transaction reporting — see Section 1.3.) Assume that when an order is placed, further review is stopped until the order arrives. This avoids the sequence of orders that otherwise would occur during the delivery lag. What impact does this modification have on the system statistics? Conjecture first, then simulate using `STOP` equal to 10 000.0 to estimate steady-state statistics.

Exercise 5.2.4   (*a*) Relative to program `msq`, provide a mathematical justification for the technique used to compute the average delay and the average number in the queue. (*b*) Does this technique require that the service node be idle at the beginning and end of the simulation for the computation of these statistics to be exact?

Exercise 5.2.5   (*a*) Implement a "selection in order" server selection rule for program `msq` and compute the statistics. (*b*) What impact does this have on the system performance statistics?

Exercise 5.2.6   Modify program `msq` so that the stopping criteria is based on "closing the door" after a fixed number of jobs have entered the service node.

Exercise 5.2.7   Modify program `msq` to allow for feedback with probability β. What statistics are produced if $\beta = 0.1$? (*a*) At what value of β does the multi-server service node saturate? (*b*) Provide a mathematical justification for why saturation occurs at this value of β.

Exercise 5.2.9   Modify program `msq` to allow for a finite capacity of r jobs in the node at one time. (*a*) Draw a histogram of the time between lost jobs at the node. (*b*) Comment on the shape of this histogram.

Exercise 5.2.10   Write a next-event simulation program that estimates the average time to complete the stochastic activity network given in Section 2.4. Compute the mean and variance of the time to complete the network.

The next-event simulation models for the single-server service node and the simple inventory system from the previous two sections have such short event lists (two events for the single-server service node and three events for the simple inventory system) that their management does not require any special consideration. There are next-event simulations, however, that may have hundreds or even thousands of events on their event list simultaneously, and the efficient management of this list is crucial. The material in this section is based on a tutorial by Henriksen (1983) and Chapter 5 of Fishman (2001).

Although the discussion in this section is limited to managing event lists, practically all of the discussion applies equally well, for example, to the management of jobs in a single-server service node. A FIFO or LIFO queue discipline results in a trivial management of the jobs in the queue: jobs arriving when the server is busy are simply added to the tail (FIFO) or head (LIFO) of the queue. A "shortest processing time first" queue discipline (which is commonly advocated for minimizing the wait time in job shops), on the other hand, requires special data structures and algorithms to efficiently insert jobs into the queue and delete jobs from the queue.

5.3.1   INTRODUCTION

An event list is the data structure that contains a list of the events that are scheduled to occur in the future, along with any ancillary information associated with these events. The list is traditionally sorted by the scheduled time of occurrence, but, as indicated in the first example in this section, this is not a requirement. The event list is also known as the calendar, future events chain, sequencing set, future event set, etc. The elements that comprise an event list are known as future events, events, event notices, transactions, records, etc. We will use the term *event notice* to describe these elements.

Why is efficient management of the event notices on the event list so important that it warrants an entire section? Many next-event simulation models expend more CPU time on managing the event list than on any other aspect (e.g., random number generation, random variate generation, processing events, miscellaneous arithmetic operations, printing reports) of the simulation.

Next-event simulations that require event list management can be broken into four categories according to these two boolean classifications:

- There is a either a *fixed* maximum or a *variable* maximum number of event notices on the event list. There are clear advantages to having the maximum number of events fixed in terms of memory allocation. All of the simulations seen thus far have had a fixed maximum number of event notices.

- The event list management technique is either being devised for one specific model or is being developed for a general-purpose simulation language. If the focus is on a single model, then the scheduling aspects of that model can be exploited for efficiency. An event list management technique designed for a general-purpose language must be robust in the sense that it performs reasonably well for a variety of simulation models.

There are two critical operations in the management of the event notices that comprise the event list. The first is the insertion, or *enqueue* operation, where an event notice is placed on the event list. This operation is also referred to as "scheduling" the event. The second is the deletion, or *dequeue* operation, where an event notice is removed from the event list. A deletion operation is performed to process the event (the more common case) or because a previously scheduled event needs to be canceled for some reason (the rare case). Insertion and deletion may occur at a prescribed position in the event list, or a search based on some criteria may need to be initiated first in order to determine the appropriate position. We will use the term *event list management scheme* to refer to the data structures and associated algorithms corresponding to one particular technique of handling event list insertions and deletions.

Of minor importance is a *change* operation, where a search for an existing event notice is followed by a change in some aspect of the event notice, such as changing its scheduled time of occurrence. Similarly, an *examine* operation searches for an existing event notice in order to examine its contents. A *count* operation is used to determine the number of event notices in the list. Due to their relative rarity in discrete-event simulation modeling and their similarity to insertion and deletion in principle, we will henceforth ignore the change, examine, and count operations and focus solely on the insertion and deletion operations.

5.3.2   EVENT LIST MANAGEMENT CRITERIA

Three criteria that can be used to assess the effectiveness of the data structures and algorithms for an event list management scheme are:

- **Speed**. The data structure and associated algorithms for inserting and deleting event notices should execute in minimal CPU time. Critical to achieving fast execution times is the efficient searching of the event list. The balance between sophisticated data structures and algorithms for searching must be weighed against the associated extraneous overhead calculations (e.g., maintaining pointers for a list or heap operations) that they require. An effective general-purpose algorithm typically bounds the number of event notices searched for inserting or deleting.

- **Robustness**. Efficient event list management should perform well for a wide range of scheduling scenarios. This is a much easier criteria to achieve if the characteristics of one particular model can be exploited by the analyst. The designer of an event list management scheme for a general-purpose language does not have this advantage.

- **Adaptability**. An effective event list management scheme should be able to adapt its searching time to account for both the length of the event list and the distribution of new events that are being scheduled. It is also advantageous for an event list management scheme to be "parameter-free" in the sense that the user should not be required to specify parameters that optimize the performance of the scheme. A "black box" approach to managing an event list is particularly appropriate for a general-purpose simulation language since users have a variety of sophistication levels.

Given these criteria, how do we know whether our event list management scheme is effective? If this is for a single model, the only way to answer this question is by running several different schemes on the same model to see which executes the fastest. It is typically not possible to prove that one particular scheme is superior to all other possible schemes, since a more clever analyst may exploit more of the structure associated with a specific model. It is, however, possible to show that one scheme dominates another in terms of execution time by making several runs with different seeds and comparing execution times.

Testing event list management schemes for a general-purpose language is much more difficult. In order to compare event list management schemes, one must consider a representative test-bed of diverse simulation models on which to test various event-list management schemes. Average and worst-case performance in terms of the CPU time tests the speed, robustness, and adaptability of various event list management schemes.

5.3.3 EXAMPLE

We consider the computer timesharing model presented in Henriksen (1983) to discuss event list management schemes. In our usual manner, we start with the conceptual model before moving to the specification and computational models.

Conceptual Model

Consider a user at a computer timesharing system who endlessly cycles from (1) thinking, to (2) typing in a command, to (3) receiving the output from the command. The user does not take any breaks, never tires, and the workstation never fails. A system diagram that depicts this behavior is given in Figure 5.3.1.

**Figure 5.3.1.
Timeshare
system
diagram.**
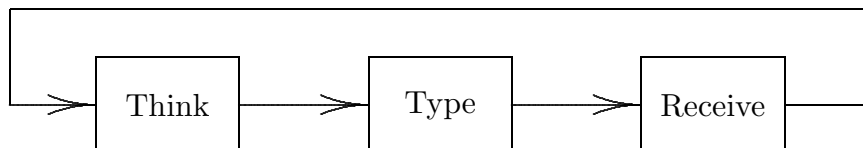
This is the first time we have encountered what is known as a "closed" system. The "open" systems considered previously are the queuing models (jobs arrive, are processed, then depart) and inventory models (inventory arrives, is purchased, then departs). In the timesharing model, there is no arrival or departure. The three-step activity loops endlessly.

The times to perform the three operations, measured in seconds, are:

- The time to think requires $Uniform(0, 10)$ seconds.

- There are an $Equilikely(5, 15)$ number of keystrokes involved in typing in a command, and each keystroke requires $Uniform(0.15, 0.35)$ seconds.

- The output from the command contains an $Equilikely(50, 300)$ number of characters, each requiring (the ancient rate of) 1/120 second to display at the workstation.

If we were to simulate just one user, this would be a rather trivial model since there is only one event on the event list: the completion of the next activity (thinking, typing a keystroke, receiving an output character). To make event list management an issue, assume that the computer timesharing system consists of n users at n terminals, each asynchronously cycling through the three-step process of thinking, typing, and receiving.

Many critiques of this model would be valid. Do all of the users really all think and type at the same rate? Does the distribution of thinking time really "cut-off" at ten seconds as the *Uniform*$(0, 10)$ thinking time implies? Are all of the *Uniform* and *Equilikely* distributions appropriate? Why doesn't the receive rate degrade when several users receive output simultaneously? Why doesn't the receiving portion of the system bog down as n increases? Because the purpose of this model is to illustrate the management of the event list, we will forgo discussion about the reasonableness and accuracy of the model. The important topic of developing "input" models that accurately mimic the system of interest is addressed in Chapter 9.

Back-of-an-Envelope Calculations

Since all of the distributions in this model are either *Uniform* or *Equilikely*, it is worthwhile to do some preliminary calculations which may provide insight into model behavior prior to the simulation. The mean of a *Uniform* or *Equilikely* random variable is, not surprisingly, the average of their two parameters. Thus the expected length of each cycle for each user is:

$$\left(\frac{0+10}{2}\right) + \left(\frac{5+15}{2}\right)\left(\frac{0.15+0.35}{2}\right) + \left(\frac{50+300}{2}\right)\left(\frac{1}{120}\right) = 5 + 10 \cdot 0.25 + 175 \cdot \frac{1}{120}$$

$$\cong 5 + 2.5 + 1.4583$$

$$= 8.9583$$

seconds. Thus if one were to observe a user at a random instant in time of a system in steady state, the probabilities that the user will be thinking, typing, and receiving are

$$\frac{5}{8.9583} \cong 0.56, \quad \frac{2.5}{8.9583} \cong 0.28, \quad \text{and} \quad \frac{1.4583}{8.9583} \cong 0.16.$$

These fractions apply to individual users, as well as the population of n users. At any particular point in simulated time, we could expect to see about 56% of the users thinking, 28% typing, and 16% receiving output.

Although it is clear from this analysis that the largest portion of a user's *simulated time* is spent thinking and the smallest portion of a user's *simulated time* is spent receiving, the opposite is true of the *number* of scheduled events. Each cycle has exactly one thinking event, an average of ten keystrokes, and an average of 175 characters received. Thus each cycle averages $1 + 10 + 175 = 186$ events. The expected fractions of events associated with thinking, typing a keystroke, and receiving an output character during a cycle are

$$\frac{1}{186} \cong 0.005, \quad \frac{10}{186} \cong 0.054, \quad \text{and} \quad \frac{175}{186} \cong 0.941.$$

The vast majority of the events scheduled during the simulation will be receiving a character. This observation will influence the probability distribution of the event times associated with event notices on the event list. This distribution can be exploited when designing an event list management scheme.

Specification Model

There are three events that comprise the activity of a user on the timesharing system:

(1) complete thinking time;

(2) complete a keystroke;

(3) complete the display of a character.

For the second two events, some ancillary information must be stored as well: the number of keystrokes in the command and the number of characters returned from the command. For all three events, an integer (1 for thinking, 2 for typing and 3 for receiving) is stored to denote the event type. Ancillary information of this type is often called an "attribute" in general-purpose simulation languages.

As with most next-event simulation models, the processing of each event triggers the scheduling of future events. For this particular model, we are fortunate that each event triggers just one future event to schedule: the next activity for the user whose event is presently being processed.

Several data structures are capable of storing the event notices for this model. Two likely candidates are an array and a linked list. For simplicity, an array will be used to store the event notices. We can use an array here because we know in advance that there will always be n events on the event list, one event notice for the next activity for each user. We begin with this very simplistic (and grossly inefficient) data structure for the event list. We will subsequently refine this data structure, and eventually outline more sophisticated schemes. We will store the times of the events in an array of length n in an order associated with the n users, and make a linear search of all elements of the array whenever the next event to be processed needs to be found. Thus the deletion operation requires searching all n elements of the event list to find the event with the smallest event time, while the insertion operation requires no searching since the next event notice for a particular user simply overwrites the current event notice.

However, there is ancillary information that must be carried with each event notice. Instead of an array of n times, the event list for this model should be organized as an array of n event structures. Each event structure consists of three fields: `time`, `type`, and `info`. The `time` field in the ith event structure stores the time of the event for the ith user ($i = 0, 1, \ldots, n-1$). The `type` field stores the event type (1, 2, or 3). The `info` field stores the ancillary information associated with a keystroke or display of a character event: the number of keystrokes remaining (for a Type 2 event) or the number of characters remaining in the output (for a Type 3 event). The `info` field is not used for a thinking (Type 1) event since no ancillary information is needed for thinking.

The initialization phase of the next-event simulation model schedules a complete thinking time event (Type 1 event) for each of the n users on the system. The choice of beginning with this event and applying the choice to all users is arbitrary*. After the initialization, the event list for a system with $n = 5$ users might look like the one presented in Figure 5.3.2. Each of the five $Uniform(0, 10)$ completion of thinking event times is placed in the `time` field of the corresponding event structure. The value in the `time` field of the third structure is the smallest, indicating that the third user will be the first to stop thinking and begin typing at time 1.305. All five of these events are completion of thinking events, as indicated by the `type` field in each event structure.

`time`	9.803	3.507	1.305	2.155	8.243
`type`	1	1	1	1	1
`info`					
	0	1	2	3	4

Figure 5.3.2. Initial event list.

The remainder of the algorithm follows the standard next-event protocol — while the terminal condition has not been met: (1) scan the event list for the most imminent event, (2) update the simulation clock accordingly, (3) process the current event, and (4) schedule the subsequent event by placing the appropriate event notice on the event list.

As the initial event (end of thinking for the third user at time 1.305) is deleted from the event list and processed, a number of keystrokes [an $Equilikely(5, 15)$ random variate which takes the value of 7 in this case — a slightly shorter than average command] and a time for the first keystroke [a $Uniform(0.15, 0.35)$ random variate which takes the value of 0.301 in this case — a slightly longer than average keystroke time] are generated. The `time` field in the third event structure is incremented by 0.301 to $1.305 + 0.301 = 1.606$, and the number of keystrokes in this command is stored in the corresponding `info` field. Subsequent keystrokes for the third user decrement the integer in the corresponding `info` field. The condition of the event list after the processing of the first event is shown in Figure 5.3.3.

`time`	9.803	3.507	1.606	2.155	8.243
`type`	1	1	2	1	1
`info`			7		
	0	1	2	3	4

Figure 5.3.3. Updated event list.

To complete the development of the specification model, we use the following notation:

- The simulation clock is t, which is measured in seconds.

- The simulation terminates when the next scheduled event is τ seconds or more.

The algorithm is straightforward, as presented in Algorithm 5.3.1.

---

* All users begin thinking simultaneously at time 0, but will behave more independently after a few cycles as the timesharing system "warms up."

Algorithm 5.3.1   This algorithm is a next-event simulation of a think-type-receive time-sharing system with n concurrent users. The algorithm presumes the existence of four functions `GetThinkTime`, `GetKeystrokeTime`, `GetNumKeystrokes`, and `GetNumCharacters` that return the random time to think, time to enter a keystroke, number of keystrokes per command, and number of characters returned from a command respectively. The function `MinIndex` returns the index of the most imminent event notice.

```
t = 0.0;                                      /* initialize system clock */
for (i = 0;  i < n;  i++) {                      /* initialize event list */
   event[i].time = GetThinkTime();
   event[i].type = 1;
}
while (t < τ) {                          /* check for terminal condition */
   j = MinIndex(event.time);            /* find index of imminent event */
   t = event[j].time;                          /* update system clock */
   if (event[j].type == 1) {          /* process completion of thinking */
      event[j].time = t + GetKeystrokeTime();
      event[j].type = 2;
      event[j].info = GetNumKeystrokes();
   }
   else if (event[j].type == 2) { /* process completion of keystroke */
      event[j].info--;     /* decrement number of keystrokes remaining */
      if (event[j].info > 0)              /* if more keystrokes remain */
         event[j].time = t + GetKeystrokeTime();
      else {                                  /* else last keystroke */
         event[j].time = t + 1.0 / 120.0;
         event[j].type = 3;
         event[j].info = GetNumCharacters();
      }
   }
   else if (event[j].type == 3) { /* process complete character rcvd */
      event[j].info--;     /* decrement number of characters remaining */
      if (event[j].info > 0)              /* if more characters remain */
         event[j].time = t + 1.0 / 120.0;
      else {                                  /* else last character */
         event[j].time = t + GetThinkTime();
         event[j].type = 1;
      }
   }
}
```

Program ttr

The think-type-receive specification model has been implemented in program `ttr`, which prints the total number of events scheduled and the average number of event notices searched for each deletion. For each value of n in the table below, the simulation was run three times with seeds 123456789, 987654321, and 555555555 for $\tau = 100$ seconds, and the averages of the three runs are reported.

number of users n	expected number of events scheduled	average number of events scheduled	average number of event notices searched
5	10 381	9 902	5
10	20 763	20 678	10
50	103 814	101 669	50
100	207 628	201 949	100

The column headed "expected number of events scheduled" is determined as follows. Since the average length of each cycle is 8.9583 seconds, each user will go through an average of

$$\frac{100}{8.9583} \cong 11.16$$

cycles in $\tau = 100$ seconds. Since the average number of events per cycle is 186, we expect to see

$$(11.16) \cdot (186) \cdot n = 2076.3n$$

total events scheduled during the simulation. These values are reported in the second column of the table. The averages in the table from the three simulations are slightly lower than the expected values due to our arbitrary decision to begin each cycle thinking, the longest event. In terms of event list management, each deletion event (required to find the next event notice) requires an exhaustive search of the `time` field in all n event structures, so the average number of event notices searched for each deletion is simply n. The simplistic event list management scheme used here sets the stage for more sophisticated schemes.

5.3.4   AN IMPROVED EVENT LIST MANAGEMENT SCHEME

Our decision in the previous example to store event times unordered is a departure from the traditional convention in simulation languages, which is to order the event notices on the event list in ascending order of event times, i.e., the event list is maintained in chronological order. If we now switch to an ordered event list, a deletion requires no searching and an insertion requires a search — just the opposite situation from the previous event list management scheme. This will be a wash time-wise for the think-type-receive model, since there is deletion for every insertion during the simulation. Every deletion associated with the scheduling of an event pulls the first event notice from the head of the list. This section is focused, therefore, on efficient algorithms for inserting event notices into the event list.

There is good and bad news associated with the move to an event list that is ordered by ascending event time. The good news is that the entire event list need not necessarily be searched exhaustively every time an insertion operation is conducted. The bad news, however, is that arrays are no longer a natural choice for the data structure due to the overhead associated with shuffling event notices down the array when an event notice is placed at the beginning or middle of the list. A singly- or doubly-linked list is a preferred data structure due to its ability to easily insert items in the middle of the list. The overhead of maintaining pointers, however, dilutes the benefit of moving to an event list that is ordered by ascending event time. Also, direct access to the array is lost and time-consuming element-by-element searches through the linked list are required.

A secondary benefit associated with switching to linked lists is that the maximum size of the list need not be specified in advance. This is of no consequence in our think-type-receive model since there are always n events in the event list. In a general-purpose discrete-event simulation language, however, linked lists expand until memory is exhausted.

Example 5.3.1   For the think-type-receive model with $n = 5$, for example, a singly-linked list, linked from head (top) to tail (bottom), to store the elements of the event list corresponding to Figure 5.3.3 is shown in Figure 5.3.4. The three values stored on each event notice are the event time, event type, and ancillary information (seven keystrokes remaining in a command for the first element in the list). The event notices are ordered by event time. A deletion now involves no search, but a search is required for each insertion.

Figure 5.3.4. Event list as a linked list.
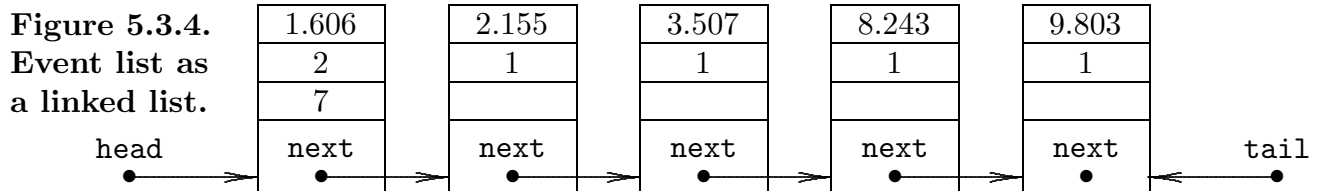
One question remains before implementing the new data structure and algorithm for the search. Should the list be searched from head to tail (top to bottom for a list with forward pointers) or tail to head (bottom to top for a list with backward pointers)? We begin by searching from tail to head and check our efficiency gains over the naive event management scheme presented in the previous subsection. The table below shows that the average number of events scheduled is identical to the previous event management scheme (as expected due to the use of identical seeds), and improvements in the average number of searches per insertion improvements range from 18.8% ($n = 100$) to 23.4% ($n = 5$).

number of users n	average number of events scheduled	average number of event notices searched
5	9 902	3.83
10	20 678	8.11
50	101 669	40.55
100	201 949	81.19

These results are certainly not stunning. The improvement in search time is slight. What went wrong? The problem here is that we ignored our earlier back-of-an envelope calculations. These calculations indicated that 94.1% of the events in the simulation would be the receipt of a character, which has a very short inter-event time. Thus we should have searched the event list from head to tail since these short events are much more likely to be inserted at or near the top of the list. We re-programmed the search to go from head to tail, and the results are given in the table below.

number of users n	average number of events scheduled	average number of event notices searched
5	9 902	1.72
10	20 678	2.73
50	101 669	10.45
100	201 949	19.81

Confirming our calculations, the forward search performs far better than the backward search.* In this case the savings in terms of the number of searches required over the exhaustive search ranges from 66% (for $n = 5$) to 80% (for $n = 100$).

The think-type-receive model with $n = 100$ highlights our emphasis on efficient event list management techniques. Even with the best of the three event list management schemes employed so far (forward linear search of a singly-linked list, averaging 19.81 searches per insertion), more time is spent on event list management than the rest of the simulation operations (e.g., random number generation, random variate generation, processing events) combined!

5.3.5 MORE ADVANCED EVENT LIST MANAGEMENT SCHEMES

The think-type-receive model represents the simplest possible case for event list management. First, the number of event notices on the event list remains constant throughout the simulation. Second, the fact that there are frequent short events (e.g., receiving a character) can be exploited in order to minimize the search time for an insertion using a forward search.

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* An interesting verification of the forward and backward searches can be made in this case since separate streams of random numbers assures an identical sequencing of events. For an identical event list of size n, the sum of the number of forward searches and the number of backward searches equals n for an insertion at the top or bottom of the list. For an insertion in the middle, however, the sum equals $n + 1$ for identical lists. Therefore, the sum of the rightmost columns of the last two tables will always lie between n and $n + 1$, which it does in this case. The sums are 5.55, 10.84, 51.00, and 101.00 for $n = 5, 10, 50$, and 100. The sum tends to $n + 1$ for large n since it is more likely to have an insertion in the middle of the list as n grows. Stated another way, when n is large it is a near certainty that several users will be simultaneously receiving characters when an event notice is placed on the event list, meaning that the event is unlikely to be placed at the front of the list.

We now proceed to a discussion of the general case where (1) the number of event notices in the event list varies throughout the simulation, (2) the maximum length of the event list is not known in advance, and (3) the structure of the simulation model is unknown so it cannot be exploited for optimizing an event list management scheme.

In order to reduce the scope of the discussion, assume that a memory allocation mechanism exists so that a memory location occupied by a deleted event notice may be immediately occupied by an event notice that is subsequently inserted into the event list. When memory space is released as soon as it becomes available in this fashion, the simulation will fail due to lack of memory only when an insertion is made to an event list that exhausts the space allocated to the event list. Many general-purpose languages effectively place all entities (e.g., event notices in the event list, jobs waiting in a queue) in the simulation in a single partitioned list in order to use memory in the most efficient manner. Data structures and algorithms associated with the allocation and de-allocation of memory are detailed in Chapter 5 of Fishman (2001).

The next four subsections briefly outline event list management schemes commonly used to efficiently insert event notices into an event list and delete event notices from an event list in a general setting: multiple linked lists, binary search trees, heaps, and hybrid schemes.

Multiple Linked Lists

One approach to reducing search time associated with insertions and deletions is to maintain multiple linear lists, each sorted by event time. Let k denote the number of such lists and n denote the number of event notices in all lists at one particular point in simulated time. Figure 5.3.5 shows $k = 2$ equal-length, singly-linked lists for $n = 10$ initial think times in the think-type-receive model. An insertion can be made into either list. If the list sizes were not equal, choosing the shortest list minimizes the search time. The time savings associated with the insertion operation is offset by (1) the overhead associated with maintaining the multiple lists, and (2) a less efficient deletion operation. A deletion now requires a search of the top (head) event notices of the k lists, and the event notice with the smallest event time is deleted.

Figure 5.3.5. Multiple linked lists.
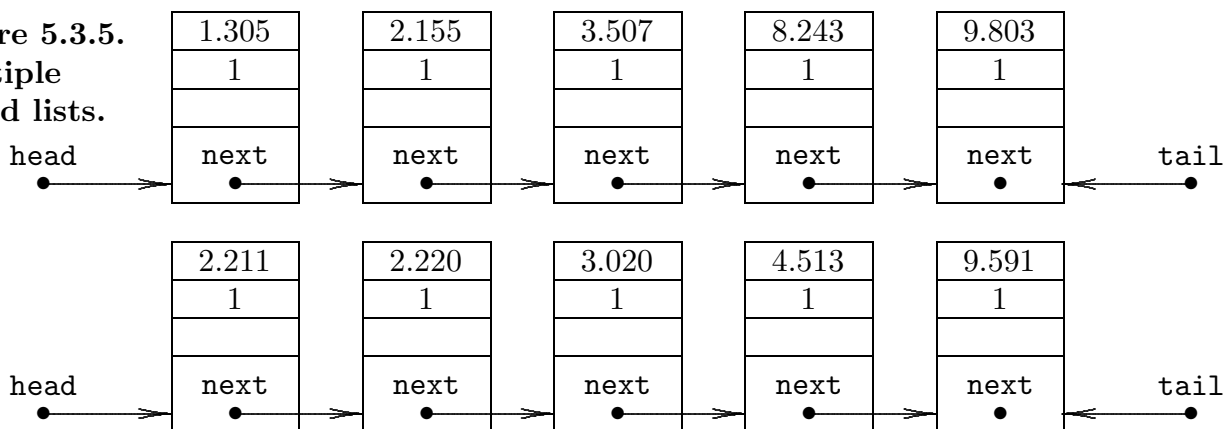
We close the discussion of multiple event lists with three issues that are important for minimizing CPU time for insertions and deletions:

- The decision of whether to fix the number of lists k throughout the simulation or allow it to vary depends on many factors, including the largest value of n throughout the simulation, the distribution of the position of insertions in the event list, and how widely n varies throughout the simulation.

- If the number of lists k is allowed to vary throughout the simulation, the modeler must determine appropriate thresholds for n where lists are split (as n increases) and combined (as n decreases).

- The CPU time associated with inserting an event notice, deleting an event notice, combining lists, and splitting lists as functions of n and k should drive the optimization of this event list management scheme.

The next two data structures, binary trees and heaps, are well-known data structures. Rather than developing the data structures and associated operations from scratch, we refer the reader to Carrano and Prichard (2002) for basic definitions, examples, and applications. Our discussion of these two data structures here will be rather general in nature.

Binary Trees

We limit our discussion of trees to *binary trees*. A binary tree consists of n nodes connected by edges in a hierarchical fashion such that a *parent* node lies above and is linked to at most two *child* nodes. The parent-child relationship generalizes to the *ancestor-descendant* relationship in an analogous fashion to a family tree. A *subtree* in a binary tree consists of a node, along with all of the associated descendants. The top node in a binary tree is the only node in the tree without a parent, and is called the *root*. A node with no children is called a *leaf*. The *height* of a binary tree is the number of nodes on the longest path from root to leaf. The *level* of a node is 1 if it is the root or 1 greater than the level of its parent if it is not the root. A binary tree of height h is *full* if all nodes at a level less than h have two children each. Full trees have $n = 2^h - 1$ nodes. A binary tree of height h is *complete* if it is full down to level $h-1$ and level h is filled from left to right. A full binary tree of height $h = 3$ with $n = 7$ nodes and a complete binary tree of height $h = 4$ with $n = 12$ nodes are displayed in Figure 5.3.6.
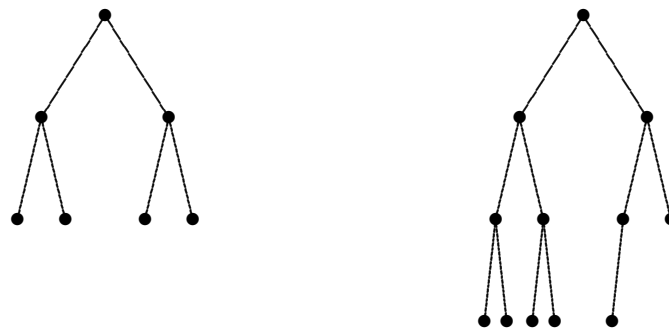
Figure 5.3.6. Full and complete binary trees.

Nodes are often associated with a numeric value. In our setting, a node corresponds to an event notice and the numeric value associated with the node is the event time. A *binary search tree* is a binary tree where the value associated with any node is greater than or equal to every value in its left subtree and less than or equal to every value in its right subtree. The "or equal to" portions of the previous sentence have been added to the standard definition of a binary search tree to allow for the possibility of equal event times.

Example 5.3.2 There are many ways to implement a binary tree. Figure 5.3.7 shows a *pointer-based* complete binary tree representation of the ten initial events (from Figure 5.3.5) for a think-type-receive model with $n = 10$ users, where each user begins with a $Uniform(0, 10)$ thinking activity. Each event notice has three fields (event time, event type, and event information) and each parent points to its child or children, if any. Every event time is greater than or equal to every event time in its left subtree and less than or equal to every event time in its right subtree. Although there are many binary search tree configurations that could contain these particular event notices, the placement of the event notices in the *complete* binary search tree in Figure 5.3.7 is unique.

Figure 5.3.7. Binary search tree with pointers for the ten initial events in the think-type-receive model.
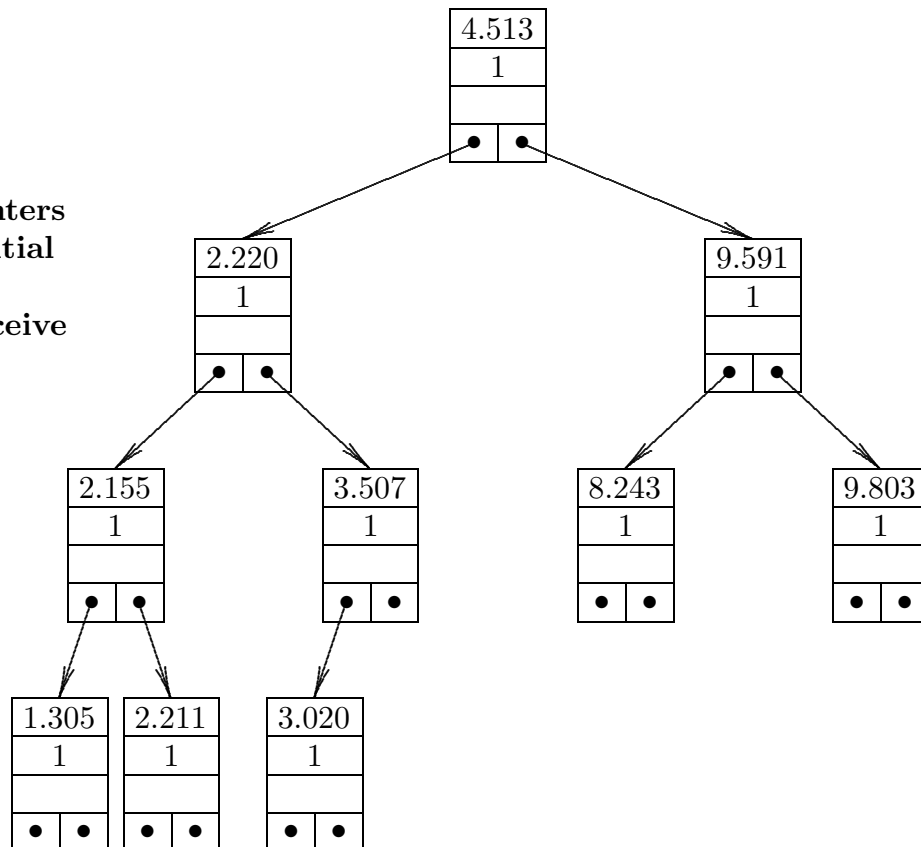
One advantage to binary search trees for storing event notices is that the "leftmost" leaf in the tree will always be the most imminent event. This makes a deletion operation fast, although it may require reconfiguring the tree after the deletion.

Insertions are faster than a linear search of a list due to the decreased number of comparisons necessary to find the appropriate insertion position. The key decision that remains for the scheme is whether the binary tree will be maintained as a complete tree (involving extra overhead associated with insertions and deletions) or allowed to evolve without the requirement that the tree is complete (which may result in an "imbalanced" tree whose height increases over time, requiring more comparisons for insertions). *Splay trees*, which require frequent rotations to maintain balance, have also performed well.

Heaps

A heap is another data structure for storing event notices in order to minimize insertion and deletion times. In our setting, a heap is a complete binary tree with the following properties: (1) the event time associated with the root is less than or equal to the event time associated with each of its children, and (2) the root has heaps as subtrees. Figure 5.3.8 shows a heap associated with the first ten events in the think-type-receive model. This heap is not unique.* An obvious advantage to the heap data structure is that the most imminent event is at the root, making deletions fast. The heap property must be maintained, however, whenever an insertion or deletion operation is performed.
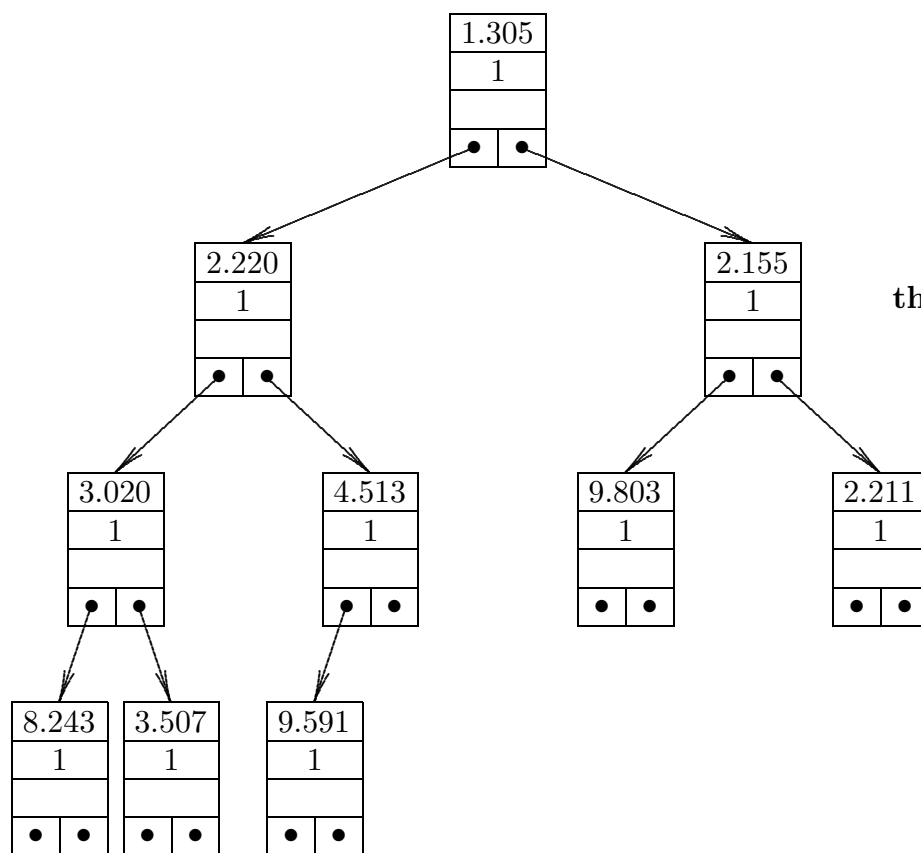
Figure 5.3.8. Heap with pointers for the ten initial events in the think-type-receive model.

---

   * If the event notices associated with times 2.220 and 2.211 in Figure 5.3.8, for example, were interchanged, the heap property would be retained.

Hybrid Schemes

The ideal event list management scheme performs well regardless of the size of the event list. Jones (1986) concludes that when there are fewer than ten event notices on an event list, a singly-linked list is optimal due to the overhead associated with more sophisticated schemes. Thus to fully optimize an event list management scheme, it may be necessary to have thresholds, similar to those that switch a thermostat on and off, that switch from one set of data structures and algorithms to another based on the number of events on the list. It is important to avoid switching back and forth too often, however, since the switch typically requires overhead processing time as well.

If a heap, for example, is used when the event list is long and a singly-linked list is used when the event list is short, then appropriate thresholds should be determined that will switch from one scheme to the other. As an illustration, when the number of event notices shrinks to $n = 5$ (e.g., n decreases from 6 to 5), the heap is converted to a singly-linked list. Similarly, when the number of event notices grows to $n = 15$ (e.g., n increases from 14 to 15), the singly-linked list is converted to a heap.

Henriksen's algorithm (Henriksen, 1983) provides adaptability to short and long event lists without alternating data structures based on thresholds. Henriksen's algorithm uses a binary search tree and a doubly-linked list simultaneously. This algorithm has been implemented in several simulation languages, including GPSS, SLX, and SLAM. At the conceptual level, Henriksen's algorithm employs two data structures:

- The event list is maintained as a single, linear, doubly-linked list ordered by event times. The list is augmented by a dummy event notice on the left with a simulated time of $-\infty$ and a dummy event notice on the right with a simulated time of $+\infty$ to allow symmetry of treatment for all real event notices.

- A binary search tree with nodes associated with a *subset* of the event notices in the event list, has nodes with the format shown in Figure 5.3.9. Leaf nodes have zeros for left and right child pointers. The leftmost node in the tree has a zero for a pointer to the next lower time tree node. This binary search tree is degenerate at the beginning of the simulation (prior to scheduling initial events).

Figure 5.3.9. Binary search tree node format.

Pointer to next lower time tree node
Pointer to left child tree node
Pointer to right child tree node
Event time
Pointer to the event notice

A three-node binary search tree associated with the ten initial events in the think-type-receive model is given in Figure 5.3.10. The binary search tree is given at the top of the figure, and the linear doubly-linked list containing the ten initial event notices (plus the dummy event notices at both ends of the list) is shown at the bottom of the figure.
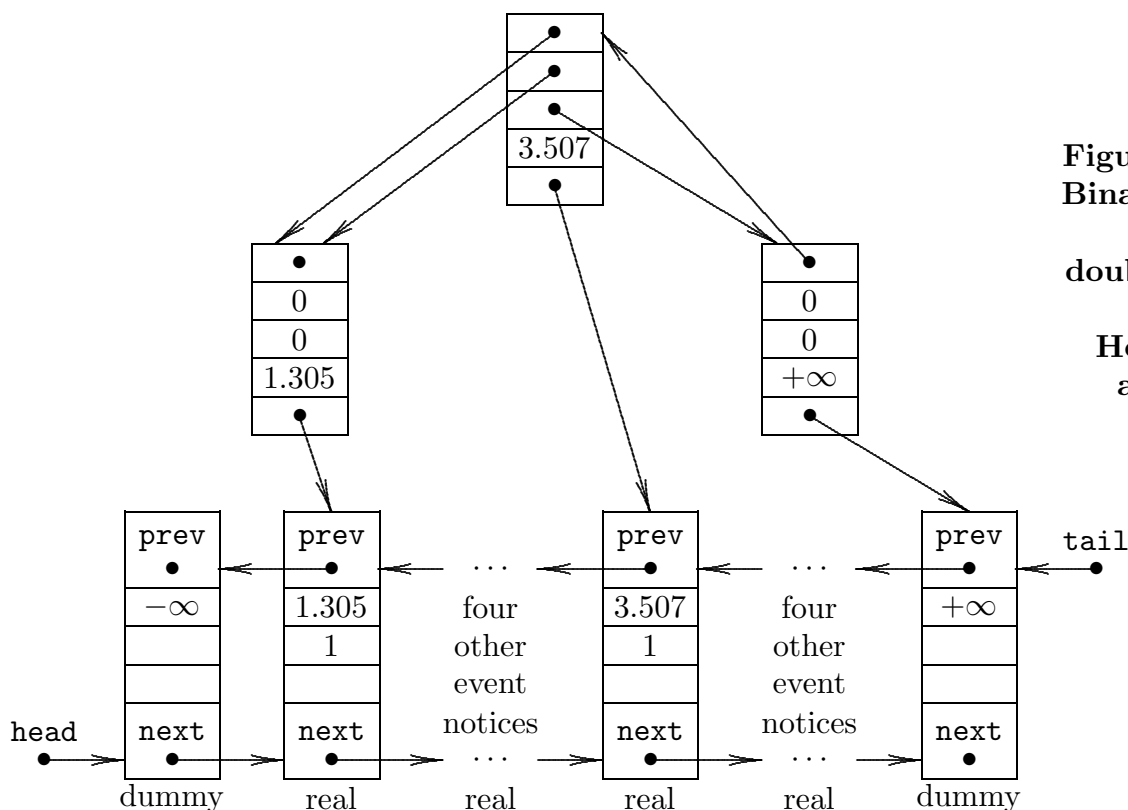
**Figure 5.3.10.
Binary search
tree and
doubly-linked
list for
Henriksen's
algorithm.**

A deletion is an $O(1)$ operation since the first real event notice in the event list is the most imminent event. To insert an event notice into the event list, the binary search tree is traversed in order to find the position of the event in the tree with the smallest event time greater than the event notice being inserted. A backward linear search of the doubly-linked event list is initiated at the event notice to the left of the one found in the binary search tree. This backward linear search continues until either:

- the appropriate insertion position is found in l or fewer searches (Henriksen recommends $l = 4$), in which case the new event notice is linked into the event list at the appropriate position, or

- the appropriate insertion position is not found in l or fewer searches, in which case a "pull" operation is attempted. The pull operation begins by examining the pointer to the event in the binary search tree with the *next lower time* relative to the one found previously. If this pointer is non-zero, its pointer is changed to the most recently examined event notice in the doubly-linked list, i.e., the lth event encountered during the search, and the search continues for another l event notices as before. If the pointer is zero, there are no earlier binary tree nodes that can be updated, so the algorithm adds a new level to the tree. The new level is initialized by setting its leftmost leaf to point to the dummy notice on the right (event time $+\infty$) and setting all other new leaves to point to the dummy notice on the left (event time $-\infty$). The binary search tree is again searched as before.

Henriksen's algorithm works quite well at reducing the average search time for an insertion. Its only drawback seems to be that the maximum search time can be quite long. Other hybrid event list management schemes may also have promise for reducing CPU times associated with insertions and deletions (e.g., Brown, 1988).

5.3.6 EXERCISES

Exercise 5.3.1   Modify program `ttr` so that the initial event for each user is the completion of the first character received as output, rather than the completion of thinking. Run the modified programs for $n = 5, 10, 50, 100$, and for initial seeds 123456789, 987654321, and 555555555. Compare the average number of events for the three simulation runs relative to the results in Section 5.3.3. Offer an explanation of why the observed average number of events goes up or down.

Exercise 5.3.2   Modify program `ttr` to include an event list that is sorted by event time and is stored in a linked list. Verify the results for a forward search given in Example 5.3.1.

Exercise 5.3.3   Assume that all events (thinking, typing a character, and receiving a character) in the think-type-receive model have deterministic durations of exactly 1/10 second. Write a paragraph describing an event list management scheme that requires *no* searching. Include the reason(s) that no searching is required.

Exercise 5.3.4   Assume that all events (thinking, typing a character, and receiving a character) in the think-type-receive model have $Uniform(0.4, 0.6)$ second durations. If you use a doubly-linked list data structure to store the event list with events stored in chronological order, would it be wiser to begin an insertion operation with a search starting at the top (head) of the list or the bottom (tail) of the list? Justify your answer.

Exercise 5.3.5[a]   Assume that all events (thinking, typing a character, and receiving a character) in the think-type-receive model have $Exponential(0.5)$ second durations. If you use a doubly-linked list data structure to store the event list with events stored in chronological order, would it be wiser to begin an insertion operation with a search starting at the top (head) of the list or the bottom (tail) of the list? Justify your answer.

Exercise 5.3.6   The *verification* process from Algorithm 1.1.1 involves checking whether a simulation model is working as expected. Program `ttr` prints the contents of the event list when the simulation reaches its terminal condition. What verification technique could be applied to this output to see if the program is executing as intended.

Exercise 5.3.7   The *verification* process from Algorithm 1.1.1 involves checking whether a simulation model is working as expected. Give a verification technique for comparing the think-type-receive model with (*a*) an unsorted event list with an exhaustive search for a deletion, and (*b*) an event list which is sorted by event time with a backward search for an insertion.

CHAPTER 6

DISCRETE RANDOM VARIABLES

Sections

This chapter begins a more thorough and methodical description of random variables, their properties, how they can be used to model the stochastic (random) components of a system of interest, and the development of algorithms for generating the associated random variates for a Monte Carlo or discrete-event simulation model. This chapter is devoted entirely to describing and generating discrete random variables. The next chapter is devoted to describing and generating continuous random variables.

Section 6.1 defines a discrete random variable and introduces four popular models: the Bernoulli, binomial, Pascal, and Poisson distributions. Section 6.2 contains an approach to generating discrete random variables that is more general than the ad hoc approaches given earlier for the $Equilikely(a, b)$ variate and the $Geometric(p)$ variate. Section 6.3 applies these variate generation techniques to the simple inventory system. Section 6.4 contains a summary of the six discrete distributions encountered thus far. Finally, Section 6.5 departs slightly from the topic of discrete random variables and considers the related topic of the development of algorithms for shuffling and drawing random samples from a group of objects.

As illustrated in previous chapters, random variables, both discrete and continuous, appear naturally in discrete-event simulation models. Because of this, it is virtually impossible to build a valid discrete-event simulation model of a *system* without a good understanding of how to construct a valid random variable model for each of the stochastic system *components*. In this chapter and the next we will develop the mathematical and computational tools for building such stochastic models. Discrete random variables are considered in this chapter; continuous random variables are considered in the next.

6.1.1 DISCRETE RANDOM VARIABLE CHARACTERISTICS

The notation and development in this section largely follows the axiomatic approach to probability. As a convention, uppercase characters X, Y, ... are used to denote random variables (discrete or continuous), the corresponding lowercase characters x, y, ... are used to denote the specific values of X, Y, ..., and calligraphic characters $\mathcal{X}$, $\mathcal{Y}$, ... are used to denote the set of all possible values (often known as the *support* of the random variable). A variety of examples are used in this section to illustrate this notation.

Definition 6.1.1   The random variable X is *discrete* if and only if its set of possible values $\mathcal{X}$ is finite or, at most, countably infinite.

In a discrete-event simulation model discrete random variables are often integers used for counting, e.g., the number of jobs in a queue or the amount of inventory demand. There is no inherent reason, however, why a discrete random variable has to be integer-valued (see Example 6.1.14).

Probability Density Function

Definition 6.1.2   A discrete random variable X is uniquely determined by its set of possible values $\mathcal{X}$ and associated *probability density function* (*pdf*), a real-valued function $f(\cdot)$ defined for each possible value $x \in \mathcal{X}$ as the probability that X has the value x

$$f(x) = \Pr(X = x).$$

By definition, $x \in \mathcal{X}$ is a possible value of X if and only if $f(x) > 0$. In addition, $f(\cdot)$ is defined so that

$$\sum_x f(x) = 1$$

where the sum is over all $x \in \mathcal{X}$.*

It is important to understand the distinction between a random variable, its set of possible values, and its pdf. The usual way to construct a *model* of a discrete random variable X is to first specify the set of possible values $\mathcal{X}$ and then, for each $x \in \mathcal{X}$, specify the corresponding probability $f(x)$. The following three examples are illustrations.

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* The pdf of a discrete random variable is sometimes called a probability *mass* function (pmf) or probability function (pf); that terminology is not used in this book.

Example 6.1.1   If the random variable X is *Equilikely*(a, b), then $\mathcal{X}$ is the set of integers between a and b inclusive. Because $|\mathcal{X}| = b - a + 1$ and each possible value is equally likely, it follows that

$$f(x) = \frac{1}{b - a + 1} \qquad x = a, a + 1, \dots, b.$$

As a specific example, if we were to roll one fair die and let the random variable X be the up face, then X would be *Equilikely*$(1, 6)$.

Example 6.1.2   Roll two fair dice. If the random variable X is the sum of the two up faces, then the set of possible values is $\mathcal{X} = \{x \mid x = 2, 3, \dots, 12\}$ and from the table in Example 2.3.1, the pdf of X is

$$f(x) = \frac{6 - |7 - x|}{36} \qquad x = 2, 3, \dots, 12.$$

Although the sum of the up faces is the usual discrete random variable for games of chance that use two dice, see Exercise 6.1.2 for an alternative.

Example 6.1.3   Suppose a coin has p as its probability of a head and suppose we agree to toss this coin until the *first* tail occurs. If X is the number of heads (i.e., the number of tosses is $X + 1$), then $\mathcal{X} = \{x \mid x = 0, 1, 2, \dots\}$ and the pdf is

$$f(x) = p^x (1 - p) \qquad x = 0, 1, 2, \dots$$

This random variable is said to be *Geometric*(p). (The coin is *fair* if $p = 0.5$.)

Because the set of possible values is *infinite*, for a *Geometric*(p) random variable some math is required to verify that $\sum_x f(x) = 1$. Fortunately, this infinite series and other similar series can be evaluated by using the following properties of *geometric series*. If $p \neq 1$ then

$$1 + p + p^2 + p^3 + \dots + p^x = \frac{1 - p^{x+1}}{1 - p} \qquad x = 0, 1, 2, \dots$$

and if $|p| < 1$ then the following three *infinite series* converge to tractable quantities:

$$1 + p + p^2 + p^3 + p^4 + \dots = \frac{1}{1 - p},$$

$$1 + 2p + 3p^2 + 4p^3 + \dots = \frac{1}{(1 - p)^2},$$

$$1 + 2^2 p + 3^2 p^2 + 4^2 p^3 + \dots = \frac{1 + p}{(1 - p)^3}.$$

Although the three infinite series converge for any $|p| < 1$, negative values of p have no meaning in Example 6.1.3. From the first of these infinite series we have that

$$\sum_x f(x) = \sum_{x=0}^{\infty} p^x (1 - p) = (1 - p)(1 + p + p^2 + p^3 + p^4 + \dots) = 1$$

as required.

As the previous examples illustrate, discrete random variables have possible values that are determined by the outcomes of a random experiment. Therefore, a Monte Carlo simulation program can be used to generate these possible values consistent with their probability of occurrence — see, for example, program `galileo` in Section 2.3. Provided the number of replications is large, a histogram of the values generated by replication should agree well with the random variable's pdf. Indeed, in the limit as the number of replications becomes infinite, the discrete-data histogram and the discrete random variable's pdf should agree *exactly*, as illustrated in Section 4.2.

Cumulative Distribution Function

Definition 6.1.3 The *cumulative distribution function (cdf)* of the discrete random variable X is the real-valued function $F(\cdot)$ defined for each $x \in \mathcal{X}$ as

$$F(x) = \Pr(X \leq x) = \sum_{t \leq x} f(t)$$

where the sum is over all $t \in \mathcal{X}$ for which $t \leq x$.

Example 6.1.4 If X is an *Equilikely*(a, b) random variable (Example 6.1.1), then the cdf is

$$F(x) = \sum_{t=a}^{x} 1/(b - a + 1) = (x - a + 1)/(b - a + 1) \qquad x = a, a + 1, \ldots, b.$$

Example 6.1.5 For the sum-of-two-dice random variable in Example 6.1.2 there is no simple equation for $F(x)$. That is of no real significance, however, because $|\mathcal{X}|$ is small enough that the cumulative pdf values are easily tabulated to yield the cdf. Figure 6.1.1 shows the pdf on the left and the cdf on the right (the vertical scale is different for the two figures). Any of the four styles for plotting a cdf given in Figure 4.2.8 is acceptable.

Figure 6.1.1. Two dice pdf and cdf.
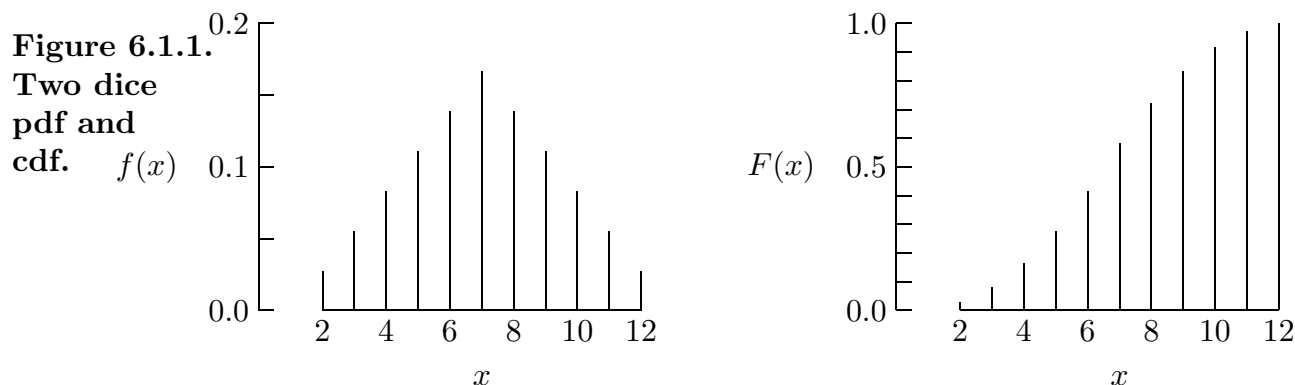
Example 6.1.6 If X is a *Geometric*(p) random variable (Example 6.1.3), then the cdf is

$$F(x) = \sum_{t=0}^{x} p^t (1 - p) = (1 - p)(1 + p + p^2 + \cdots + p^x) = 1 - p^{x+1} \qquad x = 0, 1, 2, \ldots$$

The cdf of a discrete random variable can always be generated from its corresponding pdf by recursion. If the possible values of X are the consecutive integers $x = a, a+1, \ldots, b$, for example, then

$$F(a) = f(a)$$
$$F(x) = F(x-1) + f(x) \qquad x = a+1, a+2, \ldots, b.$$

Similarly, a pdf can always be generated from its corresponding cdf by subtracting consecutive terms

$$f(a) = F(a)$$
$$f(x) = F(x) - F(x-1) \qquad x = a+1, a+2, \ldots, b.$$

Therefore, a model of a discrete random variable can be defined by specifying *either* the pdf *or* the cdf and then computing the other — there is no need to specify both.

As illustrated in Example 6.1.5, the cdf of a discrete random variable is strictly monotone increasing — if x_1 and x_2 are possible values of X with $x_1 < x_2$, then $F(x_1) < F(x_2)$. Moreover, since $F(x)$ is defined as a probability, $0 \leq F(x) \leq 1$. The monotonicity of $F(\cdot)$ is fundamentally important; in the next section we will use it as the basis for a method to generate discrete random variates.

Mean and Standard Deviation

Recall from Definition 4.2.2 that for a discrete-data sample the sample mean $\bar{x}$ and standard deviation s can be computed from the discrete-data histogram as

$$\bar{x} = \sum_x x \hat{f}(x) \qquad \text{and} \qquad s = \sqrt{\sum_x (x - \bar{x})^2 \hat{f}(x)},$$

respectively. Moreover, the histogram relative frequencies $\hat{f}(x)$ converge to the corresponding probabilities $f(x)$ as the sample size becomes infinite. These two observations motivate the following definition.

Definition 6.1.4 The (population) *mean* μ and the corresponding (population) *standard deviation* σ are

$$\mu = \sum_x x f(x) \qquad \text{and} \qquad \sigma = \sqrt{\sum_x (x - \mu)^2 f(x)},$$

where the summations are over all $x \in \mathcal{X}$. The population *variance* is σ^2. An alternative, algebraically equivalent expression for σ is

$$\sigma = \sqrt{\left(\sum_x x^2 f(x) \right) - \mu^2}.$$

The population mean μ is a fixed constant, whereas the sample mean $\bar{x}$ is a random variable that varies from sample to sample. They are fundamentally different, although analogous, quantities. Since $\bar{x}$ is a random variable, it has a pdf and cdf that describe its distribution.

Example 6.1.7   If X is an *Equilikely*(a, b) random variable (Example 6.1.1), then

$$\mu = \frac{a+b}{2} \qquad \text{and} \qquad \sigma = \sqrt{\frac{(b-a+1)^2 - 1}{12}}$$

The derivation of these two equations is left as an exercise. In particular, if X is the result of rolling one fair die, then X is *Equilikely*$(1, 6)$ so that

$$\mu = 3.5 \qquad \text{and} \qquad \sigma = \sqrt{\frac{35}{12}} \cong 1.708,$$

illustrating the fact that μ need not necessarily be a member of $\mathcal{X}$. The geometric interpretation of the mean is the horizontal center of mass of the distribution.

Example 6.1.8   If X is the sum-of-two-dice random variable (Example 6.1.2), then

$$\mu = \sum_{x=2}^{12} xf(x) = \cdots = 7 \qquad \text{and} \qquad \sigma^2 = \sum_{x=2}^{12} (x-\mu)^2 f(x) = \cdots = 35/6.$$

Therefore the population standard deviation is $\sigma = \sqrt{35/6} \cong 2.415$.

Example 6.1.9   If X is a *Geometric*(p) random variable (Example 6.1.3), then

$$\mu = \sum_{x=0}^{\infty} xf(x) = \sum_{x=1}^{\infty} xp^x(1-p) = \cdots = \frac{p}{1-p}$$

using the infinite series following Example 6.1.3, and

$$\sigma^2 = \left(\sum_{x=0}^{\infty} x^2 f(x)\right) - \mu^2 = \left(\sum_{x=1}^{\infty} x^2 p^x(1-p)\right) - \frac{p^2}{(1-p)^2} = \cdots = \frac{p}{(1-p)^2}$$

so that $\sigma = \sqrt{p}/(1-p)$. The derivation of these equations is left as an exercise. In particular, tossing a fair $(p = 0.5)$ coin until the first tail occurs generates a *Geometric*(0.5) random variable (X is the number of heads) with

$$\mu = 1 \qquad \text{and} \qquad \sigma = \sqrt{2} \cong 1.414.$$

The population mean and the population variance are two special cases of a more general notion known as "expected value".

Expected Value

Definition 6.1.5   The mean of a random variable (discrete or continuous) is also known as the *expected value*. It is conventional to denote the expected value as $E[\cdot]$. That is, the expected value of the discrete random variable X is

$$E[X] = \sum_x xf(x) = \mu$$

where the summation is over all $x \in \mathcal{X}$.*

---

  * The expected value may not exist if there are infinitely many possible values.

If we were to use a Monte Carlo simulation to generate a large random variate sample $x_1, x_2, \ldots, x_n$ corresponding to the random variable X and then calculate the sample mean $\bar{x}$, we would expect to find that $\bar{x} \to E[X] = \mu$ as $n \to \infty$. Thus, the expected value of X is really the "expected average" and in that sense the term "expected value" is potentially misleading. The expected value (the mean) is not necessarily the *most likely* possible value [which is the *mode*, the element in $\mathcal{X}$ corresponding to the largest value of $f(x)$].

Example 6.1.10   If a fair coin is tossed until the first tail appears, then the *most likely* number of heads is 0 and the *expected* number of heads is 1 (see Example 6.1.9). In this case, 0 occurs with probability 1/2 and 1 occurs with probability 1/4. Thus the most likely value (the mode) is twice as likely as the expected value (the mean). On the other hand, for some random variables the mean and mode may be the same. For example, if X is the sum-of-two-dice random variable then the expected value and the most likely value are both 7 (see Examples 6.1.5 and 6.1.8).

Definition 6.1.6   If $h(\cdot)$ is a function defined for all possible values of X, then as x takes on all possible values in $\mathcal{X}$ the equation $y = h(x)$ defines the set $\mathcal{Y}$ of possible values for a *new* random variable $Y = h(X)$. The expected value of Y is

$$E[Y] = E[h(X)] = \sum_x h(x)f(x)$$

where the sum is over all $x \in \mathcal{X}$.*

Example 6.1.11   If $y = (x - \mu)^2$ with $\mu = E[X]$, then from Definitions 6.1.4 and 6.1.6

$$E[Y] = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x) = \sigma^2.$$

That is, the variance σ^2 is the expected value of the squared difference about the mean. Similarly, if $y = x^2 - \mu^2$ then

$$E[Y] = E[X^2 - \mu^2] = \sum_x (x^2 - \mu^2)f(x) = \left(\sum_x x^2 f(x) \right) - \mu^2 = \sigma^2$$

so that

$$\sigma^2 = E[X^2] - E[X]^2.$$

This last equation demonstrates that the two operations $E[\cdot]$ and $(\cdot)^2$ do *not* commute; the expected value of X^2 is not equal to the square of $E[X]$. Indeed, $E[X^2] \geq E[X]^2$ with equality if and only if X is not really random at all, i.e., $\sigma^2 = 0$, often known as a *degenerate* distribution.

---

* Definition 6.1.6 can be established as a theorem. When presented as a definition it is sometimes called the "law of the unconscious statistician". If the set of possible values $\mathcal{X}$ is infinite then $E[Y]$ may not exist.

Example 6.1.12   If $Y = aX + b$ for constants a and b, then

$$E[Y] = E[aX + b] = \sum_x (ax + b)f(x) = a\left(\sum_x xf(x)\right) + b = aE[X] + b.$$

In particular, suppose that X is the number of heads before the first tail and that you are playing a game with a fair coin where you win \$2 for every head. Let the random variable Y be the amount you win. The possible values of Y are defined by

$$y = h(x) = 2x \qquad\qquad x = 0, 1, 2, \ldots$$

and your *expected winnings* (for each play of the game) are

$$E[Y] = E[2X] = 2E[X] = 2.$$

If you play this game repeatedly and pay more than \$2 per game to do so, then in the long run expect to lose money.

6.1.2   DISCRETE RANDOM VARIABLE MODELS

Let X be any discrete random *variable*. In the next section we will consider a unified algorithmic approach to generating possible values of X. The values so generated are random *variates*. The distinction between a random variable (discrete or continuous) and a corresponding random variate is subtle, but important. The former is an abstract, but well-defined, mathematical function that maps the outcome of an experiment to a real number (see Definition 6.1.2); the latter is an algorithmically generated possible value (realization) of the former (see Definition 2.3.2). For example, the functions `Equilikely` (Definition 2.3.4) and `Geometric` (Definition 3.1.2) generate random *variates* corresponding to *Equilikely*(a, b) and *Geometric*(p) random *variables*, respectively.

Bernoulli Random Variable

Example 6.1.13   The discrete random *variable* X with possible values $\mathcal{X} = \{0, 1\}$ is said to be *Bernoulli*(p) if $X = 1$ with probability p and $X = 0$ otherwise (i.e., with probability $1 - p$). In effect, X is a Boolean random variable with 1 as `true` and 0 as `false`. The pdf for a *Bernoulli*(p) random variable is $f(x) = p^x(1 - p)^{1-x}$ for $x \in \mathcal{X}$. The corresponding cdf is $F(x) = (1 - p)^{1-x}$ for $x \in \mathcal{X}$. The mean is $\mu = 0 \cdot (1 - p) + 1 \cdot p = p$ and the variance is $\sigma^2 = (0 - p)^2(1 - p) + (1 - p)^2 p = p(1 - p)$. Therefore, the standard deviation is $\sigma = \sqrt{p(1 - p)}$. We can generate a corresponding *Bernoulli*(p) random *variate* as follows.

```
if (Random() < 1.0 - p)
   return 0;
else
   return 1;
```

As illustrated in Section 2.3, no matter how sophisticated or computationally complex, a Monte Carlo simulation that uses n replications to estimate an (unknown) probability p is equivalent to generating an *iid* sequence of n *Bernoulli*(p) random variates.

Example 6.1.14   A popular state lottery game *Pick-3* requires players to pick a 3-digit number from the 1000 numbers between 000 and 999. It costs \$1 to play the game. If the 3-digit number picked by a player matches the 3-digit number chosen, at random, by the state then the player wins \$500, minus the original \$1 investment, for a net yield of +\$499. Otherwise, the player's yield is −\$1. (See Exercise 6.1.7 for another way to play this game.) Let the discrete random variable X represent the result of playing the game with the convention that $X = 1$ denotes a win and $X = 0$ denotes a loss. Then X is a *Bernoulli*(p) random variable with $p = 1/1000$. In addition, let the discrete random variable $Y = h(X)$ be the player's yield where

$$h(x) = \begin{cases} -1 & x = 0 \\ 499 & x = 1. \end{cases}$$

From Definition 6.1.6, the player's *expected* yield is

$$E[Y] = \sum_{x=0}^{1} h(x)f(x) = h(0)(1-p) + h(1)p = -1 \cdot \frac{999}{1000} + 499 \cdot \frac{1}{1000} = -0.5.$$

In this case, Y has just two possible values — one is 1000 times more likely than the other and neither is the expected value. Even though the support values for Y are far apart (-1 and 499), the value of $E[Y]$ shows that playing *Pick-3* is the equivalent of a voluntary 50 cent tax to the state for every dollar bet.

Because it has only two possible values, a *Bernoulli*(p) random variable may seem to have limited applicability. That is not the case, however, because this simple random variable can be used to construct more sophisticated stochastic models, as illustrated by the following examples.

Binomial Random Variable

Example 6.1.15   In the spirit of Example 6.1.3, suppose a coin has p as its probability of tossing a head and suppose we toss this coin n times. Let X be the number of heads; in this case X is said to be a *Binomial*(n, p) random variable. The set of possible values is $\mathcal{X} = \{0, 1, 2, \ldots, n\}$ and the associated pdf is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \ldots, n.$$

That is, $p^x(1-p)^{n-x}$ is the probability of x heads and $n - x$ tails and the binomial coefficient accounts for the number of different sequences in which these heads and tails can occur. Equivalently, n tosses of the coin generate an *iid* sequence X_1, X_2, ..., X_n of *Bernoulli*(p) random variables ($X_i = 1$ corresponds to a head on the ith toss) and

$$X = X_1 + X_2 + \cdots + X_n.$$

Although it may be intuitive that the pdf in Example 6.1.15 is correct, when building a discrete random variable model it is necessary to confirm that, in fact, $f(x) > 0$ for all $x \in \mathcal{X}$ (which is obvious in this case) and that $\sum_x f(x) = 1$ (which is not obvious). To verify that the pdf sum is 1, we can use the *binomial equation*

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}.$$

In the particular case where $a = p$ and $b = 1 - p$

$$1 = (1)^n = \big(p + (1-p)\big)^n = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

which is equivalent to $f(0) + f(1) + \cdots + f(n) = 1$ as desired. To determine the mean of a binomial random variable:

$$\mu = E[X] = \sum_{x=0}^{n} x f(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x! \, (n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x! \, (n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)! \, (n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)! \, (n-x)!} p^{x-1} (1-p)^{n-x}.$$

To evaluate the last sum, let $m = n - 1$ and $t = x - 1$ so that $m - t = n - x$. Then from the binomial equation

$$\mu = np \sum_{t=0}^{m} \frac{m!}{t! \, (m-t)!} p^t (1-p)^{m-t} = np\big(p + (1-p)\big)^m = np(1)^m = np.$$

That is, the mean of a *Binomial*(n, p) random variable is $\mu = np$. In a similar way, it can be shown that the variance is

$$\sigma^2 = E[X^2] - \mu^2 = \left(\sum_{x=0}^{n} x^2 f(x) \right) - \mu^2 = \cdots = np(1-p)$$

so that the standard deviation of a *Binomial*(n, p) random variable is $\sigma = \sqrt{np(1-p)}$.

Pascal Random Variable

Example 6.1.16   As a second example of using $Bernoulli(p)$ random variables to build a more sophisticated stochastic model, suppose a coin has p as its probability of a head and suppose we toss this coin until the nth tail occurs. If X is the number of heads (i.e., the number of tosses is $X + n$), then X is said to be a $Pascal(n, p)$ random variable. The set of possible values is $\mathcal{X} = \{0, 1, 2, \ldots\}$ and the associated pdf is

$$f(x) = \binom{n + x - 1}{x} p^x (1 - p)^n \qquad x = 0, 1, 2, \ldots$$

That is, $p^x(1 - p)^n$ is the probability of x heads and n tails and the binomial coefficient accounts for the number of different sequences in which these $n + x$ heads and tails can occur, given that the last coin toss must be a tail.

Although it may be intuitive that the $Pascal(n, p)$ pdf equation is correct, it is necessary to prove that the infinite pdf sum does, in fact, converge to 1. The proof of this property is based on another (negative exponent) version of the *binomial equation* — for any positive integer n

$$(1 - p)^{-n} = 1 + \binom{n}{1} p + \binom{n + 1}{2} p^2 + \cdots + \binom{n + x - 1}{x} p^x + \cdots$$

provided $|p| < 1$. (See Example 6.1.3 for $n = 1$ and $n = 2$ versions of this equation.) By using the negative-exponent binomial equation we see that*

$$\sum_{x=0}^{\infty} \binom{n + x - 1}{x} p^x (1 - p)^n = (1 - p)^n \sum_{x=0}^{\infty} \binom{n + x - 1}{x} p^x = (1 - p)^n (1 - p)^{-n} = 1$$

which confirms that $f(0) + f(1) + f(2) + \cdots = 1$. Moreover, in a similar way it can be shown that the mean is

$$\mu = E[X] = \sum_{x=0}^{\infty} x f(x) = \cdots = \frac{np}{1 - p}$$

and the variance is

$$\sigma^2 = E[X^2] - \mu^2 = \left(\sum_{x=0}^{\infty} x^2 f(x) \right) - \mu^2 = \cdots = \frac{np}{(1 - p)^2}$$

so that the standard deviation is $\sigma = \sqrt{np}/(1 - p)$. The details of this derivation are left as an exercise.

---

* Because the negative-exponent binomial equation is applicable to the mathematical analysis of its characteristics, a $Pascal(n, p)$ random variable is also known as a *negative binomial* random variable.

Example 6.1.17   As a third example of using $Bernoulli(p)$ random variables to build a more sophisticated stochastic model, a $Geometric(p)$ random variable is a special case of a $Pascal(n, p)$ random variable when $n = 1$. If instead $n > 1$ and if $X_1, X_2, \ldots, X_n$ is an iid sequence of n $Geometric(p)$ random variables, then the sum

$$X = X_1 + X_2 + \cdots + X_n$$

is a $Pascal(n, p)$ random variable. For example, if $n = 4$ and if p is large then a typical head/tail sequence might look like

$$\underbrace{hhhhhht}_{X_1 = 6} \; \underbrace{hhhhhhhhht}_{X_2 = 9} \; \underbrace{hhhht}_{X_3 = 4} \; \underbrace{hhhhhht}_{X_4 = 7}$$

where X_1, X_2, X_3, X_4 count the number of heads in each $h \ldots ht$ sequence and, in this case,

$$X = X_1 + X_2 + X_3 + X_4 = 26.$$

The number of heads in each $h \cdots ht$ $Bernoulli(p)$ sequence is an independent realization of a $Geometric(p)$ random variable. In this way we see that a $Pascal(n, p)$ random variable is the sum of n iid $Geometric(p)$ random variables. From Example 6.1.9 we know that if X is $Geometric(p)$ then $\mu = p/(1 - p)$ and $\sigma = \sqrt{p}/(1 - p)$.

Poisson Random Variable

A $Poisson(\mu)$ random variable is a limiting case of a $Binomial(n, \mu/n)$ random variable. That is, let X be a $Binomial(n, p)$ random variable with $p = \mu/n$. Fix the values of μ and x and consider what happens in the limit as $n \to \infty$. The pdf of X is

$$f(x) = \frac{n!}{x!(n - x)!} \left(\frac{\mu}{n} \right)^x \left(1 - \frac{\mu}{n} \right)^{n-x} = \frac{\mu^x}{x!} \left(\frac{n!n^x}{(n - x)!(n - \mu)^x} \right) \left(1 - \frac{\mu}{n} \right)^n$$

for $x = 0, 1, \ldots, n$. It can be shown that

$$\lim_{n \to \infty} \left(\frac{n!n^x}{(n - x)!(n - \mu)^x} \right) = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \left(1 - \frac{\mu}{n} \right)^n = \exp(-\mu)$$

so that

$$\lim_{n \to \infty} f(x) = \frac{\mu^x \exp(-\mu)}{x!}.$$

This limiting case is the motivation for defining a $Poisson(\mu)$ random variable. For large values of n, $Binomial(n, \mu/n)$ and $Poisson(\mu)$ random variables are virtually identical, particularly if μ is small. As an exercise you are asked to prove that the parameter μ in the definition of the $Poisson(\mu)$ pdf

$$f(x) = \frac{\mu^x \exp(-\mu)}{x!} \qquad\qquad x = 0, 1, 2, \ldots$$

is in fact the mean and that the standard deviation is $\sigma = \sqrt{\mu}$.

6.1.3 EXERCISES

Exercise 6.1.1 (*a*) Simulate rolling a pair of dice 360 times with five different seeds and generate five histograms of the resulting sum of the two up faces. Compare the histogram mean, standard deviation and relative frequencies with the corresponding population mean, standard deviation and pdf. (*b*) Repeat for 3600, 36 000, and 360 000 replications. (*c*) Comment.

Exercise 6.1.2 Repeat the previous exercise *except* that the random variable of interest is the absolute value of the *difference* between the two up faces.

Exercise 6.1.3 Derive the equations for μ and σ in Example 6.1.7. (See Exercise 6.1.5.)

Exercise 6.1.4 Prove that $\sum_x (x - \mu)^2 f(x) = \left(\sum_x x^2 f(x) \right) - \mu^2$.

Exercise 6.1.5 X is a discrete random variable with possible values $x = 1, 2, \ldots, n$. (*a*) If the pdf of X is $f(x) = \alpha x$ then what is α (as a function of n)? (*b*) Determine the cdf, mean, and standard deviation of X. *Hint*:

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2} \qquad \sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{x=1}^{n} x^3 = \frac{n^2(n+1)^2}{4}.$$

Exercise 6.1.6 Fill in the $= \cdots =$'s in Example 6.1.9.

Exercise 6.1.7 As an alternative to Example 6.1.14, another way to play Pick-3 is for the player to opt for a win *in any order*. That is, for example, if the player's number is 123 then the player will win (the same amount) if the state draws any of 123, 132, 231, 213, 321, or 312. Because this is an easier win, the pay-off is suitably smaller, namely $80 for a net yield of +$79. (*a*) What is the player's expected yield (per game) if this variation of the game is played? (Assume the player is bright enough to pick a 3-digit number with three different digits.) (*b*) Construct a Monte Carlo simulation to supply "convincing numerical evidence" of the correctness of your solution.

Exercise 6.1.8[a] An urn is initially filled with 1 amber ball and 1 black ball. Each time a ball is drawn, at random, if it is a black ball then it *and* another black ball are put back in the urn. Let X be the number of random draws required to find the amber ball. (*a*) What is $E[X]$? (*b*) Construct a Monte Carlo simulation to estimate $E[X]$ based on 1000, 10 000, 100 000, and 1 000 000 replications. (*c*) Comment.

Exercise 6.1.9[a] The location of an interval of fixed length $r > 0$ is selected at random on the real number line. Let X be the number of integers within the interval. Find the mean and standard deviation of X as a function of (r, p) where $p = r - \lfloor r \rfloor$.

Exercise 6.1.10 If X is $Poisson(\mu)$ random variable, prove that the mean of X is the parameter μ and that the standard deviation is $\sqrt{\mu}$.

In this section we will consider the development of correct, exact algorithms for the generation of discrete random variates. We begin with an important definition, associated theorem, and algorithm.

6.2.1 INVERSE DISTRIBUTION FUNCTION

Definition 6.2.1 Let X be a discrete random variable with cdf $F(\cdot)$. The *inverse distribution function* (*idf*) of X is the function F^*: $(0,1) \rightarrow \mathcal{X}$ defined for all $u \in (0,1)$ as

$$F^*(u) = \min_x \{x : u < F(x)\}$$

where the minimum is over all possible values $x \in \mathcal{X}$. That is, if $F^*(u) = x$ then x is the smallest possible value of X for which $F(x)$ is greater than u.*

Example 6.2.1 Figure 6.2.1 provides a graphical illustration of the idf for a discrete random variable with $\mathcal{X} = \{a, a+1, \ldots, b\}$ for two common ways of plotting the same cdf. The value of $x \in \mathcal{X}$ corresponding to a given u, in this case $u = 0.45$, is found by extending the dashed line horizontally until it strikes one of the cdf "spikes" in the left-hand cdf or one of the "risers" on the right-hand cdf. The value of $X \in \mathcal{X}$ corresponding to that cdf spike defines the value of $x = F^*(u)$.

Figure 6.2.1.
Generating
discrete
random
variables.
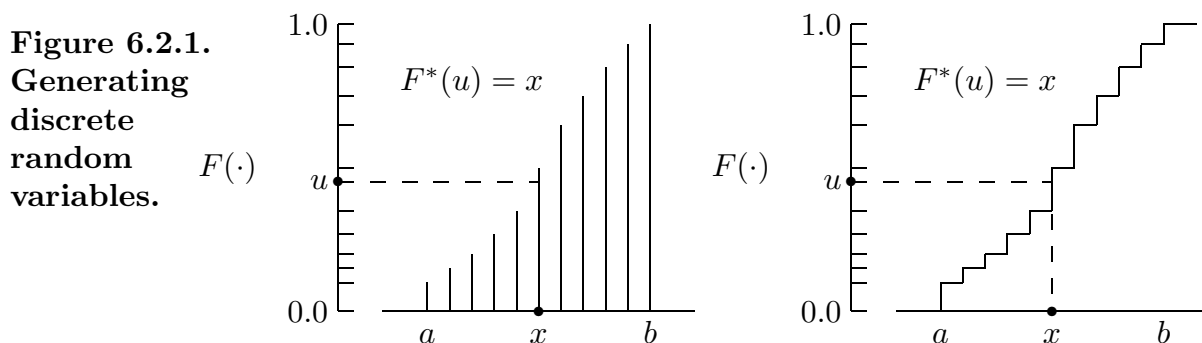
The following theorem is based on Definition 6.2.1 and the insight provided by the previous example. The significance of the theorem is that it provides an important algorithmic characterization of the idf.

Theorem 6.2.1 Let X be an integer-valued random variable with $\mathcal{X} = \{a, a+1, \ldots, b\}$ where b may be ∞ and let $F(\cdot)$ be the cdf of X. For any $u \in (0,1)$, if $u < F(a)$ then $F^*(u) = a$, else $F^*(u) = x$ where $x \in \mathcal{X}$ is the (unique) possible value of X for which $F(x-1) \le u < F(x)$.

Although Theorem 6.2.1 (and others that follow in this section) are written for *integer-valued* discrete random variables, they are easily extended to the more general case.

---

* The rather unusual definition of the inverse [and the avoidance of the notation $F^{-1}(u)$] is to account for the fact that $F(x)$ is not 1–1. A specific value of the idf is also called a *fractile*, *quantile*, or *percentile* of the distribution of X. For example, $F^*(0.95)$ is often called the 95th percentile of the distribution of X.

Algorithm 6.2.1 Let X be an integer-valued random variable with $\mathcal{X} = \{a, a+1, \ldots, b\}$ where b may be ∞ and let $F(\cdot)$ be the cdf of X. For any $u \in (0, 1)$ the following linear search algorithm defines the idf $F^*(u)$

```
x = a;
while (F(x) <= u)
   x++;
return x;                                       /* x is F*(u) */
```

Because $u < 1.0$ and $F(x) \to 1.0$ as x increases, the loop in Algorithm 6.2.1 is guaranteed to terminate. Note, however, that the search is linear and begins at the smallest possible value of X so that *average case* efficiency can be a real problem if $\mu = E[X]$ is large relative to a. More specifically, in repeated application of this algorithm using values of u generated by calls to `Random`, the average number of passes through the `while` loop will be $\mu - a$. To see this, let u be a *Uniform*$(0, 1)$ random variate and let X be the discrete random variable corresponding to the random variate x generated by Algorithm 6.2.1. If the discrete random variable Y represents the number of `while` loop passes then $Y = X - a$. Therefore, from Definition 6.1.6, $E[Y] = E[X - a] = E[X] - a = \mu - a$.

Algorithm 6.2.1 can be made more efficient by starting the search at a better (more likely) point. The best (most likely) starting point is the mode.* We then have the following more efficient version of Algorithm 6.2.1.

Algorithm 6.2.2 Let X be an integer-valued random variable with $\mathcal{X} = \{a, a+1, \ldots, b\}$ where b may be ∞ and let $F(\cdot)$ be the cdf of X. For any $u \in (0, 1)$, the following linear search algorithm defines the idf $F^*(u)$

```
x = mode;                           /* initialize with the mode of X */
if (F(x) <= u)
   while (F(x) <= u)
      x++;
else if (F(a) <= u)
   while (F(x - 1) > u)
      x--;
else
   x = a;
return x;                                       /* x is F*(u) */
```

Although Algorithm 6.2.2 is still a linear search, generally it is more efficient than Algorithm 6.2.1, perhaps significantly so, unless the mode of X is a. If $|\mathcal{X}|$ is very large, even more efficiency may be needed. In this case, a *binary* search should be considered. (See Exercise 6.2.9.)

---

* The mode of X is the value of $x \in \mathcal{X}$ for which $f(x)$ is largest. For many discrete random variables, but *not* all, $\lfloor \mu \rfloor$ is an essentially equivalent choice.

Idf Examples

In some important cases the idf $F^*(u)$ can be determined explicitly by using Theorem 6.2.1 to solve the equation $F(x) = u$ for x. The following three examples are illustrations.

Example 6.2.2   If X is *Bernoulli*(p) and $F(x) = u$ then $x = 0$ if and only if $0 < u < 1-p$ and $x = 1$ otherwise. Therefore

$$F^*(u) = \begin{cases} 0 & 0 < u < 1 - p \\ 1 & 1 - p \le u < 1. \end{cases}$$

Example 6.2.3   If X is *Equilikely*(a, b) then

$$F(x) = \frac{x - a + 1}{b - a + 1} \qquad x = a, a + 1, \ldots, b.$$

Therefore, provided $u \ge F(a)$,

$$F(x - 1) \le u < F(x) \iff \frac{(x - 1) - a + 1}{b - a + 1} \le u < \frac{x - a + 1}{b - a + 1}$$
$$\iff x - a \le (b - a + 1)u < x - a + 1$$
$$\iff x \le a + (b - a + 1)u < x + 1.$$

Thus, for $F(a) \le u < 1$ the idf is

$$F^*(u) = a + \lfloor (b - a + 1)u \rfloor.$$

Moreover, if $0 < u < F(a) = 1/(b - a + 1)$ then $0 < (b - a + 1)u < 1$ so that $F^*(u) = a$, as required. Therefore, the idf equation is valid for all $u \in (0, 1)$.

Example 6.2.4   If X is *Geometric*(p) then

$$F(x) = 1 - p^{x+1} \qquad x = 0, 1, 2, \ldots$$

Therefore, provided $u \ge F(0)$,

$$F(x - 1) \le u < F(x) \iff 1 - p^x \le u < 1 - p^{x+1}$$
$$\iff -p^x \le u - 1 < -p^{x+1}$$
$$\iff p^x \ge 1 - u > p^{x+1}$$
$$\iff x \ln(p) \ge \ln(1 - u) > (x + 1) \ln(p)$$
$$\iff x \le \frac{\ln(1 - u)}{\ln(p)} < x + 1.$$

Thus, for $F(0) \le u < 1$ the idf is

$$F^*(u) = \left\lfloor \frac{\ln(1 - u)}{\ln(p)} \right\rfloor.$$

Moreover, if $0 < u < F(0) = 1 - p$ then $p < 1 - u < 1$ so that $F^*(u) = 0$, as required. Therefore, the idf equation is valid for all $u \in (0, 1)$.

6.2.2 RANDOM VARIATE GENERATION BY INVERSION

The following theorem is of fundamental importance in random variate generation applications. Because of the importance of this theorem, a detailed proof is presented. The proof makes use of a definition and two results, listed here for reference.

- Two random variables X_1 and X_2 with corresponding pdf's $f_1(\cdot)$ and $f_2(\cdot)$ defined on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, are *identically distributed* if and only if they have a common set of possible values $\mathcal{X}_1 = \mathcal{X}_2$ and for all x in this common set $f_1(x) = f_2(x)$.

- The idf $F^*\colon (0,1) \to \mathcal{X}$ maps the interval $(0,1)$ *onto* $\mathcal{X}$.

- If U is *Uniform*$(0,1)$ and $0 \le \alpha < \beta \le 1$ then $\Pr(\alpha \le U < \beta) = \beta - \alpha$.*

Theorem 6.2.2 (Probability integral transformation) If X is a discrete random variable with idf $F^*(\cdot)$ and the continuous random variable U is *Uniform*$(0,1)$ and Z is the discrete random variable defined by $Z = F^*(U)$, then Z and X are identically distributed.

Proof Let $\mathcal{X}$, $\mathcal{Z}$ be the set of possible values for X, Z respectively. If $x \in \mathcal{X}$ then, because F^* maps $(0,1)$ onto $\mathcal{X}$, there exists $u \in (0,1)$ such that $F^*(u) = x$. From the definition of Z it follows that $x \in \mathcal{Z}$ and so $\mathcal{X} \subseteq \mathcal{Z}$. Similarly, if $z \in \mathcal{Z}$ then from the definition of Z there exists $u \in (0,1)$ such that $F^*(u) = z$. Because $F^*\colon (0,1) \to \mathcal{X}$, it follows that $z \in \mathcal{X}$ and so $\mathcal{Z} \subseteq \mathcal{X}$. This proves that $\mathcal{X} = \mathcal{Z}$. Now, let $\mathcal{X} = \mathcal{Z} = \{a, a+1, \ldots, b\}$ be the common set of possible values. To prove that X and Z are identically distributed we must show that $\Pr(Z = z) = f(z)$ for any z in this set of possible values, where $f(\cdot)$ is the pdf of X. From the definition of Z and $F^*(\cdot)$ and Theorem 6.2.1, if $z = a$ then

$$\Pr(Z = a) = \Pr\left(U < F(a)\right) = F(a) = f(a).$$

Moreover, if $z \in \mathcal{Z}$ with $z \ne a$ then

$$\Pr(Z = z) = \Pr\left(F(z-1) \le U < F(z)\right) = F(z) - F(z-1) = f(z)$$

which proves that Z and X have the same pdf and so are identically distributed.

Theorem 6.2.2 is the basis for the following algorithm by which *any* discrete random variable can be generated with just *one* call to `Random`. In Section 7.2 we will see that there is an analogous theorem and algorithm for continuous random variables. This elegant algorithm is known as the *inversion* method for random variate generation.

Algorithm 6.2.3 If X is a discrete random variable with idf $F^*(\cdot)$ then a corresponding discrete random variate x can be generated as follows

```
u = Random();
return F*(u);
```

---

* See Example 7.1.2 in the next chapter for more discussion of this result.

Inversion Examples

Random variate generation by inversion has a clear intuitive appeal and always works. It is *the* method of choice for discrete random variate generation, provided the idf can be manipulated into a form that is mathematically tractable for algorithmic implementation. We begin with an example for an arbitrary discrete distribution.

Example 6.2.5 Consider the discrete random variable X with pdf:

$$f(x) = \begin{cases} 0.1 & x = 2 \\ 0.3 & x = 3 \\ 0.6 & x = 6. \end{cases}$$

The cdf for X is plotted using two different formats in Figure 6.2.2.

Figure 6.2.2. Inversion for an arbitrary discrete random variable.
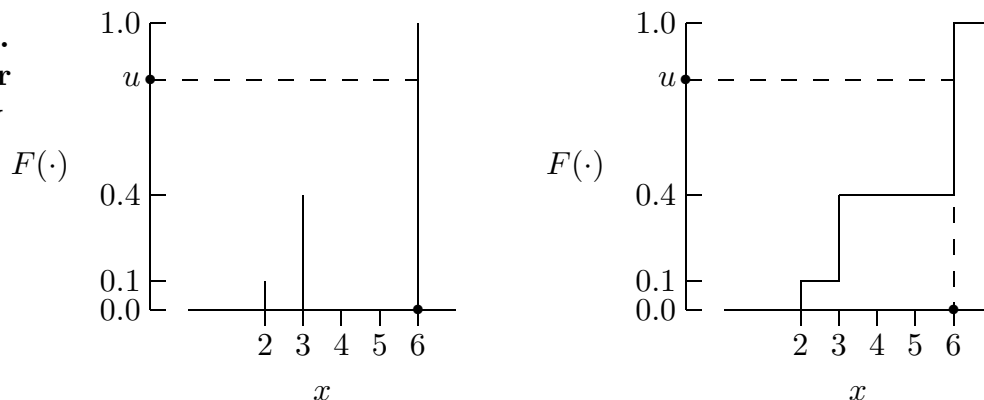
For a random number u (which strikes the vertical axes in Figure 6.2.2, for example, at $u = 0.803$), $F^*(u)$ assumes a value of 2, 3, or 6 depending on the intersection point of the associated horizontal line with the cdf. The reader is encouraged to reflect on the geometry in Figure 6.2.2 to appreciate the fact that the algorithm

```
u = Random();
if (u < 0.1)
    return 2;
else if (u < 0.4)
    return 3;
else
    return 6;
```

returns 2 with probability 0.1, returns 3 with probability 0.3, and returns 6 with probability 0.6, which corresponds to the pdf of X. The inversion method for generating discrete random variates always carries this intuitive geometric interpretation. Problems arise, however, when $|\mathcal{X}|$ is large or countably infinite. The average execution time for this particular example can be minimized by checking the ranges for u associated with $x = 6$ (the mode) first, then $x = 3$, then $x = 2$. This way, 60% of the invocations of this algorithm require the evaluation of just one condition.

Random variate generation algorithms for popular discrete probability models that include parameters follow.

Example 6.2.6   This algorithm uses inversion and the idf expression from Example 6.2.2 to generate a *Bernoulli*(p) random variate (see Example 6.1.13)

```
u = Random();
if (u < 1 - p)
    return 0;
else
    return 1;
```

Example 6.2.7   This algorithm uses inversion and the idf expression from Example 6.2.3 to generate an *Equilikely*(a, b) random variate

```
u = Random();
return a + (long) (u * (b - a + 1));
```

This is the algorithm used to define the function `Equilikely`(a, b) (see Definition 2.3.4).

Example 6.2.8   This algorithm uses inversion and the idf expression from Example 6.2.4 to generate a *Geometric*(p) random variate

```
u = Random();
return (long) (log(1.0 - u) / log(p));
```

This is the algorithm used to define the function `Geometric`(p) (see Definition 3.1.2).

Examples 6.2.6, 6.2.7, and 6.2.8 illustrate random variate generation by inversion for three of the six parametric discrete random variable models presented in the previous section. For the other three models, inversion is not so easily applied.

Example 6.2.9   If X is a *Binomial*(n, p) random variable the cdf is

$$F(x) = \sum_{t=0}^{x} \binom{n}{t} p^t (1-p)^{n-t} \qquad x = 0, 1, 2, \ldots, n.$$

It can be shown that this sum is the complement of an *incomplete beta function* (see Appendix D)

$$F(x) = \begin{cases} 1 - I(x+1, n-x, p) & x = 0, 1, \ldots, n-1 \\ 1 & x = n. \end{cases}$$

Except for special cases, an incomplete beta function cannot be inverted algebraically to form a "closed form" expression for the idf. Therefore, inversion is not easily applied to the generation of *Binomial*(n, p) random variates.

Algorithm Design Criteria

As suggested by Example 6.2.9, for many discrete random *variables* the design of a correct, exact, and efficient algorithm to generate corresponding random *variates* is significantly more complex than just a straight-forward implementation of Algorithm 6.2.3. There are some important, generally accepted algorithm design criteria listed below. With minor modifications, these criteria apply equally well to continuous random variate generation algorithms also.

- *Portability* — a random variate generation algorithm should be implementable in a high-level language in such a way that it ports easily to any common contemporary computing environment. Hardware-dependent implementations should be rejected, even if the cost to do so is an increase in execution time.

- *Exactness* — assuming a perfect random number generator, the histogram that results from generating random variates from an *exact* algorithm converges to the corresponding random variable pdf as the number of random variates generated goes to infinity. Example 6.2.10, presented later, is an illustration of a random variate generator that is approximate, not exact.

- *Robustness* — the performance of a random variate generation algorithm should be insensitive to small changes in the random variable model parameters (e.g., n and p for a $Binomial(n, p)$ model) and work properly for all reasonable values of the parameters.

- *Efficiency* — although this criteria is commonly over-rated, a random variate generation algorithm should be both time and memory efficient. Execution time often consists of two parts. *Set-up* time occurs once (for example, to compute and store a cdf array); *marginal* execution time occurs each time a random variate is generated. Ideally both times are small. If not, an implementation-dependent algorithm judgment must be made about which is more important.

- *Clarity* — if all other things are equal, a random variate generation algorithm that is easy to understand and implement is always preferred. For some people this is the most important criteria; however, for random variate generation specialists, portability, exactness, robustness, and efficiency tend to be most important.

- *Synchronization* — a random variate generation algorithm is synchronized if exactly one call to `Random` is required for each random variate generated. This property and monotonicity are often needed to implement certain *variance reduction techniques* like, for example, the *common random numbers technique*, illustrated in Example 3.1.7.

- *Monotonicity* — a random variate generation algorithm is monotone if it is synchronized and, like Algorithm 6.2.3, the transformation from u to x is monotone increasing (or monotone decreasing).

Although some of these criteria will be automatically satisfied if inversion is used, others (e.g., efficiency) may not. Generally, however, like the algorithms in Examples 6.2.6, 6.2.7, and 6.2.8, the best random variate generation algorithm is based on Algorithm 6.2.3.

Example 6.2.10 In practice, an algorithm that uses inversion but is only approximate, may be satisfactory, provided the approximation is sufficiently good. As an illustration, suppose we want to generate $Binomial(10, 0.4)$ random variates. To the $0.ddd$ precision indicated, the corresponding pdf is

x :	0	1	2	3	4	5	6	7	8	9	10
$f(x)$:	0.006	0.040	0.121	0.215	0.251	0.201	0.111	0.042	0.011	0.002	0.000

Consistent with these *approximate* pdf values, random variates can be generated by filling a 1000-element integer-valued array $a[\cdot]$ with 6 zeros, 40 ones, 121 twos, etc. Then the algorithm

```
j = Equilikely(0, 999);
return a[j];
```

can be used each time a $Binomial(10, 0.4)$ random variate is needed. The non-exactness of this algorithm is evident. A ten, for example, will never be generated, even though $f(10) = \binom{10}{10}(0.4)^{10}(0.6)^0 = 1/9765625$. It is "approximately exact," however, with an accuracy that may be acceptable in some applications. Moreover, the algorithm is portable, robust, clear, synchronized, and monotone. Marginal execution time efficiency is good. Set-up time efficiency is a potential problem, however, as is memory efficiency. For example, if $0.ddddd$ precision is desired, then it would be necessary to use a $100\,000$-element array.

Example 6.2.11 The algorithm in Example 6.2.10 for generating $Binomial(10, 0.4)$ random variates is inferior to an *exact* algorithm based on filling an 11-element floating-point array with cdf values and then using Algorithm 6.2.2 with $x = 4$ (the mode) to initialize the search.

In general, inversion can be used to generate $Binomial(n, p)$ random variates by computing a floating-point array of $n + 1$ cdf values and then using Algorithm 6.2.2 with $x = \lfloor np \rfloor$ to initialize the search. The capability provided by the library **rvms** (see Appendix D) can be used in this case to compute the cdf array by calling the cdf function **cdfBinomial**(n, p, x) for $x = 0, 1, \ldots, n$. Because this approach is inversion, in many respects it is ideal. The only drawback to this approach is some inefficiency in the sense of set-up time (to compute the cdf array) and memory (to store the cdf array), particularly if n is large.*

Example 6.2.12 The need to store a cdf array can be eliminated completely, at the expense of increased marginal execution time. This can be done by using Algorithm 6.2.2 with the cdf capability provided by the library **rvms** to compute cdf values only as needed. Indeed, in the library **rvms** this is how the idf function **idfBinomial**(n, p, u) is evaluated. Given that **rvms** provides this capability, $Binomial(n, p)$ random variates can be generated by inversion as

---

* As discussed in the next section, if n is large, the size of the cdf array can be *truncated* significantly and thereby partially compensate for the memory and set-up time inefficiency.

```
u = Random();
return idfBinomial(n, p, u);                    /* use the library rvms */
```

With appropriate modifications, the approach in Examples 6.2.11 and 6.2.12 can be used to generate $Poisson(\mu)$ and $Pascal(n, p)$ random variates. Therefore, inversion can be used to generate random variates for all six of the parametric random variable models presented in the previous section. For $Equilikely(a, b)$, $Bernoulli(p)$, and $Geometric(p)$ random variates inversion is essentially ideal. For $Binomial(n, p)$, $Pascal(n, p)$, and $Poisson(\mu)$ random variates, however, time and memory efficiency can be a problem if inversion is used. In part, this justifies the development of alternative generation algorithms for $Binomial(n, p)$, $Pascal(n, p)$, and $Poisson(\mu)$ random variates.

6.2.3   ALTERNATIVE RANDOM VARIATE GENERATION ALGORITHMS

Binomial Random Variates

As an alternative to inversion, a $Binomial(n, p)$ random variate can be generated by summing an *iid* sequence of $Bernoulli(p)$ random variates. (See Example 6.1.15.)*

Example 6.2.13   This algorithm uses a $Bernoulli(p)$ random variate generator to generate a $Binomial(n, p)$ random variate

```
x = 0;
for (i = 0;  i < n;  i++)
    x += Bernoulli(p);
return x;
```

This algorithm is portable, exact, robust, and clear. It is not synchronized or monotone and the $O(n)$ marginal execution time complexity can be a problem if n is large.

Poisson Random Variates

A $Poisson(\mu)$ random variable is the limiting case of a $Binomial(n, \mu/n)$ random variable as $n \to \infty$. Therefore, if n is large then one of these random variates can be generated as an approximation to the other. Unfortunately, because the algorithm in Example 6.2.13 is $O(n)$, if n is large we must look for other ways to generate a $Poisson(\mu)$ random variate. The $Poisson(\mu)$ cdf $F(\cdot)$ is equal to the complement of an *incomplete gamma function* (see Appendix D)

$$F(x) = 1 - P(x + 1, \mu) \qquad x = 0, 1, 2 \ldots$$

Except for special cases, an incomplete gamma function cannot be inverted to form an idf. Therefore, inversion can be used to generate a $Poisson(\mu)$ random variate, but the cdf is not simple enough to avoid the need to use a numerical approach like those in Examples 6.2.11 or 6.2.12.

---

* Random variate generation by summing an *iid* sequence of more elementary random variates is known as a *convolution method*.

There are standard algorithms for generating a $Poisson(\mu)$ random variate that do not rely on either inversion or a "large n" version of the algorithm in Example 6.2.13. The following is an example.

Example 6.2.14 This algorithm generates a $Poisson(\mu)$ random variate

```
a = 0.0;
x = 0;
while (a < μ) {
    a += Exponential(1.0);
    x++;
}
return x - 1;
```

This algorithm is portable, exact, and robust. It is neither synchronized nor monotone, and marginal execution time efficiency can be a problem if μ is large because the expected number of passes through the `while` loop is $\mu + 1$. Although the algorithm is obscure at this point, clarity will be provided in Section 7.3.

Pascal Random Variates

Like a $Binomial(n, p)$ cdf, a $Pascal(n, p)$ cdf contains an *incomplete beta function* (see Appendix D). Specifically, a $Pascal(n, p)$ cdf is

$$F(x) = 1 - I(x + 1,\, n,\, p) \qquad x = 0, 1, 2, \ldots$$

Except for special cases, an incomplete beta function cannot be inverted algebraically to form a closed-form idf. Therefore, inversion can be used to generate a $Pascal(n, p)$ random variate, but the cdf is not simple enough to avoid the need to use a numerical approach like those in Examples 6.2.11 or 6.2.12.

As an alternative to inversion, recall from Section 6.1 that the random variable X is $Pascal(n, p)$ if and only if

$$X = X_1 + X_2 + \cdots + X_n$$

where X_1, X_2, ..., X_n is an *iid* $Geometric(p)$ sequence. Therefore, a $Pascal(n, p)$ random variate can be generated by summing an *iid* sequence of n $Geometric(p)$ random variates.

Example 6.2.15 This algorithm uses a $Geometric(p)$ random variate generator to generate a $Pascal(n, p)$ random variate.

```
x = 0;
for (i = 0; i < n; i++)
    x += Geometric(p);
return x;
```

This algorithm is portable, exact, robust, and clear. It is neither synchronized nor monotone and the $O(n)$ marginal execution time complexity can be a problem if n is large.

Library rvgs

See Appendix E for the library `rvgs` (Random Variate GeneratorS). This library consists of six functions for generating discrete random variates and seven functions for generating continuous random variates (see Chapter 7). The six discrete random variate generators in the library are:

- `long Bernoulli(double` p`)` — returns 1 with probability p or 0 otherwise;

- `long Binomial(long` n`, double` p`)` — returns a $Binomial(n, p)$ random variate;

- `long Equilikely(long` a`, long` b`)` — returns an $Equilikely(a, b)$ random variate;

- `long Geometric(double` p`)` — returns a $Geometric(p)$ random variate;

- `long Pascal(long` n`, double` p`)` — returns a $Pascal(n, p)$ random variate;

- `long Poisson(double` μ`)` — returns a $Poisson(\mu)$ random variate.

These random variate generators feature minimal set-up times, in some cases at the expense of potentially large marginal execution times.

Library rvms

The $Bernoulli(p)$, $Equilikely(a, b)$, and $Geometric(p)$ generators in `rvgs` use inversion and in that sense are ideal. The other three generators do not use inversion. If that is a problem, then as an alternative to the library `rvgs`, the idf functions in the library `rvms` (Random Variable ModelS, Appendix D) can be used to generate $Binomial(n, p)$, $Pascal(n, p)$, and $Poisson(\mu)$ random variates by inversion, as illustrated in Example 6.2.12.

Because the idf functions in the library `rvms` were designed for accuracy at the possible expense of marginal execution time inefficiency, use of this approach is generally not recommended when many observations need to be generated. Instead, in that case set up an array of cdf values and use inversion (Algorithm 6.2.2), as illustrated in Example 6.2.11. This approach is considered in more detail in the next section.

6.2.4 EXERCISES

Exercise 6.2.1 Prove that $F^*(\cdot)$ is a monotone increasing function, i.e., prove that if $0 < u_1 \leq u_2 < 1$ then $F^*(u_1) \leq F^*(u_2)$.

Exercise 6.2.2 If inversion is used to generate a $Geometric(p)$ random variate x, use your knowledge of the largest and smallest possible values of `Random` to determine (as a function of p) the largest and smallest possible value of x that can be generated. (See also Exercise 3.1.3.)

Exercise 6.2.3 Find the pdf associated with the random variate generation algorithm

```
u = Random();
return ⌈3.0 + 2.0 * u²⌉;
```

Exercise 6.2.4   (*a*) Generate a *Poisson*(9) random variate sample of size 1 000 000 using the appropriate generator function in the library `rvgs` and form a histogram of the results. (*b*) Compare the resulting relative frequencies with the corresponding *Poisson*(9) pdf using the appropriate pdf function in the library `rvms`. (*c*) Comment on the value of this process as a test of correctness for the two functions used.

Exercise 6.2.5   (*a*) If X is a discrete random variable with possible values $x = 1, 2, \ldots, n$ and pdf $f(x) = \alpha x$, find an equation for the idf $F^*(u)$ as a function of n. (See Exercise 6.1.5.) (*b*) Construct an inversion function that will generate a value of X with one call to `Random`. (*c*) How would you convince yourself that this random variate generator function is correct?

Exercise 6.2.6 X is a discrete random variable with cdf $F(\cdot)$ and idf $F^*(\cdot)$. Prove or disprove the following.   (*a*) If $u \in (0, 1)$ then $F(F^*(u)) = u$.   (*b*) If $x \in \mathcal{X}$ then $F^*(F(x)) = x$.

Exercise 6.2.7   (*a*) Implement Algorithm 6.2.1 for a *Poisson*(μ) random variable and use Monte Carlo simulation to verify that the expected number of passes through the `while` loop is μ. Use $\mu = 1, 5, 10, 15, 20, 25$. (*b*) Repeat with Algorithm 6.2.2. (*c*) Comment. (Use the function `cdfPoisson` in the library `rvms` to generate the *Poisson*(μ) cdf values.)

Exercise 6.2.8   Design and then implement a consistent "invalid input" error trapping mechanism for the functions in the library `rvgs`. Write a defense of your design.

Exercise 6.2.9[a]   Suppose X is a discrete random variable with cdf $F(\cdot)$ and possible values $\mathcal{X} = \{a, a + 1, \ldots, b\}$ with both a and b finite. (*a*) Construct a binary search algorithm that will determine $F^*(u)$ for any $u \in (0, 1)$. (*b*) Present convincing numerical evidence that your algorithm is correct when used to generate *Binomial*(100, 0.2) random variates and compare its efficiency with that of algorithms 6.2.1 and 6.2.2. (*c*) Comment.

Exercise 6.2.10   (*a*) As an extension of Exercise 6.1.8, what is the idf of X? (*b*) Provide convincing numerical evidence that this idf is correct.

Exercise 6.2.11[a]   Two integers X_1, X_2 are drawn at random, without replacement, from the set $\{1, 2, \ldots, n\}$ with $n \geq 2$. Let $X = |X_1 - X_2|$. (*a*) What are the possible values of X? (*b*) What are the pdf, cdf, and idf of X? (*c*) Construct an inversion function that will generate a value of X with just one call to `Random`. (*d*) What did you do to convince yourself that this random variate generator function is correct?

Exercise 6.2.12[a]   Same as the previous exercise, except that the draw is with replacement.

6.3 Discrete Random Variable Applications

The purpose of this section is to demonstrate several discrete random variable applications using the capabilities provided by the discrete random variate generators in the library **rvgs** and the pdf, cdf, and idf functions in the library **rvms**. We begin by considering alternatives to the inventory demand models used in programs **sis2** and **sis3**.

6.3.1   ALTERNATIVE INVENTORY DEMAND MODELS

Example 6.3.1   The inventory demand model in program **sis2** is that the demand per time interval is generated as an *Equilikely*$(10, 50)$ random variate d. In this case, the mean is 30, the standard deviation is $\sqrt{140} \cong 11.8$, and the demand pdf is flat, as illustrated in Figure 6.3.1.

Figure 6.3.1.
Equilikely$(10, 50)$
demand
model
pdf.
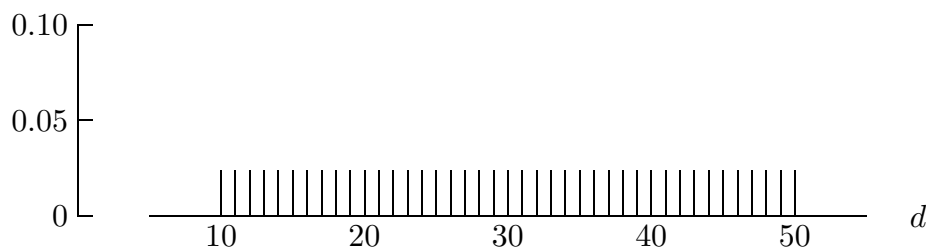
As discussed in Example 3.1.5, this is not a very realistic model.* Therefore, we consider alternative models, one of which is that there are, say, 100 instances per time interval when a demand for 1 unit *may* occur. For each of these instances the probability that the demand *will* occur is, say, 0.3 independent of what happens at the other demand instances. The inventory demand per time interval is then the sum of 100 independent *Bernoulli*(0.3) random variates or, equivalently, it is a *Binomial*$(100, 0.3)$ random variate. In this case the function **GetDemand** in the program **sis2** should be

```
long GetDemand(void)
{
    return (Binomial(100, 0.3));
}
```

The resulting random demand per time interval will have a mean of 30, a standard deviation of $\sqrt{21} \cong 4.6$, and a pdf approximately symmetric about the mean, as illustrated in Figure 6.3.2.

