II Università di Roma, Tor Vergata
Dipartimento d'Ingegneria Civile e Ingegneria Informatica
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Problem 1 *Let Ω be the sample space of a random phenomenon and let $\mathcal{E}_1$, $\mathcal{E}_2$ algebras [resp. σ-algebras] of events of Ω. May we say that the family $\mathcal{E}_1 \cup \mathcal{E}_2$ of events of Ω given by*

$$\mathcal{E}_1 \cup \mathcal{E}_2 \overset{def}{=} \{E \in \mathcal{P}(\Omega) : E \in \mathcal{E}_1 \text{ or } E \in \mathcal{E}_2\}$$

is an algebra [resp. σ-algebra]?

Solution. Clearly, since $\mathcal{E}_1$, $\mathcal{E}_2$ algebras [resp. σ-algebras] of events of Ω, the family $\mathcal{E}_1 \cup \mathcal{E}_2$ is not empty. Now, assume that an event E is in $\mathcal{E}_1 \cup \mathcal{E}_2$, then E is in $\mathcal{E}_1$ or E is in $\mathcal{E}_2$. As a consequence, E^c is in $\mathcal{E}_1$ or E^c is in $\mathcal{E}_2$. Hence, $E^c \in \mathcal{E}_1 \cup \mathcal{E}_2$. However, assuming that E and F are in $\mathcal{E}_1 \cup \mathcal{E}_2$, unless they are both in $\mathcal{E}_1$ or $\mathcal{E}_2$, there is no reason why $E \cup F$ should be in $\mathcal{E}_1 \cup \mathcal{E}_2$. This is confirmed by the following example: with reference to the die sample space $\Omega \equiv \{\omega_1, \dots, \omega_6\}$ choose

$$\mathcal{E}_1 \equiv \{\varnothing, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\}, \Omega\} \quad \text{and} \quad \mathcal{E}_2 \equiv \{\varnothing, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \Omega\}.$$

$\mathcal{E}_1$ and $\mathcal{E}_2$ are algebras of events of Ω, but

$$\mathcal{E}_1 \cup \mathcal{E}_2 = \{\varnothing, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\} \{\omega_4, \omega_5, \omega_6\}, \Omega\}$$

is not. Note that $\mathcal{E}_1 \cup \mathcal{E}_2$ is closed with respect to the complement operator, but not with respect to the union.

Problem 2 *Let Ω be the infinite sample space of a random phenomenon. The family*

$$\mathcal{E}_{count} \equiv \{E \in \mathcal{P}(\Omega) : |E| \leq \aleph_0\}$$

of all countable events of Ω is a σ-algebra of events of Ω if and only if Ω itself is countable. In this case, we have $\mathcal{E}_{count} = \mathcal{P}(\Omega)$. On the other hand, the family $\mathcal{E}_{count\text{-}cocount}$ of all events of Ω that are countable or have countable complement, in symbols

$$\mathcal{E}_{count\text{-}cocount} \equiv \{E \in \mathcal{P}(\Omega) : |E| \leq \aleph_0 \ \vee \ |E^c| \leq \aleph_0\},$$

is a σ-algebra of events of Ω.

Solution. Given any $\omega \in \Omega$ we have

$$|\{\omega\}| = 1 \leq \aleph_0.$$

Hence, $\{\omega\} \in \mathcal{E}_{\text{count}}$. Assume that $\mathcal{E}_{\text{count}}$ is a σ-algebra, then also $\{\omega\}^c \equiv \Omega - \{\omega\}$ is in $\mathcal{E}_{\text{count}}$. By definition, it follows

$$|\Omega - \{\omega\}| \leq \aleph_0,$$

which clearly implies

$$|\Omega| \leq \aleph_0.$$

Conversely, if Ω is countable, then every $E \in \mathcal{P}(\Omega)$ is countable. This implies

$$\mathcal{P}(\Omega) \subseteq \mathcal{E}_{\text{count}},$$

that is

$$\mathcal{E}_{\text{count}} = \mathcal{P}(\Omega).$$

Trivially, $\mathcal{E}_{\text{count}}$ is σ-algebra of events of Ω. As a consequence of the above argument, if Ω is not countable, that is