Figure 6.3.2.
Binomial$(100, 0.3)$
demand
model
pdf.
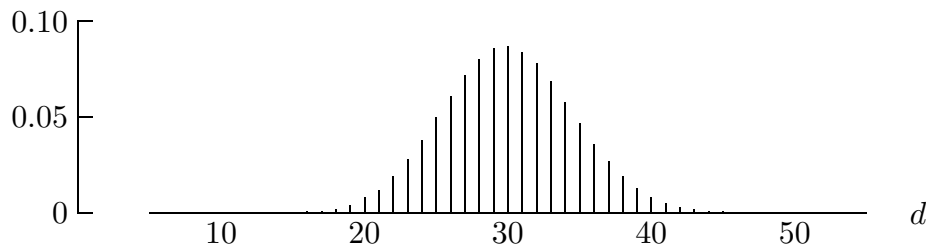
* Discussion of matching models to data collected from the system of interest is delayed until Chapter 9.

Example 6.3.2   If we believe that the $Binomial(100, 0.3)$ model in the previous example may be close to reality, then a $Poisson(30)$ model should also be considered. For this model, the random variate returned by the function `GetDemand` in program `sis2` would be

```
return (Poisson(30.0));
```

The resulting random variate demand per time interval will have a mean of 30, a standard deviation of $\sqrt{30} \cong 5.5$, and the pdf illustrated in Figure 6.3.3. Although similar to the $Binomial(100, 0.3)$ pdf in Example 6.3.1, the $Poisson(30)$ pdf has slightly "heavier" tails, consistent with the larger standard deviation.
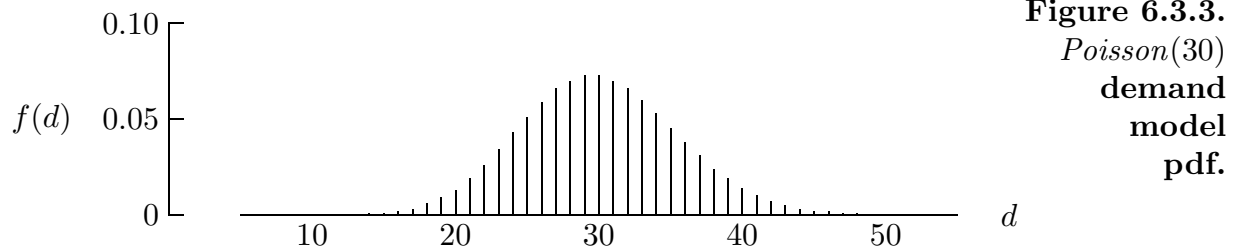
Figure 6.3.3.
$Poisson(30)$
demand
model
pdf.

For reasons discussed in Section 7.3, a traditional model for inventory systems is what was used in Example 6.3.2 — the discrete demand is a $Poisson(\lambda)$ random variable with the parameter λ matched to the expected amount of demand per time interval (the demand rate). Indeed, this is the inventory demand model used in program `sis3` with $\lambda = 30$.

Example 6.3.3   Yet another potential model for the inventory demand is that there are, say, 50 instances per time interval when a $Geometric(p)$ inventory demand will occur with p equal to, say, 0.375 and the amount of demand that actually occurs at each of these instances is independent of the amount of demand that occurs at the other instances. The demand per time interval is then the sum of 50 independent $Geometric(0.375)$ random variates or, equivalently it is a $Pascal(50, 0.375)$ random variate. The random variate returned by the function `GetDemand` in program `sis2` would then be

```
return (Pascal(50, 0.375));
```

producing a random variate demand per time interval with a mean of 30, a standard deviation of $\sqrt{48} \cong 6.9$, and the pdf illustrated in Figure 6.3.4.
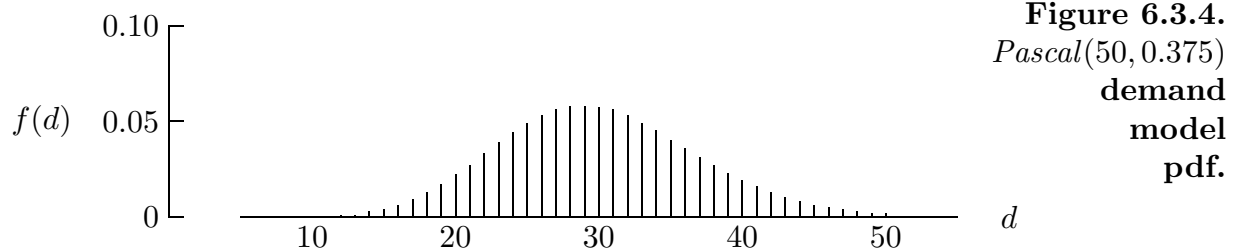
Figure 6.3.4.
$Pascal(50, 0.375)$
demand
model
pdf.

Consistent with the increase in the standard deviation from 5.5 to 6.9, this pdf has slightly heavier tails than the $Poisson(30)$ pdf.

Example 6.3.4   As an extension of the previous example, one might argue that the inventory demand model should also allow for the number of demand instances per time interval to be an independent discrete random variable as well, say $Poisson(50)$. The function `GetDemand` in program `sis2` would then be*

```
long GetDemand(void)
{
   long instances = Poisson(50.0);        /* must truncate to avoid 0 */
   return (Pascal(instances, 0.375));
}
```

With this extension, the mean of the resulting random variate demand per time interval will remain 30. As illustrated in Figure 6.3.5, the pdf will become further dispersed about the mean, consistent with an increase of the standard deviation to $\sqrt{66} \cong 8.1$.

Figure 6.3.5.
Compound
demand
model
pdf.
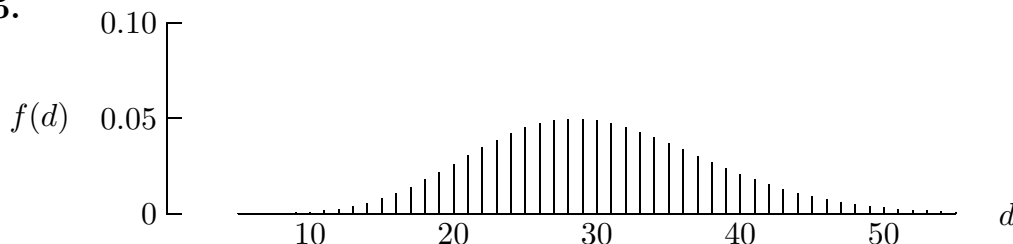
To determine the pdf of the "compound" (or "stochastic parameters") random variable corresponding to the random variates generated by the function `GetDemand` in Example 6.3.4, let the discrete random variables D and I denote the demand *amount* and number of demand *instances* per time interval, respectively. From the law of total probability it follows that

$$f(d) = \Pr(D = d) = \sum_{i=0}^{\infty} \Pr(I = i)\, \Pr(D = d \mid I = i) \qquad d = 0, 1, 2, \ldots$$

For any value of d the probability $f(d)$ can be evaluated by using the pdf capability in the library `rvms`. To do so, however, the infinite sum over possible demand instances must be *truncated* to a finite range, say $0 < a \le i \le b$. The details of how to determine a and b are discussed later in this section. Provided a and b are selected appropriately, the following algorithm can then be used to compute $f(d)$

```
double sum = 0.0;
for (i = a; i <= b; i++)                          /* use the library rvms */
   sum += pdfPoisson(50.0, i) * pdfPascal(i, 0.375, d);
return sum;                                               /* sum is f(d) */
```

These compound random variables are of interest in a branch of statistics known as Bayesian statistics.

---

* There is a potential problem with this function — see Exercise 6.3.7.

Any of the inventory demand models in the previous examples can be used in program `sis2` as a replacement for the $Equilikely(a, b)$ model. In most applications this will result in a model that more accurately reflects the system. Moreover, with some minor modification, a more accurate inventory demand model can be used in program `sis3`. The result is program `sis4`.

Program sis4

Program `sis4` is based on the next-event simulation program `sis3` but with a more realistic inventory demand model that allows for a random amount of demand at each demand instance. That is, like program `sis3` demand instances are assumed to occur at random throughout the period of operation with an average rate of λ instances per time interval so that the inter-demand time is an $Exponential(1/\lambda)$ random variate. Unlike the model on which program `sis3` is based, however, these demand instances correspond to times where a demand *may* occur. Whether or not a demand actually occurs at these demand instances is random with probability p. Moreover, to allow for the possibility of more than one unit of inventory demand at a demand instance, the demand amount is assumed to be a $Geometric(p)$ random variate. Because the expected value of a $Geometric(p)$ random variable is $p/(1-p)$ and the expected number of demand instances per time interval is λ, the expected demand per time interval is $\lambda p/(1-p)$.

Example 6.3.5   In terms of the automobile dealership example considered previously, the inventory demand model on which program `sis4` is based corresponds to an average of λ customers per week that visit the dealership with the *potential* to buy one or more automobiles. Each customer will, independently, not buy an automobile with probability $1 - p$, or they will buy one automobile with probability $(1 - p)p$, or they will buy two automobiles with probability $(1 - p)p^2$, or three with probability $(1 - p)p^3$, etc. The parameter values used in program `sis4` are $\lambda = 120.0$ and $p = 0.2$, which correspond to an expected value of 30.0 automobiles purchased per week. For future reference, note that

$$30.0 = \frac{\lambda p}{1 - p} = \lambda \sum_{x=0}^{\infty} x(1-p)p^x = \underbrace{\lambda(1-p)p}_{19.200} + \underbrace{2\lambda(1-p)p^2}_{7.680} + \underbrace{3\lambda(1-p)p^3}_{2.304} + \cdots$$

Therefore, on average, of the 120 customers that visit per week

- $\lambda(1-p) = 96.0$ do not buy anything;
- $\lambda(1-p)p = 19.200$ buy one automobile;
- $\lambda(1-p)p^2 = 3.840$ buy two automobiles;
- $\lambda(1-p)p^3 = 0.768$ buy three automobiles;
- etc.

In the remainder of this section we will deal with the issue of how to truncate a discrete random variable. By using truncation the demand model can be made more realistic by limiting the number of automobiles a customer buys to 0, 1 or 2 (see Exercise 6.3.2).

6.3.2   TRUNCATION

Let X be a discrete random variable with possible values $\mathcal{X} = \{x \mid x = 0, 1, 2, \ldots\}$ and cdf $F(x) = \Pr(X \leq x)$. A modeler might be interested in formulating a pdf that effectively restricts the possible values to a finite range of integers $a \leq x \leq b$ with $a \geq 0$ and $b < \infty$.* If $a > 0$ then the probability of X being strictly less than a is

$$\alpha = \Pr(X < a) = \Pr(X \leq a - 1) = F(a - 1).$$

Similarly, the probability of X being strictly greater than b is

$$\beta = \Pr(X > b) = 1 - \Pr(X \leq b) = 1 - F(b).$$

In general, then

$$\Pr(a \leq X \leq b) = \Pr(X \leq b) - \Pr(X < a) = F(b) - F(a - 1)$$

so that a value of X outside the range $a \leq X \leq b$ is essentially impossible if and only if $F(b) \cong 1.0$ and $F(a - 1) \cong 0.0$. There are two cases to consider.

- If a and b are specified then the cdf of X can be used to determine the left-tail and right-tail probabilities

$$\alpha = \Pr(X < a) = F(a - 1) \qquad \text{and} \qquad \beta = \Pr(X > b) = 1 - F(b)$$

  respectively. The cdf transformation from possible values to probabilities is exact.

- If instead the (positive) left-tail and right-tail probabilities α and β are specified, then the idf of X can be used to determine the possible values

$$a = F^*(\alpha) \qquad \text{and} \qquad b = F^*(1 - \beta)$$

  respectively. Because X is a discrete random variable, this idf transformation from probabilities to possible values is not exact. Instead a and b only provide *bounds* in the sense that $\Pr(X < a) \leq \alpha$ and $\Pr(X > b) < \beta$.

Example 6.3.6   For the $Poisson(50)$ random variable I in Example 6.3.4 it was necessary to determine a, b so that $\Pr(a \leq I \leq b) \cong 1.0$ with negligible error. By experimentation it was determined that, in this case, "negligible error" could be interpreted as $\alpha = \beta = 10^{-6}$. The idf capability in the random variate models library **rvms** was then used to compute

```
a = idfPoisson(50.0, α);                                    /* α = 10⁻⁶ */
b = idfPoisson(50.0, 1.0 - β);                              /* β = 10⁻⁶ */
```

to produce the results $a = 20$, $b = 87$. Consistent with the bounds produced by the (α, β) to (a, b) conversion, $\Pr(I < 20) = \mathtt{cdfPoisson}(50.0,\ 19) \cong 0.48 \times 10^{-6} < \alpha$ and $\Pr(I > 87) = 1.0 - \mathtt{cdfPoisson}(50.0,\ 87) \cong 0.75 \times 10^{-6} < \beta$.

---

* In the case of the automobile dealership example, $a = 0$ and $b = 45$ implies that there is no left-hand truncation and no week contains more than 45 customer demands.

To facilitate the evaluation of the compound pdf values in Example 6.3.4, the possible values of a *Poisson*(50) random variable were truncated to the integers 20 through 87 inclusive. As demonstrated in Example 6.3.6, this truncation is so slight as to be of no practical significance. That is, because a *Poisson*(50) pdf is effectively zero for all integers less than 20 or greater than 87, the truncated random variable and the un-truncated random variable are essentially identically distributed. In some simulation applications truncation, is more significant. There are two reasons for this.

- *Efficiency* — when discrete random variates are generated using inversion and the idf is not available as a simple algebraic expression, then a cdf search technique like Algorithm 6.2.2 must be used. To facilitate this search, cdf values are usually stored in an array. Because the range of possible values defines the size of the array, traditional computer science space/time considerations dictate that the range of possible values should be as small as possible (assuming that the model remains realistic).

- *Realism* — some discrete random variates can take on arbitrarily large values (at least in theory). If you have created a discrete random variable model with a mean of 30 and a standard deviation of 10 do you really want your random variate generator to (surprise!) return a value of 100?

When it is significant, truncation must be done correctly because the result is a *new* random variable. The four examples that follow are illustrations.

Incorrect Truncation

Example 6.3.7 As in Example 6.3.2, suppose we want to use a *Poisson*(30) demand model in program sis2 but, to avoid extreme values, we want to truncate the demand to the range $20 \leq d \leq 40$. In this case, we might be tempted to generate the demand with

```
d = Poisson(30.0);
if (d < 20)
    d = 20;
if (d > 40)
    d = 40;
return d;
```

Because the pdf values below 20 and above 40 "fold" back to 20 and 40 respectively, as the resulting pdf spikes at $d = 20$ and $d = 40$ shown in Figure 6.3.6 indicate this is *not* a correct truncation for most applications.
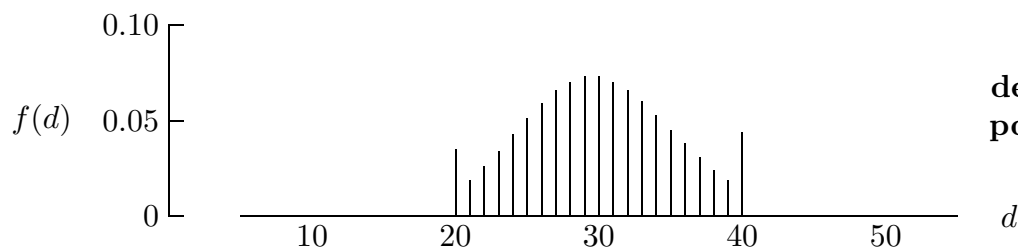
Figure 6.3.6. *Poisson*(30) **demand model pdf incorrectly truncated.**

Truncation by cdf Modification

Example 6.3.8   As in Example 6.3.7, we want to generate $Poisson(30)$ demands truncated to the range $20 \leq d \leq 40$. This time, however, we want to do it correctly. Before truncation the $Poisson(30)$ pdf is

$$f(d) = \frac{30^d \exp(-30)}{d!} \qquad d = 0, 1, 2, \ldots$$

with

$$\Pr(20 \leq D \leq 40) = F(40) - F(19) = \sum_{d=20}^{40} f(d) \cong 0.945817.$$

The truncation correction is to compute a new truncated random variable D_t with pdf $f_t(d)$. This is done by increasing the corresponding value of $f(d)$ via division by the factor $F(40) - F(19)$ as

$$f_t(d) = \frac{f(d)}{F(40) - F(19)} \qquad d = 20, 21, \ldots, 40.$$

The corresponding truncated cdf is

$$F_t(d) = \sum_{t=20}^{d} f_t(t) = \frac{F(d) - F(19)}{F(40) - F(19)} \qquad d = 20, 21, \ldots, 40.$$

A random variate correctly truncated to the range $20 \leq d \leq 40$ can now be generated by inversion using the truncated cdf $F_t(\cdot)$ and Algorithm 6.2.2 as follows

```
u = Random();
d = 30;
if (Ft(d) <= u)
    while (Ft(d) <= u)
        d++;
else if (Ft(20) <= u)
    while (Ft(d - 1) > u)
        d--;
else
    d = 20;
return d;
```

Compared to the mean and standard deviation of D, which are $\mu = 30.0$ and $\sigma = \sqrt{30} \cong 5.477$, the mean and standard deviation of D_t are

$$\mu_t = \sum_{d=20}^{40} d\, f_t(d) \cong 29.841 \qquad \text{and} \qquad \sigma_t = \sqrt{\sum_{d=20}^{40} (d - \mu_t)^2 f_t(d)} \cong 4.720.$$

The pdf of the correctly truncated $Poisson(30)$ random variable in Example 6.3.8 is illustrated in Figure 6.3.7. For reference, the incorrectly truncated pdf from Example 6.3.7 and the un-truncated pdf from Example 6.3.2 are also shown.

Figure 6.3.7. Various $Poisson(30)$ **demand model pdfs.**
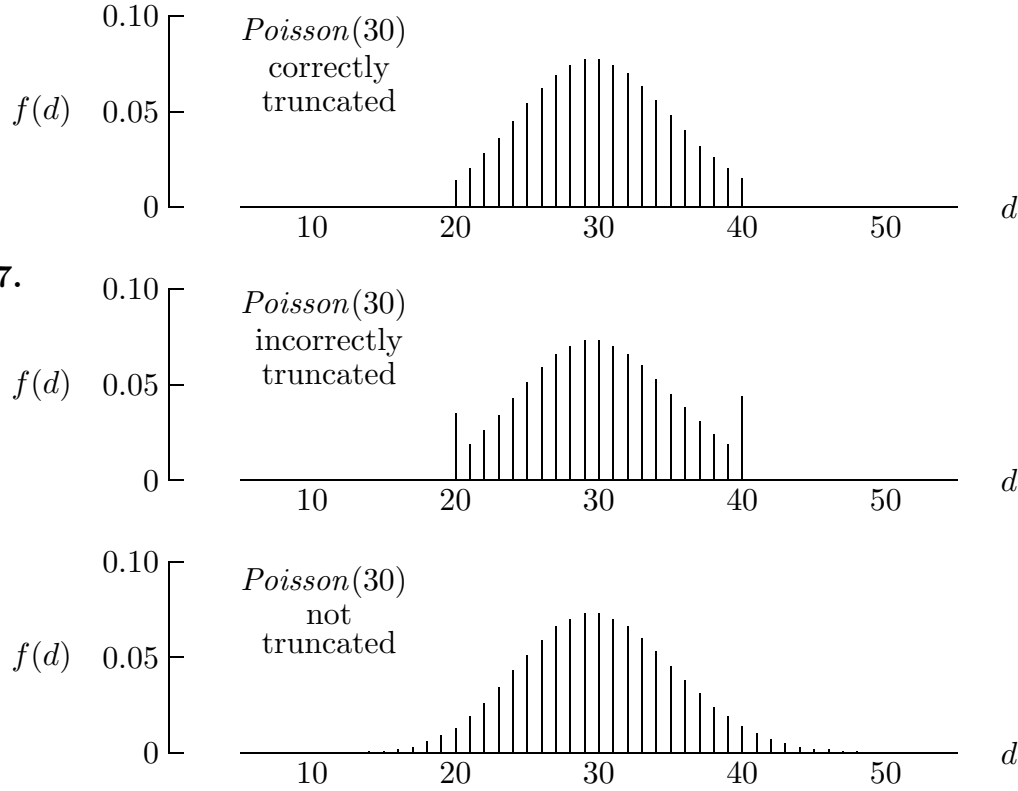
In general, if it is desired to truncate the integer-valued discrete random variable X to the set of possible values $\mathcal{X}_t = \{a, a+1, \ldots, b\} \subset \mathcal{X}$ thereby forming the new discrete random variable X_t, then the pdf and cdf of X_t are defined as

$$f_t(x) = \frac{f(x)}{F(b) - F(a-1)} \qquad x \in \mathcal{X}_t$$

and

$$F_t(x) = \frac{F(x) - F(a-1)}{F(b) - F(a-1)} \qquad x \in \mathcal{X}_t$$

where $f(\cdot)$ and $F(\cdot)$ are the pdf and cdf of X. Random values of X_t can then be generated using inversion and Algorithm 6.2.2 with $F_t(\cdot)$, as in Example 6.3.8. [The equations for $f_t(x)$ and $F_t(x)$ presume that $a-1$ is a possible value of X.]

There may be *rare* occasions where "incorrect truncation" is appropriate in a discrete-event simulation model. Decisions concerning the style of truncation are driven by the fit between the model and the real-world system.

As an alternative to truncation by cdf modification, we can use truncation by *constrained inversion*, as illustrated next. Provided the idf of X can be easily applied, this is the truncation approach of choice.

Truncation by Constrained Inversion

If $F^*(\cdot)$ is the idf of X then the following algorithm can be used to generate the random variate X_t correctly truncated to the range $a \le x \le b$

```
α = F(a − 1);            /* assumes that a - 1 is a possible value of X */
β = 1.0 - F(b);
u = Uniform(α, 1.0 - β);
x = F*(u);
return x;
```

The key here is that u is *constrained* to a subrange $(\alpha, 1 - \beta) \subset (0, 1)$ in such a way that correct truncation is automatically enforced prior to the idf inversion. This is another illustration of the elegance of random variate generation by inversion.

Example 6.3.9   The cdf and idf capabilities in the library `rvms` can be used to generate a $Poisson(30)$ random demand, correctly truncated to the range $20 \le d \le 40$, as illustrated

```
α = cdfPoisson(30.0, 19);                                    /* set-up */
β = 1.0 - cdfPoisson(30.0, 40);                              /* set-up */
u = Uniform(α, 1.0 - β);
d = idfPoisson(30.0, u);
return d;
```

This algorithm should be implemented so that α and β are `static` variables that are computed once only.

Truncation by Acceptance-Rejection

Example 6.3.10   As an alternative to the techniques in Example 6.3.8 and 6.3.9, correct truncation can also be achieved, at the potential expense of some extra calls to the function `Poisson(30)`, by using *acceptance-rejection* as follows

```
d = Poisson(30.0);
while ((d < 20) or (d > 40))
    d = Poisson(30.0);
return d;
```

Although easily remembered and easily programmed, acceptance-rejection is not synchronized or monotone even if the un-truncated generator has these properties. Generally, the cdf modification technique in Example 6.3.8 or the constrained inversion technique in Example 6.3.9 is preferable.

6.3.3   EXERCISES

Exercise 6.3.1   (*a*) Suppose you wish to use inversion to generate a *Binomial*(100, 0.1) random variate X *truncated* to the subrange $x = 4, 5, \ldots, 16$. How would you do it? Work through the details. (*b*) What is the value of the mean and standard deviation of the resulting truncated random variate?

Exercise 6.3.2   The function `GetDemand` in program `sis4` can return demand amounts outside the range $0, 1, 2$. (*a*) What is the largest demand amount that this function can return? In some applications, integers outside the range $0, 1, 2$ may not be meaningful, no matter how unlikely. (*b*) Modify `GetDemand` so that the value returned is correctly truncated to the range $0, 1, 2$. Do not use acceptance-rejection. (*c*) With truncation, what is the resulting average demand per time interval and how does that compare to the average with no truncation?

Exercise 6.3.3   Prove that the "truncation by constrained inversion" algorithm is correct. Also, draw a figure that illustrates the geometry behind the "truncation by constrained inversion" algorithm.

Exercise 6.3.4[a]   Prove that the mean of the compound demand in Example 6.3.4 is 30 and that the variance is 66.

Exercise 6.3.5[a]   (*a*) Do Exercise 6.2.10. (*b*) In addition, prove that the expected value of X is infinitely large (i.e., $E[X] = \infty$.) (*c*) Comment on the potential paradox that by using inversion just *one* call to `Random` is somehow equivalent to a direct Monte Carlo simulation of this random experiment that requires, on average, an *infinite* number of calls to `Random`. *Hint*: the pdf of X is

$$f(x) = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \qquad x = 1, 2, 3, \ldots$$

Exercise 6.3.6[a]   (*a*) Implement the function `GetDemand` in Example 6.3.4 in such a way that you can use Monte Carlo simulation to estimate the expected number of calls to `Random` per call to `GetDemand`. (*b*) Construct an alternative version of `GetDemand` that requires exactly *one* call to `Random` per call to `GetDemand` and yet is effectively equivalent to the original function in the sense that the alternative function produces a random variate whose pdf matches the pdf illustrated in Example 6.3.4. (*c*) Provide convincing numerical evidence of correctness.

Exercise 6.3.7   (*a*) Is it necessary to modify the function `GetDemand` in Example 6.3.4 so that `instances` must be positive? (*b*) If so, why and how would you do it? If not necessary, why?

For convenience, the characteristic properties of the following six discrete random variable models are summarized in this section: $Equilikely(a, b)$, $Bernoulli(p)$, $Geometric(p)$, $Pascal(n, p)$, $Binomial(n, p)$, and $Poisson(\mu)$. For more details about these models, see Sections 6.1 and 6.2; for supporting software, see the random variable models library `rvms` in Appendix D and the random variate generators library `rvgs` in Appendix E.

6.4.1 EQUILIKELY

Definition 6.4.1 The discrete random variable X is $Equilikely(a, b)$ if and only if

- the parameters a, b are integers with $a < b$

- the possible values of X are $\mathcal{X} = \{a, a + 1, \ldots, b\}$

- the pdf of X is
$$f(x) = \frac{1}{b - a + 1} \qquad x = a, a + 1, \ldots, b$$
  as illustrated in Figure 6.4.1

Figure 6.4.1.
$Equilikely(a, b)$ **pdf.**
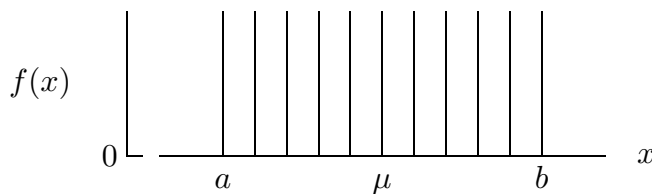
- the cdf of X is
$$F(x) = \frac{x - a + 1}{b - a + 1} \qquad x = a, a + 1, \ldots, b$$

- the idf of X is
$$F^*(u) = a + \lfloor (b - a + 1)u \rfloor \qquad 0 < u < 1$$

- the mean of X is
$$\mu = \frac{a + b}{2}$$

- the standard deviation of X is
$$\sigma = \sqrt{\frac{(b - a + 1)^2 - 1}{12}}.$$

An $Equilikely(a, b)$ random variable is used to model those situations where a discrete random variable is restricted to the integers between a and b inclusive and all values in this range are equally likely. A typical application will involve a model derived from a statement like "… an element is selected *at random* from a finite set … " An *Equilikely* random variable is also known as a *discrete uniform*, *DU*, or *rectangular* random variable — terminology that is not used in this book.

6.4.2 BERNOULLI

Definition 6.4.2 The discrete random variable X is *Bernoulli*(p) if and only if

- the real-valued parameter p satisfies $0 < p < 1$
- the possible values of X are $\mathcal{X} = \{0, 1\}$
- the pdf of X is

$$f(x) = p^x(1-p)^{1-x} \qquad x = 0, 1$$

  as illustrated in Figure 6.4.2 for $p = 0.6$
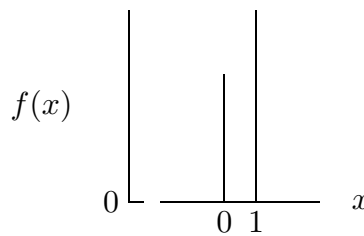
Figure 6.4.2.
Bernoulli(0.6) **pdf.**

- the cdf of X is

$$F(x) = (1-p)^{1-x} \qquad x = 0, 1$$

- the idf of X is

$$F^*(u) = \begin{cases} 0 & 0 < u < 1 - p \\ 1 & 1 - p \leq u < 1 \end{cases}$$

- the mean of X is

$$\mu = p$$

- the standard deviation of X is

$$\sigma = \sqrt{p(1-p)}.$$

A *Bernoulli*(p) random variable is used to model the Boolean situation where only two outcomes are possible — success or failure, true or false, 1 or 0, etc. The parameter p determines the probability of the two possible outcomes with the convention that

$$p = \Pr(\text{success}) = \Pr(X = 1)$$

and

$$1 - p = \Pr(\text{failure}) = \Pr(X = 0).$$

The random variables X_1, X_2, X_3, ..., define an *iid* sequence of so-called *Bernoulli*(p) *trials* if and only if each X_i is *Bernoulli*(p) and each is statistically independent of all the others. The repeated tossing of a coin (biased or not) is the classic example of an *iid* sequence — a head is equally likely on each toss (p does not change) and the coin has no memory of the previous outcomes (independence).

6.4.3   GEOMETRIC

Definition 6.4.3   The discrete random variable X is *Geometric*(p) if and only if

- the real-valued parameter p satisfies $0 < p < 1$
- the possible values of X are $\mathcal{X} = \{0, 1, 2, \ldots\}$
- the pdf of X is
$$f(x) = p^x(1 - p) \qquad\qquad x = 0, 1, 2, \ldots$$

as illustrated in Figure 6.4.3 for $p = 0.75$

Figure 6.4.3.
Geometric(0.75) **pdf.**
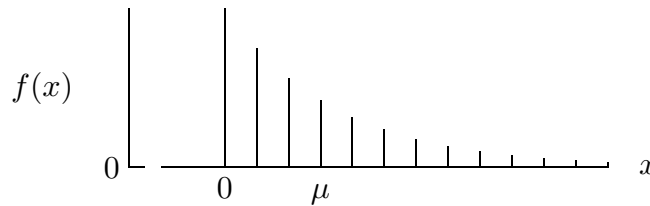
- the cdf of X is
$$F(x) = 1 - p^{x+1} \qquad\qquad x = 0, 1, 2, \ldots$$

- the idf of X is
$$F^*(u) = \left\lfloor \frac{\ln(1 - u)}{\ln(p)} \right\rfloor \qquad\qquad 0 < u < 1$$

- the mean of X is
$$\mu = \frac{p}{1 - p}$$

- the standard deviation of X is
$$\sigma = \frac{\sqrt{p}}{1 - p}.$$

A *Geometric*(p) random variable is conventionally used to model the number of "successes" (1's) before the first "failure" (0) in a sequence of independent *Bernoulli*(p) trials. For example, if a coin has p as the probability of a head (success), then X counts the number of heads before the first tail (failure). Equivalently, a *Geometric*(p) random variable can be interpreted as a model for random sampling from a urn containing balls, a fraction p of which are 1's with the remainder 0's. A *Geometric*(p) random variable counts the number of 1's that are drawn, with replacement, before the first 0. If X is *Geometric*(p) then X has the *memoryless* property — for any non-negative integer x'

$$\Pr(X \geq x + x' \,|\, X \geq x') = \Pr(X \geq x) = p^x \qquad\qquad x = 0, 1, 2, \ldots$$

independent of x'. An intuitive interpretation of this property is that a string of $x' - 1$ consecutive 1's is followed by a sequence of 0's and 1's that is probabilistically the same as a brand new sequence.

6.4.4   PASCAL

Definition 6.4.4   The discrete random variable X is $Pascal(n, p)$ if and only if

- the parameter n is a positive integer
- the real-valued parameter p satisfies $0 < p < 1$
- the possible values of X are $\mathcal{X} = \{0, 1, 2, \ldots\}$
- the pdf of X is

$$f(x) = \binom{n + x - 1}{x} p^x (1 - p)^n \qquad x = 0, 1, 2, \ldots$$

as illustrated in Figure 6.4.4 for $(n, p) = (5, \, 2/7)$
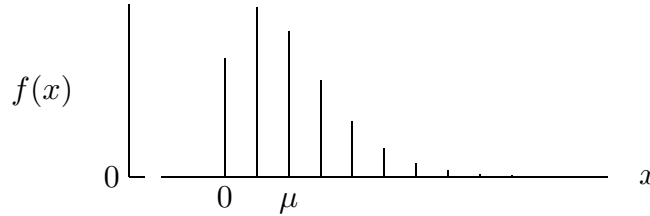
Figure 6.4.4.
$Pascal(5, 2/7)$ **pdf.**

- the cdf of X contains an incomplete beta function (see Appendix D)

$$F(x) = 1 - I(x + 1, \, n, \, p) \qquad x = 0, 1, 2, \ldots$$

- except for special cases, the idf of X must be determined by numerical inversion
- the mean of X is

$$\mu = \frac{np}{1 - p}$$

- the standard deviation of X is

$$\sigma = \frac{\sqrt{np}}{1 - p}.$$

A $Pascal(n, p)$ random variable is the number of 1's before the nth 0 in an *iid* sequence of independent $Bernoulli(p)$ trials. Therefore, a $Pascal(1, p)$ random variable and a $Geometric(p)$ random variable are equivalent. Moreover, X is a $Pascal(n, p)$ random variable if and only if

$$X = X_1 + X_2 + \cdots + X_n,$$

where X_1, X_2, …, X_n is an *iid* sequence of $Geometric(p)$ random variables. In terms of the urn model presented in Section 6.4.3, a $Pascal(n, p)$ random variable counts the number of 1's that are drawn, with replacement, before the nth 0. A $Pascal(n, p)$ random variable is also called a *negative binomial* — terminology that is not used in this book.

6.4.5   BINOMIAL

Definition 6.4.5   The discrete random variable X is $Binomial(n, p)$ if and only if

- the parameter n is a positive integer
- the real-valued parameter p satisfies $0 < p < 1$
- the possible values of X are $\mathcal{X} = \{0, 1, 2, \ldots, n\}$
- the pdf of X is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \ldots, n$$

as illustrated in Figure 6.4.5 for $(n, p) = (10, 0.3)$

Figure 6.4.5.
$Binomial(10, 0.3)$ **pdf.**
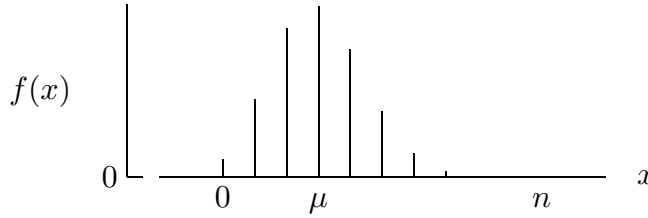
- the cdf of X contains an incomplete beta function (see Appendix D)

$$F(x) = \begin{cases} 1 - I(x + 1, \, n - x, \, p) & x = 0, 1, \ldots, n - 1 \\ 1 & x = n \end{cases}$$

- except for special cases, the idf of X must be determined by numerical inversion
- the mean of X is

$$\mu = np$$

- the standard deviation of X is

$$\sigma = \sqrt{np(1 - p)}.$$

A $Binomial(n, p)$ random variable is the number of 1's in a sequence of n independent $Bernoulli(p)$ trials. Therefore a $Binomial(1, p)$ random variable and a $Bernoulli(p)$ random variable are the same. Equivalently, X is a $Binomial(n, p)$ random variable if and only if

$$X = X_1 + X_2 + \cdots + X_n,$$

where X_1, X_2, ..., X_n is an *iid* sequence of $Bernoulli(p)$ random variables. For example, if a coin has p as the probability of a head (success), then X counts the number of heads in a sequence of n tosses. In terms of the urn model, a $Binomial(n, p)$ random variable counts the number of 1's that will be drawn, with replacement, if exactly n balls are drawn. Because drawing x 1's out of n is equivalent to drawing $n - x$ 0's it follows that X is $Binomial(n, p)$ if and only if $n - X$ is a $Binomial(n, 1 - p)$ random variable.

6.4.6 POISSON

Definition 6.4.6   The discrete random variable X is $Poisson(\mu)$ if and only if

- the real-valued parameter μ satisfies $\mu > 0$
- the possible values of X are $\mathcal{X} = \{0, 1, 2, \ldots\}$
- the pdf of X is

$$f(x) = \frac{\mu^x \exp(-\mu)}{x!} \qquad x = 0, 1, 2 \ldots$$

as illustrated in Figure 6.4.6 for $\mu = 5$
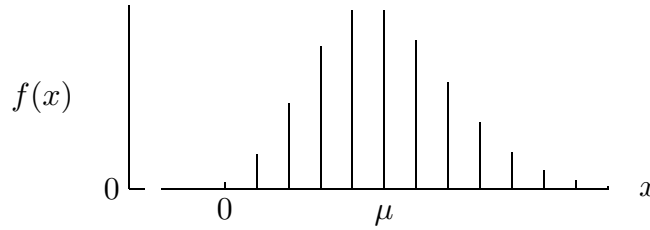
Figure 6.4.6.
$Poisson(5)$ **pdf.**

- the cdf of X contains an incomplete gamma function (see Appendix D)

$$F(x) = 1 - P(x+1, \mu) \qquad x = 0, 1, 2 \ldots$$

- except for special cases, the idf of X must be determined by numerical inversion
- the mean of X is μ
- the standard deviation of X is

$$\sigma = \sqrt{\mu}.$$

A $Poisson(\mu)$ random variable is a limiting case of a $Binomial$ random variable. That is, let X be a $Binomial(n, \mu/n)$ random variable ($p = \mu/n$). Fix the values of μ and x and consider what happens in the limit as $n \to \infty$. The pdf of X is

$$f(x) = \frac{n!}{x!(n-x)!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

$$= \frac{\mu^x}{x!} \left(\frac{n! n^x}{(n-x)!(n-\mu)^x}\right) \left(1 - \frac{\mu}{n}\right)^n.$$

It can be shown that

$$\lim_{n \to \infty} \left(\frac{n! n^x}{(n-x)!(n-\mu)^x}\right) = 1 \qquad \text{and} \qquad \lim_{n \to \infty} \left(1 - \frac{\mu}{n}\right)^n = \exp(-\mu).$$

Therefore

$$\lim_{n \to \infty} f(x) = \frac{\mu^x \exp(-\mu)}{x!}$$

which shows that for large values of n, $Binomial(n, \mu/n)$ and $Poisson(\mu)$ random variables are virtually identical, particularly if μ is small.

6.4.7   SUMMARY

Effective discrete-event simulation modeling requires that the modeler be familiar with several parametric distributions which can be used to mimic the stochastic elements of the model. In order to choose the proper distribution, it is important to know

- how these distributions arise;
- their support $\mathcal{X}$;
- their mean μ;
- their variance σ^2;
- the shape of their pdf.

Thus when a modeling situation arises, the modeler has a wide array of options for selecting a stochastic model.

It is also important to know how these distributions relate to one another. Figure 6.4.7 summarizes relationships between the six distributions considered in this text. Listed in each oval are the name, parameter(s), and support of each distribution. The solid arrows connecting the ovals denote special cases [e.g., the $Bernoulli(p)$ distribution is a special case of the $Binomial(n, p)$ distribution when $n = 1$] and transformations [e.g., the sum (convolution) of n independent and identically distributed $Bernoulli(p)$ random variables has a $Binomial(n, p)$ distribution]. The dashed line between the $Binomial(n, p)$ and $Poisson(\mu)$ distributions indicates that the limiting distribution of a binomial random variable as $n \to \infty$ has a Poisson distribution.

Figure 6.4.7.
Relationships between
discrete distributions.
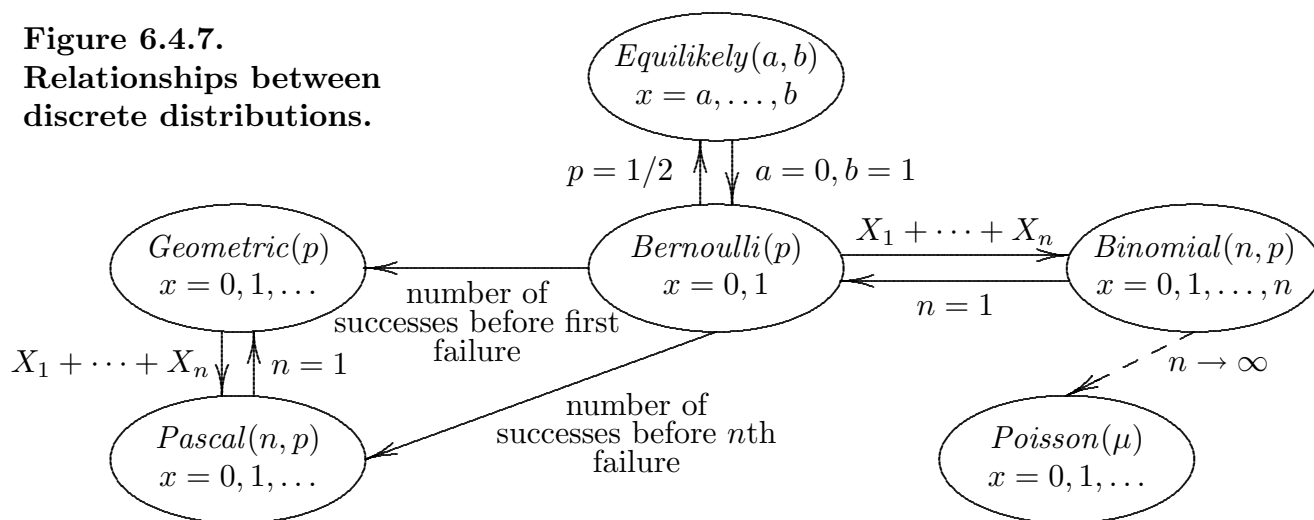
There are internal characteristics associated with these distributions that are not shown in Figure 6.4.7. One such example is that the sum of independent Poisson random variables also has a Poisson distribution. Another is that the sum of independent binomial random variables with identical parameters p also has a binomial distribution.

The mean and variance of a random variable are special cases of what are called *moments.* The table below summarizes the first four moments of the distribution of the six discrete random variables surveyed in this chapter. The first moment, the mean $\mu = E[X]$, and the second moment about the mean, the variance $\sigma^2 = E[(X - \mu)^2]$, have been defined earlier. The skewness and kurtosis are the third and fourth standardized centralized moments about the mean, defined by

$$E\left[\left(\frac{X - \mu}{\sigma}\right)^3\right] \qquad \text{and} \qquad E\left[\left(\frac{X - \mu}{\sigma}\right)^4\right].$$

The skewness is a measure of the symmetry of a distribution. A symmetric pdf has a skewness of zero. A positive skewness typically indicates that the distribution is "leaning" to the left, and a negative skewness typically indicates that the distribution is "leaning" to the right. The geometric distribution, for example, has a positive skewness for all values of its parameter p. The kurtosis is a measure of the peakedness and tail behavior of a distribution. In addition to being measures associated with particular distributions, the higher-order moments are occasionally used to differentiate between the parametric models.

Distribution	Mean	Variance	Skewness	Kurtosis
$Equilikely(a,b)$	$\dfrac{a+b}{2}$	$\dfrac{(b-a+1)^2-1}{12}$	0	$\dfrac{3}{5}\left[3 - \dfrac{4}{(b-a)(b-a+2)}\right]$
$Bernoulli(p)$	p	$p(1-p)$	$\dfrac{1-2p}{\sqrt{p(1-p)}}$	$\dfrac{1}{p(1-p)} - 3$
$Geometric(p)$	$\dfrac{p}{1-p}$	$\dfrac{p}{(1-p)^2}$	$\dfrac{1+p}{\sqrt{1-p}}$	$\dfrac{p^2+7p+1}{p}$
$Pascal(n,p)$	$\dfrac{np}{1-p}$	$\dfrac{np}{(1-p)^2}$	$\dfrac{1+p}{\sqrt{n(1-p)}}$	$3 + \dfrac{6}{n} + \dfrac{(1-p)^2}{np}$
$Binomial(n,p)$	np	$np(1-p)$	$\dfrac{1-2p}{\sqrt{np(1-p)}}$	$3 - \dfrac{6}{n} + \dfrac{1}{np(1-p)}$
$Poisson(\mu)$	μ	μ	$\dfrac{1}{\sqrt{\mu}}$	$3 + \dfrac{1}{\mu}$

Although discussion is limited here to just six discrete distributions, there are many other parametric distributions capable of modeling discrete distributions. For a complete list of common discrete and continuous parametric distributions including their pdf, cdf, idf, and moments, we recommend the compact work of Evans, Hastings, and Peacock (2000) or the encyclopedic works of Johnson, Kotz, and Kemp (1993) and Johnson, Kotz, and Balakrishnan (1994, 1995, 1997).

We conclude this section with a brief discussion of the techniques and software associated with the evaluation of the pdf, cdf, and idf of the six discrete distributions surveyed in this chapter.

6.4.8   PDF, CDF AND IDF EVALUATION

Library rvms

Pdf's, cdf's, and idf's for all six of the discrete random variable models in this section can be evaluated by using the functions in the library **rvms**, listed in Appendix D. For example, if X is $Binomial(n, p)$ then for any $x = 0, 1, 2, \ldots, n$

```
pdf = pdfBinomial(n, p, x);                                          /* f(x) */
cdf = cdfBinomial(n, p, x);                                          /* F(x) */
```

and for any $0.0 < u < 1.0$

```
idf = idfBinomial(n, p, u);                                          /* F*(u) */
```

This library also has functions to evaluate pdf's, cdf's, and idf's for all the continuous random variables cataloged in Chapter 7.

Alternative, Recursive Approaches for Calculating pdf Values

Although this approach is not used in the library **rvms** to evaluate pdf's and cdf's, if X is a discrete random variable then pdf values (and cdf values, by summation) can usually be easily generated recursively. In particular:

- if X is $Geometric(p)$ then

$$
\begin{aligned}
f(0) &= 1 - p \\
f(x) &= p f(x - 1) \qquad\qquad x = 1, 2, 3, \ldots
\end{aligned}
$$

- if X is $Pascal(n, p)$ then

$$
\begin{aligned}
f(0) &= (1 - p)^n \\
f(x) &= \frac{(n + x - 1)p}{x} f(x - 1) \qquad\qquad x = 1, 2, 3, \ldots
\end{aligned}
$$

- if X is $Binomial(n, p)$ then

$$
\begin{aligned}
f(0) &= (1 - p)^n \\
f(x) &= \frac{(n - x + 1)p}{x(1 - p)} f(x - 1) \qquad\qquad x = 1, 2, 3, \ldots, n
\end{aligned}
$$

- if X is $Poisson(\mu)$ then

$$
\begin{aligned}
f(0) &= \exp(-\mu) \\
f(x) &= \frac{\mu}{x} f(x - 1) \qquad\qquad x = 1, 2, 3, \ldots
\end{aligned}
$$

6.4.9 EXERCISES

Exercise 6.4.1 (a) Simulate the tossing of a fair coin 10 times and record the number of heads. (b) Repeat this experiment 1000 times and generate a discrete-data histogram of the results. (c) Verify numerically that the relative frequency of the number of heads is approximately equal to the pdf of a $Binomial(10, 0.5)$ random variable.

Exercise 6.4.2 Prove that a $Geometric(p)$ random variable has the memoryless property.

Exercise 6.4.3 Derive the $Geometric(p)$, $Pascal(n, p)$, $Binomial(n, p)$, and $Poisson(\mu)$ recursive pdf equations.

Exercise 6.4.4 Use the $Binomial(n, p)$ recursive pdf equations to implement the functions `pdfBinomial(`n`,`p`,`x`)` and `cdfBinomial(`n`,`p`,`x`)`. Compare your implementations, in terms of both accuracy and speed, with the corresponding functions in the library `rvms`. Use $n = 10, 100, 1000$ with $\mu = 5$ and $p = \mu/n$. The comparison can be restricted to possible values x within the range $\mu \pm 3\sigma$.

Exercise 6.4.5 Verify numerically that the pdf of a $Binomial(25, 0.04)$ random variable is virtually identical to the pdf of a $Poisson(\mu)$ random variable for an appropriate value of μ. Evaluate these pdf's in two ways: by using the appropriate pdf functions in the library `rvms` and by using the $Binomial(n, p)$ recursive pdf equations.

Exercise 6.4.6ª Prove or disprove: if X is a $Pascal(n, p)$ random variable then the cdf of X is $F(x) = I(n, x + 1, 1 - p)$.

Exercise 6.4.7ª Prove or disprove: if X is a $Binomial(n, p)$ random variable then the cdf of X is $F(x) = I(n - x, x + 1, 1 - p)$.

Exercise 6.4.8 Prove that if X is $Binomial(n, p)$ then

$$\Pr(X > 0) = \sum_{x=0}^{n-1} p(1 - p)^x.$$

What is the probability interpretation of this equation?

Exercise 6.4.9 (a) If you play Pick-3 (as in Example 6.1.14) once a day for 365 consecutive days, what is the probability that you will be ahead at the end of this period? (b) What is your expected yield (winnings) at the end of this period?

Exercise 6.4.10 Let X be a $Geometric(p)$ random variable. As an alternative to the definition used in this book, some authors define the random variable $Y = X + 1$ to be $Geometric(p)$. (a) What is the pdf, cdf, idf, mean, and standard deviation of Y? (b) Does this random variable also have the memoryless property?

The topic of primary importance in this section is *random sampling*. That is, we will focus on algorithms that use a random number generator to sample, at random, from a fixed collection (set) of objects. We begin, however, with the related topic of *random shuffling*.

6.5.1 RANDOM SHUFFLING

Definition 6.5.1 Exactly $m!$ different permutations $(a_0, a_1, \ldots, a_{m-1})$ can be formed from a finite set $\mathcal{A}$ with $m = |\mathcal{A}|$ distinct elements. A *random shuffle generator* is an algorithm that will produce any one of these $m!$ permutations in such a way that all are equally likely.

Example 6.5.1 If $\mathcal{A} = \{0, 1, 2\}$ then the $3! = 6$ different possible permutations of $\mathcal{A}$ are

$$(0, 1, 2) \quad (0, 2, 1) \quad (1, 0, 2) \quad (1, 2, 0) \quad (2, 0, 1) \quad (2, 1, 0).$$

A random shuffle generator can produce any of these six possible permutations, each with equal probability 1/6.

Algorithm 6.5.1 The intuitive way to generate a random shuffle of $\mathcal{A}$ is to draw the first element a_0 at random from $\mathcal{A}$. Then draw the second element a_1 at random from the set $\mathcal{A} - \{a_0\}$ and the third element a_2 at random from the set $\mathcal{A} - \{a_0, a_1\}$, etc. The following *in place* algorithm does that, provided the elements of the set $\mathcal{A}$ are stored (in any order) in the array $a[0], a[1], \ldots, a[m-1]$

```
for (i = 0; i < m - 1; i++) {
   j = Equilikely(i, m - 1);
   hold = a[j];
   a[j] = a[i];                          /* swap a[i] and a[j] */
   a[i] = hold;
}
```

Algorithm 6.5.1 is an excellent example of the elegance of simplicity. Figure 6.5.1 (corresponding to $m = 9$) illustrates the two indices i and j [j is an *Equilikely*$(i, m-1)$ random variate] and the state of the $a[\cdot]$ array (initially filled with the integers 0 through 8 in sorted order, although any order is acceptable) for the first three passes through the `for` loop.

Figure 6.5.1. prior to $a[0], a[3]$ swap prior to $a[1], a[6]$ swap prior to $a[2], a[4]$ swap
Permutation generation algorithm.

0	1	2	3	4	5	6	7	8

$i \quad\quad j$

3	1	2	0	4	5	6	7	8

$\quad i \quad\quad\quad\quad j$

3	6	2	0	4	5	1	7	8

$\quad\quad i \quad j$
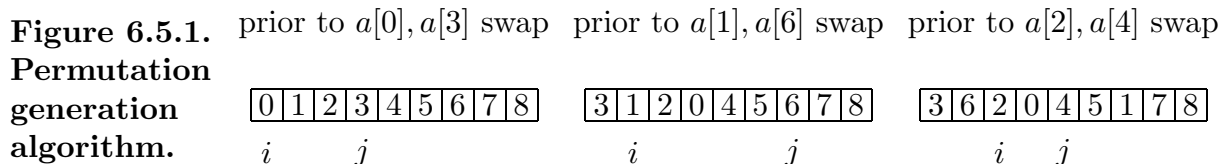
This algorithm is ideal for shuffling a deck of cards ($m = 52$ for a standard deck) — always a useful simulation skill. Moreover, as discussed later in this section, a minor modification to this algorithm makes it suitable for random sampling, without replacement.

6.5.2 RANDOM SAMPLING

We now turn to the topic of primary importance in this section, algorithms for random sampling. To provide a common basis for comparison of the algorithms, the following notation and terminology is used.

- We are given a *population* $\mathcal{P}$ of $m = |\mathcal{P}|$ elements $a_0, a_1, \ldots, a_{m-1}$. Typically m is large, perhaps so large that the population must be stored in secondary memory as a disk file, for example. If m is not large, then the population could be stored in primary memory as an array $a[0], a[1], \ldots, a[m-1]$ or a linked list. In either case, whether m is large or not, the population is a *list*. That is, $\mathcal{P}$ is ordered so that there is a first element a_0, a second element a_1, etc.

- We wish to obtain a random *sample* $\mathcal{S}$ of $n = |\mathcal{S}|$ elements $x_0, x_1, \ldots, x_{n-1}$ from $\mathcal{P}$. Like $\mathcal{P}$, the sample $\mathcal{S}$ is also a *list*. Typically n is small relative to m, but not necessarily so. If n is small then the sample can be stored in primary memory as the array $x[0], x[1], \ldots, x[n-1]$ with the data type of $x[\cdot]$ the same as the data type of the population elements. For most algorithms, however, the use of an array is not a fundamental restriction; a linked list could be used instead, for example, or the sample could be written to secondary memory as a disk file if n is large.

- The algorithms use two generic functions `Get(&z, `$\mathcal{L}$`, j)` and `Put(z, `$\mathcal{L}$`, j)` where the list $\mathcal{L}$ could be either $\mathcal{P}$ or $\mathcal{S}$ and:

    `Get(&z, `$\mathcal{L}$`, j)` returns the value of the jth element in the list $\mathcal{L}$ as z;

    `Put(z, `$\mathcal{L}$`, j)` assigns the value of z to the jth element in the list $\mathcal{L}$.

    Both $\mathcal{P}$ and $\mathcal{S}$ can be accessed by `Get` and `Put`. The use of these generic functions allows the algorithms to be presented in a form that is essentially independent of how the lists $\mathcal{P}$ and $\mathcal{S}$ are actually stored.

- In some random sampling applications it is important to preserve the order of the population in the sample; in other applications *order preservation* is not important.

- A random sampling algorithm may be *sequential*, in which case $\mathcal{P}$ is traversed once, in order, and elements are selected at random to form $\mathcal{S}$. A sequential algorithm is necessarily used when random access to $\mathcal{P}$ is not possible. In contrast, a *non-sequential* random sampling algorithm is based on the assumption that random access to $\mathcal{P}$ is possible and reasonably efficient. Note that it is access to $\mathcal{P}$, not $\mathcal{S}$, that determines whether a random sampling algorithm is sequential or non-sequential.

- A random sampling algorithm may use sampling *with replacement*, in which case the sample can contain multiple instances of the same population element. If so, then n could be larger than m. Instead, if sampling *without replacement* is used, then the sample cannot contain multiple instances of the same population element and so $n \leq m$. Sampling without replacement is the usual case in practice. For the special (trivial) case of sampling without replacement when $n = m$, $\mathcal{P} \equiv \mathcal{S}$.

Relative to random sampling algorithms, the phrase "at random" can be interpreted in at least two different (but closely related) ways: (i) each element of $\mathcal{P}$ is equally likely to be an element of $\mathcal{S}$ or (ii) each possible sample of size n is equally likely to be selected. Because these two interpretations of randomness are not equivalent, it is important to recognize what kind of samples a particular random sampling algorithm actually produces.

Non-Sequential Sampling

Algorithm 6.5.2   This $O(n)$ algorithm provides non-sequential random sampling with replacement. The value of m must be known.

```
for (i = 0;  i < n;  i++) {
   j = Equilikely(0, m - 1);
   Get(&z, P, j);                              /* random access */
   Put(z, S, i);
}
```

Because sampling is with replacement, n can be larger than m in Algorithm 6.5.2. If the elements of $\mathcal{P}$ are distinct, the number of possible samples is m^n and all samples are equally likely to be generated. The order of $\mathcal{P}$ is not preserved in $\mathcal{S}$. Algorithm 6.5.2 should be compared with Algorithm 6.5.3, which is the *without* replacement analog. For both algorithms, the $O(n)$ complexity is based on the number of random variates generated and ignores, perhaps unrealistically, the complexity of access to $\mathcal{P}$ and $\mathcal{S}$.

Example 6.5.2   In most non-sequential random sampling applications the population and sample are stored as arrays $a[0], a[1], \ldots, a[m-1]$ and $x[0], x[1], \ldots, x[n-1]$ respectively, in which case Algorithm 6.5.2 is equivalent to

```
for (i = 0;  i < n;  i++) {
   j = Equilikely(0, m - 1);
   x[i] = a[j];
}
```

Algorithm 6.5.3   This $O(n)$ algorithm provides non-sequential random sampling without replacement. The value of m must be known a priori and $n \leq m$.

```
for (i = 0;  i < n;  i++) {
   j = Equilikely(i, m - 1);                   /* note, i not 1 */
   Get(&z, P, j);                              /* random access */
   Put(z, S, i);
   Get(&x, P, i);                              /* sequential access */
   Put(z, P, i);                               /* sequential access */
   Put(x, P, j);                               /* random access */
}
```

If the elements of $\mathcal{P}$ are distinct, the number of possible samples is $m(m-1)\ldots(m-n+1) = m!/(m-n)!$ and all samples are equally likely to be generated. The order of $\mathcal{P}$ is not preserved in $\mathcal{S}$. Also, the order that exists in $\mathcal{P}$ is destroyed by this sampling algorithm. If this is undesirable, use a copy of $\mathcal{P}$ instead.

Example 6.5.3 In most non-sequential random sampling applications the population and sample are stored as arrays $a[0], a[1], \ldots, a[m-1]$ and $x[0], x[1], \ldots, x[n-1]$, respectively. In this case Algorithm 6.5.3 is equivalent to

```
for (i = 0; i < n; i++) {
   j = Equilikely(i, m - 1);
   x[i] = a[j];
   a[j] = a[i];
   a[i] = x[i];
}
```

In this form, it is clear that Algorithm 6.5.3 is a simple extension of Algorithm 6.5.1.

Sequential Sampling

The next three algorithms provide *sequential* sampling. For each algorithm the basic idea is the same — traverse $\mathcal{P}$ once, in order, and select elements to put in $\mathcal{S}$. The selection or non-selection of elements is random with probability p using the generic statements

```
Get(&z, P, j);
if (Bernoulli(p))
   Put(z, S, i);
```

For Algorithm 6.5.4, p is independent of i and j. For Algorithms 6.5.5 and 6.5.6, however, the probability of selection changes adaptively conditioned on the values of i, j, and the number of elements previously selected.

Algorithm 6.5.4 This $O(m)$ algorithm provides sequential random sampling without replacement. Each element of $\mathcal{P}$ is selected, independently, with probability p.

```
i = 0;                                  /* i indexes the sample */
j = 0;                                  /* j indexes the population */
while ( more data in P ) {
   Get(&z, P, j);                       /* sequential access */
   j++;
   if (Bernoulli(p)) {
      Put(z, S, i);
      i++;
   }
}
```

Although $p = n/m$ is a logical choice for the probability of selection, Algorithm 6.5.4 does not make use of either m or n explicitly. Note also that Algorithm 6.5.4 is not consistent with the idea of a specified sample size. Although the *expected* size of the sample is mp, the *actual* size is a $Binomial(m, p)$ random variable. That is, no matter how p is chosen (with $0 < p < 1$), there is no way to specify the exact size of the sample. It can range from 0 to m. The order of $\mathcal{P}$ is preserved in $\mathcal{S}$.

Example 6.5.4   In many sequential random sampling applications the population is stored in secondary memory as a sequential file and the sample is stored in primary memory as an array $x[0], x[1], \ldots, x[n-1]$ in which case Algorithm 6.5.4 is equivalent to

```
i = 0;
while ( more data in P ) {
   z = GetData();
   if (Bernoulli(p)) {
     x[i] = z;
     i++;
   }
}
```

The fact that Algorithm 6.5.4 does not make use of m can be an advantage in some discrete-event simulation and real-time data acquisition applications. In these applications, the objective may be to sample, at random, say 1% of the population elements, independent of the population size. Algorithm 6.5.4 with $p = 0.01$ would provide this ability, but at the expense of not being able to specify the sample size exactly. In contrast, Algorithm 6.5.5 provides the ability to specify the sample size provided m is known and Algorithm 6.5.6 provides this ability even if m is unknown.

Algorithm 6.5.5   This $O(m)$ algorithm provides sequential random sampling of n sample values from $\mathcal{P}$ without replacement, provided $m = |\mathcal{P}|$ is known a priori.

```
i = 0;                                    /* i indexes the sample */
j = 0;                                    /* j indexes the population */
while (i < n) {
   Get(&z, P, j);                              /* sequential access */
   p = (n - i) / (m - j);
   j++;
   if (Bernoulli(p)) {
     Put(z, S, i);
     i++;
   }
}
```

The key to the correctness of Algorithm 6.5.5 is that the population element a_j is selected with a probability $(n - i)/(m - j)$ that is conditioned on the number of sample elements left to be selected and the number of population elements left to be considered. Access to the population list is sequential (as is access to the sample list). Because sampling is without replacement, $n \le m$. If the elements of $\mathcal{P}$ are distinct, then the number of possible samples is the binomial coefficient $m!/(m - n)!\,n!$ (see Appendix D) and all samples are equally likely to be generated. The order of $\mathcal{P}$ is preserved in $\mathcal{S}$.

Example 6.5.5 In many sequential random sampling applications the population is stored in secondary memory as a sequential file and the sample is stored in primary memory as an array $x[0], x[1], \ldots, x[n - 1]$ in which case Algorithm 6.5.5 is equivalent to

```
i = 0;
j = 0;
while (i < n) {
   z = GetData();
   p = (n - i) / (m - j);
   j++;
   if (Bernoulli(p)) {
     x[i] = z;
     i++;
   }
}
```

Algorithm 6.5.6 This $O(m)$ algorithm provides sequential random sampling of n sample values from $\mathcal{P}$ without replacement, even if $m = |\mathcal{P}|$ is not known a priori

```
for (i = 0; i < n; i++) {
   Get(&z, P, i);                              /* sequential access */
   Put(z, S, i);
}
j = n;
while ( more data in P ) {
   Get(&z, P, j);                              /* sequential access */
   j++;
   p = n / j;
   if (Bernoulli(p)) {
     i = Equilikely(0, n - 1);
     Put(z, S, i);
   }
}
```

Algorithm 6.5.6 is based on initializing the sample with the first n elements of the population. Then, for each of the (unknown number of) additional population elements, with conditional probability $n/(j+1)$ the population element a_j (for $j \geq n$) overwrites an existing randomly selected element in the sample. Access to the population list is sequential. Access to the sample list is *not* sequential, however, and so the order in $\mathcal{P}$ is not preserved in $\mathcal{S}$. Because sampling is without replacement, $n \leq m$. If the elements of $\mathcal{P}$ are distinct, then the number of possible samples is the binomial coefficient $m!/(m-n)!\,n!$ (see Appendix D) and all samples are equally likely to be generated. There is an important caveat, however — see Example 6.5.8.

Example 6.5.6 In many sequential random sampling applications the population is stored in secondary memory as a sequential file and the sample is stored in primary memory as an array $x[0], x[1], \ldots, x[n-1]$ in which case Algorithm 6.5.6 is equivalent to

```
for (i = 0;  i < n;  i++) {
   z = GetData();
   x[i] = z;
}
j = n;
while ( more data in P ) {
   z = GetData();
   j++;
   p = n / j;
   if (Bernoulli(p)) {
     i = Equilikely(0, n − 1);
     x[i] = z;
   }
}
```

Algorithm Differences

Although Algorithms 6.5.5 and 6.5.6 have some similarities, there are two important differences.

- Algorithm 6.5.5 requires knowledge of the population size m and Algorithm 6.5.6 does not. This makes Algorithm 6.5.6 valuable in those discrete-event simulation and real-time data acquisition applications where m is not known a priori, particularly if the sample size is sufficiently small so that the sample can be stored in primary memory as an array.

- Beyond this obvious difference, there is a more subtle difference related to the number of possible samples when *order* is considered. Algorithm 6.5.5 preserves the order present in the population, Algorithm 6.5.6 does not. This difference is illustrated by the following two examples.

Example 6.5.7 If Algorithm 6.5.5 is used to select samples of size $n = 3$ from the population list $(0, 1, 2, 3, 4)$, then $m = 5$ and, because the sampling is sequential and the order of the population list is preserved in the samples, we find that exactly

$$\binom{5}{3} = \frac{5!}{2!\,3!} = 10$$

ordered samples are possible, as illustrated

$$(0, 1, 2)\ (0, 1, 3)\ (0, 1, 4)\ (0, 2, 3)\ (0, 2, 4)\ (0, 3, 4)\ (1, 2, 3)\ (1, 2, 4)\ (1, 3, 4)\ (2, 3, 4).$$

Each of these (ordered) samples will occur with equal probability $1/10$.

Example 6.5.8 If Algorithm 6.5.6 is used then 13 samples are possible, as illustrated

$$(0, 1, 2)\ (0, 1, 3)\ (0, 1, 4)\ (0, 3, 2)\ (0, 4, 2)\ \begin{matrix}(0, 3, 4)\\(0, 4, 3)\end{matrix}\ (3, 1, 2)\ (4, 1, 2)\ \begin{matrix}(3, 1, 4)\ (3, 4, 2)\\(4, 1, 3)\ (4, 3, 2)\end{matrix}$$

Each of these samples are *not* equally likely, however. Instead, each of the six samples that are pairwise alike except for permutation will occur with probability $1/20$. The other seven samples will occur with probability $1/10$. If permutations are combined (for example, by sorting and then combining like results) then, as desired, each of the resulting 10 samples will have equal probability $1/10$. Because order in $\mathcal{P}$ is not preserved in $\mathcal{S}$ by Algorithm 6.5.6, some ordered samples, like $(1, 3, 4)$, cannot occur except in permuted order, like $(3, 1, 4)$ and $(4, 1, 3)$. For this reason, Algorithm 6.5.6 produces the correct number of equally likely samples only if all alike-but-for-permutation samples are combined. This potential need for post processing is the (small) price paid for not knowing the population size a priori.

6.5.3 URN MODELS

Many discrete stochastic models are based on random sampling. We close this section with four examples. Three of these discrete stochastic models were considered earlier in this chapter, the other is new:

- *Binomial*(n, p);
- *Hypergeometric*(n, a, b);
- *Geometric*(p);
- *Pascal*(n, p).

All four of these models can be motivated by drawing, at random, from a conceptual urn initially filled with a amber balls and b black balls.

Binomial

Example 6.5.9   An urn contains a amber balls and b black balls. A total of n balls are drawn from the urn, *with* replacement. Let x be the number of amber balls drawn. A Monte Carlo simulation of this random experiment is easily constructed. Indeed, one obvious way to construct this simulation is to use Algorithm 6.5.2, as implemented in Example 6.5.2, applied to a population array of length $m = a + b$ with a 1's (amber balls) and b 0's (black balls). Then x is the number of 1's in a random sample of size n. The use of Algorithm 6.5.2 is overkill in this case, however, because an equivalent simulation can be constructed without using either a population or sample data structure. That is, let $p = a/(a + b)$ and use the $O(n)$ algorithm

```
x = 0;
for (i = 0; i < n; i++)
   x += Bernoulli(p);
return x;
```

The discrete random variate x so generated is $Binomial(n, p)$. The associated pdf of X is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, 2, \ldots, n.$$

Hypergeometric

Example 6.5.10   As a variation of Example 6.5.9, suppose that the draw from the urn is *without* replacement (and so $n \le m$). Although Algorithm 6.5.3 could be used as implemented in Example 6.5.3, it is better to use a properly modified version of the algorithm in Example 6.5.9 instead, as illustrated

```
m = a + b;
x = 0;
for (i = 0; i < n; i++) {
   p = (a - x) / (m - i);
   x += Bernoulli(p);
}
return x;
```

The discrete random variate x so generated is said to be $Hypergeometric(n, a, b)$. The associated pdf of X is

$$f(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}} \qquad x = \max\{0, n - b\}, \ldots, \min\{a, n\}.$$

The lower limit in the range of possible values of X accounts for the possibility that the sample size may exceed the number of black balls $(n > b)$. If $n > b$ then at least $n - b$ amber balls must be drawn. Similarly, the upper limit accounts for the possibility that the sample size may exceed the number of amber balls $(n > a)$. If $n > a$ then at most a amber balls can be drawn. In applications where n is less than or equal to both a and b, the range of possible values is $x = 0, 1, 2, \ldots, n$.

Geometric

Example 6.5.11   As another variation of Example 6.5.9, suppose that the draw is from the urn *with* replacement but that we draw only until the first black ball is obtained. Let x be the number of amber balls drawn. With $p = a/(a+b)$ the following algorithm simulates this random experiment

```
x = 0;
while (Bernoulli(p))
   x++;
return x;
```

The discrete random variate x so generated is *Geometric*(p). The associated pdf of X is

$$f(x) = p^x(1 - p) \qquad x = 0, 1, 2, \ldots$$

This stochastic model is commonly used in reliability studies, in which case p is usually close to 1.0 and x counts the number of successes before the first failure.

This algorithm for generating geometric variates is inferior to the inversion algorithm presented in Example 6.2.8. The algorithm presented here is (1) not synchronized, (2) not monotone, and (3) inefficient. The expected number of passes through the **while** loop is

$$1 + \frac{p}{1 - p}.$$

If p is close to 1, the execution time for this algorithm can be quite high.

Pascal

Example 6.5.12   As an extension of Example 6.5.11, suppose that we draw *with* replacement until the nth black ball is obtained. Let x be the number of amber balls drawn (so that a total of $n + x$ balls are drawn, the last of which is black). With $p = a/(a + b)$ the following algorithm simulates this random experiment.

```
x = 0;
for (i = 0; i < n; i++)
   while (Bernoulli(p))
      x++;
return x;
```

The discrete random variate x so generated is $Pascal(n, p)$. The associated pdf is

$$f(x) = \binom{n + x - 1}{x} p^x (1 - p)^n \qquad\qquad x = 0, 1, 2, \ldots$$

This stochastic model is commonly used in reliability studies, in which case p is usually close to 1.0 and x counts the number of successes before the nth failure.

Further reading on random sampling and shuffling can be found in the textbooks by Nijenhuis and Wilf (1978) and Wilf (1989) or the journal articles by McLeod and Bellhouse (1983) and Vitter (1984).

6.5.4   EXERCISES

Exercise 6.5.1   Two cards are drawn in sequence from a well-shuffled ordinary deck of 52. (a) If the draw is without replacement, use Monte Carlo simulation to verify that the second card will be higher in rank than the first with probability 8/17. (b) Estimate this probability if the draw is with replacement.

Exercise 6.5.2   Which of the four discrete stochastic models presented in this section characterizes the actual (random variate) sample size when Algorithm 6.5.4 is used?

Exercise 6.5.3   (a) For $a = 30$, $b = 20$, and $n = 10$ use Monte Carlo simulation to verify the correctness of the probability equations in Examples 6.5.9 and 6.5.10. (b) Verify that in both cases the mean of x is $na/(a + b)$. (c) Are the standard deviations the same?

Exercise 6.5.4   Three people toss a fair coin in sequence until the first occurrence of a head. Use Monte Carlo simulation to estimate (a) the probability that the first person to toss will eventually win the game by getting the first head, (b) the second person's probability of winning, and (c) the third person's probability of winning. Also, (d) compare your estimates with the analytic solutions.

Exercise 6.5.5[a]   How would the algorithm and pdf equation need to be modified if the random draw in Example 6.5.12 is *without* replacement?

Exercise 6.5.6[a]   An urn is initially filled with one amber ball and one black ball. Each time a black ball is drawn, it is replaced *and* another black ball is also placed in the urn. Let X be the number of random draws required to find the amber ball. (a) What is the pdf of X for $x = 1, 2, 3, \ldots$? (b) Use Monte Carlo simulation to estimate the pdf and the expected value of X.

Exercise 6.5.7[a]   Same as the previous exercise except that there is a reservoir of just eight black balls to add to the urn. Comment.

CHAPTER 7

CONTINUOUS RANDOM VARIABLES

Sections

The focus shifts from discrete random variables to continuous random variables in this chapter. The first four sections in this chapter have identical titles to the previous chapter with "discrete" replaced by "continuous". As in the previous chapter, we proceed with a more thorough and methodical description of continuous random variables, their properties, how they can be used to model the stochastic (random) components of a system of interest, and the development of algorithms for generating the associated random variates for a Monte Carlo or discrete-event simulation model.

Section 7.1 defines a continuous random variable and introduces eight popular models: the uniform, exponential, standard normal, normal, lognormal, Erlang, chisquare, and student distributions. Section 7.2 contains an approach to generating continuous random variables that is more general than the ad hoc approaches given earlier for the $Uniform(a, b)$ and $Exponential(\mu)$ variates. Section 7.3 applies these variate generation techniques to the generation of arrival processes and service times. Section 7.4 contains a summary of the eight continuous distributions encountered thus far. Section 7.5 considers the topic of the modeling and generation of arrivals to a system when the arrival rate varies with time. A model known as a *nonstationary Poisson process* is introduced. Finally, Section 7.6 introduces a variate generation technique known as *acceptance-rejection*, which can be used to generate continuous random variates.

The formal theory of continuous random variables closely parallels the corresponding theory of discrete random variables. In many instances little more is involved than replacing sums with integrals and differences with derivatives. With this in mind, we will cover the corresponding topics in this chapter more rapidly than in the previous chapter, pausing to elaborate only when the continuous theory is significantly different from the discrete theory.

7.1.1   CONTINUOUS RANDOM VARIABLE CHARACTERISTICS

Definition 7.1.1   The random variable X is *continuous* if and only if the cdf of X, $F(x) = P(X \leq x)$ is a *continuous* function.*

Continuous random variables are real-valued measurements like size, weight, distance, elapsed time, volume, density, etc. In practice, it is common to assume that $\mathcal{X}$ is an open interval (a, b) where a may be $-\infty$ and b may be ∞.

Probability Density Function

Definition 7.1.2   As in the discrete case, the continuous random variable X is uniquely determined by its set of possible values $\mathcal{X}$ and corresponding *probability density function* (pdf) which is a real-valued function $f(\cdot)$ defined for all $x \in \mathcal{X}$ in such a way that

$$\int_a^b f(x)\,dx = \Pr(a \leq X \leq b)$$

for any interval $(a, b) \subseteq \mathcal{X}$. By definition, $x \in \mathcal{X}$ is a possible value of X if and only if $f(x) > 0$. In addition, $f(\cdot)$ is defined so that

$$\int_x f(x)\,dx = 1$$

where the integration is over all $x \in \mathcal{X}$.

Example 7.1.1   If the random variable X is *Uniform(a, b)* then the possible values of X are all the real numbers between a and b, i.e., $\mathcal{X} = (a, b)$. Because the length of this interval is $b - a$ and all values in this interval are equally likely it follows that

$$f(x) = \frac{1}{b - a} \qquad\qquad a < x < b.$$

If X is a discrete random variable then it makes sense to evaluate a "point" probability like $\Pr(X = 1)$. However, if X is a continuous random variable, then "point" probabilities are zero. In the continuous case, probability is defined by an *area* under the pdf curve and so $\Pr(X = x) = 0$ for *any* $x \in \mathcal{X}$. For the same reason, endpoint inclusion is not an issue. That is, if X is a continuous random variable and if $[a, b] \subseteq \mathcal{X}$ then all of the probabilities $\Pr(a \leq X \leq b)$, $\Pr(a < X \leq b)$, $\Pr(a \leq X < b)$, $\Pr(a < X < b)$ are equal to $\int_a^b f(x)\,dx$.

---

  * Thus the set of possible values in $\mathcal{X}$ is continuous, *not* discrete — see Definition 6.1.1.

Cumulative Distribution Function

Definition 7.1.3   The *cumulative distribution function* (cdf) of the continuous random variable X is the continuous real-valued function $F(\cdot)$ defined for each $x \in \mathcal{R}$ as

$$F(x) = \Pr(X \le x) = \int_{t \le x} f(t)\,dt.$$

Example 7.1.2   If the random variable X is *Uniform*(a, b) then $\mathcal{X} = (a, b)$ and the cdf is

$$F(x) = \int_a^x \frac{1}{b-a}\,dt = \frac{x-a}{b-a} \qquad a < x < b.$$

In the important special case where U is *Uniform*$(0, 1)$ the cdf is

$$F(u) = \Pr(U \le u) = u \qquad 0 < u < 1.$$

Figure 7.1 illustrates the relation between a general pdf on the left and the associated cdf on the right. The two vertical axes are not drawn to the same scale. The shaded area in the graph of the pdf represents $F(x_0)$ for some arbitrary x-value x_0.

Figure 7.1.1. The pdf and cdf of a continuous random variable X.
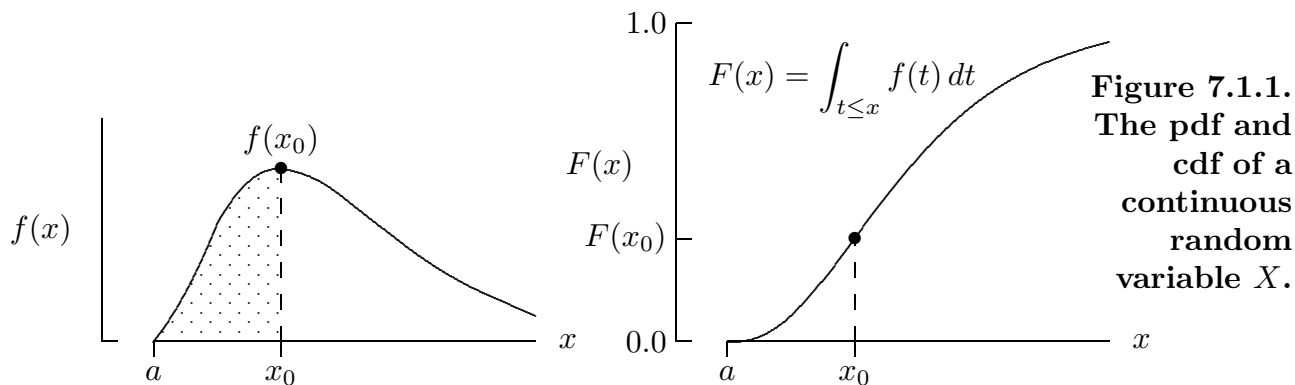
As in the discrete case, the cdf is strictly monotone increasing — if x_1 and x_2 are possible values of X with $x_1 < x_2$ and all the real numbers between x_1 and x_2 are also possible values, then $F(x_1) < F(x_2)$. Moreover $F(\cdot)$ is bounded between 0.0 and 1.0.

As illustrated in Figure 7.1.1, consistent with Definition 7.1.3 the cdf can be obtained from the pdf by integration. Conversely, the pdf can be obtained from the cdf by differentiation as

$$f(x) = \frac{d}{dx}F(x) \qquad x \in \mathcal{X}.$$

Thus, as in the discrete case, a continuous random variable model can be constructed by specifying $\mathcal{X}$ and either the pdf or the cdf. One can be determined from the other.

Example 7.1.3   In Section 3.1 an *Exponential*(μ) random variable X was defined by the transformation $X = -\mu \ln(1 - U)$ where U is *Uniform*$(0, 1)$. To determine the cdf of X from this definition, recognize that for any $x > 0$

$$
\begin{aligned}
F(x) &= \Pr(X \leq x) \\
&= \Pr\left(-\mu \ln(1 - U) \leq x\right) \\
&= \Pr\left(1 - U \geq \exp(-x/\mu)\right) \\
&= \Pr\left(U \leq 1 - \exp(-x/\mu)\right) \\
&= 1 - \exp(-x/\mu)
\end{aligned}
$$

where the last equation follows from Example 7.1.2. Therefore, the pdf of X is

$$
f(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \left(1 - \exp(-x/\mu)\right) = \frac{1}{\mu} \exp(-x/\mu) \qquad x > 0.
$$

Mean and Standard Deviation

Definition 7.1.4   The *mean* μ of the continuous random variable X is

$$
\mu = \int_x x f(x) \, dx
$$

and the corresponding *standard deviation* σ is

$$
\sigma = \sqrt{\int_x (x - \mu)^2 f(x) \, dx} = \sqrt{\left(\int_x x^2 f(x) \, dx\right) - \mu^2}
$$

where the integration is over all possible values $x \in \mathcal{X}$. The *variance* is σ^2.[*]

Example 7.1.4   If X is *Uniform*(a, b) then it can be shown that

$$
\mu = \frac{a + b}{2} \qquad \text{and} \qquad \sigma = \frac{b - a}{\sqrt{12}}.
$$

This derivation is left as an exercise.

Example 7.1.5   If X is *Exponential*(μ) then from the pdf equation in Example 7.1.3

$$
\int_x x f(x) \, dx = \int_0^\infty \frac{x}{\mu} \exp(-x/\mu) \, dx = \mu \int_0^\infty t \exp(-t) \, dt = \cdots = \mu
$$

using integration by parts. So the parameter μ is in fact the mean. Moreover, the variance is

$$
\sigma^2 = \left(\int_0^\infty \frac{x^2}{\mu} \exp(-x/\mu) \, dx\right) - \mu^2 = \left(\mu^2 \int_0^\infty t^2 \exp(-t) \, dt\right) - \mu^2 = \cdots = \mu^2.
$$

For this particular distribution, the parameter μ is also the standard deviation.

---

[*]   Compare Definition 7.1.4 with Definition 4.3.3.

Expected Value

Definition 7.1.5   As in the discrete case, the mean of a continuous random variable is also known as the *expected value* and the expected value operator is denoted as $E[\,\cdot\,]$. In particular, the mean of X is the expected value of X

$$\mu = E[X] = \int_x x f(x)\,dx$$

and the variance is the expected value of $(X - \mu)^2$

$$\sigma^2 = E[(X - \mu)^2] = \int_x (x - \mu)^2 f(x)\,dx.$$

In general, if $g(X)$ is some function of the random variable X, i.e., $Y = g(X)$, then the expected value of the random variable Y is

$$E[Y] = E[g(X)] = \int_x g(x) f(x)\,dx.$$

In all three of these equations the integration is over all $x \in \mathcal{X}$.

Example 7.1.6   Consider a circle of fixed radius r and a fixed point Q on the circumference of the circle. Suppose a second point P is selected *at random* on the circumference of the circle and let the random variable Y be the distance of the line segment (chord) joining points P and Q. To determine the expected value of Y let Θ be the angle of point P measured counter-clockwise from point Q. It can be shown that the relation between Y and Θ is $Y = 2r\sin(\Theta/2)$, as illustrated in Figure 7.1.2.
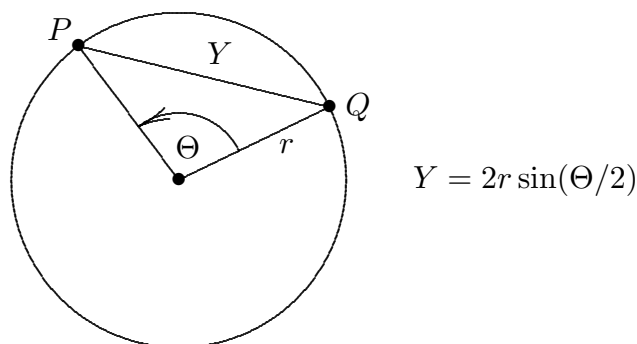
**Figure 7.1.2.
A random
chord.**

$$Y = 2r\sin(\Theta/2)$$

Therefore, if we interpret the phrase "P is selected at random" to mean that the angle Θ is *Uniform*$(0, 2\pi)$, then the pdf of Θ is $f(\theta) = 1/2\pi$ for $0 < \theta < 2\pi$ and the expected length of the chord, $E[Y]$, is

$$E[Y] = \int_0^{2\pi} 2r\sin(\theta/2) f(\theta)\,d\theta = \int_0^{2\pi} \frac{2r\sin(\theta/2)}{2\pi}\,d\theta = \cdots = \frac{4r}{\pi}.$$

Because some possible values of Y are more likely than others, Y is *not* a *Uniform*$(0, 2r)$ random variable; if it were, the expected value of Y would be r. (See Exercise 7.1.6 for an alternate approach to this example.)

Example 7.1.7   As in the discrete case, if X is a continuous random variable and Y is the continuous random variable $Y = aX + b$ for constants a and b, then

$$E[Y] = E[aX + b] = aE[X] + b.$$

7.1.2   CONTINUOUS RANDOM VARIABLE MODELS

In addition to the *Uniform*(a, b) and *Exponential*(μ) continuous random variables already discussed, in the remainder of this section we will consider several other parametric models, beginning with the so-called *Normal*$(0, 1)$ or *standard normal* continuous random variable. All students of elementary statistics will recognize this as a special case of the ubiquitous "bell-shaped curve" used by many (appropriately or not) to model virtually all things random.

Standard Normal Random Variable

Definition 7.1.6   The continuous random variable Z is said to be *Normal*$(0, 1)$ if and only if the set of all possible values is $\mathcal{Z} = (-\infty, \infty)$ and the pdf is

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) \qquad -\infty < z < \infty,$$

as illustrated in Figure 7.1.3.

Figure 7.1.3. Standard normal pdf.
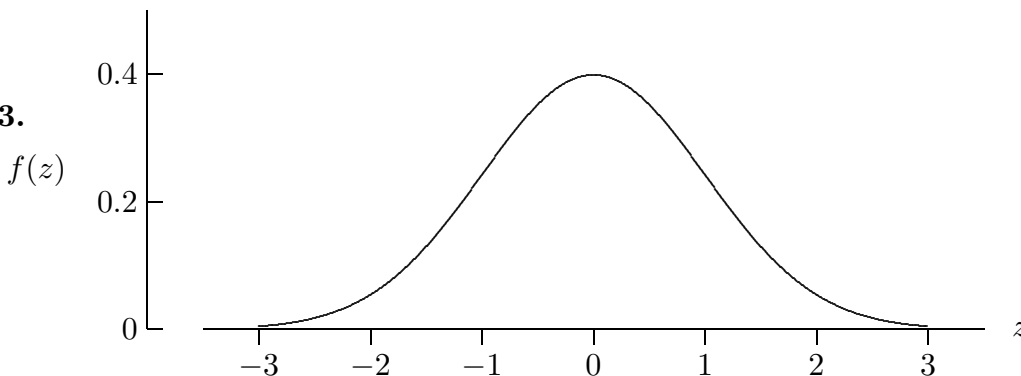
If Z is a *Normal*$(0, 1)$ random variable then Z is "standardized" (see Example 4.1.3). That is, the mean is

$$\mu = \int_{-\infty}^{\infty} z f(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp(-z^2/2)\, dz = \cdots = 0$$

and the variance is

$$\sigma^2 = \int_{-\infty}^{\infty} (z - \mu)^2 f(z)\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \exp(-z^2/2)\, dz = \cdots = 1.$$

The $\mu = 0$ derivation is easy, the $\sigma^2 = 1$ derivation is not. Both are left as an exercise. The standard deviation is $\sigma = 1$.

If Z is a $Normal(0,1)$ random variable then the corresponding cdf is

$$F(z) = \int_{-\infty}^{z} f(t)\, dt = \Phi(z) \qquad -\infty < z < \infty$$

where the special function $\Phi(\cdot)$ is defined as

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)\, dt \qquad -\infty < z < \infty.$$

As a special function, $\Phi(\cdot)$ cannot be evaluated in "closed form." It can, however, be written as a function of an incomplete gamma function $P(a, x)$, as detailed in Appendix D, and evaluated numerically. Specifically, it can be shown that if $z \geq 0$ then

$$\Phi(z) = \frac{1 + P(1/2,\, z^2/2)}{2} \qquad z \geq 0.$$

For $z < 0$, $\Phi(z)$ can be evaluated by using the identity $\Phi(-z) = 1 - \Phi(z)$. The special function $\Phi(z)$ is available in the library `rvms` as the function `cdfNormal(0.0, 1.0, z)`.

One compelling reason for the popularity of the expected value notation is that it applies equally well to both discrete and continuous random variables. For example, suppose that X is a random variable (discrete or continuous) with mean μ and standard deviation σ. For constants a and b define the new random variable $X' = aX + b$ with mean μ' and standard deviation σ'. As an extension of Example 7.1.7,

$$\mu' = E[X'] = E[aX + b] = aE[X] + b = a\mu + b$$

and

$$(\sigma')^2 = E[(X' - \mu')^2] = E[(aX - a\mu)^2] = a^2 E[(X - \mu)^2] = a^2\sigma^2.$$

Therefore

$$\mu' = a\mu + b \qquad \text{and} \qquad \sigma' = |a|\sigma.$$

(See Section 4.1 for analogous sample statistics equations.)

Example 7.1.8 Suppose that Z is a random variable with mean 0 and standard deviation 1 (often known as a *standardized random variable*) and we wish to construct a new random variable X with *specified* mean μ and standard deviation σ. The usual way to do this is to define $X = \sigma Z + \mu$. Then, because $E[Z] = 0$ and $E[Z^2] = 1$, the mean and variance of X are

$$E[X] = \sigma E[Z] + \mu = \mu \qquad \text{and} \qquad E[(X - \mu)^2] = E[\sigma^2 Z^2] = \sigma^2 E[Z^2] = \sigma^2$$

respectively, as desired. The standard deviation of X is σ. This example is the basis for the following definition.

Normal Random Variable

Definition 7.1.7   The continuous random variable X is $Normal(\mu, \sigma)$ if and only if

$$X = \sigma Z + \mu$$

where $\sigma > 0$ and the random variable Z is $Normal(0, 1)$.

From Example 7.1.7, it follows that if X is a $Normal(\mu, \sigma)$ random variable then the mean of X is μ and the standard deviation is σ. That is, for this random variable model the mean and standard deviation are *explicit* parameters. The point here is that a $Normal(\mu, \sigma)$ random variable is constructed from a $Normal(0, 1)$ random variable by "shifting" the mean from 0 to μ via the addition of μ and by "scaling" the standard deviation from 1 to σ via multiplication by σ.

The cdf of a $Normal(\mu, \sigma)$ random variable is

$$F(x) = \Pr(X \le x) = \Pr(\sigma Z + \mu \le x) = \Pr\left(Z \le (x - \mu)/\sigma\right) \qquad -\infty < x < \infty$$

so that

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right) \qquad -\infty < x < \infty$$

where the special function $\Phi(\cdot)$ is the cdf of a $Normal(0, 1)$ random variable. Because the pdf of a $Normal(0, 1)$ random variable is

$$\frac{d}{dz}\Phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2) \qquad -\infty < z < \infty$$

the associated pdf of a $Normal(\mu, \sigma)$ random variable is

$$f(x) = \frac{dF(x)}{dx} = \frac{d}{dx}\Phi\left(\frac{x - \mu}{\sigma}\right) = \cdots = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-(x - \mu)^2/2\sigma^2\right) \qquad -\infty < x < \infty$$

as illustrated in Figure 7.1.4.

Figure 7.1.4. Normal pdf.
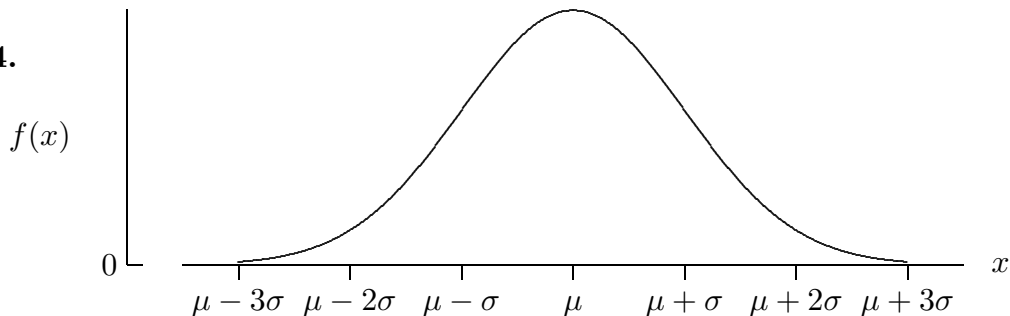
The shift parameter μ and the scale parameter σ allow the normal distribution to enjoy a wide variety of mathematical modeling applications such as the weights of newborn babies, measurements on industrial products, agricultural yields, and IQ scores. For all choices of the parameters μ and σ, the pdf of a normal random variable has the bell shape shown in Figure 7.1.4.

In a field of study known as "classical statistics," observations are assumed to be drawn from populations having a $Normal(\mu, \sigma)$ distribution which allows for conclusions to be drawn concerning μ and σ. An important result in statistical theory is known as the *central limit theorem*, which states that the limiting distribution of sums of independent and identically distributed random variables tends to the normal distribution as the number of random variables in the sum goes to infinity (Hogg, McKean, and Craig, 2005).

We summarize a few facts about normal random variables below:

- A $Normal(\mu, \sigma)$ random variable is also commonly called a *Gaussian* random variable.

- The 68–95–99.73 rule for $Normal(\mu, \sigma)$ random variables indicates that the area under the pdf within 1, 2, and 3 standard deviations of the mean μ is approximately 0.68, 0.95, and 0.9973.

- The pdf has inflection points at $\mu \pm \sigma$.

- Our choice of using $Normal(\mu, \sigma)$ to denote a normal random variable differs significantly from the more common $N(\mu, \sigma^2)$. We use the non-standard notation so that the parameters in our software are consistent with the presentation in the text.

Finally, a warning about the $Normal(\mu, \sigma)$ distribution. Since the support is $\mathcal{X} = \{x | -\infty < x < \infty\}$, the $Normal(\mu, \sigma)$ distribution is *not* an appropriate choice for modeling certain aspects (e.g., time between arrivals or service times in a queue) of a discrete-event simulation model since it could potentially move the simulation clock backward with potential disastrous consequences. Thus the $Normal(\mu, \sigma)$ distribution must be manipulated in a manner to produce only positive values. One such manipulation is the lognormal distribution presented next.

Lognormal Random Variable

Definition 7.1.8 The continuous random variable X is $Lognormal(a, b)$ if and only if

$$X = \exp(a + bZ)$$

where the random variable Z is $Normal(0, 1)$ and $b > 0$.

Like a $Normal(\mu, \sigma)$ random variable, a $Lognormal(a, b)$ random variable is also based on transforming a $Normal(0, 1)$ random variable. The transformation in this case is nonlinear and therefore the result is more difficult to analyze. However, the nonlinear transformation provides an important benefit; a $Lognormal(a, b)$ random variable is inherently positive and therefore potentially well suited to modeling naturally positive quantities like service times.

The cdf of a *Lognormal*(a, b) random variable is

$$F(x) = \Pr(X \leq x) = \Pr\big(\exp(a + bZ) \leq x\big) = \Pr\big(a + bZ \leq \ln(x)\big) \qquad x > 0$$

so that

$$F(x) = \Pr\big(Z \leq (\ln(x) - a)/b\big) = \Phi\left(\frac{\ln(x) - a}{b}\right) \qquad x > 0$$

where the special function $\Phi(\cdot)$ is the cdf of a *Normal*$(0, 1)$ random variable. Differentiation by x then establishes that the pdf of a *Lognormal*(a, b) random variable is

$$f(x) = \frac{dF(x)}{dx} = \cdots = \frac{1}{bx\sqrt{2\pi}} \exp\big(-(\ln(x) - a)^2 / 2b^2\big) \qquad x > 0$$

as illustrated in Figure 7.1.5 for $(a, b) = (-0.5, 1.0)$.

Figure 7.1.5. Lognormal pdf.
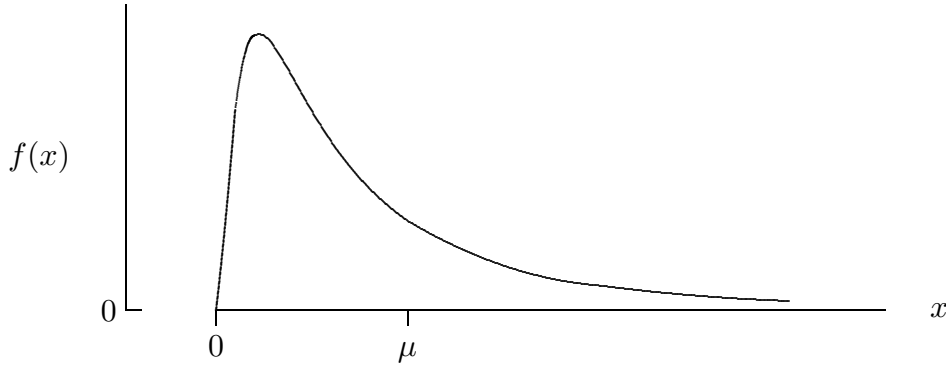
It can be verified that the mean and standard deviation of a *Lognormal*(a, b) random variable are

$$\mu = \exp\big(a + b^2/2\big) \qquad \text{and} \qquad \sigma = \exp\big(a + b^2/2\big)\sqrt{\exp(b^2) - 1}.$$

For the pdf illustrated, $\mu = 1.0$ and $\sigma = \sqrt{e - 1} \cong 1.31$.

A *Uniform*(a, b) random variable is the continuous analog of a discrete *Equilikely*(a, b) random variable. Similarly, an *Exponential*(μ) random variable is the continuous analog of a *Geometric*(p) random variable. Given that, because a *Pascal*(n, p) random variable is the sum of n *iid* *Geometric*(p) random variables, it is natural to assume that the sum of n *iid* *Exponential*(μ) random variables defines a commonly-used continuous random variable model. That is the case.

Erlang Random Variable

Definition 7.1.9 The continuous random variable X is *Erlang*(n, b) if and only if

$$X = X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \ldots, X_n$ is an *iid* sequence of *Exponential*(b) random variables.

Although the details of the derivation are beyond the scope of this book, from Definition 7.1.9 it can be shown that the pdf of an $Erlang(n, b)$ random variable is

$$f(x) = \frac{1}{b(n-1)!}(x/b)^{n-1}\exp(-x/b) \qquad x > 0$$

as illustrated in Figure 7.1.6 for $(n, b) = (3, 1.0)$. In this case $\mu = 3.0$ and $\sigma \cong 1.732$.
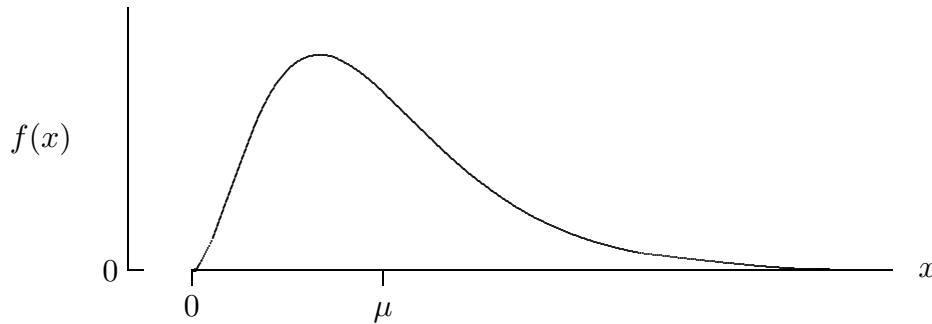
**Figure 7.1.6.
Erlang
pdf.**

The corresponding cdf is an incomplete gamma function (see Appendix D)

$$F(x) = \int_0^x f(t)\, dt = 1 - \left[\exp\left(-\frac{x}{b}\right)\right]\left(\sum_{i=0}^{n-1}\frac{(x/b)^i}{i!}\right) = P(n,\, x/b) \qquad x > 0.$$

The associated mean and standard deviation are

$$\mu = nb \qquad \text{and} \qquad \sigma = \sqrt{n}\, b.$$

Chisquare and Student Random Variables

In addition to $Erlang(n, b)$, $Exponential(\mu)$, $Lognormal(a, b)$, $Normal(\mu, \sigma)$, and $Uniform(a, b)$ random variables, two other continuous random variables commonly used in simulation applications are $Chisquare(n)$ and $Student(n)$. These two random variables are defined in the next section and summarized in Section 7.4. Although they are less likely to be used for parametric modeling in a discrete-event simulation model due to their integer parameters, they are often used for statistical inference.

7.1.3   EXERCISES

Exercise 7.1.1   The possible values of the continuous random variable X are $a < x < b$. (a) If the pdf of X is $f(x) = \alpha(b - x)$, what is α in terms of a and b? (b) Derive equations for the cdf as a function of x, in terms of a and b. (c) Derive equations for the mean and standard deviation in terms of a and b.

Exercise 7.1.2   (a) Derive the equations for μ and σ in Example 7.1.4 and compare with the corresponding equations in Example 6.1.7. (b) If σ_u^2 and σ_e^2 are the variance of a $Uniform(a, b)$ and $Equilikely(a, b)$ random variable respectively, show that $\sigma_e^2 > \sigma_u^2$ and that $\sigma_e^2/\sigma_u^2 \to 1$ as $(b - a) \to \infty$.

Exercise 7.1.3   If X is a continuous random variable with pdf $f(\cdot)$, mean μ, and variance σ^2, prove that

$$\int_x x^2 f(x)\, dx = \mu^2 + \sigma^2$$

where the integration is over all possible values of X.

Exercise 7.1.4   If X is an *Exponential*(μ) random variable and $Y = \lfloor X \rfloor$, prove that Y is a *Geometric*(p) random variable for an appropriate value of p (which depends on μ).

Exercise 7.1.5   (*a*) If $X = U/(1 - U)$ with U a *Uniform*$(0, 1)$ random variable, then what is the set of possible values $\mathcal{X}$? (*b*) What is the cdf of X? (*c*) What is the pdf of X?

Exercise 7.1.6^a   (*a*) If $Y = 2r \sin(\Theta/2)$ where the random variable Θ is *Uniform*$(0, 2\pi)$ and $r > 0$, prove that the pdf of Y is

$$f(y) = \frac{2}{\pi\sqrt{4r^2 - y^2}} \qquad 0 < y < 2r.$$

(*b*) Why is there a 2 in the numerator? (*c*) As an alternative to the derivation in Example 7.1.6, use this pdf to prove that $E[Y] = 4r/\pi$. (*d*) What is the standard deviation of Y?

Exercise 7.1.7^a   (*a*) If U is *Uniform*$(0, 1)$, if Θ is *Uniform*$(-\pi/2, \pi/2)$, if U and Θ are independent, and if $V = U + \cos(\Theta)$, prove that the possible values of V are $0 < v < 2$ and the pdf of V is

$$f(v) = \begin{cases} 1 - (2/\pi)\arccos(v) & 0 < v < 1 \\ (2/\pi)\arccos(v - 1) & 1 \le v < 2. \end{cases}$$

(*b*) How does this relate to Example 4.3.1?

Exercise 7.1.8   If X is a *Normal*(μ, σ) random variable what is the numerical value of $\Pr(|X - \mu| \le k\sigma)$ for $k = 1, 2, 3, 4$?

Exercise 7.1.9   (*a*) If $X = \exp(U)$ with U a *Uniform*(a, b) random variable then what is $\mathcal{X}$? (*b*) What is the cdf of X? (*c*) What is the pdf of X? (*d*) What is the mean of X?

Exercise 7.1.10   Show that the inflection points on the pdf for a *Normal*(μ, σ) random variable occur at $x = \mu \pm \sigma$.

Exercise 7.1.11   Let X be a random variable with finite mean μ and finite variance σ^2. Find the mean and variance of the *standardized* random variable $Y = (X - \mu)/\sigma$.

This section is the continuous random variate analog of Section 6.2. As in Section 6.2, the emphasis is on the development of inversion algorithms for random variate generation.

7.2.1   INVERSE DISTRIBUTION FUNCTION

Definition 7.2.1   Let X be a continuous random variable with cdf $F(\cdot)$ and set of possible values $\mathcal{X}$. The *inverse distribution function* (*idf*) of X is the function $F^{-1} : (0,1) \to \mathcal{X}$ defined for all $u \in (0,1)$ as
$$F^{-1}(u) = x$$
where $x \in \mathcal{X}$ is the unique possible value for which $F(x) = u$.

Because the cdf is strictly monotone increasing, there is a one-to-one correspondence between possible values $x \in \mathcal{X}$ and cdf values $u = F(x) \in (0,1)$.* Therefore, unlike the idf for a discrete random variable which is a pseudo inverse, the idf for a continuous random variable is a true inverse, as illustrated in Figure 7.2.1.
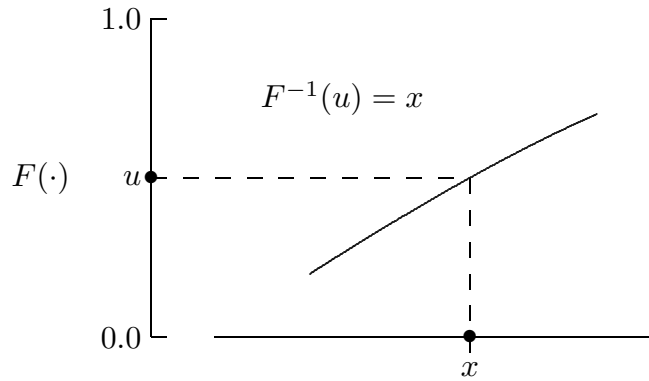
Figure 7.2.1. Generating continuous random variates.

Because the idf is a true inverse, it is sometimes possible to determine the idf in "closed form" by solving the equation $F(x) = u$ for x using common algebraic techniques. Generally this is much easier to do in the continuous case than in the discrete case because no inequality manipulations are required.

Examples

Example 7.2.1   If X is *Uniform*(a,b) then $F(x) = (x-a)/(b-a)$ for $a < x < b$. Solving the equation $u = F(x)$ for x yields the idf
$$x = F^{-1}(u) = a + (b-a)u \qquad 0 < u < 1.$$

Example 7.2.2   If X is *Exponential*(μ) then $F(x) = 1 - \exp(-x/\mu)$ for $x > 0$. Solving the equation $u = F(x)$ for x yields the idf
$$x = F^{-1}(u) = -\mu \ln(1-u) \qquad 0 < u < 1.$$

---

* The condition $f(x) > 0$ for all $x \in \mathcal{X}$ is sufficient to guarantee that $F(\cdot)$ is strictly monotone increasing — see Definition 7.1.3.

Example 7.2.3   If X is a continuous random variable with possible values $0 < x < b$ and pdf $f(x) = 2x/b^2$ then the cdf is $F(x) = (x/b)^2$. The equation $u = F(x) = (x/b)^2$ can be solved for x to establish that the idf of X is $F^{-1}(u) = b\sqrt{u}$ for $0 < u < 1$.

7.2.2   RANDOM VARIATE GENERATION BY INVERSION

As in the discrete case, the idf of a continuous random variable can be used to generate corresponding random variates by using inversion. The theoretical basis for this is the following theorem, which is known as the probability integral transformation. The proof is left as an exercise.

Theorem 7.2.1   If X is a continuous random variable with idf $F^{-1}(\cdot)$, the continuous random variable U is $Uniform(0, 1)$, and Z is the continuous random variable defined by $Z = F^{-1}(U)$, then Z and X are identically distributed.*

Algorithm 7.2.1   If X is a continuous random variable with idf $F^{-1}(\cdot)$, then a corresponding continuous random variate can be generated by inversion as follows.

```
u = Random();
return F⁻¹(u);
```

Example 7.2.4   This algorithm uses inversion and Example 7.2.1 to generate a *Uniform*(a, b) random variate.

```
u = Random();
return a + (b - a) * u;
```

This is the algorithm used to define the function $\mathtt{Uniform}(a, b)$ (see Definition 2.3.3).

Example 7.2.5   This algorithm uses inversion and Example 7.2.2 to generate an *Exponential*(μ) random variate.

```
u = Random();
return −μ * log(1 - u);
```

This is the algorithm used to define the function $\mathtt{Exponential}(\mu)$ (see Definition 3.1.1).

Note that the random variable U is $Uniform(0, 1)$ if and only if the random variable $1 - U$ is also $Uniform(0, 1)$. Therefore, it is also valid to generate an *Exponential*(μ) random variate as follows, thereby saving the essentially negligible cost of a subtraction

```
u = Random();
return −μ * log(u);
```

Because this algorithm has reverse monotonicity (e.g., large values of u correspond to small values of x and conversely), for esthetic reasons we prefer the algorithm in Example 7.2.5.

---

* Compare with Theorem 6.2.2.

The random variate generation algorithms in Examples 7.2.4 and 7.2.5 are essentially ideal. That is, both algorithms are portable, exact, robust, efficient, clear, synchronized, and monotone.

Generally it is not possible to solve for a continuous random variable idf explicitly by algebraic techniques. There are, however, two other options that may be available.

- Use a function that accurately *approximates* $F^{-1}(\cdot)$.

- Determine the idf by solving the equation $u = F(x)$ *numerically*.

We will see examples of both approaches in this section.

Approximate Inversion

If Z is a *Normal*$(0, 1)$ random variable then the cdf is the special function $\Phi(\cdot)$ and the idf $\Phi^{-1}(\cdot)$ cannot be evaluated in closed form. However, as demonstrated by Odeh and Evans (1974), the idf can be *approximated* with reasonable efficiency and essentially negligible error as the ratio of two fourth-degree polynomials. That is, for any $u \in (0, 1)$, a *Normal*$(0, 1)$ idf approximation is $\Phi^{-1}(u) \cong \Phi_a^{-1}(u)$ where

$$\Phi_a^{-1}(u) = \begin{cases} -t + p(t)/q(t) & 0.0 < u < 0.5 \\ t - p(t)/q(t) & 0.5 \leq u < 1.0, \end{cases}$$

and

$$t = \begin{cases} \sqrt{-2\ln(u)} & 0.0 < u < 0.5 \\ \sqrt{-2\ln(1-u)} & 0.5 \leq u < 1.0, \end{cases}$$

and

$$p(t) = a_0 + a_1 t + \cdots + a_4 t^4$$
$$q(t) = b_0 + b_1 t + \cdots + b_4 t^4.$$

If the ten polynomial coefficients are chosen carefully, then this approximation is accurate with an absolute error less than 10^{-9} for all $0.0 < u < 1.0$ (see Exercise 7.2.8).*

Example 7.2.6 Given that the $\Phi_a^{-1}(u)$ approximation to $\Phi^{-1}(u)$ is essentially exact and can be evaluated with reasonable efficiency, inversion can be used to generate *Normal*$(0, 1)$ random variates as illustrated.

```
u = Random();
return Φₐ⁻¹(u);
```

This algorithm is portable, essentially exact, robust, reasonably efficient, synchronized, and monotone, at the expense of clarity.

---

* The values of the ten constants are listed in the the file `rvgs.c`. The numerical analysis details of how these coefficients were originally determined are beyond the scope of this book.

Alternative Methods

For historical perspective we mention two other $Normal(0, 1)$ random variate generation algorithms, although we prefer the inversion algorithm in Example 7.2.6.

- One algorithm is based on the fact that if U_1, U_2, ..., U_{12} is an *iid* sequence of $Uniform(0, 1)$ random variables, then the random variable

$$Z = U_1 + U_2 + \cdots + U_{12} - 6$$

is approximately $Normal(0, 1)$. The mean of this random variable is exactly 0.0, the standard deviation is exactly 1.0, and the pdf is symmetric about 0. The set of possible values is $-6.0 < z < 6.0$, however, and so it is clear that some approximation is involved. The justification for this algorithm is provided by the central limit theorem — see Section 8.1. This algorithm is portable, robust, relatively efficient, and clear; it is not exact, synchronized, or monotone. We do not recommended this algorithm since it is not exact.

- The other algorithm, due to Box and Muller (1958), is based on the fact that if U_1, U_2 are independent $Uniform(0, 1)$ random variables then

$$Z_1 = \sqrt{-2\ln(U_1)}\,\cos(2\pi U_2) \qquad \text{and} \qquad Z_2 = \sqrt{-2\ln(U_1)}\,\sin(2\pi U_2)$$

will be independent $Normal(0, 1)$ random variables. If only one normal variate is needed, the second normal variate must be stored for a subsequent call or discarded. The algorithm is portable, exact, robust, and relatively efficient. It is not clear (unless the underlying theory is understood), it is not monotone, and it is only synchronized in pair-wise fashion. The polar method (Marsaglia and Bray, 1964) improves the computational efficiency.

Normal Random Variates

Consistent with Definitions 7.1.7 and 7.1.8, random variates corresponding to $Normal(\mu, \sigma)$ and $Lognormal(a, b)$ random variables can be generated by using a $Normal(0, 1)$ generator. This illustrates the importance of having a good $Normal(0, 1)$ random variate generator and helps explain why a significant amount of past research has been devoted to the development of such generators.

Example 7.2.7   This algorithm generates a $Normal(\mu, \sigma)$ random variate.

```
z = Normal(0.0, 1.0);
return  μ + σ * z;                              /* see definition 7.1.7 */
```

Lognormal Random Variates

Example 7.2.8   This algorithm generates a $Lognormal(a, b)$ random variate.

```
z = Normal(0.0, 1.0);
return exp(a + b * z);                          /* see definition 7.1.8 */
```

The algorithm in Example 7.2.6 generates a *Normal*(0, 1) random variate by inversion. Provided this algorithm is used to generate the *Normal*(0, 1) random variate, the algorithms in Examples 7.2.7 and 7.2.8 use inversion also and, therefore, are essentially ideal.

Numerical Inversion

Numerical inversion provides another way to generate continuous random variates when $F(x)$ can't be inverted algebraically (i.e., when $F^{-1}(u)$ can't be expressed in closed-form). That is, the equation $u = F(x)$ can be solved for x iteratively. Several iterative algorithms can be used. Of these, *Newton's method* provides a good compromise between rate of convergence and robustness. Given $u \in (0, 1)$, to derive Newton's method for inverting a cdf, let t be numerically close to the value of x for which $u = F(x)$. If $F(\cdot)$ is expanded in a Taylor's series about the point t, then

$$F(x) = F(t) + F'(t)(x - t) + \frac{1}{2!}F''(t)(x - t)^2 + \cdots$$

where $F'(t)$ and $F''(t)$ are the first and second derivative of the cdf evaluated at t. Because the first derivative of the cdf is the pdf, $F'(t) = f(t)$. Moreover, if $|x - t|$ is sufficiently small then the $(x - t)^2$ and higher-order terms can be ignored so that

$$u = F(x) \cong F(t) + f(t)(x - t).$$

Solving the equation $u \cong F(t) + f(t)(x - t)$ for x yields

$$x \cong t + \frac{u - F(t)}{f(t)}$$

With initial guess t_0, this last equation defines an *iterative* numerical algorithm of the form

$$t_{i+1} = t_i + \frac{u - F(t_i)}{f(t_i)} \qquad i = 0, 1, 2, \ldots$$

that solves the equation $u = F(x)$ for x in the sense that $t_i \to x$ as $i \to \infty$, as illustrated in Figure 7.2.2.
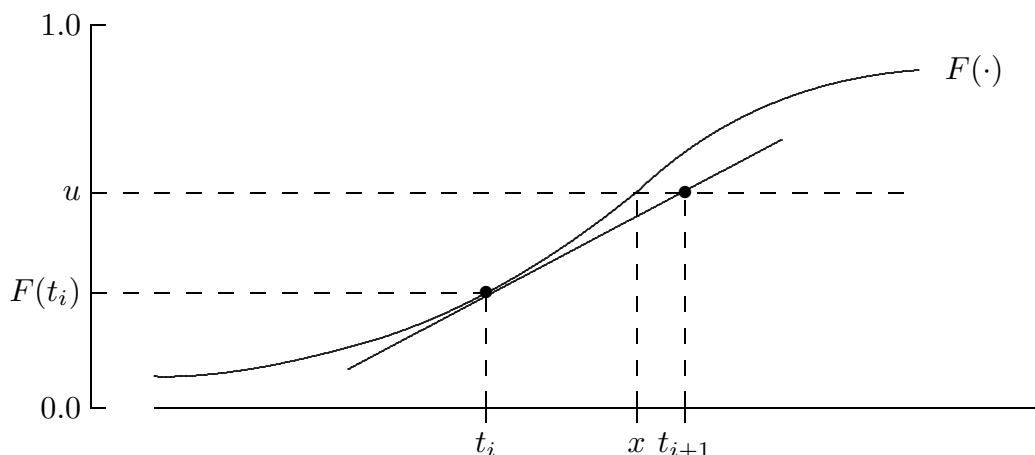

Figure 7.2.2. Numerical inversion of the cdf.

There are two remaining issues relative to Newton's method: the choice of an initial value t_0 and the test for convergence.

- Generally, as in the discrete case, the best choice for the initial value is the mode. For most of continuous random variable models considered in this book, $t_0 = \mu$ is an essentially equivalent choice.

- Generally, given a (user supplied, tiny) convergence parameter $\epsilon > 0$, the condition $|t_{i+1} - t_i| < \epsilon$ is a reasonable test for convergence.

With the initial value and convergence issues resolved in this way, Newton's method for numerical inversion is summarized by the following algorithm.

Algorithm 7.2.2   Given $u \in (0, 1)$, the pdf $f(\cdot)$, the cdf $F(\cdot)$, and a convergence parameter $\epsilon > 0$, this algorithm will solve for $x = F^{-1}(u)$

```
x = μ;                                          /* μ is E[X] */
do {
   t = x;
   x = t + (u - F(t)) / f(t);
} while (|x - t| > ε);
return x;                                       /* x is F⁻¹(u) */
```

If Algorithm 7.2.2 is used to compute the idf for an inherently positive random variable and if u is small, then a negative value of x may occur early in the iterative process. A negative value of x will cause t to be negative on the next iteration so that $F(t)$ and $f(t)$ will then be undefined. The following modification of Algorithm 7.2.2 can be used to avoid this problem.

```
x = μ;                                          /* μ is E[X] */
do {
   t = x;
   x = t + (u - F(t)) / f(t);
   if (x <= 0.0)
      x = 0.5 * t;
} while (|x - t| > ε);
return x;                                       /* x is F⁻¹(u) */
```

Algorithms 7.2.1 and 7.2.2 together provide a general-purpose inversion approach to continuous random variate generation. For example, the $Erlang(n, b)$ idf capability provided in the library **rvms**, which is based on Algorithm 7.2.2, can be used with Algorithm 7.2.1 as

```
u = Random();
return idfErlang(n, b, u);
```

7.2.3   ALTERNATIVE RANDOM VARIATE GENERATION ALGORITHMS

Properties relating parametric distributions to one another can be used to develop variate generation algorithms for some common parametric distributions. We exploited these special properties in the last chapter to devise variate generation algorithms for discrete random variables (e.g., a $Binomial(n, p)$ random variable is the sum of n independent and identically distributed $Bernoulli(p)$ random variables). We consider the Erlang, chisquare, and student distributions here.

Erlang Random Variates

As an alternative to inversion, an $Erlang(n, b)$ random variate can be generated by summing n independent and identically distributed $Exponential(b)$ random variates (see Definition 7.1.9), as follows.

```
x = 0.0;
for (i = 0;  i < n;   i++)
   x += Exponential(b);
return x;
```

This algorithm is portable, exact, robust, and clear. Because it is $O(n)$, if n is large the algorithm is not efficient. For any n, it is not synchronized or monotone. To increase computational efficiency this algorithm can be modified as follows

```
t = 1.0;
for (i = 0; i < n; i++)
   t *= (1.0 - Random());
return −b * log(t);
```

If n is large, the time saving with this modification is potentially significant because only one `log()` evaluation is required rather than n of them. Since the product of n random numbers $\prod_{i=1}^{n} U_i$ has the same distribution as $\prod_{i=1}^{n}(1 - U_i)$, still more processing time (i.e., n subtractions) can be saved by replacing the statement in the `for` loop with `t *= Random();`

Both algorithms presented here are inherently $O(n)$, however, and are thus not efficient if n is large. If floating-point arithmetic could be done with infinite precision these two algorithms would produce identical output. See Exercise 7.2.2.

Chisquare Random Variates

If n is an even positive integer then an $Erlang(n/2, 2)$ random variable is equivalent to a $Chisquare(n)$ random variable. As an alternative characterization, if n is even or odd then X is a $Chisquare(n)$ random variable if and only if $X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$ where Z_1, Z_2, ..., Z_n is an *iid* sequence of $Normal(0, 1)$ random variables. This alternative characterization can be used with a $Normal(0, 1)$ random variate generator to generate a $Chisquare(n)$ random variate as follows

```
x = 0.0;
for (i = 0; i < n; i++) {
    z = Normal(0.0, 1.0);
    x += (z * z);
}
return x;
```

This algorithm is portable, exact, robust, and clear. Because it is $O(n)$, if n is large the algorithm is not efficient. For any $n > 1$, it is not synchronized or monotone.

Student Random Variates

The random variable X is $Student(n)$ if and only if $X = Z/\sqrt{V/n}$ where Z is $Normal(0,1)$, V is $Chisquare(n)$ and Z, V are independent.* Therefore, the following algorithm uses a $Normal(0,1)$ random variate generator and a $Chisquare(n)$ random variate generator to generate a $Student(n)$ random variate as

```
z = Normal(0.0, 1.0);
v = Chisquare(n);
return z / sqrt(v / n);
```

This algorithm is exact, robust, portable, and clear. Because it is $O(n)$, if n is large the algorithm is not efficient. For any n, it is not synchronized or monotone.

7.2.4 TESTING FOR CORRECTNESS

Any continuous random variate generation algorithm should be tested for correctness. One quick method to test for *incorrectness* is to calculate the sample mean and sample standard deviation for a large sample size. Since $\bar{x} \to \mu$ and $s \to \sigma$ as $n \to \infty$, the sample mean and standard deviation should be "reasonably close" to the associated population mean and standard deviation. The meaning of "reasonably close" will be investigated further in the next two chapters, but for now we can say that if the sample mean and sample standard deviations are not closing in on the appropriate theoretical values, the algorithm is incorrect. The convergence described here is a necessary, but not a sufficient condition for the correctness of the algorithm. The reason that we can't conclude that the algorithm is correct is that two different distributions can have identical mean and variance (e.g., a $Uniform(-\sqrt{3}, \sqrt{3})$ and a $Normal(0,1)$ random variable). Two empirical tests for correctness are based on the pdf $f(x)$ and the cdf $F(x)$ which, unlike the mean and standard deviation, completely characterize the distribution.

---

* This distribution was discovered by W.S. Gossett when he was working for an Irish brewery. Since the brewery did not want its competitors to know that it was using statistical methods, they required that he use the pseudonym "Student" to publish his work. Most textbooks refer to this distribution as either the Student's-t or just the t distribution. This distribution is the basis for one of the cornerstone statistical tests — the two-sample t-test for comparing the means of two populations (Hogg, McKean, and Craig, 2005).

Comparing the histogram and $f(x)$

A natural way to test the correctness of a random variate generation algorithm at the computational level is to use the algorithm to generate a sample of n random variates and, as in Section 4.3, construct a k-bin continuous-data histogram with bin width δ. If $\mathcal{X}$ is the set of possible values, $\hat{f}(x)$ is the histogram density, and $f(x)$ is the pdf, then provided the algorithm is correct, for all $x \in \mathcal{X}$ we should find that $\hat{f}(x) \to f(x)$ as $n \to \infty$ and $\delta \to 0$. Of course, in practice, these limits cannot be achieved. Instead, we use a large but finite value of n and a small but non-zero value of δ.

Given that n is finite and δ is not zero, perfect agreement between $\hat{f}(x)$ and $f(x)$ will not be achieved. In the discrete case, this lack of perfect agreement is due to natural sampling variability only. In the continuous case, the *quantization error* associated with binning the sample is an additional factor that will contribute to the lack of agreement. That is, let $\mathcal{B} = [m - \delta/2,\ m + \delta/2)$ be a small histogram bin with width δ and midpoint m. Provided that $f(x)$ has a Taylor's series expansion at $x = m$ we can write

$$f(x) = f(m) + f'(m)(x - m) + \frac{1}{2!}f''(m)(x - m)^2 + \frac{1}{3!}f'''(m)(x - m)^3 + \cdots$$

from which it follows that

$$\Pr(X \in \mathcal{B}) = \int_{\mathcal{B}} f(x)\, dx = \cdots = f(m)\delta + \frac{1}{24}f''(m)\delta^3 + \cdots$$

If terms of order δ^4 or higher are ignored, then for all $x \in \mathcal{B}$ the histogram density is

$$\hat{f}(x) = \frac{1}{\delta}\Pr(X \in \mathcal{B}) \cong f(m) + \frac{1}{24}f''(m)\delta^2.$$

Therefore, (unless $f''(m) = 0$) depending on the sign of $f''(m)$ there is a positive or negative *bias* between the (experimental) density of the histogram bin and the (theoretical) pdf evaluated at the bin midpoint. This bias may be significant if the curvature of the pdf is large at the bin midpoint.

Example 7.2.9   If X is a continuous random variable with pdf

$$f(x) = \frac{2}{(x + 1)^3} \qquad x > 0$$

then the cdf X is

$$F(x) = \int_0^x f(t)\, dt = 1 - \frac{1}{(x + 1)^2} \qquad x > 0$$

and, by solving the equation $u = F(x)$ for x, we find that the idf is

$$F^{-1}(u) = \frac{1}{\sqrt{1 - u}} - 1 \qquad 0 < u < 1.$$

In this case, the pdf curvature is very large close to $x = 0$. Therefore, if inversion is used to generate random variates corresponding to X and if the correctness of the inversion algorithm is tested by constructing a histogram, unless δ is very small we expect that the histogram will not match the pdf well for the bins close to $x = 0$. In particular, if a histogram bin width of $\delta = 0.5$ is used then, in the limit as $n \to \infty$, for the first six histogram bins $\hat{f}(x)$ and $f(m)$ are (with $d.dddd$ precision)

m	:	0.25	0.75	1.25	1.75	2.25	2.75
$\hat{f}(x)$	:	1.1111	0.3889	0.1800	0.0978	0.0590	0.0383
$f(m)$	:	1.0240	0.3732	0.1756	0.0962	0.0583	0.0379

The "curvature bias" is evident in the first few bins. Indeed, for the first bin ($m = 0.25$) the curvature bias is

$$\frac{1}{24} f''(m)\delta^2 = 0.08192$$

which explains essentially all the difference between $\hat{f}(x)$ and $f(m)$.

Comparing the Empirical cdf and $F(x)$

Comparing the empirical cdf, introduced in Section 4.3, and the population cdf $F(x)$ is a second method for assessing the correctness of a variate generation algorithm. The quantization error associated with binning is eliminated. For large samples, the empirical cdf should converge to $F(x)$, i.e., $\hat{F}(x) \to F(x)$ as $n \to \infty$ if the variate generation algorithm is valid.

Library rvgs

All of the continuous random variate generators presented in this section are provided in the library `rvgs` as the seven functions:

- `double Chisquare(long` n`)` — returns a $Chisquare(n)$ random variate;

- `double Erlang(long` n`, double` b`)` — returns an $Erlang(n, b)$ random variate;

- `double Exponential(double` μ`)` — returns an $Exponential(\mu)$ random variate;

- `double Lognormal(double` a`, double` b`)` — returns a $Lognormal(a, b)$ random variate;

- `double Normal(double` μ`, double` σ`)` — returns a $Normal(\mu, \sigma)$ random variate;

- `double Student(long` n`)` — returns a $Student(n)$ random variate;

- `double Uniform(double` a`, double` b`)` — returns a $Uniform(a, b)$ random variate.

7.2.5   EXERCISES

Exercise 7.2.1   Prove Theorem 7.2.1.

Exercise 7.2.2   To what extent will the two $Erlang(n, b)$ random variate generation algorithms in Section 7.2.3 produce the same output? Why?

Exercise 7.2.3 What are the largest and smallest possible values that can be returned by the function `Normal(0, 1)`?

Exercise 7.2.4 A continuous random variable X is $Weibull(a, b)$ if the real-valued parameters a, b are positive, the possible values of X are $x > 0$, and the cdf is

$$F(x) = 1 - \exp\left(-(bx)^a\right) \qquad x > 0.$$

What is the pdf and idf?

Exercise 7.2.5 The continuous random variable is $Logistic(a, b)$ if the real-valued parameter a is unconstrained, the real-valued parameter b is positive, the possible values of X are $-\infty < x < \infty$, and the idf is

$$F^{-1}(u) = a - b \ln\left(\frac{1-u}{u}\right) \qquad 0 < u < 1.$$

What is the pdf and cdf?

Exercise 7.2.6ª A continuous random variable X is $Pareto(a, b)$ if the real-valued parameters a, b are positive, the possible values of X are $x > a$, and the pdf is

$$f(x) = ba^b/x^{b+1} \qquad x > a.$$

(*a*) Determine the mean and standard deviation of X, paying careful attention to any restrictions that must be imposed on a and b. (*b*) Construct a function that uses inversion to generate values of X. (*c*) Present convincing numerical evidence that this random variate generator is correct in the case $a = 1.0$, $b = 2.0$. (*d*) Comment.

Exercise 7.2.7ª (*a*) Given a circle of radius r centered at the origin of a conventional (x, y) coordinate system, construct an algorithm that will generate points *uniformly*, at random, interior to the circle. This algorithm cannot use more than two calls to `Random` per (x, y) point generated. (*b*) Explain clearly the theory behind the correctness of your algorithm. (Guessing this algorithm, even if correct, is not enough.)

Exercise 7.2.8 (*a*) If $\Phi_a^{-1}(\cdot)$ is the $Normal(0, 1)$ idf approximation as implemented in the library `rvgs` by the function `Normal` and $\Phi(\cdot)$ is the $Normal(0, 1)$ cdf as implemented in the library `rvms` by the function `cdfNormal`, use Monte Carlo simulation to *estimate* the smallest value of $\epsilon > 0$ such that $|\Phi(\Phi_a^{-1}(u)) - u| < \epsilon$ for all $u \in (0, 1)$. (Use the libraries `rngs`, `rvgs` and `rvms` directly; do not cut-and-paste from `rvgs.c` and `rvms.c` to build the Monte Carlo simulation program.) (*b*) Comment on the value of this process as a cdf/idf consistency check.

Exercise 7.2.9 (*a*) Give the *theoretical* largest and smallest standard normal random variates that can be generated by the Box–Muller method using a random number generator with $m = 2^{31} - 1$. (*b*) Compute the *actual* largest and smallest standard normal random variates that can be generated when $a = 48\,271$, $m = 2^{31} - 1$, and the random numbers U_1 and U_2 are generated by consecutive calls to `Random` using a single stream.

The purpose of this section is to demonstrate several continuous random variable applications, using the capabilities provided by the continuous random variate generators in the library `rvgs` and the pdf, cdf, and idf functions in the library `rvms`. We begin by developing an after-the-fact justification for some of the assumptions used in the discrete-event simulation models developed earlier in this book.

7.3.1   ARRIVAL PROCESS MODELS

Recall that the usual convention is to model *interarrival* times as a sequence of positive random variables $R_1, R_2, R_3, \ldots$ and then construct the corresponding *arrival* times as a sequence of random variables $A_1, A_2, A_3, \ldots$ defined by

$$A_0 = 0$$
$$A_i = A_{i-1} + R_i \qquad i = 1, 2, \ldots$$

By induction, for all i (with $A_0 = 0$)

$$A_i = R_1 + R_2 + \cdots + R_i \qquad i = 1, 2, \ldots$$

The arrival times are ordered so that $0 = A_0 < A_1 < A_2 < A_3 < \cdots$ as illustrated below.
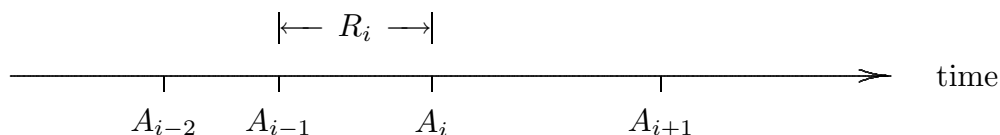
Example 7.3.1   This approach is used in programs `ssq2` and `ssq3` to generate job arrivals with the additional assumption that the random variables $R_1, R_2, R_3, \ldots$ are *Exponential*$(1/\lambda)$. In both programs, the arrival rate is equal to $\lambda = 0.5$ jobs per unit time. Similarly, this approach and an *Exponential*$(1/\lambda)$ interdemand random variable model is used in programs `sis3` and `sis4` to generate demand instances. The demand rate corresponds to an average of $\lambda = 30.0$ *actual* demands per time interval in program `sis3` and $\lambda = 120.0$ *potential* demands per time interval in program `sis4`.

Definition 7.3.1   If R_1, R_2, R_3, ... is an *iid* sequence of random positive interarrival times with common mean $1/\lambda > 0$, then the corresponding random sequence of arrival times A_1, A_2, A_3, ... is a *stationary arrival process* with *rate* λ.*

The "units" of the mean interarrival time $1/\lambda$ are time per arrival. Thus the arrival rate λ has reciprocal units of arrivals per unit time. For example, if the average interarrival *time* is 0.1 minutes, then the arrival *rate* is $\lambda = 10.0$ arrivals per minute. Although it is traditional to specify stationary arrival processes in terms of λ, it is usually more convenient for discrete-event simulation software to use $1/\lambda$ for the purpose of generating the arrival process, as is done in programs `ssq2`, `ssq3`, `sis3`, and `sis4`.

---

* A stationary arrival process is also known as a *renewal* process, which models the sequence of *event* times $A_1, A_2, A_3, \ldots$ (in our case, events are arrivals). A stationary arrival process is also called a *homogeneous* arrival process.

If the arrival rate λ *varies* with time, then the arrival process is *nonstationary*. Truly stationary arrival process are a convenient fiction.* Despite that, stationary arrival processes: (i) are important theoretically, (ii) sometimes provide a satisfactory approximation over a short time interval, and (iii) must be understood before attempting to study nonstationary arrival processes. Nonstationary arrival processes are considered in Section 7.5.

Stationary Poisson Arrival Process

Although the interarrival times R_1, R_2, R_3, ... can be any type of positive random variable, in the absence of information to the contrary both theory and practice support the hypothesis that it is usually most appropriate to assume that the interarrival times are *Exponential*$(1/\lambda)$, as is done in programs `ssq2`, `ssq3`, `sis3`, and `sis4`,

Definition 7.3.2   If R_1, R_2, R_3, ... is an *iid* sequence of *Exponential*$(1/\lambda)$ interarrival times, then the corresponding sequence A_1, A_2, A_3, ... of arrival times is a stationary *Poisson* arrival process with rate λ. Equivalently, for $i = 1, 2, 3, \ldots$ the arrival time A_i is an *Erlang*$(i, 1/\lambda)$ random variable.

Algorithm 7.3.1   Given $\lambda > 0$ and $t > 0$, this algorithm generates a realization of a stationary Poisson arrival process with arrival rate λ over the time interval $(0, t)$

```
a₀ = 0.0;                                          /* a convention */
n = 0;
while (aₙ < t) {
    aₙ₊₁ = aₙ + Exponential(1 / λ);
    n++;
}
return a₁, a₂, a₃, ..., aₙ₋₁;
```

Random Arrivals

We now turn to several fundamental theoretical results that demonstrate the interrelation between *Uniform*, *Exponential*, and *Poisson* random variables. These results will help explain why arrival processes are commonly assumed to be stationary Poisson processes. In the discussion that follows: (i) $t > 0$ defines a fixed time interval $(0, t)$, (ii) n represents the number of arrivals in the interval $(0, t)$, and (iii) $r > 0$ is the length of a small subinterval located *at random* interior to $(0, t)$.

Correspondingly, $\lambda = n/t$ is the arrival rate, $p = r/t$ is the probability that a particular arrival will be in the subinterval, and $np = nr/t = \lambda r$ is the expected number of arrivals in the subinterval. This notation and the discussion to follow is consistent with the stochastic experiment illustrated in Example 4.3.6.

---

 * Consider, for example, the rate at which jobs arrive to a fileserver or the rate at which customers arrive to a fastfood restaurant. In both cases, the arrival rate will vary dramatically during a typical 24-hour day of operation.

Theorem 7.3.1 Let A_1, A_2, …, A_n be *iid Uniform*$(0, t)$ random variables and let the discrete random variable X be the number of A_i that fall in a fixed subinterval of length $r = pt$ interior to $(0, t)$. Then X is a *Binomial*(n, p) random variable.*

Proof Each A_i is in the subinterval of length r with probability $p = r/t$. For each i define

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ is in the subinterval} \\ 0 & \text{otherwise.} \end{cases}$$

Because X_1, X_2, …, X_n is an *iid* sequence of *Bernoulli*(p) random variables and the number of A_i that fall in the subinterval is $X = X_1 + X_2 + \cdots + X_n$, it follows that X is a *Binomial*(n, p) random variable.

Random Arrivals Produce Poisson Counts

Because $p = \lambda r/n$, X is a *Binomial*$(n, \lambda r/n)$ random variable. As discussed in Chapter 6, if n is large and $\lambda r/n = r/t$ is small, then X will be indistinguishable from a *Poisson*(λr) random variable. Therefore, the previous theorem can be restated as follows.

Theorem 7.3.2 Let A_1, A_2, …, A_n be *iid Uniform*$(0, t)$ random variables and let the discrete random variable X be the number of A_i that fall in a fixed subinterval of length r interior to $(0, t)$. If n is large and r/t is small then X is indistinguishable from a *Poisson*(λr) random variable with $\lambda = n/t$.

Example 7.3.2 As in Example 4.2.2, suppose $n = 2000$ *Uniform*$(0, t)$ random variates are generated and tallied into a continuous-data histogram with 1000 bins of size $r = t/1000$. If the resulting 1000 bin counts are then tallied into a discrete-data histogram, then in this case $\lambda r = (n/t)(t/1000) = 2$ and, consistent with Theorem 7.3.2, the resulting discrete-data histogram relative frequencies will agree with the pdf of a *Poisson*(2) random variable.

To paraphrase Theorem 7.3.2, if many arrivals occur *at random* with a rate of λ then the number of arrivals X that will occur in an interval of length r is a *Poisson*(λr) random variable. Therefore, the probability of x arrivals in any interval of length r is

$$\Pr(X = x) = \frac{(\lambda r)^x \exp(-\lambda r)}{x!} \qquad x = 0, 1, 2, \ldots$$

In particular, $\Pr(X = 0) = \exp(-\lambda r)$ is the probability of *no* arrivals in an interval of length r. Correspondingly, the probability of *at least one* arrival in the interval is

$$\Pr(X > 0) = 1 - \Pr(X = 0) = 1 - \exp(-\lambda r).$$

For a fixed arrival rate λ, the probability of at least one arrival increases with increasing interval length r.

---

* As in Example 4.3.6, think of the A_i as *unsorted* arrival times.

Random Arrivals Produce Exponential Interarrivals

If R is the continuous random variable that represents the time between consecutive arrivals (the interarrival time), then the possible values of R are $r > 0$. What is the cdf of R?

Consider an arrival time A_i selected at random and a corresponding interval of length r, beginning at A_i, as illustrated below.
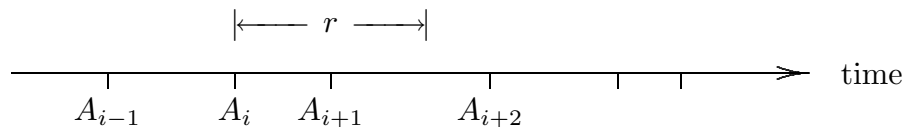
The random variable $R = A_{i+1} - A_i$ will be less than r, as illustrated, if and only if there is at least one arrival in this interval. Therefore, the cdf of R is

$$\Pr(R \le r) = \Pr(\text{at least one arrival in this interval}) = 1 - \exp(-\lambda r) \qquad r > 0.$$

From this cdf equation we see that R is an *Exponential*$(1/\lambda)$ random variable (see Definition 7.4.2). This proves the following important theorem.

Theorem 7.3.3 If arrivals occur at random (i.e., as a stationary Poisson process) with rate λ then the corresponding interarrival times form an *iid* sequence of *Exponential*$(1/\lambda)$ random variables.*

Consistent with the previous discussion, if all that is known about an arrival process is that arrivals occur at random with a constant rate λ then the function `GetArrival` in programs `ssq2` and `ssq3` is appropriate. Similarly, if all that is known about a demand process is that demand instances occur at random with a constant rate λ then the function `GetDemand` in programs `sis3` and `sis4` is appropriate.

Are there occasions when a stationary Poisson arrival process is an appropriate model? Over short time intervals, such a model may provide a good approximation. Also, a result given in Fishman (2001, page 464) states that the superposition of many general stationary arrival processes (each not necessarily Poisson) is a stationary Poisson arrival process.

The following example is based on the observation that if arrivals occur at random with rate $\lambda = 1$ (which is often referred to as a *unit* Poisson process) then the number of arrivals X in an interval of length μ will be a *Poisson*(μ) random variate. (The case $X = 4$ is illustrated below.)

---
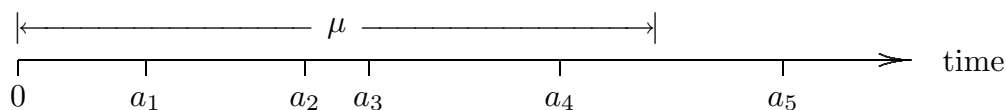
* Theorem 7.3.3 is the justification for Definition 7.3.2.

Example 7.3.3   An algorithm that generates a $Poisson(\mu)$ random variate is

```
a = 0.0;
x = 0;
while (a < μ) {
    a += Exponential(1.0);
    x++;
}
return x − 1;
```

Example 7.3.3 provides an explanation for the choice of the $Poisson(\mu)$ random variate generator presented in Example 6.2.13. The algorithm can be made a bit more efficient by replacing the calls to Exponential(1.0) with corresponding calls to Random. The details are left as an exercise.

Summary

Given a fixed time interval $(0, t)$, there are two ways of generating a realization of a stationary Poisson arrival process with rate λ over this time interval.

- Generate the number of arrivals as a $Poisson(\lambda t)$ random variate n. Then generate a $Uniform(0, t)$ random variate sample of size n and *sort* this sample to form the arrival process $0 < a_1 < a_2 < a_3 < \cdots < a_n$, as in Example 4.3.6.

- Use Algorithm 7.3.1 to generate $Exponential(1/\lambda t)$ interarrival times.

Although these two approaches are statistically equivalent, they are certainly *not* computationally equivalent. Because of the need to temporarily store the entire n-point sample and then sort it, there is no computational justification for the use of the first approach, particularly if n is large. Instead, the second approach, which is based on Theorem 7.3.3, is always preferred. Indeed, from a computational perspective, the second approach represents a triumph of theory over brute-force computing.

It is important to remember that the *mode* of the exponential distribution is 0, which means that a stationary Poisson arrival process will exhibit a clustering of arrivals. The top axis below shows the clustering associated with the arrival times of a stationary Poisson arrival process with $\lambda = 1$ and the bottom axis shows a stationary arrival process with $Erlang(4, 1/4)$ interarrival times. The top axis has many short interarrival times, which results in clusters of arrivals. The arrival pattern associated with a stationary Poisson arrival process is often referred to as "random" arrivals.
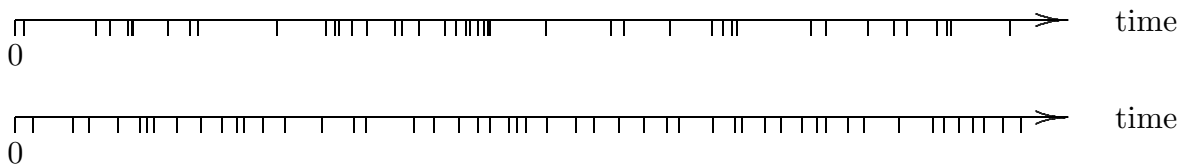
The stationary Poisson arrival process generalizes to (i) a stationary arrival process when the exponential time between arrivals is replaced by any continuous random variable with positive support, and (ii) a nonstationary Poisson process when the arrival rate λ varies with time.

This concludes our discussion of *arrival* processes. We now turn to a discussion of *service* processes.

7.3.2 SERVICE PROCESS MODELS

For a service node simulation model a default service process model is not well defined; there are only application-dependent guidelines.

- *Uniform*(a, b) service times are rarely ever an appropriate model. Service times seldom "cut off" at minimum and maximum values a and b. Service time models typically have "tails" on their pdf's.

- Because service times are inherently positive, they can *not* be *Normal*(μ, σ) unless this random variable is truncated to positive values only.

- Probability models such as the *Lognormal*(a, b) distribution generate inherently positive random variates, making them candidates for modeling service times.

- When service times consist of the sum of n *iid Exponential*(b) sub-task times, the *Erlang*(n, b) model is appropriate.

Program ssq4

Program `ssq4` is based on program `ssq3`, but with a more realistic *Erlang*$(5, 0.3)$ service time model. The corresponding service rate is 2/3. As in program `ssq3`, program `ssq4` uses *Exponential*(2) random variate interarrivals. The corresponding arrival rate is 1/2.

Example 7.3.4   The arrival rate is $\lambda = 0.5$ for both program `ssq3` and program `ssq4`. For both programs the service rate is $\nu = 2/3 \cong 0.667$. As illustrated in Figure 7.3.1, however, the distribution of service times for the two programs is very different.
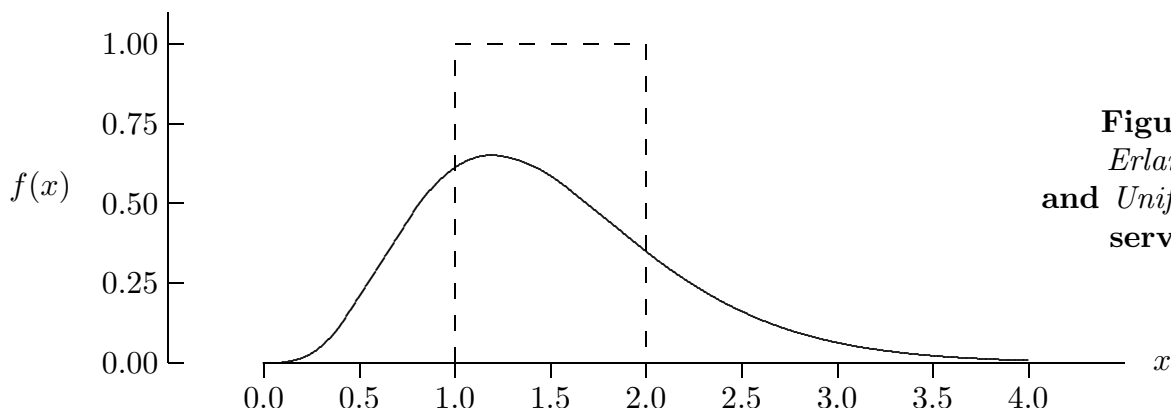

Figure 7.3.1.
Erlang$(5, 0.3)$
and *Uniform*$(1, 2)$
service time
models.

The solid line is the $Erlang(5, 0.3)$ service time pdf in `ssq4`; the dashed line represents the $Uniform(1, 2)$ pdf in `ssq3`. Both have a mean of 1.5. Because of this difference in the service time distributions, one would expect a difference in some (all?) of the steady-state service node statistics. As an exercise, you are asked to *first* conjecture how the steady-state statistics will differ for these two programs and then investigate the correctness of your conjecture via discrete-event simulation.*

Erlang Service Times

The $Erlang(n, b)$ service time model is appropriate when service processes can be naturally decomposed into a series (tandem) of independent "sub-processes" as illustrated in Figure 7.3.2.

**Figure 7.3.2.
Three
sub-process
service
time.**
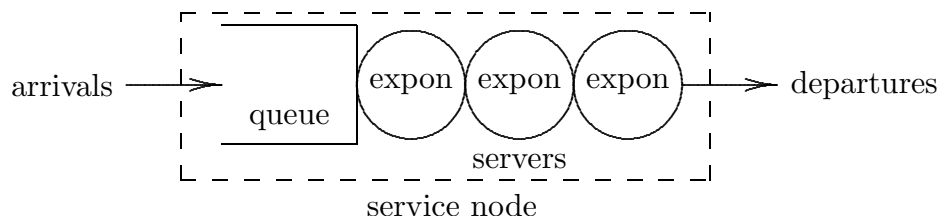
If there are n service sub-processes (the case $n = 3$ is illustrated) then the total service time will be the sum of the service times for each sub-process. If, in addition, the time to perform each service sub-process is independent of the other times then a random variate service time can be generated by generating a service time for each sub-process and then summing the sub-process service time variates. In the particular case that each of the n service sub-process is $Exponential(b)$, the total service time will be $Erlang(n, b)$. The service rate will be $1/nb$.

Truncation

Let X be a continuous random variable with the set of possible values $\mathcal{X}$ and cdf $F(x) = \Pr(X \le x)$. Consider an interval $(a, b) \subset \mathcal{X}$ and suppose we wish to restrict (truncate) the possible values of X to this interval. Truncation in this continuous-variable context is similar to, but simpler than, truncation in the discrete-variable context (see Section 6.3). By definition the probability of X being less than or equal to a is

$$\Pr(X \le a) = F(a).$$

Similarly, the probability of X being greater than or equal to b is

$$\Pr(X \ge b) = 1 - \Pr(X < b) = 1 - F(b).$$

In general then

$$\Pr(a < X < b) = \Pr(X < b) - \Pr(X \le a) = F(b) - F(a).$$

---

  * Which statistics will change, and in what way?

Relative to truncation, there are two cases to consider.

- If the left-truncation value a and the right-truncation value b are specified, then the cdf of X can be used to determine the left-tail, right-tail truncation probabilities as

$$\alpha = \Pr(X \le a) = F(a) \qquad \text{and} \qquad \beta = \Pr(X \ge b) = 1 - F(b)$$

  respectively.

- If instead α and β are specified then the idf of X can be used to determine the left-tail, right-tail truncation possible values as

$$a = F^{-1}(\alpha) \qquad \text{and} \qquad b = F^{-1}(1 - \beta)$$

  respectively.

In either case, for a continuous random variable the cdf transformation from possible values to probabilities or the idf transformation from probabilities to possible values is exact.

Example 7.3.5 As a continuation of Example 7.3.4, suppose we want to use a *Normal*(1.5, 2.0) random variable to model service times. Because service times cannot be negative, it is necessary to truncate the left tail of this random variable to non-negative service times. Moreover, to prevent extremely large service times suppose we choose to truncate the right tail to values less than 4.0. Therefore $a = 0.0$, $b = 4.0$, and the cdf capability in the library **rvms** can be used to determine the corresponding truncation probabilities α, β as

```
α = cdfNormal(1.5, 2.0, a);                    /* a is 0.0 */
β = 1.0 - cdfNormal(1.5, 2.0, b);              /* b is 4.0 */
```

The result is $\alpha = 0.2266$ and $\beta = 0.1056$. Note that the resulting *truncated Normal*(1.5. 1.0) random variable has a mean of 1.85, not 1.5, and a standard deviation of 1.07, not 2.0. The mean increases since more probability was lopped off of the left-hand tail than the right-hand tail. The standard deviation, on the other hand, decreases since the most extreme values have been eliminated.

Constrained Inversion

Once the left-tail and right-tail truncation probabilities α and β have been determined, either by specification or by computation, then the corresponding truncated random variate can be generated by using constrained inversion as

```
u = Uniform(α, 1.0 - β);
return F⁻¹(u);
```

Example 7.3.6 The idf capability in the library **rvms** can be used to generate the truncated *Normal*(1.5, 2.0) random variate in Example 7.3.5 as

```
α = 0.2266274;
β = 0.1056498;
u = Uniform(α, 1.0 - β);
return idfNormal(1.5, 2.0, u);
```

The geometry for the generation of a *Normal*(1.5, 2.0) random variate truncated on the left at $a = 0.0$ and on the right at $b = 4.0$ is shown in Figure 7.3.3. The random number in the figure is $u = 0.7090$, which corresponds to a truncated normal variate of $x = 2.601$.

Figure 7.3.3. Truncated normal constrained inversion.
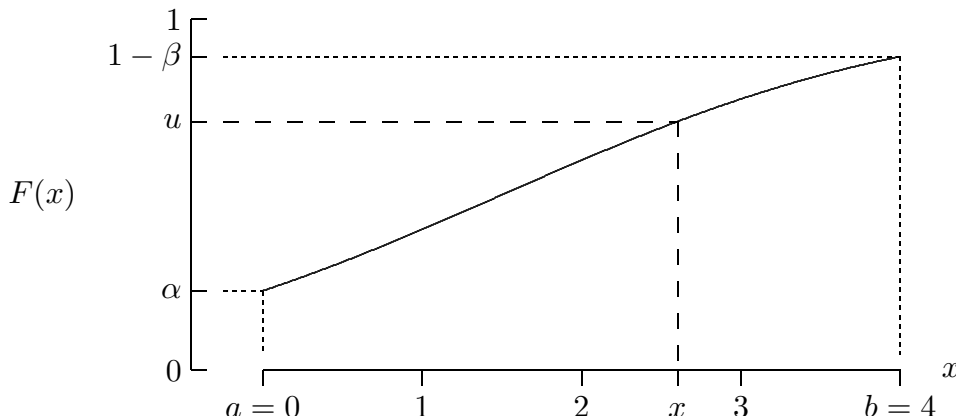
Triangular Random Variable

Although it is not one of the standard continuous random variate models summarized in the next section, the three-parameter *Triangular*(a, b, c) model is commonly used in those situations where all that is known is the range of possible values along with the most likely possible value (the mode). In this case, it is reasonable to assume that the pdf of the random variable has the shape illustrated in Figure 7.3.4.*

Figure 7.3.4. Triangular pdf.
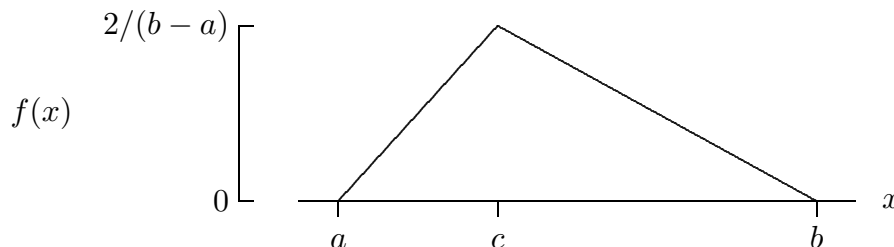
Formally, the continuous random variable X is *Triangular*(a, b, c) if and only if the real-valued parameters a, b, and c satisfy $a < c < b$, the possible values of X are $\mathcal{X} = \{x \mid a < x < b\}$, and the pdf of X is

$$f(x) = \begin{cases} \dfrac{2(x - a)}{(b - a)(c - a)} & a < x \leq c \\[2mm] \dfrac{2(b - x)}{(b - a)(b - c)} & c < x < b. \end{cases}$$

---

* See Exercise 7.3.9 for the special cases $c = a$ or $c = b$.

When modeling service times with a known finite range of possible values, the *Triangular*(a, b, c) random variable should be considered as a viable alternative to truncating one of the more traditional service time models like, for example, an *Erlang*(n, b) or *Lognormal*(a, b) random variable model. Another scenario where the triangular distribution is appropriate is the case when no data is available. One of the authors of this text was modeling the construction of the Space Station and, of course, no data existed for the various construction operations. The construction times were modeled by interviewing an expert at NASA who was able to make informed guesses about the shortest construction time a, the longest construction time b, and the most likely construction time c.

It follows from the definition of a *Triangular*(a, b, c) random variable that the mean and standard deviation are

$$\mu = \frac{a + b + c}{3} \qquad \text{and} \qquad \sigma = \frac{\sqrt{(a-b)^2 + (a-c)^2 + (b-c)^2}}{6}.$$

Moreover, by integrating the pdf, the cdf is

$$F(x) = \begin{cases} \dfrac{(x-a)^2}{(b-a)(c-a)} & a < x \leq c \\[2ex] 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & c < x < b, \end{cases}$$

and, by solving the equation $u = F(x)$ for x, the idf is

$$F^{-1}(u) = \begin{cases} a + \sqrt{(b-a)(c-a)u} & 0 < u \leq \dfrac{c-a}{b-a} \\[2ex] b - \sqrt{(b-a)(b-c)(1-u)} & \dfrac{c-a}{b-a} < u < 1. \end{cases}$$

All of these derivations are left as an exercise.

7.3.3 EXERCISES

Exercise 7.3.1 Modify the function `GetService` in program `ssq4` so that service times greater than 3.0 are not possible. Do this in two ways: (a) using acceptance-rejection; (b) using truncation by constrained inversion. (c) Comment.

Exercise 7.3.2 (a) Investigate the issue concerning steady-state statistics discussed in Example 7.3.4. Specifically, compare the steady-state statistics produced by programs `ssq3` and `ssq4`. (b) Which of these statistics are different, and why?

Exercise 7.3.3 Repeat Exercises 7.3.1 and 7.3.2 using instead the truncated random variable model from Example 7.3.5.

Exercise 7.3.4   (a) Prove that if Y is a *Normal*(μ, σ) random variable X truncated to the interval (a, b), then

$$E[Y] = \mu - \left(\frac{f(b) - f(a)}{F(b) - F(a)} \right) \sigma^2,$$

where $f(\cdot)$ and $F(\cdot)$ are the pdf and cdf of X (not Y). (b) How could you check that this result is correct?

Exercise 7.3.5   (a) Derive the equations for the mean and standard deviation of a *Triangular*(a, b, c) random variable. (b) Similarly, derive the equations for the cdf and idf.

Exercise 7.3.6[a]   (a) Use Monte Carlo simulation to demonstrate that the following algorithm is computationally more efficient than the algorithm in Example 7.3.3.

```
t = exp(μ);
x = 0;
while (t > 1.0) {
    t *= Random();
    x++;
}
return x − 1;
```

(b) Demonstrate also that the two algorithms produce identical random variates for identical streams of random numbers. (c) There is, however, a computational problem with the more efficient algorithm if μ is large. What is that problem?

Exercise 7.3.7   (a) Starting from the cdf equation for a *Triangular*(a, b, c) random variable, derive the corresponding idf equation. (b) Implement an algorithm that will generate a *Triangular*(a, b, c) random variate with just one call to `Random`. (c) In the specific case $(a, b, c) = (1.0, 7.0, 3.0)$ use Monte Carlo simulation to generate a continuous-data histogram that provides convincing evidence that your random variate generator is correct.

Exercise 7.3.8[a]   (a) As an alternative to using a truncated *Normal*$(1.5, 2.0)$ random variable in Example 7.3.5, if a *Normal*$(\mu, 2.0)$ random variable is used instead as the basis for truncation, use the capabilities in the library `rvms` to determine what value of the parameter μ is required to yield a service rate of $\nu = 2/3$. (b) Discuss your approach to solving for μ. (c) This choice of μ forces the truncated random variable to have a mean of 1.5. What are the corresponding values of α and β? (d) What is the standard deviation of the resulting truncated random variable?

Exercise 7.3.9   (a) Relative to the definition of a *Triangular*(a, b, c) random variable, verify that the parameter constraint $a < c < b$ can be relaxed to $a \leq c \leq b$, if done correctly. (b) Specifically, what are the pdf, cdf, and idf equations if $a = c < b$? (c) Same question if $a < b = c$.

The characteristic properties of the following seven continuous random variable models are summarized in this section: $Uniform(a, b)$, $Exponential(\mu)$, $Erlang(n, b)$, $Normal(\mu, \sigma)$, $Lognormal(a, b)$, $Chisquare(n)$, and $Student(n)$. For more details about these models, see Sections 7.1 and 7.2. For supporting software, see the random variable models library `rvms` in Appendix D and the random variate generators library `rvgs` in Appendix E.

7.4.1 UNIFORM

Definition 7.4.1 The continuous random variable X is $Uniform(a, b)$ if and only if

- the real-valued *location* (see Section 7.4.9) parameters a, b satisfy $a < b$

- the possible values of X are $\mathcal{X} = \{x \mid a < x < b\}$

- the pdf of X is

$$f(x) = \frac{1}{b - a} \qquad a < x < b,$$

as illustrated in Figure 7.4.1
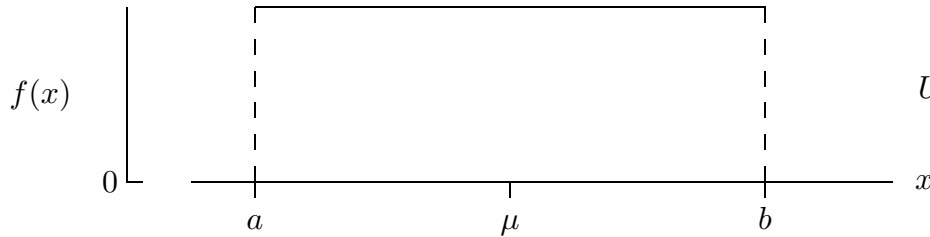
Figure 7.4.1.
Uniform(a, b) **pdf.**

- the cdf of X is

$$F(x) = \frac{x - a}{b - a} \qquad a < x < b$$

- the idf is

$$F^{-1}(u) = a + (b - a)u \qquad 0 < u < 1$$

- the mean of X is

$$\mu = \frac{a + b}{2}$$

- the standard deviation of X is

$$\sigma = \frac{b - a}{\sqrt{12}}.$$

A $Uniform(a, b)$ random variable is used to model situations where a continuous random variable is restricted to a subrange (a, b) of the real line and all values in this subrange are equally likely. Thus a $Uniform(a, b)$ random variable is the continuous analog of the discrete $Equilikely(a, b)$ random variable. A typical $Uniform(a, b)$ application will involve a model derived from a statement like "... the service time is chosen *at random* from the time interval (a, b)." The special case $a = 0$ and $b = 1$ is particularly important because virtually all random number generators are designed to generate a sequence of $Uniform(0, 1)$ random variates.

7.4.2   EXPONENTIAL

Definition 7.4.2   The continuous random variable X is *Exponential*(μ) if and only if

- the real-valued *scale* parameter μ satisfies $\mu > 0$
- the possible values of X are $\mathcal{X} = \{x \mid x > 0\}$
- the pdf of X is

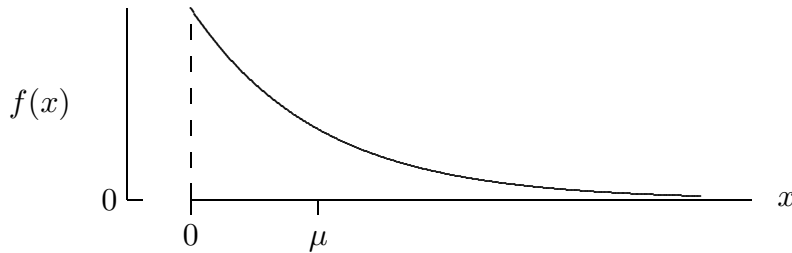$$f(x) = \frac{1}{\mu} \exp(-x/\mu) \qquad x > 0,$$

as illustrated in Figure 7.4.2

Figure 7.4.2.
Exponential(μ)
pdf.

- the cdf of X is

$$F(x) = 1 - \exp(-x/\mu) \qquad x > 0$$

- the idf is

$$F^{-1}(u) = -\mu \ln(1 - u) \qquad 0 < u < 1$$

- the mean of X is μ
- the standard deviation of X is $\sigma = \mu$.

Exponential random variables are frequently used to model the interarrival distribution associated with arrival processes. *Exponential* random variables are also used to model the service time distribution for individual service nodes in a network of queues and component lifetimes in system reliability analysis. If X is *Exponential*(μ) then X has the important *memoryless* property — for any $x' > 0$

$$\Pr\left(X \geq x + x' \mid X \geq x'\right) = \Pr(X \geq x) = \exp(-x/\mu) \qquad x > 0$$

independent of x'. If a light bulb, for example, has an exponential lifetime, then a used bulb that has been has successfully functioning for x' hours has an identical remaining lifetime pdf as that of a brand new bulb. Graphically, any right-hand tail of an exponential pdf that is rescaled so that it integrates to one is identical (except for location) to the original pdf. An *Exponential*(μ) random variable is the continuous analog of the discrete *Geometric*(p) random variable. Just as the normal distribution plays an important role in classical statistics because of the central limit theorem, the exponential distribution plays an important role in stochastic processes (particularly reliability and queueing theory) since it is the only continuous distribution with the memoryless property. The only discrete distribution with the memoryless property is the geometric distribution.

7.4.3   ERLANG

Definition 7.4.3   The continuous random variable X is $Erlang(n, b)$ if and only if

- the *shape* parameter n is a positive integer
- the real-valued *scale* parameter b satisfies $b > 0$
- the possible values of X are $\mathcal{X} = \{x \mid x > 0\}$
- the pdf of X is

$$f(x) = \frac{1}{b\,(n-1)!}\,(x/b)^{n-1}\exp(-x/b) \qquad x > 0,$$

as illustrated in Figure 7.4.3 for $n = 3$ and $b = 1$
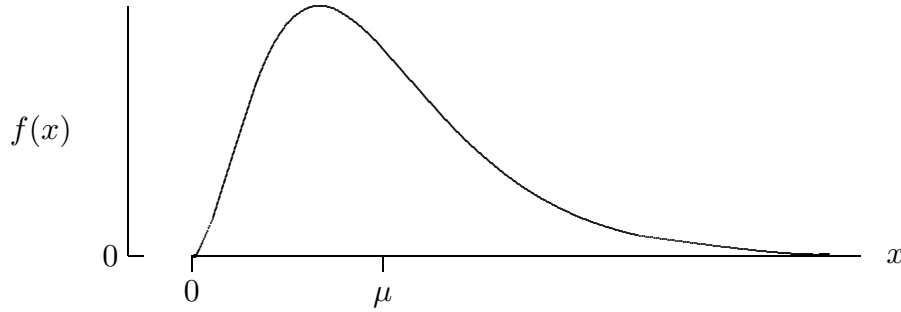
Figure 7.4.3.
$Erlang(n, b)$ **pdf.**

- the cdf of X is an incomplete gamma function (see Appendix D)

$$F(x) = P(n, x/b) \qquad x > 0$$

- except for special cases, the idf of X must be determined by numerical inversion
- the mean of X is

$$\mu = nb$$

- the standard deviation of X is

$$\sigma = \sqrt{n}\,b.$$

Erlang random variables are frequently used to model random service times, particularly when the service is defined by a series of independent sub-processes. That is consistent with the fact that a continuous random variable X is $Erlang(n, b)$ if and only if

$$X = X_1 + X_2 + \cdots + X_n,$$

where X_1, X_2, ..., X_n is an *iid* sequence of *Exponential*(b) random variables. An *Erlang*$(1, b)$ random variable and an *Exponential*(b) random variable are equivalent. An *Erlang*(n, b) random variable is the continuous analog of the discrete *Pascal*(n, p) random variable.

7.4.4   STANDARD NORMAL

Definition 7.4.4   The continuous random variable Z is $Normal(0, 1)$ (*standard normal*) if and only if

- the possible values of Z are $\mathcal{Z} = \{z \mid -\infty < z < \infty\}$
- the pdf of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) \qquad -\infty < z < \infty,$$

as illustrated in Figure 7.4.4

Figure 7.4.4.
Normal(0, 1)
pdf.
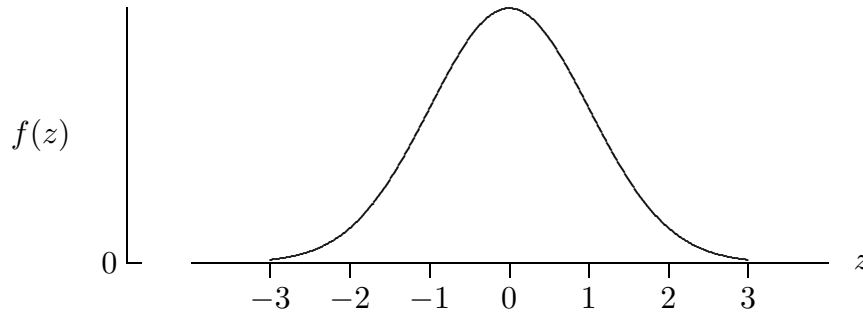
- the cdf of Z is the special function $\Phi(\cdot)$

$$F(z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)\, dt \qquad -\infty < z < \infty$$

- the idf of X is
$$F^{-1}(u) = \Phi^{-1}(u) \qquad 0 < u < 1$$

where $\Phi^{-1}(\cdot)$ is defined by an algorithm (see Section 7.2) that computes the numerical inverse of the equation $u = \Phi(x)$

- the mean of Z is $\mu = 0$

- the standard deviation of Z is $\sigma = 1$.

A $Normal(0, 1)$ random variable is a special case of a $Normal(\mu, \sigma)$ random variable. The letter Z for a $Normal(0, 1)$ random variable, and the symbols ϕ and Φ for its pdf and cdf are used because of the centrality of the normal distribution in probability and statistics. The pdf has inflection points at ± 1. For both the $Normal(0, 1)$ and $Normal(\mu, \sigma)$ models, the areas under the pdf within 1, 2, and 3 standard deviations from the mean are approximately 0.68, 0.95, and 0.997, respectively.

7.4.5   NORMAL

Definition 7.4.5   The continuous random variable X is $Normal(\mu, \sigma)$ if and only if

- the real-valued *location* parameter μ can have any value
- the real-valued *scale* parameter σ satisfies $\sigma > 0$
- the possible values of X are $\mathcal{X} = \{x \mid -\infty < x < \infty\}$
- the pdf of X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-(x-\mu)^2/2\sigma^2\right) \qquad -\infty < x < \infty,$$

  as illustrated in Figure 7.4.5
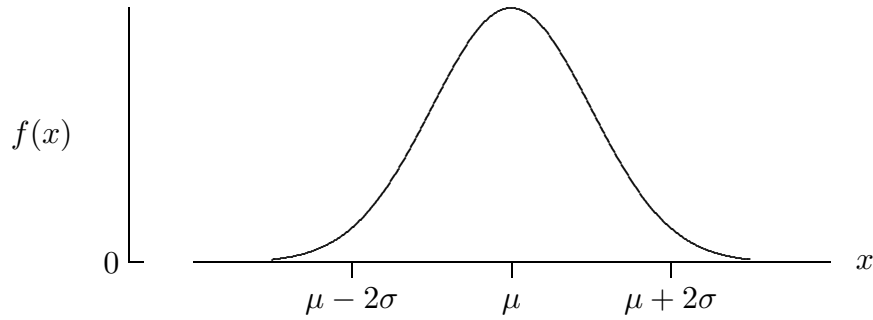
Figure 7.4.5. $Normal(\mu, \sigma)$ **pdf.**

- the cdf of X is defined by the $Normal(0, 1)$ cdf as

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad -\infty < x < \infty$$

- the idf of X is defined by the $Normal(0, 1)$ idf as

$$F^{-1}(u) = \mu + \sigma\,\Phi^{-1}(u) \qquad 0 < u < 1$$

- the mean of X is μ
- the standard deviation of X is σ.

The random variable X is $Normal(\mu, \sigma)$ if and only if

$$X = \mu + \sigma Z$$

where the random variable Z is $Normal(0, 1)$. Elementary statistics textbooks usually state this result as: if X is $Normal(\mu, \sigma)$ then $Z = (X - \mu)/\sigma$ is $Normal(0, 1)$. The primary importance of a $Normal(\mu, \sigma)$ random variable in simulation is its role as the *asymptotic* form of the sum of other random variables. That is, if X_1, X_2, ..., X_n is an *iid* sequence of random variables each with mean μ and standard deviation σ and if

$$X = X_1 + X_2 + \cdots + X_n$$

then the random variable X has a mean of $n\mu$ and a standard deviation of $\sqrt{n}\,\sigma$. In addition, X is asymptotically $Normal(n\mu, \sqrt{n}\,\sigma)$ as $n \to \infty$. This result is one form of the central limit theorem. The pdf of the $Normal(\mu, \sigma)$ has inflection points at $\mu \pm \sigma$.

7.4.6   LOGNORMAL

Definition 7.4.6   The continuous random variable X is *Lognormal*(a, b) if and only if

- the real-valued parameter a can have any value [$\exp(a)$, the median, is a scale parameter]

- the real-valued *shape* parameter b satisfies $b > 0$

- the possible values of X are $\mathcal{X} = \{x \mid x > 0\}$

- the pdf of X is

$$f(x) = \frac{1}{bx\sqrt{2\pi}} \exp\left(-(\ln(x) - a)^2/2b^2\right) \qquad x > 0,$$

    as illustrated in Figure 7.4.6 for $(a, b) = (-0.5, 1)$

Figure 7.4.6.
Lognormal(a, b)
pdf.
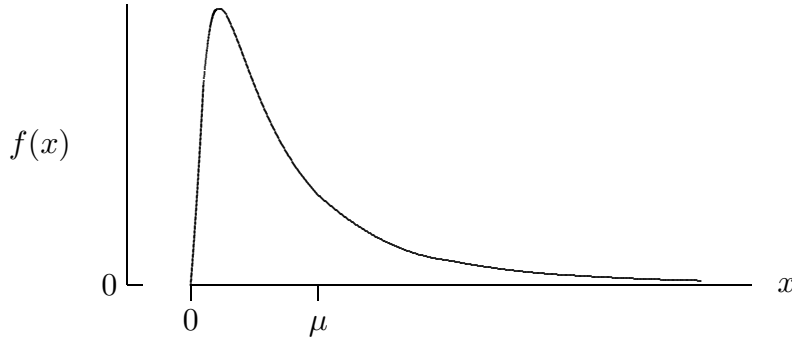
- the cdf of X is defined by the *Normal*$(0, 1)$ cdf as

$$F(x) = \Phi\left(\frac{\ln(x) - a}{b}\right) \qquad x > 0$$

- the idf of X is defined by the *Normal*$(0, 1)$ idf as

$$F^{-1}(u) = \exp\left(a + b\,\Phi^{-1}(u)\right) \qquad 0 < u < 1$$

- the mean of X is

$$\mu = \exp\left(a + b^2/2\right)$$

- the standard deviation of X is

$$\sigma = \exp\left(a + b^2/2\right)\sqrt{\exp(b^2) - 1}.$$

The random variable X is *Lognormal*(a, b) if and only if

$$X = \exp(a + bZ)$$

where the random variable Z is *Normal*$(0, 1)$. The *Lognormal* distribution is sometimes used as an alternate to the *Erlang* distribution as a model of, for example, a service time.

7.4.7   CHISQUARE

Definition 7.4.7   The continuous random variable X is $Chisquare(n)$ if and only if

- the *shape* parameter n is a positive integer, known as the "degrees of freedom"
- the possible values of X are $\mathcal{X} = \{x \mid x > 0\}$
- the pdf of V is

$$f(x) = \frac{1}{2\Gamma(n/2)}(x/2)^{n/2-1}\exp(-x/2) \qquad x > 0,$$

as illustrated in Figure 7.4.7 for $n = 5$
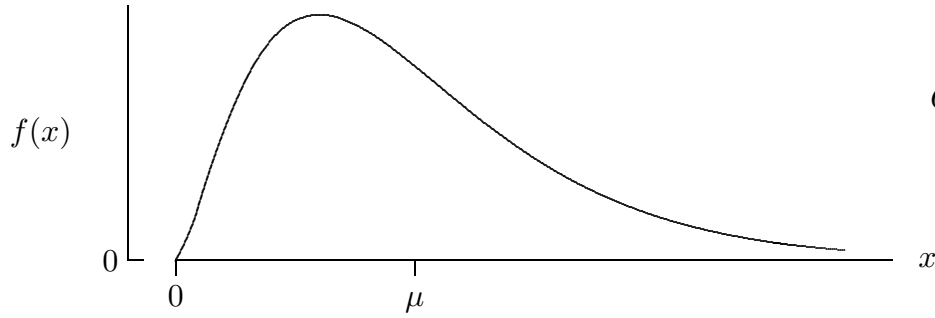
Figure 7.4.7.
$Chisquare(n)$ **pdf.**

- the cdf of X is an incomplete gamma function (see Appendix D)

$$F(x) = P(n/2, x/2) \qquad x > 0$$

- except for special cases, the idf of X must be determined by numerical inversion
- the mean of X is

$$\mu = n$$

- the standard deviation of X is

$$\sigma = \sqrt{2n}.$$

The random variable X is $Chisquare(n)$ if and only if

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

where Z_1, Z_2, ..., Z_n is an *iid* sequence of $Normal(0,1)$ random variables. The *Chisquare* random variable is commonly used in statistical "goodness-of-fit" tests (see Section 9.2). The discrete nature of the parameter n precludes the use of this distribution for model stochastic elements of a system in most applications. If n is an even number then a $Chisquare(n)$ random variable and a $Erlang(n/2, 2)$ random variable are equivalent.

7.4.8   STUDENT

Definition 7.4.8   The continuous random variable X is $Student(n)$ if and only if

- the shape parameter n is a positive integer, known as the "degrees of freedom"
- the possible values of X are $\mathcal{X} = \{x \mid -\infty < x < \infty\}$
- the pdf of X is

$$f(x) = \frac{1}{\sqrt{n}\, B(1/2,\, n/2)}\, (1 + x^2/n)^{-(n+1)/2} \qquad\qquad -\infty < x < \infty,$$

as illustrated in Figure 7.4.8 for $n = 10$, where $B(\cdot,\, \cdot)$ is the incomplete beta function defined in Appendix D

Figure 7.4.8.
Student(n)
pdf.
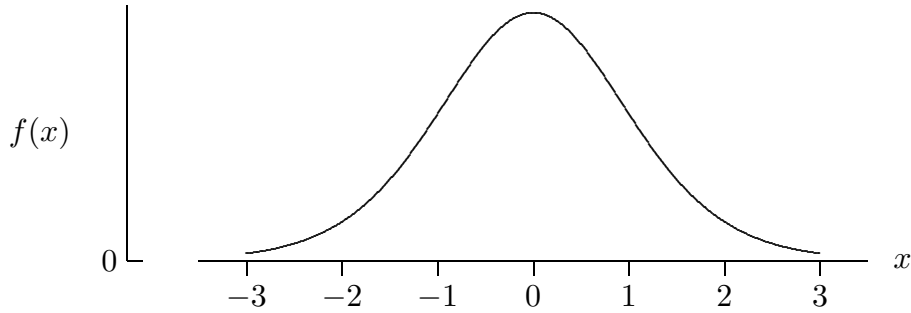
- if $x \geq 0$ then the cdf of X is a function of an incomplete beta function

$$F(x) = \frac{1 + I\big(1/2,\, n/2,\, n/(n+x^2)\big)}{2} \qquad\qquad x \geq 0$$

and if $x < 0$ then $F(x) = 1 - F(-x)$

- the idf of X must be determined by numerical inversion
- the mean of X is $\mu = 0$ for $n = 2, 3, 4 \ldots$
- the standard deviation of X is

$$\sigma = \sqrt{\frac{n}{n-2}} \qquad\qquad n = 3, 4, 5 \ldots$$

The continuous random variable X is $Student(n)$ if and only if

$$X = \frac{Z}{\sqrt{V/n}},$$

where Z is a $Normal(0, 1)$ random variable, V is a $Chisquare(n)$ random variable and Z, V are independent. The primary use of a $Student(n)$ random variable in simulation is in interval estimation. The pdf is symmetric and bell-shaped, with slightly heavier tails than the pdf of a standard normal random variable. The limiting distribution (Hogg, McKean, and Craig, 2005, Chapter 4) of a $Student(n)$ distribution as $n \to \infty$ is a $Normal(0, 1)$ distribution.

7.4.9 SUMMARY

Effective discrete-event simulation modeling requires that the modeler be familiar with several continuous parametric distributions which can be used to mimic the stochastic elements of the model. In order to choose the proper distribution, it is important to know

- how these distributions arise and how they are related to one another;
- their support $\mathcal{X}$;
- their mean μ;
- their variance σ^2;
- the shape of their pdf.

If the situation calls for a continuous model (e.g., interarrival times) a modeler can quickly examine the shape of the histogram associated with a data set to rule in and rule out certain distributions based on the shapes of their pdfs.

Parameters

Parameters in a distribution allow modeling of such diverse applications as a machine failure time, patient post-surgery survival time, and customer interarrival time by a single distribution [for example, the *Lognormal*(a, b) distribution]. In our brief overview of each of the eight continuous distributions presented in this chapter, we have used the adjectives *scale*, *shape*, and *location* to classify parameters. We now consider the meaning behind each of these terms.

Location parameters are used to shift the distribution to the left or right along the x-axis. If c_1 and c_2 are two values of a location parameter for a distribution with cdf $F(x; c)$, then there exists a real constant α such that $F(x; c_1) = F(\alpha + x; c_2)$. A familiar example of a location parameter is the mean μ of the *Normal*(μ, σ) distribution. This parameter simply translates the bell-shaped curve to the left and right for various values of μ. Location parameters are also known as "shift" parameters.

Scale parameters are used to expand or contract the x-axis by a factor of α. If λ_1 and λ_2 are two values for a scale parameter for a distribution with cdf $F(x; \lambda)$, then there exists a real constant α such that $F(\alpha x; \lambda_1) = F(x; \lambda_2)$. A familiar example of a scale parameter is μ in the *Exponential*(μ) distribution. The pdf always has the same shape, and the units on the x-axis are determined by the value of μ.

*Shape* parameters are appropriately named since they affect the shape of the pdf. A familiar example of a shape parameter is n in the *Erlang*(n, b) distribution. The pdf assumes different shapes for various values of n.

In summary, location parameters *translate* distributions along the x-axis, scale parameters *expand* or *contract* the scale for distributions, and all other parameters are shape parameters.

Relationships between distributions

It is also important to know how these distributions relate to one another. Figure 7.4.9 highlights some relationships between the eight distributions summarized in this section, plus the $Uniform(0,1)$ distribution. Listed in each oval are the name, parameter(s), and support of each distribution. Each distribution, including the standard normal distribution, is designated by the generic random variable X. The solid arrows connecting the ovals denote special cases [e.g., the $Normal(0,1)$ distribution is a special case of the $Normal(\mu, \sigma)$ distribution when $\mu = 0$ and $\sigma = 1$] and transformations [e.g., the sum (convolution) of n independent and identically distributed $Exponential(\mu)$ random variables has an $Erlang(n, \mu)$ distribution]. The dashed line between the $Student(n)$ and $Normal(0,1)$ distributions, for example, indicates that the limiting distribution of a $Student(n)$ random variable as $n \to \infty$ has a standard normal distribution.

Figure 7.4.9.
Relationships between
continuous distributions.
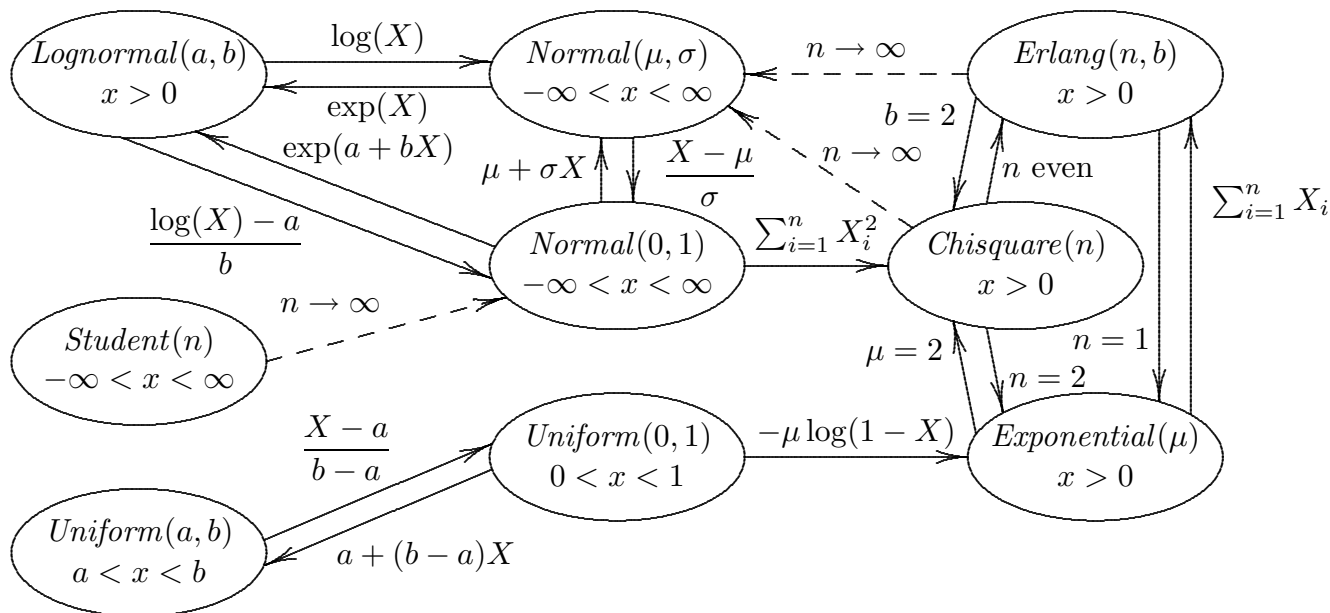
There are internal characteristics associated with these distributions that are not shown in Figure 7.4.9. One such example is that the sum of independent normal random variables also has a normal distribution. Another is that the sum of independent chi-square random variables also has chi-square distribution.

There could be an arrow drawn from the $Uniform(0,1)$ distribution to every other distribution, in theory, because of the probability integral transformation (Theorem 7.2.1). Only the transformations to the $Uniform(a, b)$ and $Exponential(\mu)$ distributions are included in the figure, however, as these are the only distributions with monotone, synchronized variate generation algorithms that do not require numerical methods.

The alert reader will notice that the sequence of arrows running from the $Uniform(0, 1)$ distribution to the $Exponential(2)$ distribution to the $Chisquare(2)$ distribution to the $Normal(0, 1)$ distribution in Figure 7.4.9 is related to the Box–Muller algorithm for generating normal variates given in Section 7.2.

Moments

As in the case of discrete random variables, the table below summarizes the first four moments of the distribution of the continuous random variables surveyed in this chapter. earlier. To save space, the $Lognormal(a, b)$ distribution uses $\omega = \exp(b^2)$. As before the mean, variance, skewness, and kurtosis are defined by

$$\mu = E[X], \quad \sigma^2 = E[(X - \mu)^2], \quad E\left[\left(\frac{X - \mu}{\sigma}\right)^3\right], \quad \text{and} \quad E\left[\left(\frac{X - \mu}{\sigma}\right)^4\right].$$

The ith moment for the $Student(n)$ distribution is only defined for $n > i$, for $i = 1, 2, 3, 4$. As with discrete distributions, the skewness is a measure of the symmetry of a distribution. The $Exponential(\mu)$ distribution, for example, has a positive skewness for all values of its parameter μ. The kurtosis is a measure of the peakedness and tail behavior of a distribution. The $Student(n)$ distribution, for example, with its heavier tails than a standard normal distribution, has a kurtosis that converges to 3 from above as $n \to \infty$.

Distribution	Mean	Variance	Skewness	Kurtosis
$Uniform(a, b)$	$\dfrac{a + b}{2}$	$\dfrac{(b - a)^2}{12}$	0	$\dfrac{9}{5}$
$Exponential(\mu)$	μ	μ^2	2	9
$Erlang(n, b)$	nb	nb^2	$\dfrac{2}{\sqrt{n}}$	$3 + \dfrac{6}{n}$
$Normal(0, 1)$	0	1	0	3
$Normal(\mu, \sigma)$	μ	σ^2	0	3
$Lognormal(a, b)$	$e^{a + b^2/2}$	$e^{2a}\omega(\omega - 1)$	$(\omega + 2)\sqrt{\omega - 1}$	$\omega^4 + 2\omega^3 + 3\omega^2 - 3$
$Chisquare(n)$	n	$2n$	$\dfrac{2^{3/2}}{\sqrt{n}}$	$3 + \dfrac{12}{n}$
$Student(n)$	0	$\dfrac{n}{n - 2}$	0	$\dfrac{3(n - 2)}{n - 4}$

Although discussion here is limited to just the continuous distributions introduced in this chapter, there are many other parametric distributions capable of modeling continuous random variables. For a complete list of common discrete and continuous parametric distributions including their pdf, cdf, idf, and moments, we recommend the compact work of Evans, Hastings, and Peacock (2000) or the encyclopedic works of Johnson, Kotz, and Kemp (1993) and Johnson, Kotz, and Balakrishnan (1994, 1995, 1997).

7.4.10   PDF, CDF AND IDF EVALUATION

Pdf's, cdf's and idf's for all seven of the continuous random variable models in this section can be evaluated by using the functions in the library `rvms` (see Appendix D). For example, if X is $Erlang(n, b)$ then for any $x > 0$

```
pdf = pdfErlang(n, b, x);                                    /* f(x) */
cdf = cdfErlang(n, b, x);                                    /* F(x) */
```

and for any $0.0 < u < 1.0$

```
idf = idfErlang(n, b, u);                                    /* F⁻¹(u) */
```

This library also has functions to evaluate pdf's, cdf's, and idf's for all the discrete random variables in Chapter 6.

7.4.11   EXERCISES

Exercise 7.4.1   Prove that the cdf of an $Erlang(n, b)$ random variable is

$$F(x) = P(n, x/b)$$

where $P(\cdot, \cdot)$ is the incomplete gamma function.

Exercise 7.4.2   Prove that if $n > 0$ is an even integer then a $Chisquare(n)$ random variable and a $Erlang(n/2, 2)$ random variable are equivalent.

Exercise 7.4.3   Prove that an $Exponential(\mu)$ random variable has the lack of memory property.

Exercise 7.4.4   From the $Normal(0, 1)$ cdf definition

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2)\, dt \qquad -\infty < z < \infty,$$

prove that $\Phi(-z) + \Phi(z) = 1$.

Exercise 7.4.5   Let X be a random variable with finite mean μ and standard deviation σ. Prove that the *standardized* random variable

$$\frac{X - \mu}{\sigma}$$

has a mean of 0 and a standard deviation of 1.

Exercise 7.4.6[a]   (*a*) Find the mean, variance, skewness, and kurtosis of

$$Z = U_1 + U_2 + \cdots + U_{12} - 6,$$

where U_1, U_2, ..., U_{12} is an *independent and identically distributed* sequence of *Uniform*$(0, 1)$ random variables. (*b*) Generate $10\,000$ variates using this approach (see Section 7.2) and provide convincing numerical evidence that the variates so generated are approximately normally distributed.

In this section we will develop two algorithms for simulating a *nonstationary* Poisson process. That is, rather than events that occur at a constant rate of occurrence λ, we now consider processes whose time-varying rate is governed by $\lambda(t)$. We introduce non-stationary Poisson processes by developing one incorrect and two correct algorithms for generating a realization of simulated event times.

7.5.1   AN INCORRECT ALGORITHM

For reference, recall that we can simulate a *stationary* Poisson process with rate λ for the time interval $0 \le t < \tau$ by using Algorithm 7.3.1, reproduced here for convenience

```
a₀ = 0.0;
n = 0;
while (aₙ < τ) {
    aₙ₊₁ = aₙ + Exponential(1 / λ);
    n++;
}
return a₁, a₂, ..., aₙ₋₁;
```

This algorithm generates a realization of a stationary Poisson process with event times $a_1, a_2, a_3, \ldots$ and constant event rate λ. Based upon this algorithm it might be conjectured that if the event rate varies with time then the following minor modification will correctly generate a *nonstationary* Poisson process with event rate function $\lambda(t)$

```
a₀ = 0.0;
n = 0;
while (aₙ < τ) {
    aₙ₊₁ = aₙ + Exponential(1 / λ(aₙ));
    n++;
}
return a₁, a₂, ..., aₙ₋₁;
```

Unfortunately, this naive modification of the correct stationary Poisson process algorithm is *not* correct. The algorithm only considers the value of the event rate function $\lambda(t)$ at the time of the previous event a_n, and hence ignores the future evolution of $\lambda(t)$ after time a_n. In general, the incorrectness of this algorithm will manifest itself as an "inertia error" that can be characterized for a rate function $\lambda(t)$ that is monotone on (a_n, a_{n+1}) as follows:

- if $\lambda(a_n) < \lambda(a_{n+1})$ then $a_{n+1} - a_n$ will tend to be too large;

- if $\lambda(a_n) > \lambda(a_{n+1})$ then $a_{n+1} - a_n$ will tend to be too small.

If $\lambda(t)$ varies slowly with t then the inertia error will be uniformly small and the incorrect algorithm may produce an acceptable approximation to what is desired. Of course this is a weak justification for using an algorithm *known* to be incorrect, particularly in light of the existence of two *correct* algorithms for simulating a nonstationary Poisson process.

Example 7.5.1   A nonstationary Poisson process with a piecewise-constant event rate function

$$\lambda(t) = \begin{cases} 1 & 0 \le t < 15 \\ 2 & 15 \le t < 35 \\ 1 & 35 \le t < 50 \end{cases}$$

was simulated using the incorrect algorithm with $\tau = 50$. Note that the expected number of events in the interval $0 \le t < 15$ is 15; in the interval $15 \le t < 35$ the expected number is $2 \times (35 - 15) = 40$; and 15 events are expected in the interval $35 \le t < 50$. Therefore, the expected number of events in the interval $0 \le t < 50$ is 70. The nonstationary Poisson process was replicated 10 000 times so that the expected number of events was 700 000, and a time-dependent relative frequency histogram was generated by partitioning the interval $0 \le t \le 50$ into 50 bins with bin size 1.0, as illustrated in Figure 7.5.1.

Figure 7.5.1. Incorrect nonstationary Poisson process generation.
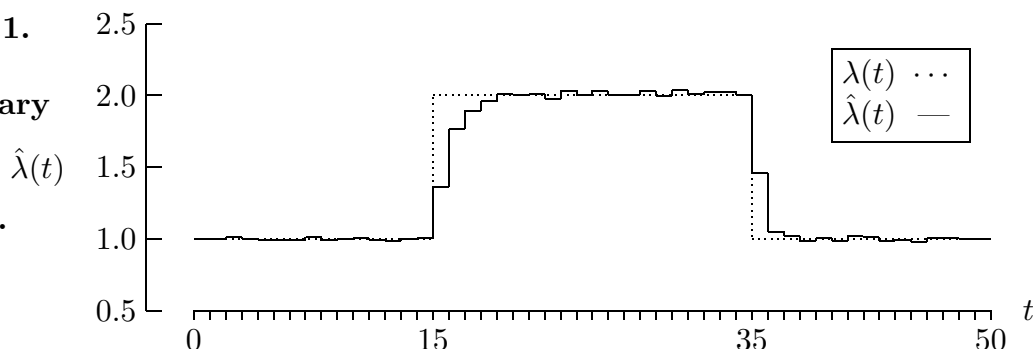
The number of events was accumulated for each bin and then divided by the number of replications (10 000) to provide an *estimate* $\hat{\lambda}(t)$ of the expected rate at time t where t corresponds to the midpoint of the bin.* Consistent with the previous discussion, the inertia error causes $\hat{\lambda}(t)$ to under-estimate $\lambda(t)$ immediately to the right of the step-increase at $t = 15$ and to over-estimate $\lambda(t)$ immediately to the right of the step-decrease at $t = 35$. Although this error is accentuated by the discontinuous nature of $\lambda(t)$, some error would remain even if the discontinuities in $\lambda(t)$ were removed. You are asked to investigate this issue further in Exercise 7.5.2.

Correct Algorithms

Two correct algorithms for generating a realization of a nonstationary Poisson process will be presented in the two subsequent subsections: thinning and inversion. Neither algorithm dominates the other in terms of performance — thinning works for a wider range of event rate functions $\lambda(t)$, but inversion is synchronized and typically more computationally efficient. The existence of these two exact algorithms is sufficient reason to never use the incorrect algorithm in practice.

---

* If the bin width had not been 1.0, to compute $\hat{\lambda}(t)$ it would have been necessary to divide by the bin width as well.

7.5.2 THINNING METHOD

The first correct algorithm for generating a nonstationary Poisson process is based upon the existence of an *upper bound* (majorizing constant) on the event rate $\lambda_{\max}$ such that $\lambda(t) \leq \lambda_{\max}$ for $0 \leq t < \tau$. As we will see, the efficiency of the algorithm is dependent upon $\lambda_{\max}$ being a tight bound. The algorithm is based upon generating a stationary Poisson process with rate $\lambda_{\max}$, but correcting for this too-high event rate and making the process nonstationary by occasionally thinning (discarding) some of the event times. The following algorithm, known as the *thinning method*, is due to Lewis and Shedler (1979). (As demonstrated in the next section, the thinning method is actually just a clever use of the acceptance-rejection method for random variate generation.)

Algorithm 7.5.1 Provided $\lambda(t) \leq \lambda_{\max}$ for $0 \leq t < \tau$, this algorithm generates a nonstationary Poisson process with event rate function $\lambda(t)$

```
a₀ = 0.0;
n = 0;
while (aₙ < τ) {
    s = aₙ;
    do {                                        /* thinning loop */
        s = s + Exponential(1 / λ_max);     /* possible next event time */
        u = Uniform(0, λ_max);
    } while ((u > λ(s)) and (s < τ));           /* thinning criteria */
    aₙ₊₁ = s;
    n++;
}
return a₁, a₂, ..., aₙ₋₁;
```

The key to the correctness of the thinning method is the rate correction defined by the `do-while` loop, as illustrated geometrically in Figure 7.5.2. If it were not for this loop, the process generated would be stationary with event rate $\lambda_{\max}$. When the event rate function $\lambda(t)$ is low (high), more (fewer) "trial" event times s are thinned out, as expected.

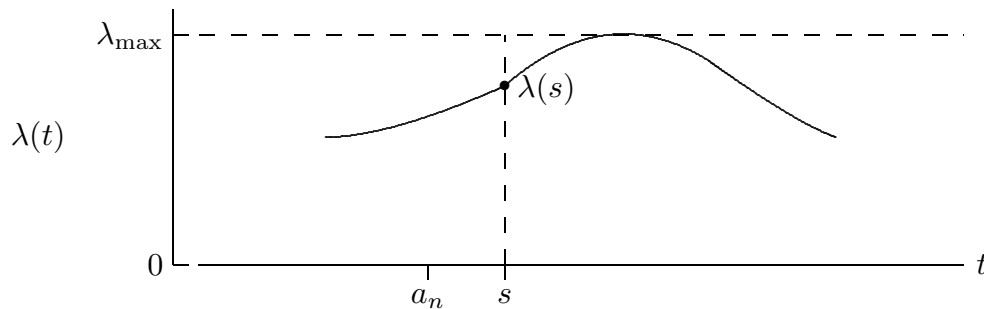
Figure 7.5.2. Thinning algorithm geometry.

Because occasional multiple "thinning" passes through the `do-while` loop are necessary to make the process nonstationary, efficiency suffers if $\lambda(s)$ differs significantly from $\lambda_{\max}$.

Example 7.5.2   Algorithm 7.5.1 was used with $\lambda_{\max} = 2.0$ to repeat the experiment in Example 7.5.1. In this case the nonstationary Poisson process is correctly simulated, as illustrated in Figure 7.5.3.

Figure 7.5.3. Thinning implementation results.
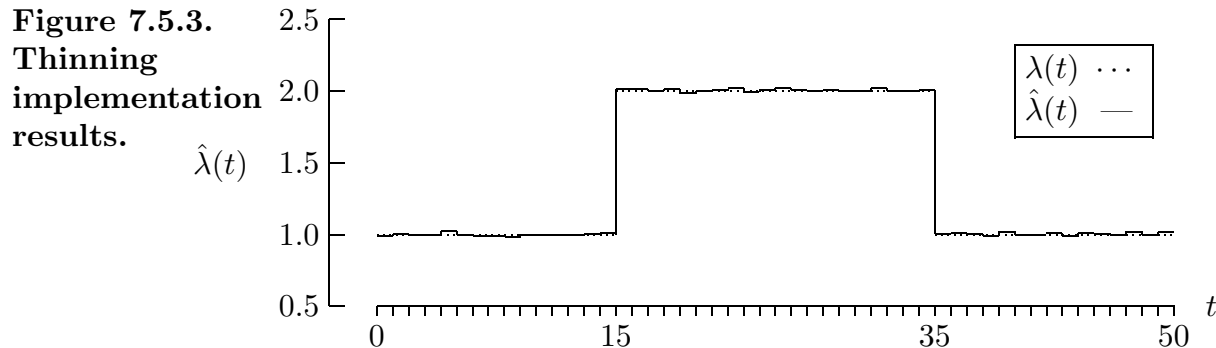
Relative to the incorrect algorithm, there is increased complexity with Algorithm 7.5.1. In particular, as compared with the time to compute the histogram in Example 7.5.1, the thinning method time was approximately 2.2 times longer.

So long as $\lambda_{\max}$ is a tight upper bound to $\lambda(t)$, resulting in infrequent thinning, Algorithm 7.5.1 is relatively efficient and, primarily because it is easy to implement, the algorithm is popular. At least two random variates are required per event, however, and even if `rngs` is used with separate streams allocated to the `Exponential` and `Uniform` calls, synchronization is a problem because multiple passes through the thinning loop can occur. The inversion method, presented next, eliminates the need for multiple calls to `Random`.

7.5.3   INVERSION METHOD

The second correct algorithm is based upon the inverse transformation method of random variate generation (see Sections 6.2 and 7.2). Because of this, and in contrast to the thinning method, this second algorithm has the desirable property that only one call to `Random` is required to generate each new event time. The algorithm is based upon the *cumulative* event rate function defined as

$$\Lambda(t) = \int_0^t \lambda(s)\, ds \qquad 0 \le t < \tau$$

which represents the expected number of events in the interval $[0, t)$. Note that $\lambda(\cdot)$ is analogous to the pdf of a continuous random variable and $\Lambda(\cdot)$ is analogous to the cdf.

We assume that $\lambda(t) > 0$ for all $t \in [0, \tau)$, except perhaps at a finite number of times where $\lambda(t)$ may be 0. Therefore, $\Lambda(t)$ is a strictly monotone increasing function with positive slope

$$\frac{d\Lambda(t)}{dt} = \lambda(t) \qquad 0 < t < \tau.$$

Because $\Lambda(\cdot)$ is a strictly monotone increasing function, there exists an *inverse* function $\Lambda^{-1}(\cdot)$, analogous to the idf of a continuous random variable. Based upon these observations, it can be shown that the following algorithm is correct.

Algorithm 7.5.2 If $\Lambda^{-1}(\cdot)$ is the inverse of the function $\Lambda(\cdot)$ defined by

$$\Lambda(t) = \int_0^t \lambda(s)\,ds \qquad 0 \le t < \tau$$

then this algorithm (due to a result by Çinlar, 1975) generates a nonstationary Poisson process with event rate function $\lambda(t)$ for $0 \le t < \tau$

```
a₀ = 0.0;                    /* initialize nonstationary Poisson process */
u₀ = 0.0;                        /* initialize unit Poisson process */
n = 0;                               /* initialize event counter */
while (aₙ < τ) {
    uₙ₊₁ = uₙ + Exponential(1.0);
    aₙ₊₁ = Λ⁻¹(uₙ₊₁);
    n++;
}
return a₁, a₂, ..., aₙ₋₁;
```

Algorithm 7.5.2 works by generating a sequence of event times $u_1, u_2, u_3, \ldots$ corresponding to a *stationary* "unit" Poisson process with rate 1.* The inverse function $\Lambda^{-1}(\cdot)$ is then used to transform the u's into a sequence of event times $a_1, a_2, a_3, \ldots$ corresponding to a *nonstationary* Poisson process with event rate function $\lambda(t)$, as illustrated in Figure 7.5.4.
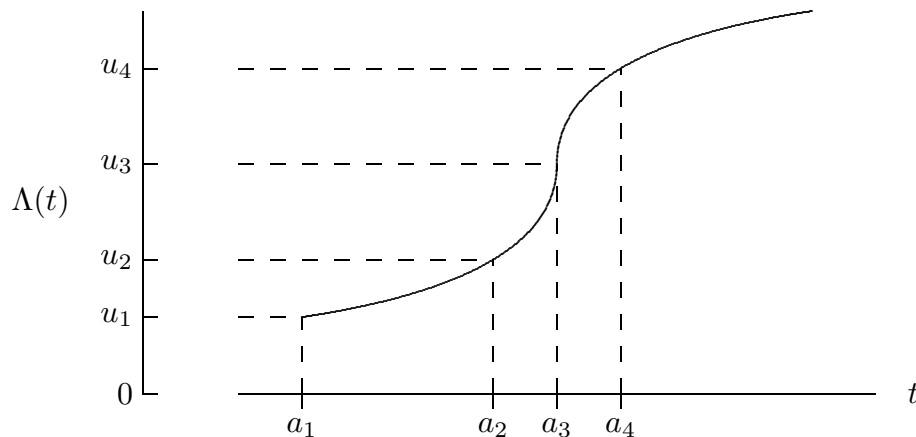
Figure 7.5.4. Inversion algorithm geometry.

The extent to which Algorithm 7.5.2 is useful is determined by the extent to which the inverse function $\Lambda^{-1}(\cdot)$ can be evaluated efficiently. As illustrated in the following example, if the event rate function $\lambda(t)$ is assumed to be piecewise constant then $\Lambda(t)$ will be piecewise linear and the evaluation of $\Lambda^{-1}(\cdot)$ is easy.

---

* From the results in Section 7.3, this is statistically equivalent to generating n points *at random* in the interval $0 < u < \Lambda(\tau)$ where $\Lambda(\tau)$ is the expected number of events in the interval $[0, \tau)$.

Example 7.5.3   If $\lambda(t)$ is the piecewise constant event rate function in Examples 7.5.1 and 7.5.2 then it follows by integration that

$$\Lambda(t) = \begin{cases} t & 0 \leq t < 15 \\ 2t - 15 & 15 \leq t < 35 \\ t + 20 & 35 \leq t < 50. \end{cases}$$

Thus, by solving the equation $u = \Lambda(t)$ for t we find that

$$\Lambda^{-1}(u) = \begin{cases} u & 0 \leq u < 15 \\ (u + 15)/2 & 15 \leq u < 55 \\ u - 20 & 55 \leq u < 70. \end{cases}$$

As illustrated in Figure 7.5.5, the nonstationary Poisson process can then be correctly simulated by inversion using Algorithm 7.5.2.

Figure 7.5.5.
Inversion
implementation
results.
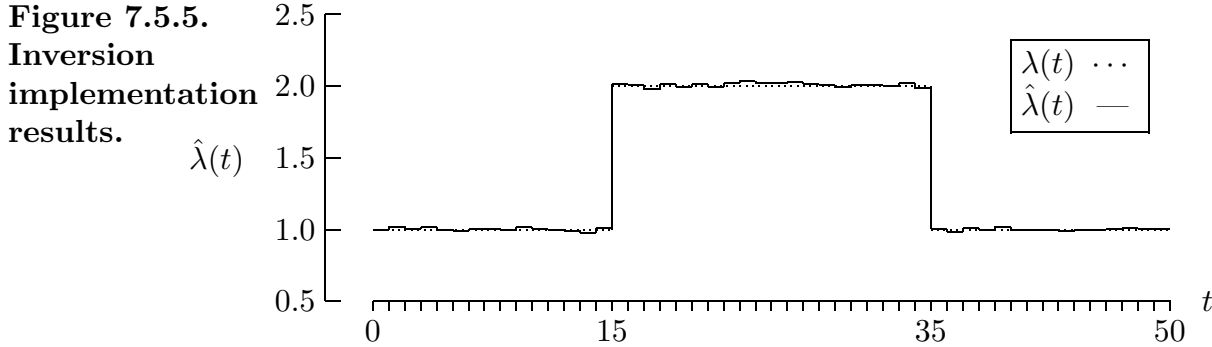
In contrast to the timing results in Example 7.5.2 for the thinning method, the time required to generate this histogram by inversion is essentially identical to the time required to generate the incorrect results in Example 7.5.1. The event rate function in this particular example has been ideal in the sense that the cumulative event rate function $\Lambda(t)$ can easily be inverted in closed form. If this is not the case, the modeler is left with three options: (i) use thinning if a majorizing constant $\lambda_{\max}$ can be found; (ii) use numerical methods to invert $\Lambda(t)$; or (iii) use an approximate algorithm.

Next-Event Simulation Orientation

The three algorithms for generating a nonstationary Poisson process (the incorrect algorithm, thinning, and inversion) all return event times $a_1, a_2, \ldots, a_{n-1}$, violating next-event simulation practice. All three, however, can easily be adapted for a next-event simulation. With inversion, for example, an arrival event being processed at time t should schedule the next arrival at time

$$\Lambda^{-1}\big(\Lambda(t) + Exponential(1.0)\big),$$

assuming that this time does not exceed the termination time τ. This procedure would be appropriate for generating the next event time a_3, given the current event time $t = a_2$, for example, in Figure 7.5.4.

What happens in a next-event simulation when the time of a scheduled event exceeds τ? If the interval $0 \leq t < \tau$ represents a single cycle that cuts off at time τ (e.g., a restaurant that opens at 11:00 AM and closes at 9:00 PM), then the event to be scheduled after time τ is simply discarded (not scheduled) and the arrival stream is cut off. Alternatively, the interval $0 \leq t < \tau$ may represent one of many cycles in a repeating cyclical process such as a hospital emergency room where $\tau = 24$ hours. Furthermore, there may be different functions $\Lambda_1(t)$, $\Lambda_2(t)$, ... for each cycle (e.g., $\Lambda_1(t)$ for Sunday, $\Lambda_2(t)$ for Monday, etc.). In this case, when

$$\Lambda_1^{-1}\big(\Lambda_1(t) + Exponential(1.0)\big)$$

exceeds τ, the residual amount of the unit stationary Poisson process,

$$\Lambda_1(t) + Exponential(1.0) - \Lambda_1(\tau)$$

is used as an argument in $\Lambda_2(t)$ to generate the first event time in the second cycle (e.g., the first patient to arrive to the emergency room on Monday).

Piecewise-Constant Versus Piecewise-Linear

In most applications, a piecewise-constant event rate function $\lambda(t)$ is not realistic. Instead, it is more likely that $\lambda(t)$ will vary continuously with t, avoiding discontinuous jumps like those at $t = 15$ and $t = 35$ in Examples 7.5.1, 7.5.2, and 7.5.3. On the other hand, because accurate estimates of $\lambda(t)$ are difficult to obtain (because of the large amount of data required), it is hard to justify "high-order" smooth polynomial approximations to $\lambda(t)$. Therefore, as a compromise, we will examine the case where $\lambda(t)$ is a piecewise-linear function (spline). That is, we will specify $\lambda(t)$ at a sequence of (t_j, λ_j) points (spline knots) and then use linear interpolation to "fill in" other values of $\lambda(t)$ as necessary.

7.5.4   PIECEWISE-LINEAR TIME-DEPENDENT EVENT RATE FUNCTIONS

As discussed in the previous paragraph, it is common to assume that the event rate $\lambda(t)$ is piecewise linear. That is, we assume that $\lambda(t)$ is specified as a piecewise-linear function defined for $0 \leq t < \tau$ by the $k + 1$ *knot pairs* (t_j, λ_j) for $j = 0, 1, \ldots, k$ with the understanding that $0 = t_0 < t_1 < \cdots < t_k = \tau$ and $\lambda_j \geq 0$ for $j = 0, 1, \ldots, k$, as illustrated for the case $k = 6$ in Figure 7.5.6.
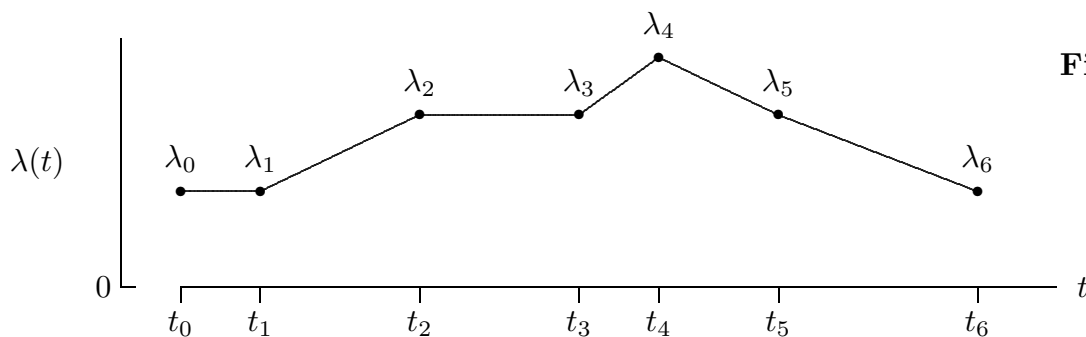
Figure 7.5.6. Piecewise-linear event rate function.

Algorithm 7.5.3   Given a piecewise-linear, time-dependent event rate function $\lambda(t)$ specified by the $k+1$ knot pairs (t_j, λ_j) for $j = 0, 1, \ldots, k$, the mathematical construction of the piecewise linear function $\lambda(t)$, the corresponding piecewise quadratic cumulative event rate function $\Lambda(t)$, and the inverse function $\Lambda^{-1}(u)$ involves four steps.

- Define the k *slopes* of the piecewise linear segments of $\lambda(t)$

$$s_j = \frac{\lambda_{j+1} - \lambda_j}{t_{j+1} - t_j} \qquad j = 0, 1, \ldots, k-1.$$

- Define the $k+1$ *cumulative event rates* $0 = \Lambda_0 \leq \Lambda_1 \leq \cdots \leq \Lambda_k$ as

$$\Lambda_j = \int_0^{t_j} \lambda(t)\, dt \qquad j = 1, 2, \ldots, k.$$

With $\Lambda_0 = 0$ the cumulative event rates can be calculated recursively as

$$\Lambda_j = \Lambda_{j-1} + \frac{1}{2}(\lambda_j + \lambda_{j-1})(t_j - t_{j-1}) \qquad j = 1, 2, \ldots, k.$$

- For each subinterval $t_j \leq t < t_{j+1}$, if $s_j \neq 0$ then

$$\lambda(t) = \lambda_j + s_j(t - t_j)$$

is a linear function of $t - t_j$. Instead, if $s_j = 0$ then $\lambda(t)$ is a constant function. Similarly, if $s_j \neq 0$ then

$$\Lambda(t) = \Lambda_j + \lambda_j(t - t_j) + \frac{1}{2}s_j(t - t_j)^2$$

is a quadratic function of $t - t_j$ and if $s_j = 0$ then $\Lambda(t)$ is a linear function of $t - t_j$.

- For each subinterval $\Lambda_j \leq u < \Lambda_{j+1}$, if $s_j \neq 0$ then the quadratic equation $u = \Lambda(t)$ can be solved for t to yield the inverse function

$$\Lambda^{-1}(u) = t_j + \frac{2(u - \Lambda_j)}{\lambda_j + \sqrt{\lambda_j^2 + 2s_j(u - \Lambda_j)}}.$$

Instead, if $s_j = 0$ then the linear equation $u = \Lambda(t)$ can be solved for t to yield the inverse function

$$\Lambda^{-1}(u) = t_j + \frac{u - \Lambda_j}{\lambda_j}.$$

The mathematical derivation of these four steps is left as an exercise.

The following algorithm is just Algorithm 7.5.2 with an additional index j to keep track of which branch of the $\Lambda^{-1}(\cdot)$ function should be used. As the event times increase, the index j of the current $\Lambda_j < u \leq \Lambda_{j+1}$ subinterval is updated accordingly, thereby eliminating the need to search for the index of the subinterval in which u_n lies.

Algorithm 7.5.4 If $\lambda(t)$ is a piecewise-linear function with components defined consistent with Algorithm 7.5.3, then this inversion algorithm generates a nonstationary Poisson process with event rate function $\lambda(t)$ for $0 \leq t < \tau$

```
a₀ = 0.0;
u₀ = 0.0;
n = 0;
j = 0;
while (aₙ < τ) {
    u_{n+1} = uₙ + Exponential(1.0);
    while ((Λ_{j+1} < u_{n+1}) and (j < k))
      j++;
    a_{n+1} = Λ⁻¹(u_{n+1});                        /* Λ_j < u_{n+1} ≤ Λ_{j+1} */
    n++;
}
return a₁, a₂, ..., a_{n-1};
```

Using Algorithm 7.5.3, the slopes s_j and cumulative event rates Λ_j can be generated once external to Algorithm 7.5.4. Or these parameters can be generated internal to Algorithm 7.5.4, as needed, each time j is incremented. In either case, the time-complexity of Algorithm 7.5.4 is essentially independent of the number of knots. When $\lambda(t)$ is piecewise-linear, Algorithm 7.5.4 is the preferred way to generate a nonstationary Poisson process.