$$|\Omega| > \aleph_0$$

or, according the continuum hypothesis,

$$|\Omega| \geq \aleph_1,$$

the family $\mathcal{E}_{\text{count}}$ cannot be a σ-algebra. On the other hand, the family $\mathcal{E}_{\text{count-cocount}}$ is. In fact, clearly $\mathcal{E}_{\text{count-cocount}} \neq \varnothing$. Furthermore, if $E \in \mathcal{E}_{\text{count-cocount}}$, according to the definition, we have two cases:

$$|E| \leq \aleph_0 \quad \text{or} \quad |E^c| \leq \aleph_0.$$

In the first case,

$$|(E^c)^c| = |E| \leq \aleph_0.$$

This implies that $E^c \in \mathcal{E}_{\text{count-cocount}}$. In the second case, we have $E^c \in \mathcal{E}_{\text{count-cocount}}$ straightforwardly. Hence, in either cases $E^c \in \mathcal{E}_{\text{count-cocount}}$. In the end, consider a sequence $(E_n)_{n\geq 1}$ of elements in $\mathcal{E}_{\text{count-cocount}}$. If $|E_n| \leq \aleph_0$ for every $n \in \mathbb{N}$, then

$$\left| \bigcup_{n\geq 1} E_n \right| \leq \aleph_0,$$

which implies $\bigcup\limits_{n\geq 1} E_n \in \mathcal{E}_{\text{count-cocount}}$. Otherwise, there exists at least $n_0 \in \mathbb{N}$ such that $|E_{n_0}| > \aleph_0$. However, in this case, since $E_{n_0} \in \mathcal{E}_{\text{count-cocount}}$, we necessarily have

$$\left|E_{n_0}^c\right| \leq \aleph_0.$$

This implies

$$\left| \left(\bigcup_{n\geq 1} E_n \right)^c \right| = \left| \bigcap_{n\geq 1} E_n^c \right| \leq \left|E_{n_0}^c\right| \leq \aleph_0.$$

Thus, it still follows that $\bigcup\limits_{n\geq 1} E_n \in \mathcal{E}_{\text{count-cocount}}$.

Problem 3 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$. *be a probability space, where* $\mathbf{P} : \mathcal{E} \to \mathbb{R}_+$ *is a countably additive probability on* Ω. *Prove that we have*

1. $\mathbf{P}\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mathbf{P}(E_k)$ *for any finite sequence* $(E_k)_{k=1}^n$ *of pairwise incompatible events in* $\mathcal{E}$*;*

2. $\mathbf{P}\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \mathbf{P}(E_n)$ *for any sequence* $(E_n)_{n\geq 1}$ *of events in* $\mathcal{E}$.

Solution.

1. Let $(E_k)_{k=1}^n$ be a finite sequence of pairwise incompatible events in $\mathcal{E}$. Then the sequence $(F_k)_{k\geq 1}$ given by

$$F_k \stackrel{\text{def}}{=} \begin{cases} E_k & \forall k = 1, \ldots, n \\ \varnothing & \forall k \geq n+1 \end{cases},$$

is a denumerable sequence of pairwise incompatible events in $\mathcal{E}$ such that

$$\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} F_k$$

and

$$\mathbf{P}(F_k) = \begin{cases} \mathbf{P}(E_k) & \forall k = 1, \ldots, n \\ 0 & \forall k \geq n+1 \end{cases}.$$

As a consequence,

$$\mathbf{P}\left(\bigcup_{k=1}^{n} E_k\right) = \mathbf{P}\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mathbf{P}(F_k) = \sum_{k=1}^{n} \mathbf{P}(F_k) = \sum_{k=1}^{n} \mathbf{P}(E_k),$$

This proves that is an additive probability

2. Let $(E_n)_{n\geq 1}$ be any sequence of events in $\mathcal{E}$. Then the sequence $(F_n)_{n\geq 1}$ given by

$$F_n \stackrel{\text{def}}{=} \begin{cases} E_1 & \text{if } n = 1 \\ E_n - \bigcup_{k=1}^{n-1} E_k & \text{if } n > 1 \end{cases},$$