7.5.5 EXERCISES

Exercise 7.5.1 Work through the details of Algorithm 7.5.3. Be sure to address the issue of how we know to always use the $+$ sign when solving the $u = \Lambda(t)$ quadratic equation and verify that as $s_j \to 0$ the $s_j \neq 0$ version of the $\Lambda^{-1}(u)$ equation degenerates into the $s_j = 0$ version.

Exercise 7.5.2 (a) Use the event rate function

$$\lambda(t) = \begin{cases} 1 & 0 \leq t < 10 \\ t/10 & 10 \leq t < 20 \\ 2 & 20 \leq t < 30 \\ (50 - t)/10 & 30 \leq t < 40 \\ 1 & 40 \leq t < 50 \end{cases}$$

and the incorrect nonstationary Poisson process algorithm to produce a figure similar to the one in Example 7.5.1. (b) Comment.

Exercise 7.5.3 (*a*) Use the event rate function in Exercise 7.5.2 and Algorithm 7.5.1 to produce a figure similar to the one in Example 7.5.2. (*b*) Comment.

Exercise 7.5.4 (*a*) Use the event rate function in Exercise 7.5.2 and Algorithm 7.5.4 to produce a figure similar to the one in Example 7.5.3. (*b*) Comment.

Exercise 7.5.5 Prove that if $\lambda(t)$ is constant with value $\lambda_{\max}$ then Algorithm 7.5.1 reduces to Algorithm 7.3.1.

Exercise 7.5.6 Prove that if $\lambda(t)$ is constant with value λ then Algorithm 7.5.2 reduces to Algorithm 7.3.1.

Exercise 7.5.7[a] Use Algorithm 7.5.4 with $\tau = 2000$ to construct a finite-horizon simulation of an initially idle single-server service node with a nonstationary Poisson arrival process. Assume that the arrival rate function $\lambda(t)$ is the piecewise-linear function illustrated and that the service time distribution is *Erlang*(4, 0.25).
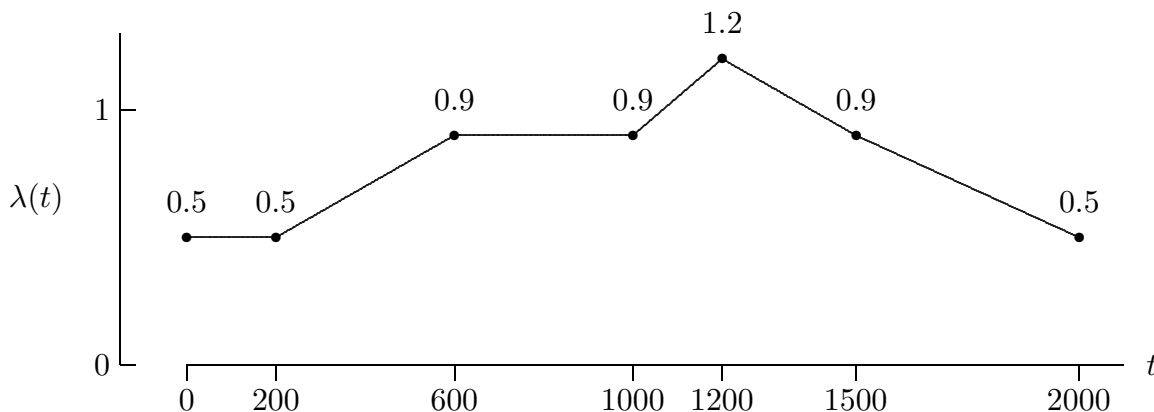
(*a*) The simulation should be replicated 64 times to estimate the mean of the instantaneous (snapshot) number in the node at each of the $\lambda(t)$ time knots (other than $t_0 = 0$). (*b*) Comment. (*c*) If you were to approximate the nonstationary Poisson arrival process with an "equivalent" stationary arrival process with constant rate $\bar{\lambda}$, what would the numerical value of $\bar{\lambda}$ be?

Exercise 7.5.8 Draw $\Lambda(t)$ associated with $\lambda(t)$ from Figure 7.5.6. Show the geometry associated with Algorithm 7.5.4 on your graph.

Exercise 7.5.9 Construct an algorithm for generating a realization of a nonstationary Poisson process with a piecewise constant event rate function $\lambda(t)$.

Exercise 7.5.10 Thinning can be performed in a more general setting than presented in this section. A *majorizing function* $\lambda_{\max}(t)$ can replace the majorizing constant $\lambda_{\max}$. Reformulate the thinning algorithm in its more general setting.

The thinning method was presented in the previous section as a technique for generating a realization of a nonstationary Poisson arrival process. The thinning method is based on a more general technique, acceptance-rejection, commonly used in non-uniform random variate generation. We introduce the acceptance-rejection technique in this section, and illustrate its use by developing an efficient algorithm for generating a $Gamma(a, b)$ random variate.

7.6.1   BACKGROUND

Definition 7.6.1   *Acceptance-rejection* is a technique for generating non-uniform random variates. The technique is most commonly applied to generating *continuous* random variates as follows. Let X be a continuous random variable with an associated set of possible values $\mathcal{X}$ and pdf $f(x)$ defined for all $x \in \mathcal{X}$. Choose a *majorizing* pdf $g(x)$ with the same set of possible values $x \in \mathcal{X}$ and an associated real-valued constant $c > 1$ such that

- $f(x) \leq c\, g(x)$ for all $x \in \mathcal{X}$;

- the idf $G^{-1}(\cdot)$ associated with the majorizing pdf can be evaluated *efficiently*.

The basic idea behind acceptance-rejection (von Neumann, 1951) is to use the inverse transformation method to generate a possible value $x \in \mathcal{X}$ consistent with the majorizing pdf $g(\cdot)$ and then either accept or reject x dependent on how well $c\, g(x)$ (the dashed line in Figure 7.6.1) approximates $f(x)$ (the solid line in Figure 7.6.1).
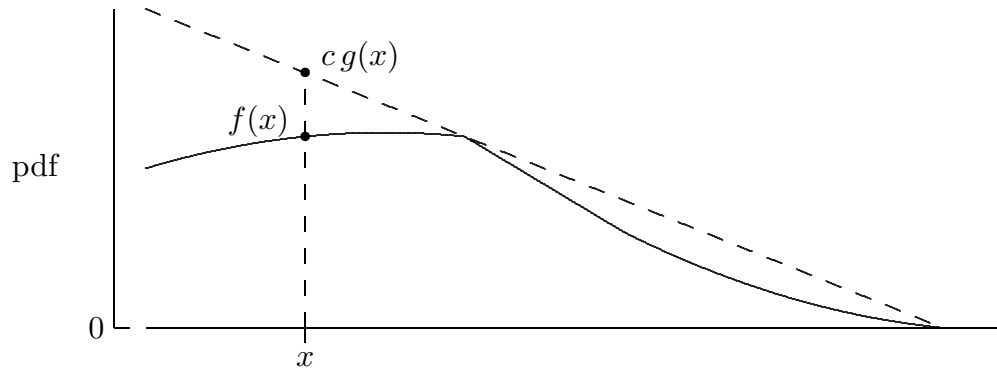
Figure 7.6.1. Acceptance-rejection geometry.

Algorithm 7.6.1   Given the two pdfs $f(\cdot)$ and $g(\cdot)$ with common domain $\mathcal{X}$, given a constant $c > 1$ such the $f(x) \leq c\, g(x)$ for all $x \in \mathcal{X}$, and given the idf $G^{-1}(\cdot)$ corresponding to $g(\cdot)$, then the following acceptance-rejection algorithm generates a random variate X with pdf $f(\cdot)$

```
do {
    u = Random();
    x = G⁻¹(u);
    v = Random();
} while (c * g(x) * v > f(x));
return x;
```

Algorithm Correctness

To prove that Algorithm 7.6.1 is correct, define the *acceptance region*

$$\mathcal{A} = \left\{ (t, v) \,\middle|\, v \le \frac{f(t)}{c\, g(t)} \right\} \subset \mathcal{X} \times (0, 1)$$

and then write Algorithm 7.6.1 in the equivalent form

```
do {
    u = Random();
    t = G⁻¹(u);                                    /* generate T */
    v = Random();                                  /* generate V */
} while ((t,v) ∉ A);
x = t;                                             /* generate X */
return x;
```

This formulation helps make it clear that Algorithm 7.6.1 works by repeatedly generating samples (t, v) of the bivariate random variable (T, V) until a sample falls in the acceptance region. When this occurs $x = t$ is a sample of the univariate random variable X.

Proof   What we must prove is that $\Pr(X \le x) = F(x)$ for all $x \in \mathcal{X}$ where

$$F(x) = \int_{t \le x} f(t)\, dt.$$

From the definition of conditional probability

$$\Pr(X \le x) = \Pr\left(T \le x \mid (T, V) \in \mathcal{A}\right) = \frac{\Pr\left((T \le x) \text{ and } (T, V) \in \mathcal{A}\right)}{\Pr\left((T, V) \in \mathcal{A}\right)},$$

where the numerator in the last expression is

$$\Pr\left((T \le x) \text{ and } (T, V) \in \mathcal{A}\right) = \int_{t \le x} \Pr\left((T, V) \in \mathcal{A} \mid T = t\right) g(t)\, dt,$$

and the denominator $\Pr\left((T, V) \in \mathcal{A}\right)$ is this same integral integrated over all $t \in \mathcal{X}$. The correctness of Algorithm 7.6.1 then follows from the fact that V is *Uniform*$(0, 1)$ and so

$$\Pr\left((T, V) \in \mathcal{A} \mid T = t\right) = \Pr\left(V \le \frac{f(t)}{c\, g(t)}\right) = \frac{f(t)}{c\, g(t)}.$$

That is

$$\Pr\left((T \le x) \text{ and } (T, V) \in \mathcal{A}\right) = \int_{t \le x} \frac{f(t)}{c\, g(t)} g(t)\, dt = \frac{F(x)}{c},$$

and $\Pr\left((T, V) \in \mathcal{A}\right) = 1/c$ so that $\Pr(X \le x) = F(x)$, as desired.

Note that, in addition to calling `Random` twice, each `do-while` loop iteration in Algorithm 7.6.1 involves

- one evaluation of $G^{-1}(u)$;

- one evaluation of $f(x)$;

- one evaluation of $g(x)$.

Therefore, the extent to which Algorithm 7.6.1 is efficient is determined, in part, by the efficiency of these three function evaluations. In addition, the expected number of loop iterations is also important and this is determined by how well $c\,g(x)$ fits $f(x)$.*

The design of a good acceptance-rejection algorithm, customized to a specific pdf $f(\cdot)$, is usually a mathematically-oriented exercise. Important trade-off choices must be made between the expected number of `do-while` loop iterations and the computational simplicity of $g(\cdot)$ and $G^{-1}(\cdot)$ function evaluations.

Example 7.6.1 Suppose that the pdf $f(x)$ (the solid line in Figure 7.6.2 defined on $0 < x < w$) is a smoothed histogram of many service times. Suppose also that $f(x)$ can be fit well with a *triangular* majorizing pdf $c\,g(x)$, also defined on $0 < x < w$ with parameters (h, w). (The dashed line is $c\,g(x)$, not $g(x)$.)
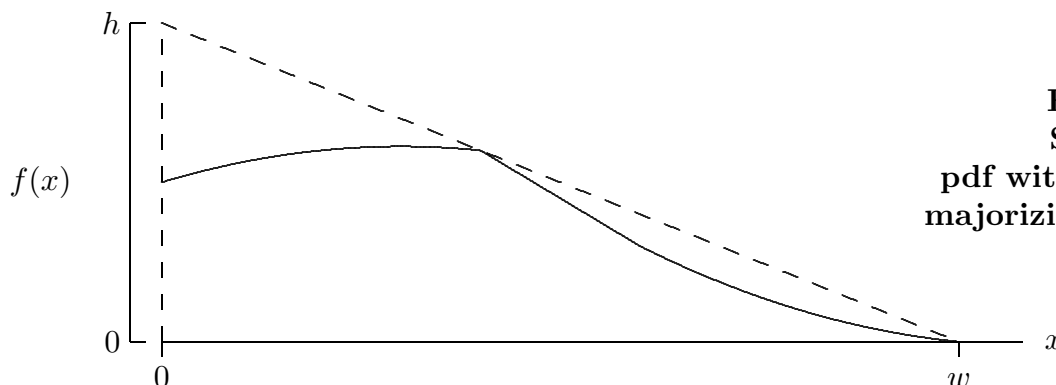
**Figure 7.6.2.
Service time
pdf with triangular
majorizing function.**

Appropriate values for (h, w) can be determined by inspection. The triangular majorizing pdf can then be written as

$$c\,g(x) = (1 - x/w)h \qquad 0 < x < w$$

and the constant c can be determined from the pdf requirement $\int_0^w g(x)\,dx = 1$. That is

$$c = c\int_0^w g(x)\,dx = \int_0^w c\,g(x)\,dx = \int_0^w (1 - x/w)h\,dx = \cdots = \frac{1}{2}hw$$

and so we find that, in this case, a majorizing pdf is

$$g(x) = \left(1 - \frac{x}{w}\right)\frac{2}{w} \qquad 0 < x < w.$$

---

* See Exercise 7.6.1.

The associated majorizing cdf is

$$G(x) = \int_0^x g(s)\, ds = \frac{2}{w} \int_0^x \left(1 - \frac{s}{w}\right) ds = \cdots = \frac{2x}{w}\left(1 - \frac{x}{2w}\right) \qquad 0 < x < w.$$

The attractive feature of this triangular majorizing pdf is that the associated cdf is quadratic and so the idf can be determined from the quadratic formula. That is, we can write $u = G(x)$ and solve for u to find that the idf of the majorizing pdf is

$$x = G^{-1}(u) = w\left(1 - \sqrt{1 - u}\right) \qquad 0 < u < 1.$$

The resulting acceptance-rejection algorithm is

```
do {
   u = Random();
   x = w * (1.0 - sqrt(1.0 - u));
   v = Random();
} while (h * (1.0 - x / w) * v > f(x));
return x;
```

Example 7.6.2   As an alternate to the triangular majorizing pdf in Example 7.6.1, suppose that we choose an *Exponential*(a) majorizing pdf

$$g(x) = \frac{1}{a} \exp(-x/a) \qquad 0 < x < \infty,$$

where the parameters (a, c) are selected so that $f(x) \le c\, g(x)$ for all $x > 0$. Then

$$G^{-1}(u) = -a \ln(1 - u) \qquad 0 < u < 1$$

and the resulting acceptance-rejection algorithm is

```
do {
   x = Exponential(a);
   v = Random();
} while ((c / a) * exp(-x / a) * v > f(x));
return x;
```

Relative to Examples 7.6.2 and 7.6.3, the choice of whether to use the triangle or exponential majorizing pdf is determined by the shape of $f(\cdot)$ and the resulting value of c. By necessity, c is always greater than 1; the closer c is to 1, the better, since the expected number of passes through the `do-while` loop (rejections) is smaller. For more insight, see Exercise 7.6.1.

7.6.2 GAMMA RANDOM VARIATES

We now use the acceptance-rejection technique to develop an efficient algorithm for generating $Gamma(a, b)$ random variates. We begin with the following definition.

Definition 7.6.2 The continuous random variable X is $Gamma(a, b)$ if and only if

- the real-valued shape parameter a satisfies $a > 0$
- the real-valued scale parameter b satisfies $b > 0$
- the possible values of X are $\mathcal{X} = \{x | x > 0\}$
- the pdf of X is

$$f(x) = \frac{1}{b\Gamma(a)}(x/b)^{a-1}\exp(-x/b) \qquad x > 0$$

- the cdf of X is an incomplete gamma function

$$F(x) = P(a, x/b) \qquad x > 0$$

- the mean of X is
$$\mu = ab$$

- the standard deviation of X is
$$\sigma = b\sqrt{a}.$$

As shown in Figure 7.6.3, a $Gamma(a, b)$ random variable is an important generalization of three common continuous random variables.
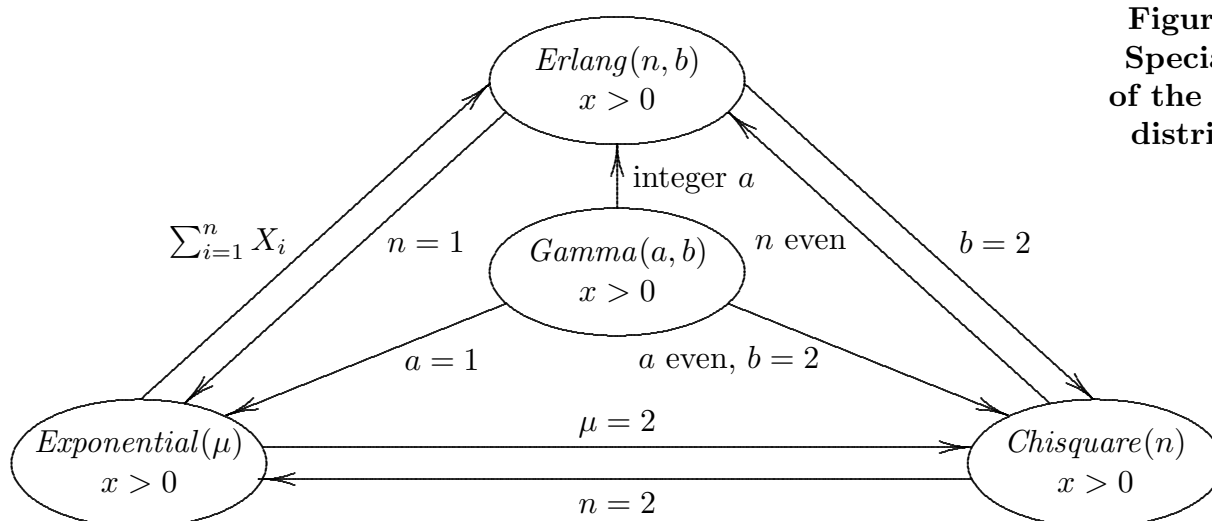
**Figure 7.6.3.
Special cases
of the gamma
distribution.**

In specific,

- A $Gamma(1, b)$ random variable is an $Exponential(b)$ random variable.

- If n is a positive integer, then a $Gamma(n, b)$ random variable is an $Erlang(n, b)$ random variable.

- If n is a positive integer, then a $Gamma(n/2,\ 2)$ random variable is a $Chisquare(n)$ random variable.

Except for special cases (in particular, $a = 1$), because the cdf of a $Gamma(a, b)$ random variable is an incomplete gamma function, there is no hope of determining a closed-form equation for the idf $F^{-1}(\cdot)$. Therefore, the inverse transformation method is not directly applicable. However, as demonstrated in the remainder of this section, the acceptance-rejection technique is applicable. The reason for this is based, in part, on the following theorem, which effectively reduces the problem of generating a two-parameter gamma random variate to generating a one-parameter gamma random variate. The proof is left as an exercise.

Theorem 7.6.1   The random variable X is $Gamma(a, 1)$ if and only if the random variable $Y = bX$ is $Gamma(a, b)$, for all $a > 0$ and $b > 0$.

The significance of Theorem 7.6.1 is that it is sufficient to develop an efficient algorithm to generate a $Gamma(a, 1)$ random variate; a $Gamma(a, b)$ random variate can then be generated via multiplication by b. Therefore, in the discussion that follows we assume that the pdf to be majorized is the $Gamma(a, 1)$ pdf

$$f(x) = \frac{1}{\Gamma(a)} x^{a-1} \exp(-x) \qquad\qquad x > 0.$$

As illustrated in Figure 7.6.4, the x^{a-1} term in $f(x)$ causes the shape of the pdf to be fundamentally different in the two cases $a > 1$ and $a < 1$. The pdf for $a < 1$ has a vertical asymptote at $x = 0$. Because of this, it should be no surprise to find that two different majorizing pdfs are required, one for each case.

Figure 7.6.4.
$Gamma(2.0, 1)$
and
$Gamma(0.5, 1)$
pdfs.
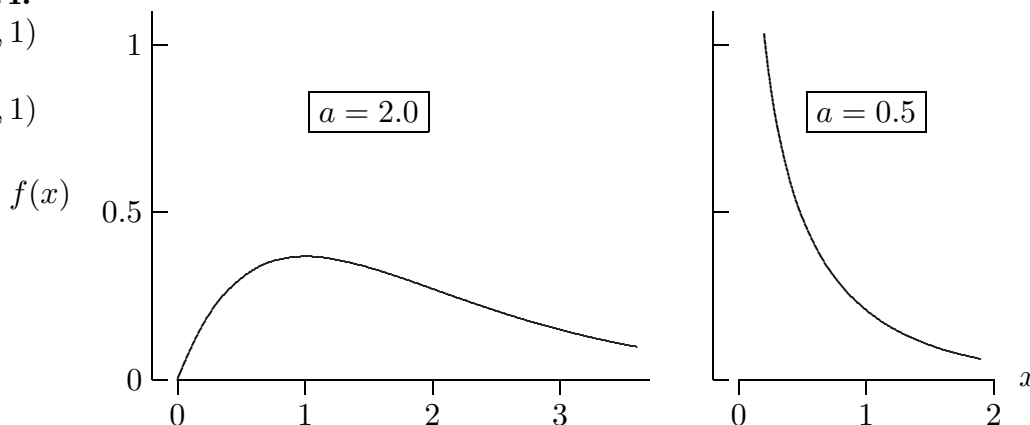
Cheng's Algorithm

We begin with the more common case, $a > 1$, and develop a majorizing pdf using an approach developed by Cheng (1977). The majorizing cdf that Cheng used is

$$G(x) = \frac{x^t}{x^t + a^t} \qquad x > 0$$

which has the associated pdf

$$g(x) = \frac{dG(x)}{dx} = \frac{ta^t x^{t-1}}{(x^t + a^t)^2} \qquad x > 0$$

and idf

$$G^{-1}(u) = a \left(\frac{u}{1-u} \right)^{1/t} \qquad 0 < u < 1.$$

In these three equations, the real-valued parameter t is positive but otherwise arbitrary.

Note that $G(x)$ is, in fact, a valid cdf for a positive, continuous random variable. That is, $G(x)$ is a continuous function that increases monotonically from a value of 0 at $x = 0$ to a value of 1 as $x \to \infty$. This cdf was chosen, in part, so that the idf could be easily evaluated.

The key to Cheng's approach is a clever choice of t and the following theorem. The proof is left as a (challenging) exercise.

Theorem 7.6.2 If $f(x)$ is the pdf of a $Gamma(a, 1)$ random variable, if $g(x)$ is the pdf in Definition 7.6.3, and if $t = \sqrt{2a - 1}$ with $a > 1$, then for all $x > 0$

$$\frac{f(x)}{g(x)} = \frac{x^{a-t}(x^t + a^t)^2 \exp(-x)}{ta^t \Gamma(a)} \leq \frac{f(a)}{g(a)} = \frac{4a^a \exp(-a)}{t\Gamma(a)}$$

That is, the largest possible value of the ratio $f(x)/g(x)$ occurs at $x = a$.

Note that if $t = \sqrt{2a - 1}$ with $a > 1$ then $1 < t < a$. In addition, if

$$c = \frac{f(a)}{g(a)} = \frac{4a^a \exp(-a)}{\sqrt{2a - 1}\,\Gamma(a)}$$

then it can be verified numerically that we have the following table

a :	1	1.5	2	3	5	10	20	50	∞
c :	1.47	1.31	1.25	1.20	1.17	1.15	1.14	1.13	1.13

Because c is the largest possible value of $f(x)/g(x)$ it follows that

$$f(x) \leq c\, g(x) \qquad x > 0.$$

The fact that c is only slightly greater than 1.0 for all $a > 1$ is important because it means that $g(x)$ is a good majorizing pdf (i.e., few rejections) for $f(x)$ for all values of of $a > 1$.*

---

* Cheng's algorithm becomes increasingly more efficient as $a \to \infty$.

If $f(x)$ is the pdf of a $Gamma(a, 1)$ random variable and $g(x)$ is the pdf in Definition 7.6.3, then Cheng's acceptance-rejection algorithm for generating a $Gamma(a, 1)$ random variate is

```
c = f(a) / g(a);
t = sqrt(2.0 * a - 1.0);
do {
   u = Random();
   s = log(u / (1.0 - u));
   x = a * exp(s / t);
   v = Random();
} while (c * g(x) * v > f(x));
return x;
```

Efficiency Improvement

Most acceptance-rejection methods can be made more efficient by analyzing the computational details of the implementation in an attempt to eliminate unnecessary and redundant calculations. The following example is an illustration of this.

Example 7.6.3   The $c\,g(x)\,v > f(x)$ test at the bottom of the `do-while` loop in Cheng's $Gamma(a, 1)$ variate generation algorithm can be simplified considerably by observing that if

$$x = G^{-1}(u) = a \left(\frac{u}{1 - u} \right)^{1/t}$$

then

$$x^t = \frac{a^t u}{1 - u} \qquad \text{and} \qquad x^t + a^t = \frac{a^t}{1 - u}.$$

The inequality test can then be written as

$$c\,g(x)\,v > f(x) \iff \left(\frac{4a^a \exp(-a)}{t\Gamma(a)} \right) \left(\frac{ta^t x^{t-1}}{(x^t + a^t)^2} \right) v > \frac{1}{\Gamma(a)} x^{a-1} \exp(-x)$$

$$\iff 4va^a a^t x^t \exp(-a) > x^a (x^t + a^t)^2 \exp(-x)$$

$$\iff 4va^a a^t \left(\frac{a^t u}{1 - u} \right) \exp(-a) > a^a \left(\frac{u}{1 - u} \right)^{a/t} \left(\frac{a^t}{(1 - u)} \right)^2 \exp(-x)$$

$$\iff 4vu(1 - u) > \left(\frac{u}{1 - u} \right)^{a/t} \exp(a - x).$$

All this symbol manipulation yields the following simple yet efficient acceptance-rejection algorithm. The correctness of this algorithm can be verified (for particular values of $a > 1$ and b) by generating, say 100 000, values and comparing the resulting continuous data histogram or empirical cdf with the corresponding $Gamma(a, b)$ pdf or cdf, respectively.

Algorithm 7.6.2   If $a > 1$, then for any $b > 0$ this algorithm generates a $Gamma(a, b)$ random variate without calculating c directly

```
t = sqrt(2.0 * a - 1.0);
do {
   u = Random();
   s = log(u / (1.0 - u));
   x = a * exp(s / t);
   v = Random();
} while (4.0 * v * u * (1.0 - u) > exp(a - x + a * s / t));
x = b * x;
return x;
```

Ahrens and Dieter's Algorithm

We now turn to the case $0 < a \leq 1$ and develop a majorizing pdf using an approach developed by Ahrens and Dieter (1974). The development begins with the observation that if $a \leq 1$ and if we define

$$c\,g(x) = \begin{cases} x^{a-1}/\Gamma(a) & 0 < x \leq 1 \\ \exp(-x)/\Gamma(a) & 1 < x < \infty \end{cases}$$

then by inspection

$$f(x) = \frac{x^{a-1}\exp(-x)}{\Gamma(a)} \leq c\,g(x) \qquad 0 < x < \infty.$$

To determine the constant c write

$$c = c\int_0^\infty g(x)\,dx = \int_0^1 \frac{x^{a-1}}{\Gamma(a)}\,dx + \int_1^\infty \frac{\exp(-x)}{\Gamma(a)}\,dx = \cdots = \frac{1}{t\,\Gamma(a+1)}$$

where the parameter $t < 1$ is

$$t = \frac{e}{e+a} \qquad (e = \exp(1) = 2.71828\ldots)$$

It can be verified numerically that $c > 1$ for all $0 < a \leq 1$ and, indeed, we have the following representative table

a	:	0	1/32	1/16	1/8	1/4	1/2	1
c	:	1.00	1.03	1.06	1.11	1.20	1.34	1.37

Therefore, the majorizing function is

$$g(x) = \begin{cases} atx^{a-1} & 0 < x \leq 1 \\ at\exp(-x) & 1 < x < \infty \end{cases}$$

which has the associated cdf

$$G(x) = \begin{cases} tx^a & 0 < x \leq 1 \\ 1 - at\exp(-x) & 1 < x < \infty. \end{cases}$$

The idf can be expressed in closed-form as

$$G^{-1}(u) = \begin{cases} (u/t)^{1/a} & 0 < u \le t \\ -\ln\left((1-u)/at\right) & t < u < 1. \end{cases}$$

As in Cheng's algorithm, the choice of majorizing pdf is dictated, in part, by the simplicity of $G^{-1}(u)$. Moreover, because c is uniformly close to 1 for all $0 < a \le 1$, Ahrens and Dieter's algorithm is efficient, becoming more so as $a \to 0$.

Finally, we observe that the $c\,g(x)v > f(x)$ inequality in Ahrens and Dieter's algorithm can be written (depending on whether $x < 1$ or $x > 1$) as

$$c\,g(x)\,v > f(x) \iff \begin{cases} v > \exp(-x) & 0 < x \le 1 \\ v > x^{a-1} & 1 < x < \infty, \end{cases}$$

which is summarized by the following algorithm.

Algorithm 7.6.3   If $0 < a \le 1$ then for any $b > 0$ this algorithm generates a $Gamma(a, b)$ random variate.

```
t = e / (e + a);                          /* e = exp(1) ≅ 2.718281828459··· */
do {
   u = Random();
   if (u ≤ t) {                           /* 0 < x ≤ 1 */
     x = exp(log(u / t) / a);
     s = exp(-x);
   }
   else {                                 /* 1 < x < ∞ */
     x = -log((1.0 - u) / (a * t));
     s = exp((a - 1.0) * log(x));
   }
   v = Random();
} while (v > s);
x = b * x;
return x;
```

This completes our discussion of the algorithms for generating a $Gamma(a, b)$ random variate. The problem was initially reduced to generating generating a $Gamma(a, 1)$ random variate. If $a = 1$ (the exponential case), then an $Exponential(1)$ variate is generated. If $a > 1$, then Cheng's Algorithm (Algorithm 7.6.2) is executed. If $a < 1$, then Ahrens and Dieter's Algorithm (Algorithm 7.6.3) is executed. Finally, in all three cases, the $Gamma(a, 1)$ random variate is multiplied by b to produce a $Gamma(a, b)$ random variate. We end this section with a review of the performance of the two acceptance-rejection algorithms.

Performance

Example 7.6.4   Algorithms 7.6.2 and 7.6.3 were implemented and compared with the corresponding generators in the library `rvgs`. The times (in seconds) required to generate 1000 random observations are as illustrated for selected values of a

a :	0.5	1.0	2.0	4.0	8.0	16.0	32.0	64.0
algorithms 7.6.2 and 7.6.3 :	2.8	2.7	2.8	2.7	2.6	2.6	2.5	2.5
library `rvgs` :	1.5	0.7	1.6	3.2	6.2	12.3	24.5	48.8

The second row of the table corresponds to Algorithms 7.6.2 and 7.6.3. As expected, the times are essentially independent of a. The third row used functions from the library `rvgs` as follows: for $a = 0.5$, *Chisquare*(1); for $a = 1.0$, *Exponential*(1.0); for all other values of a, *Erlang*($a, 1.0$). Because the algorithm used in `rvgs` to generate an *Erlang*(a, b) random variate is $O(a)$, it should be avoided if a is larger than, say, 4; instead use Algorithm 7.6.2.

Further Reading

This section has provided an elementary introduction to the acceptance-rejection technique for generating random variates, along with a fairly sophisticated application (i.e., generating a *Gamma*(a, b) random variate). We recommend Devroye's (1986) text for a more thorough treatment.

7.6.3   EXERCISES

Exercise 7.6.1   Let the discrete random variable R with possible value $r = 0, 1, 2, \ldots$ count the number of rejections (per return) by Algorithm 7.6.1. (a) Find the pdf of R. (b) Prove that $E[R] = c - 1$.

Exercise 7.6.2   Work through the details of Examples 7.6.1 and 7.6.2. How do you know to choose the '$-$' sign when using the quadratic equation to determine $G^{-1}(u)$?

Exercise 7.6.3   Prove Theorem 7.6.1.

Exercise 7.6.4[a]   Prove Theorem 7.6.2. Hint: Consider the first derivative of the ratio $f(x)/g(x)$ and verify that this first derivative is zero at some point $x > 0$ if and only if $s(x) = 0$ where

$$s(x) = (a - x - t)a^t + (a - x + t)x^t \qquad x > 0.$$

Then show that $s(x) = 0$ if and only if $x = a$.

Exercise 7.6.5   Implement the *Gamma*(a, b) generator defined by Algorithms 7.6.2 and 7.6.3. Present convincing numerical evidence that your implementation is correct in the three cases $(a, b) = (0.5, 1.0)$, $(2.0, 1.0)$, and $(10.0, 1.5)$.

CHAPTER 8

OUTPUT ANALYSIS

Sections

This chapter contains a more thorough presentation of techniques that can be applied to the output produced by a Monte Carlo or discrete-event simulation. In the case of discrete-event simulation models, these techniques are complicated by the presence of serial (auto-) correlation — which means that the standard statistical techniques which are often based on the assumption of independence must be modified to account for the autocorrelation.

Section 8.1 contains a graphical demonstation of the fundamentally important central limit theorem. It also contains an introduction to *interval estimation*, or *confidence intervals* as they are more commonly known. These intervals give an indication of the accuracy or precision associated with a *point estimate* of a performance measure (such as a mean, variance, or fractile). Section 8.2 applies the concepts associated with interval estimation to Monte Carlo simulation. The purpose is to augment the point estimate with an interval which contains the true value of the parameter of interest with high "confidence". Section 8.3 introduces the concept of finite-horizon and infinite-horizon statistics as two classifications of statistics associated with a system. The M/G/1 queue is used to illustrate the difference between the two types of statistics. Section 8.4 introduces "batch means", which is a popular technique for reducing the effect of autocorrelation when analyzing simulation output. Section 8.5 provides insight into the mathematics of traditional queueing theory as applied to the steady-state statistics associated with the single-server service node.

This chapter is concerned with simulation applications based on the fundamentally important *central limit theorem*, and its relation to interval estimation. That is, because discrete-event simulation models produce statistical estimates (means, relative frequencies, etc.) that are inherently uncertain, some technique must be used to quantify that uncertainty; many such techniques are based on the central limit theorem.

8.1.1 CENTRAL LIMIT THEOREM

Although a rigorous proof of the central limit theorem is beyond the scope of this book (see Hogg, McKean, and Craig, 2005, for a proof), we can use Monte Carlo simulation to demonstrate the theorem's correctness and, by so doing, gain insight. We begin with a statement of the theorem.

Theorem 8.1.1 If X_1, X_2, ..., X_n are *iid* random variables with a common mean finite μ and common finite standard deviation σ and if $\bar{X}$ is the *mean* of these random variables

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

then the mean of $\bar{X}$ is μ and the standard deviation of $\bar{X}$ is $\sigma/\sqrt{n}$. Furthermore, $\bar{X}$ approaches a *Normal*$(\mu, \sigma/\sqrt{n})$ random variable as $n \to \infty$.

Sample Mean Distribution

Example 8.1.1 Suppose we choose one of the random variate generators in the library `rvgs` — it does not matter which one — and use it to generate a sequence of random variate samples, each of fixed sample size $n > 1$. As each of the n-point samples indexed $j = 1, 2, \ldots$ is generated, the corresponding sample mean $\bar{x}_j$ and sample standard deviation s_j can be calculated using Algorithm 4.1.1 (Welford's Algorithm), as illustrated

$$\underbrace{x_1, x_2, \ldots, x_n}_{\bar{x}_1, s_1}, \underbrace{x_{n+1}, x_{n+2}, \ldots, x_{2n}}_{\bar{x}_2, s_2}, \underbrace{x_{2n+1}, x_{2n+2}, \ldots, x_{3n}}_{\bar{x}_3, s_3}, x_{3n+1}, \ldots$$

A continuous-data histogram can then be created using program `cdh`, as illustrated in Figure 8.1.1.
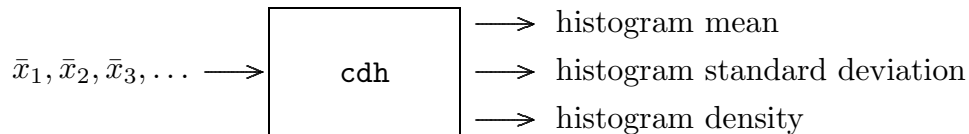
Figure 8.1.1. Calculating a histogram of the sample means.

Let μ and σ denote the theoretical (population) mean and standard deviation of the random variates. Consistent with Theorem 8.1.1, the resulting *sample mean histogram* will have the following three properties:

- Independent of n, the histogram mean will be approximately μ.

- Independent of n, the histogram standard deviation will be approximately $\sigma/\sqrt{n}$.

- The histogram approaches the pdf of a $Normal(\mu, \sigma/\sqrt{n})$ random variable as $n \to \infty$.

Example 8.1.2 To illustrate Example 8.1.1 in more detail, the library `rvgs` was used to generate 10 000 $Exponential(\mu)$ random variate samples of size $n = 9$ and 10 000 samples of size $n = 36$ (using the default `rngs` stream and initial seed in both cases). These histograms are shown in Figure 8.1.2.
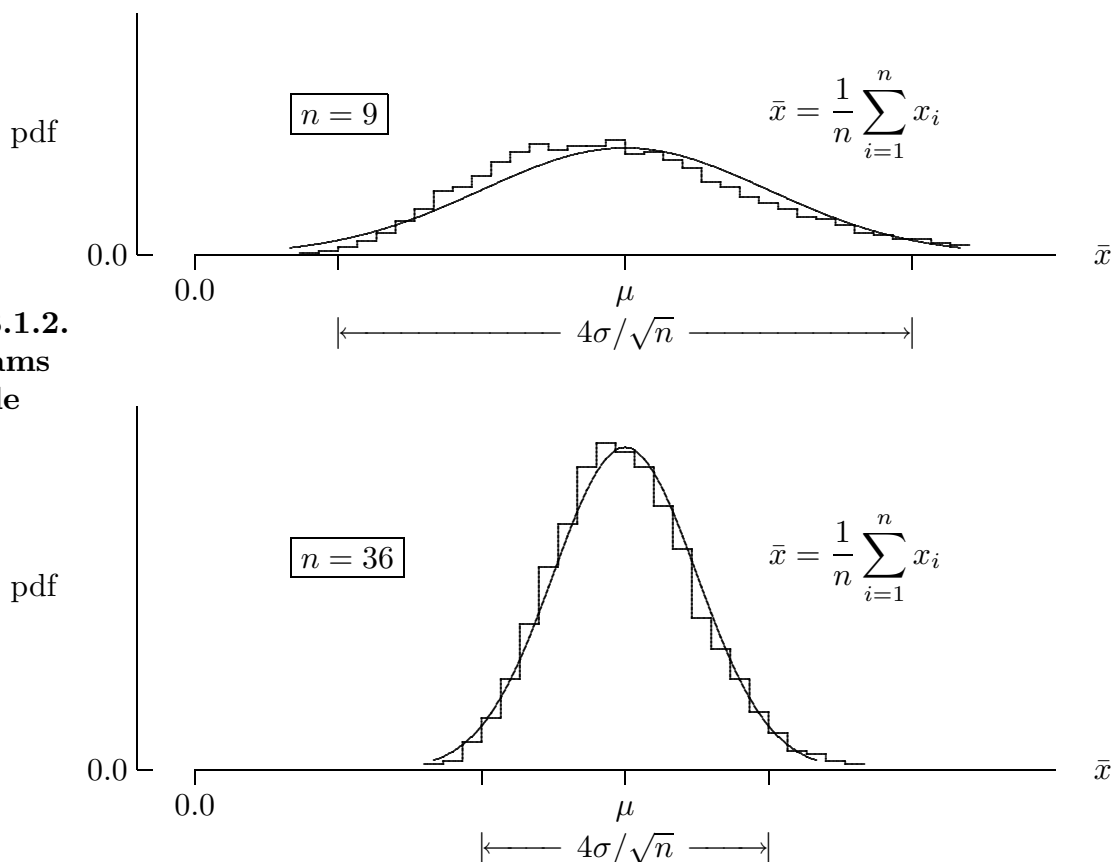
Figure 8.1.2. Histograms of sample means.

Consistent with the central limit theorem, for both cases the histogram mean and standard deviation are approximately μ and $\sigma/\sqrt{n}$ respectively. (Because $Exponential(\mu)$ random variate samples were used, the population standard deviation is $\sigma = \mu$ in both cases.) Moreover, the histogram density corresponding to the 36-point sample means (the stairstep curve) is matched almost exactly by the pdf of a $Normal(\mu, \sigma/\sqrt{n})$ random variable (the continuous curve). This agreement illustrates that, for $Exponential(\mu)$ samples, $n = 36$ is large enough for the sample mean to be approximately a $Normal(\mu, \sigma/\sqrt{n})$ random variate. Similarly, the histogram density corresponding to the 9-point sample means matches relatively well, but with a skew to the left. This skew is caused by the highly asymmetric shape of the $Exponential(\mu)$ pdf and the fact that, in this case, $n = 9$ is not a large enough sample — the sample mean histogram retains a vestige of the asymmetric population pdf.

As stated in Example 8.1.1 and illustrated in Example 8.1.2, the standard deviation of the sample mean histogram is $\sigma/\sqrt{n}$. Moreover, in the sense of Chebyshev's inequality, this standard deviation defines an *interval* within which essentially all the sample means lie. Because the existence of this interval is so important as the theoretical basis for the development of an interval estimation technique, for emphasis we provide the following two related summary points.

- Essentially all (about 95%) of the sample means are within an interval of width of $4\sigma/\sqrt{n}$ centered about μ.

- Because $\sigma/\sqrt{n} \to 0$ as $n \to \infty$, if n is large then essentially all of the sample means will be arbitrarily close to μ.

The random variate samples in Examples 8.1.1 and 8.1.2 are drawn from a population with a fixed pdf. As illustrated in Example 8.1.2, the accuracy of the $Normal(\mu, \sigma/\sqrt{n})$ pdf approximation of the distribution of $\bar{X}$ is dependent on the shape of this population pdf. If the samples are drawn from a population with a highly asymmetric pdf (like the $Exponential(\mu)$ pdf for example) then n may need to be as large as 30 or more for the $Normal(\mu, \sigma/\sqrt{n})$ pdf to be a good fit to the histogram density. If instead the samples are drawn from a population with a pdf symmetric about the mean (like the $Uniform(a, b)$ pdf for example) then values of n as small as 10 or less may produce a satisfactory fit.

Standardized Sample Mean Distribution

Example 8.1.3 As an extension of Example 8.1.1, there is another way to characterize the central limit theorem. In addition to computing the sample means $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$ we can standardize each sample mean by subtracting μ and dividing the result by $\sigma/\sqrt{n}$ to form the *standardized* sample means $z_1, z_2, z_3, \ldots$ defined by

$$z_j = \frac{\bar{x}_j - \mu}{\sigma/\sqrt{n}} \qquad j = 1, 2, 3, \ldots$$

We can then generate a continuous-data histogram for the standardized sample means by using program `cdh`, as illustrated in Figure 8.1.3.

$z_1, z_2, z_3, \ldots \longrightarrow$ `cdh` $\longrightarrow$ histogram mean
$\longrightarrow$ histogram standard deviation
$\longrightarrow$ histogram density
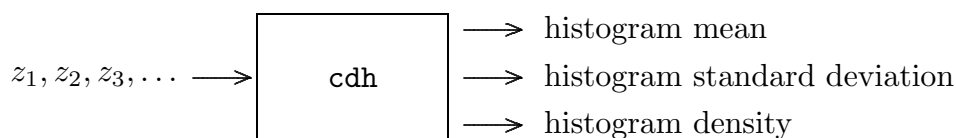
Figure 8.1.3. Calculating a histogram of the standardized sample means.

The resulting *standardized sample mean histogram* will have the following three properties.

- Independent of n, the histogram mean will be approximately 0.

- Independent of n, the histogram standard deviation will be approximately 1.

- Provided n is sufficiently large, the histogram density will approximate the pdf of a $Normal(0, 1)$ random variable.

Example 8.1.4 To illustrate Example 8.1.3 in more detail, the sample means from Example 8.1.2 were standardized to form the two histograms illustrated in Figure 8.1.4.
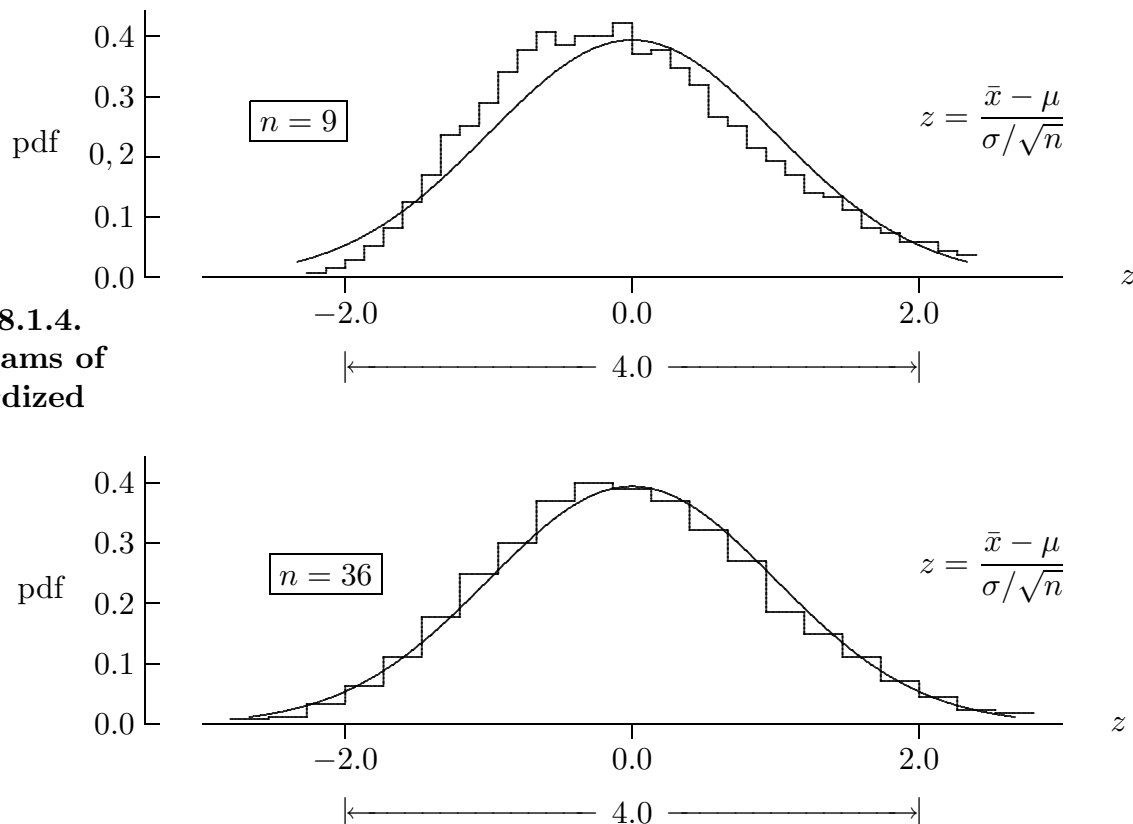
Figure 8.1.4.
Histograms of
standardized
sample
means.

Consistent with Example 8.1.3, in both cases the histogram mean and standard deviation are approximately 0.0 and 1.0 respectively. Moreover, the histogram density corresponding to the 36-point samples (the stairstep curve) matches the pdf of a *Normal*$(0, 1)$ random variable (the continuous curve) almost exactly. Similarly, the histogram density corresponding to the 9-point samples also matches, but not quite as well because it retains a vestige of the population's asymmetric pdf.

t-Statistic Distribution

The next issue to consider is the effect of replacing the *population* standard deviation σ with the *sample* standard deviation s_j in the equation

$$z_j = \frac{\bar{x}_j - \mu}{\sigma/\sqrt{n}} \qquad j = 1, 2, 3, \ldots$$

But first, a definition and some discussion.

Definition 8.1.1 Each sample mean $\bar{x}_j$ is a *point estimate* of μ, each sample variance s_j^2 is a *point estimate* of σ^2, and each sample standard deviation s_j is a *point estimate* of σ.

Relative to Definition 8.1.1 and the discussion in Section 4.1, the following points address the issue of *bias* as it relates to the sample mean and sample variance.

- The sample mean is an *unbiased* point estimate of μ. That is, the mean of the sample means $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, ... will converge to μ as the number of sample means is increased. Indeed, this is guaranteed by the central limit theorem.

- The sample variance is a *biased* point estimate of σ^2. Specifically, the mean of the sample variances s_1^2, s_2^2, s_3^2, ... will converge to $(n-1)\sigma^2/n$, not σ^2. To remove this $(n-1)/n$ bias it is conventional to multiply the sample variance by a *bias correction* factor $n/(n-1)$. The result is an *unbiased* point estimate of σ^2.*

The square root of the unbiased point estimate of σ^2 is a point estimate of σ. The corresponding point estimate of $\sigma/\sqrt{n}$ is then

$$\frac{\left(\sqrt{\dfrac{n}{n-1}}\right) s_j}{\sqrt{n}} = \frac{s_j}{\sqrt{n-1}}$$

The following two examples illustrate what changes in Examples 8.1.3 and 8.1.4 if $\sigma/\sqrt{n}$ is replaced with $s_j/\sqrt{n-1}$ in the definition of z_j.

Example 8.1.5 As an extension of Example 8.1.3, instead of computing z_j for each n-point sample we can calculate the t-statistic

$$t_j = \frac{\bar{x}_j - \mu}{s_j/\sqrt{n-1}} \qquad j = 1, 2, 3, \ldots$$

and then generate a continuous-data histogram using program `cdh`, as illustrated in Figure 8.1.5.

$t_1, t_2, t_3, \ldots \longrightarrow$ `cdh` $\longrightarrow$ histogram mean
$\longrightarrow$ histogram standard deviation
$\longrightarrow$ histogram density
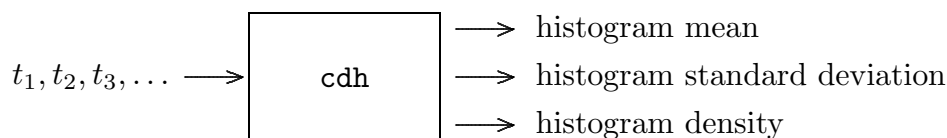
Figure 8.1.5. Calculating a histogram of sample t-statistics.

The resulting *t-statistic histogram* will have the following three properties.

- If $n > 2$, the histogram mean will be approximately 0.

- If $n > 3$, the histogram standard deviation will be approximately $\sqrt{(n-1)/(n-3)}$.

- Provided n is sufficiently large, the histogram density will approximate the pdf of a *Student*$(n-1)$ random variable.

---

* If the "divide by $n-1$" definition of the sample variance had been used in Section 4.1 then multiplication by the $n/(n-1)$ bias correction factor would not be necessary.

Example 8.1.6   To illustrate Example 8.1.5 in more detail, the sequence of sample means and sample standard deviations from Example 8.1.2 were processed as t-statistics to form the two histograms illustrated in Figure 8.1.6.
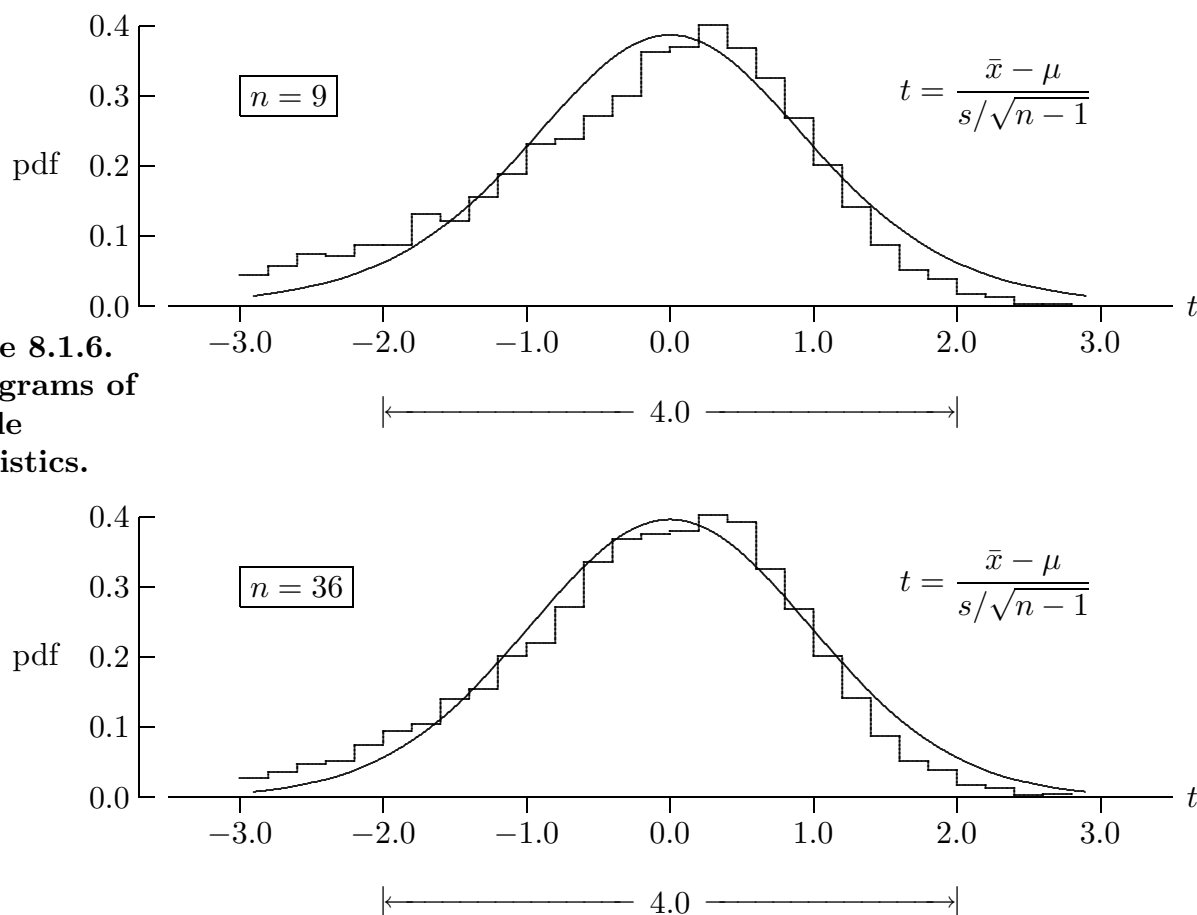
**Figure 8.1.6.
Histograms of
sample
t-statistics.**

Consistent with Example 8.1.5, in both cases the histogram mean is approximately 0.0 and the histogram standard deviation is approximately $\sqrt{(n-1)/(n-3)} \cong 1.0$. Moreover, the histogram density corresponding to the 36-point samples (the stairstep curve) matches the pdf of a *Student*(35) random variable (the continuous curve) relatively well. Similarly, the histogram density corresponding to the 9-point samples matches the pdf of a *Student*(8) random variable, but not quite as well.

8.1.2 INTERVAL ESTIMATION

The discussion to this point has been focused on the central limit theorem, leading to Examples 8.1.5 and 8.1.6 which provide an experimental justification for the following theorem. In turn, this theorem is the theoretical basis for the important idea of *interval estimation* which is the application focus of this chapter.

Theorem 8.1.2 If x_1, x_2, ..., x_n is an (independent) random sample from a "source" of data with *unknown* finite mean μ, if $\bar{x}$ and s are the mean and standard deviation of this sample, and if n is large, then it is approximately true that

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$

is a *Student*$(n-1)$ random variate.*

The significance of Theorem 8.1.2 is that it provides the justification for estimating an *interval* that is quite likely to contain the mean μ. That is, if T is a *Student*$(n-1)$ random variable and if α is a "confidence parameter" with $0.0 < \alpha < 1.0$ (typically $\alpha = 0.05$), then there exists a corresponding positive real number t^* such that

$$\Pr(-t^* \le T \le t^*) = 1 - \alpha$$

as illustrated in Figure 8.1.7.
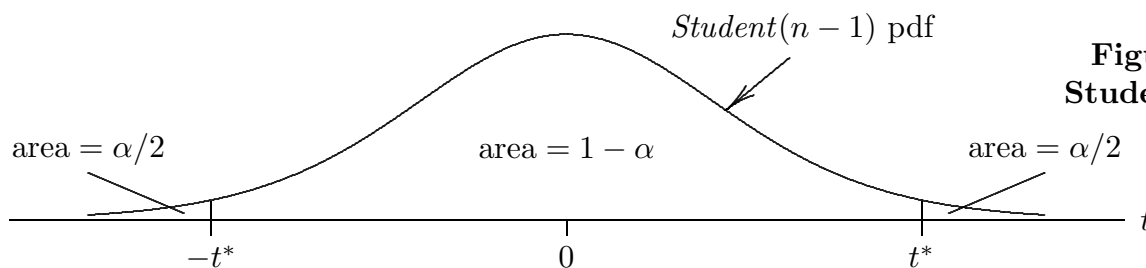
Figure 8.1.7.
Student$(n-1)$
pdf.

Suppose that μ is *unknown*. From Theorem 8.1.2, since t is a *Student*$(n-1)$ random variate the inequality

$$-t^* \le \frac{\bar{x} - \mu}{s/\sqrt{n-1}} \le t^*$$

will be (approximately) true with probability $1 - \alpha$. Multiplying by $s/\sqrt{n-1}$ yields

$$-\frac{t^* s}{\sqrt{n-1}} \le \bar{x} - \mu \le \frac{t^* s}{\sqrt{n-1}}.$$

Subtracting $\bar{x}$ and negating, the inequality

$$\bar{x} - \frac{t^* s}{\sqrt{n-1}} \le \mu \le \bar{x} + \frac{t^* s}{\sqrt{n-1}}$$

will be (approximately) true with probability $1 - \alpha$. This proves that there is an interval within which μ lies with probability $1 - \alpha$, as summarized by the following theorem.

---

* Recall from Figure 7.4.9 that as $n \to \infty$ a *Student*$(n-1)$ random variate becomes statistically indistinguishable from a *Normal*$(0, 1)$ random variate. Although Theorem 8.1.2 is an exact result if the random sample is drawn from a *Normal*(μ, σ) population, this theorem is stated here only as an approximate result.

Theorem 8.1.3   If x_1, x_2, ..., x_n is an (independent) random sample from a "source" of data with finite *unknown* mean μ, if $\bar{x}$ and s are the mean and standard deviation of this sample, and if n is large, then given a confidence parameter α with $0.0 < \alpha < 1.0$ there exists an associated positive real number t^* such that

$$\Pr\left(\bar{x} - \frac{t^*s}{\sqrt{n-1}} \leq \mu \leq \bar{x} + \frac{t^*s}{\sqrt{n-1}}\right) \cong 1 - \alpha.$$

Example 8.1.7   From Theorem 8.1.3, if $\alpha = 0.05$ then we are 95% confident that μ lies *somewhere* in the interval defined by the endpoints

$$\bar{x} - \frac{t^*s}{\sqrt{n-1}} \qquad \text{and} \qquad \bar{x} + \frac{t^*s}{\sqrt{n-1}}.$$

For a fixed sample size n and level of confidence $1 - \alpha$ we can use the idf capability in the library `rvms` to determine the critical value as $t^* = \mathtt{idfStudent}(n-1, 1-\alpha/2)$. For example, if $n = 30$ and $\alpha = 0.05$ then the value of t^* is `idfStudent(29, 0.975)` which is approximately 2.045.

Definition 8.1.2   The interval defined by the two endpoints

$$\bar{x} \pm \frac{t^*s}{\sqrt{n-1}}$$

is a $(1 - \alpha) \times 100\%$ confidence *interval estimate* for μ. The parameter $1 - \alpha$ is the *level of confidence* associated with this interval estimate and t^* is the *critical value* of t.

Algorithm 8.1.1   The following steps summarize the process of calculating an *interval estimate* for the unknown mean μ of that much larger population from which a random sample x_1, x_2, x_3, ..., x_n of size $n > 1$ was drawn.

- Pick a level of confidence $1 - \alpha$ (typically $\alpha = 0.05$, but $\alpha = 0.20, 0.10$ and 0.01 are also commonly used).

- Calculate the sample mean $\bar{x}$ and the sample standard deviation s (Definition 4.1.1).

- Calculate the critical value $t^* = \mathtt{idfStudent}(n-1, 1-\alpha/2)$.

- Calculate the interval endpoints

$$\bar{x} \pm \frac{t^*s}{\sqrt{n-1}}.$$

If n is sufficiently large you are then $(1 - \alpha) \times 100\%$ confident that μ lies somewhere in the interval defined by these two endpoints. The associated point estimate of μ is the mid-point of this interval (which is $\bar{x}$).

Example 8.1.8   The data

$$1.051 \quad 6.438 \quad 2.646 \quad 0.805 \quad 1.505 \quad 0.546 \quad 2.281 \quad 2.822 \quad 0.414 \quad 1.307$$

is a random sample of size $n = 10$ from a population with unknown mean μ. The sample mean and standard deviation are $\bar{x} = 1.982$ and $s = 1.690$.

- To calculate a 90% confidence interval estimate we first evaluate

$$t^* = \texttt{idfStudent}(9,\, 0.95) \cong 1.833$$

  and then construct the interval estimate as

$$1.982 \pm (1.833)\frac{1.690}{\sqrt{9}} = 1.982 \pm 1.032.$$

  Therefore, we are approximately 90% confident that the (unknown) value of μ is somewhere between 0.950 and 3.014. This is typically reported as the confidence interval $0.950 \leq \mu \leq 3.014$.

- Similarly, to calculate a 95% confidence interval estimate

$$t^* = \texttt{idfStudent}(9,\, 0.975) \cong 2.262$$

  and thus the interval is

$$1.982 \pm (2.262)\frac{1.690}{\sqrt{9}} = 1.982 \pm 1.274.$$

  Therefore, we are approximately 95% confident that the (unknown) value of μ is somewhere in the interval $0.708 \leq \mu \leq 3.256$.

- And, to calculate a 99% confidence interval estimate

$$t^* = \texttt{idfStudent}(9,\, 0.995) \cong 3.250$$

  and thus the interval is

$$1.982 \pm (3.250)\frac{1.690}{\sqrt{9}} = 1.982 \pm 1.832.$$

  Therefore, we are approximately 99% confident that the (unknown) value of μ is somewhere in the interval $0.150 \leq \mu \leq 3.814$.

In each case, the modifier "approximately" is used because n is not large. If we had knowledge that the 10 data values were drawn from a $Normal(\mu,\, \sigma)$ population distribution, then the "approximately" modifier could be dropped and the confidence interval would be known as an "exact" confidence interval.

Tradeoff — Confidence Versus Sample Size

Example 8.1.8 illustrates the classic interval estimation tradeoff. For a given sample size, more confidence can be achieved *only* at the expense of a larger interval and a smaller interval can be achieved *only* at the expense of less confidence. The only way to make the interval smaller without lessening the level of confidence is to increase the sample size — collect *more* data!

Collecting more data in simulation applications is usually easy to do. The question then is how much more data is enough? That is, how large should n be in Algorithm 8.1.1 to achieve an interval estimate $\bar{x} \pm w$ for a *user-specified* interval width w? The answer to this question is based on using Algorithm 4.1.1 in conjunction with Algorithm 8.1.1 to iteratively collect data until a specified interval width is achieved. This process is simplified by the observation that if n is large then the value of t^* is essentially independent of n, as illustrated in Figure 8.1.8.

Figure 8.1.8. Sample size considerations.

$$t^* = \texttt{idfStudent}(n - 1, 1 - \alpha/2)$$

$$2.576 \ (\alpha = 0.01)$$
$$1.960 \ (\alpha = 0.05)$$
$$1.645 \ (\alpha = 0.10)$$
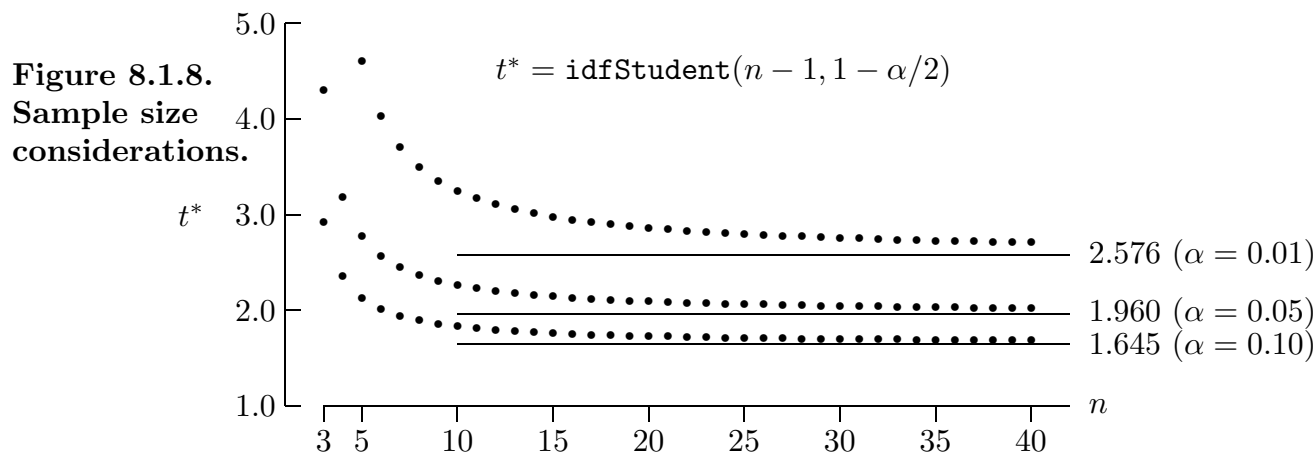
The asymptotic (large n) value of t^* indicated is

$$t^*_\infty = \lim_{n \to \infty} \texttt{idfStudent}(n - 1, 1 - \alpha/2) = \texttt{idfNormal}(0.0, 1.0, 1 - \alpha/2).$$

Unless α is very close to 0.0, if $n > 40$ then the asymptotic value t^*_∞ can be used. In particular, if $n > 40$ and we wish to construct a 95% confidence interval estimate for μ then $t^*_\infty = 1.960$ can be used for t^* in Algorithm 8.1.1.

Example 8.1.9 Given a reasonable guess for s and a user-specified *half-width* parameter w, if t^*_∞ is used in place of t^* the value of n can be determined by solving the equation

$$w = \frac{t^*_\infty s}{\sqrt{n - 1}}$$

for n. The result is

$$n = \left\lfloor \left(\frac{t^*_\infty s}{w} \right)^2 \right\rfloor + 1$$

provided $n > 40$ (which will surely be the case if w/s is small). For example, if $s = 3.0$, $\alpha = 0.05$, and $w = 0.5$, then a sample size of $n = 139$ should be used.

Example 8.1.10   Similarly, if a reasonable guess for s is not available, then w can be specified as a proportion of s thereby eliminating s from the previous equation. For example, if w is 10% of s (so that $w/s = 0.1$) and if 95% confidence is desired, then a value of $n = 385$ should be used to estimate μ to within $\pm w$.

Program estimate

Based on Algorithm 8.1.1, program `estimate` automates the interval estimation process. A typical application of this program is to estimate the value of an unknown population mean μ by using n replications to generate an *independent* random variate sample $x_1, x_2, \ldots, x_n$. Let `Generate()` represent a discrete-event or Monte Carlo simulation program configured as a function that returns a random variate output x. If n is specified then the algorithm

```
for (i = 1; i <= n; i++)
    x_i = Generate();
return x_1, x_2, ..., x_n;
```

generates a random variate sample. Given a user-defined level of confidence $1-\alpha$, program `estimate` can then be used with x_1, x_2, ..., x_n, to compute an interval estimate for μ.

As an alternative to using program `estimate` directly, if the desired interval half-width w is specified instead of the sample size n then repeated calls to `Generate()` can be used in conjunction with Algorithm 4.1.1 to iteratively collect data until the $2w$ interval width is achieved. This process is summarized by the following algorithm which uses the t^* asymptotic approximation discussed previously to eliminate needless calls to `idfStudent`$(n - 1, 1 - \alpha/2)$ inside the `while` loop.

Algorithm 8.1.2   Given an interval half-width w and level of confidence $1 - \alpha$, this algorithm computes the (large sample size) interval estimate for μ of the form $\bar{x} \pm w$.

```
t = idfNormal(0.0, 1.0, 1 - α/2);                    /* this is t*_∞ */
x = Generate();
n = 1;
v = 0.0;
x̄ = x;
while ((n < 40) or (t * sqrt(v / n) > w * sqrt(n - 1)) {
    x = Generate();
    n++
    d = x - x̄;
    v = v + d * d * (n - 1) / n;
    x̄ = x̄ + d / n;
}
return n, x̄;
```

Welford's Algorithm (Algorithm 4.1.1) is contained in Algorithm 9.1.2 so that only one pass is made through the data and the likelihood of overflow is minimized.

It is important to appreciate the need for sample *independence* in Algorithms 8.1.1 and 8.1.2. The problem here is that both of these algorithms can be used naively even if the sample is *not* independent. However, the resulting interval estimate is valid only if the sample is independent. If the sample is not independent then the interval "coverage" will be incorrect, perhaps dramatically so. For example, an interval alleged to be 95% confident may be actually only, say, 37% confident.

The Meaning of Confidence

One common mistake that occurs when novices begin working with confidence intervals is reflected in the following statements:

> "I made n runs of the simulation and created a 95% confidence interval for μ. The probability that the true mean μ is contained in my confidence interval is 0.95."

The problem with this statement is that the (unknown) true value of μ either *is* or *is not* in the confidence interval. A probability statement can be made, although it is painfully tedious:

> "I made n runs of the simulation and created a 95% confidence interval for μ. This particular interval may or may not contain μ. However, if I were to create many confidence intervals like this one many times over, then approximately 95% of these intervals would cover the true mean μ."

This discussion indicates why the phrase "confidence interval" is used rather than the phrase "probability interval."

Example 8.1.11 One hundred 95% confidence intervals associated with data drawn from a $Normal(6, 3)$ population with sample size $n = 9$ are shown in Figure 8.1.9, each displayed as a vertical line. A horizontal line is drawn at the population mean $\mu = 6$. Three of the confidence intervals "miss low" (e.g., the upper limit is less than $\mu = 6$) and two of the confidence intervals "miss high" (e.g., the lower limit is greater than $\mu = 6$). The lower and upper *limits* of these confidence intervals are random — whereas μ is fixed.

Figure 8.1.9. One hundred confidence intervals.
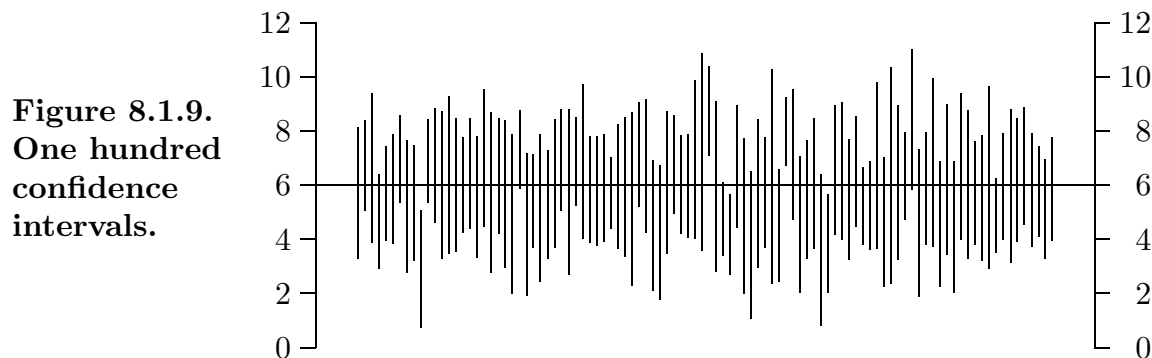
8.1.3 EXERCISES

Exercise 8.1.1   This exercise illustrates the central limit theorem. (a) Generate 100 000 *Uniform*(2, 6) random variate samples of size $n = 4$. For each sample calculate the sample mean and then generate a 20-bin continuous-data histogram of the 100 000 sample means. (Use 10 bins on each side of 4.0 and choose the other histogram parameters appropriately.) Plot the histogram and overlay the theoretical *Normal* pdf that should approximate it. (b) Repeat with $n = 9$. (c) Repeat with $n = 25$. (d) Comment.

Exercise 8.1.2   To illustrate that the central limit theorem is an approximation, prove that for an *iid* sequence of *Exponential*(μ) random variables X_1, X_2, ..., X_n the pdf of the $\bar{X}$ is

$$f(\bar{x}) = \frac{n}{\mu(n-1)!} \left(\frac{n\bar{x}}{\mu} \right)^{n-1} \exp(-n\bar{x}/\mu) \qquad \bar{x} > 0.$$

Exercise 8.1.3   (a) Generate 100 000 *Poisson*(10) random variate samples of size 4, for each sample calculate the variance, and then calculate the mean of these 100 000 sample variances. (b) Compare this mean with the theoretical variance of a *Poisson*(10) random variable. (c) Explain why there is a difference in these two variances.

Exercise 8.1.4   This exercise is a continuation of Exercise 8.1.1. (a) Generate 100 000 *Uniform*(2, 6) random variate samples of size $n = 4$. For each sample calculate both the sample mean $\bar{x}$, the standard deviation s, and

$$t = \frac{\bar{x} - 1}{s/\sqrt{n-1}}.$$

Then generate a 20-bin histogram of the 100 000 t's. (Use 10 bins on each side of 0.0 and choose the other histogram parameters appropriately.) Plot the histogram and overlay the theoretical *Student* pdf that should approximate it. (b) Repeat with $n = 9$. (c) Repeat with $n = 25$. (d) Comment.

Exercise 8.1.5   This exercise illustrates interval estimation for the mean. (a) Generate 100 000 *Exponential*(1) random variate samples of size $n = 4$ and for each sample calculate a 95% confidence interval estimate of the mean. Compute the relative frequency of times (out of 100 000) that the interval estimate includes 1.0. (b) Repeat with $n = 9$, 25, 49, 100, and 1000. (c) Comment on how the "coverage" depends on the sample size.

Exercise 8.1.6   Repeat Exercise 8.1.5 using *Uniform*(2, 6) random variates instead.

Exercise 8.1.7   Repeat Exercise 8.1.5 using *Geometric*(0.8) random variates instead.

Exercise 8.1.8   How does Exercise 8.1.6 relate to Exercises 8.1.1 and 8.1.4? Be specific.

Exercise 8.1.9[a]   Do Exercise 7.5.7, but calculate 95% confidence interval estimates for the instantaneous (snapshot) number in the node at each of the $\lambda(t)$ time knots.

8.2 Monte Carlo Estimation

One of the primary applications of parameter estimation is Monte Carlo simulation. That is, Monte Carlo simulation is based upon the use of many independent replications of the same static, stochastic system to generate an estimate of some parameter of interest. If, as in Section 2.3, this estimate is just a *point* estimate then there is no associated measure of precision. How close is the point estimate to the true (unknown) value of the parameter being estimated? If the Monte Carlo estimate is an *interval* estimate, however, then the precision of the estimate has been quantified by establishing an interval in which the true value of the parameter lies with high probability.

8.2.1 PROBABILITY ESTIMATION

Example 8.2.1 The program `craps` was run once with $n = 100$ replications and a `rngs` initial seed of 12345. There were 56 wins and 44 losses; the corresponding *point* estimate of the probability of winning is 0.56. Recall from Section 2.4.2 that the true (theoretical) probability of winning at Craps is $244/495 \cong 0.493$ and so this *point* estimate is not particularly accurate. However, by using *interval* estimation it can be shown that 56 wins in 100 tries corresponds to a 95% confidence *interval* estimate of 0.56 ± 0.10 (see Example 8.2.3, to follow). Therefore we are 95% confident that the true probability of winning is somewhere between 0.46 and 0.66. There are two important points in this case:

- the 0.56 ± 0.10 interval estimate is "correct" in the sense that it contains the true probability of winning;

- the interval is quite wide indicating that (many) more than 100 replications should have been used.

The general interval estimation technique developed in the previous section applies to the specific case of Monte Carlo simulation used to estimate a probability. That is, recall from Section 4.1 that if x_1, x_2, ..., x_n is a random sample of size n with the property that each x_i is either 0 (a "loss") or 1 (a "win") then the sample mean is

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{\text{the number of } 1's}{n}$$

and the sample standard deviation is

$$s = \sqrt{\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right) - \bar{x}^2} = \sqrt{\bar{x} - \bar{x}^2} = \sqrt{\bar{x}(1 - \bar{x})}.$$

From Algorithm 8.1.1, if n is sufficiently large the corresponding interval estimate for the (unknown) population mean is

$$\bar{x} \pm t^* \sqrt{\frac{\bar{x}(1 - \bar{x})}{n - 1}}$$

where the critical value $t^* = \texttt{idfStudent}(n - 1, 1 - \alpha/2)$ is determined by the sample size and the level of confidence $1 - \alpha$.

Example 8.2.1 not withstanding, when Monte Carlo simulation is used to estimate a probability, the sample size (the number of replications) is almost *always* large. Moreover, the conventional level of confidence is 95%. Consistent with the discussion in Section 8.1, if n is sufficiently large and if $\alpha = 0.05$ then

$$t^* = t_\infty^* = \texttt{idfNormal}(0.0, 1.0, 0.975) \cong 1.960 \cong 2.0.$$

In addition, if n is large, then the distinction between n and $n-1$ in the interval estimate equation is insignificant. Therefore, the approximate 95% confidence interval estimate can be written as $\bar{x} \pm 1.96\sqrt{\bar{x}(1-\bar{x})/n}$, or $\bar{x} - 1.96\sqrt{\bar{x}(1-\bar{x})/n} < p < \bar{x} + 1.96\sqrt{\bar{x}(1-\bar{x})/n}$.

Algorithm

If p denotes the (unknown) value of the probability to be estimated, then it is conventional to use the notation $\hat{p}$, rather than $\bar{x}$, to denote the point estimate of p. Using this notation, Monte Carlo probability interval estimation can be summarized by the following algorithm. [If p is very close to either 0.0 or 1.0, then more sophisticated approaches to interval estimation, such as those given in Hogg, McKean, and Craig (2005), are required.]

Algorithm 8.2.1 To estimate the unknown probability p of an event $\mathcal{A}$ by Monte Carlo simulation, replicate n times and let

$$\hat{p} = \frac{\text{the number of occurrences of event } \mathcal{A}}{n}$$

be the resulting point estimate of $p = \Pr(\mathcal{A})$. We are then 95% confident that p lies *somewhere* in the interval with endpoints

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Example 8.2.2 Program `craps` was run 200 times with $n = 100$. For each run a 95% confidence interval estimate was computed using Algorithm 8.2.1. Each of these 200 interval estimates are illustrated as vertical lines in Figure 8.2.1; the horizontal line corresponds to the true probability $p \cong 0.493$. (The left-most vertical line corresponds to the 0.56 ± 0.10 interval estimate in Example 8.2.1.)

Figure 8.2.1. Confidence intervals for craps.
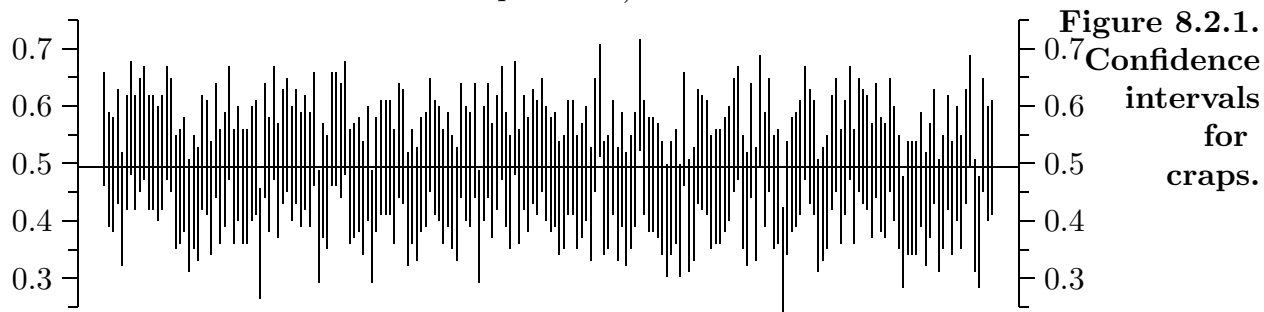
Consistent with the probability interpretation of a 95% confidence interval, we see that in this case 95.5% (191 of 200) of the intervals include p.

Approximate Interval Estimation

If the point estimate $\hat{p}$ is close to 0.5 then $\sqrt{\hat{p}(1 - \hat{p})}$ is also close to 0.5. In this case the 95% confidence interval estimate reduces to (approximately)

$$\hat{p} \pm \frac{1}{\sqrt{n}}.$$

The significance of this approximate interval estimate is that it is easily remembered, it is conservative, and it clearly highlights the "curse" of $\sqrt{n}$. In particular, (unless p is close to 0.0 or 1.0) approximately:

- 100 replications are necessary to estimate p to within ± 0.1;

- 10 000 replications are necessary to estimate p to within ± 0.01;

- 1 000 000 replications are necessary to estimate p to within ± 0.001.

Therefore, the general result is seen clearly — to achieve 1 more decimal digit of precision in the interval estimate, be prepared to do 100 times more work! (This is a worst case scenario — see Example 8.2.4.)

Example 8.2.3   For a random sample of size $n = 100$, $\sqrt{n} = 10$. Therefore, if $\hat{p} = 0.56$ then the corresponding 95% confidence interval estimate is approximately 0.56 ± 0.10.

Specified Precision

Example 8.2.4   When using Monte Carlo simulation to estimate a probability p it is frequently the case that we want to know *a priori* how many replications n will be required to achieve a specified precision $\pm w$. Specifically, if it is desired to estimate p as $\hat{p} \pm w$ with 95% confidence then, replacing 1.96 with 2, the equation

$$w = 2\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

can be solved for n to yield

$$n = \left\lfloor \frac{4\hat{p}(1 - \hat{p})}{w^2} \right\rfloor.$$

A small run with say $n = 100$ can be used to get a *preliminary* value for $\hat{p}$. This preliminary value can then be used in the previous equation to determine the approximate number of replications required to estimate p to within $\pm w$. In particular, suppose a small trial run produced a preliminary estimate of $\hat{p} = 0.2$ and suppose we wish to estimate p to within ± 0.01. Then $w = 0.01$ and, in this case, the approximate number of replications required to achieve a 95% confidence interval estimate is

$$n = \left\lfloor \frac{4(0.2)(0.8)}{(0.01)^2} \right\rfloor = 6400$$

instead of the more conservative 10 000 replications suggested previously.

The sample size determinations considered here also apply to political polls used to determine the support for a particular issue or candidate. Monte Carlo estimation has two advantages over this more classical statistical estimation problem: (i) observations tend to be very cheap and can be collected quickly, and (ii) there is no concern over *sampling bias* that might be generated by non-representative sampling.* Sampling bias can be introduced in classical statistics by factors such as dependent sampling.

Algorithm 8.2.2, properly modified, can be used to automate the process illustrated in Example 8.2.4. In this case the function `Generate()` represents a Monte Carlo program configured to estimate the probability $p = \Pr(\mathcal{A})$ by returning a value of $x = 1$ or $x = 0$ depending on whether event $\mathcal{A}$ occurred or not. The result is the following algorithm.

Algorithm 8.2.2 Given a specified interval half-width w and level of confidence $1 - \alpha$, the following algorithm computes a confidence interval estimate for p to at least $\hat{p} \pm w$ precision.

```
t = idfNormal(0.0, 1.0, 1 - α / 2);
n = 1;
x = Generate();                              /* returns 0 or 1 */
p̂ = x;
while ((n < 40) or (t * sqrt(p̂ * (1 - p̂)) > w * sqrt(n)) {
   n++;
   x = Generate();                           /* returns 0 or 1 */
   p̂ = p̂ + (x − p̂) / n;
}
return n, p̂;
```

8.2.2 MONTE CARLO INTEGRATION

As a second application of interval estimation we will now consider the use of Monte Carlo simulation to estimate the value of the definite integral

$$ I = \int_a^b g(x)\, dx. $$

---

* One of the more famous cases of biased sampling occurred after the pre-election polls of 1948 which predicted that Thomas Dewey would defeat incumbent Harry Truman in the U.S. presidential election. Three separate polls incorrectly predicted that Dewey would prevail. What went wrong? All three polls used a technique known as *quota sampling*, which resulted in somewhat more likely access to Republicans, who were more likely to have permanent addresses, own telephones, etc. (Trosset, 2004). A photograph of the victorious and jubilant Truman holding a copy of the *Chicago Tribune* with the headline "Dewey defeats Truman" was the result of biased sampling.

This is a classic application with considerable intuitive appeal, but limited practical value.*
Let X be a $Uniform(a, b)$ random variable and use the function $g(\cdot)$ to define the random
variable

$$Y = (b - a)g(X).$$

Because the pdf of X is $f(x) = 1/(b - a)$ for $a < x < b$, from Definition 7.1.5 the expected
value of Y is

$$E[Y] = \int_a^b (b - a)g(x)f(x)\, dx = \int_a^b g(x)\, dx = I.$$

Therefore $I = E[Y]$ and so an interval estimate $\hat{I} \pm w$ for I can be calculated by using a
slightly modified version of Algorithm 8.1.2 to determine an interval estimate for $E[Y]$, as
summarized by Algorithm 8.2.3.

Algorithm 8.2.3   Given a specified interval half-width w, level of confidence $1 - \alpha$, and
a need to evaluate the definite integral

$$I = \int_a^b g(x)\, dx,$$

the following algorithm computes an interval estimate for I to at least $\hat{I} \pm w$ precision.

```
t = idfNormal(0.0, 1.0, 1 - α / 2);
n = 1;
x = Uniform(a, b);                          /* modify as appropriate */
y = (b - a) * g(x);                         /* modify as appropriate */
ȳ = y;
v = 0.0;
while ((n < 40) or (t * sqrt(v / n) > w * sqrt(n - 1))) {
   n++;
   x = Uniform(a, b);                       /* modify as appropriate */
   y = (b - a) * g(x);                      /* modify as appropriate */
   d = y - ȳ;
   v = v + d * d * (n - 1) / n;
   ȳ = ȳ + d / n;
}
return n, ȳ;                                /* ȳ is Î */
```

---

* One dimensional Monte Carlo integration has limited practical value because there are
much more efficient deterministic algorithms for doing the same thing (e.g., the trapezoid
rule and Simpson's rule) — see Example 8.2.5.

Example 8.2.5 Algorithm 8.2.3 was used to evaluate the integral

$$I = \frac{1}{\sqrt{2\pi}} \int_{-3}^{4} \exp(-x^2/2) \, dx$$

with $w = 0.01$ and $\alpha = 0.05$. With a `rngs` initial seed of 12345 and

$$g(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \qquad -3 < x < 4,$$

the number of replications $n = 37\,550$ produced the interval estimate

$$\hat{I} \pm w = 0.997 \pm 0.01.$$

That is, we are 95% confident that $0.987 < I < 1.007$. This interval estimate is consistent with the correct value of I which is $I = \Phi(4) - \Phi(-3) \cong 0.9986$. Recognize, however, that $37\,550$ function evaluations were required to achieve this modest precision. In contrast, the trapezoid rule with just 30 function evaluations evaluates I to ± 0.0001 precision. To achieve this precision with Monte Carlo integration, approximately 3.8×10^8 function evaluations would be required (start early, your mileage may vary).

Algorithm 8.2.3 uses

$$\hat{I} = \frac{(b-a)}{n} \sum_{i=1}^{n} g(x_i) \qquad \text{as a point estimate of} \qquad I = \int_{a}^{b} g(x) \, dx$$

where $x_1, x_2, \ldots, x_n$ is a *Uniform*(a, b) random variate sample. As illustrated in Example 8.2.5, a huge sample size may be required to achieve reasonable precision. What can be done to improve things?

Importance Sampling

Relative to Example 8.2.5, it should be clear that the slow convergence is due, in part, to the use of *Uniform*(a, b) random variates. That is, it is wasteful to repeatedly sample and sum $g(x)$ for values of x where $g(x) \cong 0$. Instead, the integration process should concentrate on sampling and summing $g(x)$ for values of x where $g(x)$ is large because those values contribute most to the integral. What is needed to accomplish this is a random variable X whose pdf is similar to $g(x)$. This particular *variance reduction technique* is known as importance sampling.

In general, to derive a generalization of Algorithm 8.2.3 we can proceed as illustrated previously *but* with X a general random variable with pdf $f(x) > 0$ defined for all possible values $x \in \mathcal{X} = (a, b)$. Then define the new random variable

$$Y = g(X)/f(X)$$

and recognize that the expected value of Y is

$$E[Y] = \int_{a}^{b} \frac{g(x)}{f(x)} f(x) \, dx = \int_{a}^{b} g(x) \, dx = I.$$

As before, an interval estimate $\hat{I} \pm w$ for $I = E[Y]$ can be calculated by using a slightly modified version of Algorithm 8.2.3 to determine an interval estimate for $E[Y]$. The key feature here is a clever choice of X or, equivalently, the pdf $f(\cdot)$.

Example 8.2.6 In the spirit of Example 8.2.5, suppose we wish to evaluate the integral

$$I = \int_0^4 \exp(-x^2/2)\, dx.$$

If X is a *Uniform*(a,b) random variate then this is, essentially, Example 8.2.5 all over again, complete with slow convergence. If instead X is an *Exponential*(1) random variate, truncated to the interval $\mathcal{X} = (0,4)$, then better convergence should be achieved. In this case, the pdf of X is

$$f(x) = \frac{\exp(-x)}{F(4)} \qquad 0 < x < 4$$

where $F(4) = 1 - \exp(-4) \cong 0.9817$ is the cdf of an un-truncated *Exponential*(1) random variable. Algorithm 8.2.3, with the two indicated x, y assignments modified accordingly, can be used to compute an interval estimate for I. The details are left as an exercise.

8.2.3 TIME-AVERAGED STATISTICS

Recall from Section 1.2 that, by definition, the time-averaged number in a single-server service node is

$$\bar{l} = \frac{1}{\tau} \int_0^\tau l(t)\, dt$$

where $l(t)$ is the number in the service node at time t and the average is over the interval $0 < t < \tau$. The following example, an application of Monte Carlo integration, clarifies the statement made in Section 1.2 that "if we were to observe (sample) the number in the service node at many different times chosen *at random* between 0 and τ and then calculate the arithmetic average of all these observations, the result should be close to $\bar{l}$."

Example 8.2.7 Let the random variable T be *Uniform*$(0, \tau)$ and define the new (discrete) random variable

$$L = l(T).$$

By definition the pdf of T is $f(t) = 1/\tau$ for $0 < t < \tau$; therefore the expected value of L is

$$E[L] = \int_0^\tau l(t)f(t)\, dt = \frac{1}{\tau} \int_0^\tau l(t)\, dt = \bar{l}.$$

Now, let the observation times $t_1, t_2, \ldots, t_n$ be generated as an n-point random variate sample of T. Then $l(t_1), l(t_2), \ldots, l(t_n)$ is a random sample of L, and the mean of this random sample is an (unbiased) *point* estimate of the expected value $E[L] = \bar{l}$. If the sample size is large, the sample mean ("arithmetic average") should be close to $\bar{l}$.*

---

* There is a notational conflict in this example. Do not confuse the n (sample size) in

Random Sampling

The previous example applies equally well to any other function defined for $0 < t < \tau$. For example, the utilization can be estimated by sampling $x(t)$. Moreover, the random times t_i at which the $l(t_i)$ [or $x(t_i)$] samples are taken can be generated by the same algorithm used to generate a stationary Poisson arrival process (Algorithm 7.3.1). That is, to generate n random sample times (in sorted order) with an average inter-sample time of δ we can use the algorithm

```
t₀ = 0;
for (i = 0;  i < n;  i++)
    tᵢ₊₁ = tᵢ + Exponential(δ);
return t₁, t₂, . . . , tₙ;
```

8.2.4   EXERCISES

Exercise 8.2.1   (a) Modify program `buffon` so that it will use Algorithm 8.2.1 to produce a 95% confidence interval estimate for the probability of a line crossing. Then run this modified program 10 000 times (with a different initial `rngs` initial seed for each run), use $n = 100$ replications per run, and count the proportion of times out of 10 000 that the true (theoretical) probability of a line crossing falls within the 95% confidence interval predicted. (b) Comment.

Exercise 8.2.2[a]   Work through the details of Example 8.2.6, estimating the value of I as $\hat{I} \pm w$ with 95% confidence and $w = 0.01$.

Exercise 8.2.3   How many replications are required to estimate a probability whose value is expected to be 0.1 to with ± 0.001 with 95% confidence?

Exercise 8.2.4   (a) Use Algorithm 8.2.3 to estimate the value of the integral

$$I = \int_0^2 x^7 (2 - x)^{13} \, dx$$

to within 1% of its true (unknown) value with 95% confidence. (b) Calculate the true value of I. (c) Comment.

Exercise 8.2.5   (a) Evaluate the integral in Exercise 8.2.4 using the trapezoid rule. (b) Comment.

Exercise 8.2.6   Modify program `ssq4` to use random sampling to generate 95% confidence interval estimates for $\bar{l}$, $\bar{q}$ and $\bar{x}$. Sample at the rate 2λ (where λ is the arrival rate).

Example 8.2.7 with the n (number of jobs) in the single-server service node model presented in Section 1.2. The two numbers may be the same, but that is not necessarily the case.

The discussion in this section is motivated by the following type of question and answer. Consider a single-server service node that processes a *large* number of jobs arriving as a stationary Poisson process with arrival rate $\lambda = 1/2$. If the service time is an *Erlang*$(5, 0.3)$ random variable, so that the service rate is $\nu = 2/3$, then what is the average wait in the node, the time-averaged number in the node, and the utilization? This question can be answered by using program `ssq4`. To do so, however, we must make some judgment about what constitutes a "large" number of jobs. Moreover, we should be concerned about the possible effects on the simulation output of forcing the service node to begin and end in an idle state, as is done in program `ssq4`.*

8.3.1 EXPERIMENTAL INVESTIGATION

Definition 8.3.1   *Steady-state* system statistics are those statistics, if they exist, that are produced by simulating the operation of a *stationary* discrete-event system for an effectively infinite length of time.

Example 8.3.1   If program `sis4` is used to simulate a large number of time intervals the resulting average inventory level per time interval is an estimate of a steady-state statistic. In particular, for a `rngs` initial seed of 12345 and an increasing number of time intervals n, program `sis4` yields the following sequence of average inventory level estimates $\bar{l} = \bar{l}^+ - \bar{l}^-$.

n	:	20	40	80	160	320	640	1280	2560
$\bar{l}$	:	25.98	26.09	25.49	27.24	26.96	26.36	27.19	26.75

As $n \to \infty$ it appears that $\bar{l}$ will converge to a steady-state value of approximately 26.75.

Since the $n \to \infty$ convergence in Example 8.3.1 can only be approximated, a better approach to estimating the steady-state value of $\bar{l}$ would be to develop an *interval* estimation technique applicable when n is large, but finite. One way to do this is to use the method of *batch means* — which will be introduced in Section 8.4.

The "if they exist" phrase in Definition 8.3.1 is important. For some systems steady-state statistics do not exist.

Example 8.3.2   If we run the simulation program `ssq4` with an arrival rate of $1/2$ and the *Erlang*(n, b) parameters specified so that the service rate is $1/2$ or less, then we will find that the average wait and the average number in the service node will tend to increase without limit as the number of jobs is increased. The reason for this should be intuitive — unless the service rate is greater than the arrival rate then, on average, the server will not be able to keep up with the demand resulting in an average queue length that continues to grow as more jobs arrive. In this case the steady-state average wait and the average number in the node are both infinite.

---

*   As an alternative, if we interpret "large" as *infinite* then this question can also be answered by applying the mathematical tools of *steady-state* queuing theory. Some of these theoretical tools are summarized in Section 8.5.

Infinite-Horizon Versus Finite-Horizon Statistics

Steady-state statistics are also known as *infinite-horizon* statistics. The idea here is that an infinite-horizon discrete-event simulation is one for which the simulated operational time is effectively infinite. In contrast, a *finite-horizon* discrete-event simulation is one for which the simulated operational time is finite. One of the characteristics of an infinite-horizon simulation is that the initial conditions are not important — as simulated time becomes infinite, the choice of the initial state of the simulated system on system statistics is irrelevant, i.e., the simulated system loses all memory of its initial state. For finite-horizon simulations this is not true — the initial state of the system is important.

Definition 8.3.2 *Transient* system statistics are those statistics that are produced by a finite-horizon discrete-event simulation.

Example 8.3.3 With minor modifications program `ssq4` can be used to simulate an initially idle $M/M/1$ service node processing a small number of jobs (say 100) and with a relatively high traffic intensity (say 0.8).* If this program is executed multiple times varying *only* the `rngs` initial seed from replication to replication then the average wait in the node will vary significantly from replication to replication. Moreover, for most (perhaps all) replications, the average wait will *not* be close to the steady-state (infinite-horizon) average wait in an $M/M/1$ service node. On the other hand, if a relatively large number of jobs (say 10 000) are used then the replication to replication variability of the average wait in the node will be much less significant and for most replications the average wait will be close to the steady-state average number in an $M/M/1$ service node.

Consistent with Example 8.3.3, if the number of jobs processed (or equivalently, the simulated operational time) is small then the average wait in the service node will be strongly biased by the initial idle state of the system. However, as the number of jobs becomes infinite this initial condition bias disappears.

In addition to the importance of the initial condition bias, there is another important distinction between steady-state and transient statistics. In an infinite-horizon simulation it is virtually always true that the system "environment" is assumed to remain *static* (stationary). Thus, for example, if the system is a single-server service node then both the arrival rate and the service rate are assumed to remain constant in time. Most real-world systems do not operate in a static environment for an extended period of time and for those systems steady-state statistics may have little meaning. In a finite-horizon simulation, there is no need to assume a static environment — natural temporal (dynamic) changes in the environment can be incorporated into the discrete-event simulation and the transient statistics so produced will then properly reflect the influence of this dynamic environment.

---

* Recall that an $M/M/1$ service node is a single-server service node with a Poisson arrival process and *Exponential* service times.

The system dictates the choice between using steady-state or transient statistics:

- Consider a bank that opens at 9 AM and closes at 5 PM. A finite-horizon simulation over the 8-hour period produces *transient* statistics which might be valuable in determining the optimal staffing of tellers throughout the day.

- Consider a fast food restaurant with a drive-up window that experiences a lunch rush period between 11:45 AM and 1:15 PM with an arrival rate that remains constant over the rush period. This 90-minute period could be simulated for a much longer time period, producing *steady-state* statistics which might be valuable for estimating the average wait time at the drive-up window.

Steady-state statistics are better understood because they are easier to analyze *mathematically* using analytic methods as, for example, in Section 8.5. Transient statistics are important because steady-state is often a convenient fiction — most real systems do not operate long enough in a stationary environment to produce steady-state statistics. Depending on the application, both transient and steady-state statistics may be important. For that reason, one of the most important skills that must be developed by a discrete-event simulation specialist is the ability to decide, on a system-by-system basis, which kind of statistics best characterizes the system's performance.

Initial and Terminal Conditions

Finite-horizon discrete-event simulations are also known as *terminating* simulations. This terminology emphasizes the point that there must be some terminal conditions that are used to define the simulated period of system operation. Terminal conditions are typically stated in terms of either a fixed number of events (e.g., processed jobs), or a fixed amount of (simulated) elapsed time. In any case, for a finite-horizon discrete-event simulation model it is important to clearly define the desired terminal system state.

Example 8.3.4 In program `ssq4`, the system (service node) state is idle at the beginning and at the end of the simulation. The terminal condition is specified by the "close the door" time. Similarly, in program `sis4` the state of the inventory system is the current and on-order inventory levels; these states are the same at the beginning and at the end of the simulation. The terminal condition is specified by the number of time intervals.

Infinite-horizon (steady-state) discrete-event simulations must be terminated. However, in an infinite-horizon simulation the state of the system at termination should not be important and so it is common to use whatever stopping conditions are most convenient. The basic idea here is that steady-state statistics are based on such a huge amount of data that a few "non-steady-state" data points accumulated at the beginning and the end of the simulation should have no significant impact (bias) on output statistics. Recognize, however, that there is a fair amount of wishful thinking involved here and a significant amount of research [e.g., Schruben (1982), Gallagher, Bauer, and Maybeck (1996)] has been devoted to the problem of eliminating initial condition (and terminal condition) bias from simulation-generated steady-state statistics.

8.3.2 FORMAL REPRESENTATION

To present the previous discussion in a more formal analytic setting, consider a discrete-event system whose state is characterized by a time-varying random variable $X(t)$ with associated realizations denoted by $x(t)$. The state variable $X(t)$ can be either discrete or continuous and the initial state $X(0)$ can be either specified as a fixed value or modeled as a random variable with a specified pdf. The state variable $X(\cdot)$ is known formally as a *stochastic process*.*

- The typical objective of a finite-horizon simulation of this system would be to estimate the time-averaged *transient* statistic

$$\bar{X}(\tau) = \frac{1}{\tau} \int_0^\tau X(t)\, dt,$$

where $\tau > 0$ is the terminal time.

- In contrast, the typical objective of an infinite-horizon simulation of this system would be to estimate the time-averaged *steady-state* statistic

$$\bar{x} = \lim_{\tau \to \infty} \bar{X}(\tau) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau X(t)\, dt.$$

As the notation suggests, $\bar{X}(\tau)$ is a random variable but $\bar{x}$ is not. That is, to estimate $\bar{X}(\tau)$ it is necessary to replicate a finite-horizon simulation many times, each time generating one estimate of $\bar{X}(\tau)$. In contrast, to estimate $\bar{x}$ it is, in principle, only necessary to run the infinite-horizon simulation once, but for a very loooong time.

Definition 8.3.3 If a discrete-event simulation is repeated, varying *only* the `rngs` initial states from run to run, then each run of the simulation program is a *replication* and the totality of replications is said to be an *ensemble*.

Replications are used to generate *independent* estimates of the same transient statistic. Therefore, the initial seed for each replication should be chosen so that there is no replication-to-replication overlap in the sequence of random numbers used. The standard way to prevent this overlap is to use the final state of each `rngs` stream from *one* replication as the initial state for the *next* replication. This is automatically accomplished by calling `PlantSeeds` once *outside* the main replication loop. The set of estimated time-averaged statistics is then an independent sample that can be used to produce an interval estimate, as in Section 8.2.

---

* For a single-server service node with stochastic arrival and service times, $X(t)$ might be the number in the node at time t. For a simple inventory system with stochastic demand instances and delivery lags, $X(t)$ might be the inventory level at time t. For a stochastic arrival process, $X(t)$ might be the number of arrivals in the interval $(0, t)$.

Independent Replications and Interval Estimation

Suppose the finite-horizon simulation is replicated n times, each time generating a state time history $x_i(t)$ and the corresponding time-averaged transient statistic

$$\bar{x}_i(\tau) = \frac{1}{\tau} \int_0^\tau x_i(t) \, dt$$

where $i = 1, 2, \ldots, n$ is the replication index within the ensemble. Each data point $\bar{x}_i(\tau)$ is then an independent observation of the random variable $\bar{X}(\tau)$. If n is large enough the pdf of $\bar{X}(\tau)$ can be estimated from a histogram of the $\bar{x}_i(\tau)$ data. In practice, however, it is usually only the expected value $E[\bar{X}(\tau)]$ that is desired and a *point* estimate of this transient statistic is available as an *ensemble average*, even if n is not large. That is,

$$\frac{1}{n} \sum_{i=1}^n \bar{x}_i(\tau)$$

is a point estimate of $E[\bar{X}(\tau)]$. Moreover, an *interval* estimate for $E[\bar{X}(\tau)]$ can be calculated from the ensemble (sample) mean and standard deviation of the $\bar{x}_i(\tau)$ data by using Algorithm 8.1.1. Independence is assured by the careful selection of seeds. Normality follows from the central limit theorem since $\bar{x}_i(\tau)$ is an average.

Example 8.3.5   As an illustration of the previous discussion, a modified version of program `sis4` was used to produce 21 replications. For each replication, 100 time intervals of operation were simulated with $(s, S) = (20, 80)$. The 21 transient time-averaged inventory levels that were produced are illustrated.
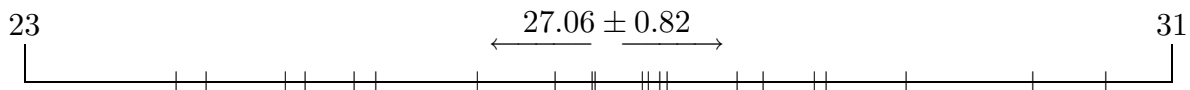
If $\bar{L}(\tau)$ denotes the time-averaged inventory level (with $\tau = 100$) then each of these 21 numbers is a *realization* of $\bar{L}(\tau)$. The mean and standard deviation of this sample are 27.06 and 1.76 respectively. If a 95% confidence interval estimate is desired then the corresponding critical value of t is

$$t^* = \texttt{idfStudent}(20, 0.975) = 2.086$$

Therefore, from program `estimate` the 95% confidence interval estimate is

$$27.06 \pm \frac{(2.086)(1.76)}{\sqrt{20}} = 27.06 \pm 0.82$$

as illustrated above. We are 95% confident that $E[\bar{L}(\tau)]$ is somewhere between the interval endpoints $26.24 = 27.06 - 0.82$ and $27.88 = 27.06 + 0.82$. That is, we are 95% confident that if program `sis4` were used to generate a *large* ensemble then the ensemble average of the resulting transient time-averaged inventory levels would be somewhere between 26.24 and 27.88.

In some discrete-event simulations the terminal time is not fixed. Instead, the terminal time is determined by other terminal conditions. For example, in program ssq2 the terminal condition is specified by the number of jobs processed, not the stopping time, and because of this the terminal time will vary randomly from replication to replication. The idea of multiple replications, with nothing varied except the initial state of the rngs stream, is still valid; each replication will produce an independent observation of the transient statistic of interest and these observations can be combined to produce a point and interval estimate.

Example 8.3.6 A modified version of program ssq2 was used to produce 20 replications of a simulation for which 100 jobs are processed through an initially idle $M/M/1$ service node with arrival rate $\lambda = 1.0$ and service rate $\nu = 1.25$. The resulting 20 observations of the average wait in the node are illustrated.

From program estimate the resulting 95% confidence interval estimate is 3.33 ± 1.04. Thus we are 95% confident that if we were to do thousands of replications, the ensemble average of the resulting transient time-averaged number in the node would be somewhere between $2.29 = 3.33 - 1.04$ and $4.37 = 3.33 + 1.04$.

In Section 8.5 it is shown that if the arrival rate is $\nu = 1.00$ and the service rate is $\nu = 1.25$ for an $M/M/1$ service node then the steady-state average wait in the node is

$$\frac{1}{\nu - \lambda} = \frac{1}{1.25 - 1.00} = 4.0.$$

This exact steady-state value determined by analytic methods is indicated in Example 8.3.6 with a '•'. Therefore, we see that it is *possible* that 100 jobs are enough to produce steady-state statistics. To explore this conjecture, the modified version of program ssq2 was used to produce 60 more replications. Consistent with the $\sqrt{n}$ rule this four-fold increase in the number of replications (from 20 to 80) should result in an approximate two-fold decrease in the width of the interval estimate and if the ensemble mean remains close to 3.33, then 4.0 will no longer be interior to the interval estimate.

Example 8.3.7 Based upon 80 replications, the resulting 95% confidence interval estimate was 3.25 ± 0.39, as illustrated.
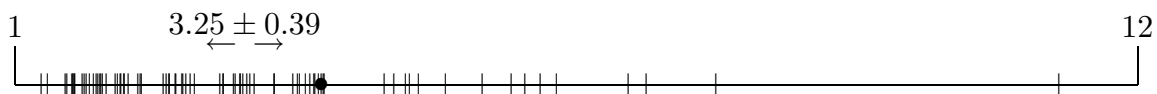
Now we see that in this case 100 jobs are clearly not enough to produce steady-state statistics — the bias of the initially idle state is still evident in the transient statistic.