is a sequence of pairwise incompatible events in $\mathcal{E}$ such that

$$\bigcup_{n\geq 1} E_n = \bigcup_{n\geq 1} F_n.$$

In fact, clearly $F_n \subseteq E_n$, for every $n \geq 1$. Hence,

$$\bigcup_{n\geq 1} F_n \subseteq \bigcup_{n\geq 1} E_n.$$

Conversely, if $x \in \bigcup_{n\geq 1} E_n$, then the set $N_x \equiv \{n \in \mathbb{N} : x \in E_n\} \neq \varnothing$. Write $\hat{n}_x \equiv \min(N_x)$. We have $x \in E_{\hat{n}_x}$. In case $\hat{n}_x = 1$, by definition we have $x \in F_1$. In case $\hat{n}_x > 1$, we have $x \notin E_k$ for every $k = 1, \ldots, \hat{n}_x - 1$. Hence, $x \notin \bigcup_{k=1}^{\hat{n}_x - 1} E_k$ and, again by definition, $x \in F_{\hat{n}_x}$. Therefore, in any case, we obtain $x \in F_{\hat{n}_x}$. This implies that $x \in \bigcup_{n\geq 1} F_n$.
Now, given $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$, we have

$$F_{n_1} \cap F_{n_2} = \varnothing.$$

In fact, assuming for instance $n_1 < n_2$, we have

$$F_{n_1} \subseteq E_{n_1} \quad \text{and} \quad F_{n_2} \cap E_{n_1} = \left(E_{n_2} - \bigcup_{k=1}^{n_2 - 1} E_k\right) \cap E_{n_1} = \varnothing.$$

This implies that the events of the sequence $(F_n)_{n\geq 1}$ are pairwise incompatible. As a consequence of the above arguments, it follows

$$\mathbf{P}\left(\bigcup_{n\geq 1} E_n\right) = \mathbf{P}\left(\bigcup_{n\geq 1} F_n\right) = \sum_{n\geq 1} \mathbf{P}(F_n) \leq \sum_{n\geq 1} \mathbf{P}(E_n),$$

which is the desired result.

Problem 4 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *a probability space and let* $E, F \in \mathcal{E}$ *such that*

$$\mathbf{P}(E) + \mathbf{P}(F) \geq 1. \tag{1}$$

Prove that

$$\mathbf{P}(E) + \mathbf{P}(F) - 1 \leq \mathbf{P}(E \cap F) \leq \min\{\mathbf{P}(E), \mathbf{P}(F)\} \tag{2}$$

Determine a similar lower and upper bound for $\mathbf{P}(E \cap F)$ *under the assumption*

$$\mathbf{P}(E) + \mathbf{P}(F) < 1. \tag{3}$$

Solution. First, observe that

$$E \cap F \subseteq E \qquad \text{and} \qquad E \cap F \subseteq F. \tag{4}$$

This implies

$$\mathbf{P}(E \cap F) \subseteq \mathbf{P}(E) \qquad \text{and} \qquad \mathbf{P}(E \cap F) \subseteq \mathbf{P}(F).$$

It clearly follows

$$\mathbf{P}(E \cap F) \leq \min\{\mathbf{P}(E), \mathbf{P}(F)\}. \tag{5}$$

Second, we have

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F), \tag{6}$$

which implies

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cup F). \tag{7}$$

On the other hand,

$$\mathbf{P}(E \cup F) \leq 1. \tag{8}$$

Combining (7) and (8) we then obtain

$$\mathbf{P}(E) + \mathbf{P}(F) - 1 \leq \mathbf{P}(E \cap F). \tag{9}$$

From (5) and (9) the desired (2) immedately follows. Note that in case

$$E \cap F = \varnothing \qquad \text{and} \qquad E \cup F = \Omega,$$

that is

$$F = E^c,$$

we have both

$$\mathbf{P}(E) + \mathbf{P}(F) - 1 = 0 \qquad \text{and} \qquad \mathbf{P}(E \cap F) = 0.$$

Therefore, the lower bound in (2) is achieved. Instead, in case

$$E \subseteq F \qquad \text{or} \qquad F \subseteq E,$$

we have

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) \quad \text{and} \quad \min\{\mathbf{P}(E), \mathbf{P}(F)\} = \mathbf{P}(E)$$

or

$$\mathbf{P}(E \cap F) = \mathbf{P}(F) \quad \text{and} \quad \min\{\mathbf{P}(E), \mathbf{P}(F)\} = \mathbf{P}(F).$$

Hence, the upper bound in (2) is achieved. Now, under Assumption (3), combining (7) and (5) we obtain

$$0 \leq \mathbf{P}(E \cap F) \leq \min\{\mathbf{P}(E), \mathbf{P}(F)\},$$

where the lower bound is achieved when $E \cap F = \varnothing$ while the upper bound $\min\{\mathbf{P}(E), \mathbf{P}(F)\}$ is still achieved when $E \subseteq F$ or $F \subseteq E$. On the other hand, still under Assumption (3), from Equation (7) it follows

$$\mathbf{P}(E \cap F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cup F) < 1 - \mathbf{P}(E \cup F). \tag{10}$$

Therefore, one wonders whether a sharper bound for $\mathbf{P}(E \cap F)$ is given by $\min\{\mathbf{P}(E), \mathbf{P}(F)\}$ or $1 - \mathbf{P}(E \cup F)$. Considering the possibility

$$\min\{\mathbf{P}(E), \mathbf{P}(F)\} < 1 - \mathbf{P}(E \cup F), \tag{11}$$

in case $F = E^c$, we obtain

$$\mathbf{P}(E \cup F) = 1.$$

Thus, Equation (11) would imply

$$\min\{\mathbf{P}(E), \mathbf{P}(F)\} < 0.$$

which is clearly false. However, considering the possibility

$$1 - \mathbf{P}(E \cup F) < \min\{\mathbf{P}(E), \mathbf{P}(F)\}, \tag{12}$$

in case $E \subseteq F$, we obtain

$$1 - \mathbf{P}(E \cup F) = 1 - \mathbf{P}(F)$$

and

$$\min\{\mathbf{P}(E), \mathbf{P}(F)\} = \mathbf{P}(E).$$

Hence, from (12) we would obtain

$$1 - \mathbf{P}(F) < \mathbf{P}(E),$$

that is

$$1 < \mathbf{P}(E) + \mathbf{P}(F),$$

which is also clearly false. As a consequence, under Assumption (3), both $\min\{\mathbf{P}(E), \mathbf{P}(F)\}$ and $1 - \mathbf{P}(E \cup F)$ are superior bounds for $\mathbf{P}(E \cap F)$. This implies that a sharper superior bound is given by $\min\{\mathbf{P}(E), \mathbf{P}(F), 1 - \mathbf{P}(E \cup F)\}$. Summarizing, under Assumption (3), we have

$$0 \leq \mathbf{P}(E \cap F) \leq \min\{\mathbf{P}(E), \mathbf{P}(F), 1 - \mathbf{P}(E \cup F)\}.$$

Problem 5 *Five Italian players are playing poker. The deck of poker cards contains* 36 *cards of the usual ranks* ($6, 7, 8, 9, 10, J, Q, K, A$) *and of the usual suites (hearts* $\heartsuit$*, clubs* $\spadesuit$*, diamonds* $\diamondsuit$*, flowers* $\clubsuit$*).*

1. *How many hands are possible by a random deal?*
2. *How many hands give a straight flush by a random deal?*
3. *How many hands give a four of a kind by a random deal?*
4. *How many hands give a flush by a random deal?*
5. *How many hands give a full house by a random deal?*

6. *How many hands give a straight by a random deal?*

7. *How many hands give a three of a kind by a random deal?*

8. *How many hands give two pair by a random deal?*

9. *How many hands give one pair by a random deal?*

10. *How many hands give no pair by a random deal?*

11. *How many hands fail to give any of the above combinations by a random deal?*

12. *What about if the players are Americans? In this case the deck of poker card contains* 56 *cards of the usual ranks* $(1, \ldots, 10, J, Q, K, A)$ *and of the usual suites.*

Solution.

1. Since a poker hand is indifferent to the order in which is arranged by the deal, the number of all possible hands is just the number of all possible subsets of 5 elements that can be selected from a set of 36 elements. Hence,
$$\binom{36}{5}$$
is the number of all possible hands.

2. According to the (Italian) poker rules, there are 6 possibilities for choosing the rank of the first card of a straight. The ranks of the other cards are then consequently determined. That is the ony possible straights in a deck of 36 cards are
$$A, 6, 7, 8, 9; \quad 6, 7, 8, 9, 10; \quad ; \ldots ; \quad 10, J, Q, K, A.$$
We have a straight flush when the cards of the straight have all the same suits. We can choose the suit for the straight flush in 4 differeent ways. Hence, we have
$$4 \cdot 6$$
possible hands giving a straight flush by a random deal.