Example 8.3.8   As a continuation of Example 8.3.6, the number of jobs per replication was increased from 100 to 1000. As before, 20 replications were used to produce 20 observations of the average wait in the node, as illustrated.
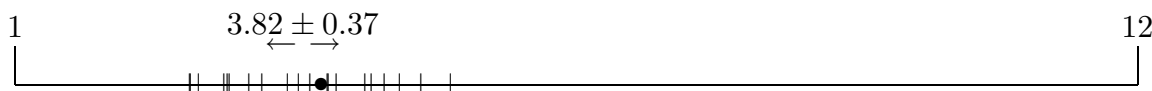
With this increase in the number of processed jobs, the 95% confidence interval estimate shifted to the right toward the steady-state average wait and shrunk to 3.82 ± 0.37. The steady-state average wait '•' indicator is now interior to the interval estimate, suggesting that in this case 1000 jobs are enough to achieve statistics that are close to their steady-state values.

Relative to the 100-jobs per replication results in Example 8.3.6, reproduced here for convenience, note that the 1000-jobs per replication results in Example 8.3.8 exhibit a sample distribution that is much more symmetric about the sample mean.

That is, the 1000-jobs per replication results are more consistent with the underlying theory of interval estimation, dependent as it is on a sample mean distribution that is approximately $Normal(\mu, \sigma/\sqrt{n})$ and centered on the (unknown) population mean.

8.3.3   EXERCISES

Exercise 8.3.1   (*a*) Repeat the experiment presented in Example 8.3.5 using program `sis4` modified so that estimates of $\bar{L}(10)$, $\bar{L}(20)$, $\bar{L}(40)$, and $\bar{L}(80)$ can also be produced. (*b*) Comment.

Exercise 8.3.2   (*a*) Repeat the 100-jobs per replication experiment in Examples 8.3.6 and 8.3.7 using program `ssq4` but modified so that estimates of the average number in the node can be produced for 10, 20, 40, 80, 160, and 320 replications. (*b*) For the 320 replication case, construct a continuous-data histogram of the average waits. (*c*) Comment.

Exercise 8.3.3   (*a*) Repeat the 1000-jobs per replication experiment presented in Examples 8.3.8 using program `ssq4` but modified so that estimates of the average number in the node can be produced for 10, 20, 40, 80, 160, and 320 replications. (*b*) For the 320 replication case, construct a continuous-data histogram of the average waits. (*c*) Comment.

Exercise 8.3.4   Illustrate graphically how well the two histograms in Exercise 8.3.2 and 8.3.3 can be approximated by a $Normal(\mu, \sigma)$ pdf for appropriately chosen values of (μ, σ).

As discussed in the previous section, there are two types of discrete-event simulation models, transient (finite horizon) and steady-state (infinite horizon). In a transient simulation, *interval* estimates can be constructed for the statistics of interest by replication. In a steady-state simulation, the general advice is to run the simulation for a *long* time in order to (hopefully) achieve a good *point* estimate for each statistic of interest. The last observation leads to an obvious question — is there a way to obtain *interval* estimates for steady-state statistics?

The answer is yes, there are several ways to obtain interval estimates for steady-state statistics. The most common way to do so is by using the so-called method of *batch means*.

8.4.1   EXPERIMENTAL INVESTIGATION

Example 8.4.1   To investigate the issue of interval estimates for steady-state statistics, program `ssq2` was modified to simulate an $M/M/1$ single-server service node with $\lambda = 1.0$, $\nu = 1.25$. In this case the steady-state utilization is $\rho = \lambda/\nu = 0.8$ and, from the results in Section 8.5, the expected steady-state wait is $E[W] = 4$. We can attempt to eliminate the initial state bias by initializing the program variable `departure` to a non-zero value, in this case 4.2. That is, in Exercise 8.4.1 you are asked to prove that this initialization causes the simulation to begin in its *expected* steady-state condition in the sense that the first job's expected wait is 4. The multi-stream capability of the library `rngs` was used (with initial seed 12345) to create 16 independent replications, with the first using stream 0 for both arrival times and service times, the second using stream 1 for both, etc. The *finite-horizon* interval estimation technique from the previous section was used to compute average wait point and interval estimates for $8, 16, 32, \ldots, 1024$ processed jobs. The results are illustrated in Figure 8.4.1.
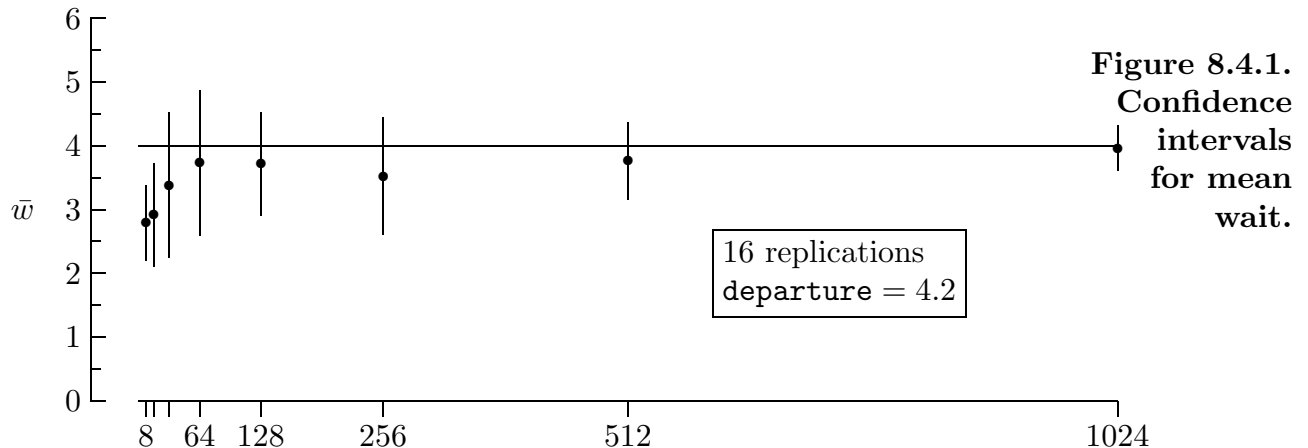
Figure 8.4.1.
Confidence
intervals
for mean
wait.

The horizontal line is the steady-state wait $E[W] = 4$. The vertical lines are the 95% confidence *interval* estimates for a sample size of 16 and the • denotes the *point* estimate $\bar{w}$. As hoped, we see that unless the number of jobs is very small the *finite-horizon* interval estimates are accurate *steady-state* estimates, becoming increasingly more accurate as the number of jobs increases.

Example 8.4.2   As a continuation of Example 8.4.1, in an attempt to better understand the use of replicated finite-horizon statistics to estimate a steady-state statistic, two other initial values of `departure` were tried, 0.0 and 6.0. The initial value 0.0 can be justified on the basis that the *most likely* steady-state condition is an *idle* node. Indeed, in this case an arriving job finds the service node idle with probability 0.2. The resulting interval estimates are presented in the plot on the left-hand side of Figure 8.4.2. The initial value 6.0 was chosen by experimentation and justified on the basis that it produced interval estimates consistent with the expected steady-state wait even when the number of jobs was small. The resulting interval estimates are on the right-hand side of Figure 8.4.2.

Figure 8.4.2.
Confidence
intervals
for mean
wait.
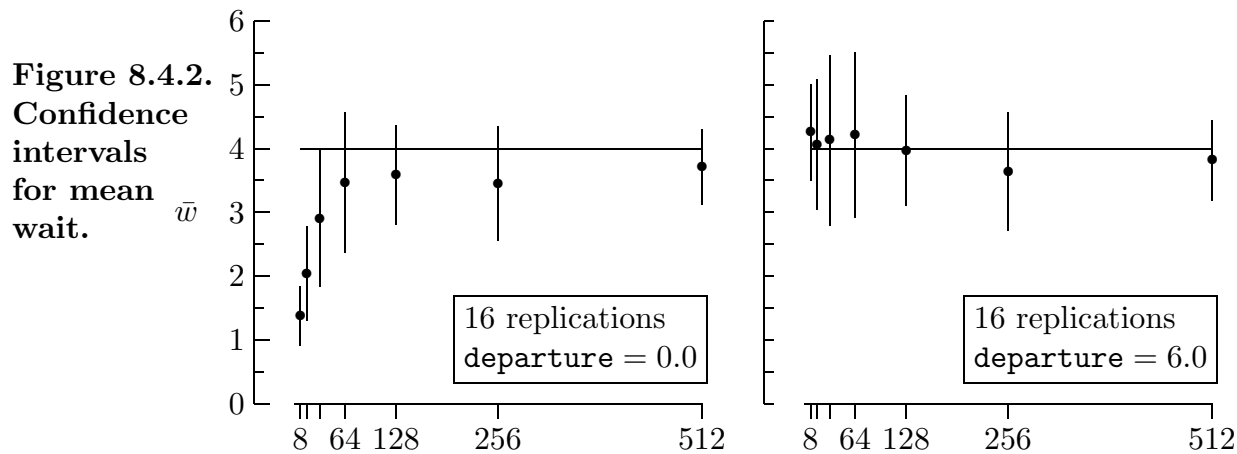
The graph on the left illustrates the potential significant impact of the initial state bias on the estimates when the number of jobs is small. The graph on the right illustrates that, at least in principle, it is possible to choose an initial state for the service node that minimizes the initial state bias. As in Example 8.4.1, when the number of jobs is large, the estimates are very good and essentially *independent* of the initial state of the service node. Of course, this is as it should be if they are to be steady-state estimates.

Replications

Examples 8.4.1 and 8.4.2 illustrate that one way to obtain interval estimates for steady-state statistics is to use the replication-based finite-horizon interval estimation technique from the previous section with each replication corresponding to a long simulated period of operation. To do so, three interrelated issues must be resolved.

- What is the initial state of the system?

- How many replications should be used?

- What is the length of the simulated period of operation (per replication).

Figures 8.4.1 and 8.4.2 provide insight into the first and third of these issues. The initial state values 4.2, 0.0, and 6.0 for `departure` illustrated the initial states. The length of each simulation varied from 8 to 1024 jobs. The number of replications, however, was always fixed at 16. The following example provides insight into the second issue.

Example 8.4.3   To better understand the importance of the number of replications of the simulation model, the simulations in Examples 8.4.1 and 8.4.2 were repeated using 64 replications. Except for this 4-fold increase in replications (using the first 64 `rngs` streams) all other parameters in the model were held at the same levels as before. By increasing the number of replications, the width of the interval estimates should be reduced and the estimates thereby improved. Specifically, because of the square root rule (Section 8.1), we expect that a 4-fold increase in the number of replications should reduce the interval width by approximately 1/2.

Figure 8.4.3. Confidence intervals for mean wait.
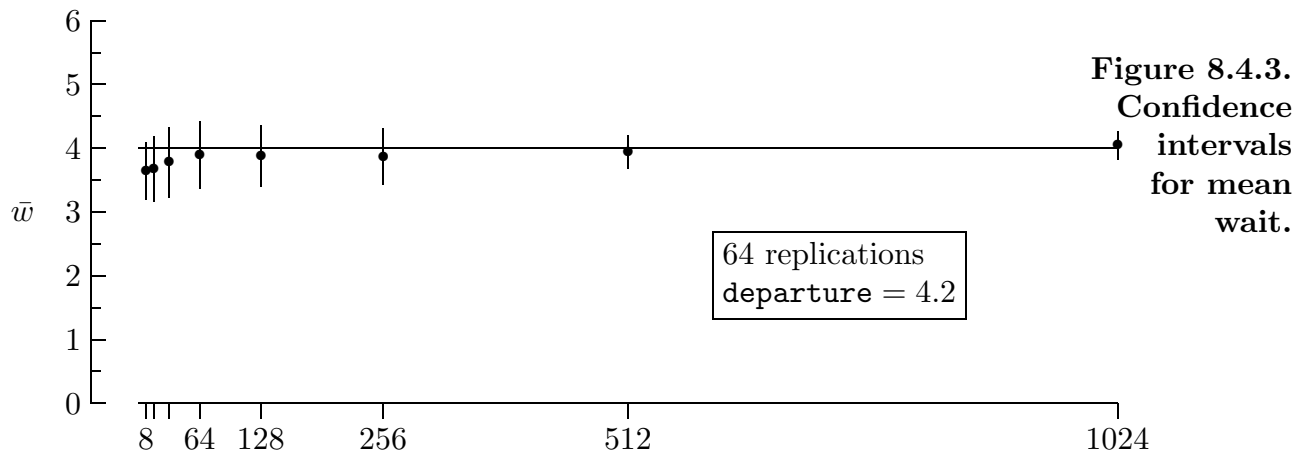
As illustrated in Figure 8.4.3, when compared to the results displayed in Figure 8.4.1, the interval estimates have been cut approximately in half. Moreover, even for a small number of jobs the interval estimates include the steady-state expected value, indicating that the initial condition bias has been largely eliminated by initializing the program variable `departure` to 4.2. That is, the initial state bias that *seemed* to be evident in Figure 8.4.1 based on 16 replications was actually just natural small-sample variation. Another run with a different seed may just as likely produce a bias that is above the theoretical steady-state value. A valid 95% confidence interval procedure will contain the true value 95% of the time. It is quite possible that the two confidence intervals in Figure 8.4.1 based on 16 replications that fell wholly below the steady state value were among the 5% of the confidence intervals that did not contain the true value of 4.

Example 8.4.4   As a continuation of the previous example, when compared to the corresponding results in Example 8.4.2 based on 16 replications, these figures based on 64 replications reveal that again the width of the interval estimates have been halved approximately. Moreover, from the graph on the right-hand side of Figure 8.4.4 we see that choosing the initial value of `departure` to be 6.0, which *seemed* to work with just 16 replications, actually introduces some initial state bias, causing the estimates (for a small number of jobs) to be larger than the theoretical steady-state value. That 6.0 seemed right from Figure 8.4.2 was a false impression based on misleading small-sample statistics.

Figure 8.4.4. Confidence intervals for mean wait.
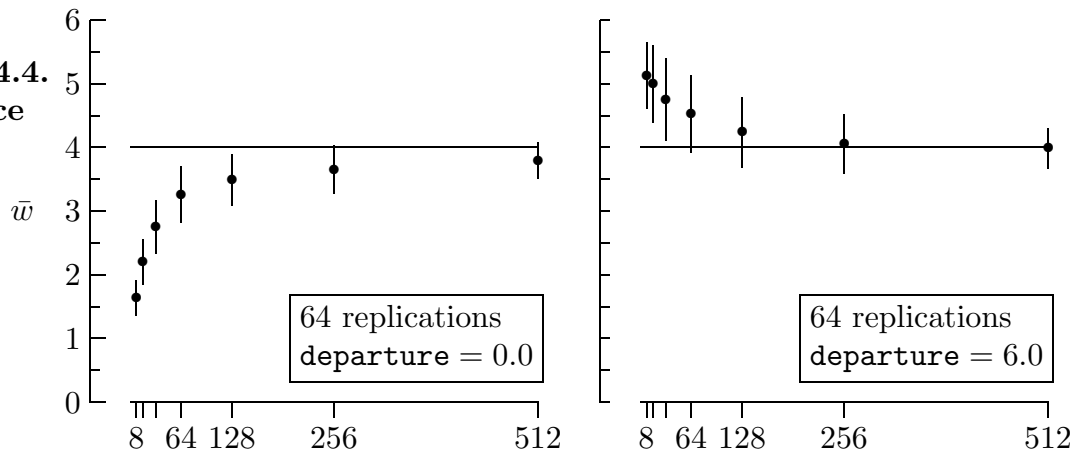
To summarize, replicated finite-horizon statistics can be used to generate accurate infinite-horizon (steady-state) interval estimates. However, there is an initial bias problem with this technique. Therefore, we need another interval estimation technique that avoids the initial state bias problem.* [Interval estimation techniques other than batch means are outlined in Alexopoulos and Seila (1998) and Goldsman and Nelson (1998).]

8.4.2   METHOD OF BATCH MEANS

The reason the initial state bias was an issue in the previous examples was because each replication was initialized *with the same system state.* Why not, instead, make *one* long run with, say, $64 \cdot 512$ jobs and partition the wait times into 64 consecutive *batches* each of length 512? An average wait can be computed for each batch of 512 jobs and an interval estimate can be computed from the 64 batch means. This is the method of *batch means.* With this method, the initial state bias is eliminated because the statistics for each batch (other than the first) are naturally initialized to the state of the system at the time the statistical counters for the batch are reset.

Algorithm 8.4.1   The method of batch means applied to the sequence $x_1, x_2, \ldots, x_n$ consists of six steps. (Only the first two steps are unique to the method of batch means. Except for notation, the last four steps are identical to Algorithm 8.1.1.)

- Select a *batch size* $b > 1$. The number of batches is then $k = \lfloor n/b \rfloor$. (If b is not a divisor of n, discard the last $n \bmod b$ data points.)

- Group the sequence into k batches:

$$\underbrace{x_1, x_2, \ldots, x_b}_{\text{batch 1}}, \underbrace{x_{b+1}, x_{b+2}, \ldots, x_{2b}}_{\text{batch 2}}, \underbrace{x_{2b+1}, x_{2b+2}, \ldots, x_{3b}}_{\text{batch 3}}, \ldots$$

---

* We were able to avoid the initial state bias problem in Example 8.4.3 (with `departure` initialized to 4.2) only because we already knew the steady-state result! In practice, the steady-state result is seldom known, for if it were there would be no point in simulating the system to compute the steady-state estimates.

and for each batch calculate the *batch mean*

$$\bar{x}_j = \frac{1}{b} \sum_{i=1}^{b} x_{(j-1)b+i} \qquad j = 1, 2, \dots, k.$$

• Compute the mean $\bar{x}$ and standard deviation s of the batch means $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$.

• Pick a *level of confidence* $1 - \alpha$ (typically $\alpha = 0.05$).

• Calculate the critical value $t^* = \mathtt{idfStudent}(k - 1, 1 - \alpha/2)$.

• Calculate the interval endpoints $\bar{x} \pm t^* s / \sqrt{k - 1}$.

You then claim to be $(1 - \alpha) \times 100\%$ confident that the true (unknown) steady-state mean lies somewhere in this interval. If the batch size b is large, your claim is probably true *even if the sample is autocorrelated* (see Section 4.4 and Appendix F). The method of batch means reduces the impact of autocorrelated data since only the trailing observations of one batch are significantly correlated with the initial observation of the subsequent batch.

Provided no points are discarded, the "mean of the means" is the same as the "grand sample mean":

$$\bar{x} = \frac{1}{k} \sum_{j=1}^{k} \bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

This is an important observation because it demonstrates that the choice of the (b, k) parameters has *no* impact on the *point* estimate of the mean. Only the width of the *interval* estimate is affected by the choice of (b, k).

Example 8.4.5 A modified version of program $\mathtt{ssq2}$ was used to generate $n = 32768$ consecutive waits for an initially idle $M/M/1$ service node with $(\lambda, \nu) = (1, 1.25)$. The batch means estimate of the steady-state wait $E[W] = 4.0$ was computed for several different (b, k) values with the following results.

(b, k)	$(8, 4096)$	$(64, 512)$	$(512, 64)$	$(4096, 8)$
$\bar{w}$	3.94 ± 0.11	3.94 ± 0.25	3.94 ± 0.29	3.94 ± 0.48

As illustrated, the point estimate 3.94 is independent of (b, k) but the width of the interval estimate clearly is not.

The variable width result in Example 8.4.5 is typical of the method of batch means. The obvious question then is, for a fixed value of $n = bk$ what is the best choice of (b, k)? The answer to this question is conditioned on the following important observations relative to the theory upon which the method of batch means is based.

• The validity of the $\bar{x} \pm t^* s / \sqrt{k - 1}$ interval estimate is based on the assumption that the batch means $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$ are (at least approximately) an *independent*, identically distributed sequence of *Normal* random variates.

- The method of batch means becomes increasingly more valid as the batch size b is increased. This occurs for two related reasons: (i) the mean of b random variables, independent or not, tends to become more *Normal* as b is increased and (ii) the autocorrelation that typically exists in time-sequenced data is reduced in the associated time-sequenced batch means, becoming effectively zero if b is sufficiently large.

Guidelines

If the value of the product $n = bk$ is fixed, then the batch size b cannot be too large for if it is the number of batches k will be too small resulting in a wide interval estimate width (see Figure 8.1.8). The following guidelines are offered for choosing (b, k).

- Schmeiser (1982) recommends a number of batches between $k = 10$ and $k = 30$, although this advice should be tempered by the two other guidelines.

- Pegden, Shannon, and Sadowski (1995, page 184) recommend a batch size that is at least ten times as large as the largest lag for which the autocorrelation function remains significant (see Section 4.4).

- Banks, Carson, Nelson, and Nicol (2001, page 438) recommend that the batch size be increased until the lag one autocorrelation between batch means is less than 0.2.

The following examples provide additional insight into the choice of (b, k).

Example 8.4.6 An extrapolation of the autocorrelation results in Example 4.4.3 suggests that for an $M/M/1$ service node with steady-state utilization $\lambda/\nu = 0.8$ the effective cut-off lag is approximately 100. Figure 8.4.5 illustrates that autocorrelation is present in a sequence of batch means for batch sizes $b = 4, 8, 16, 32, 64, 128$.* Although the second guideline is violated with $b = 256$, Figure 8.4.5 shows that the third guideline will be satisfied with $b = 256$.

Figure 8.4.5. Sample autocorrelation function.
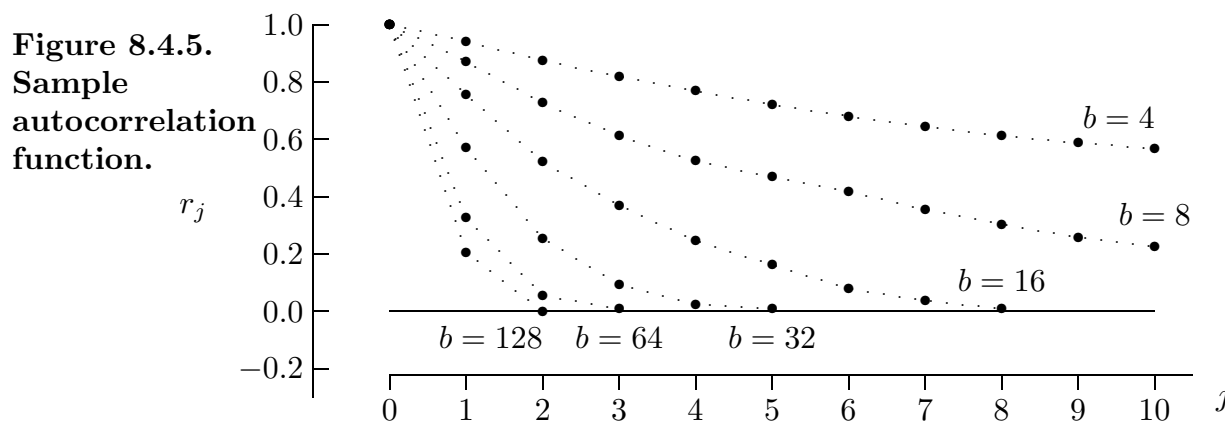
As b is increased the autocorrelation in the batch means is reduced. For $b = 128$ the reduction is so complete that there is no residual autocorrelation for lags greater than 1.

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* For each value of b, the results in this figure were produced by piping 1000 batch means into program `acs`.

To summarize, we expect that if $k = 64$ batches each of size $b = 256$ are used with Algorithm 8.4.1, then the batch mean interval estimate of the steady-state wait for jobs in an $M/M/1$ service node with steady-state utilization $\lambda/\nu = 0.8$ will contain $E[W]$ with probability $1 - \alpha$. Therefore, in this case $n = bk = 16384$ jobs must be processed to produce this estimate. If λ/ν were larger, a larger value of n would be required.

Iteration

Since Algorithm 8.4.1 can be applied for *any* reasonable pair of (b, k) values, even if the underlying theory is not valid, it is common to find the method of batch means presented as a *black box* algorithm with an associated recommendation to experiment with increasing values of b until "convergence" is achieved. In this context, convergence means significant overlap in a sequence of increasingly more narrow interval estimates produced, typically, by using $k = 64$ batches and systematically doubling the batch size b. Although they were computed in a different way, the results produced by fixing $k = 64$ and systematically doubling b will be similar to those illustrated in Examples 8.4.3 and 8.4.4.

Example 8.4.7 The method of batch means, with $(b, k) = (256, 64)$, was applied to a simulation of a $M/M/1$ service node with $(\lambda, \nu) = (1.00, 1.25)$ to produce interval estimates of the steady-state wait $E[W] = 4$. Ten 95% confidence interval estimates were produced, corresponding to the first ten `rngs` streams.

Figure 8.4.6. Confidence intervals for mean wait.
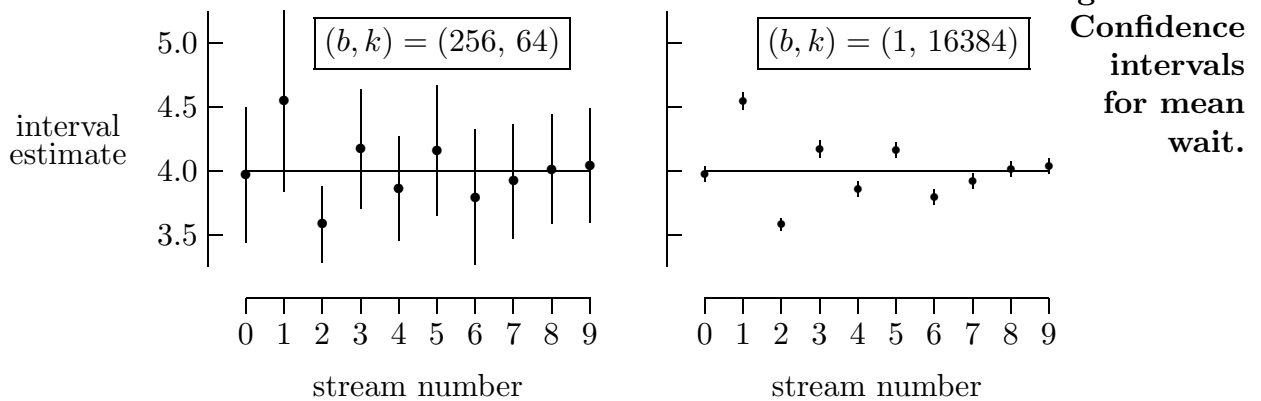
As illustrated on the left-hand-side of Figure 8.4.6, all but one of these interval estimates (the one corresponding to stream 2) brackets the true steady-state value, resulting in an *actual* coverage of 90%. In contrast, as illustrated on the right-hand-side of Figure 8.4.6, if the interval estimates are computed naively and incorrectly by *not* batching but, instead, just piping all $n = 16384$ waits directly into program `estimate`, then the intervals are *far* too narrow and only three out of ten estimates bracket the true steady-state value, resulting in an actual coverage of 30%. The message is clear — failure to correctly batch the data can lead to unrealistically small variance estimates, resulting in a false sense of confidence about the accuracy of the interval estimate. For example, if $(b, k) = (1, 16384)$, then a confidence interval estimate alleged to have 95% confidence appears to have a coverage of approximately 30%.

8.4.3   EXERCISES

Exercise 8.4.1[a]   In the notation of Section 1.2, let c_0 denote the departure time of the (virtual) 0th job (i.e., the initial value of `departure` in program `ssq2`). Assume a Poisson arrival process with rate λ, assume service times are $Exponential(1/\nu)$, and let W_1 be the associated random wait of the first job. (a) Prove that

$$E[W_1] = c_0 + \frac{1}{\nu} - \frac{1}{\lambda}\left(1 - \exp(-\lambda c_0)\right)$$

(b) How does this relate to Examples 8.4.1 and 8.4.3?

Exercise 8.4.2 (a) Repeat Example 8.4.5 except with observations obtained by $n = 32768$ calls to the function `Exponential(4.0)`. (b) Comment. (c) Why are the interval estimates so much more narrow? Do not just conjecture, explain.

Exercise 8.4.3   Produce a figure like Figure 8.4.6, but for the two cases $(b, k) = (64, 256)$ and $(b, k) = (16, 1024)$.

Exercise 8.4.4   The stochastic process $X_1, X_2, X_3, \ldots$ is said to be *covariance stationary* provided there is a common mean μ and variance σ^2

$$E[X_i] = \mu \quad \text{and} \quad E[(X_i - \mu)^2] = \sigma^2 \qquad\qquad i = 1, 2, 3 \ldots$$

and the covariance is independent of i so that

$$\frac{E[(X_i - \mu)(X_{i+j} - \mu)]}{\sigma^2} = \rho(j) \qquad\qquad j = 1, 2, 3, \ldots$$

For a integer batch size b, define the k batch means $\bar{X}_1$, $\bar{X}_2$, ..., $\bar{X}_k$ as in Algorithm 8.4.1. The grand sample mean is then

$$\bar{X} = \frac{\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_k}{k}.$$

(a) Prove that, for any value of b, the mean of $\bar{X}$ is

$$E[\bar{X}] = \mu.$$

(b) Prove that if b is large enough to make $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k$ independent, then the variance of $\bar{X}$ is

$$E[(\bar{X} - \mu)^2] = \frac{\sigma^2}{n}\xi_b$$

where $n = bk$ and

$$\xi_b = 1 + 2\sum_{j=1}^{b-1}(1 - j/b)\,\rho(j).$$

(c) What is the significance of this result?

The purpose of this section is to provide insight into the mathematics of traditional queuing theory as it applies to characterizing the *steady-state* statistics for a single-server service node. That is, in this section we will develop mathematical characterizations of what happens to job-averaged statistics and time-averaged statistics in the limit as the number of processed jobs becomes infinite. This discussion is based on the single-server service node model developed in Section 1.2.

8.5.1   GENERAL CASE

In all the discussion in this section we assume that the arrival process and service process are *stationary*. That is, random variate interarrival times can be generated by sampling from a fixed interarrival time distribution with mean $1/\lambda > 0$. Therefore, consistent with Definition 7.3.1, the arrival process is stationary with rate λ. (Later in the section, as necessary, we will assume a *Poisson* arrival process.) Similarly, service times can be generated by sampling from a fixed service time distribution with mean $1/\nu > 0$. Analogous to the arrival process terminology, it is conventional to say that the service process is stationary with rate ν.*

Effective Service Rate

Although it is conventional to refer to the reciprocal of the average service time as the "service rate" (see Definition 1.2.5), some explanation is necessary. Suppose we monitor jobs departing the service node.

- During a period of time when the server is always busy — as soon as one job departs another enters the server — the average time between departures will be $1/\nu$ and the departure times will form a stationary process with rate ν. Therefore, during a continuously busy period, the service process *is* stationary with rate ν.

- During a period of time when the server is always idle there are no departures and so the service rate is 0. Therefore, during a continuously idle period, the service process is stationary with rate 0.

Recall that the server utilization $\bar{x}$ is the proportion of time (probability) that the server is busy and so $1 - \bar{x}$ is the proportion of time (probability) that the server is idle. Therefore, an estimate of the *expected* service rate is $\bar{x} \cdot \nu + (1 - \bar{x}) \cdot 0 = \bar{x}\nu$. That is, because $\bar{x}$ is an estimate of the probability that the server is busy, the service rate is analogous to a discrete random variable with possible values ν (when busy) and 0 (when idle) and associated probabilities $\bar{x}$ and $1 - \bar{x}$. The expected value of this random variable is $\bar{x}\nu$. The point here is that ν is actually the *maximum* possible service rate. Because the server is occasionally idle the *effective* service rate is $\bar{x}\nu$.

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* For historical reasons, in queuing theory it is conventional to use μ to denote the service rate. Unfortunately, this notational convention conflicts with the common use of μ to denote a mean or expected value. Because of this notational conflict, ν is used consistently throughout this book to denote a service rate.
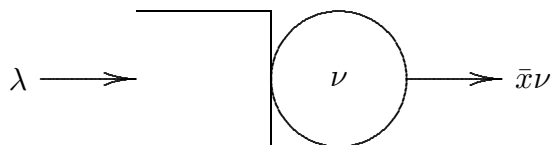
Flow Balance

Definition 8.5.1   The *traffic intensity* ρ is the ratio of the arrival rate to the service rate. That is, $\rho = \lambda/\nu$. (See Section 1.2.)

The traffic intensity is particularly relevant in steady-state queuing theory because of the following important theorem. Although a rigorous proof of this very general result is beyond the scope of this book, a specific instance of the theorem is proved later in this section.

Theorem 8.5.1   Steady-state (infinite-horizon) statistics exist for a stochastic, conservative single-server service node with infinite capacity and *any* queue discipline if and only if $\rho < 1$.

Assume that $\rho = \lambda/\nu < 1$ and let $\bar{x}$ be the steady-state utilization. Since the arrival rate is less than the service rate, it is intuitive that in this case the service node is "flow balanced" or equivalently, that there is a "conservation of jobs." That is, as illustrated in Figure 8.5.1, the average rate of flow of jobs *into* the service node is λ, the average rate that jobs flow *out* of the service node is $\bar{x}\nu$, and "flow in = flow out" so that $\lambda = \bar{x}\nu$.

**Figure 8.5.1.
Single-server
service node.**

If the flow balance equation $\lambda = \bar{x}\nu$ is compared with Definition 8.5.1 we see that it is equivalent to the equation $\bar{x} = \rho$. That is, if $\rho = \lambda/\nu < 1$ then ρ is the *steady-state* server utilization. If instead $\rho \geq 1$ then $\bar{x}$ will approach 1.0 as the number of jobs increases and the average queue length will grow without bound. If $\rho \geq 1$ then steady-state values for the statistics $\bar{d}$, $\bar{w}$, $\bar{q}$, and $\bar{l}$ are arbitrarily large.

Example 8.5.1   Use a modified version of program `ssq4` to simulate processing a large number of jobs, graphing the accumulated value of $\bar{x}$ every, say, 20 jobs (as in Example 3.1.3). If $\rho < 1$ then this sequence of $\bar{x}$'s will converge to ρ. If $\rho \geq 1$ then the convergence will be to 1.0.

Steady-state flow balance is a *long term* property — if $\rho < 1$ then in the long term as many jobs will flow into the service node as flow out. Flow balance is generally *not* true in the short term. Even if the mean arrival rate and service rate are constant, there will be periods of time when the *effective* arrival rate and/or service rate are significantly different from their mean values and during these periods "flow in $\neq$ flow out." During periods when the flow in is greater than the flow out, the queue length will build. When the opposite is true, the queue length will diminish. In either case there will be a natural ebb and flow *that will persist forever*. Steady-state does *not* mean that the state of the service node becomes constant — it means that the initial state of the system is not relevant.

Definition 8.5.2 Let $\bar{w}_n$ be the average wait experienced by the first n jobs to leave the service node

$$\bar{w}_n = \frac{1}{n} \sum_{i=1}^{n} w_i$$

and let $\bar{l}_n$ be the time-averaged number in the node over the time interval $(0, \tau_n)$

$$\bar{l}_n = \frac{1}{\tau_n} \int_0^{\tau_n} l(t)\, dt,$$

where τ_n is the departure time of the nth job to leave the service node. Let the average delay $\bar{d}_n$, average service time $\bar{s}_n$, time-averaged number in the queue $\bar{q}_n$, and the server utilization $\bar{x}_n$ be defined similarly.

Note that there is a subtle but significant difference between Definition 8.5.2 and the corresponding definitions in Section 1.2. In Definition 8.5.2 the job index is referenced to jobs *leaving* the service node.

- If the queue discipline is FIFO then $\tau_n = c_n$ and Definition 8.5.2 is identical to those used in Section 1.2.*

- If the queue discipline is not FIFO then the order of arrival is different than the order of departure (for at least some jobs) so τ_n may differ from c_n.

Concentrating as it does on departing jobs, Definition 8.5.2 allows for *any* type of queue discipline (see Exercise 1.2.7).

Using arguments similar to those in Section 1.2, it can be shown that the six statistics in Definition 8.5.2 are related as follows

$$\bar{w}_n = \bar{d}_n + \bar{s}_n$$
$$\bar{l}_n = \bar{q}_n + \bar{x}_n$$
$$\bar{l}_n \cong \left(\frac{n}{\tau_n} \right) \bar{w}_n$$
$$\bar{q}_n \cong \left(\frac{n}{\tau_n} \right) \bar{d}_n$$
$$\bar{x}_n \cong \left(\frac{n}{\tau_n} \right) \bar{s}_n$$

where, for any finite value of n, the $\cong$ is an equality if and only if the service node is idle at $t = 0$ and at $t = \tau_n$. Note that, if the service node is idle at τ_n then

$$\tau_n = \max\{c_1, c_2, \ldots, c_n\}.$$

What we will discuss next is what happens to $\bar{w}_n$, $\bar{l}_n$, $\bar{d}_n$, $\bar{q}_n$, $\bar{s}_n$, and $\bar{x}_n$ in the limit as $n \to \infty$, provided $\rho < 1$.

---

* Recall that c_n is the departure time of the nth job to *enter* the service node.

Definition 8.5.3   Assume that $\rho < 1$ (so that Theorem 8.5.1 applies) and let

$$\bar{w} = \lim_{n \to \infty} \bar{w}_n = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} w_i$$

and

$$\bar{l} = \lim_{n \to \infty} \bar{l}_n = \lim_{n \to \infty} \frac{1}{\tau_n} \int_0^{\tau_n} l(t)\, dt.$$

Define $\bar{d}$, $\bar{s}$, $\bar{q}$, and $\bar{x}$ similarly. Each of these quantities represents a steady-state statistic for a single-server service node.

Steady-State Equations

Given a knowledge of λ and ν with $\rho = \lambda/\nu < 1$, we want to determine equations that relate the steady-state statistics in Definition 8.5.3 to λ, ν, and ρ. Since $\bar{s}_n$ is an unbiased point estimate of the expected service time

$$\bar{s} = \lim_{n \to \infty} \bar{s}_n = 1/\nu.$$

In addition, we already argued that if $\rho < 1$ then

$$\bar{x} = \lim_{n \to \infty} \bar{x}_n = \rho.$$

Therefore, we know two of the six steady-state statistics in Definition 8.5.3. Moreover, if $\rho < 1$ then

$$\lim_{n \to \infty} \frac{n}{\tau_n} = \lambda.$$

This result can be argued intuitively based on the observation that in the time interval from 0 to τ_n the expected number of jobs to *enter* the service node is $\lambda \tau_n$. If n is sufficiently large then, because $\rho < 1$, this product is also the expected number of jobs to *leave* the service node. Therefore, $n \cong \tau_n \lambda$ or equivalently $n/\tau_n \cong \lambda$ with equality in the limit as $n \to \infty$. From this result we see that in the limit as $n \to \infty$ the "equation"

$$\bar{x}_n \cong \left(\frac{n}{\tau_n}\right) \bar{s}_n$$

becomes

$$\bar{x} \cong \frac{\lambda}{\nu}.$$

However, $\rho = \lambda/\nu$ and so as $n \to \infty$ we see that the $\cong$ will become an equality. It can be shown that the same comment also applies to the other two "equations"

$$\bar{l}_n \cong \left(\frac{n}{\tau_n}\right) \bar{w}_n \qquad \text{and} \qquad \bar{q}_n \cong \left(\frac{n}{\tau_n}\right) \bar{d}_n.$$

Therefore, we have the following theorem that summarizes much of the previous discussion.

Theorem 8.5.2 For a conservative single-server service node with infinite capacity and *any* queue discipline, if $\rho = \lambda/\nu < 1$ then the steady-state server utilization and average service time are

$$\bar{x} = \rho \qquad \text{and} \qquad \bar{s} = 1/\nu.$$

Moreover, the four steady state statistics $\bar{w}$, $\bar{l}$, $\bar{d}$, $\bar{q}$ are related by the four linear equations*

$$\bar{w} = \bar{d} + 1/\nu$$
$$\bar{l} = \bar{q} + \rho$$
$$\bar{l} = \lambda\bar{w}$$
$$\bar{q} = \lambda\bar{d}.$$

At first glance it might appear that the four linear equations in Theorem 8.5.2 relating $\bar{w}$, $\bar{l}$, $\bar{d}$, and $\bar{q}$ can be solved to determine unique values for these four steady-state statistics. That is *not* the case, however. These four equations can be written in matrix form as

$$\begin{bmatrix} \nu & 0 & -\nu & 0 \\ 0 & 1 & 0 & -1 \\ \lambda & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 \end{bmatrix} \begin{bmatrix} \bar{w} \\ \bar{l} \\ \bar{d} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} 1 \\ \rho \\ 0 \\ 0 \end{bmatrix}$$

and by applying elementary row operations it can be shown that the coefficient matrix has only *three* linearly independent rows. That is, three of the four steady-state statistics can be determined by solving these equations but the fourth statistic must be determined some other way — either by discrete-event simulation or, in special cases, by analytic methods.

Example 8.5.2 Suppose $\lambda = 0.5$ (so that $1/\lambda = 2.0$) and $\nu = 0.625$ (so that $1/\nu = 1.6$). Then the steady-state utilization is $\bar{x} = \rho = 0.8$ and $\bar{s} = 1.6$. Suppose also that we have used discrete-event simulation to determine that the steady-state average wait is $\bar{w} = 8.0$. Then the other three steady-state statistics are

$$\bar{d} = \bar{w} - 1/\nu = 8.0 - 1.6 = 6.4$$
$$\bar{l} = \lambda\bar{w} = 8.0/2 = 4.0$$
$$\bar{q} = \bar{l} - \rho = 4.0 - 0.8 = 3.2.$$

The discussion and results to this point have been "distribution free." That is, no distribution-specific stochastic assumptions were made about the interarrival times and service times. (In addition, the discussion and results to this point are valid for *any* queue discipline.) Given this generality, little more can be said without making additional distributional assumptions, as we will now do.

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* The last two equations in Theorem 8.5.2 are known as *Little's Equations* — they are an important part of the folklore of steady-state queuing theory.

8.5.2   M/M/1 SERVICE NODE

In the previous discussion we saw that the four steady-state single-server service node statistics $\bar{w}$, $\bar{l}$, $\bar{d}$, and $\bar{q}$ can be evaluated *provided* any one of the four can be evaluated. We will now consider an important *analytic* method whereby $\bar{l}$ (and thus $\bar{w}$, $\bar{d}$, and $\bar{q}$) can be determined — provided we are willing to make appropriate assumptions about the distribution of interarrival times and service times.

Definition 8.5.4   An $M/M/1$ service node has interarrival times and service times that are *iid* sequences of *Exponential*$(1/\lambda)$ and *Exponential*$(1/\nu)$ random variables, respectively.

Notation

The first M in the $M/M/1$ notation refers to the distribution of the interarrival times, the second M refers to the distribution of the service times, and 1 denotes the number of parallel servers. M refers to the "memoryless" (Markov) property of the *Exponential* distribution (see Section 7.4). The standard notation used for probability distributions in queuing theory are:

- M for *Exponential*;
- E for *Erlang*;
- U for *Uniform*;
- D for deterministic (not stochastic);
- G for general (a random variable with positive support not listed above).

Example 8.5.3   A $E/U/1$ service node has a single server with *Erlang* interarrival times and *Uniform* service times. Similarly, a $M/G/3$ service node has *Exponential* interarrival times and three statistically identical servers operating in parallel, each with the same general service time distribution. Program `ssq1` simulates a $G/G/1$ service node. Programs `ssq2` and `ssq3` simulate a $M/U/1$ service node. Program `ssq4` simulates a $M/E/1$ service node. Program `msq` simulates a $M/U/3$ service node.

M/M/1 Steady-State Characteristics

The *steady-state* characteristics of an $M/M/1$ service node have been extensively analyzed. One common approach to this analysis is based on the following definition.

Definition 8.5.5   Let the discrete random variable $L(t)$ denote the number of jobs in an $M/M/1$ service node at time $t > 0$. The possible values of $L(t)$ are $l = 0, 1, 2, \ldots$ and the associated pdf is
$$f(l, t) = \Pr(L(t) = l).$$

That is, the function $f(l, t)$ represents the pdf of the number of jobs at time t.

The pdf $f(l, t)$ can be *estimated* provided we have a a single-server service node next-event simulation program (like program `ssq4`) with the capability to report the number in the system at a specified time t. By using *many* replications and varying *only* the initial seed from replication to replication we can generate many independent samples of $L(t)$. From these samples, we can form a discrete-data histogram that will estimate $f(l, t)$. In particular, if we simulate an $M/M/1$ service node in this way, if the steady-state utilization $\rho = \lambda/\nu$ is less than 1, and if t is very large then, from the following theorem (stated without proof), the histogram will approximate the pdf of a *Geometric*(ρ) random variable,

Theorem 8.5.3   For an $M/M/1$ service node with $\rho = \lambda/\nu < 1$

$$\lim_{t \to \infty} f(l, t) = (1 - \rho)\rho^l \qquad l = 0, 1, 2, \ldots$$

That is, if L is the steady-state number of jobs in an $M/M/1$ service node then L is a *Geometric*(ρ) random variable.*

M/M/1 Steady-State Equations

Before we construct a proof of Theorem 8.5.3, observe that if L is a *Geometric*(ρ) random variable (see Section 6.4) then the steady-state average (expected) number in the service node is

$$\bar{l} = E[L] = \frac{\rho}{1 - \rho}.$$

Using this result and the steady-state equations from Theorem 8.5.2, it follows that

$$\bar{w} = \frac{\bar{l}}{\lambda} = \frac{1}{\nu - \lambda} \qquad \bar{q} = \bar{l} - \rho = \frac{\rho^2}{1 - \rho} \qquad \bar{d} = \frac{\bar{q}}{\lambda} = \frac{\rho}{\nu - \lambda}.$$

That is, these four equations, along with the three additional equations

$$\bar{x} = \rho = \frac{\lambda}{\nu} \qquad \bar{r} = \frac{1}{\lambda} \qquad \bar{s} = \frac{1}{\nu}$$

represent the *steady-state* statistics for an $M/M/1$ service node.

Example 8.5.4   If we simulate an $M/M/1$ service node for thousands of jobs, printing intermediate statistics every, say, 20 jobs, and if $\rho < 1$ then as $n \to \infty$ we will see the intermediate statistics $\bar{l}_n$, $\bar{w}_n$, $\bar{q}_n$, $\bar{d}_n$, $\bar{x}_n$, $\bar{r}_n$, and $\bar{s}_n$ all eventually converge to the values given by the $M/M/1$ steady-state equations.

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* The capacity must be infinite and the queue discipline cannot make use of any service time information. That is, relative to Definition 1.2.2, Theorem 8.5.3 is valid for FIFO, LIFO, and SIRO (service in random order) queue disciplines but is *not* valid for a priority queue discipline that is based on a knowledge of the service times.

Example 8.5.5   For selected values of ρ the steady-state* average number in the node, queue, and in service for an $M/M/1$ service node are

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\bar{l}$	0.11	0.25	0.43	0.67	1.00	1.50	2.33	4.00	9.00	∞
$\bar{q}$	0.01	0.05	0.13	0.27	0.50	0.90	1.63	3.20	8.10	∞
$\bar{x}$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0

Thus, for example, an $M/M/1$ service node operating with a steady-state utilization of 0.8 will contain a steady-state average of 3.2 jobs in the queue and 0.8 jobs in service for a total of 4.0 jobs in the service node. As illustrated in Figure 8.5.2, $\bar{l}$ (as well as $\bar{w}$, $\bar{q}$, and $\bar{d}$) becomes infinite as ρ approaches 1. This is consistent with the observation that steady-state is possible only if $\rho < 1$.

Figure 8.5.2. Average number of jobs in queue.

$$\bar{l} = \rho/(1-\rho)$$

Traffic Intensity (ρ)
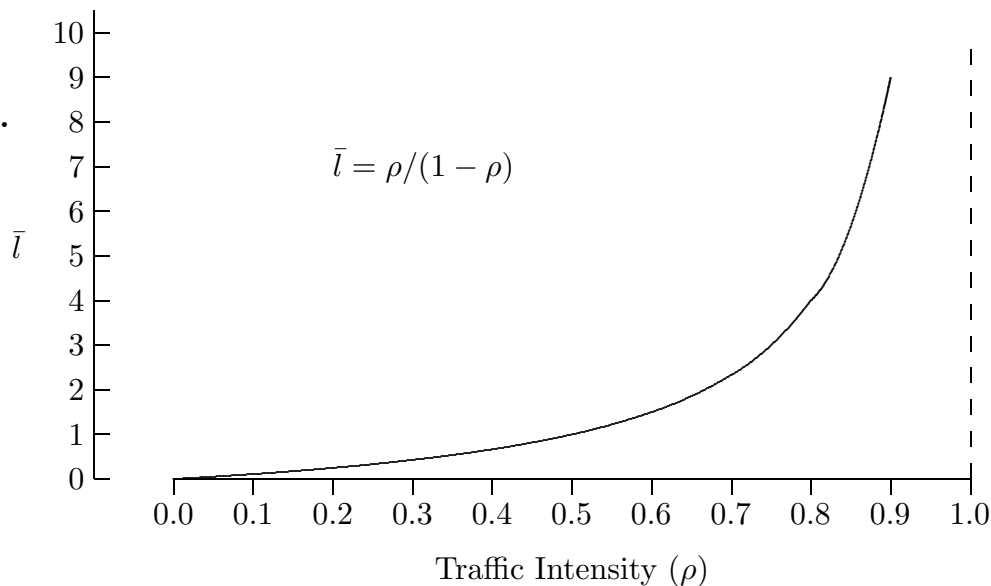
Example 8.5.6   If L is a *Geometric*(ρ) random variable then

$$\Pr(L = l) = (1 - \rho)\rho^l \qquad\qquad l = 0, 1, 2, \ldots$$

Thus, even though the *expected* number of jobs becomes infinite as $\rho \to 1$, if $\rho < 1$ then it is always true for an $M/M/1$ service node that the steady-state pdf is a monotone *decreasing* function of l. That is, for an $M/M/1$ service node with $\rho < 1$, the *most likely* steady-state number in the node is 0, the next most likely number is 1, etc. This is a counter-intuitive result, very much dependent on the memoryless property of *Exponential* random variables.

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   * Although the emphasis in this section is on steady-state behavior of the statistics associated with a single-server queue, Kelton and Law (1985), for example, consider the transient behavior of the statistics.

Proof of Theorem 8.5.3

To prove that the steady-state number of jobs in an $M/M/1$ service node is a *Geometric*(ρ) random variable, we begin by observing that the *arrivals* to an $M/M/1$ service node are a stationary Poisson process with rate λ where $1/\lambda > 0$ is the mean interarrival time. Let $\delta > 0$ be the width of a small interval of time and recall from Section 7.3 that the number of arrivals in this time interval is a *Poisson*$(\delta\lambda)$ random variable. Therefore

$$\Pr(x \text{ arrivals in interval of width } \delta) = \frac{\exp(-\delta\lambda)(\delta\lambda)^x}{x!} = \left(1 - \delta\lambda + \frac{(\delta\lambda)^2}{2!} - \cdots\right)\frac{(\delta\lambda)^x}{x!}$$

for $x = 0, 1, 2, \ldots$. It turns out that in the expansion of these *Poisson*$(\delta\lambda)$ probabilities, terms of order $(\delta\lambda)^2$ or higher can be ignored because we will ultimately derive the pdf of $L(t)$ by using a limit as $\delta \to 0$. Therefore, if $\delta\lambda$ is small

$$\Pr(0 \text{ arrivals in } \delta) \cong 1 - \delta\lambda$$
$$\Pr(1 \text{ arrival in } \delta) \cong \delta\lambda$$
$$\Pr(2 \text{ or more arrivals in } \delta) \cong 0.$$

Similarly, during a period when the server is continuously busy then, because the service times are *Exponential*$(1/\nu)$, the *departures* from the service node are a stationary Poisson process with rate ν where $1/\nu > 0$ is the mean service time. It follows from the previous discussion that if $L(t) > 0$ and if $\delta\nu$ is small relative to 1 then

$$\Pr(0 \text{ departures in } \delta) \cong 1 - \delta\nu$$
$$\Pr(1 \text{ departure in } \delta) \cong \delta\nu$$
$$\Pr(2 \text{ or more departures in } \delta) \cong 0.$$

These probabilities are valid provided there is at least one job in the node. Instead, if $L(t) = 0$ then the server is idle, no departure is possible, and so

$$\Pr(0 \text{ departures in } \delta) \cong 1$$
$$\Pr(1 \text{ or more departures in } \delta) \cong 0.$$

Let $f(l, t')$ denote the pdf of the random variable $L(t')$ with $t' = t + \delta$. That is, for $l = 0, 1, 2, \ldots$

$$f(l, t) = \Pr\big(L(t) = l\big) \qquad \text{and} \qquad f(l, t') = \Pr\big(L(t') = l\big).$$

The small-$\delta\lambda$ and small-$\delta\nu$ probability equations derived previously are the key to proving Theorem 8.5.3 by determining how $f(l, t)$ and $f(l, t')$ are related.

Transient Rate Equations

We now use the arrival and departure probabilities established previously to derive the relationship between $f(l, t')$ and $f(l, t)$. We do so by assuming that δ is sufficiently small so that both $\delta\nu$ and $\delta\lambda$ are sufficiently small and so terms of order δ^2 or smaller can be ignored.

First consider $f(l, t')$ for $l = 0$. This is the probability that the service node is idle at $t' = t + \delta$. If $L(t') = 0$ then it must be true that at time t:

- either $L(t) = 0$ and there was no arrival in the next δ;

- or $L(t) = 1$ and there was one departure in the next δ.

Therefore, by conditioning on the two possible states of the service node at time t, the probability that $L(t') = 0$ is

$$f(0, t') \cong f(0, t)(1 - \delta\lambda) + f(1, t)\delta\nu.$$

Equivalently, this equation can be written

$$\frac{f(0, t + \delta) - f(0, t)}{\delta} \cong \nu f(1, t) - \lambda f(0, t) \qquad l = 0.$$

In the limit as $\delta \to 0$ the $\cong$ becomes an equality and the left-hand side of this equation represents the first derivative of $f(0, t)$ with respect to t.

Now, consider $f(l, t')$ for $l > 0$. If $L(t') = l$ then it must be true that at time t:

- either $L(t) = l$ and there were no arrivals or departures in the next δ;

- or $L(t) = l + 1$ and there was one departure in the next δ;

- or $L(t) = l - 1$ and there was one arrival in the next δ.

Therefore, by conditioning on the three possible states of the service node at time t, the probability that $L(t') = l$ is

$$f(l, t') \cong f(l, t)(1 - \delta\lambda)(1 - \delta\nu) + f(l + 1, t)(1 - \delta\lambda)(\delta\nu) + f(l - 1, t)(\delta\lambda)(1 - \delta\nu).$$

If terms of order δ^2 or smaller are ignored, this equation can be rewritten as

$$f(l, t') \cong f(l, t)(1 - \delta\lambda - \delta\nu) + f(l + 1, t)\delta\nu + f(l - 1, t)\delta\lambda$$

which is equivalent to

$$\frac{f(l, t + \delta) - f(l, t)}{\delta} \cong \nu f(l + 1, t) - (\lambda + \nu)f(l, t) + \lambda f(l - 1, t) \qquad l = 1, 2, \ldots$$

As in the $l = 0$ case, in the limit as $\delta \to 0$ the $\cong$ becomes an equality and the left-hand side of this equation represents the first derivative of $f(l, t)$ with respect to t.

Steady-State Rate Equations

If $\rho < 1$ then as $t \to \infty$ the pdf $f(l, t)$ will lose its "memory" of the *initial* state of the queue (at $t = 0$) and converge to a steady-state pdf. That is, if t is large then the instantaneous rate of change of the pdf will become zero resulting in a pdf $f(l)$ that is *independent* of t and characterized by the equations

$$0 = \nu f(1) - \lambda f(0) \qquad\qquad l = 0$$
$$0 = \nu f(l + 1) - (\lambda + \nu)f(l) + \lambda f(l - 1) \qquad\qquad l = 1, 2, \dots$$

These equations can be solved for $f(\cdot)$ as follows.

- First divide by ν, then write the equations as

$$f(1) = \rho f(0) \qquad\qquad l = 0$$
$$f(l + 1) = (1 + \rho)f(l) - \rho f(l - 1) \qquad\qquad l = 1, 2, \dots$$

- From these equations we can argue inductively that

$$f(2) = (1 + \rho)f(1) - \rho f(0) = \rho^2 f(0)$$
$$f(3) = (1 + \rho)f(2) - \rho f(1) = \rho^3 f(0)$$
$$\vdots$$
$$f(l) = \dots = \rho^l f(0) \qquad\qquad l = 0, 1, 2, \dots$$

- Finally, by applying the usual pdf normalizing convention

$$1 = \sum_{l=0}^{\infty} f(l) = f(0) \sum_{l=0}^{\infty} \rho^l = f(0)(1 + \rho + \rho^2 + \cdots) = \frac{f(0)}{1 - \rho}$$

for $\rho < 1$, we find that the probability of an idle server is

$$f(0) = 1 - \rho$$

which is consistent with the fact that $1 - \rho$ is the steady-state probability of an idle server. Therefore,
$$f(l) = (1 - \rho)\rho^l \qquad\qquad l = 0, 1, 2, \dots$$

which proves Theorem 8.5.3 — the steady-state number in an $M/M/1$ service node is a *Geometric*(ρ) random variable.

Steady-State Waiting Time

Let the random variable W denote the steady-state waiting time in an $M/M/1$ service node. From results established earlier we know that the expected value of W is

$$\bar{w} = \frac{1}{\nu - \lambda}$$

but, what is the pdf of W?

If ρ is very small then most jobs will experience little or no delay so that their only wait will be their service time. Therefore, if ρ is small it seems intuitive that W will be *Exponential*$(1/\nu)$. On the other hand, if ρ is close to 1 then most jobs will experience large delays and their wait will be determined primarily by their delay time. In this case it is *not* intuitive that W should be *Exponential*. It is, however — provided the queue discipline is FIFO. Indeed, we can prove the following theorem.

Theorem 8.5.4 If $\rho = \lambda/\nu < 1$ then the steady-state wait in a FIFO $M/M/1$ service node with infinite capacity is an *Exponential*$(\bar{w})$ random variable with $\bar{w} = 1/(\nu - \lambda)$.

Proof Let l be the number in the service node at the instant a new job arrives. Since the steady-state number in the node is a *Geometric*(ρ) random variable, this occurs with probability $(1-\rho)\rho^l$. There are now $l+1$ jobs in the node and, because the queue discipline is FIFO, the time this new job will spend in the node is the *sum* of its service time and the service time of all those l jobs ahead of it. Each of the l jobs in the queue (including the job that just arrived) has a service time that is *Exponential*$(1/\nu)$ and, because of the memoryless property of the *Exponential* distribution, the *remaining* service time of the job in service is also *Exponential*$(1/\nu)$. Thus the wait experienced by a job that arrives to find l jobs in a FIFO $M/M/1$ service node is the sum of $l+1$ *iid Exponential*$(1/\nu)$ random variables. That is, this wait is an *Erlang*$(l + 1, 1/\nu)$ random variable W with possible values $w > 0$ and pdf

$$\frac{\nu}{l!}(\nu w)^l \exp(-\nu w) \qquad\qquad w > 0$$

(see Section 7.4). To determine the pdf of W it is necessary to condition on all possible values of l. That is, from the law of total probability the pdf of W is

$$\sum_{l=0}^{\infty} \frac{\nu(1-\rho)\rho^l}{l!}(\nu w)^l \exp(-\nu w) = \nu(1-\rho)\exp(-\nu w)\sum_{l=0}^{\infty}\frac{1}{l!}(\rho\nu w)^l$$

$$= (\nu - \lambda)\exp(-\nu w)\exp(\lambda w)$$

$$= \frac{1}{\bar{w}}\exp(-w/\bar{w}) \qquad\qquad w > 0$$

which proves that W is an *Exponential*$(\bar{w})$ random variable.

Caveat

It is important to avoid becoming too taken with the elegance of $M/M/1$ steady-state queuing theory results. These results provide valuable discrete-event simulation benchmarks and useful system performance tools, but their applicability to real systems is limited for at least three reasons.

- The assumption of an $Exponential(1/\nu)$ server is not valid in many practical applications — not even approximately. This is particularly true of service time distributions, which *rarely* have a mode of 0 in real-word systems.

- The assumption that both the arrival and service processes are *stationary* may not be valid, particularly if people are an integral part of the system.

- The number of jobs required to achieve statistics close to their steady-state value may be very large, particularly if the traffic intensity is close to 1.0.

Relative to the last point, see Exercise 8.5.4.

8.5.3   EXERCISES

Exercise 8.5.1   Prove that the coefficient matrix

$$\begin{bmatrix} \nu & 0 & -\nu & 0 \\ 0 & 1 & 0 & -1 \\ \lambda & -1 & 0 & 0 \\ 0 & 0 & \lambda & -1 \end{bmatrix}$$

has rank 3.

Exercise 8.5.2   If Q is the steady-state number in the queue of an $M/M/1$ service node (with $\rho < 1$) then what is the pdf of Q?

Exercise 8.5.3   Simulate an initially idle $M/M/1$ service node with $\lambda = 1$ and $\nu = 1.25$. Generate the histograms corresponding to $f(l, t)$ for $t = 10$, $t = 100$, and $t = 1000$. Repeat if the service times are $Erlang(4, 0.2)$. Comment on the shape of the histograms.

Exercise 8.5.4   Verify that if an $M/M/1$ service node is simulated for thousands of jobs, printing intermediate statistics every 50 jobs, and if $\rho < 1$, then as $n \to \infty$ the intermediate statistics $\bar{l}_n$, $\bar{w}_n$, $\bar{q}_n$, $\bar{d}_n$, $\bar{x}_n$, $\bar{r}_n$, and $\bar{s}_n$ will all eventually converge to the values given by the seven $M/M/1$ steady-state equations. Consider the cases $\rho = 0.3$, 0.6, and 0.9 and comment on the *rate* of convergence.

Exercise 8.5.5   An $M/M/1$ service node produced a steady-state value of $\bar{l} = 5$. What was ρ?

Exercise 8.5.6[a]   If D is the steady-state delay in the queue of an $M/M/1$ service node (with $\rho < 1$) then what is the pdf of D?

CHAPTER 9

INPUT MODELING

This chapter, which introduces procedures for developing credible input models, completes all of the building blocks for the modeling of a system via discrete-event simulation. We recommend reading Appendix G prior to reading this chapter for a discussion of the sources of error in discrete-event simulation modeling and for a development of a high-level framework for the modeling process.

Sections

In the various simulation models presented thus far in the text, both the choice of the distribution and the associated parameters have been "pulled out of thin air." Program `ssq2`, for example, uses $Uniform(a, b)$ for the service time distribution with parameters $a = 1$ and $b = 2$. These parameters have been fictitious so as to concentrate our efforts on developing algorithms for the modeling of the system, rather than determining the origins of their values. In practice, representative "input models" are determined by gathering data from the system under consideration, and analyzing that data to determine an appropriate probability distribution to mimic some aspect (e.g., service times) of the real-world system. This chapter introduces the input modeling process in an example-driven fashion.

Input modeling can be subdivided into stationary models, whose probabilistic mechanism does not vary with time, and nonstationary models, where time plays a substantive role in the model. Nonparametric (trace-driven) methods for determining stationary input models are introduced in Section 9.1, along with a discussion concerning the collecting of data. Parametric methods for determining stationary models are taken up in Section 9.2. We introduce the *method of moments* and *maximum likelihood* as two techniques for estimating parameter values for a particular distribution. Section 9.3 considers methods for determining nonstationary input models. The introduction to input modeling presented here is rather cursory, so the reader is encouraged to consult the more complete treatments in, for example, Banks, Carson, Nelson, and Nicol (2001, Chapter 9) Bratley, Fox, and Schrage (1987, Chapter 4), Fishman (2001, Chapter 10), Law and Kelton (2000, Chapter 6), and Lewis and Orav (1989, Chapters 6 and 7), and more advanced input modeling techniques in Nelson and Yaminitsky (1998).

Most discrete-event simulation models have stochastic elements that mimic the probabilistic nature of the system under consideration. A close match between the input model and the true underlying probabilistic mechanism associated with the system is required for successful input modeling. The general question considered in this section and the next two sections is how to model an element (e.g., arrival process, service times) in a discrete-event simulation model given a data set collected on the element of interest. For brevity, we assume that there is an existing system from which data can be drawn. The example-driven approach used here examines only introductory techniques for input modeling.

There are five basic questions to be answered in the following order:

- Have the data been sampled in an appropriate fashion?

- Should a *trace-driven* model or *parametric* probability model be selected as an input model? If the latter is chosen, the following three questions arise.

  — What type of distribution seems to "fit" the sample? Equivalently, what type of random variate generator seems to have produced the data? Does the $Exponential(\mu)$, $Gamma(a, b)$, or $Lognormal(a, b)$, for example, most adequately describe the data?

  — What are the value(s) of the parameter(s) that characterize the distribution? If the distribution is $Exponential(\mu)$, for example, then what is the value of μ?

  — How much confidence do we have in our answers to the two previous questions?

In statistical jargon the processes involved in answering these five questions are known as:

- assuring that appropriate *statistical sampling* procedures have been used for data collection;

- choosing between a *nonparametric* or *parametric* input model;

  — *hypothesizing* a distribution;

  — *estimating* the distribution's parameter(s);

  — *testing* for *goodness of fit*.

We will consider these five processes, in order.

9.1.1 DATA COLLECTION

There are two approaches that arise with respect to the collection of data. The first is the classical approach, where a designed experiment is conducted to collect the data. The second is the exploratory approach, where questions are addressed by means of existing data that the modeler had no hand in collecting. The first approach is better in terms of control and the second approach is generally better in terms of cost.

Collecting data on the appropriate elements of the system of interest is one of the initial and pivotal steps in successful input modeling. An inexperienced modeler, for example, improperly collects delay times on a single-server service node when the mean delay time is the performance measure of interest. Although these delay times are valuable for model validation, they do not contribute to the input model. The appropriate data elements to collect for an input model for a single-server service node are typically arrival and service times. The analysis of sample data collected on stationary processes is considered in this section and the analysis of sample data collected on nonstationary processes is considered in the next section.

Even if the decision to sample the appropriate element is made correctly, Bratley, Fox, and Schrage (1987) warn that there are several things that can be "wrong" with a data set. Vending machine sales will be used to illustrate the difficulties.

- Wrong amount of aggregation. We desire to model daily sales, but have only monthly sales.

- Wrong distribution in time. We have sales for this month and want to model next month's sales.

- Wrong distribution in space. We want to model sales at a vending machine in location A, but only have sales figures on a vending machine at location B.

- Insufficient distribution resolution. We want the distribution of the number of soda cans sold at a particular vending machine, but our data is given in cases, effectively rounding the data up to the next multiple of 24, the number of cans in a case.

- Censored data. We want to model *demand*, but we only have *sales* data. If the vending machine ever sells out, this constitutes a right-censored observation. The reliability and biostatistics literature contains techniques for accommodating censored data sets (Lawless, 2003).

All of these rather sophisticated depictions of data problems exclude the issue of *data validity*. Henderson (2004), for example, developed a trace-driven simulation model for ambulance planning where the ambulance took just one second to load the patient into the ambulance. How did the modeler arrive at such unusually efficient emergency personnel? The source of the problem was in the data collection procedures. The ambulance personnel had a "computer-aided dispatch" system which included a button situated on the dashboard of the ambulance to collect the arrival time to and departure time from the scene. The ambulance driver often (understandably) got busy and forgot to record event times by pressing the button. Later, when the error was discovered, the driver would press the button several times in a row in order to "catch up" on missed events. This would result in data values collected at the scene as short as one second, which, of course, turned up in the trace-driven simulation. The tiny event times were sampled occasionally, resulting in the unusually efficient performance. These errors not only corrupt the recorded time at the scene, but also corrupt any surrounding travel times.

There is a lot more that could be written about the careful collection of data and the assortment of pitfalls that await the collector. We end our short and sketchy discussion here and refer the interested reader to Cochran (1977) for further reading.

9.1.2 PRELIMINARY DATA ANALYSIS

We now begin a discussion of some preliminary steps that can be taken to analyze a data set. Following convention, the sample values are denoted by $x_1, x_2, \ldots, x_n$, and may be sampled from a discrete or continuous population. The following elementary example (which will be referenced throughout this section) is used to illustrate the types of decisions that often arise in input modeling for a stationary process.

Example 9.1.1 Consider a data set of $n = 23$ service times collected on a server in a single-server service node to determine an input model in a discrete-event simulation model. The service times in seconds are:

105.84	28.92	98.64	55.56	128.04	45.60	67.80	105.12	48.48

51.84	173.40	51.96	54.12	68.64	93.12	68.88	84.12	68.64

41.52	127.92	42.12	17.88	33.00.

The data values are collected in the order that they are displayed. [Although these service times actually come from the life testing literature (Caroni, 2002; Lawless, 2003), the same principles apply to both input modeling and survival analysis.]

The first step is to assess whether the observations are independent and identically distributed. The data must be given in the order collected for independence to be assessed. Situations where the identically distributed assumption would *not* be valid include:

- A new teller has been hired at a bank and the 23 service times represent a task that has a steep learning curve. The expected service time is likely to decrease as the new teller learns how to perform the task more efficiently.

- The service times represent 23 times to completion of a physically demanding task during an 8-hour shift. If fatigue is a significant factor, the expected time to complete the task is likely to increase with time.

If a simple linear regression of the service times versus the observation numbers shows a significant nonzero slope, then the identically distributed assumption is probably *not* appropriate, and a nonstationary model is appropriate. If the slope of the regression line does not statistically differ from zero, then a stationary model is appropriate. This simple linear regression differs slightly from those considered in Section 4.4 in that the observation number, which ranges from 1 to 23 and is known by statisticians as the *independent variable*, is fixed, rather than random.

Assume that there is a suspicion that a learning curve is present, which makes us suspect that the service times are decreasing. The scatterplot and least-squares regression line shown in Figure 9.1.1 indicate a slight downward trend in the service times. The regression line has a slope of -1.3 and an y-intercept of 87.9 seconds. But is this negative slope significantly different from zero? A hypothesis test* shows that the negative slope is more likely to be due to sampling variability, rather than a systematic decrease in service times over the 23 values collected. For the remainder of this section, we will assume that a *stationary* model is appropriate, and thus the observations can be treated as 23 identically distributed observed service times.

Figure 9.1.1. Ordered service times and regression line.

Assume that there is a suspicion that a learning curve is present

---
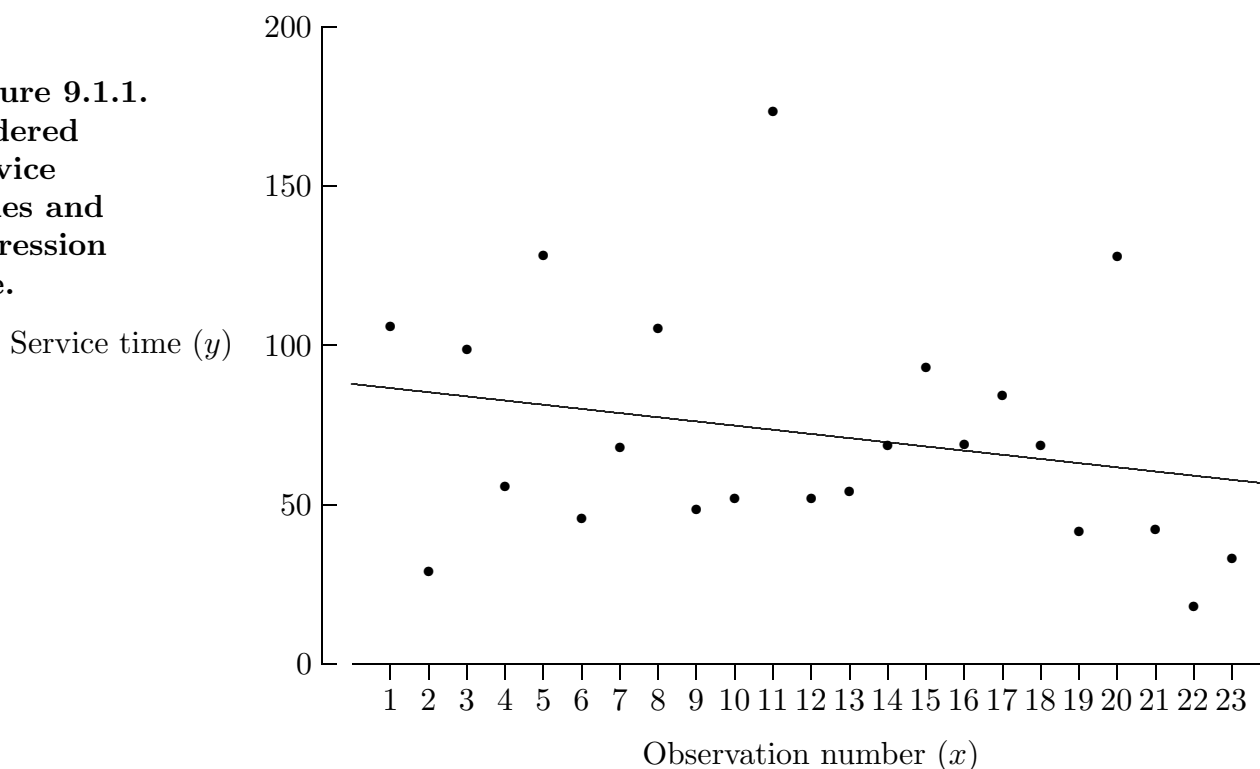
 * The reader should consult any introductory statistics textbook if hypothesis testing terminology and procedures are not familiar. Although beyond the prerequisites for this text, here are some of the details for the reader with a background in statistics. The null and alternative hypotheses are $H_0 : \beta_1 = 0$ and $H_1 : \beta_1 < 0$ associated with the linear model $Y = \beta_0 + \beta_1 X + \epsilon$, where X is the (fixed) observation number, Y is the associated (random) service time, β_0 is the intercept, β_1 is the slope, and ϵ is an error term [see Kutner, Nachtsheim, Neter, Wasserman (2003) for details]. The p-value associated with the hypothesis test is 0.14, which is insufficient evidence to conclude that there is a statistically significant learning curve present. The negative slope is probably due to sampling variability. The p-value may be small enough, however, to warrant further data collection.

There are a number of graphical and statistical methods for assessing independence. These include analysis of the sample autocorrelation function associated with the observations and a scatterplot of adjacent observations. The sample autocorrelation function for the service times is plotted in Figure 9.1.2 for the first eight lags. The sample autocorrelation function value at lag 1, for example, is the sample correlation for adjacent service times. The sample autocorrelation function value at lag 4, for example, is the sample correlation for service times four customers apart. We plot these autocorrelation values as spikes, rather than just points as we did in Figure 4.4.4, although this is largely a matter of taste. The horizontal dashed lines at $\pm 2/\sqrt{n}$ are 95% bounds used to determine whether the spikes in the autocorrelation function are statistically significant. Since none of the spikes strayed significantly outside of these bounds for this particular small data set, we are satisfied that the observations are truly independent and identically distributed. Since only $n = 23$ data values were collected and there seems to be no pattern to the sample autocorrelation function, we are likely to dismiss the spike at lag six falling slightly above the limits as being due to random sampling variability. Even if these were truly independent observations, we would expect 1 in 20 of the observations to fall outside of the 95% bounds. See Chatfield (2004) for further discussion of sample autocorrelation functions.
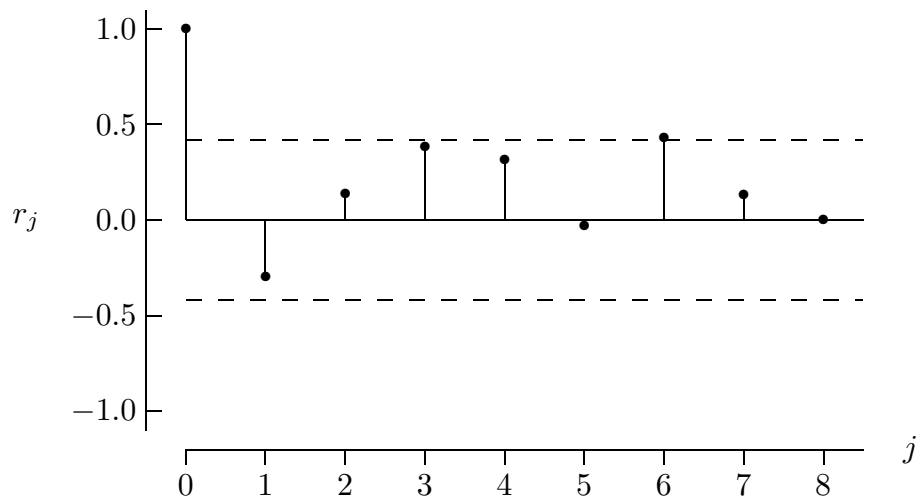
Figure 9.1.2. Sample autocorrelation function of the first eight lags.

The next decision that needs to be made is whether a parametric or nonparametric input model should be used. We discuss these two options in the following two subsections.

9.1.3 TRACE-DRIVEN (NONPARAMETRIC) MODELING

The trace-driven, or nonparametric approach allows a data set to "stand on its own", as opposed to being fit to a parametric model, such as the *Exponential*(μ) distribution. The decision of whether to use a nonparametric model or a parametric model can rest on the sample size n, the number of unusual "gaps" that occur in the data that should be smoothed over, and the importance and use of the input model in the simulation. We discuss discrete and continuous distributions separately.

Discrete distributions

When the simulation modeling situation calls for a discrete model (e.g., number of cars at a dealership, number of operations in a service time, or number of puppies in a litter), the modeling and associated variate generation algorithms using the trace-driven approach are straightforward. Let $x_1, x_2, \ldots, x_n$ be the data values collected on a random variable that is discrete by its nature. One simple trace-driven input model for variate generation would be to repeatedly select one of the data values with probability $1/n$. This corresponds to an "empirical" pdf

$$\hat{f}(x) = \frac{1}{n} \qquad x = x_i; \ i = 1, 2, \ldots, n$$

if all of the data values are distinct. With a discrete random variable, of course, it is likely that there are tied data values. If d data values, for example, are tied at a particular x_i, then the pdf at that particular x_i is $\hat{f}(x) = d/n$. The empirical cdf can be defined by

$$\hat{F}(x) = \frac{N(x)}{n} \qquad x = x_i; \ i = 1, 2, \ldots, n,$$

where $N(x)$ is the number of data values that are less than or equal to x.

Variate generation in this case is straightforward. An algorithm for generating variates for a discrete-event simulation requires only two lines. Algorithm 9.1.1 is synchronized, efficient, clear, and, if the data values are sorted, monotone. The algorithm has reasonable storage requirements for moderate sample sizes n, which is typically the case for data collected by hand.

Algorithm 9.1.1   Given the (sorted or unsorted) data values $x_1, x_2, \ldots, x_n$ stored in x[1], x[2], ..., x[n], the following algorithm returns one of the data values with proba-bility $1/n$ for use in a trace-driven input model.

```
i = Equilikely(1, n);
return (x[i]);
```

Two disadvantages to the trace-driven approach are the interpolation and extrapola-tion problems, as illustrated in the following example.

Example 9.1.2   Consider the number of door panels that contain painting defects in a lot of ten panels. Such a random variable is naturally modeled by a discrete distribution since it is a counting variable. Furthermore, it is clear that the support of this random variable is $\mathcal{X} = \{x | x = 0, 1, \ldots, 10\}$. The interpolation and extrapolation problems can be seen in the following scenario. Assume that $n = 50$ lots are sampled, yielding the data shown below.

x	:	0	1	2	3	4	5	6	7	8	9	10
counts	:	14	11	8	0	7	4	5	1	0	0	0

The empirical pdf is

$$\hat{f}(x) = \begin{cases} 14/50 & x = 0 \\ 11/50 & x = 1 \\ 8/50 & x = 2 \\ 7/50 & x = 4 \\ 4/50 & x = 5 \\ 5/50 & x = 6 \\ 1/50 & x = 7. \end{cases}$$

There is a "hole" in the data set at $x = 3$. Surprisingly, none of the 50 lots sampled contains exactly three door panels with defects. If we decide to take the trace-driven approach, the discrete-event simulation would never generate a lot that contains exactly three defects, which is almost certainly undesirable. This is known as the *interpolation* problem with the trace-driven approach. Likewise, it is not possible to generate exactly 8, 9, or 10 panels with defects using the trace-driven approach, which also may be undesirable. This is known as the *extrapolation* problem with the trace-driven approach. These two problems often lead modelers to consider the parametric approach for small data sets or those that exhibit significant random sampling variability. The parametric approach will smooth over the rough spots and fill the gaps that may be apparent in the data. The parametric $Binomial(10, p)$ model, for example, might be a more appropriate model for this particular data set. Estimating an appropriate value of p is a topic taken up later in this section.

Continuous distributions

When the simulation modeling situation calls for a continuous model (e.g., interarrival times or service times at a service node), the trace-driven approach is analogous to the discrete case. Again let $x_1, x_2, \ldots, x_n$ be the data values collected on a random variable that is continuous by its nature. One simple trace-driven input model for variate generation would be to repeatedly select one of the data values with probability $1/n$, just as in the discrete case. This again corresponds to an "empirical" pdf

$$\hat{f}(x) = \frac{1}{n} \qquad x = x_i; \ i = 1, 2, \ldots, n$$

if all of the data values are distinct. As before, if d data values are tied at a particular $x = x_i$ value, then the pdf at that particular x_i is $\hat{f}(x) = d/n$. The empirical cdf can be defined by

$$\hat{F}(x) = \frac{N(x)}{n} \qquad x = x_i; \ i = 1, 2, \ldots, n,$$

where $N(x)$ is the number of data values that are less than or equal to x. Variate generation is identical to the discrete case. The algorithm for generating variates for a discrete-event simulation is repeated here.

Algorithm 9.1.2   Given the (sorted or unsorted) data values $x_1, x_2, \ldots, x_n$ stored in x[1], x[2], ..., x[n], the following algorithm returns one of the data values with probability $1/n$ for use in a trace-driven input model.

```
i = Equilikely(1, n);
return (x[i]);
```

Example 9.1.3   Returning to the $n = 23$ service times from Example 9.1.1, the empirical cdf is plotted in Figure 9.1.3. The geometric interpretation of Algorithm 9.1.2 is as follows: the random number generated in the call to *Equilikely*(1, *n*) has a height on the $\hat{F}(x)$ axis that corresponds to a returned x-value associated with the stairstep empirical cdf.

Figure 9.1.3. Empirical cdf for the service times.
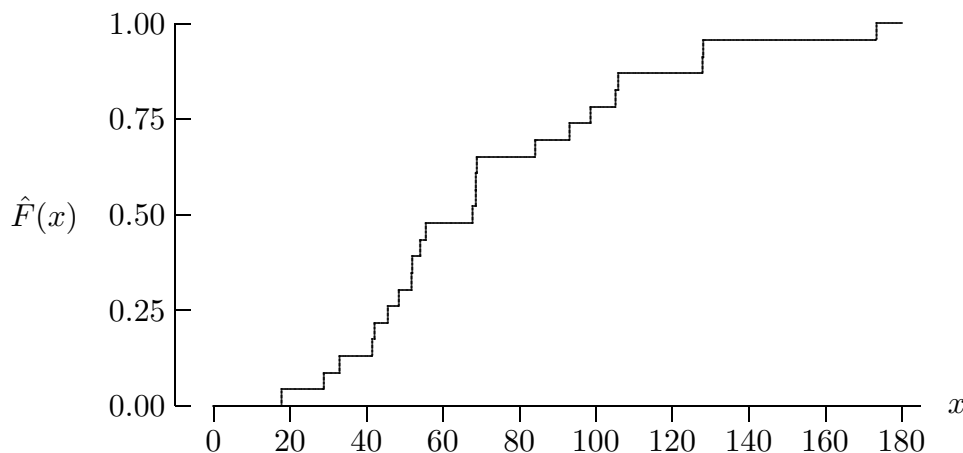
The nonparametric model that repeatedly selects one of the service times with probability 1/23 has several drawbacks for this data set. First, the small size of the data set indicates that there will be significant sampling variability. Second, the tied value, 68.64 seconds, and the observation in the far right-hand tail of the distribution, 173.40 seconds, tend to indicate that the smoothing associated with a parametric analysis may be more appropriate. When these are combined with the interpolation and extrapolation problems inherent to trace-driven simulations, a modeler might be pushed toward a parametric analysis.

There is a way to overcome the interpolation problem for a continuous data set. Using the $n - 1$ "gaps" created by the n data values, a piecewise-linear empirical cdf can be derived. These $n - 1$ gaps can each have a linear cdf that rises $1/(n - 1)$ between each of the sorted data values. Thus using the "rise over run" definition of the slope of a line, each piecewise-linear segment will have a slope of

$$\frac{1/(n-1)}{x_{(i+1)} - x_{(i)}}$$

on $x_{(i)} \leq x < x_{(i+1)}$ for $i = 1, 2, \ldots, n - 1$, where $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ are the (distinct) sorted data values.*

---

* The values $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ are referred to as "order statistics" by statisticians. See David (2004) for an introduction and comprehensive survey of the literature.

Since each of the piecewise segments must pass through the point $(x_{(i)}, (i-1)/(n-1))$, for $i = 1, 2, \ldots, n-1$, some algebra yields the piecewise-linear empirical cdf

$$\hat{F}(x) = \frac{i-1}{n-1} + \frac{x - x_{(i)}}{(n-1)(x_{(i+1)} - x_{(i)})} \qquad x_{(i)} \le x < x_{(i+1)},$$

for $i = 1, 2, \ldots, n-1$, and $\hat{F}(x) = 0$ for $x < x_{(1)}$, and $\hat{F}(x) = 1$ for $x \ge x_{(n)}$.* This estimator is still considered "nonparametric" because no parameters need to be estimated in order to arrive at the estimator.

Example 9.1.4 Returning to the $n = 23$ service times, the piecewise-linear empirical cdf is plotted in Figure 9.1.4. The empirical cdf has 22 linear segments, and, not surprisingly, has a shape that is similar to $\hat{F}(x)$ from Figure 9.1.3. This cdf allows x-values of any real value between $x_{(1)} = 17.88$ and $x_{(23)} = 173.40$, as opposed to just the data values, which was the case with the empirical cdf plotted in Figure 9.1.3. The cdf is not a function (but is plotted as a vertical line) at the tied data value 68.64. The piecewise-linear empirical cdf effectively solves the interpolation problem presented earlier.
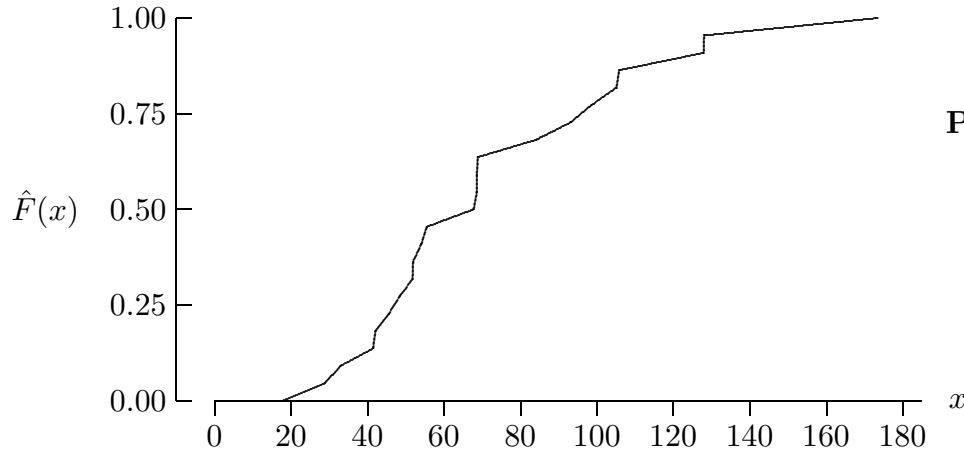
**Figure 9.1.4.
Piecewise-linear
empirical
cdf for
the service
times.**

We now consider generating variates from the piecewise-linear empirical cdf. Since this cdf is linear, it can be inverted in closed form for distinct data values. When this equation is solved for x (replacing $\hat{F}(x)$ with u), the following equation is appropriate for generating a variate on segment i

$$x = x_{(i)} + \big((n-1)u - (i-1)\big)(x_{(i+1)} - x_{(i)}) \qquad \frac{i-1}{n-1} \le u < \frac{i}{n-1},$$

for $i = 1, 2, \ldots, n-1$. A variate generation algorithm must first find the appropriate piecewise segment, then invert the linear function as shown above.

---

* Notice that this estimator is undefined for tied data values. Fortunately, as seen later, this does not create problems with the associated variate generation algorithm.

Algorithm 9.1.3   A one-time sort of the data values $x_1, x_2, \ldots, x_n$ into the order statistics $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ must be performed prior to executing this variate generation algorithm. These sorted values are stored in `x[1]`, `x[2]`, ..., `x[n]`. This algorithm returns a variate associated with the piecewise-linear empirical cdf for use in a trace-driven input model.

```
u = Random();
i = ⌈(n - 1) * u⌉;
return (x[i] + ((n - 1) * u - (i - 1)) * (x[i + 1] - x[i]));
```

Algorithm 9.1.3 is monotone, synchronized, clear, and efficient. Further, the use of the piecewise-linear empirical cdf eliminates the interpolation problem associated with Algorithm 9.1.2. The geometry behind the three steps in Algorithm 9.1.3 are: (i) a random number u is generated and plotted on the $\hat{F}(x)$ axis; (ii) the appropriate piecewise-linear segment corresponding to the random number is determined; (iii) the appropriate piecewise-linear segment is inverted yielding a variate in the range $x_{(1)} < x < x_{(n)}$.*

How does Algorithm 9.1.3 handle tied data values? These tied data values will be generated with probability d/n, where d is the number of values tied at a particular x-value. This is precisely what we would hope for in the case of tied values.

The piecewise-linear empirical cdf has solved the interpolation problem. Solving the extrapolation problem is difficult and requires more assumptions concerning the population.

This concludes our discussion of trace-driven modeling of stationary processes. The simple nonparametric techniques presented here let the data "speak for itself" rather than approximating the distribution with a parametric model such as the *Exponential*(μ) or *Normal*(μ, σ) distribution.

Considering the modeling framework developed in Appendix G, trace-driven simulations suffer from three sources of error: (i) error due to a finite data set embodied in $\mathcal{C}_r$; (ii) error due to a finite simulation run length embodied in $\mathcal{G}_r$; and (iii) error due to incorrect assumptions about the system embodied in $\mathcal{A}$. The parametric approach, presented next, suffers from these three sources of error and more. Barton and Schruben (2001) list five additional disadvantages of the parametric approach:

- Parametric modeling requires more effort and expertise than nonparametric modeling;

- Error is introduced when an inaccurate parametric model is selected;

- Additional error is introduced in the estimation of parameters;

- Significant serial correlation and between-variable dependencies can oftentimes be lost;

- The parametric approach is typically more difficult to "sell" to management.

---

* These are strict equalities on $x_{(1)} < x < x_{(n)}$ because we have assumed that the random number generator will not generate exactly 0.0 or 1.0.

In spite of these problems, parametric modeling remains significantly more popular than nonparametric modeling in the simulation community. A novel mix of the two approaches using Bézier distributions is given by Wagner and Wilson (1995). With their approach, the modeler determines whether a "notch" in the empirical cdf, such as the one between $x = 60$ and $x = 70$ in Figure 9.1.3, is part of the target cdf of interest (and should be included in the model) or just random sampling variability (which should be smoothed over in the model).

9.1.4 EXERCISES

Exercise 9.1.1 Consider the (tiny) fictitious data set with just $n = 3$ values collected: 1, 2, and 6. (a) Plot the empirical cdf. (b) Plot the piecewise-linear empirical cdf. (c) Generate 1000 samples of size $n = 3$ from the empirical cdf and generate 1000 samples of size $n = 3$ from the piecewise-linear empirical cdf, and calculate the sample means and variances of each data set of 1000 values. (d) Comment.

Exercise 9.1.2 Derive the equation

$$\hat{F}(x) = \frac{i-1}{n-1} + \frac{x - x_{(i)}}{(n-1)(x_{(i+1)} - x_{(i)})} \qquad x_{(i)} \leq x < x_{(i+1)},$$

for $i = 1, 2, \ldots, n-1$, associated with the piecewise-linear empirical cdf.

Exercise 9.1.3 Prove that

$$x = x_{(i)} + \big((n-1)u - (i-1)\big)(x_{(i+1)} - x_{(i)}) \qquad \frac{i-1}{n-1} \leq u < \frac{i}{n-1},$$

for $i = 1, 2, \ldots, n-1$, is the value of the idf associated with the piecewise-linear empirical cdf.

Exercise 9.1.4 Consider program ssq2. (a) Modify the program to compute and print the average delay times for 50 replications with $Exponential(2.0)$ interarrival times and $Uniform(1.0, 2.0)$ service times. (b) Modify the program to compute and print the average delay times for 50 replications with $Uniform(1.0, 2.0)$ service times, and interarrival times which are drawn from an empirical cdf of $n = 4$ $Exponential(2.0)$ random variates (use 50 such empirical cdfs). This is analogous to a trace-driven simulation when the service time distribution is known but only 4 interarrival times have been collected. (c) Compute the means and standard deviation of the two sets of numbers generated in parts (a) and (b). Plot an empirical cdf of the two sets of numbers. (d) Comment on the two empirical cdfs. (e) Would it have been a good or bad idea to have used a single random number stream to generate the service times in parts (a) and (b) of this problem? Why?

We now describe the parametric approach to the analysis of a stationary data set. The good news associated with the parametric approach is that the smoothing, interpolation, and extrapolation problems all vanish. The bad news is that more sophistication in probability and statistics is required (e.g., there is a wide array of parametric models, far beyond those described in this text, to choose from) and additional error is introduced when a parametric model is fitted to a data set. Like making a copy of a copy on a copying machine, we are one more layer removed from the data values when we fit a model.

9.2.1 HYPOTHESIZING A DISTRIBUTION

In some respects the most difficult aspect of fitting a distribution to data is the first step — hypothesizing a parametric model. This process is less "mechanical" than the others and consequently it is more dependent on insight, theory, and experience. At best, there are just guidelines and questions. The answers to the questions posed below can be used to rule in or rule out certain probability models.

- What is the *source* of the data? Is the data inherently discrete or continuous? Is the data bounded? Is it non-negative? Is the data produced by a stationary arrival process? Is the data the sum or product of random variables?

- Generate a histogram and look at its shape — frequently this will eliminate many distributions from further consideration. Is the histogram "flat"? Is it symmetric about the mean? The shape of the histogram can change drastically for different histogram parameters, and you may need to experiment with different values for the histogram parameters. Obviously, this is best done interactively in a computer graphics environment where the histogram parameters can be varied easily.

- Calculate some simple sample statistics in order to help thin the field of potential input models. For example, (i) a coefficient of variation $s/\bar{x}$ close to 1, along with the appropriate histogram shape, indicates that the $Exponential(\mu)$ distribution is a potential input model; (ii) a sample skewness close to 0 indicates that a symmetric distribution [e.g., a $Normal(\mu, \sigma)$ or $Uniform(a, b)$ distribution] is a potential input model.

At this point, if you are not a statistician, consider finding one to guide you through this process.

If the collection of data values is either expensive, difficult, or time consuming, you may have to address the question of what to do if you do not have enough data to generate a meaningful histogram. The answer is, not much — unless you can confidently hypothesize the distribution based primarily on a knowledge of the *source* of the data. For example, if the data are interarrival times and you have information that the process is stationary and likely to be a Poisson process, then the interarrival distribution is likely to be $Exponential(\mu)$ and you only need enough data to accurately estimate the mean interarrival time μ.

Example 9.2.1 A histogram with $k = 9$ cells for the $n = 23$ service times from Example 9.1.1 is given in Figure 9.2.1.
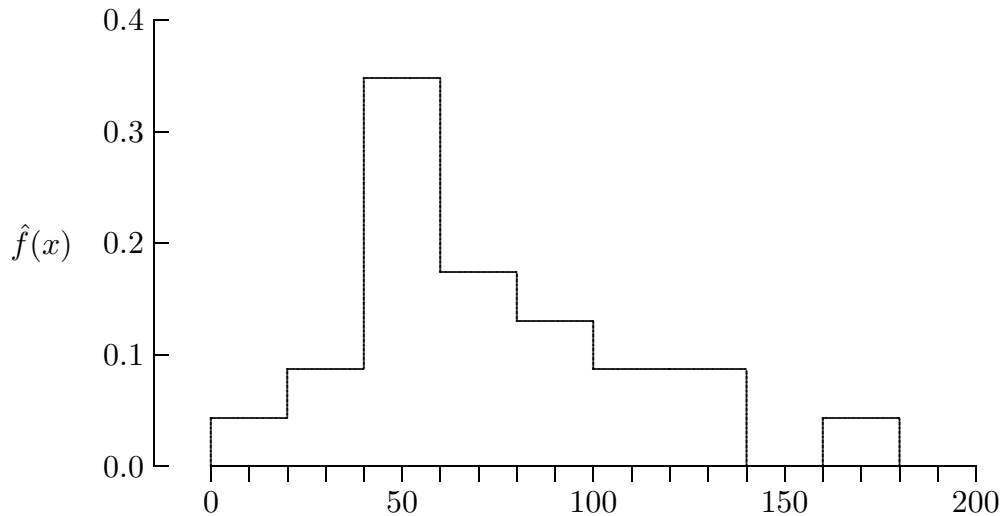

Figure 9.2.1. Histogram of service times.

It is very difficult to draw any meaningful conclusions with a data set that is this small. This histogram indicates that a skewed, bell-shaped pattern is appropriate for a pdf. The largest observation lies in the far right-hand tail of the distribution, so care must be taken to assure that it is representative of the population. The sample mean, standard deviation, coefficient of variation, and skewness are

$$\bar{x} = 72.22 \qquad s = 37.49 \qquad \frac{s}{\bar{x}} = 0.52 \qquad \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s}\right)^3 = 0.88.$$

Since the data set is inherently non-negative (since these are service times), a short list of the potential models from the text would include the $Lognormal(a, b)$ and the $Gamma(a, b)$ distributions.

Another graphical device for distinguishing one distribution from another is a plot of the skewness on the vertical axis versus the coefficient of variation on the horizontal axis. This approach has the advantage of viewing several competing parametric models simultaneously. The *population* coefficient of variation $\gamma_2 = \sigma/\mu$ and the *population* skewness $\gamma_3 = E[(X-\mu)^3/\sigma^3]$ can be plotted for several parametric distributions. The $Gamma(a, b)$ distribution, for example, has mean $\mu = ab$, standard deviation $\sigma = b\sqrt{a}$, coefficient of variation $\gamma_2 = 1/\sqrt{a}$, and skewness $\gamma_3 = 2/\sqrt{a}$. Thus a plot of γ_2 vs. γ_3 is linear. The special cases of the $Gamma(a, b)$ distribution, namely the $Chisquare(n)$, $Erlang(n, b)$, and $Exponential(\mu)$ distributions, all lie on this line. Some distributions are represented as *points*, while others lie in *lines*, while others fall in *regions*. A reasonable criterion for a hypothesized model is that the *sample* values for γ_2 and γ_3, calculated in the previous example as $\hat{\gamma}_2 = 0.52$ and $\hat{\gamma}_3 = 0.88$, should lie reasonably close to the point, curve, or region associated with the *population* values.

Example 9.2.2   A plot of γ_2 vs. γ_3 is given in Figure 9.2.2 for the *Exponential*(μ), *Gamma*(a, b), *Lognormal*(a, b), and *Weibull*(a, b) distributions. The *Weibull*(a, b) distribution will be introduced after this example. The point plotted for the service time data is at $(\hat{\gamma}_2, \hat{\gamma}_3) = (0.52, 0.88)$ and lies reasonably close (based on the small sample size) to the *Gamma*, *Lognormal*, and *Weibull* models. All three are potential parametric input models.

Figure 9.2.2.
Coefficient
of variation
vs. skewness.
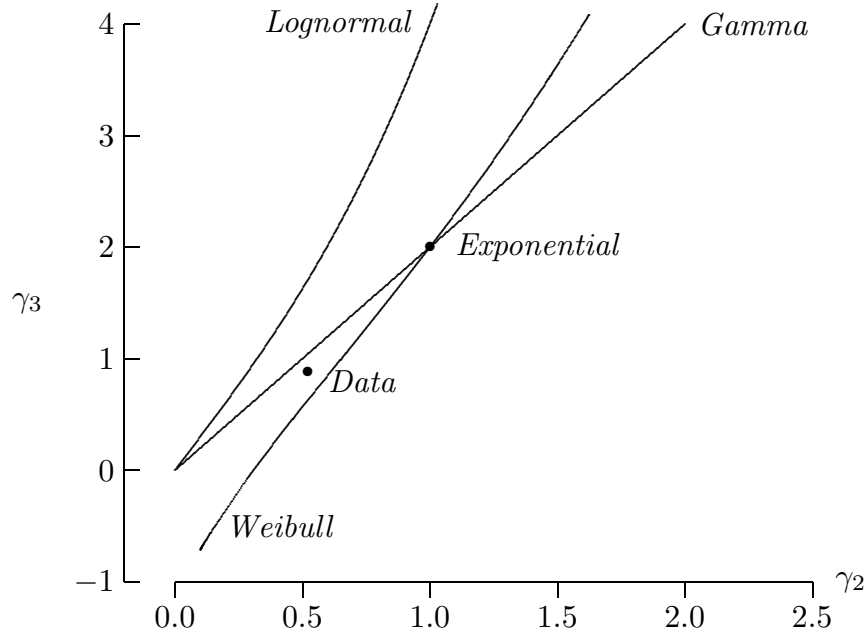
Weibull Random Variates

Since the *Lognormal*, *Gamma*, and *Weibull* models are candidates for modeling the service times, we briefly diverge from our discussion of input modeling to introduce the *Weibull* distribution.

Definition 9.2.1   The continuous random variable X is *Weibull*(a, b) if and only if

- the real-valued shape parameter a satisfies $a > 0$

- the real-valued scale parameter b satisfies $b > 0$

- the possible values of X are $\mathcal{X} = \{x | x > 0\}$

- the pdf of X is
$$f(x) = b^a a x^{a-1} \exp\left(-(bx)^a\right) \qquad x > 0$$

- the cdf of X can be expressed in closed form as*
$$F(x) = 1 - \exp\left(-(bx)^a\right) \qquad x > 0$$

---

* Beware! There is no standard parameterization for the *Weibull* distribution. Both $F(x) = 1 - \exp\left(-(x/b)^a\right)$ and $F(x) = 1 - \exp(-bx^a)$ are also legitimate *Weibull* cdfs.

- the mean of X is

$$\mu = \frac{1}{b} \, \Gamma\left(1 + \frac{1}{a}\right)$$

- the standard deviation of X is

$$\sigma = \frac{1}{b} \sqrt{\Gamma\left(1 + \frac{2}{a}\right) - \left[\Gamma\left(1 + \frac{1}{a}\right)\right]^2}.$$

The cdf of the *Weibull*(a, b) random variable can be inverted in closed form, resulting in efficient variate generation [unlike the *Gamma*(a, b) random variable, which requires an acceptance-rejection algorithm]. The *Weibull*(a, b) distribution reduces to the *Exponential*(μ) distribution when $a = 1$. The pdf is monotone decreasing for $a < 1$ and somewhat bell-shaped for $a > 1$.

The next step in the process of input modeling is to determine appropriate parameters for the hypothesized model — a process known as parameter estimation.

9.2.2 PARAMETER ESTIMATION

We will consider two basic approaches to estimating distribution parameters — the *method of moments* and *maximum likelihood estimation* techniques. For many of the distributions considered in this book, the two approaches yield similar or identical results. The general setting is as follows. We have collected data values denoted by $x_1, x_2, \ldots, x_n$ and used the techniques from the previous section to decide on one or more potential input models. Let q denote the number of unknown parameters [e.g., $q = 1$ for the *Exponential*(μ) model and $q = 2$ for the *Lognormal*(a, b) model].

Definition 9.2.2   The *method of moments* is an intuitive algebraic method for estimating the parameters of a distribution by equating the first q *population* moments defined by

$$E[X^k]$$

for $k = 1, 2, \ldots, q$ with the corresponding first q *sample* moments defined by

$$\frac{1}{n} \sum_{i=1}^{n} x_i^k$$

for $k = 1, 2, \ldots, q$, and solving the $q \times q$ set of equations for the q unknown parameters.

Although it may seem initially awkward to set a fixed quantity (the *population* moments) to a random quantity (the *sample* moments), the intuition associated with the method of moments technique is compelling — a fitted model will match the first q sample moments. We use hats to denote estimators. The method is easily learned by example.

Example 9.2.3   To estimate the single ($q = 1$) parameter μ for a $Poisson(\mu)$ model, equate the population mean to the sample mean,

$$\mu = \bar{x}$$

and solve for μ yielding the method of moments estimator $\hat{\mu} = \bar{x}$.

Example 9.2.4   To estimate the single ($q = 1$) parameter μ for a $Exponential(\mu)$ model, equate the population mean to the sample mean,

$$\mu = \bar{x}$$

and solve for μ yielding the method of moments estimator $\hat{\mu} = \bar{x}$. Even though the $Exponential(\mu)$ model was not on our short list of potential input models for the $n = 23$ service times from our analysis in the previous section, we will go through the process of fitting the distribution to make the point that the fitting process can be entered into blindly, with corresponding results. Since the sample mean is $\bar{x} = 72.22$, the method of moments parameter estimate is $\hat{\mu} = 72.22$. The fitted cdf is $\hat{F}(x) = 1 - \exp(-x/72.22)$ for $x > 0$. This fitted cdf is plotted along with the empirical cdf in Figure 9.2.3.

Figure 9.2.3. Fitted exponential cdf and empirical cdf.
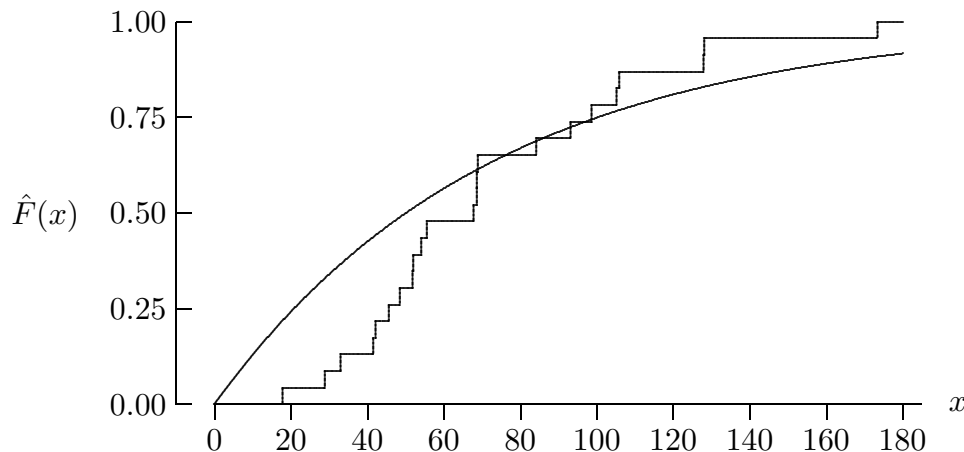
We can see that this is a very poor fit. The fitted *Exponential* cdf is much higher than the empirical cdf for small service times. This means that a discrete-event simulation using the *Exponential*(72.22) model will generate many more short service times than it should. Likewise, the fitted *Exponential* cdf is much lower than the empirical cdf for large service times. Using statistical jargon, this indicates that the fitted pdf has a heavier right-hand "tail" than the empirical. A discrete-event simulation using the *Exponential*(72.22) model will generate more long service times that it should. Fitting the *Exponential* model in this case has resulted in a fitted model with significantly higher variability than the data would suggest (the fitted model has standard deviation 72.22 whereas the sample standard deviation is 37.49). Using this particular input model will induce significant error into the discrete-event simulation. The accurate modeling of the right-hand tail in particular is important for queuing simulations since these values constitute the unusually long service times that cause congestion.

Example 9.2.5   To estimate the two ($q = 2$) parameters μ and σ for a $Normal(\mu, \sigma)$ model, equate the first two population moments to the first two sample moments,

$$E[X] = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$E[X^2] = \frac{1}{n} \sum_{i=1}^{n} x_i^2.$$

For compactness, denote the right-hand sides of these two equations by m_1 and m_2. Using the relationship $\sigma^2 = E[X^2] - \mu^2$, the equations can be rewritten as

$$\mu = m_1$$

$$\sigma^2 + \mu^2 = m_2.$$

Solving for μ and σ yields the method of moments estimators

$$\hat{\mu} = m_1 = \bar{x}$$

$$\hat{\sigma} = \sqrt{m_2 - m_1^2} = s.$$

Thus the method of moments technique estimates the population mean by the sample mean and the population standard deviation by the sample standard deviation.

In the previous three examples the population moments and the sample moments match up in a way that is intuitive. The next three examples illustrate the less intuitive case where oftentimes more algebra is required to arrive at the method of moments estimators.

Example 9.2.6   To estimate the single ($q = 1$) parameter p for the $Geometric(p)$ model, start by equating the population mean and the sample mean

$$\frac{p}{1 - p} = \bar{x}.$$

Then solve for p to yield the estimate

$$\hat{p} = \frac{\bar{x}}{1 + \bar{x}}.$$

The parameter p for the $Geometric(p)$ model must satisfy $0 < p < 1$. Does the method of moments estimator satisfy this constraint on the parameter? If the data is drawn from a $Geometric(p)$ population, then the data must assume the values $\mathcal{X} = \{0, 1, 2, \ldots\}$. This means that $\bar{x} \geq 0$. The only case where the method of moments experiences difficulty is a data set with $\bar{x} = 0$, which corresponds to a data set of all zeros.

Example 9.2.7   To estimate the two ($q = 2$) parameters a and b for the $Gamma(a, b)$ model, introduced in Section 7.6, equate the first two population moments to the first two sample moments,

$$ab = m_1$$

$$ab^2 + (ab)^2 = m_2$$

using the relationship $\sigma^2 = E[X^2] - \mu^2$. Solving for a and b yields
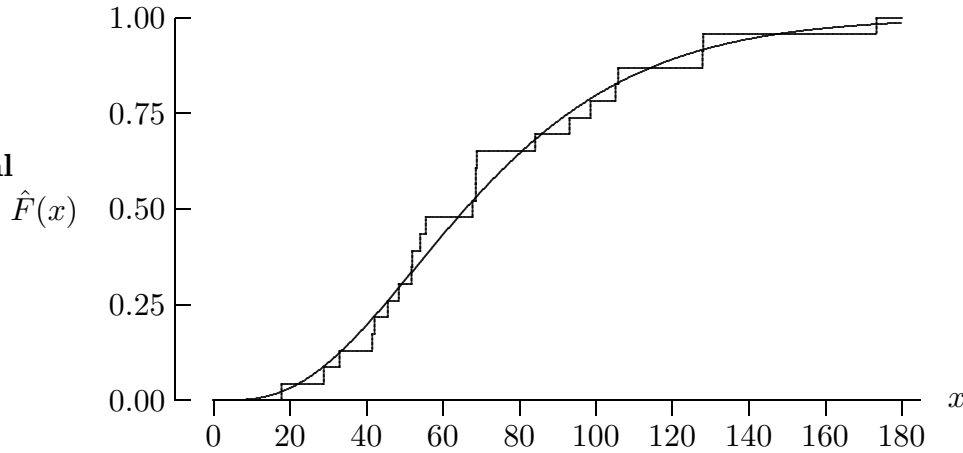
$$\hat{a} = \frac{m_1^2}{m_2 - m_1^2} = \frac{\bar{x}^2}{s^2} \qquad \text{and} \qquad \hat{b} = \frac{m_2 - m_1^2}{m_1} = \frac{s^2}{\bar{x}}.$$

Since the $Gamma(a, b)$ distribution was on our short list of potential input models for the service time data, we estimate the parameters as:

$$\hat{a} = \frac{\bar{x}^2}{s^2} = \frac{72.22^2}{37.49^2} = 3.7 \qquad \text{and} \qquad \hat{b} = \frac{s^2}{\bar{x}} = \frac{37.49^2}{72.22} = 19.46$$

yielding the fitted cdf displayed in Figure 9.2.4.

Figure 9.2.4. Fitted gamma cdf and empirical cdf.

The fitted $Gamma(a, b)$ model is *far* superior to the fitted $Exponential(\mu)$ model illustrated in Figure 9.2.3. Both tails and the body display better fits and, since there are now *two* parameters in the fitted models, both the mean and the variance are now appropriate. It is up to the modeler to determine whether the "notch" in the empirical cdf between $x = 60$ and $x = 70$ is: (i) an inherent and important part of the distribution which may require switching to a trace-driven approach or (ii) just sampling variability, in which case the $Gamma(a, b)$ model appropriately smooths over the notch.

Since the $Exponential(\mu)$ distribution is a special case of the $Gamma(a, b)$ distribution when $a = 1$, the $Gamma$ distribution will typically "appear" to fit a data set better, even when the data is drawn from an $Exponential$ population. The extra parameter parameter gives the $Gamma$ distribution more "flexibility" than the more rigid one-parameter $Exponential$. The extra parameter allows for an "S-shape" for the cdf, as illustrated in Figure 9.2.4, which is not possible for the cdf of an $Exponential$ random variable.

Example 9.2.8   To estimate the two ($q = 2$) parameters a and b for the $Uniform(a, b)$ distribution start by equating the first two population and sample moments

$$\frac{a + b}{2} = m_1$$

$$\frac{(b - a)^2}{12} + \left(\frac{a + b}{2}\right)^2 = m_2.$$

Then solve for a and b to yield the estimates

$$\hat{a} = \bar{x} - \sqrt{3}\, s$$

$$\hat{b} = \bar{x} + \sqrt{3}\, s.$$

The a, b estimates for the $Uniform(a, b)$ distribution provide a good illustration of the difference between method of moments estimates and maximum likelihood estimates. It is intuitive that *if* the data is truly $Uniform(a, b)$ then all the data must fall between a and b. The $\hat{a}$ and $\hat{b}$ values given above do not necessarily satisfy this criterion. The maximum likelihood estimators for a and b are the smallest and largest data values. When maximum likelihood estimates differ from method of moments estimates, it is generally considered desirable to use the former. Computational considerations are also important, however, and when there is a difference, method of moments estimates are often easier to calculate.

Definition 9.2.3   The *maximum likelihood estimators* are the parameter values associated with a hypothesized distribution that correspond to the distribution that is the most likely to have produced the data set $x_1, x_2, \ldots, x_n$. If $\theta = (\theta_1, \theta_2, \ldots, \theta_q)'$ is a vector of unknown parameters, then the maximum likelihood estimators $\hat{\theta}$, often abbreviated MLE, maximize the likelihood function

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

The vector of unknown parameters θ has been added to the pdf in order to emphasize the dependence of the likelihood function on θ.

Since the observations are independent, the likelihood function, $L(\theta)$, is the product of the pdf evaluated at each data value, known to statisticians as the *joint* pdf. The maximum likelihood estimator $\hat{\theta}$ is found by maximizing $L(\theta)$ with respect to θ, which typically involves some calculus. Thus the maximum likelihood estimator corresponds to the particular value(s) of the parameters that are most likely to have produced the data values $x_1, x_2, \ldots, x_n$.

In practice, it is often easier to maximize the log likelihood function, $\ln L(\theta)$, to find the vector of maximum likelihood estimators, which is valid because the logarithm function is monotonic. The log likelihood function is

$$\ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i, \theta)$$

and is asymptotically normally distributed by the central limit theorem, since it consists of the sum of n random, independent terms.

Example 9.2.9   Let $x_1, x_2, \ldots, x_n$ be a random sample from an *Exponential*(μ) population. The likelihood function is

$$L(\mu) = \prod_{i=1}^{n} f(x_i, \mu) = \prod_{i=1}^{n} \frac{1}{\mu} \exp(-x_i/\mu) = \mu^{-n} \exp\left(-\sum_{i=1}^{n} x_i/\mu\right).$$

The log likelihood function is

$$\ln L(\mu) = -n \ln \mu - \sum_{i=1}^{n} x_i/\mu.$$

In order to maximize the log likelihood function, differentiate with respect to μ.

$$\frac{\partial \ln L(\mu)}{\partial \mu} = -\frac{n}{\mu} + \frac{\sum_{i=1}^{n} x_i}{\mu^2}.$$

Equating to zero and solving for μ yields the maximum likelihood estimator

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i,$$

which is the sample mean $\bar{x}$. In this particular case, the method of moments and maximum likelihood estimators are identical. This is the case for many, but not all, of the distributions introduced in this text.

Example 9.2.10   Figure 9.2.5 illustrates the maximum likelihood estimator for μ for an *Exponential* population drawn from a (fictitious) sample size of $n = 3$

$$1 \qquad 2 \qquad 6.$$

(One could argue that these data values look more discrete than continuous, but we use integers to make the arithmetic easy.) Consider all the allowable μ values for an *Exponential*(μ) distribution (i.e., $\mu > 0$). Of the infinite number of $\mu > 0$, the $\hat{\mu} = \bar{x} = 3$ value depicted in Figure 9.2.5 (shown in the associated pdf) is that which maximizes the product of the pdf values at the data points, i.e., the product of the lengths of the vertical dashed lines.
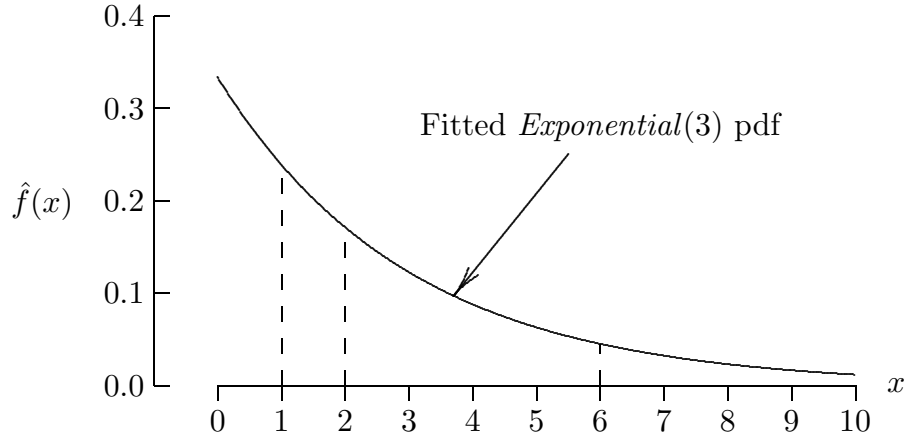
**Figure 9.2.5.
Maximum
likelihood
estimation
geometry.**

Any other choice of μ results in a smaller value of $L(\mu)$. The pdf plotted in Figure 9.2.5 is the one most likely to produce the observed data values. Although the next example violates one of the pre-requisites for the text (single-variable calculus), we include it to illustrate how two-parameter distributions are handled.

Example 9.2.11 The *Weibull*(a, b) distribution was also listed as a potential model for the $n = 23$ service times. We derive the $q = 2$ maximum likelihood estimates in the general case, then determine their numerical values associated with the service times. The *Weibull* distribution has pdf

$$f(x) = b^a a x^{a-1} \exp\left(-(bx)^a\right) \qquad x > 0,$$

where b is a positive scale parameter and a is a positive shape parameter. Let $x_1, x_2, \ldots, x_n$ denote the data values. The likelihood function is a function of the unknown parameters a and b

$$L(a, b) = \prod_{i=1}^{n} f(x_i) = b^{an} a^n \left[\prod_{i=1}^{n} x_i\right]^{a-1} \exp\left(-\sum_{i=1}^{n} (bx_i)^a\right).$$

The mathematics is typically more tractable for maximizing a log likelihood function, which, for the *Weibull* distribution, is

$$\ln L(a, b) = n \ln a + an \ln b + (a - 1) \sum_{i=1}^{n} \ln x_i - b^a \sum_{i=1}^{n} x_i^a.$$

Differentiating these equations with respect to the unknown parameters a and b and equating to zero yields the 2×2 set of nonlinear equations

$$\frac{\partial \ln L(a, b)}{\partial a} = \frac{n}{a} + n \ln b + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} (bx_i)^a \ln bx_i = 0$$

and

$$\frac{\partial \ln L(a, b)}{\partial b} = \frac{an}{b} - ab^{a-1} \sum_{i=1}^{n} x_i^a = 0.$$

These simultaneous equations have no closed-form solution for $\hat{a}$ and $\hat{b}$. To reduce the problem to a single unknown, the second equation can be solved for b in terms of a

$$b = \left(\frac{n}{\sum_{i=1}^{n} x_i^a} \right)^{1/a}.$$

Law and Kelton (2000, page 305) give an initial estimate for a and Qiao and Tsokos (1994) give a fixed-point algorithm for calculating $\hat{a}$ and $\hat{b}$. Their algorithm is guaranteed to converge for any positive initial estimate for a.

For the 23 service times, the fitted *Weibull* distribution has maximum likelihood estimators $\hat{a} = 2.10$ and $\hat{b} = 0.0122$. The log likelihood function evaluated at the maximum likelihood estimators is $\ln L(\hat{b}, \hat{a}) = -113.691$. Figure 9.2.6 shows the empirical cdf along with the cdf of the *Weibull* fit to the data.

Figure 9.2.6. Fitted Weibull cdf and empirical cdf.
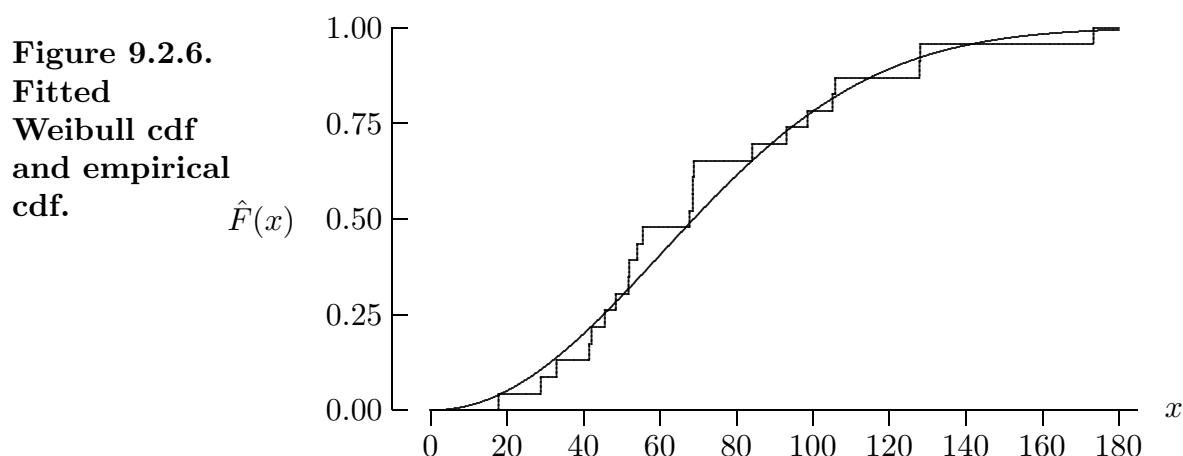
The difference between the fitted $Gamma(a, b)$ cdf displayed in Figure 9.2.4 and the fitted $Weibull(a, b)$ cdf displayed in Figure 9.2.6 may look so insignificant that the modeler might conclude that the two are interchangeable. This is an example of where a good knowledge of probability and statistics can help to differentiate between the two models. The $Gamma(a, b)$ distribution has "exponential" right-hand tails, which means that for large service times that have not been completed, the remaining service time is approximately exponentially distributed via the memoryless property. Is this an appropriate property for service times? That all depends on the situation. It is enough to say that the $Gamma(a, b)$ model has a heavier right-hand tail and that the $Weibull(a, b)$ model has a lighter right-hand tail for the particular fitted distribution given here. There may be specific information about the system that would help determine which model is superior. As mentioned earlier, precise modeling in the right-hand tail of a service time distribution is crucial to the effective modeling of the service node.

The method of moments and maximum likelihood estimators are special cases of what is known more generally in statistics as "point estimates." Hogg, McKean, and Craig (2005) and Casella and Berger (2002), for example, survey the properties of point estimators known as unbiasedness, minimum variance, efficiency, and consistency.

Accuracy of Point Estimators

Many argue that a point estimator is of little use by itself since it does not contain any information about the accuracy of the estimator. Statisticians often use *interval estimation* to quantify the precision of estimates. When there is a single ($q = 1$) parameter, a confidence interval is used to measure the precision of the point estimate. When there are several ($q > 1$) parameters, a "confidence region" is used to measure the precision of the point estimates. Interval estimators incorporate the variability of the point estimator to indicate the accuracy of the point estimator. Not surprisingly, these intervals and regions tend to narrow as the sample size n increases.

Although the specifics of interval estimation are beyond the prerequisites for this text, we present an example of such an interval estimate and its associated interpretation for the service time example.

Example 9.2.12   Returning to the $n = 23$ service times that were fitted by the *Weibull* model, the fact that the "likelihood ratio test statistic," defined by Casella and Berger (2003) as $2[\ln L(\hat{b}, \hat{a}) - \ln L(b, a)]$, is asymptotically *Chisquare*(2), means that a 95% confidence region for the parameters is the region containing all b and a satisfying

$$2[-113.691 - \ln L(b, a)] < 5.99,$$

where the 95th percentile of the *Chisquare*(2) distribution is 5.99. The maximum likelihood estimators and 95% confidence region are shown in Figure 9.2.7. The maximum likelihood estimators are plotted at the point $(\hat{a}, \hat{b}) = (2.10, 0.0122)$. The line $a = 1$ (when the *Weibull* distribution collapses to the *Exponential* distribution) is not interior to the confidence region, which gives a third confirmation that the *Exponential* distribution is not an appropriate model for the service time data set.
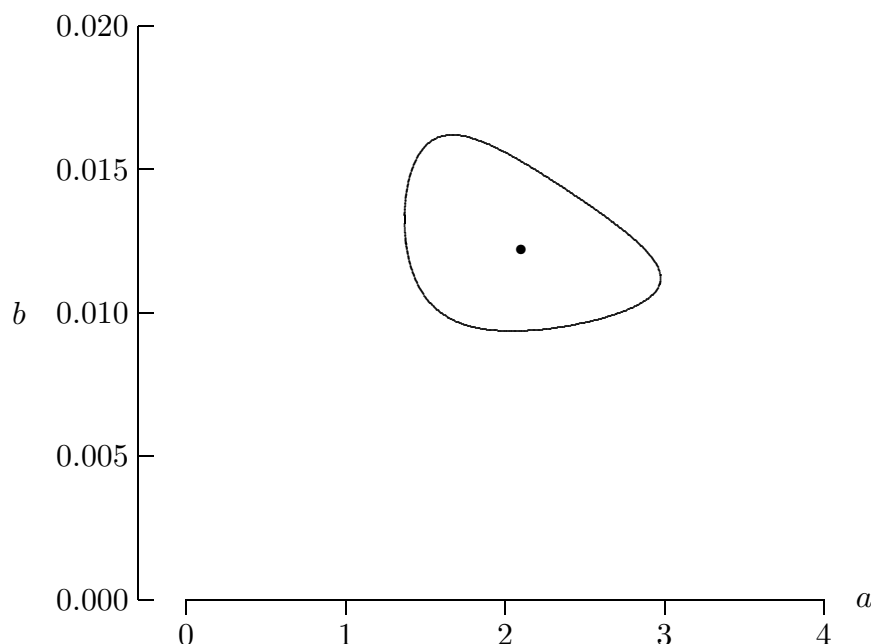

Figure 9.2.7. Confidence region for the estimated Weibull parameters.

A confidence region like the one plotted in Figure 9.2.7 can be quite helpful in determining the effect of the sampling variability associated with the data set drawn from the system on the simulation output. Assume that the $Weibull(a, b)$ model is selected as an input model. It would be reasonable to make simulation runs at the values of the maximum likelihood estimates $(\hat{a}, \hat{b}) = (2.10, 0.0122)$, as well as a few selected points along the boundary of the confidence region. In this manner the effect of how far the parameters of the input model could stray from the maximum likelihood estimators can be used to assess the impact on the output values from the simulation.

9.2.3   GOODNESS OF FIT

There are two approaches to testing for goodness of fit — visual and analytic. The visual method has been emphasized thus far. A *visual* test for goodness of fit is an educated inspection of how well the histogram or empirical cdf is approximated (*fitted*) by a hypothesized distribution pdf or cdf whose parameters have been estimated. Visually comparing the fitted cdf of the $Exponential(72.22)$ model with the empirical cdf in Figure 9.2.3, for example, allowed us to conclude that the $Exponential$ model was not an appropriate service time model. Other methods for visually comparing distributions include "P–P" (probability–probability) and "Q–Q" (quantile–quantile) plots as described, for example, in Law and Kelton (2000). A P–P plot is the fitted cumulative distribution function at the ith order statistic $x_{(i)}$, $\hat{F}(x_{(i)})$, versus the adjusted empirical cumulative distribution function, $\tilde{F}(x_{(i)}) = (i - 0.5)/n$, for $i = 1, 2, \ldots, n$. A plot where the points fall close to the line passing through the origin and (1, 1) indicates a good fit.

Example 9.2.13   For the $n = 23$ service times, a P–P plot for the $Weibull$ fit is shown in Figure 9.2.8, along with a line connecting $(0, 0)$ and $(1, 1)$. P–P plots should be constructed for all competing models.
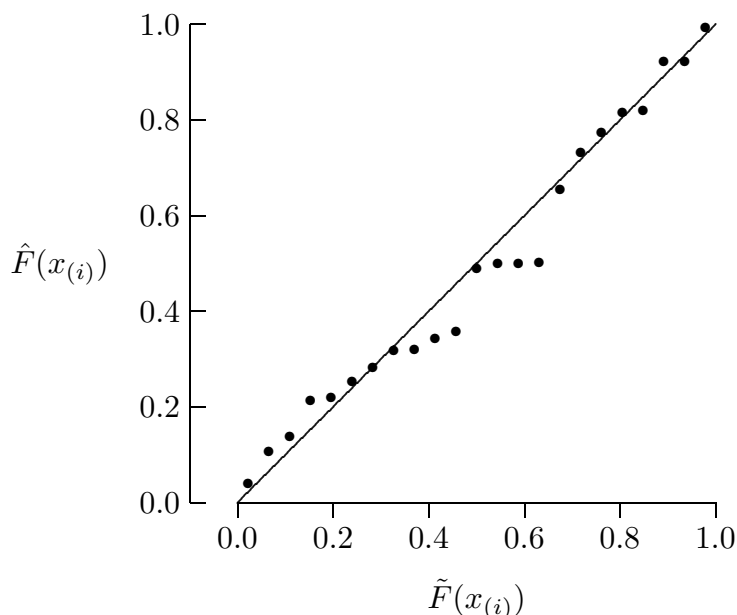

Figure 9.2.8. P–P plot for the fitted Weibull model.

We would like to have an objective technique assessing fit since visual methods are subjective. The analytic approach begins with two standard statistical tests: *chi-square* for discrete models and *Kolmogorov–Smirnov* for continuous models.

- If the data is discrete, the heights of the mass values on a discrete data histogram are compared with the corresponding mass values associated with the fitted distribution. The chi-square goodness-of-fit test statistic is a function of the squared differences between these heights.

- If the data is continuous, the Kolmogorov–Smirnov goodness-of-fit test statistic is the largest vertical distance between the empirical and fitted cdfs. This largest difference is traditionally denoted by D_n.

Example 9.2.14   The Kolmogorov–Smirnov test statistics for four of the models considered in this chapter, all fitted by maximum likelihood, are given below. A large vertical difference in cdfs for the *Exponential* model, and smaller vertical differences for the three two-parameter models, confirm our visual goodness-of-fit results from Figures 9.2.3, 9.2.4, and 9.2.6.

Distribution	D_{23}
$Exponential(\mu)$	0.307
$Weibull(a, b)$	0.151
$Gamma(a, b)$	0.123
$Lognormal(a, b)$	0.090

The Cramer–von Mises and Anderson–Darling goodness-of-fit tests (Lawless, 2003; D'Agostino and Stephens, 1986) are improvements upon the Kolmogorov–Smirnov test. These tests are often more sensitive to lack of fit in the tails of a distribution, which can often be critical in developing an input model.

Many of the discrete-event simulation packages and several stand-alone software packages automate all of the processes considered in this chapter with little or no expert intervention. These packages calculate parameter estimates for dozens of popular parametric distributions, and perform goodness-of-fit tests so that the distribution that best fits the data set can quickly be determined.

In both simulation education and practice, it is typically the case that the input model for the discrete-event simulation is considered "exact," and output analysis (considered in Chapter 9) estimates performance measures based on this incorrect assumption. The correct (and more difficult) approach is to somehow combine the error associated with the input modeling process with the error involved in simulating the system in the estimation of performance measures. Chick (2001), for example, incorporates random sampling variability from the data set in the analysis of the output from a discrete-event simulation.

9.2.4 EXERCISES

Exercise 9.2.1 Consider program `ssq2`. (*a*) Modify the program to compute and print the average delay times for 50 replications with *Exponential*(2.0) interarrival times and *Uniform*(1.0, 2.0) service times. (*b*) Modify the program to compute and print the average delay times for 50 replications with *Exponential*($\hat{\mu}$) interarrival times and *Uniform*(1.0, 2.0) service times, where $\hat{\mu}$ is the sample mean of $n = 4$ *Exponential*(2.0) random variates (use 50 such sample means). (*c*) Compute the means and standard deviation of the two sets of numbers generated in parts (*a*) and (*b*). Plot an empirical cdf of the two sets of numbers. (*d*) Comment.

Exercise 9.2.2 Explain why the *Gamma*(*a*, *b*) distribution is generally preferred to the *Erlang*(*n*, *b*) distribution for modeling a continuous random variable.

Exercise 9.2.3 Explain why you would be nervous about the validity of a discrete-event simulation model with a *Uniform*(*a*, *b*) service time.

Exercise 9.2.4 Perform the algebra necessary to arrive at the method of moments estimators for *a* and *b* in Example 9.2.8.

Exercise 9.2.5 Use a modified version of program `ssq2` to estimate the steady-state wait for customers entering a $M/M/1$ service node with arrival rate $\lambda = 1$, and service rate $\nu = 1.25$. Which of the continuous distributions we have studied seems to fit this data best? Comment.

Exercise 9.2.6 Derive the *Lognormal*(*a*, *b*) method of moments parameter estimation equations.

Exercise 9.2.7[a] Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from an *inverse Gaussian* (Wald) population with positive parameters λ and μ, and pdf (Chikkara and Folks, 1989)

$$f(x) = \sqrt{\frac{\lambda}{2\pi}} x^{-3/2} \exp\left(-\frac{\lambda}{2\mu^2 x}(x - \mu)^2\right) \qquad x > 0.$$

Find the maximum likelihood estimators for μ and λ.

Exercise 9.2.8 Find the maximum likelihood estimators for μ when n independent and identically distributed observations are drawn from a *Poisson*(μ) population.

Exercise 9.2.9 Use a modified version of program `ssq3` to supply convincing numerical evidence that the steady-state number of customers in a $M/M/1$ service node is a *Geometric*(ρ) random variable.

Exercise 9.2.10 Use a modified version of program `msq` to estimate the pdf of the steady-state number of customers in a $M/M/20$ service node. What discrete distribution fits this histogram best?

There are modeling situations that arise where one of the stationary input models from the previous section does not adequately describe a stochastic element of interest. A *nonstationary* model is needed to model the arrival times of patients to an emergency room, the failure times of an automobile, or the arrival times of customers to a fast food restaurant. Any stochastic model that varies with time requires a nonstationary model. Therefore, accurate input modeling requires a careful evaluation of whether a stationary (no time dependence) or nonstationary model is appropriate.

9.3.1   AN EXAMPLE

Modeling customer arrival times to a lunch wagon will be used throughout this section to illustrate the decision-making process involved in determining a nonstationary input model.

Example 9.3.1   Customer arrival times to a lunch wagon between 10:00 AM and 2:30 PM are collected on three days. These realizations were generated from a hypothetical arrival process given by Klein and Roberts (1984). A total of 150 arrival times were observed, comprising 56 on the first day, 42 on the second day, and 52 on the third day. Defining $(0, 4.5]$ to be the time interval of interest (in hours), the three realizations are

$$0.2153 \quad 0.3494 \quad 0.3943 \quad 0.5701 \quad 0.6211 \quad \ldots \quad 4.0595 \quad 4.1750 \quad 4.2475,$$

$$0.3927 \quad 0.6211 \quad 0.7504 \quad 0.7867 \quad 1.2480 \quad \ldots \quad 3.9938 \quad 4.0440 \quad 4.3741,$$

and

$$0.4499 \quad 0.5495 \quad 0.6921 \quad 0.9218 \quad 1.2057 \quad \ldots \quad 3.5099 \quad 3.6430 \quad 4.3566.$$

One preliminary statistical issue concerning this data is whether the three days represent processes drawn from the same population. External factors such as the weather, day of the week, advertisement, and workload should be fixed. For this particular example, we assume that these factors have been fixed and the three realizations are drawn from a representative target arrival process.

Since the arrival times are realizations of a continuous-time, discrete-state stochastic process, the remaining question concerns whether or not the process is stationary. If the process proves to be stationary, the techniques from the previous section, such as drawing a histogram, and choosing a parametric or nonparametric model for the *interarrival* times, are appropriate. On the other hand, if the process is nonstationary, a nonstationary Poisson process, which was introduced in Section 7.5, might be an appropriate input model. Recall that a nonstationary Poisson process is governed by an event rate function $\lambda(t)$ that gives an arrival rate [e.g., $\lambda(2) = 10$ means that the arrival rate is 10 customers per hour at time 2] that can vary with time. Although we place exclusive emphasis on the nonstationary Poisson process model in this section, there are dozens of other nonstationary models that are appropriate in other modeling situations. As in the previous section on the modeling of stationary processes, input models can be divided into trace-driven (nonparametric) and parametric models. We begin with trace-driven models.

9.3.2 TRACE-DRIVEN MODELING

This subsection describes three nonparametric (trace-driven) procedures for estimating the event rate function $\lambda(t)$ or, equivalently, the cumulative event rate function

$$\Lambda(t) = \int_0^t \lambda(\tau)d\tau$$

from k realizations sampled from a "target" nonstationary Poisson process. The term *target* refers to the population process that we want to estimate and simulate. The first procedure can be used on "count" data. The second and third procedures are appropriate for "raw" data. In each of the procedures, we (i) find point estimates for the cumulative event rate function, (ii) find interval estimates for the cumulative event rate function, and (iii) develop algorithms for generating event times from the estimated process.

Count Data

The event rate function $\lambda(t)$ or cumulative event rate function $\Lambda(t)$ is to be estimated on $(0, S]$, where S is a known constant. The interval $(0, S]$ may represent the time a system allows arrivals (e.g., 9:00 AM to 5:00 PM at a bank) or one period of a cycle (e.g., one day at an emergency room). There are k representative realizations collected on the target process on $(0, S]$.

For systems with high arrival rates (e.g., busy call centers or web sites), there is often so much data that *counts* of events that occur during *bins* (subintervals) are available, rather than the raw event times. Although this is less preferable than having the raw data, it is still possible to construct an estimate of the event rate function and generate variates for a discrete-event simulation model. The time interval $(0, S]$ can be partitioned into m subintervals

$$(a_0, a_1], (a_1, a_2], \ldots, (a_{m-1}, a_m],$$

where $a_0 = 0$ and $a_m = S$. The subintervals do not necessarily have equal widths. Let $n_1, n_2, \ldots, n_m$ be the total number of observed events in the subintervals over the k realizations.

Example 9.3.2 Consider the $m = 9$ equal-width subintervals associated with the lunch wagon example

$$(0.0, 0.5], (0.5, 1.0], \ldots, (4.0, 4.5].$$

The counts during each half-hour subinterval are

subinterval number, i :	1	2	3	4	5	6	7	8	9
event count, n_i :	5	13	12	24	32	27	21	10	6

Since the counts range from $n_1 = 5$ arrivals in the first subinterval to $n_5 = 32$ arrivals in the fifth subinterval, there is a strong suspicion that a nonstationary model is appropriate and we can proceed toward fitting a nonstationary Poisson process to the data set of counts.

To simplify the estimation process, assume that the target nonstationary Poisson process has an event rate function $\lambda(t)$ that is piecewise constant on each subinterval of the partition $(a_0, a_1], (a_1, a_2], \ldots, (a_{m-1}, a_m]$. Since the average event rate function on the interval $(a_{i-1}, a_i]$ is the rate per unit time of the events that occur on that interval, the maximum likelihood estimator is the average number of events that occurred on the interval, normalized for the length of the interval

$$\hat{\lambda}(t) = \frac{n_i}{k(a_i - a_{i-1})} \qquad a_{i-1} < t \le a_i$$

for $i = 1, 2, \ldots, m$.

Example 9.3.3 For the lunch wagon count data, there are five arrivals in the first subinterval. The estimated event rate during this subinterval is

$$\hat{\lambda}(t) = \frac{5}{3(0.5 - 0.0)} = 10/3 \qquad a_{i-1} < t \le a_i$$

customers per hour. Figure 9.3.1 shows the event rate function for all nine of the subintervals calculated in a similar manner.
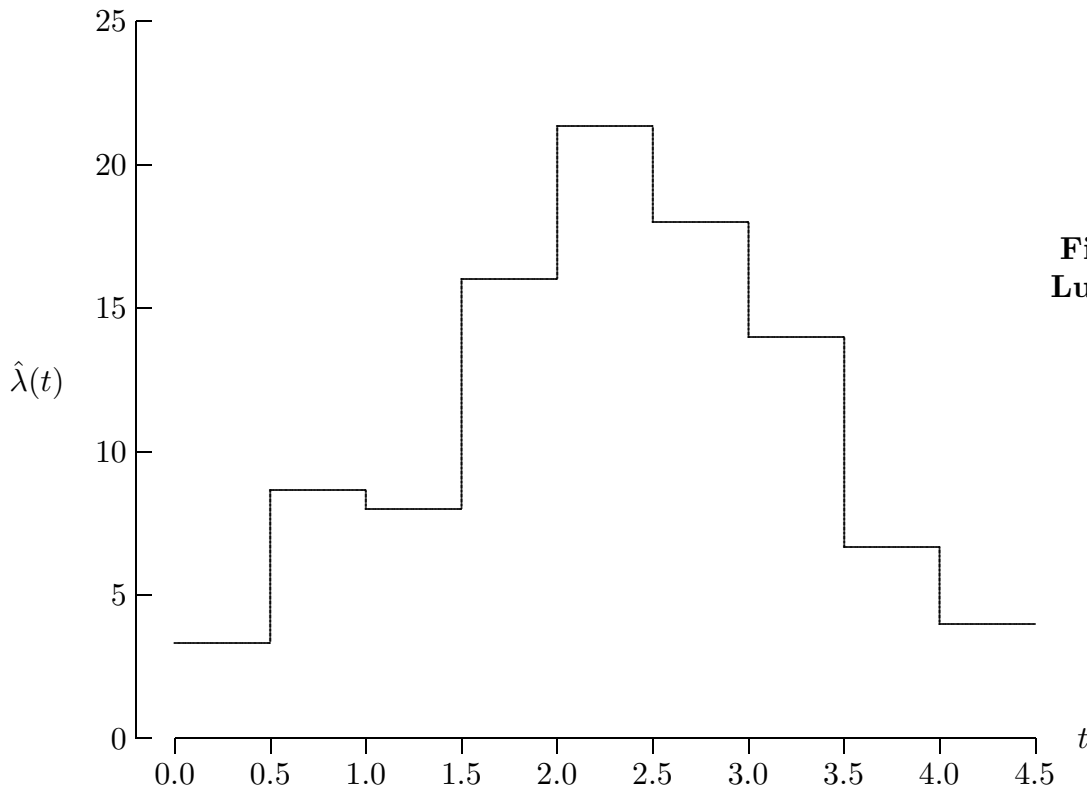

Figure 9.3.1. Lunch wagon event rate function estimate.

We have connected the piecewise constant segments with vertical lines, although this is largely a matter of taste. Even though the diagram in Figure 9.3.1 looks tantalizingly close to a histogram, it should not be called a "histogram" because of the time dependence.

Since the event rate estimator is piecewise constant, the associated cumulative event rate function estimator is a continuous, piecewise-linear function on $(0, S]$

$$\hat{\Lambda}(t) = \int_0^t \hat{\lambda}(\tau)d\tau = \left(\sum_{j=1}^{i-1} \frac{n_j}{k}\right) + \frac{n_i(t - a_{i-1})}{k(a_i - a_{i-1})} \qquad a_{i-1} < t \le a_i$$

for $i = 1, 2, \ldots, m$.* This estimator passes through the points $\left(a_i, \sum_{j=1}^i n_j/k\right)$ for $i = 1, 2, \ldots, m$. A graph of a generic version of this estimator is shown in Figure 9.3.2. The tick marks on the horizontal axis correspond to the subinterval endpoints. The tick marks on the vertical axis are determined by the count data. In the unlikely case that the piecewise-linear segments are all parallel [i.e., the estimator is a line connecting $(0,0)$ with $(S, \sum_{i=1}^m n_i/k)$], the model reduces from a nonstationary Poisson process to a stationary (homogeneous) Poisson process. Asymptotic properties of this estimator in the case of equal-width subintervals are considered by Henderson (2003).

Figure 9.3.2. Cumulative event rate function estimate.
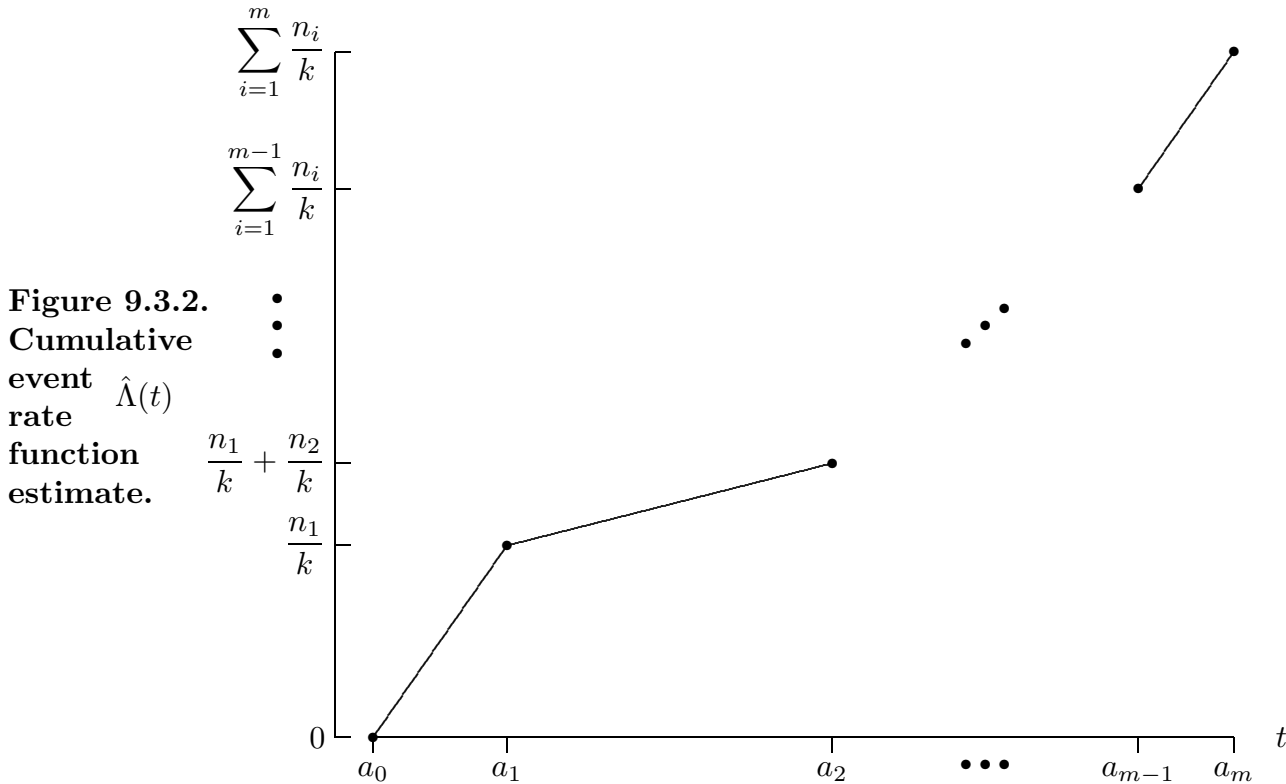
* If there are no events observed on interval i, i.e., $n_i = 0$, then the event rate function estimate is zero on interval i and the cumulative event rate function estimate is constant on interval i. In the variate generation algorithm to be described subsequently, no events will be generated for such an interval. This is useful for modeling an interval where no events should occur, e.g., lunch breaks.

Example 9.3.4 Figure 9.3.3 contains the piecewise-linear cumulative event rate function estimate associated with the lunch wagon arrival counts. The estimator begins at $(0, 0)$ and ends at $(4.5, 150/3)$, as expected. Recalling that the interpretation of the cumulative event rate function is the expected number of events by time t, we can say that the data suggests that there will be $n/k = 150/3 = 50$ arrivals to the lunch wagon per day between the hours of 10:00 AM and 2:30 PM. The S-shape for the cumulative event rate function estimator indicates that fewer customers tend to arrive at the beginning and end of the observation period.
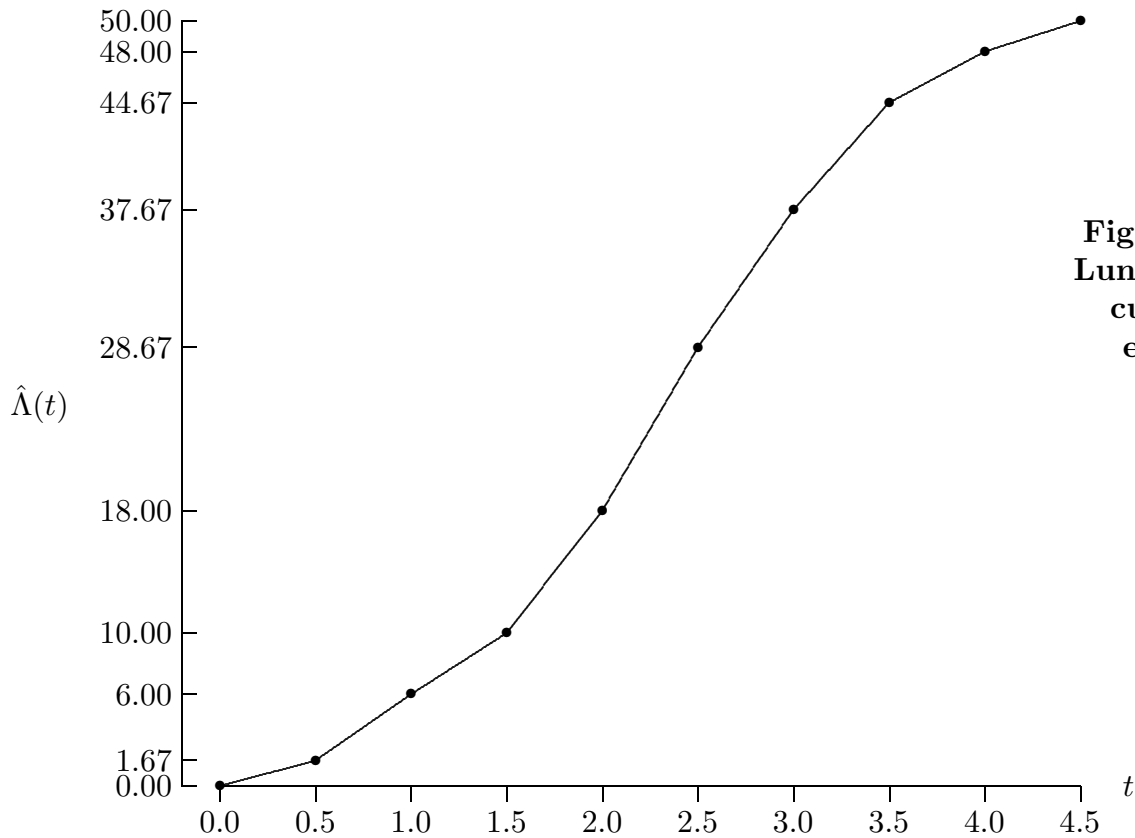
Figure 9.3.3. Lunch wagon cumulative event rate function estimate.

As in the previous section, it is important to assess the accuracy of the estimators developed thus far, which is typically done via confidence intervals. Based on the fact that the number of events by time t has a $Poisson\big(\Lambda(t)\big)$ distribution, an approximate, two-sided $(1 - \alpha)100\%$ confidence interval for $\Lambda(t)$ is

$$\hat{\Lambda}(t) - z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}} < \Lambda(t) < \hat{\Lambda}(t) + z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}},$$

for $0 < t \leq S$, where $z_{\alpha/2}$ is the $1 - \alpha/2$ fractile of the standard normal distribution. The interval is always asymptotically exact at the endpoints, but asymptotically exact for all t in $(0, S]$ only when the target event rate function $\lambda(t)$ is piecewise constant over each subinterval $(a_{i-1}, a_i]$ in the arbitrary partition of $(0, S]$. In most applications, this rather restrictive assumption is *not* satisfied.

Example 9.3.5   Figure 9.3.4 illustrates the cumulative event rate function estimator and
the associated 95% confidence limits. Not surprisingly, these limits expand with time since
we are less certain of the number of arrivals by time 4.0, for example, than we are of the
number of arrivals by time 1.0. Increasing the number of realizations k will narrow the
width of the confidence limits.

**Figure 9.3.4.
Lunch
wagon
cumulative
event rate
function
estimator
and
confidence
limits.**
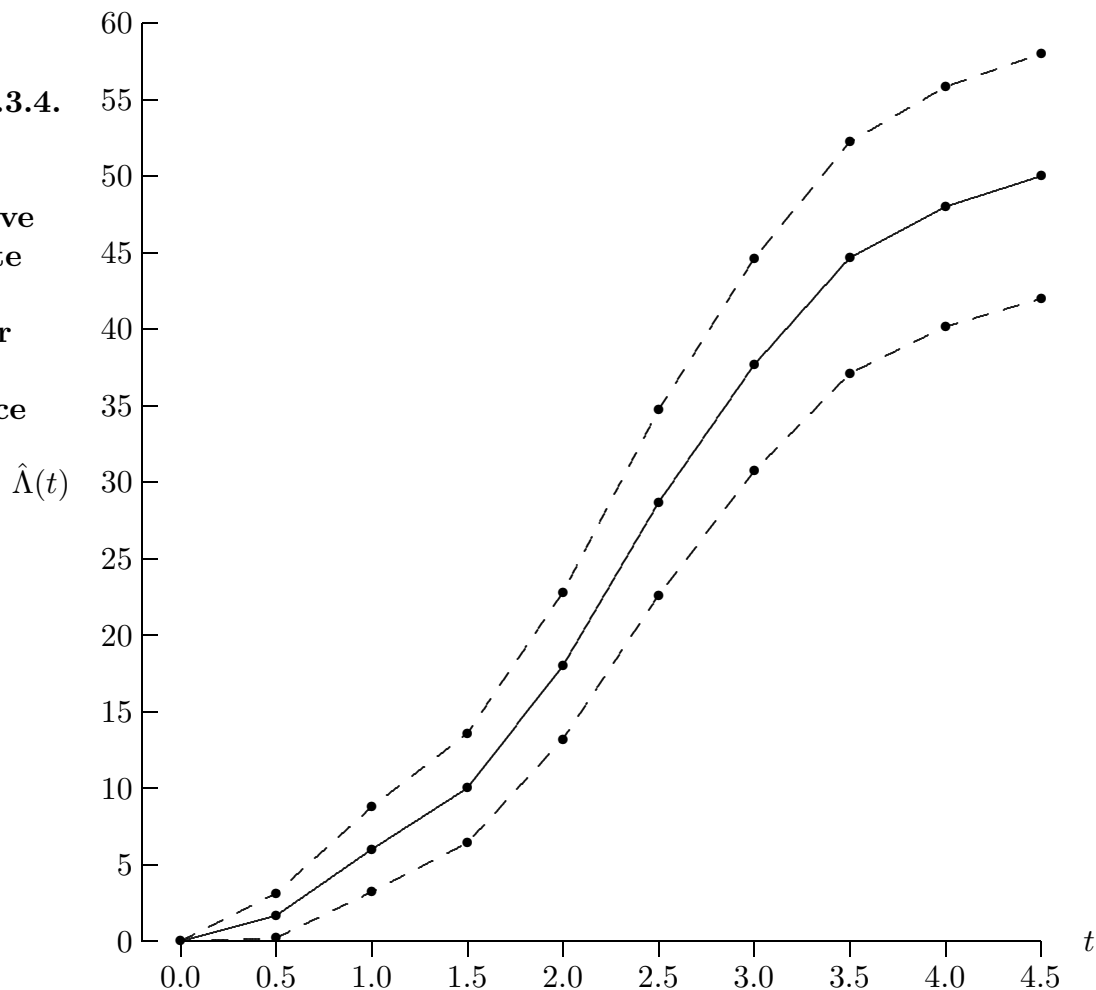
At this point the input model for the nonstationary Poisson process has been deter-
mined. A realization of a nonstationary Poisson process can be generated from the cumula-
tive event rate function estimate for modeling in a discrete-event simulation by inversion.
Let $E_1, E_2, \ldots$ be the event times of a unit *stationary* Poisson process. Let $T_1, T_2, \ldots$
denote the associated event times for the nonstationary Poisson process with cumulative
event rate function $\hat{\Lambda}(t)$ generated on $(0, S]$. Algorithm 9.3.1 generates the nonstationary
Poisson process event times from the inputs $a_0, a_1, a_2, \ldots, a_m$; $n_1, n_2, \ldots, n_m$; and k. A
key element of this algorithm is determining the appropriate subinterval (denoted in the
algorithm by i) in which to perform the inversion.

Algorithm 9.3.1 Given the number of realizations collected k; subinterval boundaries $a_0, a_1, a_2, \ldots, a_m$; and event counts $n_1, n_2, \ldots, n_m$, the following algorithm returns a realization of a nonstationary Poisson process via inversion associated with the cumulative event rate estimator $\hat{\Lambda}(t)$.

```
Λmax = Σ(i=1 to m) ni/k;                       /* upper bound for HPP */
i = 1;                                    /* initialize interval counter */
j = 1;                                    /* initialize variate counter */
Λ = ni/k;                            /* initialize cumulative event rate */
Ej = Exponential(1.0);                    /* generate first HPP event time */
while (Ej <= Λmax) {                      /* while more events to generate */
   while (Ej > Λ) {                            /* while in wrong interval */
      i = i + 1;                              /* increment interval counter */
      Λ = Λ + ni/k;
   }
   Tj = ai - (Λ - Ej) * k * (ai - ai-1) / ni;
   j = j + 1;
   Ej = Ej-1 + Exponential(1.0);
}
return (T1, T2, ... , Tj-1);                     /* return event times */
```

The geometry associated with Algorithm 9.3.1 is as follows. The *stationary* (homogeneous) unit Poisson process values $E_1, E_2, \ldots$ are generated along the vertical axis in Figure 9.3.3. Each E_j is associated with some subinterval i, $i = 1, 2, \ldots, m$. The appropriate piecewise linear segment is inverted, yielding the associated *nonstationary* Poisson process value T_j. Algorithm 9.3.1 is monotone, synchronized, clear, and efficient.

Algorithm 9.3.1 is valid only when the target process has an event rate function $\lambda(t)$ that is piecewise constant, a dubious assumption in a real-world setting. Any departure from this assumption results in an approximate estimator $\hat{\Lambda}(t)$ and an associated approximate variate generation algorithm between the subinterval endpoints. The binning of the data into subintervals typically results in missed trends that occur between the subinterval endpoints. As seen subsequently, this problem is overcome when working with raw data.

One problem with Algorithm 9.3.1 is that there is no limit on the number of nonstationary Poisson process event times generated. The algorithm must be modified to accommodate a *next-event* simulation model where the scheduling of the next event occurs when a current event is being processed. Algorithm 9.3.2 uses the next-event approach to schedule the next event when the current event is being processed. The algorithm has the same static inputs ($a_0, a_1, \ldots, a_m$; $n_1, n_2, \ldots, n_m$, and k) as Algorithm 9.3.1, except this algorithm returns the next event time given that the current event occurs at the dynamic input time $T \in (0, S]$. The algorithm returns the time of the next nonstationary Poisson process event $\hat{\Lambda}^{-1}\big(\hat{\Lambda}(T) + E\big)$, where E is an *Exponential*(1.0) variate.

Algorithm 9.3.2   Given the number of realizations collected k; subinterval boundaries $a_0, a_1, a_2, \ldots, a_m$; event counts $n_1, n_2, \ldots, n_m$; and the time of the current event T, the following algorithm returns the time of the next event in a nonstationary Poisson process via inversion associated with the cumulative event rate estimator $\hat{\Lambda}(t)$.

```
Λ_max = Σ_{j=1}^m n_j/k;                             /* maximum cumulative event rate */
j = 1;                                               /* initialize interval index */
while (T > a_j) {                                    /* while wrong interval */
   j = j + 1;                                        /* find interval index */
}
Λ_T = Σ_{i=1}^{j-1} n_i / k + n_j * (T - a_{j-1}) / (k * (a_j - a_{j-1}));
Λ = Σ_{i=1}^j n_i / k;
Λ_T = Λ_T + Exponential(1.0);
if (Λ_T > Λ_max)
   return(-1);                  /* -1 indicates no more events to generate */
else {
   while (Λ_T > Λ) {
      j = j + 1;
      Λ = Λ + n_j / k;
   }
   return(a_j - (Λ - Λ_T) * k * (a_j - a_{j-1}) / n_j);
}
```

The algorithm returns the next event time or -1 if there are no further events to be generated. The variable Λ_T initially contains the cumulative event rate function associated with the time of the current nonstationary Poisson process event T, i.e., $\hat{\Lambda}(T)$. This variable is updated to contain the cumulative event rate function associated with the time of the next unit stationary Poisson process event, $\hat{\Lambda}(T) + Exponential(1.0)$. At the end of the execution of this algorithm, the variable Λ contains the cumulative event rate function value at the right subinterval endpoint associated with the returned event time.

A more sophisticated implementation of this next-event algorithm would store $\Lambda_{\max}$, j, Λ, and Λ_T between the generation of events, effectively eliminating the first seven lines of the algorithm. The procedure would then begin with the updating of the Λ_T value, saving substantial execution time for large m.

Raw Data I

We now proceed to the case where *raw event times* are available for the k realizations collected on $(0, S]$. The meaning of n_i now changes from the count data case. We now let n_i, $i = 1, 2, \ldots, k$ be the number of observations in the ith realization, $n = \sum_{i=1}^k n_i$, and let $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ be the order statistics of the superposition of the event times in the k realizations, $t_{(0)} = 0$ and $t_{(n+1)} = S$.

Example 9.3.6 For the lunch wagon example, the realizations were collected between time 0 (10:00 AM) and $S = 4.5$ (2:30 PM), yielding $n_1 = 56$, $n_2 = 42$, and $n_3 = 52$ observations on the $k = 3$ days. The superposition consists of $n = 150$ arrival times: $t_{(0)} = 0$, $t_{(1)} = 0.2153$, $t_{(2)} = 0.3494$, $t_{(3)} = 0.3927$, $t_{(4)} = 0.3943$, ..., $t_{(150)} = 4.3741$, and $t_{(151)} = 4.5$. The superposition of the arrival times to the lunch wagon is plotted in Figure 9.3.5 [$t_{(0)} = 0$ and $t_{(151)} = 4.5$ are not plotted]. A cursory visual inspection of the arrival pattern reveals a concentration of arrivals near the noon hour.

Figure 9.3.5. Lunch wagon arrival times.
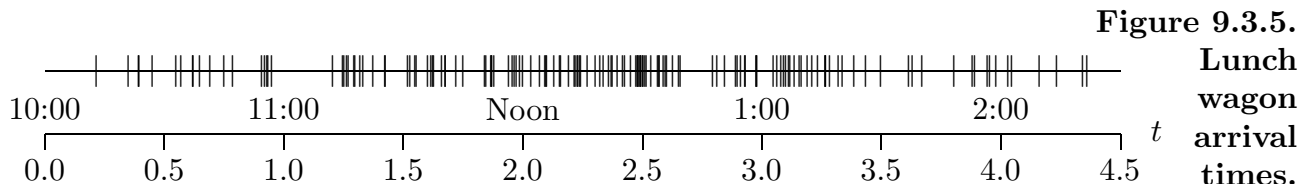
We now introduce the standard nonparametric estimate of $\Lambda(t)$, which is often known as the "step-function" estimator. For the jth independent replication of the target nonstationary Poisson process, $j = 1, 2, \ldots, n$, let $N_j(t)$ denote the number of events observed in the time interval $(0, t]$ and let

$$N_k^*(t) = \sum_{j=1}^{k} N_j(t) \qquad 0 < t \leq S$$

denote the aggregated counting (or superposition) process so that $n = N_k^*(S)$. The step function $N_j(t)$ is an appropriate estimator for the cumulative event rate function for realization j only, $j = 1, 2, \ldots, k$. The step-function estimator of $\Lambda(t)$ which averages all of the k replications is

$$\hat{\Lambda}(t) = \frac{1}{k} \sum_{i=1}^{k} N_j(t) = \frac{N_k^*(t)}{k} \qquad 0 < t \leq S.$$

Using a result from Leemis (1991), the $N_1(t)$, $N_2(t)$, ... , $N_j(t)$ are independent and identically distributed $Poisson\big(\Lambda(t)\big)$ random variables. Also, for any arbitrary t in the interval $(0, S]$, we can conclude that

$$\lim_{k \to \infty} \hat{\Lambda}(t) = \Lambda(t)$$

with probability one. In addition, we can construct the following asymptotically exact $100(1 - \alpha)\%$ confidence interval for $\Lambda(t)$ whose form is identical to the event count data case[*]

$$\hat{\Lambda}(t) - z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}} < \Lambda(t) < \hat{\Lambda}(t) + z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}}.$$

[*] This interval is asymptotically exact for all t, unlike the interval for the count data.

The standard step-function estimator takes upward steps of height $1/k$ only at the event times in the superposition of the k realizations collected from the target process. This means that a variate generation algorithm will produce realizations that contain only event times equal to those collected in one of the k realizations. This is exactly the problem that we ran into in Section 9.1 which we called the "interpolation" problem. In a very similar fashion, we can overcome this problem using a piecewise-linear cumulative event rate function estimator. Thus we skip the development of a variate generation algorithm and move directly to the solution of the interpolation problem.

Raw Data II

There are $n+1$ "gaps" on $(0, S]$ created by the superposition $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$. Setting $\hat{\Lambda}(S) = n/k$ yields a process where the expected number of events by time S is the average number of events in k realizations, since $\Lambda(S)$ is the expected number of events by time S. The piecewise-linear cumulative event rate function estimate rises by $n/[(n+1)k]$ at each event time in the superposition. Thus the piecewise-linear estimator of the cumulative event rate function between the time values in the superposition is

$$\hat{\Lambda}(t) = \frac{in}{(n+1)k} + \left[\frac{n(t - t_{(i)})}{(n+1)k(t_{(i+1)} - t_{(i)})} \right] \qquad t_{(i)} < t \leq t_{(i+1)}; i = 0, 1, 2, \ldots, n.$$

This estimator passes through the points $\left(t_{(i)}, in/(n+1)k \right)$, for $i = 1, 2, \ldots, n+1$. This interval estimator was developed in Leemis (1991) and extended to overlapping intervals in Arkin and Leemis (2000).

In a similar fashion to the previous two estimators, a $(1 - \alpha)100\%$ asymptotically exact (as $k \to \infty$) confidence interval for $\Lambda(t)$ can be determined as

$$\hat{\Lambda}(t) - z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}} < \Lambda(t) < \hat{\Lambda}(t) + z_{\alpha/2}\sqrt{\frac{\hat{\Lambda}(t)}{k}},$$

where $z_{\alpha/2}$ is the $1 - \alpha/2$ fractile of the standard normal distribution.

Example 9.3.7   Figure 9.3.6 contains point and interval estimates for $\Lambda(t)$ on $(0, 4.5]$ for the lunch wagon arrival times. The solid line denotes $\hat{\Lambda}(t)$ and the dotted lines denote the 95% confidence interval bounds. Lower confidence interval bounds that fall below zero are equated to zero. The cumulative event rate function estimator at time 4.5 is $150/3 = 50$, which is the point estimator for the expected number of arriving customers per day. If $\hat{\Lambda}(t)$ is linear, a stationary model is appropriate. As before, since customers are more likely to arrive to the lunch wagon between 12:00 $(t = 2.0)$ and 12:30 $(t = 2.5)$ than at other times and the cumulative event rate function estimator has an S-shape, a nonstationary model is indicated. The point and interval estimates appear more jagged than in the case of the count data because each of the three curves represents 151 (tiny) linear segments.

Figure 9.3.6.
Lunch wagon
cumulative
event rate
function
estimate and
confidence
interval limits.
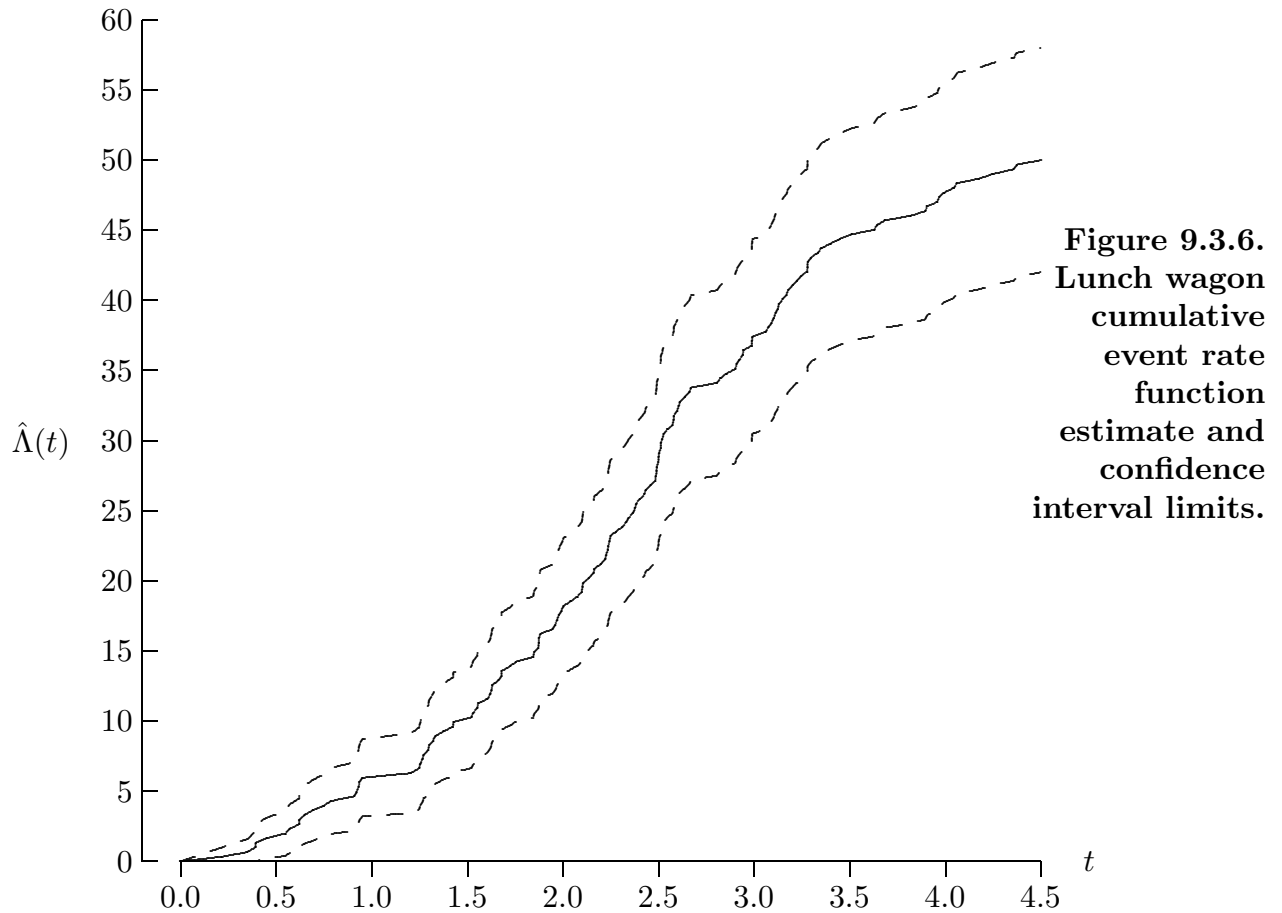
   The fact that each of the piecewise segments rises the same height can be exploited to create an efficient variate generation algorithm.

Algorithm 9.3.3   Given the number of realizations collected k; the number of observed events n; the interval length S; and superposition times $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$, the following algorithm returns a realization of a nonstationary Poisson process via inversion associated with the piecewise-linear cumulative event rate estimator $\hat{\Lambda}(t)$.

```
i = 1;
Eᵢ = Exponential(1.0);
while (Eᵢ <= n / k) {
    m = ⌊(n + 1) * k * Eᵢ / n⌋;
    Tᵢ = t₍ₘ₎ + (t₍ₘ₊₁₎ - t₍ₘ₎) * ((n + 1) * k * Eᵢ / n - m);
    i = i + 1;
    Eᵢ = Eᵢ₋₁ + Exponential(1.0);
}
return (T₁, T₂, ... , Tᵢ₋₁);
```

Algorithm 9.3.3 shows that it is a straightforward procedure to obtain a realization of $i - 1$ events on $(0, \ S]$ from i calls to *Exponential*(1.0). The values of $E_1, E_2, \ldots$ denote the values in the unit Poisson process, m denotes the index of the appropriate piecewise-linear segment to invert, and $T_1, T_2, \ldots, T_{i-1}$ denote the returned event times from the simulated nonstationary Poisson process. Inversion has been used to generate this non-stationary Poisson process, so certain variance reduction techniques, such as antithetic variates or common random numbers, may be applied to simulation output. Tied values in the superposition pose no problem to this algorithm although there may be tied values in the generated realization. As n increases, the amount of memory required increases, but the amount of execution time required to generate a realization depends only on the ratio n/k, the average number of events per realization. Thus collecting more realizations (resulting in narrower confidence intervals) increases the amount of memory required, but does not impact the expected execution time for generating a realization. Converting Algorithm 9.3.3 to a next-event orientation is left as an exercise.

To summarize, this piecewise-linear cumulative event rate function estimator is in some sense ideal in that

- it uses raw data which avoids the loss of accuracy imposed by breaking $(0, S]$ into arbitrary time subintervals;

- the point and interval estimates of $\Lambda(t)$ are closed-form and easily computed;

- the point estimate of $\Lambda(t)$ is consistent, i.e., $\lim_{k \to \infty} \hat{\Lambda}(t) = \Lambda(t)$;

- the interval estimate of $\Lambda(t)$ is asymptotically exact as $k \to \infty$;

- the variate generation algorithm which can be used to simulate a realization from $\hat{\Lambda}(t)$ is efficient, monotone, and synchronized.

This completes our discussion of trace-driven modeling of nonstationary processes. We now consider parametric modeling of nonstationary processes.

9.3.3   PARAMETRIC MODELING

We will continue to focus on our lunch wagon arrival times and the fitting of a non-stationary Poisson process as we switch the emphasis to parametric models. Maximum likelihood can again be used for estimating the parameters in a parametric nonstationary Poisson process model. The likelihood function for estimating the vector of unknown parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_q)'$ from a single realization of event times $t_1, t_2, \ldots, t_n$ drawn from a nonstationary Poisson process with event rate $\lambda(t)$ on $(0, S]$ is

$$L(\theta) = \left[\prod_{i=1}^{n} \lambda(t_i)\right] \exp\left[-\int_0^S \lambda(t)dt\right].$$

Maximum likelihood estimators can be determined by maximizing $L(\theta)$ or its logarithm with respect to the q unknown parameters. Confidence regions for the unknown parameters can be found in a similar manner to the service time example in Section 9.2.

Because of the additive property of the event rate function for multiple realizations, the likelihood function for the case of k realizations is

$$L(\theta) = \left[\prod_{i=1}^{n} k\lambda(t_i) \right] \exp\left[- \int_0^S k\lambda(t)dt \right].$$

There are many potential parametric models for nonstationary Poisson processes. We limit our discussion to procedures for fitting a *power law process*, where the event rate function is

$$\lambda(t) = b^a a t^{a-1} \qquad t > 0,$$

for shape parameter $a > 0$ and scale parameter $b > 0$. This event rate function can assume monotone increasing $(a > 1)$ and monotone decreasing $(a < 1)$ shapes. These event rate functions can be used to model sequences of events which occur with increasing (decreasing) frequency. The event rate function is constant when $a = 1$, which corresponds to an ordinary (stationary) Poisson process. The likelihood function for k realizations is

$$L(b, a) = k^n b^{na} a^n \exp\left(- k(bS)^a \right) \prod_{i=1}^{n} t_i^{a-1}.$$

The log likelihood function is

$$\ln L(b, a) = n \ln(ka) - na \ln b - k(bS)^a + (a - 1) \sum_{i=1}^{n} \ln t_i.$$

Differentiating with respect to a and b and equating to zero yields

$$\frac{\partial \ln L(b, a)}{\partial a} = n \ln b + \frac{n}{a} + \sum_{i=1}^{n} \ln t_i - k(bS)^a \ln(bS) = 0$$

and

$$\frac{\partial \ln L(b, a)}{\partial b} = \frac{an}{b} - kS^a a b^{a-1} = 0.$$

Unlike the Weibull distribution from the previous section, these equations can be solved in closed form. The analytic expressions for b and a are

$$\hat{a} = \frac{n}{n \ln S - \sum_{i=1}^{n} \ln t_i},$$

$$\hat{b} = \frac{1}{S} \left(\frac{n}{k} \right)^{1/a}.$$

We now apply this model to the lunch wagon arrival time data.

Example 9.3.8   Substituting the $n = 150$ lunch wagon arrival times into these formulas yields maximum likelihood estimators $\hat{a} = 1.27$ and $\hat{b} = 4.86$. The cumulative event rate function for the power law process

$$\Lambda(t) = (bt)^a \qquad\qquad t > 0,$$

is plotted along with the nonparametric estimator in Figure 9.3.7. Note that due to the peak in customer arrivals around the noon hour, the power law process is *not* an appropriate model since it is unable to adequately approximate the event rate function. It is unable to model a nonmonotonic $\lambda(t)$, and hence is a poor choice for a model. In fact, the fitted cumulative event rate function is so close to linear that the process would look almost like a stationary Poisson process — completely missing the noon hour rush period! Other models can be fit in a similar fashion.

Figure 9.3.7. Lunch wagon cumulative event rate function estimate and fitted power law process.
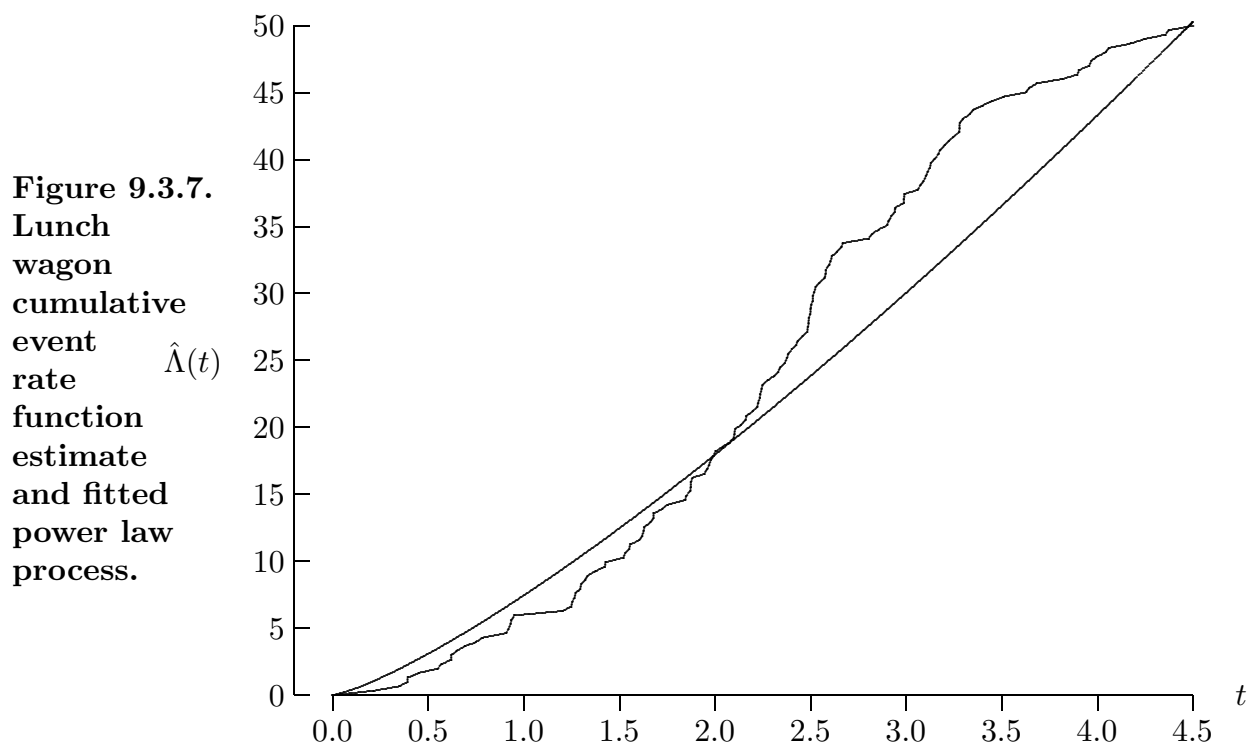
A more appropriate two-parameter distribution to consider would be one with an event rate function that increases initially, then decreases. The log-logistic process, for example, with event rate function (Meeker and Escobar, 1998)

$$\lambda(t) = \frac{ba(bt)^{a-1}}{1 + (bt)^a} \qquad t > 0,$$

for shape parameter $a > 0$ and scale parameter $b > 0$, would certainly be more appropriate than the power law process.

The EPTMP (exponential-polynomial-trigonometric function with multiple periodicities) model, originally given by Lee, Wilson, and Crawford (1991) and generalized by Kuhl, Damerdji, and Wilson (1998) with event rate function

$$\lambda(t) = \exp\left[\sum_{i=0}^{m} \alpha_i t^i + \sum_{j=1}^{p} \gamma_j \sin(\omega_j t + \phi_j)\right] \qquad t > 0$$

can also model a nonmonotonic event rate function. The exp forces the event rate function to be positive, the first summation models polynomial trend, and the second summation models sinusoidal periodicities in the event rate. The cyclic portion of the model has been used in discrete-event simulation applications to model the times that storms occur in the Arctic Sea and arrivals of livers for transplantation by donors. Kuhl, Sumant, and Wilson (2004) generalize this approach to model periodic effects that are not necessarily modeled by a trigonometric function. Goodness-of-fit tests associated with the fitted models are given in Rigdon and Basu (2000) and Ross (2002b).

9.3.4   EXERCISES

Exercise 9.3.1   The lunch wagon example was generated from the cumulative event rate function from Klein and Roberts (1984)

$$\Lambda(t) = \begin{cases} 5t^2 + t & 0 < t \leq 1.5 \\ 16t - 11.25 & 1.5 < t \leq 2.5 \\ -3t^2 + 31t - 30 & 2.5 < t \leq 4.5. \end{cases}$$

(*a*) Find the rate function $\lambda(t)$. (*b*) Generate 10 realizations via the thinning algorithm and 10 realizations using inversion (both algorithms were introduced in Section 7.5) and provide convincing numerical evidence that the two algorithms were implemented correctly.

Exercise 9.3.2   Derive the cumulative event rate function estimator from the event rate function estimator in the case of count data.

Exercise 9.3.3   Perform the algebra necessary to confirm the appropriate assignment of T_j in Algorithm 9.3.1.

Exercise 9.3.4   Draw a diagram that illustrates the geometry associated with Algorithm 9.3.2.

Exercise 9.3.5   Use Algorithm 9.3.2 to determine the time of the next event in a nonstationary Poisson process when $m = 3$; $a_0 = 0$, $a_1 = 2$, $a_2 = 6$, $a_3 = 7$; $n_1 = 10$, $n_2 = 3$, $n_3 = 11$; $k = 2$; $T = 1.624$; $E = 5.091$.

Exercise 9.3.6   Convert Algorithm 9.3.3 to a next-event orientation, e.g., given a current event that occurs at time T, when should the next event be scheduled?

CHAPTER 10

PROJECTS

Sections

Projects related to the discrete-event simulation topics considered in this text comprise this final chapter. These projects may be used as launching pads for a term project or can be used as miscellaneous topics at the end of the course.

Section 10.1 introduces six empirical test of randomness for a random number generator. These six tests, along with the Kolmogorov–Smirnov goodness-of-fit test, are applied to the Lehmer random number generator $(a, m) = (48271, 2^{31} - 1)$. Section 10.2 introduces a *birth-death* process, which is a mathematical model for systems that can be characterized by a state that changes only by either increasing by one or decreasing by one. Population models are a popular example of birth-death models used by biologists. The number of jobs in a single-server service node is another example of a birth-death process. Section 10.3 introduces a finite-state *Markov chain*, which uses a state transition matrix to define the probability mechanism for transitions from state to state. These states can represent a very broad range of real-world system states, e.g., the state of the weather on a particular day, the position of a job in a job shop, the social class of an individual, or some genetic characteristic of an individual. Finally, Section 10.4 extends the single-server service node which has been used throughout the book to a network of single-server service nodes.

Although the $(a, m) = (16807, 2^{31} - 1)$ minimal standard Lehmer random number generator introduced in Chapter 2 is less than ideal, there is no need to subject it to any of the standard tests of randomness — that has been done many times before. In contrast, the $(a, m) = (48271, 2^{31} - 1)$ generator used in this book is less well established in the literature and so there is merit in testing it for randomness. Moreover, testing for randomness is such an important topic that it must be included in any comprehensive study of discrete-event simulation. With this in mind, empirical tests of randomness will be illustrated in this section by applying six of the standard tests to the output produced by the library `rngs`.

10.1.1   EMPIRICAL TESTING

Definition 10.1.1   An *empirical test* of randomness is a statistical test of the hypothesis that repeated calls to a random number generator will produce an *iid* sample from a *Uniform*$(0, 1)$ distribution. An empirical test of randomness consists of three steps.

- Generate a *sample* by repeated call to the generator.

- Compute a *test statistic* whose statistical distribution (pdf) is known when the random numbers are truly *iid Uniform*$(0, 1)$ random variates.

- Assess the likelihood of the observed (computed) value of the test statistic relative to the theoretical distribution from which it is assumed to have been drawn.

The test in Definition 10.1.1 is *empirical* because it makes use of actual generated data and produces a conclusion that is only valid in a local, statistical sense. In contrast, *theoretical* tests are not statistical. Instead, as illustrated in Section 2.2, theoretical tests use the numerical value of the generator's modulus and multiplier to assess the global randomness of the generator.*

Chi-Square Statistic

Definition 10.1.2   Let X' be a random variable (discrete or continuous) and suppose the possible values of X' are partitioned into a finite set of possible values (states) $\mathcal{X}$ to form a new discrete random variable X. Typically these states correspond to the bins of a histogram. Based on a large random sample, for each possible state $x \in \mathcal{X}$ define

$$e[x] = \text{the } \textit{expected} \text{ number of times state } x \text{ will occur}$$

$$o[x] = \text{the } \textit{observed} \text{ number of times state } x \text{ occurred.}$$

The resulting non-negative quantity

$$v = \sum_x \frac{\left(o[x] - e[x]\right)^2}{e[x]}$$

is known as a *chi-square statistic*. The sum is over all $x \in \mathcal{X}$.

---

* See Knuth (1998) for a comprehensive discussion of theoretical and empirical tests of randomness.

All the empirical tests of randomness in this section make use of some form of the chi-square statistic. The following test is typical.

Test of Uniformity

Algorithm 10.1.1   The uniformity of a random number generator's output can be tested by making n calls to `Random` and tallying this random variate sample into an equally-spaced k-bin histogram. In this case x is the histogram bin index and the bin counts $o[0], o[1], \ldots, o[k-1]$ can be tallied as follows.

```
for (x = 0; x < k; x++)
   o[x] = 0;
for (i = 0; i < n; i++) {
   u = Random();
   x = ⌊ u * k ⌋;
   o[x]++;
}
```

Because the expected number of points per bin is $e[x] = n/k$ for all x, in this case the chi-square statistic is

$$v = \sum_{x=0}^{k-1} \frac{\left(o[x] - n/k\right)^2}{n/k}.$$

The *Test of Uniformity* (frequency test), based on Algorithm 10.1.1, provides a statistical answer to the question "Is the histogram flat?". More precisely, is the variability among the histogram heights sufficiently small so as to conclude that the n random numbers were drawn from a $Uniform(0,1)$ population? The underlying theory behind this test is that *if* the random sample is a truly *iid* $Uniform(0,1)$ random variate sample and *if* n/k is sufficiently large, say 10 or greater, then the test statistic v is (approximately) a $Chisquare(k-1)$ random variate. Therefore, if

$$v_1^* = \texttt{idfChisquare}(k-1, \alpha/2) \qquad \text{and} \qquad v_2^* = \texttt{idfChisquare}(k-1, 1-\alpha/2)$$

then $\Pr(v_1^* \le v \le v_2^*) \cong 1 - \alpha$. To apply the test of uniformity, pick $k \ge 1000$, $n \ge 10k$, and then do the following.

- Use Algorithm 10.1.1 to calculate a k-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1-\alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1 - \alpha$ for a legitimate random number generator).

Just as in a jury trial, where people are assumed innocent until proven guilty, a random number generator is assumed to be "good" until proven "bad." Therefore, if $v_1^* \leq v \leq v_2^*$ we haven't really proven that the generator is good — that was already assumed. Instead, we have only failed to prove it bad. Equivalently, an empirical test is never passed; instead it is either "failed" or "not failed." This is an important statistical point. Equivalent statistical jargon is "reject" the null hypothesis and "fail to reject" the null hypothesis.

Example 10.1.1   As an illustration of the test of uniformity, the library `rngs` was used (with an initial seed of 12345) to create the data for 256 tests, one for each possible stream.* For each test the number of histogram bins was $k = 1000$ and the number of observations was $n = 10\,000$. Therefore the expected number of observations per bin was $n/k = 10$. The resulting 256 chi-square statistics (the v's) are plotted in Figure 10.1.1 along with the corresponding critical values $v_1^* = 913.3$ and $v_2^* = 1088.5$.
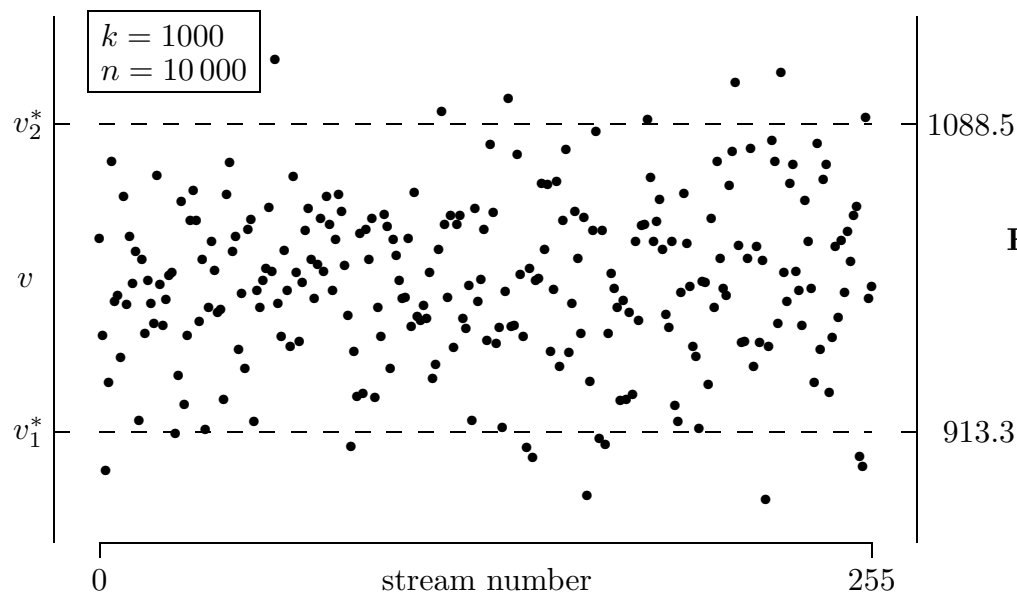
Figure 10.1.1.
Test of
uniformity.

Consistent with the (hoped for) goodness of fit of the random number generator in `rngs`, most of the 256 tests were not failed. Indeed, there were just seven "the histogram is not uniform" ($v > v_2^*$) failures and 11 "the histogram is too uniform" ($v < v_1^*$) failures.

Two-Tailed Versus One-Tailed Tests

We have formulated the test of uniformity as a *two-tailed* test. That is, as illustrated in Example 10.1.1 the test could be failed in two ways. If v is too small ($v < v_1^*$) then the observed and expected data agree too well — the histogram is too flat. At the other extreme, if v is too large ($v > v_2^*$) then the observed and expected data don't agree well at all. Intuitively $v > v_2^*$ failures seem to be worse and, consequently, some authors argue that v can't be too small. Instead, these authors advocate a *one-tailed* test based only on a check to see if v is too large. Knuth (1998) advocates a two-tailed test; we agree.

---

* Test 0 corresponds to stream 0, test 1 to stream 1, etc.

As the name suggests, the test of uniformity is nothing more than a test for histogram flatness — does the generator tend to fill each histogram bin equally? The *order* (or lack thereof) in which the bins are filled is not tested. Because of this, the test of uniformity is a statistically weak test of randomness. Moreover, it is perhaps the most obvious test to apply to a *Uniform*(0, 1) random number generator. If you are tempted to use a generator whose quality is unknown (to you), then it would be prudent to first apply the test of uniformity. However, don't be surprised to find that the generator will produce an acceptably flat histogram (at least for selected values of k) and, in that case, do not jump to the conclusion that the generator is necessarily "good."

But what about the 18 failures in Example 10.1.1, do these failures indicate problems? The answer is no. In fact, if the generator is good then approximately $\alpha \times 100\%$ of the tests *should* be failed. In particular, in 256 tests with a 95% level of confidence, we would expect to have approximately $256 \cdot 0.05 \cong 13$ failures; significantly more *or less* than this number would be an indication of problems.

Test of Extremes

The next empirical test is similar, in spirit, to the test of uniformity. It is a more powerful test, however, with theoretical support based on the following theorem.

Theorem 10.1.1   If U_0, U_1, ..., U_{d-1} is an *iid* sequence of *Uniform*(0, 1) random variables and if
$$R = \max\{U_0, U_1, \ldots, U_{d-1}\}$$
then the random variable $U = R^d$ is *Uniform*(0, 1).*

Proof   Pick $0 < r < 1$. Since $R \leq r$ if and only if $U_j \leq r$ for $j = 0, 1, \ldots, d - 1$,
$$
\begin{aligned}
\Pr(R \leq r) &= \Pr\left((U_0 \leq r) \text{ and } (U_1 \leq r) \text{ and } \ldots \text{ and } (U_{d-1} \leq r)\right) \\
&= \Pr(U_0 \leq r)\Pr(U_1 \leq r)\ldots\Pr(U_{d-1} \leq r) \\
&= r \cdot r \cdot \ldots \cdot r \\
&= r^d
\end{aligned}
$$
and so the cdf of the random variable R is $\Pr(R \leq r) = r^d$. However, $R \leq r$ if and only if $R^d \leq r^d$ and therefore
$$\Pr(R^d \leq r^d) = \Pr(R \leq r) = r^d.$$
Finally, let $U = R^d$ and let $u = r^d$ so that the cdf of the random variable U is
$$\Pr(U \leq u) = \Pr(R^d \leq r^d) = r^d = u \qquad\qquad 0 < u < 1.$$
Since $\Pr(U \leq u) = u$ is the cdf of a *Uniform*(0, 1) random variable, the theorem is proven.

---

* Don't misinterpret Theorem 10.1.1. The random variable $U = R^d$ is *Uniform*(0, 1), the random variable R is *not Uniform*(0, 1).

Algorithm 10.1.2 The "extreme" behavior of a random number generator can be tested by grouping (batching) the generator's output d terms at a time, finding the maximum of each batch, raising this maximum to the dth power, and then tallying all the maxima so generated as follows.

```
for (x = 0; x < k; x++)
   o[x] = 0;
for (i = 0; i < n; i++) {
   r = Random();
   for (j = 1; j < d; j++) {
     u = Random();
     if (u > r)
        r = u;
   }
   u = exp(d * log(r));                          /* u is r^d */
   x = ⌊ u * k ⌋;
   o[x]++;
}
```

Because the expected number of points per bin is $e[x] = n/k$ for all x, in this case the chi-square statistic is

$$v = \sum_{x=0}^{k-1} \frac{\left(o[x] - n/k\right)^2}{n/k}$$

with critical values $v_1^* = \mathtt{idfChisquare}(k-1, \alpha/2)$ and $v_2^* = \mathtt{idfChisquare}(k-1, 1-\alpha/2)$.

The *Test Of Extremes* is based on Algorithm 10.1.2. To apply this test pick $k \geq 1000$, $n \geq 10k$, $d \geq 2$ and then do the following.

- Use Algorithm 10.1.2 to calculate a k-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1 - \alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1-\alpha$ for a legitimate random number generator). The test is typically applied for a range of values, say $d = 2, 3, \ldots, 6$. For $d = 1$ the test of extremes reduces to the test of uniformity.

As formulated, the test of extremes is actually a test of *maxima*. To make the test of extremes into a test of *minima* involves only a minor extension to the theory, i.e.,

$$S = \min\{U_0, U_1, \ldots, U_{d-1}\}$$

and $(1 - S)^d$ are *Uniform*$(0, 1)$ random variates. The details are left as an exercise.

Example 10.1.2   As in Example 10.1.1, the library `rngs` was used (with an initial seed of 12345) to create the data for 256 tests, one for each `rngs` stream. For each test, the number of histogram bins was $k = 1000$ and the number of observations was $n = 10\,000$, where each observation was determined by the maximum of $d = 5$ random numbers. As in Example 10.1.1, the critical values denoted by horizontal lines in Figure 10.1.2 are $v_1^* = 913.3$ and $v_2^* = 1088.5$.

**Figure 10.1.2.
Test of
extremes.**
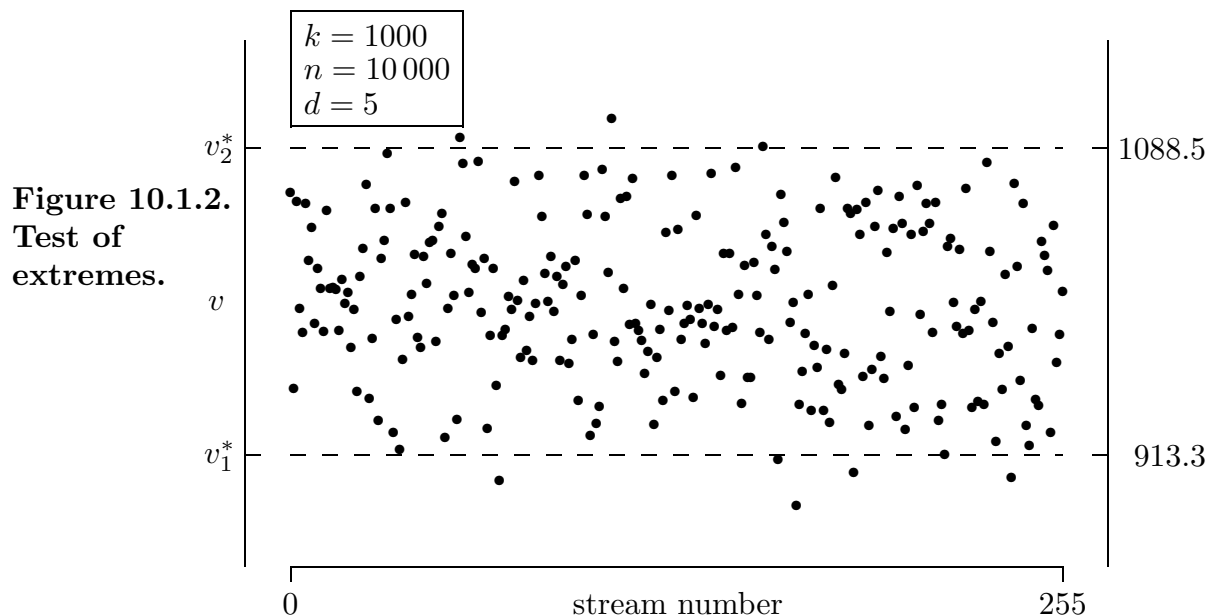
Consistent with expectations, most of the tests were not failed. The number of $v > v_2^*$ failures was three and there were five $v < v_1^*$ failures as well. These results are further evidence of the goodness of the generator. Indeed, because of its multi-variate character, the test of extremes (with $d > 1$) is statistically stronger than the test of uniformity and so these results are more confidence inspiring than are those in Example 10.1.1.

Runs-Up Test of Independence

The next empirical test we will consider is the runs-up test of independence. Although the underlying theory is simple, the test is considered to be relatively powerful, particularly when applied to Lehmer generators with poorly chosen multipliers.

Definition 10.1.3   Let u_0, u_1, u_2, ... be a sequence of numbers produced by repeated calls to a random number generator. A *run-up of length x* is a subsequence of length x with the property that each term in the subsequence is greater than its predecessor. By convention, each run-up is extended to its largest possible length and then, to make each run length independent, the next element is "thrown away" before beginning to compute the length of the next run-up. For example, the sequence of numbers

$$0.21, \ 0.39, \ 0.47, \ \underline{0.12}, \ 0.87, \ \underline{0.21}, \ 0.31, \ 0.38, \ 0.92, \ 0.93, \ \underline{0.57}, \ \ldots$$

consists of a run-up of length 3, followed by a run-up of length 1, followed by a run-up of length 4, etc. The "thrown away" points are underlined.

The following theorem provides theoretical support for the runs-up test of independence. The proof of this theorem is left as an exercise.

Theorem 10.1.2   If U_0, U_1, U_2, ... is an *iid* sequence of $Uniform(0,1)$ random variables then, beginning with U_0, the probability of a run-up of length x is

$$p[x] = \frac{x}{(x+1)!} \qquad x = 1, 2, \ldots$$

The corresponding probability of a run-up of length k or greater is

$$\sum_{x=k}^{\infty} p[x] = \frac{1}{k!}.$$

Algorithm 10.1.3   This algorithm uses repeated calls to Random to tally n runs-up into a k-bin histogram of runs of length $x = 1, 2, \ldots, k$.*

```
for (x = 1; x <= k; x++)
   o[x] = 0;
for (i = 0; i < n; i++) {
   x = 1;
   u = Random();
   t = Random();
   while (t > u) {
      x++;
      u = t;
      t = Random();
   }
   if (x > k)
      x = k;
   o[x]++;
}
```

Based on the probabilities in Theorem 10.1.2, the appropriate chi-square statistic in this case is

$$v = \sum_{x=1}^{k} \frac{\left(o[x] - np[x]\right)^2}{np[x]}$$

with $e[x] = np[x]$ for all x. The test statistic v is (approximately) a $Chisquare(k-1)$ random variate, and the critical values of v are

$$v_1^* = \texttt{idfChisquare}(k-1, \alpha/2) \qquad \text{and} \qquad v_2^* = \texttt{idfChisquare}(k-1, 1-\alpha/2).$$

---

* Note that $o[0]$ is not used and that $o[k]$ counts runs of length k *or greater*.

A run-up of length $x > 6$ is so unlikely that it is conventional to choose $k = 6$ and consolidate all runs-up of length 6 or greater, as in Algorithm 10.1.3. The resulting probabilities are shown below.

$$\begin{array}{ccccccc} x & : & 1 & 2 & 3 & 4 & 5 & 6 \text{ or greater} \\ p[x] & : & 1/2 & 1/3 & 1/8 & 1/30 & 1/144 & 1/720 \end{array}$$

For n runs-up the expected number of runs-up of length x is $np[x]$. Consistent with this, if $k = 6$ then $n = 7200$ runs-up are required to raise $np[6]$ to 10.

The *Runs-Up Test of Independence* is based on Algorithm 10.1.3 with $n \geq 7200$ and $k = 6$. To apply this test, do the following.

- Use Algorithm 10.1.3 to calculate a k-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1 - \alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1 - \alpha$ for a legitimate random number generator).

Example 10.1.3   As in Examples 10.1.1 and 10.1.2 (with initial `rngs` seed 12345 and $k = 6$), the results of 256 runs-up tests are plotted in Figure 10.1.3. For each test, the number of runs-up was $n = 7200$. The critical values are $v_1^* = 0.83$ and $v_2^* = 12.83$.

Figure 10.1.3. Runs-up test of independence.
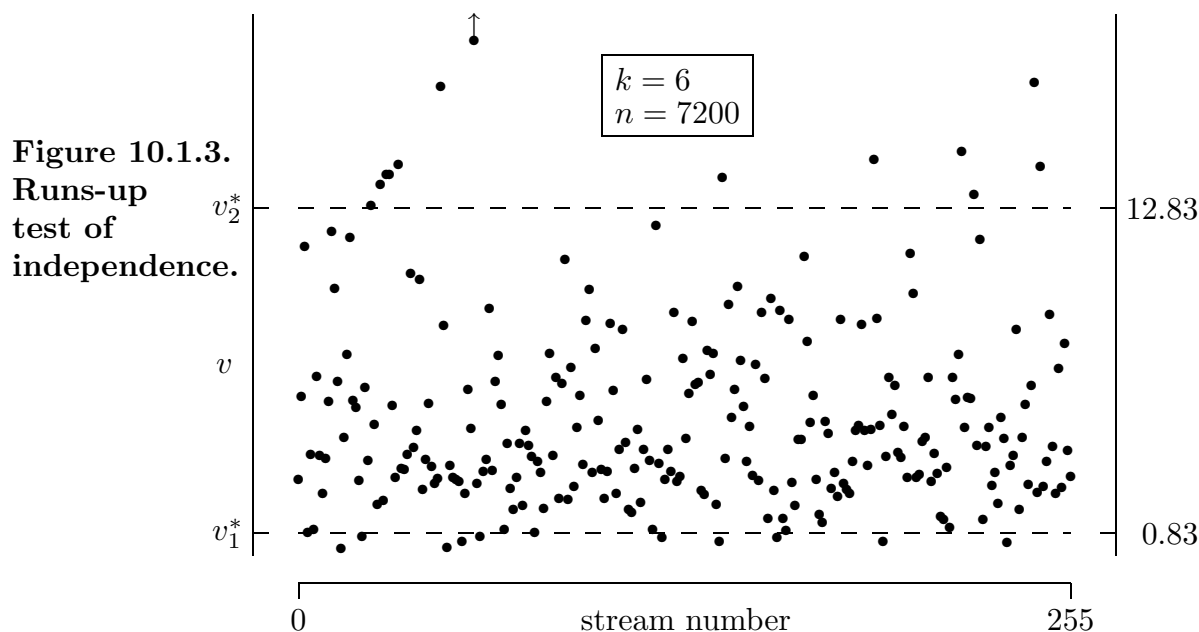
There were ten $v < v_1^*$ failures and thirteen $v > v_2^*$ failures, including the one failure indicated with a $\uparrow$ that was a spectacular off-scale $v = 26.3$ for stream number 58. This total of 23 failures is uncomfortably larger than expected indicating possible problems with these particular portions of the generator.

Use of the chi-square statistic

$$v = \sum_{x=1}^{k} \frac{\left(o[x] - np[x]\right)^2}{np[x]}$$

in the runs-up test of independence is questionable for two (related) reasons.

- The expected number of observations per histogram bin is dramatically *non*-uniform. Because the expected number of long runs is so small, the division by $np[x]$ causes the natural sample variation associated with long run lengths to have a much larger impact (bias) on v than does the natural sample variation associated with short run lengths. For the runs-up test this bias is undesirable but inevitable. The effect of the bias can be minimized only at the expense of using a large value of n.

- For this test, the chi-square statistic is only *approximately* a *Chisquare*$(k-1)$ random variate. The approximation becomes better as the expected number of observations per histogram bin is increased. This is particularly true because the number of bins $(k = 6)$ is so small.

Example 10.1.4 The tests illustrated in Figure 10.1.3 were repeated with n doubled to $n = 14\,400$. This doubles the expected number of observations in each histogram bin. The same initial **rngs** seed (12345) was used as before and so the net effect was to add an additional 7200 observations to those generated previously.
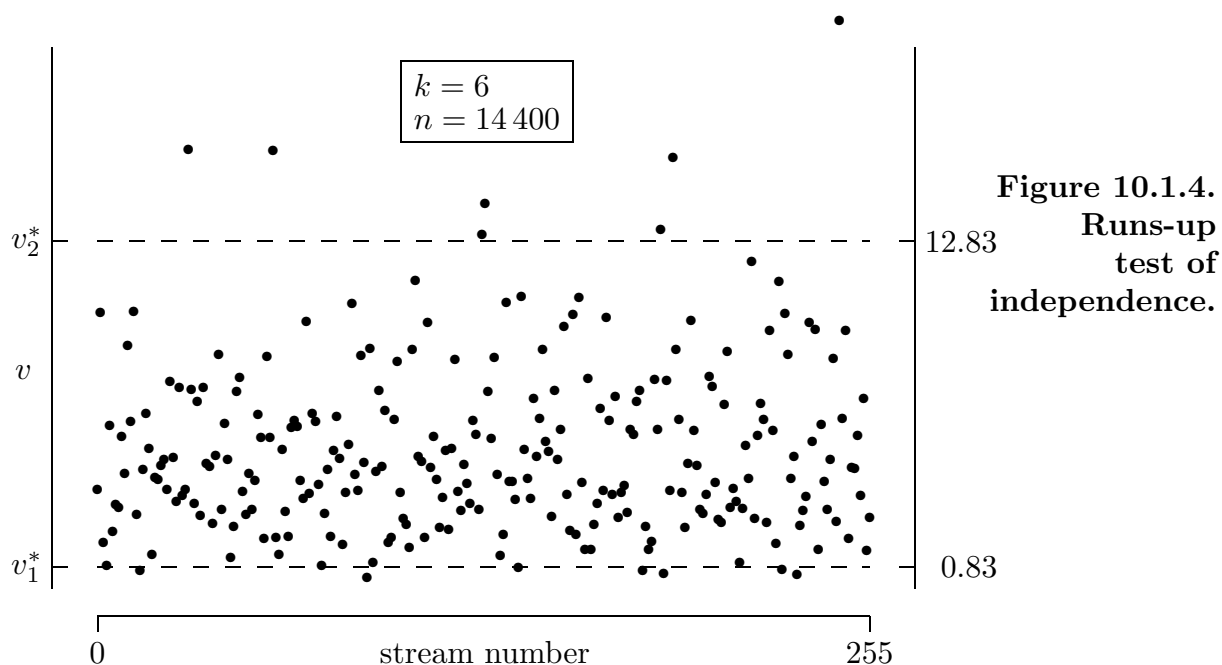

Figure 10.1.4. Runs-up test of independence.

In this case, consistent with expectations, the number of $v > v_2^*$ failures was seven and there were also seven $v < v_1^*$ failures. Our confidence in the goodness of the **rngs** generator is restored.

Three additional empirical tests of randomness will now be presented, the first of which is quite similar to the runs-up test. We will then consider the statistical question of how to use each of these six tests to answer the question, "Is the random number generator good?".

Gap Test of Independence

The gap test is defined by two real-valued parameters a, b chosen so that $0 \le a < b \le 1$. These two parameters define a subinterval $(a, b) \subset (0, 1)$ with length $\delta = b - a$.

Given an *iid* sequence $U_0, U_1, U_2, \ldots$ of $Uniform(0, 1)$ random variables, the term U_i will be in the interval (a, b) with probability δ. If $U_i \in (a, b)$ then a *gap* of length 0 occurs if $U_{i+1} \in (a, b)$. This occurs with probability $f(0) = \delta$. If instead $U_{i+j} \notin (a, b)$ for $j = 1, 2, \ldots, x$ and $U_{i+j} \in (a, b)$ for $j = x + 1$ then a gap of length x has occurred. This occurs with probability $f(x) = \delta(1 - \delta)^x$. If X is the discrete random variable that counts gap lengths, it follows that X is $Geometric(1 - \delta)$. (See Section 6.4.3.)

Because X is $Geometric(1 - \delta)$, the expected gap length is $E[X] = (1 - \delta)/\delta$. Similar to the runs-up test it is conventional to pick a positive integer k (which depends on δ) and consolidate all gaps of length $k - 1$ or greater. This consolidation occurs with probability

$$\Pr(X \ge k - 1) = \delta(1 - \delta)^{k-1}\left(1 + (1 - \delta) + (1 - \delta)^2 + \cdots\right) = \frac{\delta(1 - \delta)^{k-1}}{1 - (1 - \delta)} = (1 - \delta)^{k-1}.$$

Algorithm 10.1.4   Given two real-valued parameters a, b with $0.0 \le a < b < 1.0$, this algorithm uses repeated calls to `Random` to tally n gaps into a k-bin histogram corresponding to gaps of length $x = 0, 1, 2, \ldots, k - 1$ as follows.

```
for (x = 0; x < k; x++)
    o[x] = 0;
for (i = 0; i < n; i++) {
    x = 0;
    u = Random();
    while (u ∉ (a, b)) {
        x++;
        u = Random();
    }
    if (x > k - 1)
        x = k - 1;
    o[x]++;
}
```

Similar to the runs-up algorithm, $o[k - 1]$ counts gaps of length $k - 1$ or greater.

Because the gap length is a *Geometric*$(1 - \delta)$ random variate, if a total of n gaps are generated then the expected number of gaps of length x is

$$e[x] = \begin{cases} n\delta(1 - \delta)^x & x = 0, 1, 2, \ldots, k - 2 \\ n(1 - \delta)^{k-1} & x = k - 1. \end{cases}$$

The smallest value of $e[x]$ occurs when $x = k - 2$. Therefore, to make this expected number of gaps (and all others) be at least 10 it is necessary to choose k so that

$$e[k - 2] = n\delta(1 - \delta)^{k-2} \geq 10.$$

Solving for k yields the inequality

$$k \leq 2 + \left\lfloor \frac{\ln(10/n\delta)}{\ln(1 - \delta)} \right\rfloor.$$

Typically $\delta = b - a$ is small, resulting in a relatively large value for k. For example, if $n = 10\,000$ and $\delta = 0.05$ then $k \leq 78$.*

The chi-square statistic corresponding to the gap test of independence is

$$v = \sum_{x=0}^{k-1} \frac{\left(o[x] - e[x]\right)^2}{e[x]}$$

and v is (approximately) a *Chisquare*$(k - 1)$ random variate. The critical values of v are

$$v_1^* = \texttt{idfChisquare}(k - 1, \alpha/2) \qquad \text{and} \qquad v_2^* = \texttt{idfChisquare}(k - 1, 1 - \alpha/2).$$

The *Gap Test Of Independence* is based on Algorithm 10.1.4. To apply this test, pick $n \geq 10\,000$, $0 \leq a < b \leq 1$, and $k \leq 2 + \lfloor \ln(10/n\delta)/ \ln(1 - \delta) \rfloor$ with $\delta = b - a$. Then do the following.

- Use Algorithm 10.1.4 to calculate the k-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1 - \alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1 - \alpha$ for a legitimate random number generator).

---

* The a, b parameters are usually chosen so that the subinterval (a, b) lies at one end of the $(0, 1)$ interval or the other. The reason for this is that when the inverse transformation method is used to generate random variates, points near the ends of the $(0, 1)$ interval are mapped to the tails of the variates's distribution. Because extreme points tend to have low probability but high "impact" in a discrete-event simulation, it is considered particularly important to test that points in the tail of the distribution are generated correctly.

As for the three empirical tests considered previously, the library `rngs` can be used to generate the data for 256 gap tests and a figure similar to Figures 10.1.1, 10.1.2, 10.1.3, and 10.1.4 can be created. Doing so is left as an exercise. Similarly, the construction of a corresponding figure for the following empirical test of randomness is left as an exercise.

Test of Bivariate Uniformity

The test of uniformity can be extended to pairs of points. That is, pairs of randomly generated points can be tallied into a two-dimensional $k \times k$-bin histogram using the following algorithm.