3. There are 9 possibility for choosing the rank of card for the four of a kind, once the card has been chosen there is no room for the choice of the suites. Then, there are 8 possibilities for choosing the rank of fifth card and for each rank there are $\binom{4}{1} = 4$ possibilities for choosing the suits. As a consequence, we have
$$9 \cdot 8 \cdot 4$$
possible hands giving a four of a kind by a random deal.

4. There are $\binom{4}{1} = 4$ possibilities for choosing the suits of the flush and $\binom{9}{5} - 6$ possibilities for choosing the rank of the cards in the flush discarding the 6 possible straights (which is necessary to discard to avoid straight flushes). Therefore, we have
$$4 \cdot \left(\binom{9}{5} - 6\right)$$
possible hands giving a flush by a random deal.

5. There are 9 possibilities for choosing the rank of the card for the three of a kind in a full house and, for each choice of this rank, we have $\binom{4}{3}$ possible choices for the suites. Hence, we have 8 possibilities for choosing the rank of the card for the pair in a full house and, for each choice of this rank, we have $\binom{4}{2}$ possible choices for the suites. In the end, we have

$$9 \cdot \binom{4}{3} \cdot 8 \cdot \binom{4}{2}$$

possible hands giving a full house by a random deal.

6. As discusse above, following the (Italian) poker rules, there are 6 possibilities for choosing the rank of the first card of a straight. The ranks of the other cards are then consequently determined. Then choosing for each card of the straight one of the possible suites we build all possible straights, including the straight flushes, which have to be discarded. Therefore. we have

$$6 \cdot 4^5 - 24$$

possible hands giving a straight wich is not a straight flush by a random deal.

7. There are 9 possibilities for choosing the rank of the card for a three of a kind and for each choice of this rank, we have $\binom{4}{3}$ possible choices for the suites of the cards constituting the three of a kind. Now, we need to choose two additional ranks for the two remaining cards which should not constitute a pair. This can be done in $\binom{8}{2}$ ways, and we can choose the suits of each remaining card in 4 ways. Therefore there are

$$9 \cdot \binom{4}{3} \cdot \binom{8}{2} \cdot 4^2$$

possible hands giving a three of a kind by a random deal.

8. There are $\binom{9}{2}$ possibilities for choosing the rank of the cards for the two pairs and for each choice of these ranks we have $\binom{4}{2}$ possible choices for the suites. The rank of the last card of the hand can be chosen among the remaining 7 ranks and the suits of the last card can be chosen in 4 ways. Hence, we have

$$\binom{9}{2} \cdot \binom{4}{2}^2 \cdot 7 \cdot 4$$

possible hands giving two pair by a random deal.

9. There are 9 possibilities for choosing the rank of the cards for the pair and for each choice of this rank we have $\binom{4}{2}$ possible choices for the suites. Then the rank of the remaining three cards have to be chosen to be different, which can be done in $\binom{8}{3}$ ways, and the suits of each of the remaining three cards can be chosen in 4 different. Therefore, we have

$$9 \cdot \binom{4}{2} \cdot \binom{8}{3} \cdot 4^3$$

possible hands giving a pair by a random deal.

Problem 6 *An urn contains N distinguishable balls of which M are white, with $1 \leq M < N$, and the remaining $N - M$ are black. The urn is shaken and the balls are drawn from the urn one after the other without replacement. How many of the possible drawn sequences show the first white ball at the kth draw?*

Ans.

$$\binom{N-M}{k-1} (k-1)! M (N-k)!$$

Solution. Since the balls are distinguishable it is convenient to think in terms of permutations. In fact, we can assume that the white balls are numbered from 1 to M and the white balls are numbered from $M+1$ to N. If we want the first white ball at the kth attempt, we have to start drawing $k-1$ black balls, and this can be done in $\binom{N-M}{k-1}$ ways. The chosen balls can be arranged in $(k-1)!$ different orders. The red ball at the kth place can be choosen in M different ways and the remaining $N-k$ balls can be arranged in $(n-k)!$ different orders.

Problem 7 *An urn contains N balls of which M are white, with $1 \leq M < N$, and the remaining $N-M$ are black. The urn is shaken and the balls are drawn one after the other without replacement. Suppose that both the white balls and the black ones are undistinguishable among them. How many of the possible drawn sequences show the first white ball at the kth draw?*

Ans.

$$\binom{N-k}{M-1}$$

Solution. The drawn sequences which contain the first white ball at the kth draw are distinguishable just for the position of the other $M-1$ white balls after the kth draw. So each of them identifies with the choice of $M-1$ places from the $N-k$ available ones.

Problem 8 *An urn contains N distinguishable balls of which M are white, with $1 \leq M < N$, and the remaining $N-M$ are black. The urn is shaken and k balls are drawn without replacement. If $k \leq M$, how many of the possible unordered samples contains $j \leq k$ white balls and $k-j$ black balls?*

Ans.

$$\binom{M}{j}\binom{N-M}{k-j}.$$

Solution. .

Problem 9 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $E, F \in \mathcal{E}$. Show that E and F are independent if and only if:*

1. *E^c and F^c are independent;*
2. *E and F^c are independent;*
3. *E^c and F are independent.*

Solution. We need to prove that

$$\begin{aligned}
\mathbf{P}(E \cap F) = \mathbf{P}(E)\mathbf{P}(F) & \\
& \Leftrightarrow \mathbf{P}\left(E^c \cap F^c\right) = \mathbf{P}\left(E^c\right)\mathbf{P}\left(F^c\right) \\
& \Leftrightarrow \mathbf{P}\left(E \cap F^c\right) = \mathbf{P}\left(E\right)\mathbf{P}\left(F^c\right) \\
& \Leftrightarrow \mathbf{P}\left(E^c \cap F\right) = \mathbf{P}\left(E^c\right)\mathbf{P}\left(F\right).
\end{aligned}$$

First, assume that $\mathbf{P}(E \cap F) = \mathbf{P}(E)\mathbf{P}(F)$ and consider $\mathbf{P}(E^c \cap F^c)$. On account of the properties of the probability function, we can write

$$\begin{aligned}
\mathbf{P}(E^c \cap F^c) &= \mathbf{P}((E \cup F)^c) \\
&= 1 - \mathbf{P}(E \cup F) \\
&= 1 - (\mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)) \\
&= 1 - (\mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E)\mathbf{P}(F)) \\
&= 1 - \mathbf{P}(E) - \mathbf{P}(F)(1 - \mathbf{P}(E)) \\
&= (1 - \mathbf{P}(E))(1 - \mathbf{P}(F)) \\
&= \mathbf{P}(E^c)\mathbf{P}(F^c).
\end{aligned}$$

Second, assume that $\mathbf{P}(E^c \cap F^c) = \mathbf{P}(E^c)\mathbf{P}(F^c)$ and consider $\mathbf{P}(E \cap F^c)$. We can write

$$\mathbf{P}(F^c) = \mathbf{P}(\Omega \cap F^c) = \mathbf{P}((E \cup E^c) \cap F^c) = \mathbf{P}((E \cap F^c) \cup (E^c \cap F^c)) = \mathbf{P}(E \cap F^c) + \mathbf{P}(E^c \cap F^c).$$

Therefore,

$$\mathbf{P}(F^c) = \mathbf{P}(E \cap F^c) + \mathbf{P}(E^c)\mathbf{P}(F^c).$$

It then follows

$$\mathbf{P}(E \cap F^c) = (1 - \mathbf{P}(E^c))\mathbf{P}(F^c) = \mathbf{P}(E)\mathbf{P}(F^c).$$

Third, assume that $\mathbf{P}(E \cap F^c) = \mathbf{P}(E)\mathbf{P}(F^c)$ and consider $\mathbf{P}(E^c \cap F)$. We can write

$$\begin{aligned}
\mathbf{P}(E^c \cap F) &= 1 - \mathbf{P}((E^c \cap F)^c) = 1 - \mathbf{P}(E \cup F^c) = 1 - (\mathbf{P}(E) + \mathbf{P}(F^c) - \mathbf{P}(E \cap F^c)) \\
&= 1 - (\mathbf{P}(E) + \mathbf{P}(F^c) - \mathbf{P}(E)\mathbf{P}(F^c)) = 1 - (\mathbf{P}(E)(1 - \mathbf{P}(F^c)) + \mathbf{P}(F^c)) \\
&= 1 - \mathbf{P}(E)(1 - \mathbf{P}(F^c)) - \mathbf{P}(F^c) = (1 - \mathbf{P}(E))(1 - \mathbf{P}(F^c)) = \mathbf{P}(E^c)\mathbf{P}((F^c)^c) \\
&= \mathbf{P}(E^c)\mathbf{P}(F).
\end{aligned}$$