Algorithm 10.1.5 This algorithm uses the results of n pairs of calls to `Random` to tally a two-dimensional $k \times k$-bin histogram as follows.

```
for (x₁ = 0; x₁ < k; x₁++)
    for (x₂ = 0; x₂ < k; x₂++)
      o[x₁, x₂] = 0;
for (i = 0; i < n; i++) {
    u₁ = Random();
    x₁ = ⌊ u₁ * k ⌋;
    u₂ = Random();
    x₂ = ⌊ u₂ * k ⌋;
    o[x₁, x₂]++;
}
```

The expected number of counts per bin is $e[x_1, x_2] = n/k^2$ and so the appropriate chi-square statistic is

$$v = \sum_{x_1=0}^{k-1} \sum_{x_2=0}^{k-1} \frac{\left(o[x_1, x_2] - n/k^2\right)^2}{n/k^2}.$$

The underlying theory is that the test statistic v is a $Chisquare(k^2 - 1)$ random variate. Therefore, the critical values of v are

$$v_1^* = \texttt{idfChisquare}(k^2 - 1, \alpha/2) \qquad \text{and} \qquad v_2^* = \texttt{idfChisquare}(k^2 - 1, 1 - \alpha/2).$$

The *Test Of Bivariate Uniformity* is based on Algorithm 10.1.5. To apply this test, pick $k \geq 100$, $n \geq 10k^2$ and then do the following.

- Use Algorithm 10.1.5 to calculate a $k \times k$-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1 - \alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1 - \alpha$ for a legitimate random number generator).

The test of bivariate uniformity can be extended in an obvious way to triples, quadruples, etc. Doing so with meaningful values of k and n, however, results in a requirement for a large d-dimensional array (with k^d elements) and an enormous number of u's. For example, even if d is only 2 and if $k = 100$, then the bivariate histogram is a 100×100 array and at least $n = 2 \cdot 10 \cdot 100^2 = 200,000$ calls to `Random` are required *each* time the test is applied. Of course, smaller values of k can be used but the resulting histogram bin grid will be too coarse, resulting in a test that has little power to find bad random number generators.

Permutation Test of Independence

The permutation test of independence makes use of the following result, which Knuth (1998) calls the "factorial number system." Let t be a positive integer and consider the set of $t!$ integers between 0 and $t! - 1$. For any such integer x there is an associated unique set of $t - 1$ integer coefficients $c_1, c_2, \ldots, c_{t-1}$ with $0 \le c_j \le j$ such that

$$x = c_{t-1}(t - 1)! + \cdots + c_2 2! + c_1 1!$$

For example, if $t = 5$ then $t! = 120$ and for the following integers between 0 and 119 we have

$$33 = 1 \cdot 4! + 1 \cdot 3! + 1 \cdot 2! + 1 \cdot 1!$$
$$47 = 1 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$$
$$48 = 2 \cdot 4! + 0 \cdot 3! + 0 \cdot 2! + 0 \cdot 1!$$
$$119 = 4 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$$

Given an *iid* sequence $U_0, U_1, \ldots, U_{t-1}$, Knuth cleverly uses the factorial number system to associate an integer x between 0 and $t! - 1$ uniquely with the sequence, as follows.

- Let $0 \le c_{t-1} \le t - 1$ be the index of the largest element of the sequence. Remove this largest element to form a new sequence with $t - 1$ terms.

- Let $0 \le c_{t-2} \le t - 2$ be the index of the largest element of the new sequence. Remove this largest element to form a new sequence with $t - 2$ terms.

- Repeat until all the coefficients $c_{t-1}, c_{t-2}, \ldots, c_1$ are determined (and the resulting sequence has just one term), then "encode" the original sequence with the integer

$$x = c_{t-1}(t - 1)! + \cdots + c_2 2! + c_1 1!$$

For the purposes of describing an algorithm for implementing the permutation test of independence, we refer to x as a "permutation index."

The corresponding theory behind the permutation test of independence is that the original sequence can have $t!$ equally likely possible relative orderings (permutations). Accordingly, the permutation index x is equally likely to have any value between 0 and $t! - 1$. That is, each possible values of x has probability $1/t!$.

Algorithm 10.1.6   To implement the permutation test of independence group (batch) `Random`'s output t terms at a time, calculate a permutation index x for each batch, and then tally all the indices so generated into a histogram with $k = t!$ bins as follows.

```
for (x = 0; x < k; x++)
  o[x] = 0;
for (i = 0; i < n; i++) {
  for (j = 0; j < t; j++)              /* build the permutation array */
    u[j] = Random();
  r = t - 1;
  x = 0;
  while (r > 0) {
    s = 0;
    for (j = 1; j <= r; j++)           /* max of u[0], u[1], ..., u[r] */
      if (u[j] > u[s])
        s = j;
    x = (r + 1) * x + s;
    temp = u[s];                                   /* swap u[s] and u[r] */
    u[s] = u[r];
    u[r] = temp;
    r--;
  }
  o[x]++;
}
```

The expected number of counts per bin is $e[x] = n/k$ and so the appropriate chi-square statistic is

$$v = \sum_{x=0}^{k-1} \frac{\left(o[x] - n/k\right)^2}{n/k}.$$

The test statistic v is a *Chisquare*$(k-1)$ random variate. Therefore, the critical values of v are $v_1^* = \mathtt{idfChisquare}(k-1, \alpha/2)$ and $v_2^* = \mathtt{idfChisquare}(k-1, 1-\alpha/2)$.

The *Permutation Test of Independence* is based on Algorithm 10.1.6. To apply this test, pick $t > 3$, $k = t!$, $n \geq 10k$, and then do the following.

- Use Algorithm 10.1.6 to calculate a k-bin histogram.

- Calculate the chi-square statistic v.

- Pick a $(1 - \alpha) \times 100\%$ level of confidence (typically $\alpha = 0.05$).

- Determine the critical values v_1^* and v_2^*.

If $v < v_1^*$ or $v > v_2^*$ the test *failed* (this occurs with probability $1 - \alpha$ for a legitimate random number generator).

Summary

We have now introduced six empirical tests of randomness — how do we use them?

- If the generator to be tested is truly awful (some are) then it may be sufficient to use just one test, for example the runs-up test of independence, a few times with $\alpha = 0.05$ and check to see if failures are consistently produced. If so, repeat with one or two other tests and if failures again occur consistently, the generator should be discarded.

- If the generator to be tested is better than truly awful, but still not good, then *proving* (in a statistical sense) that it isn't good may require a lot of testing and some relatively careful analysis of the results. The recommended way to conduct this analysis is with the Kolmogorov–Smirnov statistic, introduced in the next subsection. Unfortunately, if the generator to be tested isn't good there is no *guarantee* that any of the six empirical tests presented will detect the generator's deficiency, no matter how sophisticated the analysis of the test results. If the generator isn't good then (by definition) there is *some* empirical test that will reveal the deficiency. There are, however, no simple rules for finding this silver bullet test.

- If the generator to be tested is good then it *should* fail any of these tests approximately $\alpha \times 100\%$ of the time. In this case, it is relatively meaningless to select just one test and apply it a few times. If in 10 tests you observe 0, 1 or even 2 failures, what have you really learned? Extensive testing is required to conclude that a generator is good.

10.1.2 KOLMOGOROV–SMIRNOV ANALYSIS

The Kolmogorov–Smirnov test statistic measures the largest *vertical* distance between an empirical cdf calculated from a data set and a hypothesized cdf. Although the focus in this section has been on testing random numbers for uniformity, the KS statistic can be applied to distributions other than the $Uniform(0, 1)$ distribution, as indicated by the general definition given below.

Definition 10.1.4 Let X_1, X_2, ..., X_n be an *iid* sequence of continuous random variables with a common set of possible values $\mathcal{X}$ drawn from a population with *theoretical* (or hypothesized) cdf $F(x)$. For x, define the associated *empirical* cdf as

$$F_n(x) = \frac{\text{the number of } X_i\text{'s that are } \leq x}{n}.$$

The *Kolmogorov–Smirnov* (KS) statistic is

$$D_n = \sup_x |F(x) - F_n(x)|$$

where the sup* is taken over all x.

---

* sup is an abbreviation for suprememum, which plays an analogous role to maximum.

Example 10.1.5 We compute the KS test statistic D_6 for the first six random numbers produced by library `rngs` with an initial seed of 12345 and stream 0. The first $n = 6$ random numbers, shown to *d.ddd* precision, are:

$$0.277 \quad 0.726 \quad 0.698 \quad 0.941 \quad 0.413 \quad 0.720.$$

The first step in determining the empirical cdf is to sort the data values, yielding:

$$0.277 \quad 0.413 \quad 0.698 \quad 0.720 \quad 0.726 \quad 0.941.$$

A plot of the empirical cdf (which takes an upward step of height $1/6$ at each data value) and the theoretical cdf [the cdf of a *Uniform*$(0, 1)$ random variable] are plotted in Figure 10.1.5.

Figure 10.1.5. Empirical and theoretical cdfs.
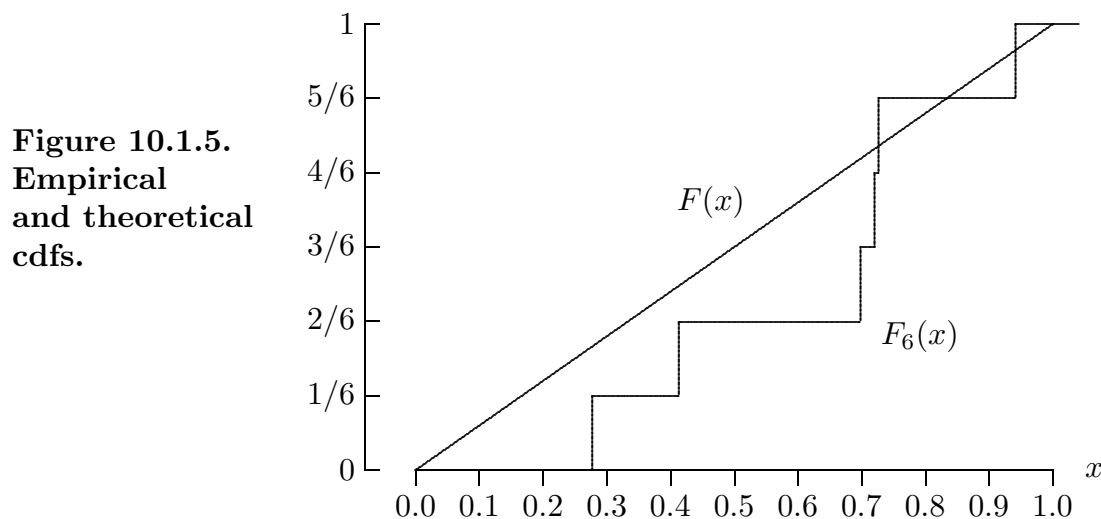
Where does the largest vertical distance between $F(x)$ and $F_6(x)$ occur? It occurs just to the left (this is why sup, rather than max, is used in the definition of D_n) of 0.720. For this particular data set, $D_6 \cong 0.720 - 2/6 \cong 0.387$. Large values of D_n indicate poor agreement between the theoretical cdf $F(x)$ and the empirical cdf $F_6(x)$.

As illustrated in Figure 10.1.5, because $F(\cdot)$ is continuous monotone increasing function and $F_n(\cdot)$ is monotone increasing in a piecewise constant fashion, the largest difference between $F(x)$ and $F_n(x)$ must occur at one of the data values. This means that a *computational* formula for determining the value of D_n is

$$D_n = \max_{i=1,2,\ldots,n} \left\{ \left| F(x_i) - \frac{i}{n} \right|, \left| F(x_i) - \frac{i-1}{n} \right| \right\}.$$

Therefore, D_n can be determined by a simple linear search over the $2n$ values given in the computational formula.

The next question that must be addressed is how large D_n must be for the KS test to fail. This will require the distribution of D_n.

The Distribution of D_n

Remarkably, it is known that the random variable (statistic) D_n depends *only* on n (Ross, 2002b, page 203). That is, for a fixed value of n, the shape of D_n's pdf (and associated cdf) is *independent* of $F(\cdot)$. For this reason, the KS statistic is said to be *distribution free*. Recall that we saw an analogous result back in Chapter 8.1; the Central Limit Theorem guarantees that for large sample sizes the sample mean is *Normal*, independent of the shape of the "parent" distribution. Unlike the Central Limit Theorem, which is an asymptotic (large sample size) result, the distribution free characteristic of the KS statistic is an *exact* result for *all* values of $n \geq 1$.*

For small to moderate values of n, Drew, Glen, and Leemis (2000) devised an algorithm that returns the cdf of D_n, which consists of nth order polynomials defined in a piecewise fashion. A "perfect fit" occurs when $F(x)$ intersects each riser of the stairstep function $F_n(x)$ at its midpoint, resulting D_n value of $1/(2n)$. We illustrate the results of the algorithm for $n = 6$ in the next example.

Example 10.1.6 The cdf associated with D_6 is given by sixth order polynomials defined in a piecewise fashion:

$$F_{D_6}(y) = \begin{cases} 0 & y < \frac{1}{12} \\ \frac{5}{324}(12y-1)^6 & \frac{1}{12} \leq y < \frac{1}{6} \\ 2880w^6 - 4800w^5 + 2360w^4 - \frac{1280}{3}w^3 + \frac{235}{9}w^2 + \frac{10}{27}w - \frac{5}{81} & \frac{1}{6} \leq y < \frac{1}{4} \\ 320w^6 + 320w^5 - \frac{2600}{3}w^4 + \frac{4240}{9}w^3 - \frac{785}{9}w^2 + \frac{145}{27}w - \frac{35}{1296} & \frac{1}{4} \leq y < \frac{1}{3} \\ -280w^6 + 560w^5 - \frac{1115}{3}w^4 + \frac{515}{9}w^3 + \frac{1525}{54}w^2 - \frac{565}{81}w + \frac{5}{16} & \frac{1}{3} \leq y < \frac{5}{12} \\ 104w^6 - 240w^5 + 295w^4 - \frac{1985}{9}w^3 + \frac{775}{9}w^2 - \frac{7645}{648}w + \frac{5}{16} & \frac{5}{12} \leq y < \frac{1}{2} \\ -20w^6 + 32w^5 - \frac{185}{9}w^3 + \frac{175}{36}w^2 + \frac{3371}{648}w - 1 & \frac{1}{2} \leq y < \frac{2}{3} \\ 10w^6 - 38w^5 + \frac{160}{3}w^4 - \frac{265}{9}w^3 - \frac{115}{108}w^2 + \frac{4651}{648}w - 1 & \frac{2}{3} \leq y < \frac{5}{6} \\ -2w^6 + 12w^5 - 30w^4 + 40w^3 - 30w^2 + 12w - 1 & \frac{5}{6} \leq y < 1. \end{cases}$$

This cdf is plotted in Figure 10.1.6, along with two important points highlighted. The first value corresponds to the KS test statistic from the last example: $(0.387, 0.743)$, which indicates that although the test statistic from the previous example seemed rather large, it was only at the 74th percentile of the distribution of D_6. The second value highlighted is $(0.519, 0.950)$, which corresponds to the D_6 value associated with the 95th percentile of the distribution. Thus a KS test with a sample size of $n = 6$ at significance level $\alpha = 0.05$ fails when the test statistic D_6 exceeds the critical value $d_6^* = 0.519$ (which will occur 5% of the time for a valid random number generator).

---

* One other advantage that the KS test statistic possesses over the chi-square statistics is that it requires no *binning* from the modeler. All of the chi-square tests discussed thus far calculated a k-bin histogram (or a $k \times k$ bin in the test of bivariate uniformity), where k was determined arbitrarily. There is no such arbitrary parameter in the case of the KS statistic.

$$F_{D_6}(y)$$

Figure 10.1.6.
Cumulative
distribution
function of
D_6.
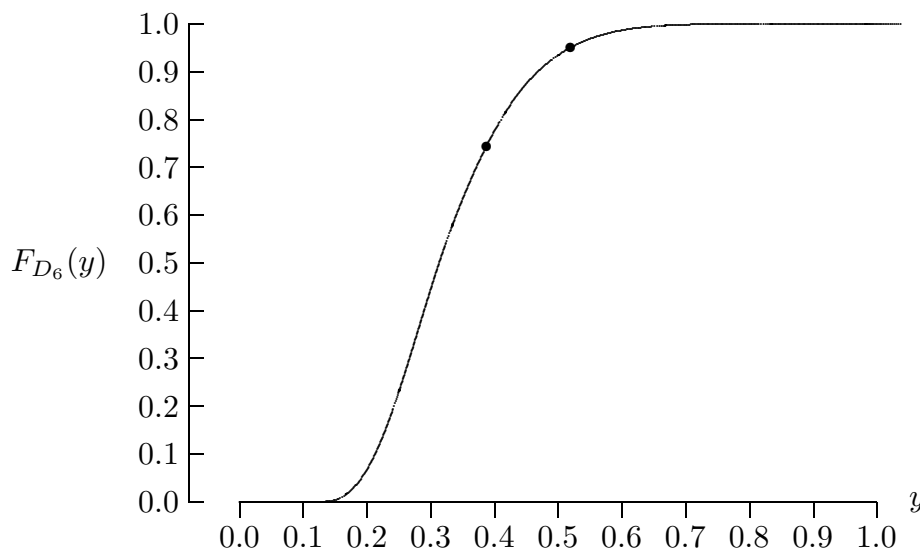
For larger values of n, approximations have been developed in order to approximate particular critical values of the distribution of D_n. The following theorem describes an approximation developed by Stephens (1974).

Theorem 10.1.3   If D_n is a KS random variable (statistic) and if $\Pr(D_n \leq d_n^*) = 1 - \alpha$ then the (one-tailed) critical value d_n^* is given by

$$d_n^* = \frac{c_{1-\alpha}}{\sqrt{n} + 0.12 + 0.11/\sqrt{n}}$$

where, for selected values of α, the constant $c_{1-\alpha}$ is

α	:	0.100	0.050	0.025	0.010
$c_{1-\alpha}$	:	1.224	1.358	1.480	1.628

Example 10.1.7   If $n = 6$ and $\alpha = 0.05$ then

$$d_6^* = \frac{1.358}{\sqrt{6} + 0.12 + 0.11/\sqrt{6}} \cong 0.519,$$

which is identical to the exact value from the previous example. Therefore

$$\Pr(D_6 \leq 0.519) = 0.95.$$

Example 10.1.8   If $n = 256$ and $\alpha = 0.05$ then

$$d_{256}^* = \frac{1.358}{16 + 0.12 + 0.007} \cong 0.084.$$

Therefore

$$\Pr(D_{256} \leq 0.084) = 0.95$$

and so the probability that a *single* realization of the KS random variable D_{256} will be less than 0.084 is 0.95. We fail the KS test when our computed test statistic D_{256} exceeds the critical value $d_{256}^* = 0.084$.

Example 10.1.8 suggests how to combine $n = 256$ empirical chi-square test statistics (the x's from the first part of this section) into one KS statistic d_{256} and, depending on whether the KS statistic is less than 0.084 or greater, with 95% confidence conclude that the source of the empirical test statistics — the random number generator in question — is random or not. This can be done independently for each empirical test. The details follow.

Let x_1, x_2, ..., x_n be a random variate sample that is assumed to be *Chisquare*$(k-1)$ for an appropriate value of k.* Let d_n be the associated sample statistic defined by

$$d_n = \sup_x |F(x) - F_n(x)|$$

where $F(x)$ is the *Chisquare*$(k-1)$ cdf, the max is over all $x \in \mathcal{X}$, and

$$F_n(x) = \frac{\text{the number of } x_i\text{'s that are } \leq x}{n}.$$

That is, d_n is a realization of the KS random variable D_n.

Example 10.1.9   The 256 runs-up test chi-square statistics illustrated in Figure 10.1.4 were sorted and used to generate $F_{256}(x)$. A portion of this empirical cdf is illustrated (as the piecewise-constant curve) along with the associated theoretical cdf $F(x)$ in Figure 10.1.5.

Figure 10.1.7.
Empirical and
theoretical
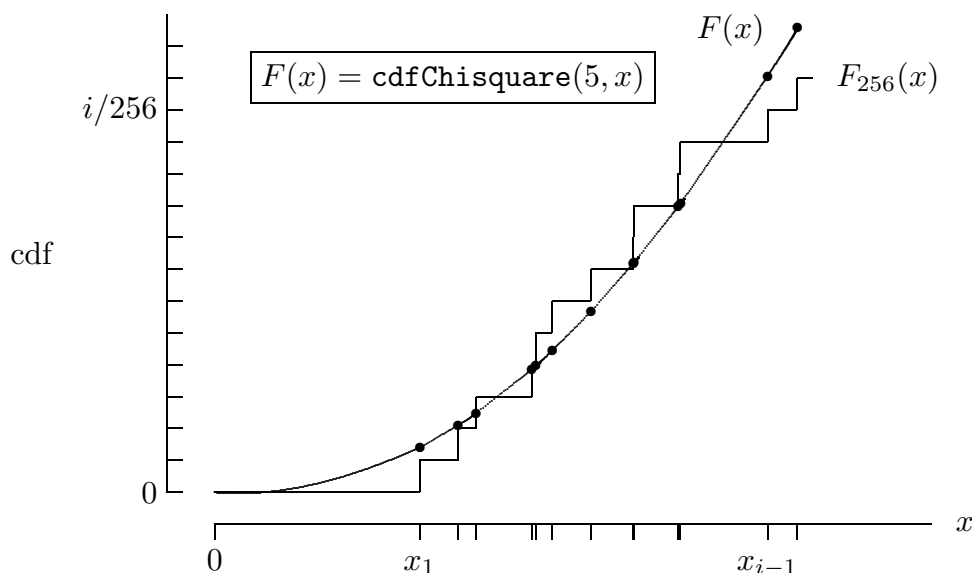chi-square cdf.

As indicated, in this case the hypothesis is that the chi-square statistics are samples of a *Chisquare*(5) random variable. The empirical cdf and hypothesized theoretical cdf fit well. For reference, the $\bullet$'s indicate the value of $F(x_i)$ for $i = 1, 2, \ldots, 13$.

---

* For example, for the test of uniformity $k-1$ is one less than the number of histogram bins.

Example 10.1.10   For the $n = 256$ chi-square statistics in Example 10.1.4, the computed KS statistic was $d_{256} = 0.044$, which is a value comfortably less than $d_{256}^* = 0.084$. This is strong evidence that the runs-up test of independence has detected *no* deficiencies in the `rngs` generator. Similarly, for the chi-square statistics in Examples 10.1.1 and 10.1.2, the computed KS statistic was 0.035 and 0.047 respectively; this is strong evidence that the test of uniformity and test of extremes also detected no deficiencies in the `rngs` generator.*

Most specialists consider the best random number generator empirical testing strategy to consist of multiple applications ($n = 100$ or more) of multiple chi-square tests (at least five) using the KS statistic with a 95% (or 99%) level of confidence as the ultimate empirical test for randomness. Most of these specialists also have unrestricted access to computers, hardworking graduate students, and much patience.

This section has contained a brief overview of random number testing. We recommend Law and Kelton (2000, pages 417–427), Banks, Carson, Nelson, and Nicol (2001, pages 264–284), and Fishman (2001, Chapter 9) for alternate, introductory treatments of the topic.

10.1.3   EXERCISES

Exercise 10.1.1   (*a*) Implement the test of uniformity (with $\alpha = 0.05$) and apply it to each of the 256 streams of random numbers supplied by `rngs`. Use any nine-digit number as initial seed for the generator. (*b*) For how many streams did the test fail? (*c*) Comment.

Exercise 10.1.2   Same procedure as in the previous exercise except using the test of extremes with $d = 5$.

Exercise 10.1.3   (*a*) Prove that if U_0, U_1, …, U_{d-1} is an *iid* sequence of d *Uniform*$(0, 1)$ random variables and if
$$S = \min\{U_0, U_1, \ldots, U_{d-1}\}$$
then $(1 - S)^d$ is a *Uniform*$(0, 1)$ random variable. (*b*) Use this result to modify the test of extremes accordingly and then repeat Exercise 10.1.2.

Exercise 10.1.4   (*a*) Prove Theorem 10.1.2. *Hint*: first prove that the probability a run-up will be longer than x is $1/(x + 1)!$. (*b*) What is the expected number of calls to `Random` to produce n runs?

---

* For the chi-square statistics in Example 10.1.3 the KS statistic was 0.056. Therefore, even though the number of test failures in this example was unexpectedly high, when all 256 tests results are viewed in the aggregate, there is no significant evidence of non-randomness.

Exercise 10.1.5 (*a*) Repeat the procedure in Exercise 10.1.1 using the runs-up test of independence. (*b*) How would you turn this test into a runs-down test?

Exercise 10.1.6 Relative to the equation in Definition 10.1.2, if n is the number of observations prove that an equivalent equation for v is

$$v = \sum_x \left(\frac{o^2[x]}{e[x]} \right) - n.$$

Exercise 10.1.7 (*a*) Implement the gap test of independence, generate 256 chi-square statistics and compute the corresponding KS statistic. (*b*) Comment.

Exercise 10.1.8 Repeat the previous exercise for the bivariate test of uniformity.

Exercise 10.1.9 Relative to the gap test of independence, what is the expected number of call to `Random` to produce n gaps?

Exercise 10.1.10 Is the runs-up test of independence valid if the *iid* samples are drawn from another continuous distribution? Experiment with draws from an *Exponential*(1) distribution.

Exercise 10.1.11 Repeat Exercise 10.1.7 for the permutation test of independence.

Exercise 10.1.12 Repeat Exercise 10.1.10 for the permutation test of independence.

Exercise 10.1.13 (*a*) Explain why one random number is skipped between runs in the runs-up test of independence. (*b*) If the random number between runs is *not* skipped, do you expect the value of the test statistic v to increase or decrease? Explain your reasoning.

Exercise 10.1.14 What is the set of possible values $\mathcal{X}$ for the KS random variable D_n corresponding to sample size n?

Exercise 10.1.15[a] What is the distribution of D_2, the KS statistic in the case of a sample size of $n = 2$?

Exercise 10.1.16[a] If 256 goodness-of-fit tests are conducted with a 95% level of confidence, then one expects $256 \cdot 0.05 \cong 13$ failures. How far away from 13 failures (high and low) would arouse suspicions concerning an empirical test of randomness?

The goal of this section is to introduce a particular type of stochastic process known as a *birth-death process.* Because the time evolution of such a process is usually defined in terms of an *Exponential* random variable, we begin with a brief review of the characteristics of this random variable from Section 7.4.2, with the cosmetic change of X replaced by T in order to emphasize the time dependence.

10.2.1 EXPONENTIAL RANDOM VARIABLES

Definition 10.2.1   If the random variable T is *Exponential*(μ) then

- the real-valued scale parameter μ satisfies $\mu > 0$

- the possible values of T are $\mathcal{T} = \{t \mid t > 0\}$

- the pdf of T is

$$f(t) = \frac{1}{\mu} \exp(-t/\mu) \qquad t > 0$$

- the cdf of T is

$$F(t) = \Pr(T \le t) = 1 - \exp(-t/\mu) \qquad t > 0$$

- the idf of T is

$$F^{-1}(u) = -\mu \ln(1 - u) \qquad 0 < u < 1$$

- the mean and standard deviation of T are both μ.

In reliability analysis, if the random variable T represents the *lifetime* or *failure time* of a process — the elapsed time between the initialization of a process and its time of (first) failure — then it is common to assume that T is *Exponential*(μ). In this application the parameter μ represents the expected lifetime and the probability that the lifetime will exceed some threshold value t is $\Pr(T > t) = 1 - \Pr(T \le t) = \exp(-t/\mu)$.*

Example 10.2.1   If the lifetime T of a process (in seconds) is *Exponential*(100) then its expected lifetime is 100 seconds and, for example, the probability that this process will last more that 120 seconds is $\Pr(T > 120) = \exp(-120/100) \cong 0.30$, as illustrated in Figure 10.2.1.

Figure 10.2.1.
Exponential(100)
pdf.
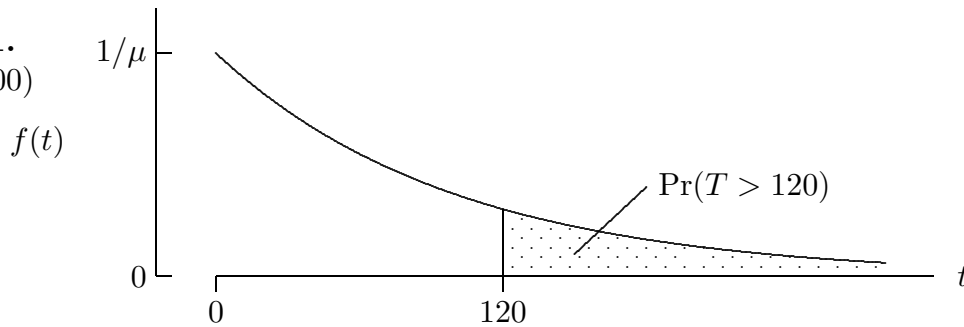
* $S(t) = \Pr(T > t)$ is known as the "survivor function" in reliability analysis.

Exponential Random Variables Are Memoryless

Theorem 10.2.1   If T is *Exponential*(μ) then T has the *lack of memory* property. That is, for any $\tau > 0$ and $t > 0$

$$\Pr(T > t + \tau \mid T > \tau) = \Pr(T > t) = \exp(-t/\mu).$$

Proof   As a consequence of the definition of conditional probability,

$$
\begin{aligned}
\Pr(T > t + \tau \mid T > \tau) &= \frac{\Pr\big((T > t + \tau) \text{ and } (T > \tau)\big)}{\Pr(T > \tau)} \\
&= \frac{\Pr(T > t + \tau)}{\Pr(T > \tau)} \\
&= \frac{\exp\big(-(t + \tau)/\mu\big)}{\exp(-\tau/\mu)} \\
&= \exp(-t/\mu) \\
&= \Pr(T > t).
\end{aligned}
$$

To paraphrase Theorem 10.2.1, $\Pr(T > t + \tau \mid T > \tau)$ is *independent* of τ for all $\tau > 0$. That is, T has *no* memory of its past. Equivalently, T is said to be *Markovian* — the *future* stochastic behavior of T is determined uniquely by its *current* state. In most respects, people are decidedly non-Markovian; we remember our past and that memory dictates, in part, our future behavior. To a surprising extent, however, many common natural stochastic processes are Markovian.

Example 10.2.2   As a continuation of Example 10.2.1, if the lifetime T of a process (in seconds) is *Exponential*(100) and if, for example, $t = 120$, $\tau = 200$ then

$$\Pr(T > 120 + 200 \mid T > 200) = \Pr(T > 120) \cong 0.30.$$

Therefore, this process is just as likely to survive for at least 120 more seconds when it has survived 200 seconds as it was to survive for at least 120 seconds when it was *new*.

It can be shown that the *Exponential* random variable is the *only* continuous random variable with the lack of memory property. Indeed, the memoryless property can be viewed as the defining characteristic of this random variable. That is, it can be proven that if T is a positive continuous random variable with mean $\mu = E[T]$ and if

$$\Pr(T > t + \tau \mid T > \tau) = \Pr(T > t)$$

for all $t > 0$, $\tau > 0$ then T is *Exponential*(μ). The proof is left as an exercise.*

---

* In the discrete case, a similar result applies to a *Geometric*(p) random variable.

The theory we have covered so far is a review of previous material. The following important theorem is new. The proof is deferred until later in the section.

Theorem 10.2.2   If T_1, T_2, ..., T_n are independent *Exponential*(μ_i) random variables (the μ_i may be different) and if T is the smallest of these, i.e.,

$$T = \min\{T_1, T_2, \ldots, T_n\},$$

then T is *Exponential*(μ) where

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \cdots + \frac{1}{\mu_n}.$$

In addition, the probability that T_i is the smallest of T_1, T_2, ..., T_n is

$$\Pr(T = T_i) = \frac{\mu}{\mu_i} \qquad\qquad i = 1, 2, \ldots, n.$$

Example 10.2.3   Suppose that T_1 and T_2 are independent *Exponential* random variables with means $\mu_1 = 2$ and $\mu_2 = 4$ respectively. Then $\mu = 4/3$ and the random variable $T = \min\{T_1, T_2\}$ is *Exponential*$(4/3)$. Moreover,

$$\Pr(T = T_1) = \frac{2}{3} \qquad\qquad \text{and} \qquad\qquad \Pr(T = T_2) = \frac{1}{3}.$$

If you were to generate many (t_1, t_2) samples of (T_1, T_2) then t_1 would be smaller than t_2 approximately $2/3$ of the time. Moreover, a histogram of $t = \min\{t_1, t_2\}$ would converge to the pdf of an *Exponential*$(4/3)$ random variable for a large number of replications.

Example 10.2.4   A computer uses 16 RAM chips, indexed $i = 1, 2, \ldots, 16$. If these chips fail independently (by natural aging rather than, say, because of a power surge) what is the expected time of the first failure?

- If each chip has an lifetime T_i (in hours) that is *Exponential*(10000), then the first failure is determined by $T = \min\{T_1, T_2, \ldots, T_{16}\}$ which is an *Exponential*(μ) random variable with $\mu = 10000/16$. Therefore, the expected time of the *first* failure is $E[T] = 10000/16 = 625$ hours.

- If instead one of the 16 chips has a lifetime that is *Exponential*(1000) then $E[T]$ is

$$\frac{1}{\mu} = \frac{15}{10000} + \frac{1}{1000} = \frac{25}{10000} = \frac{1}{400}$$

  and so the expected time of the first failure is 400 hours. Moreover, when the first failure occurs, the probability that it will be caused by the *Exponential*(1000) chip is $400/1000 = 0.4$.

Example 10.2.5   As a continuation of the previous example, suppose there are 17 identical *Exponential*(10000) RAM chips, including one spare. If the spare is installed as a replacement at the time of the first failure, then because of the memoryless property, the 15 old chips are statistically identical to the one new chip. Therefore, the expected time to the *next* failure is also 625 hours.

Proof of Theorem 10.2.2

To prove the first result in Theorem 10.2.2, let $T = \min\{T_1, T_2, \ldots, T_n\}$, where T_1, T_2, ..., T_n are independent *Exponential*(μ_i) random variables. For any $t > 0$, the cdf of T is

$$
\begin{aligned}
F(t) &= \Pr(T \le t) \\
&= 1 - \Pr(T > t) \\
&= 1 - \Pr\big((T_1 > t) \text{ and } (T_2 > t) \text{ and } \ldots \text{ and } (T_n > t)\big) \\
&= 1 - \Pr(T_1 > t)\Pr(T_2 > t)\ldots\Pr(T_n > t) \\
&= 1 - \exp(-t/\mu_1)\exp(-t/\mu_2)\ldots\exp(-t/\mu_n) \\
&= 1 - \exp(-t/\mu),
\end{aligned}
$$

where

$$
\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \cdots + \frac{1}{\mu_n}.
$$

This proves that T is *Exponential*(μ).

To prove the second part of Theorem 10.2.2, let $T' = \min\{T_2, T_3, \ldots, T_n\}$. From the previous result, T' is *Exponential*(μ') where

$$
\frac{1}{\mu'} = \frac{1}{\mu_2} + \frac{1}{\mu_3} + \cdots + \frac{1}{\mu_n}.
$$

Moreover, T_1 and T' are independent. Now recognize that $T_1 = T$ if and only if $T_1 \le T'$ and that by conditioning on the possible values of T'

$$
\Pr(T_1 = T) = \Pr(T_1 \le T') = \int_0^\infty \Pr(T_1 \le t')f(t')\,dt'
$$

where

$$
f(t') = \frac{1}{\mu'}\exp(-t'/\mu') \qquad t' > 0
$$

is the pdf of T' and $\Pr(T_1 \le t') = 1 - \exp(-t'/\mu_1)$ is the cdf of T_1. Therefore

$$
\Pr(T_1 = T) = \frac{1}{\mu'}\int_0^\infty \big(1 - \exp(-t'/\mu_1)\big)\exp(-t'/\mu')\,dt'
$$

$$
\vdots
$$

$$
= 1 - \frac{\mu}{\mu'} = \mu\left(\frac{1}{\mu} - \frac{1}{\mu'}\right) = \frac{\mu}{\mu_1}.
$$

This proves that $\Pr(T_i = T) = \mu/\mu_i$ for $i = 1$. Because the argument is equally valid for any i, the second part of Theorem 10.2.2 is proven.

Theorem 10.2.2 has an equivalent formulation expressed in terms of *rates* $\lambda_i = 1/\mu_i$ rather than expected times μ_i. The rate formulation is more intuitive and, for that reason, more easily remembered.

Theorem 10.2.3 If T_1, T_2, ..., T_n is an independent set of *Exponential*$(1/\lambda_i)$ random variables (all the λ_i may be different) and if T is the smallest of these, i.e.

$$T = \min\{T_1, T_2, \ldots, T_n\}$$

then T is *Exponential*$(1/\lambda)$ where

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

In addition, the probability that T_i is the smallest of T_1, T_2, ..., T_n is

$$\Pr(T_i = T) = \frac{\lambda_i}{\lambda} \qquad i = 1, 2, \ldots, n.$$

Example 10.2.6 Consider n independent streams of Poisson arrivals, with arrival rates $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, that merge to form a single stream, as illustrated in Figure 10.2.2.

Figure 10.2.2. Merging n independent arrival streams.
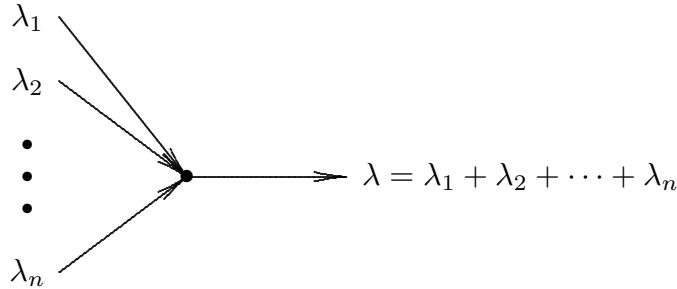
From Theorem 10.2.3, the single stream is a Poisson arrival process with rate λ. Moreover, the next arrival at the merge point will come from stream i with probability λ_i/λ.*

Example 10.2.7 Consider a multi-server service node with c independent servers all of which are busy at time t. Assume that for $s = 1, 2, \ldots, c$ server s is *Exponential*$(1/\nu_s)$ with service rate ν_s. From Theorem 10.2.3, the next departure from the service node will be at $t + T$ where T is an *Exponential*$(1/\nu)$ random variable and

$$\nu = \nu_1 + \nu_2 + \cdots + \nu_c$$

is the sum of the individual service rates. Moreover, this next departure will come from server s with probability ν_s/ν.

---

* Similarly, if a single stream of Poisson arrivals with rate λ splits into n streams independently with probability p_1, p_2, ..., p_n, then each of the streams so formed are Poisson processes with rate $p_i\lambda$ for $i = 1, 2, \ldots, n$.

10.2.2 CONTINUOUS-TIME BIRTH-DEATH PROCESSES

A birth-death processes is defined in terms of a discrete state-space $\mathcal{X}$ that may be either finite or infinite. Without loss of generality, we assume throughout that if the state-space is *finite* then

$$\mathcal{X} = \{0, 1, 2, \dots, k\}$$

for some positive integer k, else if the state-space is *infinite* then

$$\mathcal{X} = \{0, 1, 2, \dots\}.$$

Definition 10.2.2   Let X be a discrete random variable, indexed by time t as $X(t)$, that evolves in time as follows.

- At any time $t > 0$, $X(t) \in \mathcal{X}$.

- If $x \in \mathcal{X}$ is the state of X at time t then X will remain in this state for a random time T that is *Exponential* with mean $\mu(x)$.

- When a state transition occurs then, independent of the time spent in the current state, the random variable X will either shift from its current state x to state $x+1 \in \mathcal{X}$ with probability $p(x)$ or shift from state x to state $x-1 \in \mathcal{X}$ with probability $1 - p(x)$.

The shift from state x to $x+1$ is called a *birth*, the shift from x to $x-1$ is called a *death*. The stochastic process defined by $X(t)$ is called a *continuous-time birth-death process*.*

A continuous-time birth-death process is completely characterized by the initial state $X(0) \in \mathcal{X}$ at $t = 0$ and the two functions $\mu(x)$ and $p(x)$ defined for all $x \in \mathcal{X}$ so that

$$\mu(x) > 0 \qquad \text{and} \qquad 0 \le p(x) \le 1$$

with the understanding that $p(0) = 1$ because a death cannot occur when X is in the state $x = 0$. Moreover, if the state-space is finite then $p(k) = 0$ because a birth cannot occur when X is in the state $x = k$.

Continuous-Time Birth-Death Processes Are Memoryless

It is important to recognize that because of the very special memoryless property of the *Exponential* random variable, a birth-death process has no memory of its past — this is the Markovian assumption. At any time, the future time evolution of X depends *only* on the current state. The length of (past) time X has been in this state, or any other state, is completely irrelevant in terms of predicting when the next (future) state transition will occur and whether it will be a birth or a death. As we will see, for any $c \ge 1$ an $M/M/c$ service node represents an important special case of a continuous-time birth-death process with $X(t)$ corresponding to the number in the node at time t and births and deaths corresponding to arrivals and departures respectively.

---

* Discrete-time birth-death processes are considered later in this section.

We can use the fact that the function `Bernoulli(`p`)` in the library `rvgs` will return the value 1 or 0 with probability p and $1 - p$ respectively to simulate the time evolution of a birth-death process with Algorithm 10.2.1. Each replication of this simulation will generate one *realization* $x(t)$ of the birth-death process $X(t)$; repeated replication generates an *ensemble* of realizations.

Algorithm 10.2.1   Given the state-space $\mathcal{X}$, the initial state $X(0) \in \mathcal{X}$, and the functions $\mu(x)$, $p(x)$ defined for all $x \in \mathcal{X}$ with $\mu(x) > 0$ and $0 \le p(x) \le 1$, this algorithm simulates the time evolution of the associated continuous-time birth-death process for $0 < t < \tau$.

```
t = 0;
x = X(0);
while (t < τ) {
   t += Exponential(μ(x));
   x += 2 * Bernoulli(p(x)) - 1;
}
```

Birth Rates and Death Rates

Continuous-time birth-death processes are frequently characterized in terms of *rates*

$$\lambda(x) = \frac{1}{\mu(x)} \qquad\qquad x \in \mathcal{X}$$

rather than the corresponding expected *times* $\mu(x)$. The reason for this is that

$$b(x) = \lambda(x)p(x) \qquad\qquad \text{and} \qquad\qquad d(x) = \lambda(x)\big(1 - p(x)\big)$$

can then be thought of as *birth* and *death* rates respectively when X is in the state x. Note that $\lambda(x) = b(x) + d(x)$ is the rate at which the process will "flow" *out* of state x, to one of the adjoining states, $x + 1$ or $x - 1$.*

Because a birth-death process is stochastic, it is a mistake to think of $b(x)$ and $d(x)$ as *instantaneous* birth and death rates. Instead they represent *infinite-horizon* rates. Over a long period of time, $b(x)$ and $d(x)$ represent the average rate of flow from state x into the states $x + 1$ and $x - 1$ respectively. This flow analogy is particularly compelling when the processes is represented by a directed graph, as illustrated in Figure 10.2.3.

**Figure 10.2.3.
Rate diagram
for a birth-death
process.**

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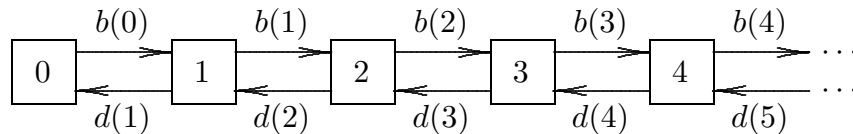
* If $\lambda(x) = 0$ then state x is said to be an *absorbing* state, a state from which the birth-death process can't escape.

Example 10.2.8 An $M/M/1$ service node with infinite capacity, arrival rate λ, and service rate ν defines an infinite-state continuous-time birth-death process. In this case $X(t)$ represents the number in the node at time $t > 0$ and the births and deaths correspond to arrivals and departures respectively.

- If $X(t) = 0$ then the (idle) node will remain in this state until the next arrival. Therefore, in this state only births are possible and they occur at the rate λ so that

$$p(0) = 1 \qquad \text{and} \qquad \lambda(0) = \lambda.$$

- If $X(t) = x > 0$ then the transition to the next state will occur at the rate

$$\lambda(x) = \lambda + \nu \qquad x = 1, 2, \ldots$$

and it will be an arrival (birth) with probability

$$p(x) = \frac{\lambda}{\lambda + \nu} \qquad x = 1, 2, \ldots$$

or a departure (death) with probability $1 - p(x) = \nu/(\lambda + \nu)$.

Therefore, the birth rate is

$$b(x) = \lambda(x)p(x) = \lambda \qquad x = 0, 1, 2, \ldots$$

and the death rate is

$$d(x) = \lambda(x) - b(x) = \begin{cases} 0 & x = 0 \\ \nu & x = 1, 2, \ldots \end{cases}$$

The corresponding (infinite) directed graph representation is shown in Figure 10.2.4.
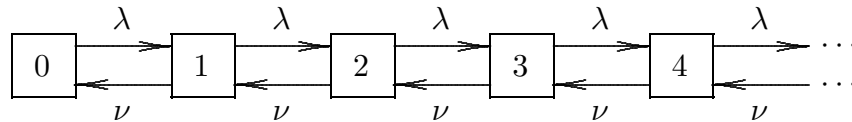
**Figure 10.2.4.
Rate diagram
for an M/M/1
service node.**

Example 10.2.9 As a continuation of the previous example, the notation $M/M/1/k$ denotes an M/M/1 service node with *finite* capacity k. Note that $k \geq 1$ is the maximum possible number of jobs in the service node, *not* in the queue. In this case, the equations in Example 10.2.8 must be modified slightly to reflect the fact that the state-space is $\mathcal{X} = \{0, 1, 2, \ldots, k\}$. The associated (finite) directed graph representation is shown in Figure 10.2.5.
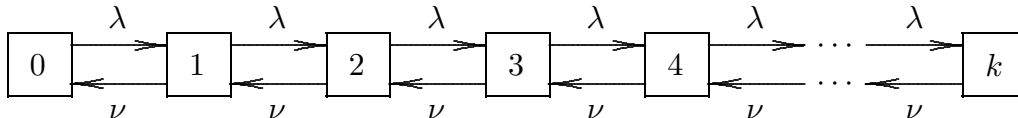

**Figure 10.2.5.
Rate diagram
for an M/M/1/k
service node.**

10.2.3   STEADY-STATE CHARACTERIZATION

Definition 10.2.3   Let $f(x,t) = \Pr(X(t) = x)$ denote the probability (pdf) that a continuous-time birth-death process $X(t)$ is in the state $x \in \mathcal{X}$ at time $t > 0$. The *steady-state* pdf (if it exists) is defined by

$$f(x) = \lim_{t \to \infty} f(x,t) \qquad x \in \mathcal{X}.$$

Steady-State Flow Balance

As always, when dealing with probabilities it must be true that $f(x) \geq 0$ for all $x \in \mathcal{X}$ and

$$\sum_x f(x) = 1$$

where the sum is over all $x \in \mathcal{X}$. Recall that when the birth-death process is in state x the rate of flow into the states $x+1$ and $x-1$ is $b(x)$ and $d(x)$ respectively. Because $f(x)$ represents the steady-state probability that the process is in state x, it follows that for each $x \in \mathcal{X}$:

- $b(x)f(x)$ is the steady-state rate of flow from state x to state $x+1$;

- $d(x)f(x)$ is the steady-state rate of flow from state x to state $x-1$.

In steady-state the net rate of flow *into* state x must be balanced by the net rate of flow *out* of this state, as illustrated in Figure 10.2.6.

Figure 10.2.6.
Generic flow
diagram about
node x.
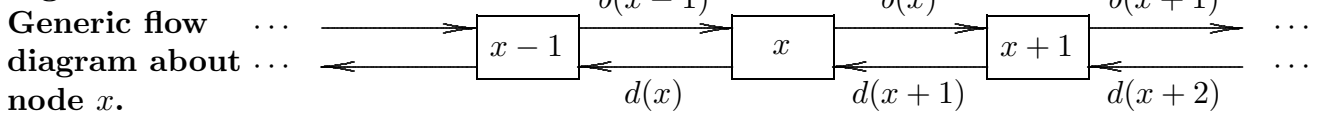
That is, for state $x \in \mathcal{X}$, steady-state is characterized by the *flow balance equation*

$$\text{(flow in)} \qquad d(x+1)f(x+1) + b(x-1)f(x-1) = b(x)f(x) + d(x)f(x) \qquad \text{(flow out)}.$$

This equation is *linear* in the unknown steady-state probabilities. Moreover, there is one equation for each $x \in \mathcal{X}$ and so the steady-state probabilities can be determined by solving a (possibly infinite) system of linear equations.

Example 10.2.10   From Example 10.2.8, for an $M/M/1$ service node with infinite capacity the balance equations are

$$\nu f(1) = \lambda f(0)$$
$$\nu f(x+1) + \lambda f(x-1) = (\lambda + \nu)f(x) \qquad x = 1, 2, \ldots$$

Recognize that, except for cosmetic differences, these equations are identical to the steady-state equations developed in Section 8.5.

By looking at the steady-state rate of flow into and out of each state we can write the flow balance equations as

$$\text{flow in} = \text{flow out}$$

$$d(1)f(1) = b(0)f(0) \qquad\qquad \text{state } 0$$

$$d(2)f(2) + b(0)f(0) = b(1)f(1) + d(1)f(1) \qquad\qquad \text{state } 1$$

$$d(3)f(3) + b(1)f(1) = b(2)f(2) + d(2)f(2) \qquad\qquad \text{state } 2$$

$$\vdots$$

$$d(x+1)f(x+1) + b(x-1)f(x-1) = b(x)f(x) + d(x)f(x) \qquad\qquad \text{state } x$$

$$\vdots$$

If the first $x + 1$ flow balance equations are added and then like terms are canceled, it can be shown that the resulting equation is

$$d(x+1)f(x+1) = b(x)f(x) \qquad\qquad x = 0, 1, 2, \ldots$$

Note that, as illustrated in Figure 10.2.7, this last equation could have been written "by inspection" if we had applied flow balance to the digraph *arc* connecting states x and $x+1$ rather than to the digraph *node* x.
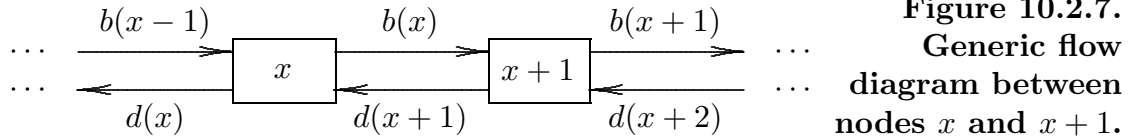
Figure 10.2.7. Generic flow diagram between nodes x and $x+1$.

Steady-State Probabilities

As summarized by the following theorem, the equations

$$d(x+1)f(x+1) = b(x)f(x) \qquad\qquad x = 0, 1, 2, \ldots$$

provides a recursive algorithm to solve for $f(1)$, $f(2)$, $f(3)$, ... in terms of $f(0)$.

Theorem 10.2.4 For a continuous-time birth-death process with $\mathcal{X} = \{0, 1, 2, \ldots, k\}$ where k may be infinite, the steady-state probabilities (if they exist) can be calculated recursively in terms of $f(0)$ as

$$f(x+1) = \frac{b(x)b(x-1)\ldots b(1)b(0)}{d(x+1)d(x)\ldots d(2)d(1)}f(0) \qquad\qquad x = 0, 1, 2, \ldots, k-1.$$

The remaining steady-state probability $f(0)$ is determined by the equation

$$\sum_x f(x) = 1$$

where the sum is over all $x \in \mathcal{X}$.

Algorithm 10.2.2   Based on Theorem 10.2.4 with $\mathcal{X} = \{0, 1, 2, \ldots, k\}$ where k may be infinite, the following three-step algorithm computes the steady-state probabilities (if they exist) for a continuous-time birth-death process.

- With $f'(0) = 1$ define the *un-normalized* steady-state probabilities $f'(x)$ recursively as

$$f'(x + 1) = \frac{b(x)}{d(x + 1)} f'(x) \qquad\qquad x = 0, 1, 2, \ldots, k - 1.$$

- Sum over all possible states $x \in \mathcal{X}$ to calculate the normalizing constant

$$\alpha = \sum_x f'(x) > 0.$$

- Compute the (normalized) steady-state probabilities as*

$$f(x) = \frac{f'(x)}{\alpha} \qquad\qquad x = 0, 1, 2, \ldots, k.$$

Finite State Space

If the number of possible states is *finite* then Algorithm 10.2.2 is always valid provided $b(k) = 0$ and $d(x) \neq 0$ for $x = 1, 2, \ldots, k$ where the finite state space is $\mathcal{X} = \{0, 1, 2, \ldots, k\}$. That is, steady-state will always be achieved as $t \to \infty$.

Example 10.2.11   As a continuation of Example 10.2.10, for an $M/M/1/k$ service node with capacity $1 \leq k < \infty$, we have (using l in place of x)

$$f'(l + 1) = \rho f'(l) \qquad\qquad l = 0, 1, \ldots, k - 1$$

where $\rho = \lambda/\nu$. If follows that

$$f'(l) = \rho^l \qquad\qquad l = 0, 1, 2, \ldots, k$$

and

$$\alpha = \sum_{l=0}^{k} f'(l) = 1 + \rho + \cdots + \rho^k = \frac{1 - \rho^{k+1}}{1 - \rho} \qquad\qquad (\rho \neq 1).$$

Therefore, the steady-state pdf for an $M/M/1/k$ service node is

$$f(l) = \left(\frac{1 - \rho}{1 - \rho^{k+1}} \right) \rho^l \qquad\qquad l = 0, 1, 2, \ldots, k.$$

provided $\rho \neq 1$. If $\rho = 1$ then $f(l) = 1/(k + 1)$ for $l = 0, 1, 2, \ldots, k$.

---

* Note that $\alpha = 1/f(0)$.

The results in Example 10.2.11 are summarized by the following theorem. Note that if $\rho < 1$ then $\rho^{k+1} \to 0$ as $k \to \infty$. Therefore, as expected, if $\rho < 1$ then as the service node capacity becomes infinite Theorem 10.2.5 reduces to Theorem 9.5.3

Theorem 10.2.5 For an $M/M/1/k$ service node with $\rho = \lambda/\nu \neq 1$, the steady-state probability of l jobs in the service node is

$$f(l) = \left(\frac{1-\rho}{1-\rho^{k+1}} \right) \rho^l \qquad l = 0, 1, 2, \ldots, k.$$

If $\rho = 1$ then $f(l) = 1/(k+1)$ for $l = 0, 1, 2, \ldots, k$.

Example 10.2.12 As the equation in Theorem 10.2.5 indicates, steady-state can *always* be achieved for an $M/M/1/k$ service node, even if $\rho = \lambda/\nu > 1$. The reason for this is that if the service node capacity is finite then jobs will be rejected if they arrive to find the service node full (holding k jobs).

Figure 10.2.8. Steady-state probabilities for the $M/M/1/k$ queue.
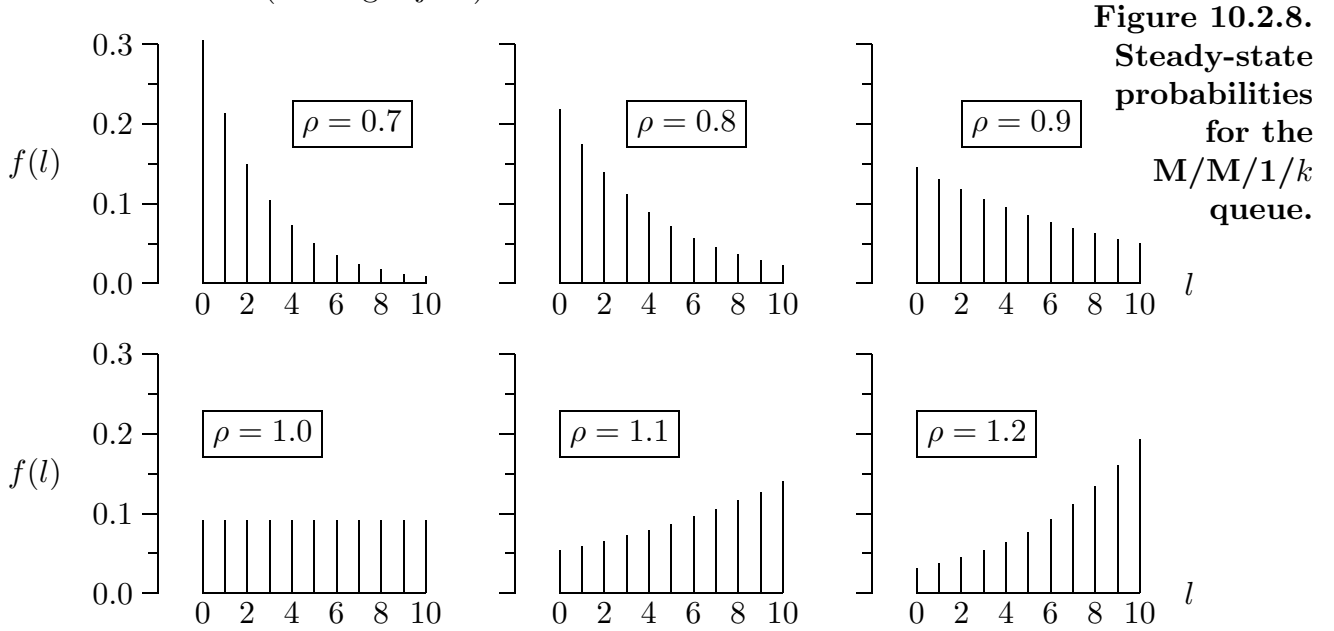
As illustrated in Figure 10.2.8 for $k = 10$, if $\rho < 1$ then $f(l)$ is a monotone decreasing function of l. When $\rho = 1$ all of the $k+1$ possibility service node states are equally likely. For values of $\rho > 1$, $f(l)$ increases monotonically with increasing l. The probability of rejection is

$$f(k) = \left(\frac{1-\rho}{1-\rho^{k+1}} \right) \rho^k$$

and the probability of an idle service node is

$$f(0) = \left(\frac{1-\rho}{1-\rho^{k+1}} \right).$$

As ρ increases, the probability of rejection increases and the probability of an idle service node decreases.

Steady-State Statistics

From the equation in Theorem 10.2.5, for an $M/M/1/k$ service node with $\rho \neq 1$ the steady-state expected number in the service node $E[L] = \lim_{t \to \infty} \bar{l}$ is

$$E[L] = \sum_{l=0}^{k} l\, f(l) = \sum_{l=1}^{k} l\, f(l) = \left(\frac{1-\rho}{1-\rho^{k+1}} \right) (\rho + 2\rho^2 + \cdots + k\rho^k).$$

From the identity

$$\rho + 2\rho^2 + \cdots + k\rho^k = \rho \frac{d}{d\rho} \left(1 + \rho + \rho^2 + \cdots + \rho^k \right) = \frac{\rho(1-\rho^{k+1}) - (k+1)\rho^{k+1}(1-\rho)}{(1-\rho)^2}$$

which is valid for all $\rho \neq 1$, it follows that

$$E[L] = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}}.$$

If $\rho = 1$ then $E[L] = k/2$.

Similarly, the steady-state expected number in the queue $E[Q] = \lim_{t \to \infty} \bar{q}$ is

$$E[Q] = \sum_{l=1}^{k} (l-1)f(l) = \sum_{l=1}^{k} l\, f(l) - \sum_{l=1}^{k} f(l) = \cdots = E[L] - (1 - f(0)).$$

Little's equations can be used to determine the corresponding expected wait and delay. To do so, note that because rejection occurs when the service node is full, the rate at which jobs flow into the service node is $(1 - f(k))\lambda$, not λ. Therefore, the steady-state expected wait $E[W] = \lim_{t \to \infty} \bar{w}$ is

$$E[W] = \frac{E[L]}{(1 - f(k))\lambda}$$

and the steady-state expected delay $E[D] = \lim_{t \to \infty} \bar{d}$ is

$$E[D] = \frac{E[Q]}{(1 - f(k))\lambda}.$$

In the three equations for $E[Q]$, $E[W]$, and $E[D]$ the probabilities $f(0)$ and $f(k)$ are given in Example 10.2.12.

Infinite State Space

If the number of possible states for a continuous-time birth-death process is *infinite* then, even if $d(x) \neq 0$ for all x, it is possible that the sum of un-normalized steady-state probabilities may not converge — the parameter α may be infinite. In this case Algorithm 10.2.2 will fail because the continuous-time birth-death process can never achieve steady-state. It turns out that the condition $\alpha < \infty$ is both a necessary and sufficient condition for steady-state. As we will see, the $\alpha < \infty$ condition is the continuous-time birth-death process analog of the familiar $\rho < 1$ steady-state condition for infinite-capacity, single-server and multi-server service nodes.

Example 10.2.13 As a continuation of Examples 10.2.11 and 10.2.12, consider the limiting case of an $M/M/1/k$ service node with $k = \infty$, i.e., a conventional infinite capacity $M/M/1$ service node. In this case

$$\alpha = \sum_{l=0}^{\infty} f'(l) = 1 + \rho + \rho^2 + \cdots$$

and (since $\rho > 0$) this harmonic series converges if and only if $\rho < 1$. That is, if $\rho < 1$ then this series converges to

$$\alpha = \frac{1}{1-\rho}$$

so that the steady-state probability of l jobs in an $M/M/1$ service node is

$$f(l) = \frac{f'(l)}{\alpha} = (1-\rho)\rho^l \qquad l = 0, 1, 2, \ldots$$

That is, in steady-state the number of jobs in an $M/M/1$ service node is a *Geometric*(ρ) random variable. This is Theorem 8.5.3.

Multi-Server Service Node

Let us now consider the more mathematically challenging case of an infinite-capacity $M/M/c$ service node with arrival rate λ and c identical servers, each with service rate ν. In this case (using l in place of x) it can be shown that an $M/M/c$ service node defines a continuous-time birth-death process with

$$p(l) = \begin{cases} \dfrac{\lambda}{\lambda + l\nu} & l = 0, 1, 2, \ldots, c-1 \\[2mm] \dfrac{\lambda}{\lambda + c\nu} & l = c, c+1, c+2, \ldots \end{cases}$$

and

$$\lambda(l) = \begin{cases} \lambda + l\nu & l = 0, 1, 2, \ldots, c-1 \\ \lambda + c\nu & l = c, c+1, c+2, \ldots \end{cases}$$

Therefore, the birth rate is

$$b(l) = p(l)\lambda(l) = \lambda \qquad l = 0, 1, 2, \ldots$$

and the (state-dependent) death rate is

$$d(l) = \lambda(l) - b(l) = \begin{cases} l\nu & l = 0, 1, 2, \ldots, c-1 \\ c\nu & l = c, c+1, c+2, \ldots \end{cases}$$

Algorithm 10.2.2 can then be applied, as illustrated in the following example.

Example 10.2.14   For an $M/M/c$ service node, define $\rho = \lambda/c\nu$. Then,

$$\frac{b(l-1)}{d(l)} = \begin{cases} c\rho/l & l = 1, 2, \ldots, c-1 \\ \rho & l = c, c+1, c+2, \ldots \end{cases}$$

and, with $f'(0) = 1$, from Algorithm 10.2.2

$$f'(l) = \begin{cases} (c\rho)^l/l! & l = 0, 1, 2, \ldots, c-1 \\ (c\rho)^c \rho^{l-c}/c! & l = c, c+1, c+2, \ldots \end{cases}$$

If $\rho < 1$ then the results in the following theorem are based on evaluating the constant

$$\alpha = \frac{1}{f(0)}$$

$$= \sum_{l=0}^{\infty} f'(l)$$

$$= \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} + \sum_{l=c}^{\infty} \frac{(c\rho)^c \rho^{l-c}}{c!}$$

$$= \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{c!}(1 + \rho + \rho^2 + \cdots)$$

$$= \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} + \frac{(c\rho)^c}{(1-\rho)c!}$$

and calculating

$$f(l) = \frac{f'(l)}{\alpha} \qquad l = 0, 1, 2, \ldots$$

Theorem 10.2.6   For an $M/M/c$ service node with $c > 1$ and $\rho = \lambda/c\nu < 1$, the steady-state probability of l jobs in the service node is $f(l)$ where, for $l = 0$

$$f(0) = \left(\frac{(c\rho)^c}{(1-\rho)\,c!} + \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} \right)^{-1}$$

and for $l = 1, 2, \ldots$

$$f(l) = \begin{cases} \dfrac{(c\rho)^l}{l!} f(0) & l = 1, 2, \ldots, c-1 \\ \dfrac{(c\rho)^c}{c!} \rho^{l-c} f(0) & l = c, c+1, c+2, \ldots \end{cases}$$

If $c = 1$ this theorem reduces to Theorem 9.5.3.

Steady-State Statistics

From the equations in Theorem 10.2.6, for an $M/M/c$ service node the steady-state expected number in the queue is

$$
\begin{aligned}
E[Q] &= \sum_{l=c}^{\infty} (l-c) f(l) \\
&= \sum_{l=c}^{\infty} (l-c) \frac{(c\rho)^c}{c!} \rho^{l-c} f(0) \\
&= \frac{(c\rho)^c}{c!} f(0) \sum_{l=c}^{\infty} (l-c) \rho^{l-c} \\
&= \frac{(c\rho)^c}{c!} f(0) \left(\rho + 2\rho^2 + 3\rho^3 + \cdots \right) \\
&\;\;\vdots \\
&= \frac{(c\rho)^c}{c!} \cdot \frac{\rho}{(1-\rho)^2} f(0).
\end{aligned}
$$

In particular, it can be shown that

$$
\lim_{\rho \to 1} E[Q] = \frac{\rho}{1-\rho}.
$$

Therefore, as in the single-server case, we see that as the steady-state utilization of each server approaches 1 the average number in the queue (and thus in the service node) approaches infinity. This is consistent with the simulations in Figure 1.2.9.

As in Section 8.5, for an $M/M/c$ service node with $\rho = \lambda/c\nu$ the four steady-state statistics $E[W]$, $E[L]$, $E[D]$, and $E[Q]$ are related by the four equations

$$
E[W] = E[D] + \frac{1}{\nu} \qquad E[L] = E[Q] + c\rho \qquad E[L] = \lambda E[W] \qquad E[Q] = \lambda E[D]
$$

and any three of these can be calculated provided the fourth is known. Therefore, by using the fact that the steady-state average number in the queue is

$$
E[Q] = \frac{(c\rho)^c}{c!} \frac{\rho}{(1-\rho)^2} \left(\frac{(c\rho)^c}{(1-\rho)\, c!} + \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} \right)^{-1},
$$

the other steady-state statistics for an $M/M/c$ service node can be calculated as

$$
\begin{aligned}
E[L] &= E[Q] + c\rho & &\text{number in the node} \\
E[D] &= E[Q]/\lambda & &\text{delay in the queue} \\
E[W] &= E[L]/\lambda & &\text{wait in the node.}
\end{aligned}
$$

If $\rho \geq 1$ then steady-state statistics do not exist.

Example 10.2.15   For an $M/M/4$ service node with $\lambda = 1.0$ and $\rho = 0.75$ (so that the service rate for each server is $\nu = 1/3$), the steady-state probability of an idle node is

$$f(0) = \left(\frac{(c\rho)^c}{(1-\rho)\,c!} + \sum_{s=0}^{c-1} \frac{(c\rho)^s}{s!} \right)^{-1} = \left(\frac{3^4}{(1/4)\,4!} + \sum_{s=0}^{3} \frac{3^s}{s!} \right)^{-1} = \cdots = \frac{2}{53}$$

and the steady-state average number in the queue is

$$E[Q] = \frac{(c\rho)^c}{c!} \cdot \frac{\rho}{(1-\rho)^2} f(0) = \frac{3^4(3/4)}{4!\,(1/4)^2} \left(\frac{2}{53} \right) = \frac{81}{53} \cong 1.53.$$

Therefore, because $c\rho = 3.0$,

$$E[L] = \frac{81}{53} + 3 = \frac{240}{53} \cong 4.53$$

$$E[D] = \frac{81}{53} \cong 1.53$$

$$E[W] = \frac{240}{53} \cong 4.53.$$

Algorithm 10.2.3   Given the arrival rate λ, service rate ν (per server), and the number of servers c, then if $\rho = \lambda/c\nu < 1$ this algorithm computes the first-order, steady-state statistics for an $M/M/c$ service node. The symbols $\bar{q}$, $\bar{l}$, $\bar{d}$, and $\bar{w}$ denote $E[Q]$, $E[L]$, $E[D]$, and $E[W]$.

```
ρ = λ / (c * ν);
sum = 0.0;
s = 0;
t = 1.0;
while (s < c) {
    sum += t;
    s++;
    t *= (c * ρ / s);
}                                          /* at this point t = (cρ)ᶜ/c! */
sum += t / (1.0 - ρ);                       /* and sum is 1/f(0) */
q̄ = t * ρ / ((1.0 - ρ) * (1.0 - ρ) * sum);
l̄ = q̄ + (c * ρ);
d̄ = q̄ / λ;
w̄ = l̄ / λ;
return q̄, l̄, d̄, w̄, ρ;
```

Finite Capacity Multi-Server Service Node

     An $M/M/c$ service node with *finite* capacity $c \le k < \infty$ is denoted $M/M/c/k$. As in the special case $c = 1$, k is the capacity of the service node, not the queue. A proof of the following theorem is left as an exercise.

Theorem 10.2.7 For an $M/M/c/k$ service node with $1 < c \leq k$ and $\rho = \lambda/c\nu \neq 1$, the steady-state probability of l jobs in the service node is $f(l)$ where for $l = 0$

$$f(0) = \left(\frac{(c\rho)^c}{(1 - \rho)\, c!} \left(1 - \rho^{k-c+1}\right) + \sum_{l=0}^{c-1} \frac{(c\rho)^l}{l!} \right)^{-1}$$

and for $l = 1, 2, \ldots, k$

$$f(l) = \begin{cases} \dfrac{(c\rho)^l}{l!} f(0) & l = 1, 2, \ldots, c - 1 \\[2ex] \dfrac{(c\rho)^c}{c!} \rho^{l-c} f(0) & l = c, c+1, c+2, \ldots, k. \end{cases}$$

If $\rho = 1$ then

$$f(0) = \left(\frac{c^c}{c!}\, (k - c + 1) + \sum_{l=0}^{c-1} \frac{c^l}{l!} \right)^{-1}$$

and

$$f(l) = \begin{cases} \dfrac{c^l}{l!} f(0) & l = 1, 2, \ldots, c - 1 \\[2ex] \dfrac{c^c}{c!} f(0) & l = c, c+1, c+2, \ldots, k. \end{cases}$$

If $c = 1$ then this theorem reduces to Theorem 10.2.5.

Steady-State Statistics

Analogous to the case of an $M/M/c$ service node, for an $M/M/c/k$ service node steady-state statistics can be computed from the equations in Theorem 10.2.7. That is, the steady-state number in the service node is

$$E[L] = \sum_{l=0}^{k} l\, f(l)$$

and the steady-state number in the queue is

$$E[Q] = \sum_{l=c}^{k} (l - c)\, f(l).$$

These two equations can be evaluated numerically for specific values of ρ, c, and k. The equations can also be algebraically simplified somewhat, but the resulting expressions are too complex to be useful for most applications.

Given that $E[L]$ and $E[Q]$ have been evaluated, either numerically or algebraically, the corresponding steady-state wait in the service node and delay in the queue are

$$E[W] = \frac{E[L]}{(1 - f(k))\lambda} \qquad \text{and} \qquad E[D] = \frac{E[Q]}{(1 - f(k))\lambda}$$

respectively.

Infinitely Many Servers

The $M/M/c$ steady-state results in Theorem 10.2.6 are valid for any $c \geq 1$. If the number of servers is (effectively) infinite ($c \to \infty$) or, equivalently, if the jobs are *self-serving*, each with a expected service time of $1/\nu$, then

$$\frac{b(l-1)}{d(l)} = \frac{\lambda}{l\nu} \qquad l = 1, 2, 3, \ldots$$

and, with $f'(0) = 1$, from Algorithm 10.2.2

$$f'(l) = \frac{(\lambda/\nu)^l}{l!} \qquad l = 0, 1, 2, \ldots$$

In this case, the normalizing constant is

$$\alpha = \frac{1}{f(0)} = \sum_{l=0}^{\infty} f'(l) = \sum_{l=0}^{\infty} \frac{(\lambda/\nu)^l}{l!} = \exp(\lambda/\nu).$$

Therefore, in steady-state the probability of l jobs in an $M/M/\infty$ service node is

$$f(l) = \frac{\exp(-\lambda/\nu)(\lambda/\nu)^l}{l!} \qquad l = 0, 1, 2, \ldots$$

That is, we have proven the following theorem, valid for *any* value of λ/ν.

Theorem 10.2.8   In steady-state the number of jobs in an $M/M/\infty$ service node is a *Poisson*(λ/ν) random variable.

The $M/M/\infty$ terminology is potentially confusing. An $M/M/\infty$ "queue" will in fact *never* have a queue; the number of servers is infinite and so *any* arrival rate, no matter how large, can be accommodated. The steady-state expected number of jobs in an $M/M/\infty$ service node is $E[L] = \lambda/\nu$ and this ratio can be arbitrarily large.

It is interesting to compare the two limiting cases associated with an M/M/c service node: $c = 1$ and $c = \infty$.

- If $\lambda/\nu < 1$, then the steady-state number in an $M/M/1$ service node is a *Geometric*(λ/ν) random variable.

- For any value of λ/ν, the steady-state number in an $M/M/\infty$ service node is a *Poisson*(λ/ν) random variable.

Except for these two limiting cases, there is no simple discrete random variable characterization for the steady-state number of jobs in an $M/M/c$ service node.

10.2.4   DISCRETE-TIME BIRTH-DEATH PROCESSES

For some stochastic models time is a *continuous* variable with, for example, possible values $t \geq 0$; for other stochastic models time is a *discrete* variable with, for example, possible values $t = 0, 1, 2, \ldots$ To make the distinction between discrete-time and continuous-time models more clear, this section concludes with a discussion of the discrete-time analog of the continuous-time birth-death process model discussed previously.

As in the continuous-time case, a discrete-time birth-death processes is defined in terms of a discrete state-space $\mathcal{X}$ that can be either finite or infinite. Without loss of generality, we assume that if the state-space is *finite* then

$$\mathcal{X} = \{0, 1, 2, \ldots, k\}$$

for some positive integer k, else if the state-space is *infinite* then

$$\mathcal{X} = \{0, 1, 2, \ldots\}.$$

Formulation

There are two equivalent formulations of a discrete-time birth-death process. We begin with a three-state "birth-death-remain" formulation that is intuitive, but not directly analogous to Definition 10.2.2. We then show that this formulation is equivalent to an alternative formulation that is analogous to the definition of a continuous-time birth-death process.

Definition 10.2.4   Let X be a discrete random variable, indexed by time t as $X(t)$, that evolves in time as follows.

- $X(t) \in \mathcal{X}$ for all $t = 0, 1, 2, \ldots$

- State transitions can occur only at the discrete times $t = 0, 1, 2, \ldots$

- When a state transition *can* occur then the random variable X *will* either shift from its current state x to state $x + 1 \in \mathcal{X}$ with probability $b(x)$ or shift from state x to state $x - 1 \in \mathcal{X}$ with probability $d(x)$ or remain in its current state x with probability $c(x) = 1 - b(x) - d(x)$.

The shift from state x to $x + 1$ is called a *birth*, the shift from x to $x - 1$ is called a *death*. The stochastic process defined by $X(t)$ is called a *discrete-time birth-death process*.

A discrete-time birth-death process is completely characterized by the initial state $X(0) \in \mathcal{X}$ at $t = 0$ and the functions $b(x)$, $c(x)$, $d(x)$ defined for all $x \in \mathcal{X}$ so that

$$b(x) \geq 0 \qquad c(x) \geq 0 \qquad d(x) \geq 0 \qquad b(x) + c(x) + d(x) = 1$$

with the understanding that $d(0) = 0$ because a death can't occur when X is in the state $x = 0$. Moreover, if the state-space is finite then $b(k) = 0$ because a birth can't occur when X is in the state $x = k$.

Because the $b(x)$, $c(x)$, $d(x)$ probabilities sum to 1, it is sufficient to specify any two of them and then solve for the third. So, for example, one could specify the birth probability $b(x)$ and death probability $d(x)$ only. The "remain in state" probability $c(x)$ is then defined by

$$c(x) = 1 - b(x) - d(x)$$

provided, of course, that $c(x) \geq 0$. The significance of $c(x)$ is discussed later.

As in the continuous-time case, a discrete-time birth-death process has no memory of its past — this is the Markovian assumption. That is, the value of the future states $X(t+1)$, $X(t+2)$, ..., depend *only* on the value of the current state $X(t)$. Moreover, we can use a modified version of Algorithm 10.2.1 to simulate a discrete-time birth-death process as follows.

Algorithm 10.2.4   Given the state-space $\mathcal{X}$, the initial state $X(0) \in \mathcal{X}$, and the probabilities $b(x) \geq 0$, $c(x) \geq 0$, $d(x) \geq 0$ defined for all $x \in \mathcal{X}$ with $b(x) + c(x) + d(x) = 1$, this algorithm simulates the time evolution of the associated discrete-time birth-death process for all $t = 0, 1, 2, \ldots, \tau$.

```
t = 0;
x = X(0);
while (t < τ) {
  t++;
  u = Random();
  if (u < d(x))
    s = -1;
  else if (u < d(x) + c(x))
    s = 0;
  else
    s = 1;
  x += s;
}
```

Alternative Formation

Note that if $c(x) = 0$, then when the discrete-time birth-death process enters state x it will remain in this state for exactly 1 time unit. That is, if $c(x) = 0$ and $X(t) = x$ then, with probability one, $X(t+1) \neq x$. If, however, $c(x) > 0$ then, once state x is entered, the discrete-time birth-death process will remain in this state for a (discrete) random amount of time. Indeed, because of independence, the probability of remaining in this state for exactly $n + 1$ time units is

$$c^n(x) \left(1 - c(x)\right) \qquad\qquad n = 0, 1, 2, \ldots$$

Therefore, if $c(x) > 0$ and $X(t) = x$ then the amount of time the discrete-time birth-death process will remain in state x is $1 + T$ where T is a $Geometric(c(x))$ random variable (see Definition 6.4.3). It follows in this case that when the process enters state x the *expected* time the process will remain in state x is

$$\begin{aligned}
\mu(x) &= E[1 + T] \\
&= 1 + E[T] \\
&= 1 + \frac{c(x)}{1 - c(x)} \\
&= \frac{1}{1 - c(x)}.
\end{aligned}$$

Equivalently, $1 - c(x) = b(x) + d(x)$ and so the expected time in state x is

$$\mu(x) = \frac{1}{b(x) + d(x)}.$$

Moreover, when a state change does occur, it will be a birth with probability

$$p(x) = \frac{b(x)}{b(x) + d(x)}$$

or a death with probability $1 - p(x)$.*

The following theorem summarizes the previous discussion. Recognize that, as an alternative to Definition 10.2.4, this theorem provides an equivalent characterization of a discrete-time birth-death process. The conceptual advantage of this alternative characterization is that it parallels the *continuous-time* birth-death process definition (Definition 10.2.2) and illustrates nicely that the (memoryless) *Geometric* random variable is the discrete analog of the continuous *Exponential* random variable.

---

* It is important to appreciate the difference between $p(x)$ and $b(x)$. By definition

$$b(x) = \Pr\left(X(t + 1) = x + 1 \mid X(t) = x\right)$$

is the probability of a state transition from state x at time t to state $x + 1$ at time $t + 1$. Instead,

$$p(x) = \Pr\left(X(t + 1) = x + 1 \mid (X(t) = x) \text{ and } (X(t + 1) \neq x)\right)$$

is $b(x)$ *conditioned* on the knowledge that there was a state change. This second probability can't be smaller than the first and, indeed, if $c(x) > 0$ then $p(x) > b(x)$.

Theorem 10.2.9   Given $p(x) = b(x)/(b(x) + d(x))$ and $\mu(x) = 1/(b(x) + d(x))$ for all $x \in \mathcal{X}$, if X is a discrete random variable that evolves in time as follows then the stochastic process defined by $X(t)$ is a discrete-time birth-death process.

- At any time $t = 0, 1, 2, \ldots$, $X(t) \in \mathcal{X}$.

- State transitions can occur only at the discrete times $t = 0, 1, 2, \ldots$ and if x is the state of X at time t then X will remain in this state for a random time $1 + T$ where T is *Geometric* with mean $\mu(x) - 1$.

- When a state transition occurs then, independent of the time spent in the current state, the random variable X will either shift from its current state x to state $x + 1 \in \mathcal{X}$ with probability $p(x)$ or shift from state x to state $x - 1 \in \mathcal{X}$ with probability $1 - p(x)$.

   If the process is specified in terms of $b(x)$ and $d(x)$, as in Definition 10.2.4, then $p(x)$ and $\mu(x)$ are defined by the equations in Theorem 10.2.9. On the other hand, if the process is specified in terms of $p(x)$ and $\mu(x)$ directly, then $b(x)$ and $d(x)$ are defined by

$$ b(x) = \frac{p(x)}{\mu(x)} \qquad\qquad d(x) = \frac{1 - p(x)}{\mu(x)}. $$

The following algorithm is an implementation of Theorem 10.2.9. This algorithm, which is the discrete analog of Algorithm 10.2.1, is stochastically equivalent to Algorithm 10.2.4. Note that because the parameter that defines a *Geometric* random variable is a probability, not a mean, Algorithm 10.2.5 is stated in terms of the "remain in state" probability $c(x)$, which can be shown to be related to $\mu(x)$ by the equation

$$ c(x) = \frac{\mu(x) - 1}{\mu(x)} \qquad\qquad x \in \mathcal{X} $$

with the understanding that $\mu(x) \geq 1$ for all $x \in \mathcal{X}$.

Algorithm 10.2.5   Given the state-space $\mathcal{X}$, the initial state $X(0) \in \mathcal{X}$, and the probabilities $p(x)$, $c(x)$ defined for all $x \in \mathcal{X}$ by the equations $p(x) = b(x)/(b(x) + d(x))$ and $c(x) = 1 - b(x) - d(x)$, this algorithm simulates the time evolution of the associated discrete-time birth-death process for all $t = 0, 1, 2, \ldots, \tau$.

```
t = 0;
x = X(0);
while (t < τ) {
  t += 1 + Geometric(c(x));
  x += 2 * Bernoulli(p(x)) - 1;
}
```

Example 10.2.16   As an example of a discrete-time birth-death process with an infinite state space, consider a data structure (e.g., a linked list) for which one item is either inserted or deleted each time the data structure is accessed. For each access the probability of an insertion is p, independent of past accesses; the probability of a deletion is $q = 1 - p$, provided the data structure is not empty. If accesses are indexed $t = 0, 1, 2, \ldots$, and $X(t)$ is the number of items in the data structure after the tth access, then $X(t)$ is a discrete-time birth-death process. We can use the characterization of this stochastic process in Definition 10.2.4 to write

$$b(x) = p \qquad x = 0, 1, 2, \ldots \qquad \text{and} \qquad d(x) = \begin{cases} 0 & x = 0 \\ q & x = 1, 2, \ldots \end{cases}$$

Equivalently, the characterization of this stochastic process in Theorem 10.2.9 can be used to write

$$p(x) = \begin{cases} 1 & x = 0 \\ p & x = 1, 2, \ldots \end{cases} \qquad \text{and} \qquad \mu(x) = \begin{cases} 1/p & x = 0 \\ 1 & x = 1, 2, \ldots \end{cases}$$

In either case,

$$c(x) = \begin{cases} q & x = 0 \\ 0 & x = 1, 2, \ldots \end{cases}$$

Flow Balance Equations

As in the continuous-time case, the expected times $\mu(x)$ can be converted to rates via the equation

$$\lambda(x) = \frac{1}{\mu(x)} \qquad x \in \mathcal{X}.$$

These rates can then be multiplied by $p(x)$ and $1 - p(x)$ respectively to define the birth and death *rates*

$$b(x) = \lambda(x)\, p(x) \qquad \text{and} \qquad d(x) = \lambda(x)\, (1 - p(x))$$

which, for a discrete-time birth-death process, are numerically equivalent to the birth and death *probabilities*, as specified in Definition 10.2.4. It follows that the stochastic process can be represented by a labeled directed graph as illustrated in Figure 10.2.9.
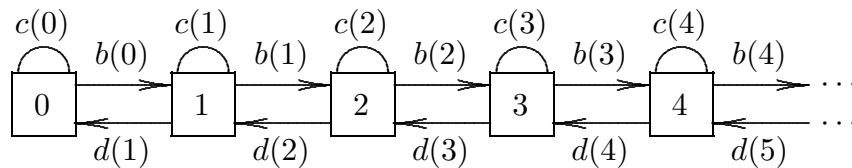

Figure 10.2.9. Transition probabilities.

The steady-state probability $f(x)$ of finding the process in state x can be determined, as in the continuous-time case, by solving the flow-balance equations

$$d(x + 1)f(x + 1) = b(x)f(x) \qquad x \in \mathcal{X}.$$

Example 10.2.17   As a continuation of Example 10.2.16, consider the labeled directed graph representation of the corresponding discrete-time birth-death stochastic process given in Figure 10.2.10.

**Figure 10.2.10.
Transition
probabilities.**
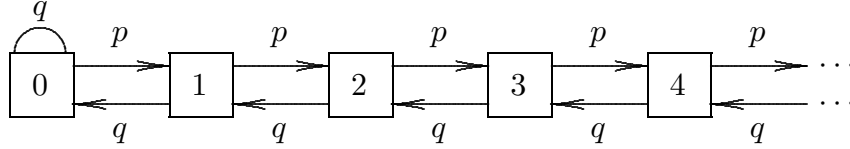
The corresponding flow-balance equations are

$$q\,f(x+1) = p\,f(x) \qquad\qquad x = 0, 1, 2, \ldots$$

Except for cosmetic differences $(p \leftrightarrow \lambda,\ q \leftrightarrow \nu)$ these are *exactly* the same equations as those characterizing the steady-state number in an $M/M/1$ service node. Therefore, it follows that, provided $\rho < 1$

$$\rho = \frac{p}{q} = \frac{p}{1-p}$$

and, provided $\rho < 1$

$$f(x) = (1 - \rho)\,\rho^x \qquad\qquad x = 0, 1, 2, \ldots$$

Thus we see that, provided the probability of insertion p is less than the probability of deletion q, the steady-state number of items in the data structure is a *Geometric*(ρ) random variable. In particular, in this case the expected number of items in the data structure is

$$E[X] = \frac{\rho}{1-\rho} = \frac{p}{q-p}.$$

As a specific example, if $(p, q) = (0.48,\, 0.52)$ then $E[X] = 12$ and the steady-state distribution of the size of the data structure is as illustrated in Figure 10.2.11.

**Figure 10.2.11.
Steady-state
size of
the data
structure.**
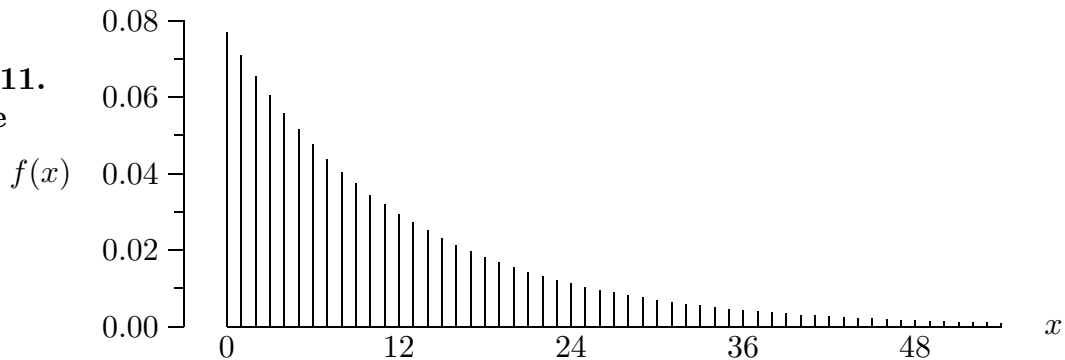
If p is greater than or equal to q, then steady-state can not be achieved because the expected length of the data structure will grow without bound. If, however, the size of the data structure is physically restricted to be no larger than $k < \infty$, then the state-space is finite and, analogous to an $M/M/1/k$ service node model, steady-state will be possible.

10.2.5   EXERCISES

Exercise 10.2.1   Suppose that T_1 and T_2 are independent *Exponential* random variables with means μ_1 and μ_2 respectively. Define $T = \max\{T_1, T_2\}$. (*a*) What is the pdf of T? (*b*) What is $E[T]$? (*c*) Use Monte Carlo simulation to verify the correctness of your math for the case $\mu_1 = 1$, $\mu_2 = 2$.

Exercise 10.2.2   Complete the derivation of Theorem 10.2.2 by proving that

$$\frac{1}{\mu'} \int_0^\infty \left(1 - \exp(-t'/\mu_1)\right) \exp(-t'/\mu') \, dt' = \frac{\mu}{\mu_1}.$$

Exercise 10.2.3   (*a*) If $T_1, T_2, \ldots, T_n$ is an independent sequence of *Exponential*(μ) random variables and $T = \max\{T_1, T_2, \ldots, T_n\}$ what is the cdf and pdf of T? (*b*) Use Monte Carlo simulation to verify the correctness of this pdf for the case $n = 5$, $\mu = 1$. (*c*) Construct an algorithm that will generate the random variate T with just one call to `Random`.

Exercise 10.2.4   (*a*) Use Algorithm 10.2.1 and Example 10.2.8 to construct (yet another) program to simulate an $M/M/1$ service node. (*b*) How would you verify that this program is correct? (*c*) Illustrate for the case $\lambda = 3$, $\nu = 4$.

Exercise 10.2.5   (*a*) Use Algorithm 10.2.1 and Example 10.2.9 to construct a program to simulate an $M/M/1/k$ service node. This program should estimate steady-state values for the expected number in the service node and the steady-state probabilities $f(l)$ for $l = 0, 1, \ldots, k$. (*b*) How would you verify that the program is correct? (*c*) Illustrate for the case $\lambda = 3$, $\nu = 4$, $k = 4$.

Exercise 10.2.6   (*a*) Use Algorithm 10.2.3 to construct a program to calculate the first-order $M/M/c$ steady-state statistics given values of λ, ν and a range of c values. (*b*) Calculate and print a table of these four steady-state statistics for the case $\lambda = 30$, $\nu = 2$ and $c = 20, 19, 18, 17, 16, 15$. (*c*) Comment.

Exercise 10.2.7   If $\rho = 1$ derive the steady-state pdf for an $M/M/1/k$ service node.

Exercise 10.2.8   Let $Q(t)$ be the number of jobs in the *queue* of an $M/M/c$ service node at time $t > 0$. Derive an equation for the steady-state $(t \to \infty)$ pdf $\Pr(Q(t) = q)$ for $q = 0, 1, 2, \ldots$

Exercise 10.2.9   Use Algorithm 10.2.1 to verify by simulation that the results in Theorem 10.2.7 are correct in the case $c = 4$, $\lambda = 8$, $\nu = 2$, $k = 10$.

Exercise 10.2.10   (*a*) Determine an easily computed condition involving c, λ and ν that will characterize when the steady-state distribution of the number in an $M/M/c$ service node and an $M/M/\infty$ service node are essentially identical. (The condition "c is real big" is not enough.) (*b*) Present simulation data in support of your condition for the case $\lambda = 10$, $\nu = 1$.

Exercise 10.2.11   A classic example* of a discrete-time finite-state birth-death process is based on a game of chance played as follows. Two players, $\mathcal{P}$ and $\mathcal{Q}$, start with a fixed amount of money, k (dollars), with a belonging to $\mathcal{P}$ and $k - a$ to $\mathcal{Q}$. On each play of the game, $\mathcal{P}$ will win one dollar (from $\mathcal{Q}$) with probability p, independent of the result of previous games, else $\mathcal{Q}$ will win the dollar with probability $q = 1 - p$. *Play continues until one player has no more money.* Let $f(x)$ represent the steady-state probability that player $\mathcal{P}$ has x dollars. (*a*) Set up the flow-balance equations for this process and argue why $f(x) = 0$ for all $x = 1, 2, \ldots, k - 1$. (*b*) For the case $k = 1000$ with $p = 0.51$ estimate $f(0)$ and $f(k)$ by simulation to within ± 0.01 with 95% confidence for each of the cases $a = 10, 20, 30, 40, 50$. (*c*) Also, estimate the expected number of plays. (*d*) Comment.

Exercise 10.2.12   As a continuation of Exercise 10.2.11, derive equations for $f(0)$ and $f(k)$. Hint: To derive the equations, define $\Pr(a) = f(k)$ as the probability that player $\mathcal{P}$ ultimately ends up with all the money given that $\mathcal{P}$ has a dollars initially and then argue that the 3-term recursion equation

$$\Pr(a) = q \Pr(a - 1) + p \Pr(a + 1) \qquad a = 1, 2, \ldots, k - 1$$

is valid, with the boundary conditions $\Pr(0) = 0$, $\Pr(k) = 1$.

Exercise 10.2.13   Prove Theorem 10.2.7.

Exercise 10.2.14   Consider the following entities to determine whether an *Exponential*(μ) distribution is an appropriate probabilistic model for their lifetimes: (*a*) fuse, (*b*) candle, (*c*) cat, (*d*) light bulb, (*e*) software company. Base your discussion on the appropriateness of an exponential model on the memoryless property (i.e., would a used entity have an identical lifetime distribution to a new entity?).

---

   * This problem is well known in probability as the *gambler's ruin* problem. A special case of the gambler's ruin problem known as the problem of *duration of play* was proposed to Dutch mathematician Christian Huygens (1629–1695) by French mathematician Pierre de Fermat (1601–1665) in 1657. The general form of the gambler's ruin problem was solved by mathematician Jakob Bernoulli (1654–1705) and published eight years after his death. Ross (2002a, pages 90–94) considers the problem using the analytic approach to probability.

This section introduces a particular type of stochastic process known as a *finite-state Markov chain.* As we will see, a finite-state birth-death process (see Section 10.2) is a special case of a finite-state Markov chain. The primary distinction is that in a birth-death process, state transitions are restricted to neighboring states while in a Markov chain, there is no such restriction. Because a much larger set of state transitions are possible, a Markov chain is a much richer stochastic process model. There is, however, a price to pay for this increased modeling flexibility — an added level of mathematical sophistication is required to analyze steady-state behavior.

10.3.1   DISCRETE-TIME FINITE-STATE MARKOV CHAINS

In reverse order of presentation relative to Section 10.2, in this section we consider *discrete-time* Markov chains before *continuous-time* Markov chains.

Definition 10.3.1   Let X be a discrete random variable, indexed by time t as $X(t)$, that evolves in time as follows.

- $X(t) \in \mathcal{X}$ for all $t = 0, 1, 2, \ldots$

- State transitions can occur only at the discrete times $t = 0, 1, 2, \ldots$ and at these times the random variable $X(t)$ will shift from its current state $x \in \mathcal{X}$ to another state, say $x' \in \mathcal{X}$, with fixed probability $p(x, x') = \Pr(X(t+1) = x' \mid X(t) = x) \geq 0$.

If $|\mathcal{X}| < \infty$ the stochastic process defined by $X(t)$ is called a *discrete-time, finite-state Markov chain.* Without loss of generality, throughout this section we assume that the finite state space is $\mathcal{X} = \{0, 1, 2, \ldots, k\}$ where k is a finite, but perhaps quite large, integer.

A discrete-time, finite-state Markov chain is completely characterized by the initial state at $t = 0$, $X(0)$, and the function $p(x, x')$ defined for all $(x, x') \in \mathcal{X} \times \mathcal{X}$. When the stochastic process leaves the state x the transition must be either to state $x' = 0$ with probability $p(x, 0)$, or to state $x' = 1$ with probability $p(x, 1)$, ..., or to state $x' = k$ with probability $p(x, k)$, and the sum of these probabilities must be 1. That is,

$$\sum_{x'=0}^{k} p(x, x') = 1 \qquad x = 0, 1, \ldots, k.$$

Because $p(x, x')$ is independent of t for all (x, x'), the Markov chain is said to be *homogeneous* or *stationary.*

Definition 10.3.2   The *state transition probability* $p(x, x')$ represents the probability of a transition *from* state x *to* state x'. The corresponding $(k+1) \times (k+1)$ matrix

$$\mathbf{p} = \begin{bmatrix} p(0,0) & p(0,1) & \cdots & p(0,k) \\ p(1,0) & p(1,1) & \cdots & p(1,k) \\ \vdots & \vdots & \ddots & \vdots \\ p(k,0) & p(k,1) & \cdots & p(k,k) \end{bmatrix}$$

with elements $p(x, x')$ is called the *state transition matrix.*

The elements of the state transition matrix **p** are non-negative and the elements of each row sum to 1.0. That is, each row of the state transition matrix defines a pdf for a discrete random variable. (A matrix with these properties is said to be a *stochastic* matrix.) The diagonal elements of the state transition matrix need not be zero. If $p(x, x) > 0$ then when $X(t) = x$ it will be possible for the next state "transition" to leave the state of the system unchanged, i.e., $X(t + 1) = x$. This is sometimes called *immediate feedback*.

Example 10.3.1   Consider a Markov chain with 4 possible states ($k = 3$, $\mathcal{X} = \{0, 1, 2, 3\}$) and state transition matrix

$$\mathbf{p} = \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.1 \\ 0.0 & 0.5 & 0.3 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.4 & 0.0 & 0.5 & 0.1 \end{bmatrix}.$$

For example, the probability of a transition from state 2 to 3 is $p(2, 3) = 0.6$ and the probability of immediate feedback when the system is in state 1 is $p(1, 1) = 0.5$.

As illustrated in the following example, a finite-state Markov chain can be represented as a directed graph. That is:

• The nodes of this graph are the possible states (the state space $\mathcal{X}$).

• The labeled arcs, including the immediate feedback circular arcs, are defined by the state transition probabilities $p(x, x')$.

The sum of the probabilities associated with all of the arcs emanating from any node including the circular (feedback) arc, must be 1.0. Why?

Example 10.3.2   A "probability" directed graph representation of the discrete-time Markov chain in Example 10.3.1 is shown in Figure 10.3.1.

Figure 10.3.1. State transition diagram for a discrete-time, finite-state Markov chain.
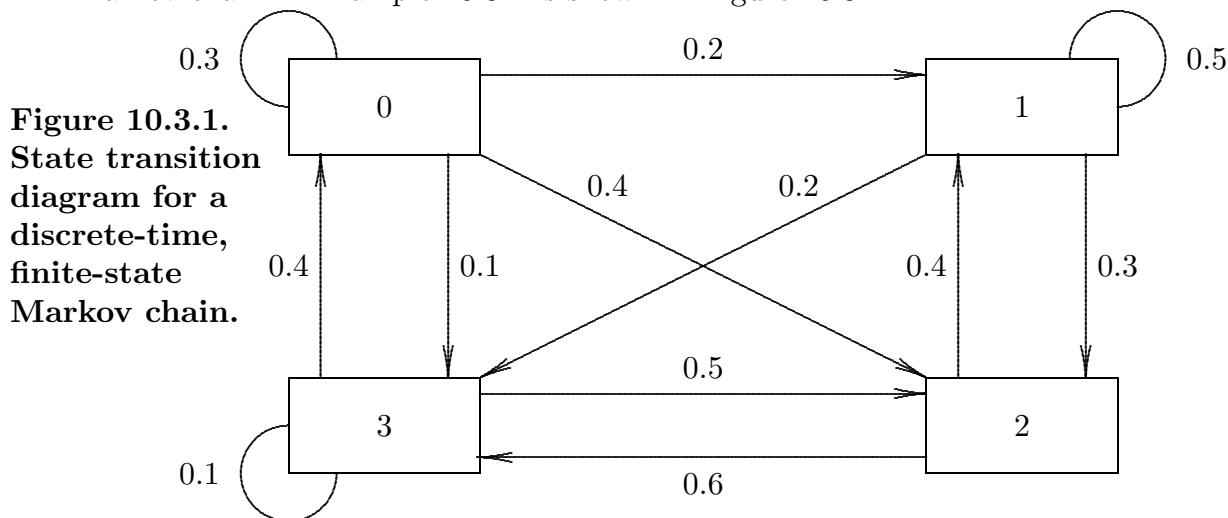
Definition 10.3.3 Define the *cumulative* state transition function $P(x, x')$ as

$$P(x, x') = p(x, 0) + p(x, 1) + \cdots + p(x, x') \qquad (x, x') \in \mathcal{X} \times \mathcal{X}.$$

That is, $P(x, x')$ is the probability of a transition *from* state x *to* a new state between 0 and x' inclusive. The associated $(k + 1) \times (k + 1)$ matrix

$$\mathbf{P} = \begin{bmatrix} P(0,0) & P(0,1) & \cdots & 1.0 \\ P(1,0) & P(1,1) & \cdots & 1.0 \\ \vdots & \vdots & \ddots & \vdots \\ P(k,0) & P(k,1) & \cdots & 1.0 \end{bmatrix}$$

with elements $P(x, x')$ is called the *cumulative* state transition matrix.

Example 10.3.3 Relative to Example 10.3.2, the state transition matrix and cumulative state transition matrix are

$$\mathbf{p} = \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.1 \\ 0.0 & 0.5 & 0.3 & 0.2 \\ 0.0 & 0.4 & 0.0 & 0.6 \\ 0.4 & 0.0 & 0.5 & 0.1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{P} = \begin{bmatrix} 0.3 & 0.5 & 0.9 & 1.0 \\ 0.0 & 0.5 & 0.8 & 1.0 \\ 0.0 & 0.4 & 0.4 & 1.0 \\ 0.4 & 0.4 & 0.9 & 1.0 \end{bmatrix}.$$

Given $\mathbf{p}$, the following algorithm can be used to compute the cumulative state transition matrix.

Algorithm 10.3.1 This algorithm constructs the *cumulative* state transition matrix $\mathbf{P}$ from the state transition matrix $\mathbf{p}$.

```
for (x = 0;  x < k + 1;  x++) {                              /* loop over rows */
  P[x, 0] = p[x, 0];                        /* first column elements identical */
  for (x' = 1;  x' < k;  x'++)                             /* loop over columns */
    P[x, x'] = P[x, x' - 1] + p[x, x'];                       /* pdf → cdf */
  P[x, k] = 1.0;                                     /* last column elements */
}
return P;
```

Note that the elements in each row of a cumulative state transition matrix are monotone increasing. Also $P[x, k] = 1.0$ for all $x \in \mathcal{X}$, a condition that is enforced in Algorithm 10.3.1 to guard against accumulated floating-point round-off error yielding a computed value of $P[x, k]$ that is close to, but slightly less than, 1.0. This is important because a computed value of $P[x, k] < 1.0$ would create the possibility that Algorithm 10.3.2 (to follow) might fail.

If **p** is a stochastic matrix then each row represents the pdf of a discrete random variable. Specifically, the x-row of **p** is the pdf of the "to" state when the "from" state is x. Correspondingly, the rows of the cumulative state transition matrix **P** are cdf's. Because of this, the cumulative state transition matrix can be used with the discrete idf method of random variable generation (Algorithm 6.2.1) to construct the following algorithm for the function NextState.

Algorithm 10.3.2   Given the cumulative state transition matrix **P** and $x \in \mathcal{X}$, this algorithm is an implementation of the function $x' = \texttt{NextState}(x)$ using inversion.

```
u = Random();
x' = 0;
while (P[x, x'] <= u)
   x'++;
return x';
```

Because of the simple linear search in Algorithm 10.3.2, the $O(k)$ time complexity of the function NextState may be a problem if $|\mathcal{X}| = k + 1$ is large. For larger values of k, Algorithm 6.2.2 or, perhaps, a binary search should be used instead. Also, if k is large it is quite likely that the state transition matrix **p** will be *sparse* — each node in the associated digraph will have only a few adjacent nodes. If k is large and **p** is sparse, then a two-dimensional array is *not* the right data structure to use. Instead, a more appropriate data structure, for example a one-dimensional array of pointers to linked lists, and associated idf search algorithm are required. In any case, given the function NextState, the following algorithm simulates the time evolution of the corresponding discrete-time, finite-state Markov chain.

Algorithm 10.3.3   Given the initial state $X(0) \in \mathcal{X}$, this algorithm simulates the time evolution of the associated discrete-time, finite-state Markov chain for $t = 0, 1, 2, \ldots, \tau$.

```
t = 0;
x = X(0);
while (t < τ) {
  t++;
  x = NextState(x);
}
```

Example 10.3.4   With the state transition matrix from Example 10.3.3, the initial state $X(0) = 1$, and $\tau = 100\,000$, Algorithm 10.3.3 produced the output shown below.

state	:	0	1	2	3
proportion	:	0.160	0.283	0.284	0.273

The second row of this table is the proportion of time spent in each state. Equivalently, because τ is large, this row is an *estimate* of the steady-state probability of finding the Markov chain in each state.

For all $x \in \mathcal{X}$ let

$$f(x, t) = \Pr\left(X(t) = x\right) \qquad t = 0, 1, 2, \ldots$$

be the probability that the Markov chain is in state x at time t. That is, $f(x, t)$ is the pdf of $X(t)$. We assume that the *initial* distribution (the pdf of $X(0)$)

$$f(x, 0) = \Pr\left(X(0) = x\right) \qquad x \in \mathcal{X}$$

is known. The issue then is how to use this initial distribution and the state transition matrix $\mathbf{p}$ to determine the pdf of $X(t)$ for $t = 1, 2, \ldots$ To do this, it is convenient (and conventional) to define these pdf's as a sequence of *row* vectors

$$\mathbf{f}_t = [\, f(0, t) \quad f(1, t) \quad \ldots \quad f(k, t)\,] \qquad t = 0, 1, 2, \ldots$$

In particular, $\mathbf{f}_0$ is the known initial distribution.

Chapman–Kolmogorov Equation

By using matrix/vector notation, from the law of total probability we can write the pdf of $X(1)$ as

$$
\begin{aligned}
f(x', 1) &= \Pr(X(1) = x') \\
&= \sum_{x=0}^{k} \Pr(X(0) = x)\, \Pr(X(1) = x' \mid X(0) = x) \\
&= \sum_{x=0}^{k} f(x, 0)\, p(x, x') \qquad x' \in \mathcal{X}
\end{aligned}
$$

which is equivalent to the matrix equation

$$\mathbf{f}_1 = \mathbf{f}_0\, \mathbf{p}.$$

Similarly, it follows that the pdf of $X(2)$ is

$$\mathbf{f}_2 = \mathbf{f}_1\, \mathbf{p} = \left(\mathbf{f}_0\, \mathbf{p}\right) \mathbf{p} = \mathbf{f}_0\, \mathbf{p}^2$$

and (by induction) the pdf of $X(t)$ is

$$\mathbf{f}_t = \mathbf{f}_0\, \mathbf{p}^t \qquad t = 1, 2, 3, \ldots$$

This last matrix/vector equation, know as the *Chapman–Kolmogorov equation*, is of fundamental importance — the pdf of $X(t)$ is determined by raising the state transition matrix to the tth power, then multiplying the result, on the left, by the row vector corresponding to the pdf of $X(0)$.

Example 10.3.5   As a continuation of Examples 10.3.1 and 10.3.2, it can be verified
that

$$\mathbf{p} = \begin{bmatrix} 0.300 & 0.200 & 0.400 & 0.100 \\ 0.000 & 0.500 & 0.300 & 0.200 \\ 0.000 & 0.400 & 0.000 & 0.600 \\ 0.400 & 0.000 & 0.500 & 0.100 \end{bmatrix}.$$

$$\mathbf{p}^2 = \begin{bmatrix} 0.130 & 0.320 & 0.230 & 0.320 \\ 0.080 & 0.370 & 0.250 & 0.300 \\ 0.240 & 0.200 & 0.420 & 0.140 \\ 0.160 & 0.280 & 0.210 & 0.350 \end{bmatrix}$$

$$\mathbf{p}^4 = \begin{bmatrix} 0.149 & 0.296 & 0.274 & 0.282 \\ 0.148 & 0.297 & 0.279 & 0.277 \\ 0.170 & 0.274 & 0.311 & 0.245 \\ 0.150 & 0.295 & 0.269 & 0.287 \end{bmatrix}$$

$$\mathbf{p}^8 = \begin{bmatrix} 0.155 & 0.290 & 0.284 & 0.272 \\ 0.155 & 0.290 & 0.284 & 0.271 \\ 0.156 & 0.289 & 0.285 & 0.270 \\ 0.155 & 0.290 & 0.284 & 0.272 \end{bmatrix}.$$

If, as in Example 10.3.4, the initial state is $X(0) = 1$ with probability 1, then from the
Chapman–Kolmogorov equation it follows that

$$\mathbf{f}_0 = \begin{bmatrix} 0.000 & 1.000 & 0.000 & 0.000 \end{bmatrix}$$

$$\mathbf{f}_1 = \mathbf{f}_0\,\mathbf{p}^1 = \begin{bmatrix} 0.000 & 0.500 & 0.300 & 0.200 \end{bmatrix}$$

$$\mathbf{f}_2 = \mathbf{f}_0\,\mathbf{p}^2 = \begin{bmatrix} 0.080 & 0.370 & 0.250 & 0.300 \end{bmatrix}$$

$$\mathbf{f}_4 = \mathbf{f}_0\,\mathbf{p}^4 = \begin{bmatrix} 0.148 & 0.297 & 0.279 & 0.277 \end{bmatrix}$$

$$\mathbf{f}_8 = \mathbf{f}_0\,\mathbf{p}^8 = \begin{bmatrix} 0.155 & 0.290 & 0.284 & 0.271 \end{bmatrix}.$$

If, instead the initial state could be any of the four possible values $X(0) = 0, 1, 2, 3$ with
equal probability, then

$$\mathbf{f}_0 = \begin{bmatrix} 0.250 & 0.250 & 0.250 & 0.250 \end{bmatrix}$$

$$\mathbf{f}_1 = \mathbf{f}_0\,\mathbf{p}^1 = \begin{bmatrix} 0.175 & 0.275 & 0.300 & 0.250 \end{bmatrix}$$

$$\mathbf{f}_2 = \mathbf{f}_0\,\mathbf{p}^2 = \begin{bmatrix} 0.153 & 0.293 & 0.278 & 0.278 \end{bmatrix}$$

$$\mathbf{f}_4 = \mathbf{f}_0\,\mathbf{p}^4 = \begin{bmatrix} 0.154 & 0.290 & 0.283 & 0.273 \end{bmatrix}$$

$$\mathbf{f}_8 = \mathbf{f}_0\,\mathbf{p}^8 = \begin{bmatrix} 0.155 & 0.290 & 0.284 & 0.271 \end{bmatrix}.$$

In either case, we see that for increasing t the pdf vector $\mathbf{f}_t$ converges rapidly to a pdf that
is consistent with the estimated steady-state pdf in Example 10.3.4.

Matrix Power Method

It follows immediately from the Chapman–Kolmogorov equation that if the steady-state pdf vector $\lim_{t \to \infty} \mathbf{f}_t = \mathbf{f}_\infty$ exists then so also must the matrix

$$\mathbf{p}^\infty = \lim_{t \to \infty} \mathbf{p}^t$$

exist. As illustrated in Example 10.3.5 the matrix $\mathbf{p}^\infty$ has a very special structure — all rows are equal to the steady-state pdf vector. The reason for this is that the equation

$$\mathbf{f}_\infty = \mathbf{f}_0 \, \mathbf{p}^\infty$$

must be valid for *any* initial pdf vector $\mathbf{f}_0$. In particular if the initial state of the Markov chain is $X(0) = 0$ with probability 1 then the initial pdf vector is

$$\mathbf{f}_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\mathbf{f}_0 \, \mathbf{p}^\infty = \mathbf{f}_\infty$ will be the *first* (0th) row of $\mathbf{p}^\infty$. Similarly, if $X(0) = 1$ with probability 1 then

$$\mathbf{f}_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

and we see that the *second* row of $\mathbf{p}^\infty$ must also be $\mathbf{f}_\infty$. This argument applies to all rows of $\mathbf{p}^\infty$ which proves the following theorem.

Theorem 10.3.1   For a discrete-time, finite-state Markov chain with state transition matrix $\mathbf{p}$, if a unique steady-state pdf vector $\mathbf{f}_\infty = \begin{bmatrix} f(0) & f(1) & \cdots & f(k) \end{bmatrix}$ exists then

$$\mathbf{p}^\infty = \lim_{t \to \infty} \mathbf{p}^t = \begin{bmatrix} f(0) & f(1) & \cdots & f(k) \\ f(0) & f(1) & \cdots & f(k) \\ \vdots & \vdots & & \vdots \\ f(0) & f(1) & \cdots & f(k) \end{bmatrix}.$$

Algorithm 10.3.4   If a unique steady-state pdf vector exists, then Theorem 10.3.1 provides a numerical algorithm for computing it by computing $\mathbf{p}^t$ for increasing values of t until all rows of $\mathbf{p}^t$ converge to a common vector of non-negative values that sum to 1.

Example 10.3.6   If the successive powers $\mathbf{p}$, $\mathbf{p}^2$, $\mathbf{p}^3$, $\mathbf{p}^4$, ... are computed for the state transition matrix in Example 10.3.1 and displayed with *d.dddd* precision we find that for $t > 16$ there is *no* change in the displayed values of $\mathbf{p}^t$. Thus we conclude that in this case

$$\mathbf{p}^\infty = \begin{bmatrix} 0.1549 & 0.2895 & 0.2844 & 0.2712 \\ 0.1549 & 0.2895 & 0.2844 & 0.2712 \\ 0.1549 & 0.2895 & 0.2844 & 0.2712 \\ 0.1549 & 0.2895 & 0.2844 & 0.2712 \end{bmatrix}$$

and that the steady-state pdf vector for this Markov chain is

$$\mathbf{f}_\infty = \begin{bmatrix} 0.1549 & 0.2895 & 0.2844 & 0.2712 \end{bmatrix}.$$

Eigenvector Method

Algorithm 10.3.4 is easily understood and easily implemented. The primary drawback to this algorithm is that the multiplication of one $(k+1) \times (k+1)$ matrix by another requires $(k + 1)^3$ multiplications and to compute $\mathbf{p}^t$ sequentially this process must be repeated t times. Thus the time complexity of this algorithm is $O(t\,k^3)$. Some efficiency can be gained by using "successive squaring" to compute, instead, the sequence

$$\mathbf{p}, \mathbf{p}^2, \mathbf{p}^4, \mathbf{p}^8, \ldots$$

But this algorithm is still $O(\log(t)\,k^3)$ and so efficiency is an issue if the state space is large. Accordingly, we consider an alternative approach based on the fact that

$$\mathbf{f}_{t+1} = \mathbf{f}_t\,\mathbf{p} \qquad\qquad t = 0, 1, 2, \ldots$$

From this equation it follows that, in the limit as $t \to \infty$, a steady-state pdf vector must satisfy the eigenvector equation

$$\mathbf{f}_\infty = \mathbf{f}_\infty\,\mathbf{p}.$$

That is, we have the following theorem.

Theorem 10.3.2   For a discrete-time, finite-state Markov chain with state transition matrix $\mathbf{p}$, if a steady-state pdf vector $\mathbf{f}_\infty = [\,f(0) \quad f(1) \quad \cdots \quad f(k)\,]$ exists then $\mathbf{f}_\infty$ is a (left) eigenvector of $\mathbf{p}$ with eigenvalue 1, i.e.,

$$\mathbf{f}_\infty\,\mathbf{p} = \mathbf{f}_\infty.$$

Equivalently, the steady-state pdf vector is characterized by the following set of $(k + 1)$ linear *balance* equations

$$\sum_{x=0}^{k} f(x)\,p(x, x') = f(x') \qquad\qquad x' = 0, 1, \ldots, k.$$

Algorithm 10.3.5   If a unique steady-state pdf vector exists, then Theorem 10.3.2 provides a second obvious algorithm for computing it.

- Solve the balance equations or, equivalently, compute a left eigenvector of $\mathbf{p}$ with eigenvalue 1.0.

- Check that all elements of the eigenvector are non-negative.

- Normalize the eigenvector so that the elements sum to 1.0.

Not only is Algorithm 10.3.5 inherently more efficient that Algorithm 10.3.4, it is also the preferred way to produce *symbolic* (algebraic) steady-state solutions to finite-state Markov chain models, provided such solutions are possible. The following example is an illustration.

Example 10.3.7   The balance equations for the 4-state Markov chain in Example 10.3.1 (redisplayed in Figure 10.3.2 for convenience) are
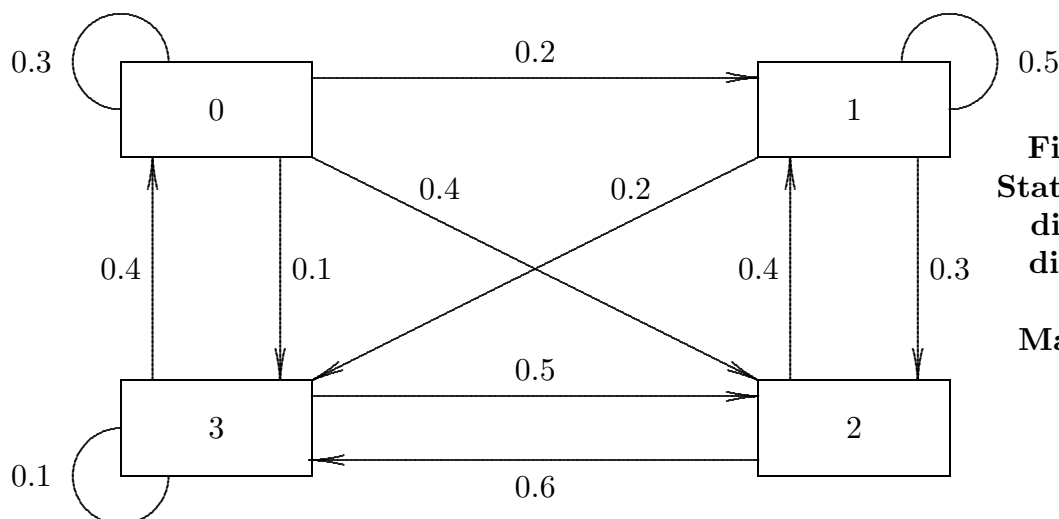

Figure 10.3.2. State transition diagram for a discrete-time, finite-state Markov chain.

$$
\begin{aligned}
0.3f(0) &&&+ 0.4f(3) &= f(0) \\
0.2f(0) + 0.5f(1) + 0.4f(2) &&&&= f(1) \\
0.4f(0) + 0.3f(1) &&&+ 0.5f(3) &= f(2) \\
0.1f(0) + 0.2f(1) + 0.6f(2) &&&+ 0.1f(3) &= f(3).
\end{aligned}
$$

By inspection $f(0) = f(1) = f(2) = f(3) = 0$ is a solution to the balance equations. Therefore, if steady-state is possible this linear system of equations must be singular for, if not, $f(0) = f(1) = f(2) = f(3) = 0$ would be the *only* possible solution. Consistent with their singular nature, the *four* balance equations row-reduce to the equivalent *three* equations

$$
\begin{aligned}
f(0) && - \frac{152}{266}f(3) &= 0 \\
f(1) && - \frac{284}{266}f(3) &= 0 \\
f(2) &- \frac{279}{266}f(3) &= 0.
\end{aligned}
$$

By back-substitution $f(2)$, $f(1)$, and $f(0)$ can be expressed in terms of $f(3)$ and then combined with the normalization equation $f(0) + f(1) + f(2) + f(3) = 1$ to yield the steady-state probabilities

$$
f(0) = \frac{152}{981} \qquad f(1) = \frac{284}{981} \qquad f(2) = \frac{279}{981} \qquad f(3) = \frac{266}{981}.
$$

To the accuracy displayed in Example 10.3.6, these probabilities agree exactly with the steady-state pdf vector.

Flow Balance Equations

Discrete-time Markov chains can be characterized in terms of expected times*

$$\mu(x) = \frac{1}{1 - p(x, x)} \qquad x \in \mathcal{X}$$

or, equivalently in terms of *rates*

$$\lambda(x) = \frac{1}{\mu(x)} = 1 - p(x, x) \qquad x \in \mathcal{X}.$$

When $X(t) = x$, the term $\mu(x)$ represents the expected time the Markov chain will remain in state x; equivalently $\lambda(x)$ represents the rate of "flow" out of this state. By accounting for immediate feedback in this way, it is important to recognize that the corresponding state transition matrix must be modified accordingly. That is, the state transition matrix is now

$$\mathbf{q} = \begin{bmatrix} 0 & q(0, 1) & \cdots & q(0, k) \\ q(1, 0) & 0 & \cdots & q(1, k) \\ \vdots & \vdots & \ddots & \vdots \\ q(k, 0) & q(k, 1) & \cdots & 0 \end{bmatrix},$$

where the modified state transition probabilities (which exclude immediate feedback) are

$$q(x, x') = \begin{cases} \dfrac{p(x, x')}{1 - p(x, x)} & x' \neq x \\ 0 & x' = x. \end{cases}$$

Then, when $X(t) = x$ the product

$$\lambda(x)\, q(x, x') = \begin{cases} p(x, x') & x' \neq x \\ 0 & x' = x \end{cases}$$

represents the rate of flow out of state x, into state x'. Moreover, the balance equations

$$f(x')\, p(x', x') + \sum_{x \neq x'} f(x)\, p(x, x') = f(x') \qquad x' = 0, 1, \ldots, k$$

can then be expressed as

$$\sum_{x \neq x'} f(x)\, \lambda(x)\, q(x, x') = f(x')\big(1 - p(x', x')\big) \qquad x' = 0, 1, \ldots, k.$$

That is, the balance equations in Theorem 10.3.2 are equivalent to the balance equations

$$\underbrace{\sum_{x \neq x'} f(x)\, \lambda(x)\, q(x, x')}_{\text{flow in}} = \underbrace{f(x')\, \lambda(x')}_{\text{flow out}} \qquad x' = 0, 1, \ldots, k.$$

---

* This assumes that no states are *absorbing*. That is, $p(x, x) < 1$ for all $x \in \mathcal{X}$.

When $X(t) = x$, the product $\lambda(x)\,q(x, x') = p(x, x')$ represents the *rate* at which the flow out of state x goes into state $x' \neq x$. Consistent with this observation, as an alternate to a directed graph representation like that in Example 10.3.7, a usually superior directed graph representation of a discrete-time, finite-state Markov chain is:

- Each node $x \in \mathcal{X}$ is labeled by $\lambda(x)$.

- The arcs are labeled by the state transition *rates* $p(x, x') = \lambda(x)\,q(x, x')$.

For this representation, at each node x the sum of all the labeled out-arcs is $\lambda(x)$. There are no feedback arcs.

Example 10.3.8   As an alternative to the approach in Example 10.3.7, for the "flow rate" directed graph representation, shown in Figure 10.3.2,
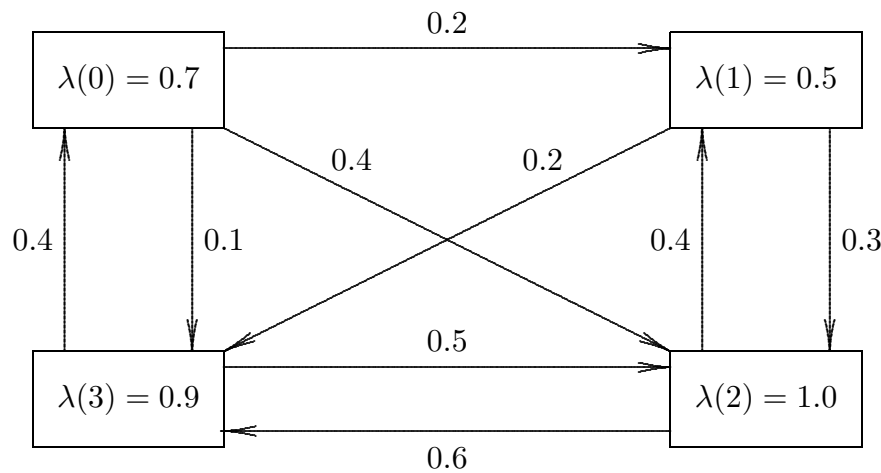

Figure 10.3.3. State transition diagram without self-loops for a discrete-time, finite-state Markov chain.

the balance equations are

$$
\begin{aligned}
+\,0.4f(3) &= 0.7f(0) \\
0.2f(0) \qquad\quad +\,0.4f(2) \qquad\quad &= 0.5f(1) \\
0.4f(0) + 0.3f(1) \qquad\quad +\,0.5f(3) &= 1.0f(2) \\
0.1f(0) + 0.2f(1) + 0.6f(2) \qquad\quad &= 0.9f(3).
\end{aligned}
$$

By inspection, these equations are equivalent to those in Example 10.3.7

Existence and Uniqueness

One important theoretical, question remains — what conditions are necessary and sufficient to guarantee that a finite-state Markov chain has a unique steady-state pdf vector? Although it is beyond the scope of this book to answer this question in complete generality, the following general 2-state Markov chain example illustrates the kinds of situations that can arise.

Example 10.3.9   The most general 2-state ($k = 1$) Markov chain has state transition matrix

$$\mathbf{p} = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix},$$

where α, β are non-negative parameters no larger than 1. Consider the following cases.

- If $\alpha = \beta = 0$ then the Markov chain will remain forever in either state 0 or state 1, depending on the initial state. In this case the Markov chain is said to be *reducible* and there is more than one steady-state pdf vector.

- If either $0 < \alpha < 1$ or $0 < \beta < 1$ or both, then there is exactly one steady-state pdf vector

$$\mathbf{f}_\infty = \begin{bmatrix} \dfrac{\beta}{\alpha + \beta} & \dfrac{\alpha}{\alpha + \beta} \end{bmatrix}.$$

- If $\alpha = \beta = 1$ then the Markov chain will cycle forever between states 0 and 1 with period 2. In this case the Markov chain is said to be *periodic* and there is no steady-state pdf vector.

The three cases illustrated in Example 10.3.9 generalize to *any* finite-state, discrete-time Markov chain as follows. Let $p^t[x, x']$ denote the $[x, x']$ element of the matrix $\mathbf{p}^t$.

- If $p^t[x, x'] > 0$ for some $t > 0$, the state x' is said to be *accessible* from state x (in t steps).

- If x' is accessible from x and, in addition, x is accessible from x' then states x and x' are said to *communicate*.

- State communication defines an *equivalence relation* on the state space $\mathcal{X}$. That is, the state space naturally partitions into disjoint sets of communicating states.

- If there is only one set in the state space partition defined by state communication, then the Markov chain is said to be *irreducible*. That is, a Markov chain is irreducible if and only if all states communicate. A *sufficient* condition for a Markov chain to be irreducible is the existence of some $t > 0$ such that $p^t[x, x'] > 0$ for all $[x, x'] \in \mathcal{X} \times \mathcal{X}$.

- If there is a partition of the state space $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_r$ with the property that the Markov chain will transition forever as indicated

$$\cdots \to \mathcal{X}_1 \to \mathcal{X}_2 \to \cdots \to \mathcal{X}_r \to \mathcal{X}_1 \to \cdots$$

  then the Markov chain is said to be *periodic* with period $r > 1$. If no such partition exists, the Markov chain is said to *aperiodic*. A *sufficient* condition for a Markov chain to be aperiodic is $p[x, x] > 0$ for at least one $x \in \mathcal{X}$.

The following theorem provides an important condition sufficient to guarantee the existence of a unique steady-state vector.

Theorem 10.3.3   An irreducible, aperiodic, finite-state Markov chain has one and only one steady-state pdf vector.

10.3.2 CONTINUOUS-TIME, FINITE-STATE MARKOV CHAINS

In the remainder of this section we consider *continuous-time* Markov chains. The emphasis is on demonstrating that "within" any continuous-time Markov chain is an associated "embedded" discrete-time Markov chain. As in the discrete-time case, we assume a finite state space $\mathcal{X}$ which is, without loss of generality,

$$\mathcal{X} = \{0, 1, 2, \ldots, k\}.$$

The state space $\mathcal{X}$ remains the same as before, but now the dwell time in a state is *continuous*, rather than discrete.

Definition 10.3.4 Let X be a discrete random variable, indexed by time t as $X(t)$, that evolves in time as follows.

- At any time $t > 0$, $X(t) \in \mathcal{X}$.

- If $x \in \mathcal{X}$ is the state of X at time t then X will remain in this state for a random time T that is *Exponential* with mean $\mu(x)$.

- When a state transition occurs then, independent of the time spent in the current state, the random variable X will shift from its current state x to a different state, say $x' \neq x$, with fixed probability $p(x, x')$.

The stochastic process defined by $X(t)$ is called a *continuous-time, finite-state Markov chain.**

A continuous-time, finite-state Markov chain is completely characterized by the initial state $X(0) \in \mathcal{X}$, the expected time-in-state function $\mu(x)$ defined for all $x \in \mathcal{X}$ so that $\mu(x) > 0$, and the state-transition probability function defined so that

$$p(x, x') \geq 0 \qquad (x, x') \in \mathcal{X} \times \mathcal{X}$$

with $p(x, x) = 0$ for all $x \in \mathcal{X}$ and

$$\sum_{x'=0}^{k} p(x, x') = 1 \qquad x \in \mathcal{X}.$$

Equivalently, because $p(x, x) = 0$, the last equation can be written as

$$\sum_{x' \neq x} p(x, x') = 1 \qquad x \in \mathcal{X}$$

where the summation is over all $x' \in \mathcal{X}$ except state x.

---

* Compare with Definitions 10.2.2 and 10.3.1.

As in the discrete-time case, the analysis of the steady-state behavior of a continuous-time, finite-state Markov chain is based on the state transition matrix $\mathbf{p}$, defined as follows. (Compare with Definition 10.3.2.)

Definition 10.3.5   The *state transition probability* $p(x, x')$ represents the probability of a transition *from* state x *to* state x'. The corresponding $(k + 1) \times (k + 1)$ matrix

$$\mathbf{p} = \begin{bmatrix} 0.0 & p(0,1) & \cdots & p(0,k) \\ p(1,0) & 0.0 & \cdots & p(1,k) \\ \vdots & \vdots & \ddots & \vdots \\ p(k,0) & p(k,1) & \cdots & 0.0 \end{bmatrix}$$

with elements $p(x, x')$ is called the *state transition matrix.*

The elements of the state transition matrix $\mathbf{p}$ are non-negative and the elements of each row sum to 1.0. That is, each row of the state transition matrix defines the pdf for a discrete random variable. By convention, immediate feedback at each state is accounted for by the *Exponential* time-in-state model and so the diagonal elements of the state transition matrix are zero.

Continuous-time, finite-state Markov chains are usually characterized in terms of *rates*

$$\lambda(x) = \frac{1}{\mu(x)} \qquad\qquad x \in \mathcal{X}$$

rather than the corresponding expected *times* $\mu(x)$. The reason for this is that, when $X(t) = x$, $\lambda(x)$ represents the infinite-horizon rate at which X will "flow" out of state x. Therefore, when $X(t) = x$, the product $\lambda(x)\, p(x, x')$ represents the rate of flow out of state x, into state $x' \neq x$. The sum of these rates of flow is $\lambda(x)$.

Example 10.3.10   Consider a Markov chain with 4 possible states ($k = 3$), state transition matrix

$$\mathbf{p} = \begin{bmatrix} 0.0 & 0.3 & 0.6 & 0.1 \\ 0.0 & 0.0 & 0.6 & 0.4 \\ 0.0 & 0.3 & 0.0 & 0.7 \\ 0.4 & 0.0 & 0.6 & 0.0 \end{bmatrix}$$

and rates

$$\lambda(0) = 1 \qquad\qquad \lambda(1) = 2/3 \qquad\qquad \lambda(2) = 1/2 \qquad\qquad \lambda(3) = 2.$$

For example, the probability of a transition from state 2 to 3 is $p(2,3) = 0.7$ and when $X(t) = 2$ the expected rate of flow into state 3 is $\lambda(2)\, p(2,3) = (0.5)(0.7) = 0.35$.

As illustrated in the following example, a continuous-time, finite-state Markov chain can be represented as a directed graph.

- The nodes of this graph are the possible states (the state space $\mathcal{X}$). Each node $x \in \mathcal{X}$ is labeled by $\lambda(x)$.

- The labeled arcs are defined by the state transition probabilities.

This kind of labeled digraph provides a convenient graphical representation of a (small) continuous-time Markov chain. For each node (state), the sum of all the labeled *out*-arcs is 1.0.

Example 10.3.11 The Markov chain in Example 10.3.10 can be represented as the labeled digraph shown in Figure 10.3.4.
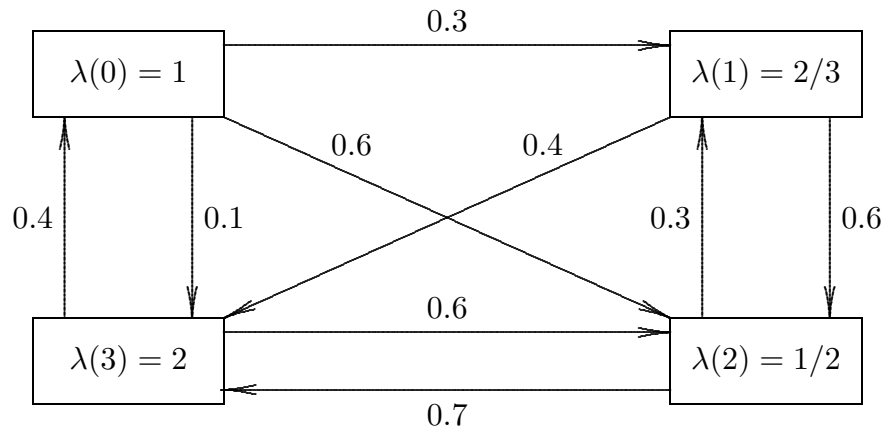

Figure 10.3.4. State transition diagram for a continuous-time, finite-state Markov chain.

The row elements of the state transition matrix **p** can be summed to form the *cumulative* state transition matrix **P** (see Algorithm 10.3.1). This matrix can then be used, as in Algorithm 10.3.2, to defined the function NextState(x) which returns a random state $x' \in \mathcal{X}$ (with $x' \neq x$) consistent with the pdf defined by the x-row of the state transition matrix. This function can then be used to simulate the time evolution of a continuous-time Markov chain (for τ time units) with the following algorithm. (Compare with Algorithm 10.3.3.)

Algorithm 10.3.6 Given the initial state $X(0) \in \mathcal{X}$ and the functions $\mu(x)$ defined for all $x \in \mathcal{X}$, this algorithm simulates the time evolution of the associated continuous-time, finite-state Markov chain for $0 < t < \tau$.

```
t = 0;
x = X(0);
while (t < τ) {
  t += Exponential(μ(x));
  x = NextState(x);
}
```

As illustrated by the following example, Algorithm 10.3.6 will produce a *piecewise-constant* state time history $x(t)$ with step discontinuities when the state transitions occur.

Example 10.3.12 Algorithm 10.3.6 applied to the Markov chain in Examples 10.3.11 (with $X(0) = 0$ and $\tau = 25$) yields the following state time history.*

Figure 10.3.5.
Realization
of a
continuous-
time,
discrete-
state,
Markov,
chain.
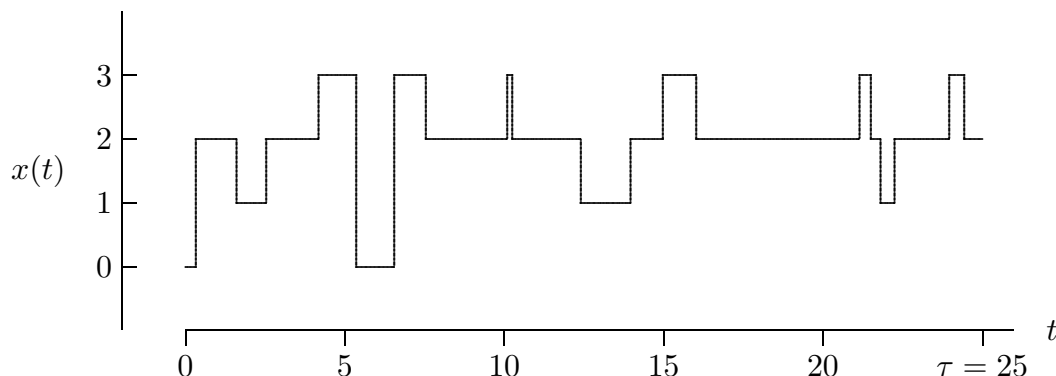
Example 10.3.13 An implementation of Algorithm 10.3.6 was run with $\tau = 10\,000$, initial state $X(0) = 0$, and an **rngs** initial seed of 12345. The resulting proportion of time spent in each state is shown below.

$$
\begin{array}{llcccc}
\text{state} & : & 0 & 1 & 2 & 3 \\
\text{proportion} & : & 0.100 & 0.186 & 0.586 & 0.128
\end{array}
$$

As we will see, the proportions in Example 10.3.13 form a (good) estimate of the steady-state pdf for the continuous-time Markov chain in Examples 10.3.11. For future reference, note that (with *d.ddd* precision)

$$
\mathbf{p}^\infty = \begin{bmatrix} 0.135 & 0.153 & 0.375 & 0.337 \\ 0.135 & 0.153 & 0.375 & 0.337 \\ 0.135 & 0.153 & 0.375 & 0.337 \\ 0.135 & 0.153 & 0.375 & 0.337 \end{bmatrix},
$$

where $\mathbf{p}$ is the state transition matrix for this Markov chain (Example 10.3.10). Clearly the common rows of this matrix, which represent the steady-state distribution of *something*, are not in agreement with the result in Example 10.3.13. The question then is how, if at all, are the two pdf vectors

$$
\begin{bmatrix} 0.100 & 0.186 & 0.586 & 0.128 \end{bmatrix}
$$

and

$$
\begin{bmatrix} 0.135 & 0.153 & 0.375 & 0.337 \end{bmatrix}
$$

related?

---

* This time history was generated by using the **rngs** initial seed 12345. Of course, with a different initial seed, the time history would have been different.

Embedded Chain

The answer to the question just posed is based on the observation that "within" any *continuous*-time Markov chain is an associated *embedded discrete*-time Markov chain. This is made clear by comparing Algorithm 10.3.6, reproduced here for convenience, which simulates a continuous-time Markov chain

```
t = 0;
x = X(0);
while (t < τ) {
    t += Exponential(μ(x));              /* continuous-time increment */
    x = NextState(x);
}
```

with Algorithm 10.3.3, which simulates the associated embedded discrete-time Markov chain

```
t = 0;
x = X(0);
while (t < τ) {
    t += 1;                              /* discrete-time increment /*
    x = NextState(x);
}
```

The numerical value of τ in these two algorithms may differ. The only other difference is the time-increment assignment. Therefore, with a proper choice of τ's and the use of multiple `rngs` streams, the state-to-state transitions for both the continuous-time Markov chain and its associated embedded discrete-time Markov chain would be *exactly* the same. The only difference would be the *Exponential*$(\mu(x))$ amount of time the continuous-time Markov chain spends in each state, before the next state transition occurs.

This embedded chain observation suggests that the answer to the question posed previously is that the steady-state pdf vector $\mathbf{f}_\infty$ for the *embedded* discrete-time Markov chain is related to the steady-state pdf vector $\mathbf{f}$ for the corresponding continuous-time Markov chain by the equation

$$f(x) = \frac{f_\infty(x)}{\alpha\,\lambda(x)} \qquad x \in \mathcal{X}$$

where

$$\alpha = \sum_x \frac{f_\infty(x)}{\lambda(x)}$$

is the real-valued parameter required to produce the normalization

$$\sum_x f(x) = 1$$

and the sums are over all $x \in \mathcal{X}$.

Example 10.3.14   As discussed previously, the steady-state pdf vector for the embedded discrete-time Markov chain corresponding to the continuous-time Markov chain in Example 10.3.10 is (with *d.ddd* precision)

$$\mathbf{f}_\infty = \begin{bmatrix} 0.135 & 0.153 & 0.375 & 0.337 \end{bmatrix}.$$

Starting with the un-normalized pdf definition

$$f'(x) = \frac{f_\infty(x)}{\lambda(x)} \qquad x = 0, 1, 2, 3$$

it follows that

$$f'(0) \cong \frac{0.135}{1.0000} \qquad f'(1) \cong \frac{0.153}{0.6667} \qquad f'(2) \cong \frac{0.375}{0.5000} \qquad f'(3) \cong \frac{0.337}{2.0000}.$$

In this case, the numerical value of the parameter α is

$$\alpha = \sum_{x=0}^{3} f'(x) \cong 1.283$$

and so the (normalized) steady-state pdf vector for the continuous-time Markov chain is

$$\mathbf{f} = \frac{1}{\alpha} \mathbf{f}' \cong \begin{bmatrix} 0.105 & 0.179 & 0.585 & 0.131 \end{bmatrix}$$

which compares well with the steady-state *estimate*

$$\begin{bmatrix} 0.100 & 0.186 & 0.586 & 0.128 \end{bmatrix}$$

in Example 10.3.13.

10.3.3   EXERCISES

Exercise 10.3.1   (*a*) Use Algorithms 10.3.2 and 10.3.3 to construct a program `dtmc` that simulates one realization $x(t)$ of a discrete-time, finite-state Markov chain. Program `dtmc` should estimate the proportion of time spent in each state and the statistic

$$\bar{x} = \frac{1}{\tau} \int_0^\tau x(t)\, dt.$$

(*b*) How does this statistic relate to the computed proportions? (*c*) Comment. (Simulate the Markov chain in Example 10.3.1 with $X(0) = 1$ and $\tau = 10\,000$.)

Exercise 10.3.2 (a) As an extension of Exercise 10.3.1, modify program `dtmc` to produce multiple realizations (replications). (b) With $X(0) = 1$, $\tau = 1, 2, 4, 8, 16, 32, 64$ and using 100 replications, compute a 95% confidence interval estimate for the transient statistic

$$\bar{X}(\tau) = \frac{1}{\tau} \int_0^\tau X(t)\, dt.$$

(c) How does this statistic relate to the statistic $\bar{x}$ computed in Exercise 10.3.1? (d) Comment.

Exercise 10.3.3 (a) As an alternate to always using $X(0) = 1$, repeat Exercise 10.3.2 except that for each replication draw the value of $X(0)$ at random consistent with the initial distribution

$$\mathbf{f}_0 = \begin{bmatrix} 0.155 & 0.290 & 0.284 & 0.271 \end{bmatrix}.$$

(b) Comment.

Exercise 10.3.4 (a) Use Algorithms 10.3.2 and 10.3.6 to construct a program `ctmc` that simulates one realization $x(t)$ of a continuous-time, finite-state Markov chain. Program `ctmc` should estimate the proportion of time spent in each state and the statistic

$$\bar{x} = \frac{1}{\tau} \int_0^\tau x(t)\, dt.$$

(b) How does this statistic relate to the computed proportions? (c) Comment. (Simulate the Markov chain in Example 10.3.10 with $X(0) = 1$ and $\tau = 10\,000$.)

Exercise 10.3.5 (a) As an extension of Exercise 10.3.4, modify program `ctmc` to produce multiple realizations (replications). (b) With $X(0) = 1$, $\tau = 1, 2, 4, 8, 16, 32, 64$ and using 100 replications, compute a 95% confidence interval estimate for the transient statistic

$$\bar{X}(\tau) = \frac{1}{\tau} \int_0^\tau X(t)\, dt.$$

(c) How does this statistic relate to the statistic $\bar{x}$ computed in Exercise 10.3.4? (d) Comment.

Exercise 10.3.6[a] (a) Use program `ctmc` (Exercise 10.3.4) to estimate the steady-state pdf of the number in a $M/M/2/5$ service node if $\lambda = 2.5$, the two servers are distinguishable with $\nu_1 = 1$, $\nu_2 = 2$ and the server discipline is to always choose the fastest available server. *Hint*: represent the state of the service node as a 3-tuple of the form (x_1, x_2, q) where x_1, x_2 represent the number in server 1, 2 respectively and q represents the number in the queue. (b) What did you do to convince yourself that your results are correct?

Exercise 10.3.7   Work through the details of Example 10.3.7. Keep the constants as fractions, rather than converting to decimal so that your solutions will be expressed as

$$f(0) = \frac{152}{981} \qquad f(1) = \frac{284}{981} \qquad f(2) = \frac{279}{981} \qquad f(3) = \frac{266}{981}.$$

Exercise 10.3.8   If $0 < \alpha < 1$ and $0 < \beta < 1$ then the discrete-time Markov $X(t)$ chain defined by

$$\mathbf{p} = \begin{bmatrix} 0 & 1-\alpha & \alpha & 0 \\ \beta & 0 & 0 & 1-\beta \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is irreducible and aperiodic. (a) Determine its steady-state pdf vector and the steady-state expected value

$$\lim_{t\to\infty} E[X(t)].$$

(b) If $X(0) = 2$ with probability 1, determine $E[X(t)]$ for $t = 1, 2, 3, 4$.

Exercise 10.3.9   If $0 < \alpha_x < 1$ for $x = 0, 1, \ldots, k$, the discrete-time Markov chain defined by

$$\mathbf{p} = \begin{bmatrix} 1-\alpha_0 & \alpha_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & \alpha_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1-\alpha_{k-1} & \alpha_{k-1} \\ \alpha_k & 0 & 0 & 0 & \cdots & 0 & 0 & 1-\alpha_k \end{bmatrix}$$

is irreducible and aperiodic. (a) Determine its steady-state pdf vector. *Hint*: express your answer in terms of the parameters

$$\mu_x = \frac{1}{\alpha_x} \qquad x = 0, 1, \ldots, k.$$

(b) Comment.

Exercise 10.3.10   The discrete-time Markov chain defined by

$$\mathbf{p} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has *two* steady-state pdf vectors. What are they?

We discuss how to construct a discrete-event simulation model of a network of $k \geq 1$ single-server service nodes in this section. This simulation will be based upon the following model.

- The single-server service nodes are indexed $s = 1, 2, \ldots, k$. The index $s = 0$ is reserved for the "super node" that represents the *exterior* of the network — the *source* of jobs flowing into the network and the *sink* for jobs flowing out of the network. The set of service nodes is denoted $\mathcal{S} = \{1, 2, \ldots, k\}$ with $\mathcal{S}_0 = \{0\} \cup \mathcal{S} = \{0, 1, 2, \ldots, k\}$.

- There is a $(k+1) \times (k+1)$ *node transition matrix* $\mathbf{p}$ defined in such a way that each job leaving node $s \in \mathcal{S}_0$ will transition to node $s' \in \mathcal{S}_0$ with probability

$$p[s, s'] = \Pr(\text{transition from node } s \text{ to node } s').$$

By convention $p[0, 0] = 0.0$.

- Each service node has its own queue with its own type of queueing discipline (FIFO, LIFO, etc.), its own service time distribution (*Exponential*, *Erlang*, etc.), and infinite capacity. The service rate of node $s \in \mathcal{S}$ is ν_s.

- The net flow of jobs *into* the network — the *external* arrivals — is assumed to be a Poisson process with rate $\lambda_0(t)$. External arrivals occur at node $s' \in \mathcal{S}$ with probability $p[0, s']$ and so

$$p[0, s'] \, \lambda_0(t)$$

is the external arrival rate at node $s' \in \mathcal{S}$, provided $p[0, s'] > 0$. Consistent with this assumption, the external arrivals at each service node form a Poisson process.

- As the notation suggests, the external arrival rate $\lambda_0(t)$ may vary with time to form a nonstationary arrival process.* However, for each $s' \in \mathcal{S}$, the probability $p[0, s']$ is constant in time. Indeed, all the $p[s, s']$ probabilities are constant in time.

Definition 10.4.1   If $p[0, s'] = 0$ for all $s' \in \mathcal{S}$ and $p[s, 0] = 0$ for all $s \in \mathcal{S}$ then the network is said to be *closed*. That is, a closed network is one for which the number of jobs in the network is fixed. A network that is not closed is said to be *open*.

By allowing for a time-varying external arrival rate $\lambda_0(t)$, the discrete-event simulation model we are developing will apply equally well to both open and closed networks. To simulate a closed network, it is sufficient to prevent departures from the network by choosing the node transition probabilities so that $p[s, 0] = 0$ for all $s \in \mathcal{S}$ and by choosing the external arrival rate so that $\lambda_0(t)$ becomes zero when the expected (or actual) number of jobs in the network reaches a desired level. Thus, from a simulation perspective, a closed network is just a special case of an open network.

---

* Techniques for simulating a *nonstationary* Poisson arrival process are discussed in Section 7.5. Techniques for fitting a nonstationary Poisson arrival process to a data set of arrival times are discussed in Section 9.3.

There is an appealing conceptual simplicity to modeling external arrivals as transitions *from* the 0th node and departures from the network as transitions *to* the 0th node. In this way, by viewing the *exterior* of the network as one (super) node, all transitions, including external arrivals and network departures, can be neatly characterized by a single $(k+1) \times (k+1)$ node transition matrix **p** with the structure indicated. This matrix defines the *topology* of the network in the sense that $p[s, s'] > 0$ if and only if jobs can transition directly from node s to s'. The $p[0, s']$ row of this matrix represents the external arrival probabilities; the $p[s, 0]$ column represents the network departure probabilities.*

$$\mathbf{p} = \begin{bmatrix} 0.0 & p[0,1] & p[0,2] & p[0,3] & \cdots & p[0,k] \\ p[1,0] & p[1,1] & p[1,2] & p[1,3] & \cdots & p[1,k] \\ p[2,0] & p[2,1] & p[2,2] & p[2,3] & \cdots & p[2,k] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p[k,0] & p[k,1] & p[k,2] & p[k,3] & \cdots & p[k,k] \end{bmatrix}.$$

Each row of **p** must sum to 1.0. Just as in a simulation of a Markov chain (Section 10.3), we can convert **p** to a *cumulative* node transition matrix **P** (Algorithm 10.3.1). This cumulative node transition matrix can then be used to simulate the node-to-node transitions of jobs as they move into, through, and out of the network (Algorithm 10.3.2).
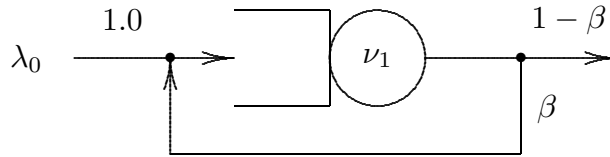
Examples

The following four examples illustrate the node transition matrix associated with simple networks of single-server service nodes. The graph-like network representation illustrated in these examples is relatively standard. Given that, it is important to be able to translate this representation into the corresponding node transition matrix.

Example 10.4.1   The $k = 1$ single-server model upon which the `ssq` series of programs is based corresponds to the node transition matrix

$$\mathbf{p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As an extension, this single-server model with feedback probability β from Section 3.3 is illustrated in Figure 10.4.1.

Figure 10.4.1.
Single-server service node with feedback probability β.

corresponds to

$$\mathbf{p} = \begin{bmatrix} 0 & 1 \\ 1 - \beta & \beta \end{bmatrix}.$$

---

* As an alternative approach, if the network is *closed* then the first row and column of **p** are null so that, in this case, **p** is effectively a $k \times k$ matrix.

Small networks like those in the following three examples are commonly used to model *job shops*. The idea is that raw material enters the job shop at one (or more) nodes where it is processed, then moved to other network nodes for further processing, etc., until eventually a finished product leaves the network. Travel time from node to node is assumed to be negligible and hence is not modeled explicitly.

Example 10.4.2 The conventional *tandem-server* network model consists of k servers in series (sequence), as illustrated in Figure 10.4.2.
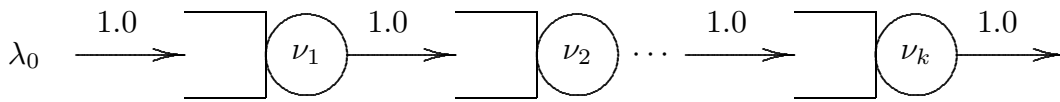

Figure 10.4.2. Tandem-server network.

corresponds to the node transition matrix

$$\mathbf{p} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

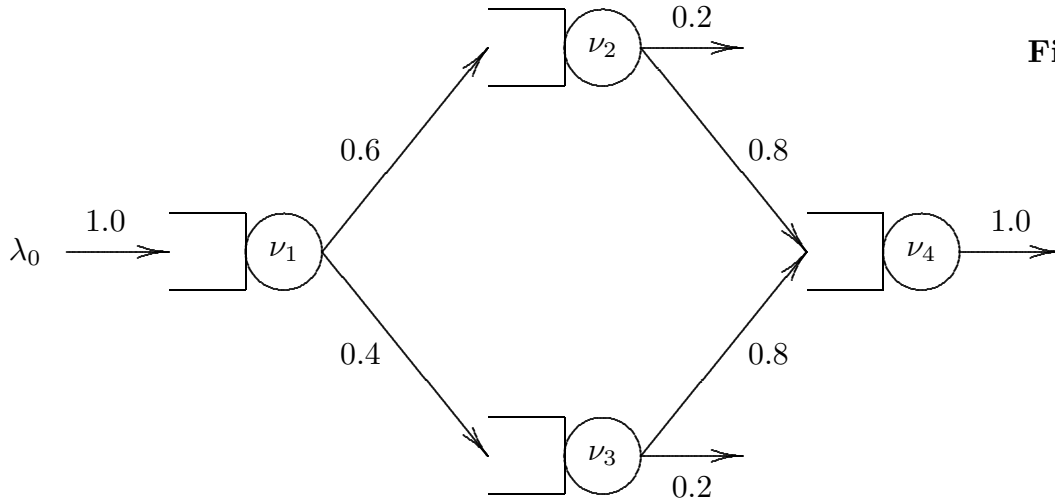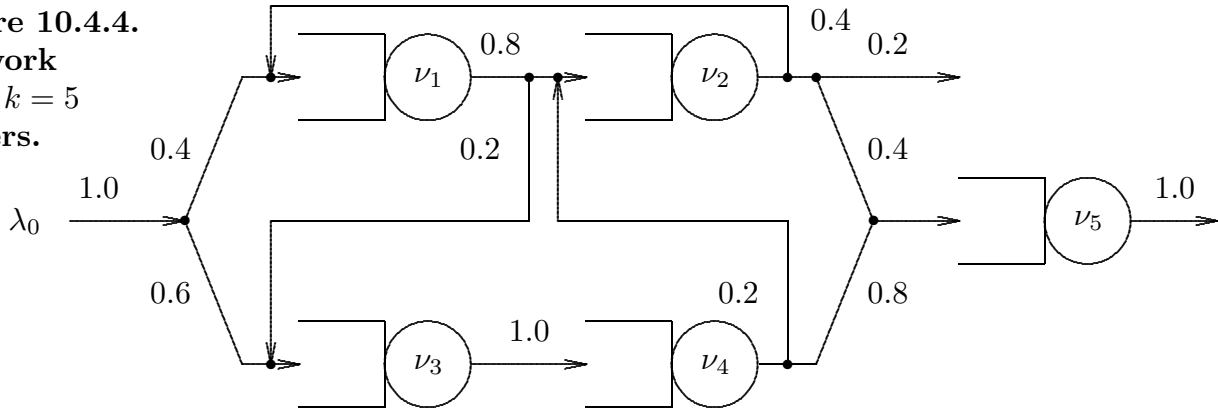**Example 10.4.3** The network with $k = 4$ servers shown in Figure 10.4.3

Figure 10.4.3. Network with $k = 4$ servers.

corresponds to the node transition matrix

$$\mathbf{p} = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.6 & 0.4 & 0.0 \\ 0.2 & 0.0 & 0.0 & 0.0 & 0.8 \\ 0.2 & 0.0 & 0.0 & 0.0 & 0.8 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

Example 10.4.4   The network with $k = 5$ servers in Figure 10.4.4

Figure 10.4.4.
Network
with $k = 5$
servers.

corresponds to the node transition matrix

$$\mathbf{p} = \begin{bmatrix} 0.0 & 0.4 & 0.0 & 0.6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & 0.2 & 0.0 & 0.0 \\ 0.2 & 0.4 & 0.0 & 0.0 & 0.0 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.8 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

As indicated by the relatively large number of non-zero elements in $\mathbf{p}$, the topology of this network is significantly more complex than the previous three networks.

- Although there is no *immediate* feedback (i.e., the diagonal elements of $\mathbf{p}$ are all 0.0), feedback is certainly possible. For example, a job leaving node 1 will go directly to node 2 with probability 0.8 and from there will be fed back to node 1 with probability 0.4 so that the probability of this two-step $1 \rightarrow 2 \rightarrow 1$ feedback is 0.32.

- More complex feedback paths are possible. For example, a job could circulate along the path $1 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1$ and, because of independence, having done so once, traverse the same path again, and again . . .   The probability of this path is so low (0.016), however, that multiple cycles on this path by any job will virtually never occur.

States and Events

   To characterize the state of a network of single-server service nodes at any time we need to know the number of jobs (if any) in each of the k service nodes. Consistent with this, the events — those things that cause the state of the network to change — are:

- the external arrivals;

- the service node departures.

Event List

Consistent with the previous discussion, the associated list of next-event times

next external arrival

next departure from node 1

$$\vdots$$

next departure from node k

consists of (at most) $k + 1$ elements, one for each event type. The detailed structure of this event list (array, linked list, binary tree, heap, etc.) is an (important) implementation issue that we need not resolve at this modeling level. Note that at any event time t

- the number of jobs (if any) in each of the k service node

and

- the contents of the event list,

together provide a comprehensive snapshot of the system. That is, provided all this information is recorded, a next-event simulation of a network of single-server service nodes could be stopped at an event time and then re-started later by using *only* this recorded information to re-initialize the simulation. This stop-restart process would produce dynamic system behavior that would be statistically indistinguishable from that produced by the system if it were not stopped. This kind of conceptual check is an important part of discrete-event system modeling.

In addition to the state variables and the event list, to allow for various queueing disciplines, it will be necessary to maintain k queue data structures, one for each service node. Again, the detailed structure of these queues (circular arrays, linked lists, etc.) is an (important) implementation detail that need not be resolved at this point.

Next-Event Simulation

Consistent with the model just constructed at the conceptual level, we can now construct, at the specification level, a next-event simulation of a network of single-server service nodes. Recognize that, at least in principle:

- the number of network service nodes can be arbitrarily large;
- the network topology can be arbitrarily complex.

That is, the next-event approach to this discrete-event simulation places no inherent bound on either the size or topological complexity of the network. Of course there will be practical bounds, but these will be imposed by hardware limitations and the choice of data structures and supporting algorithms, not the next-event approach to the simulation.

The following algorithm makes use of an auxiliary function `ProcessArrival(`s'`)` that handles the arrival of a job at node $s' \in \mathcal{S}$. Specifically:

- if the server at service node s' is idle, this function will place the arriving job in service and schedule the corresponding departure from the service node;

- else, if the server is busy, this function will place the arriving job in the queue.

Algorithm 10.4.1   The main loop of a next-event simulation of a network of single-server service nodes is

```
while ( some stopping criteria is not met ) {
   s = DetermineNextEvent();
   if (s == 0)
     CreateJob();
   else
     ProcessDeparture(s);
}
```

where the action of each function can be characterized as follows.

The function `DetermineNextEvent()` will:

- check the event list to determine the next (most imminent) event type $s \in \mathcal{S}_0$ and associated time t;

- advance the clock to t.

The function `CreateJob()` will:

- create an *external* arrival by using the 0th row of the cumulative transition matrix $\mathbf{P}$ to determine at which service node $s' \in \mathcal{S}$ this arrival will occur and then call the function `ProcessArrival(`s'`)`;

- schedule the next external arrival.

The function `ProcessDeparture(`s`)` will:*

- remove a job from service node $s \in \mathcal{S}$, use the sth row of the cumulative transition matrix $\mathbf{P}$ to determine where this job goes next (the new node $s' \in \mathcal{S}_0$), and then (unless $s' = 0$ in which case the job exits the network) call the function `ProcessArrival(`s'`)` to process the arrival of this job at service node $s' \in \mathcal{S}$;

- update the status of service node s by selecting a job from the queue at node s, if one exists, or otherwise setting the server's status to idle;

- if the server at node s is not idle, schedule the next departure from this node.

---

* The processing of a job that experiences immediate feedback must be implemented carefully in the first two steps of `ProcessDeparture(`s`)`.

10.4.1 STEADY-STATE

Flow balance can be used to characterize steady-state for a network of single-server service nodes with constant arrival rates. This characterization is based upon the following definition and Theorem 10.4.1.

Definition 10.4.2 For each service node $s \in \mathcal{S}$ define λ_s to be the *total* arrival rate. That is, λ_s is determined by summing all the *internal* arrival rates — jobs from other network service nodes including, possibly, s itself — plus the *external* arrival rate.

If the external arrival rate λ_0 is constant, from Definition 10.4.2 it follows that steady-state flow balance at all k service nodes is characterized by the k "flow in equals flow out" equations

$$\underbrace{\lambda_0 p[0, s']}_{\text{external}} + \underbrace{\lambda_1 p[1, s'] + \cdots + \lambda_{s'} p[s', s'] + \cdots + \lambda_k p[k, s']}_{\text{internal}} = \lambda_{s'} \qquad s' = 1, 2, \ldots, k.$$

Moreover, it is intuitive that flow balance can be achieved only if $\lambda_s < \nu_s$ for $s = 1, 2, \ldots, k$. This is summarized by the following theorem.

Theorem 10.4.1 If steady-state is possible then:

- The total flow rate $\lambda_1, \lambda_2, \ldots, \lambda_k$ into and out of each service node is characterized by the k linear *balance equations**

$$\sum_{s=0}^{k} \lambda_s \, p[s, s'] = \lambda_{s'} \qquad s' = 1, 2, \ldots, k.$$

- The steady-state *utilization* of each server is

$$\rho_s = \frac{\lambda_s}{\nu_s} < 1 \qquad s = 1, 2, \ldots, k.$$

Example 10.4.5 As a continuation of Example 10.4.1, for a single-server service node with external arrival rate λ_0, service rate ν_1, and feedback probability β shown in Figure 10.4.5,
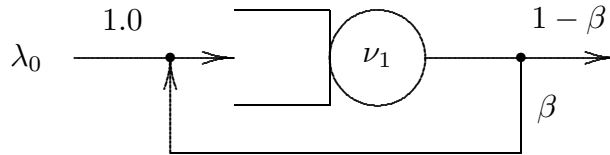
Figure 10.4.5. Single-server service node with feedback probability β.

the single balance equation is

$$\lambda_0 + \lambda_1 \beta = \lambda_1.$$

Therefore $\lambda_1 = \lambda_0/(1 - \beta)$ and we see that the steady-state utilization of a single-server service node with feedback is $\rho_1 = \lambda_1/\nu_1 = \lambda_0/(1 - \beta)\nu_1$ provided $\rho_1 < 1$.

---

* There are k equations and k unknowns; the constant external arrival rate λ_0 is *not* unknown.

Example 10.4.6   As a continuation of Example 10.4.2, for the tandem-server network illustrated in Figure 10.4.6,

Figure 10.4.6.
Tandem-server
network.

the balance equations are

$$\lambda_{s-1} = \lambda_s \qquad\qquad s = 1, 2, \ldots, k$$

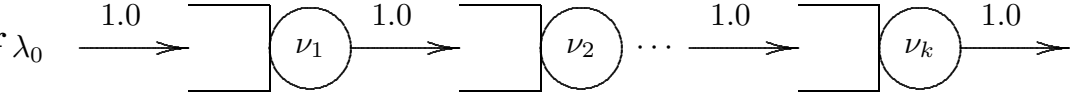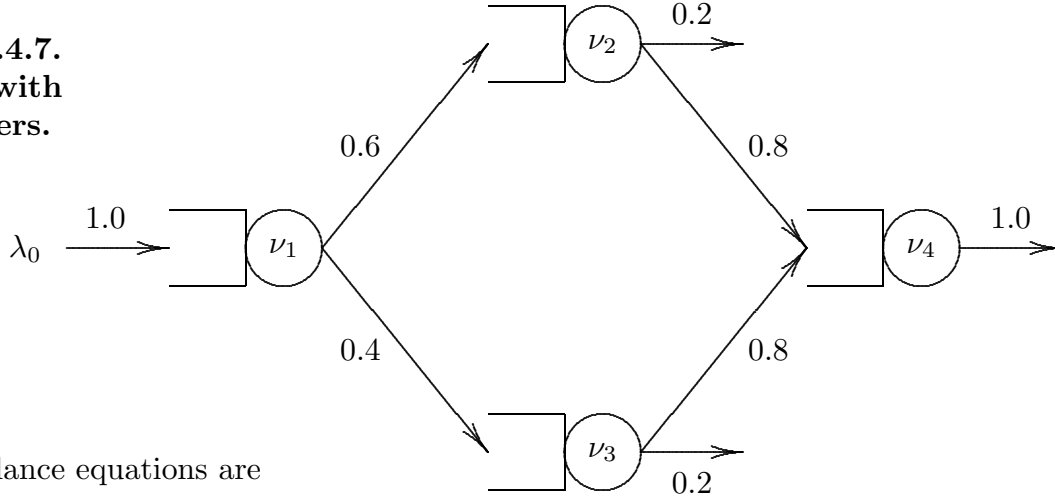and so $\lambda_s = \lambda_0$ for all $s = 1, 2, \ldots, k$. Therefore, the steady-state utilization of each service node in a tandem-server network is $\rho_s = \lambda_0/\nu_s$ provided $\rho_s < 1$ for $s = 1, 2, \ldots, k$.

Example 10.4.7   As a continuation of Example 10.4.3, for the network in Figure 10.4.7,

Figure 10.4.7.
Network with
$k = 4$ servers.

the balance equations are

$$
\begin{aligned}
\lambda_0 &= \lambda_1 && \text{node 1} \\
0.6\lambda_1 &= \lambda_2 && \text{node 2} \\
0.4\lambda_1 &= \lambda_3 && \text{node 3} \\
0.8\lambda_2 + 0.8\lambda_3 &= \lambda_4 && \text{node 4.}
\end{aligned}
$$

It is easily verified that the (unique) solution to these four equations is

$$\lambda_1 = \lambda_0 \qquad\qquad \lambda_2 = 0.6\lambda_0 \qquad\qquad \lambda_3 = 0.4\lambda_0 \qquad\qquad \lambda_4 = 0.8\lambda_0$$

which defines the steady-state utilization of each service node as

$$\rho_1 = \frac{\lambda_0}{\nu_1} \qquad\qquad \rho_2 = 0.6\,\frac{\lambda_0}{\nu_2} \qquad\qquad \rho_3 = 0.4\,\frac{\lambda_0}{\nu_3} \qquad\qquad \rho_4 = 0.8\,\frac{\lambda_0}{\nu_4}$$

provided $\rho_s < 1$ for $s = 1, 2, 3, 4$.
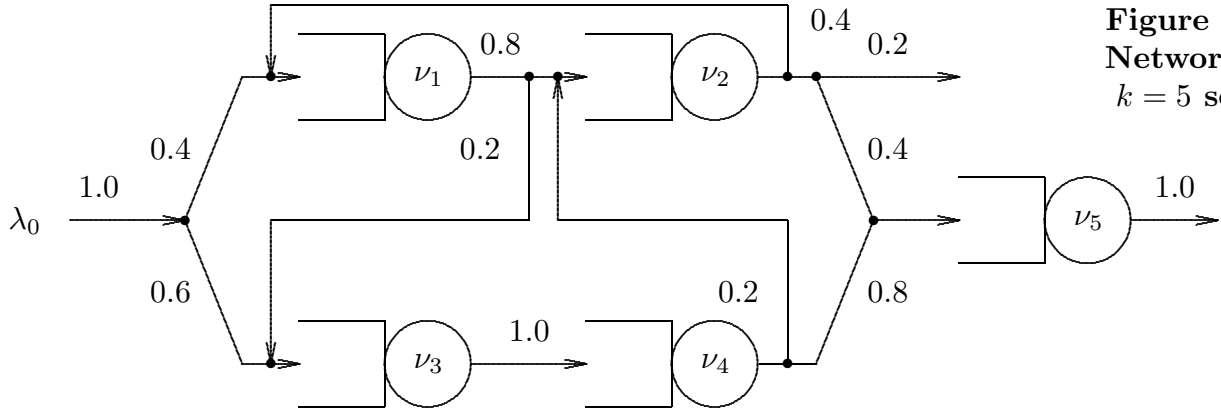
Example 10.4.8   As a continuation of Example 10.4.4,

Figure 10.4.8. Network with $k = 5$ servers.

the balance equations are

$$0.4\lambda_0 + 0.4\lambda_2 = \lambda_1 \qquad \text{node 1}$$
$$0.8\lambda_1 + 0.2\lambda_4 = \lambda_2 \qquad \text{node 2}$$
$$0.6\lambda_0 + 0.2\lambda_1 = \lambda_3 \qquad \text{node 3}$$
$$\lambda_3 = \lambda_4 \qquad \text{node 4}$$
$$0.4\lambda_2 + 0.8\lambda_4 = \lambda_5 \qquad \text{node 5.}$$

It can be verified that the (unique) solution to these five equations is

$$\lambda_1 = \frac{560}{830}\lambda_0 \qquad \lambda_2 = \frac{570}{830}\lambda_0 \qquad \lambda_3 = \frac{610}{830}\lambda_0 \qquad \lambda_4 = \frac{610}{830}\lambda_0 \qquad \lambda_5 = \frac{716}{830}\lambda_0$$

which defines the steady-state utilization of each service node as

$$\rho_1 = \frac{560}{830} \cdot \frac{\lambda_0}{\nu_1} \qquad \rho_2 = \frac{570}{830} \cdot \frac{\lambda_0}{\nu_2} \qquad \rho_3 = \frac{610}{830} \cdot \frac{\lambda_0}{\nu_3} \qquad \rho_4 = \frac{610}{830} \cdot \frac{\lambda_0}{\nu_4} \qquad \rho_5 = \frac{716}{830} \cdot \frac{\lambda_0}{\nu_5}$$

provided $\rho_s < 1$ for $s = 1, 2, 3, 4, 5$.

Consistency Check

In addition to the k flow balance equations in Theorem 10.4.1, there is one additional equation

$$\sum_{s=1}^{k} \lambda_s p[s, 0] = \lambda_0$$

which characterizes flow balance for the 0th node. Although this may seem to be a new equation, independent of the others, it is not. That is, it can be shown that this equation can be derived from the k equations in Theorem 10.4.1. The proof of this result is left as an exercise. The point is that the 0th node balance equation provides no *new* information. However, it does provide a good consistency check on the correctness of the steady-state λ's.

Example 10.4.9   To illustrate the previous discussion:

- for Example 10.4.1, the 0th node balance equation is $\lambda_1(1 - \beta) = \lambda_0$;

- for Example 10.4.2, the 0th node balance equation is $\lambda_k = \lambda_0$;

- for Example 10.4.3, the 0th node balance equation is $0.2\lambda_2 + 0.2\lambda_3 + \lambda_4 = \lambda_0$;

- for Example 10.4.4, the 0th node balance equation is $0.2\lambda_2 + \lambda_5 = \lambda_0$.

It can verified that in each case, this equation is satisfied.

Jackson Networks

Definition 10.4.3   A network of single-server service nodes is a *Jackson network* if and only if:

- the external arrival process is a stationary Poisson process (λ_0 is constant in time);

- the service time distribution at each service node is *Exponential*($1/\nu_s$) (the service rate ν_s is constant in time);

- all service times processes and the external arrival process are statistically independent;

- at each service node, the queue discipline assumes no knowledge of service times (e.g., FIFO, LIFO, and SIRO, which were defined in Section 1.2, are valid queue disciplines, but SJF is not).

If we are willing to make all the assumption associated with a Jackson network then the following remarkable theorem, *Jackson's theorem*, is valid. The proof of this theorem is beyond the scope of this text.

Theorem 10.4.2   Each service node in a Jackson network operates *independently* as an $M/M/1$ service node in steady-state.*

By combining Theorems 10.4.1 and 10.4.2 we have the following algorithm for analyzing the steady-state behavior of an *open* Jackson network of single-server service nodes.

Algorithm 10.4.2   The steady-state behavior of an open Jackson network of k single-server service nodes can be analyzed as follows.

- Set up and solve the k balance equations for $\lambda_1, \lambda_2, \ldots, \lambda_k$.

- Check that the 0th node balance equation is satisfied.

- Check that steady-state can occur, i.e., $\rho_s = \lambda_s/\nu_s < 1$ for all $s = 1, 2, \ldots, k$.

- Analyze each node as a statistically independent $M/M/1$ service node with arrival rate λ_s, service rate ν_s, and utilization ρ_s.

---

* This theorem is valid for both *open* and *closed* Jackson networks.

Example 10.4.10 As a continuation of Example 10.4.5, for an external Poisson arrival process with rate λ_0, an *Exponential*$(1/\nu_1)$ server, and feedback probability β
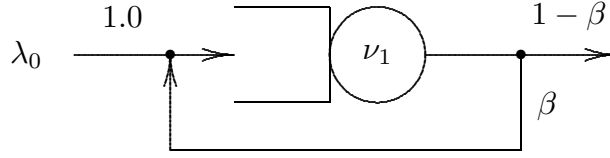
the steady-state number in the service node, L_1, is a *Geometric*(ρ) random variable provided $\rho_1 = \lambda_0/(1 - \beta)\nu_1 < 1$.* Therefore, in particular, if $\lambda_0 = 1$, $\nu_1 = 3$, and $\beta = 0.5$, then $\rho_1 = 2/3$ and it follows that the expected value of L_1 is

$$E[L_1] = \frac{\rho_1}{1 - \rho_1} = \frac{2/3}{1 - 2/3} = 2.$$

Note that:

- If there is no feedback ($\beta = 0$) then $\rho_1 = 1/3$ and $E[L_1] = 0.5$.

- If the feedback probability approaches $2/3$ ($\beta \to 2/3$) then $\rho_1 \to 1$ and $E[L_1] \to \infty$.

Example 10.4.11 As a continuation of Example 10.4.6, for an external Poisson arrival process with rate λ_0 and a tandem-server network of k independent *Exponential*$(1/\nu_s)$ servers
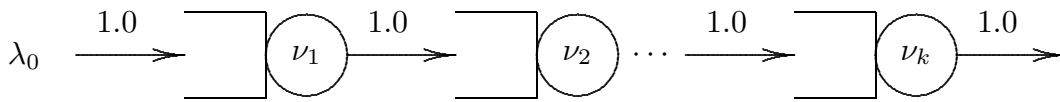
the steady-state number in each service node, L_s, is a *Geometric*(ρ_s) random variable provided $\rho_s = \lambda_0/\nu_s < 1$ for all $s = 1, 2, \ldots, k$. The steady-state number in the network is $L = L_1 + L_2 + \cdots + L_k$. Therefore the expected value of L is

$$E[L] = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \cdots + \frac{\rho_k}{1 - \rho_k}.$$

Moreover, if all the service rates are *equal*, i.e., $\nu_s = \nu$ for all s (and if $\rho = \lambda_0/\nu < 1$) then

$$E[L] = \frac{k\rho}{1 - \rho}$$

Because L is a sum of k independent *Geometric*(ρ) random variables, we have the following theorem.

Theorem 10.4.3 Given a stationary Poisson arrival process with rate λ to a tandem-server network of k independent *Exponential*$(1/\nu)$ servers with $\rho = \lambda/\nu < 1$ then the steady-state number in the network is a *Pascal*(k, ρ) random variable.

---

* The arrival, service, and feedback processes must be statistically independent.

In general for any open Jackson network with k service nodes the steady-state expected number in the network will be

$$E[L] = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \cdots + \frac{\rho_k}{1 - \rho_k}.$$

The $E[L] = \lambda E[W]$ form of Little's equation is applicable to the entire network and so the expected *wait* (time spent in the network) is

$$E[W] = \frac{E[L]}{\lambda_0}.$$

Example 10.4.12   As a continuation of Example 10.4.7, if the network depicted in Figure 10.4.11

Figure 10.4.11.
Network with
$k = 4$ **servers.**
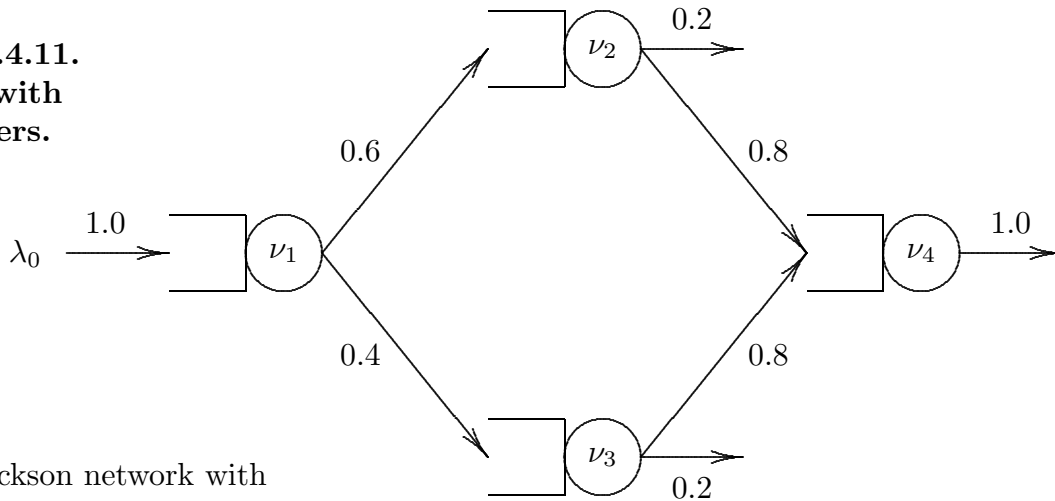
is a Jackson network with

$$\lambda_0 = \frac{1}{120} \qquad \nu_1 = \frac{1}{84} \qquad \nu_2 = \frac{1}{170} \qquad \nu_3 = \frac{1}{225} \qquad \nu_4 = \frac{1}{120}$$

then it can be shown that

$$E[L] = \frac{7}{3} + \frac{17}{3} + 3 + 4 = 15.$$

It follows that the expected wait in the network is

$$E[W] = \frac{E[L]}{\lambda_0} = 120 \cdot 15 = 1800.$$

Exercise 10.4.4 illustrates another way to compute $E[W]$.

10.4.2   EXERCISES

Exercise 10.4.1   Consider an open Jackson network with node transition matrix

$$\mathbf{p} = \begin{bmatrix} 0.00 & 1.00 & 0.00 & 0.00 \\ 0.20 & 0.00 & 0.72 & 0.08 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \end{bmatrix}$$

and parameters

$$\lambda_0 = 0.15 \qquad \nu_1 = 1.00 \qquad \nu_2 = 0.90 \qquad \nu_3 = 0.12.$$

(a) Verify that this system can achieve steady-state and determine the steady-state utilization of each service node. (b) Verify that the 0th node balance equation is satisfied. (c) Compute $E[L]$ and $E[W]$.

Exercise 10.4.2   Do the same as in Exercise 10.4.1 except for the Jackson network defined by

$$\mathbf{p} = \begin{bmatrix} 0.00 & 0.50 & 0.00 & 0.50 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.25 & 0.25 & 0.50 \\ 1.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

with parameters

$$\lambda_0 = 2 \qquad \nu_1 = 2 \qquad \nu_2 = 3 \qquad \nu_3 = 3.$$

Exercise 10.4.3   (a) As an extension of Theorem 10.4.3, prove that if the queue discipline at each service node is FIFO then the steady-state wait in a k-node tandem-server network is an $Erlang(k, 1/(\nu - \lambda))$ random variable. (b) Is this result consistent with Example 10.4.11 and Little's equation $E[L] = \lambda_0 E[W]$?

Exercise 10.4.4   (a) As a continuation of Example 10.4.12, compute the expected steady-state wait $E[W_s]$ for $s = 1, 2, 3, 4$ and then verify that

$$E[W] = E[W_1] + 0.6\, E[W_2] + 0.4\, E[W_3] + 0.8\, E[W_4] = 1800.$$

(b) What special property of the network in Example 10.4.12 facilitates this approach to computing $E[W]$?

Exercise 10.4.5   Construct a next-event simulation of the network in Exercise 10.4.1. Assume all queues are FIFO.

Exercise 10.4.6   Construct a next-event simulation of the network in Exercise 10.4.2. Assume all queues are FIFO.

A. Simulation Languages

This text emphasizes the use of a high-level algorithmic language for the implementation of discrete-event simulation models. Many discrete-event simulation modelers prefer to use a simulation programming language to save on model development time. This appendix surveys the history of these simulation programming languages and illustrates the use of one of these languages on the single-server service node model from Section 1.2.

A.1 HISTORY

The use of a general-purpose simulation programming language (SPL) expedites model development, input modeling, output analysis, and animation. In addition, SPLs have accelerated the use of simulation as an analysis tool by bringing down the cost of developing a simulation model. Nance (1993) gives a history of the development of SPLs from 1955 to 1986. He defines six elements that must be present in an SPL:

- random number generation

- variate generation

- list processing capabilities so that objects can be created, altered, and deleted

- statistical analysis routines

- summary report generators

- a timing executive or event calendar to model the passage of time

These SPLs may be (i) a set of subprograms in a high-level algorithmic language such as FORTRAN, Java, or C that can be called to meet the six requirements, (ii) a preprocessor that converts statements or symbols to lines of code in a high-level algorithmic language, or (iii) a conventional programming language.

The historical period is divided into five distinct periods. The names of several languages that came into existence in each period (subsequent versions of one particular language are not listed) are:

- 1955–1960. The period of search: GSP

- 1961–1965 The advent: CLP, CSL, DYNAMO, GASP, GPSS, MILITRAN, OPS, QUIKSCRIPT, SIMSCRIPT, SIMULA, SOL

- 1966–1975. The formative period: AS, BOSS, Q-GERT, SLANG, SPL

- 1971–1978. The expansion period: DRAFT, HOCUS, PBQ, SIMPL

- 1979–1986. Consolidation and regeneration: INS, SIMAN, SLAM

The General Purpose System Simulator (GPSS) was first developed on various IBM computers in the early 1960's (Karian and Dudewicz, 1991). Its block semantics were ideally suited for queueing simulations. Algol-based SIMULA was also developed in the 1960's and had features that were ahead of its time, including abstract data types, inheritance, the co-routine concept, and quasi-parallel execution.

SIMSCRIPT was developed by the RAND Corporation with the purpose of decreasing model and program development times. SIMSCRIPT models are described in terms of entities, attributes, and sets. The syntax and program organization was influenced by FORTRAN. The Control and Simulation Language (CSL) takes an "activity scanning" approach to language design, where the activity is the basic descriptive unit. The General Activity Simulation Program (GASP), as with several of the other languages, used flow-chart symbols to bridge the gap between personnel unfamiliar with programming and programmers unfamiliar with the application area. Although originally written in Algol, GASP provided FORTRAN subroutines for list-processing capabilities (e.g., queue insertion). GASP was a forerunner to the Simulation Language for Alternative Modeling (SLAM) and SIMulation ANalysis (SIMAN) languages. SLAM (Pritsker, 1995) was the first language to include three modeling perspectives in one language: network (process orientation), discrete-event, and continuous (state variables). SIMAN was the first major SPL executable on an IBM PC.

Languages that have been developed since Nance's consolidation and regeneration period include Witness, Csim, ProModel (Harrell, Ghosh, and Bowden, 2000), SIGMA (Schruben, 1992), @RISK (Seila, Ceric, Tadikamalla, 2003), and Arena (a combination of SIMAN and the animator Cinema, see Kelton, Sadowski, and Sadowski, 2002). A survey of the current SPLs on the market is given by Swain (2003). More detail on current popular SPLs is given in Chapter 3 of Banks, Carson, Nelson, and Nicol (2001).

A.2 SAMPLE MODEL

Although the choice of languages is arbitrary, we illustrate the use of the SLAM (Pritsker, 1995) language, which came into prominence between the eras of SPLs consisting entirely of subprograms and the current "point-and-click" GUIs now available with most SPLs. The language's market share peaked in the late 1980's and and early 1990's. We model the single-server service node in SLAM. The single-server service node model was presented in Section 1.2 using a "process interaction" world view and later presented in Section 5.1 (program `ssq3`) using a "next-event" world view. Like program `ssq3`, SLAM uses a next-event world view.

SLAM Nodes

The simplest way to model a single-server service node in SLAM is to use a *network* orientation with *nodes* and *branches* used to represent the system. SLAM uses the term *entity* to generically refer to objects passing through the network. Other languages often use the term *transaction*. Depending on the model, an entity could be a job, a person, an object, a unit of information, etc. Branches, or activities, connect the nodes in a network diagram and represent the paths along which entities move. Activities are used to model the passage of time. Nodes can be used for creating entities, delaying entities at queues, assigning variables, etc. For the single-server service node model, an entity is a job. There are 20 types of nodes in the SLAM language. We will use three of these node types to model a single-server service node.

A `CREATE` node that is used to create entities is illustrated in Figure A.1.
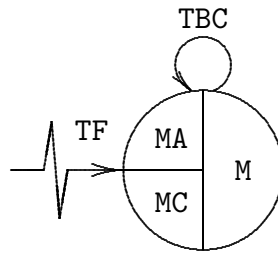
The node parameters are:

- `TF`, the time of the first entity creation (default: 0);

- `TBC`, the time between entity creations (default: ∞);

- `MA`, the attribute number associated with the entity where the creation time is saved, known as the "mark attribute" (default: don't save the creation time);

- `MC`, maximum number of entities created at this node (default: ∞);

- `M`, maximum number of emanating branches to which a created entity can be routed (default: ∞).

Entities are created using a next-event approach with a deterministic or probabilistic time between creations, as specified by the `TBC` parameter. Each entity has a user-specified number of *attributes* that it maintains as it passes through the network. Typical attributes are the arrival time to the system, routing information, and service time characteristics. The `MC` parameter determines when the `CREATE` node will stop creating entities. If the entity is to be "cloned" and duplicate copies sent out over several branches emanating from the `CREATE` node, the `M` parameter designates the maximum number of branches that can be taken.

A `QUEUE` node that is used to store entities that are waiting for a server is illustrated in Figure A.2.

The node parameters are:

- `IQ`, initial number in the queue (default: 0);

- `QC`, queue capacity (default: ∞);

- `IFL`, a file number used by the SLAM filing system (no default).

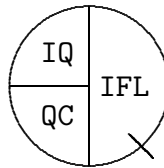The QUEUE node is used for delaying entities prior to a "service" activity. If the initial number in the queue is positive, SLAM will place entities in the queue at the beginning of the simulation run and begin the simulation run with all servers busy. If the queue capacity is finite, three things can happen to an entity that arrives to a full queue: the arriving entity can be destroyed; the arriving entity can balk to another node; the entity can block another server from performing their activity due to lack of space in the queue. The file number IFL indicates where SLAM will store entities that are delayed because of busy server(s). SLAM maintains a filing system which is a FORTRAN-based doubly-linked list to store entities waiting in queues, as well as calendar events.

The TERMINATE node is illustrated in Figure A.3.

Figure A.3.
TERMINATE
node.
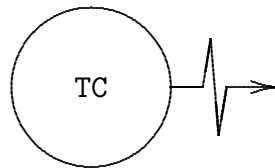
The node parameter TC specifies a simulation termination condition (default: ∞). When the TC$^{\text{th}}$ entity arrives to the TERMINATE node, the simulation ends and a SLAM output report is printed.

SLAM Network Discrete-Event Simulation Model

Our attention now turns to using these three nodes to model a single-server service node. Recall that the times between arrivals to the node were independent *Exponential*(2.0) random variates, and service times were independent *Uniform*(1.0, 2.0) random variates. A SLAM network diagram for a single-server service node is illustrated in Figure A.4.

Figure A.4.
Single-server
service node
model.
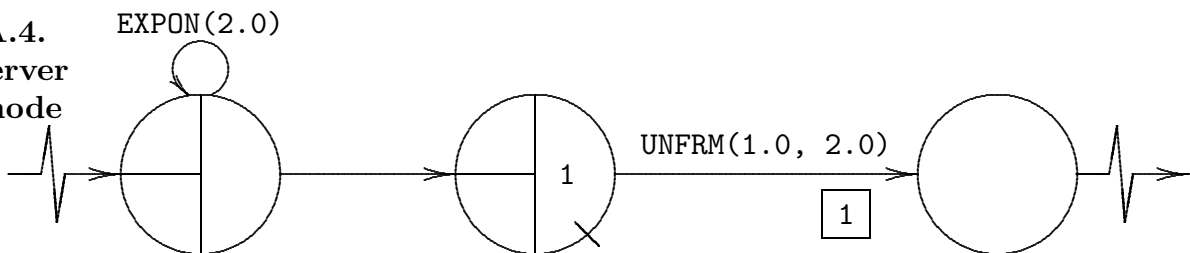
The first job is created at time zero; subsequent jobs are created with an inter-creation time that is exponentially distributed with a mean of 2 minutes.* Jobs traverse the branch (activity) between the CREATE and QUEUE node in 0 time units. In the QUEUE node, jobs wait in file 1 (numbered for identification on the output report) for the server. The server is modeled by the activity (branch) number 1 (numbered for output report identification) following the QUEUE node. The time to traverse this branch is *Uniform*(1, 2) and corresponds to a service time. The TERMINATE node is where the entity representing the job is "destroyed."

---

* SLAM uses a Lehmer random number generator with 10 streams and changeable initial seed values. SLAM also provides random variate generators for several popular distributions (e.g., exponential, normal, Poisson, Weibull).

The SLAM philosophy is that models are constructed as network diagrams at the on-screen iconic level. For most implementations of SLAM, software is available for automatically converting an iconic model (network diagram) to executable FORTRAN-like statements. The executable SLAM statements corresponding to the single-server service node network diagram are shown below.

```
GEN, LEEMIS, SSQ3, 9/30/04, 1, Y, Y, Y, Y, Y, 72;
LIMITS, 1, 2, 100;
NETWORK;
    CREATE, EXPON(2.0);                        CREATE CUSTOMERS
    QUEUE(1);                             WAIT FOR THE SERVER HERE
    ACTIVITY/1, UNFRM(1.0, 2.0);                  SERVICE TIME
    TERMINATE;                          CUSTOMERS LEAVE SYSTEM
    ENDNETWORK;
INIT, 0, 10000;                         RUN FOR 10 000 MINUTES
FIN;
```

The statements that begin in column 1 are called SLAM control statements. SLAM provides 20 control statements to create initial or terminal conditions, allocate memory, run traces, etc. Five such statements are illustrated in this example. The statements that are indented correspond to the network diagram. All statements end in a semicolon, followed by optional comments.

The GEN(erate) control statement has a parameter list that begins with the modeler's name, model name, and a date. The 1 following the date indicates the number of replications (called "runs" in SLAM) of the simulation model. The fields following the number of replications (with Y parameters indicating "YES") indicate that defaults are to be taken for printing an echo report, input statements, etc. The last parameter tells SLAM to print a 72 column output report.

The LIMITS control statement has three parameters: the number of files used, the number of attributes per entity and the maximum number of concurrent entities in the system. SLAM uses a file system based on static allocation of memory at compilation time. The NETWORK control statement indicates the beginning of the SLAM network statements. The INIT control statement indicates that the simulation should be run between times 0 and 10,000. The FIN control statement indicates the end of the SLAM control statements.

SLAM provides the user with an "echo" report, which gives information such as ranking criterion in files (the FIFO default is used here), random number streams and associated seed values, model initialization options, termination criteria, and memory allocation. The echo report has been omitted since its contents simply reflect that the model has been properly specified.

The SLAM summary report follows.

```
         S L A M   I I   S U M M A R Y   R E P O R T

   SIMULATION PROJECT SSQ3                    BY LEEMIS
   DATE  9/30/2004                            RUN NUMBER    1 OF    1


   CURRENT TIME    0.1000E+05
   STATISTICAL ARRAYS CLEARED AT TIME  0.0000E+00


           **FILE STATISTICS**

   FILE                     AVERAGE  STANDARD   MAXIMUM  CURRENT AVERAGE
   NUMBER   LABEL/TYPE      LENGTH   DEVIATION  LENGTH   LENGTH  WAIT TIME

      1         QUEUE        1.181     1.839      16        0     2.349
      2        CALENDAR      1.756     0.429       3        1     1.273


           **SERVICE ACTIVITY STATISTICS**

   ACT ACT LABEL OR  SER AVERAGE    STD  CUR AVERAGE MAX IDL MAX BSY  ENT
   NUM START NODE    CAP  UTIL      DEV  UTIL BLOCK  TME/SER TME/SER  CNT

    1       QUEUE     1  0.756     0.43   0    0.00   16.75   138.05  5026
```

The report echos the model name, modeler's name and date, then indicates the simulation clock time (current time) when this report was generated. The file statistics section shows that there were two files used in the model: the first for queuing up jobs and the second to hold the event calendar. The statistics for the queue show that the average number in the queue (length of the queue) was 1.181, with a standard deviation of 1.839. The queue length reached a maximum of 16 jobs The average delay (wait) time for a job was 2.349 minutes. The queue is currently (at time 10,000) empty. The statistics for the calendar show that there were an average of 1.756 events on the calendar during the simulation, with a standard deviation of 0.429. There were a maximum of three events on the calendar at one time: the next arrival, an end-of-service event, and the pseudo-event (see Lecture 13) at time 10,000 to end the simulation. There is currently one event left on the calendar (which must be the next arrival since the pseudo-event has been processed and the server is idle at the end of the simulation). Finally, the average calendar event stays on the calendar for 1.273 time units.

The service activity statistics indicate that the server utilization was 0.756. The server is currently (at time 10,000) idle. The longest time that the server was idle during the simulation was 16.75 minutes, while the longest time the server was busy during the simulation was 138.05. A total of 5026 jobs were served during the simulation, which is consistent with an arrival rate of 0.5.

A.3   CONCLUSIONS

There are advantages to using SLAM, or any other SPL. First, many lines of code are saved over implementing a model in a high-level algorithmic language. This may be helpful in terms of debugging. Second, an SPL has routines for managing files and the event calendar which allow the modeler to focus on *modeling*, as opposed to sequencing events and establishing queue priorities. Third, standard summary reports, including histograms and other statistical displays, are automatically generated. Fourth, animation of a model, a powerful tool for communication with management, is typically included as a part of modern simulation languages. This would be a custom and rather intricate programming project for someone using a high-level algorithmic language. Templates are now available in many languages that customize an animation to a particular application area (e.g., banking, medicine, manufacturing). Fifth, most ordinary embellishments (e.g., multiple servers or a new queue discipline) are an easy matter in a simulation language. SLAM, for example, allows a modeler to write FORTRAN or C code for any embellishments that are not built into the language.

There are several disadvantages to using an SPL. First, simulation languages tend to have a slant toward one particular type of model. SLAM, for example, is very strong in modeling queuing-type models (e.g., inventory models). It has not been designed well for modeling most reliability systems, however, and a modeler might need to write code for modeling series and parallel systems in an high-level algorithmic language. Second, the modeler does not have complete control over the data structures that are used in a simulation language. In some models, this may be burdensome. Third, memory may be wasted in a simulation language. SLAM, for example, requires that all entities have the same number of attributes — something that a programmer may prefer to avoid. Finally, since the modeler does not have access to the internal algorithms contained in an SPL, a modeler can easily make an incorrect assumption about the behavior of the language under certain circumstances, which can result in incorrect conclusions. Schriber and Brunner (1998) survey the internal assumptions and associated algorithms present in most modern SPLs.

Recent advances in SPLs include object-oriented simulation languages (Joines and Roberts, 1998), web-based simulation languages (Kilgore, 2002), and parallel and distributed simulation (Fujimoto, 1998).

A.4   EXERCISES

Exercise A.1   (*a*) Program the M/G/1 queue described in this section in the language of your choice. (*b*) If possible, control the random number generator seed so that your output matches that of program `ssq3`.

Exercise A.2   (*a*) Use an Internet search engine to identify SPLs that support discrete-event simulation. (*b*) Classify these languages based on modeling features of your choice.

B. Integer Arithmetic

The purpose of this appendix is to summarize the arithmetic and mathematical properties of integers that are most relevant to an understanding of random number generation. The primary emphasis is on integer division and factoring.

Terminology

Because some properties summarized in this appendix apply to, for example, the positive integers only, we begin with the following standard clarifying terminology:

- the *positive integers* are $1, 2, 3, \ldots$;

- the *non-negative integers* (positive and zero) are $0, 1, 2, 3, \ldots$;

- the *negative integers* are $\ldots, -3, -2, -1$;

- the *integers* (positive, negative and zero) are $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

So, for example, the positive integers have the property that they are closed under addition and multiplication, but not under subtraction. In contrast, the integers are closed under all three operations.

Theory

There are two (related) theoretical properties of the positive integers that are listed here for completeness:

- *well ordering* — any non-empty set of positive integers has a smallest element;

- *mathematical induction* — S is a set of positive integers. If $1 \in S$ and if $n + 1 \in S$ for each $n \in S$, then *all* the positive integers are in S.

All of the standard integer arithmetic existence theorems are based on one or both of these properties. The following important theorem is an example.

Integer Division

Theorem B.1   *Division Theorem* — if b is an integer and a is a positive integer then there exists a unique pair of integers q, r with $0 \le r < a$ such that $b = aq + r$.

Example B.1   The division theorem is an *existence* theorem. That is, the integer pair (q, r) is guaranteed to exist. The theorem does not address at all the algorithmic question of *how* to determine (q, r). Of course, we were all taught to do division as children and, in that sense, the algorithm for finding (q, r) is second nature — divide a into b to get the quotient q with remainder r.

- If $(a, b) = (7, 17)$ then $(q, r) = (2, 3)$. That is, $17 = 7 \cdot 2 + 3$.

- If $(a, b) = (7, -17)$ then $(q, r) = (-3, 4)$. That is, $-17 = 7 \cdot (-3) + 4$.

In terms of implementation, the computation of (q, r) is directly related to the following two functions.
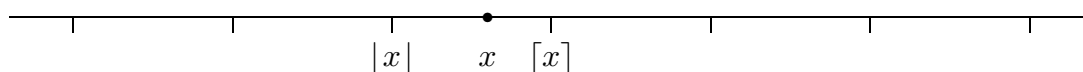
Floor and Ceiling Functions

Definition B.1   For any real-valued number x

- the value of the *floor* function $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$;

- the value of the *ceiling* function $\lceil x \rceil$ is the smallest integer n such that $x \leq n$.

Example B.2   For example, $\lfloor 7.2 \rfloor = \lfloor 7.7 \rfloor = 7$ and $\lfloor -7.2 \rfloor = \lfloor -7.7 \rfloor = -8$. Similarly $\lceil 7.2 \rceil = \lceil 7.7 \rceil = 8$ and $\lceil -7.2 \rceil = \lceil -7.7 \rceil = -7$.

The floor ("round to the left") and ceiling ("round to the right") functions are related. (This figure represents the real number line with tick marks at the integers.)

Specifically, the floor and ceiling functions satisfy the following properties.

- If x is an integer then $\lceil x \rceil = \lfloor x \rfloor$.

- If x is not an integer then $\lceil x \rceil = \lfloor x \rfloor + 1$.

- In general $\lceil x \rceil = -\lfloor -x \rfloor$.

Because of these properties, it is sufficient for a programming language to provide just one of these functions, usually the floor function.

Example B.3   In ANSI C both the floor and ceiling functions are provided, albeit in a potentially confusing way. That is, a floating point value is converted to an integer value by "truncation of any fractional part." So, for example, if x is a `double` and n is a `long` then the assignment `n = x` and the type conversion coerced by the cast `(long) x` will both produce $\lfloor x \rfloor$ if $x \geq 0.0$ or $\lceil x \rceil$ if $x < 0.0$. Therefore, one has to be careful about the sign of x if a float-to-int conversion is used to compute the floor or ceiling function. As an alternative, although the functions `floor(x)` and `ceil(x)` in the standard library `<math.h>` return a floating point result, they are otherwise a correct implementation of $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively, even if x is negative.

An important characterization that is used at several places in the body of this text is that for any number x, integer or not

- $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

So, for example, $3 = \lfloor 3.4 \rfloor \leq 3.4 < \lfloor 3.4 \rfloor + 1 = 4$. Moreover, for any integer n

- $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

but, in general,

- $\lfloor nx \rfloor \neq n\lfloor x \rfloor$.

So, for example, $\lfloor 3.4 + 8 \rfloor = \lfloor 3.4 \rfloor + 8 = 11$ but $\lfloor 8 \cdot 3.4 \rfloor = \lfloor 27.2 \rfloor = 27 \neq 8 \cdot \lfloor 3.4 \rfloor = 24$.

Modulus Function

Definition B.2 Relative to Theorem B.1 and Definition B.1, if b is an integer and a is a positive integer then the *remainder* is $r = b - \lfloor b/a \rfloor a$ where the *quotient* is $q = \lfloor b/a \rfloor$. Equivalently, the *modulus* (mod) function is effectively defined by the division theorem, with an implementation based on the floor function as

$$b \bmod a = b - \lfloor b/a \rfloor a.$$

Example B.4 As in Example B.1, if $(a, b) = (7, 17)$ then the quotient is $q = \lfloor 17/7 \rfloor = 2$ and the remainder is $r = 17 \bmod 7 = 3$. Provided a is a positive integer and b is a non-negative integer, in ANSI C the computation of q and r is direct as

```
q = b / a;                                          /* q is ⌊b/a⌋ */
r = b % a;                                          /* r is b mod a */
```

If, however, b (or a) is a negative integer then the ANSI C standard allows the computation of q and r to be implementation dependent. That is, in this case the value of q and r may be different from the mathematical definition of $\lfloor b/a \rfloor$ and $b \bmod a$.*

Although the following theorem is not as fundamental as either Theorem B.1 or B.3 (presented later), it is an important result that relates directly to the implementation of Lehmer random number generators introduced in Section 2.1. From an implementation perspective, the significance of this theorem is that the mod function can be applied term-by-term to help prevent integer overflow in intermediate calculations.

Theorem B.2 If a is a positive integer and $b_1, b_2, \ldots, b_n$ are integers then

- $(b_1 + b_2 + \cdots + b_n) \bmod a = \big((b_1 \bmod a) + (b_2 \bmod a) + \cdots + (b_n \bmod a)\big) \bmod a$;

- $(b_1 b_2 \cdots b_n) \bmod a = \big((b_1 \bmod a)\,(b_2 \bmod a) \cdots (b_n \bmod a)\big) \bmod a$.

Example B.5 If a is a positive integer and b, c are integers then because $a \bmod a = 0$ and $0 \le b \bmod a < a$ it follows that

$$
\begin{aligned}
(b + ac) \bmod a &= ((b \bmod a) + (ac \bmod a)) \bmod a \\
&= ((b \bmod a) + ((a \bmod a)(c \bmod a) \bmod a)) \bmod a \\
&= (b \bmod a) \bmod a \\
&= b \bmod a.
\end{aligned}
$$

---

* It is the case, however, that the division and remainder operations are consistent in that the value of `(b / a) * a + (b % a)` is required to be equal to b for all integer values of a and b, provided a is not zero.

Divisors and Primes

Definition B.3   If r is zero in Theorem B.1, so that $b = aq$, then a is said to be a *divisor* (or *factor*) of b. Equivalently, a is said to *divide* b. (Although not used in this book, the notation $a \mid b$ is commonly used to denote "a divides b.")

Example B.6   Given the positive integers a, b, and c:

- if a divides b and a divides c then a divides $b + c$;

- if a divides b and a divides c with $b > c$ then a divides $b - c$;

- if a divides b or a divides c then a divides bc;

- if a divides b and b divides c then a divides c.

Definition B.4   A positive integer $p > 1$ is *prime* if and only if the only positive integers that divide p are p and 1.*

Example B.7   By convention, 1 is not a prime integer. The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ... The program `sieve`, described subsequently, implements the classic *prime sieve of Eratosthenes*, a simple algorithm for finding all the prime numbers between 2 and a user-specified upper limit.

Theorem B.3   *Fundamental Theorem of Arithmetic* — any positive integer $n > 1$ can be uniquely written as
$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$
where $p_1 < p_2 < \cdots < p_r$ are the r distinct prime integer divisors of n with corresponding positive integer exponents k_1, k_2, ..., k_r.

Example B.8   Relative to Theorem B.3, if n is prime then $r = 1$ and $p_1 = n$ with $k_1 = 1$. If n is not prime then $r \geq 2$. For example

$$38 = 2 \cdot 19$$
$$39 = 3 \cdot 13$$
$$40 = 2^3 \cdot 5$$
$$41 = 41 \qquad (41 \text{ is prime})$$
$$60 = 2^2 \cdot 3 \cdot 5$$
$$2100 = 2^2 \cdot 3 \cdot 5^2 \cdot 7$$

In general, given a table of primes between 2 and $\lfloor \sqrt{n} \rfloor$ (see program `sieve`), the prime factors of n can be determined by systematically trying 2, 3, 5, 7, 11, ..., in order.

---

* A positive integer that is not prime is said to be *composite*.

Definition B.5   If a, b are positive integers then the positive integer d is a *common divisor* if and only if d divides both a and b. The largest such common divisor is called the *greatest common divisor*, denoted $\gcd(a, b)$. In general, $\gcd(a, b) = \gcd(b, a)$.

Example B.9   The two integers 12, 30 have 2, 3, and 6 as common divisors. The largest of these is 6 and so $\gcd(12, 30) = 6$. Note that $\gcd(12, 24) = 12$.

Definition B.6   The two positive integers a, b are *relatively prime* if and only if a, b have no common prime divisors, or, equivalently, $\gcd(a, b) = 1$.

Example B.10   The integers 10, 18 are not relatively prime because they have 2 as a common prime factor. Indeed, $\gcd(10, 18) = 2$. The integers 10, 21 are relatively prime because they have no common prime factors, i.e., $\gcd(10, 21) = 1$.

Example B.11   If either of the distinct positive integers a, b (or both) are prime then a, b are relatively prime, i.e., $\gcd(a, b) = 1$.

   The definition of the greatest common divisor of two positive integers a and b is clear. Determining the greatest common divisor is not. The *Euclidean Algorithm* can be used to compute the greatest common divisor of a and b.

Algorithm B.1   *Euclidean Algorithm* — Given two positive integers a, b this algorithm computes the greatest common division $\gcd(a, b)$

```
r = a % b;
while (r > 0) {
    a = b;
    b = r;
    r = a % b;
};
return b;
```

Theorem B.4   (Fermat's "little" theorem) If m is prime and a is an integer such that a/m is not an integer, then
$$a^{m-1} \bmod m = 1.$$

A small variation allows this theorem to be stated with a simpler hypothesis.

Theorem B.5   (Fermat's "little" theorem, second statement) If m is prime, then

$$a^m \bmod m = a \bmod m$$

for all integers a.

Sieve of Eratosthenes

The discussion thus far concerning prime numbers has focused on the definition of a prime number and associated results rather than how to find prime numbers. Named after a Greek scientist who devised the algorithm, the sieve of Eratosthenes finds all prime numbers between 2 and some specified positive integer N. We begin with a conceptual description of the sieve.

Assume that we want a list of all the prime numbers between 2 and N. One way to proceed is as follows. For illustration, assume that $N = 100$. First, write all the integers between 2 and 100 in order:

$$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad \ldots \quad 100$$

The sieve works from left to right on this string of integers in the following fashion. A pointer initially points to 2. We know that 2 is prime, but all other even numbers can't be prime, so we cross out $4, 6, 8, \ldots, 100$.

$$\underset{2}{\downarrow} \quad 3 \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad \cancel{8} \quad 9 \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \quad \cancel{14} \quad 15 \quad \ldots \quad \cancel{100}$$

The pointer now advances to the next integer that has not been crossed out: 3, which is prime. The multiples of 3, namely $6, 9, 12, \ldots, 99$ can't be prime, so they are crossed out. (This will result in the even multiples of 3 being crossed out twice.)

$$2 \quad \underset{3}{\downarrow} \quad \cancel{4} \quad 5 \quad \cancel{6} \quad 7 \quad \cancel{8} \quad \cancel{9} \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \quad \cancel{14} \quad \cancel{15} \quad \ldots \quad \cancel{100}$$

The pointer now advances to 5, the next integer that has not been crossed out. The integers $10, 15, 20, \ldots, 100$ can't be prime, so they are crossed out.

$$2 \quad 3 \quad \cancel{4} \quad \underset{5}{\downarrow} \quad \cancel{6} \quad 7 \quad \cancel{8} \quad \cancel{9} \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \quad \cancel{14} \quad \cancel{15} \quad \ldots \quad \cancel{100}$$

Finally, the pointer is advanced to 7, and the integers $14, 21, 28, \ldots, 98$ are crossed out.

$$2 \quad 3 \quad \cancel{4} \quad 5 \quad \cancel{6} \quad \underset{7}{\downarrow} \quad \cancel{8} \quad \cancel{9} \quad \cancel{10} \quad 11 \quad \cancel{12} \quad 13 \quad \cancel{14} \quad \cancel{15} \quad \ldots \quad \cancel{100}$$

The sieve only needs to advance the pointer to $\sqrt{N} = \sqrt{100} = 10$ to cross out all of the composite integers between 2 and 100. The 25 prime numbers that remain are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.$$

At the specification level, we define an array `prime` that will eventually contain 0 for composite numbers, and 1 for prime numbers. To initialize, `prime[0]` and `prime[1]` are set to zero, and `prime[2]` through `prime[N]` are set to one. The pointer from the conceptual development is the `for` loop index n. When ever a prime number n is encountered, `prime[2 * n]`, `prime[3 * n]`, etc. are set to zero.

Algorithm B.2 Given some positive integer N, this algorithm implements the sieve of Eratosthenes, assigning the array `prime` to binary values (0 for composite, 1 for prime) in order to indicate all of the prime numbers between 2 and N.

```
prime[0] = 0;                                    /* zero is composite */
prime[1] = 0;                                    /* one is composite */
for (n = 2;  n <= N;  n++)
   prime[n] = 1;
for  (n = 2; n <= √N; n++)
   if (prime[n])
       for (s = 2;  s <= (N/n);  s++)
          prime[s * n] = 0;
```

Program `sieve` implements this algorithm and prints the primes between 2 and N in a tabular format.

The operation of the sieve of Eratosthenes suggests that primes may be rarer for larger integers. Experimentation indicates that this may indeed be the case. There are 25 primes between 1 and 100, 21 primes between 101 and 200, 16 primes between 1001 and 1100, 11 primes between 10 001 and 10 100, and only 6 primes between 100 001 and 100 100. Do we ever "run out" of primes (i.e., is there a "largest" prime)? The answer is no. There are an infinite number of primes, as proved by Euclid. The distribution of primes over the positive integers is a central topic in "analytic number theory." The *prime number theorem*, which was proved independently by Hadamard and de la Vallée–Poussin, states that

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln(x)} = 1$$

where $\pi(x)$ is the number of primes less than of equal to x.

The presentation of integer arithmetic given in this appendix is intentionally elementary. For more information, we recommend the following texts, in order of increasing level of sophistication: Chapter 3 of Epp (1990), Vanden Eynden (2001), Chapters 3 and 4 of Graham, Knuth, and Patashnik (1989), and Niven, Zuckerman, and Montgomery (1991).

B.1 EXERCISES

Exercise B.1 Implement the Euclidean Algorithm (Algorithm B.1). Test the algorithm with (*a*) $a = 12$ and $b = 30$. (*b*) $a = 48\,271$ and $b = 2^{31} - 1$.

Exercise B.2 Modify program `sieve` to *count* the number of primes between 1 and x, defined earlier as $\pi(x)$, in order to check the validity of the prime number theorem for $x = 10, 100, 1000, 10\,000, 100\,000$, and $1\,000\,000$.

C. Parameter Estimation Summary

This appendix contains a summary of the parameter estimates for the discrete and continuous stationary parametric models presented in this text. See

- Section 9.2.1 for a discussion of how to hypothesize a model (e.g., only use the *Poisson*(μ) distribution when $\bar{x}/s^2 \cong 1.0$);

- Section 9.2.2 for a general discussion of parameter estimation techniques (i.e., the method of moments and maximum likelihood estimation techniques whose results are listed in this appendix);

- Sections 9.2.3 and 10.1.2 for a brief introduction to goodness-of-fit tests which should be conducted after a model has been selected and fitted (e.g., the Kolmogorov–Smirnov test).

Discrete Random Variables

We have considered six discrete distributions in this book, two of which are special cases of the others. The following is a summary of parameter estimates (denoted with hats) for all six of these discrete distributions. In each case, $\bar{x}$ and s denote the sample mean and standard deviation, respectively. The integer parameter n in the *Binomial*(n, p) and the *Pascal*(n, p) distributions is not necessarily the same as the sample size.

- To estimate the two parameters in a *Binomial*(n, p) distribution use

$$\hat{n} = \left\lfloor \frac{\bar{x}^2}{\bar{x} - s^2} + 0.5 \right\rfloor \qquad \text{and} \qquad \hat{p} = \frac{\bar{x}}{\hat{n}}.$$

These estimates are valid only if $n \geq 1$ and $0.0 < p < 1.0$.

- Since the *Bernoulli*(p) distribution corresponds to the special case $n = 1$ of a *Binomial*(n, p) distribution, use

$$\hat{p} = \bar{x}.$$

- To estimate the two parameters in a *Pascal*(n, p) distribution use

$$\hat{n} = \left\lfloor \frac{\bar{x}^2}{s^2 - \bar{x}} + 0.5 \right\rfloor \qquad \text{and} \qquad \hat{p} = \frac{\bar{x}/\hat{n}}{1 + \bar{x}/\hat{n}}.$$

These estimates are valid only if $n \geq 1$ and $0.0 < p < 1.0$.

- Since the *Geometric*(p) distribution corresponds to the special case $n = 1$ of a *Pascal*(n, p) distribution, use

$$\hat{p} = \frac{\bar{x}}{1 + \bar{x}}.$$

- To estimate the one parameter in a *Poisson*(μ) distribution use

$$\hat{\mu} = \bar{x}.$$

- To estimate the two parameters in an *Equilikely*(a, b) distribution use

$$\hat{a} = \min\{x_1, x_2, \ldots, x_n\} \qquad \text{and} \qquad \hat{b} = \max\{x_1, x_2, \ldots, x_n\}.$$

Continuous Random Variables

We have considered seven continuous distributions in this book, one of which is a special case of another. The following is a summary of parameter estimates (denoted by hats) for all seven of these continuous distributions. In each case, $\bar{x}$ and s denote the sample mean and standard deviation, respectively. The integer parameter n in the $Erlang(n, b)$, $Chisquare(n)$, and the $Student(n)$ distributions is not necessarily the same as the sample size.

- To estimate the two parameters in an $Erlang(n, b)$ distribution use

$$\hat{n} = \left\lfloor \frac{\bar{x}^2}{s^2} + 0.5 \right\rfloor \qquad \text{and} \qquad \hat{b} = \frac{\bar{x}}{\hat{n}}.$$

These estimates are valid only if $n \geq 1$ and $b > 0.0$.

- Since the $Exponential(\mu)$ distribution corresponds to the special case $n = 1$ of a $Erlang(n, b)$ distribution, use

$$\hat{\mu} = \bar{x}.$$

- To estimate the two parameters in a $Normal(\mu, \sigma)$ distribution use

$$\hat{\mu} = \bar{x} \qquad \text{and} \qquad \hat{\sigma} = s.$$

- To estimate the two parameters in a $Lognormal(a, b)$ distribution use

$$\hat{a} = \frac{1}{2} \ln\left(\frac{\bar{x}^4}{\bar{x}^2 + s^2} \right) \qquad \text{and} \qquad \hat{b} = \sqrt{\ln\left(\frac{\bar{x}^2 + s^2}{\bar{x}^2} \right)}.$$

- To estimate the one parameter in a $Chisquare(n)$ distribution use

$$\hat{n} = \lfloor \bar{x} + 0.5 \rfloor.$$

This estimate is valid only if $n \geq 1$ and $s^2/\bar{x} \cong 2.0$.

- To estimate the one parameter in a $Student(n)$ distribution use

$$\hat{n} = \left\lfloor \frac{2s^2}{s^2 - 1} + 0.5 \right\rfloor.$$

This estimate is valid only if $\bar{x} \cong 0.0$ and $s > 1.0$.

- To estimate the two parameters in a $Uniform(a, b)$ distribution use

$$\hat{a} = \min\{x_1, x_2, \ldots, x_n\} \qquad \text{and} \qquad \hat{b} = \max\{x_1, x_2, \ldots, x_n\}.$$

The page number(s) in brackets after each reference refers to the page number where the reference is cited.

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