Fourth, assume that $\mathbf{P}(E^c \cap F) = \mathbf{P}(E^c)\mathbf{P}(F)$ and consider $\mathbf{P}(E \cap F)$. We can write

$$\begin{aligned}
\mathbf{P}(F) &= \mathbf{P}(F \cap \Omega) = \mathbf{P}(F \cap (E \cup E^c)) = \mathbf{P}((E \cap F) \cup (E^c \cap F)) = \mathbf{P}(E \cap F) + \mathbf{P}(E^c \cap F) \\
&= \mathbf{P}(E \cap F) + \mathbf{P}(E^c)\mathbf{P}(F).
\end{aligned}$$

Hence,

$$\mathbf{P}(F) = \mathbf{P}(E \cap F) + \mathbf{P}(E^c)\mathbf{P}(F).$$

As a consequence,

$$\mathbf{P}(E \cap F) = (1 - \mathbf{P}(E^c))\mathbf{P}(F) = \mathbf{P}((E^c)^c)\mathbf{P}(F) = \mathbf{P}(E)\mathbf{P}(F).$$

This case closes the chain of implications which allow us to conclude that the desired equivalences hold true. □

Problem 10 *A urn, say Urn A, contains* 5 *white balls and* 10 *black balls. Another urn, say Urn B, contains* 3 *white balls and* 12 *black balls. A fair coin is tossed. If it shows heads [resp. tails] a ball is drawn from Urn A [resp. B]. Suppose that this random experiment has been done and we know that a white ball has been drawn. What is the probability that the ball has been drawn from Urn A? What is the probability that the ball has been drawn from Urn B?*

Solution. .

Problem 11 *In a large town, after a robbery, a thief jumped into a cab and disappeared. An eyewitness on the crime scene told the police that the cab was yellow. Having some doubt on the reliability of the eyewitness, the police consulted a mathematician. Assuming that*

1. 20% *of the cabs in the town are yellow;*
2. *from the past experience police knows that an eyewitness is* 80% *accurate, that is an eyewitness identifies correctly whather the colour of a taxi is yellow or not* 8 *out* 10 *times.*

   Compute the probability that the information reported by the eyewitness is true. That is the probability that the cab was yellow given that the eyewitness said so. Do you think this information is useful?

   Hint: consider the events "the cab was yellow", "the cab was not yellow", "the eyewitness says the cab was yellow", and "the eyewitness says the cab was not yellow" and formulate in terms of conditional probability the accuracy of the eyewitness.

Solution. Consider the events "the cab was yellow" and "the cab was not yellow", which we may denote by $\{C = 1\}$ and $\{C = 0\}$, respectively. We have

$$\mathbf{P}(C = 1) = 0.2 \quad \text{and} \quad \mathbf{P}(C = 0) = 0.8.$$

Hence consider the events "the eyewitness says the cab was yellow" and "the eyewitness says the cab was not yellow", which we may denote by $\{E = 1\}$ and $\{E = 0\}$, respectively. The 80% of accuracy of the eyewitness means that

$$\mathbf{P}(E = 1 \mid C = 1) = \mathbf{P}(E = 0 \mid C = 0) = 0.8.$$

Our goal is to compute

$$\mathbf{P}(C = 1 \mid E = 1).$$

By the symmetry formula, we have

$$\mathbf{P}(C = 1 \mid E = 1) = \frac{\mathbf{P}(E = 1 \mid C = 1)\,\mathbf{P}(C = 1)}{\mathbf{P}(E = 1)}.$$

Now, on account of the total probability form, we have

$$\begin{aligned}
\mathbf{P}(E = 1) &= \mathbf{P}(E = 1 \mid C = 1)\,\mathbf{P}(C = 1) + \mathbf{P}(E = 1 \mid C = 0)\,\mathbf{P}(C = 1) \\
&= \mathbf{P}(E = 1 \mid C = 1)\,\mathbf{P}(C = 1) + (1 - \mathbf{P}(E = 0 \mid C = 0))\,\mathbf{P}(C = 0) \\
&= 0.8 \cdot 0.2 + (1 - 0.8) \cdot 0.8 \\
&= 0.32
\end{aligned}$$

In the end, we obtain

$$\mathbf{P}(C = 1 \mid E = 1) = \frac{0.8 \cdot 0.2}{0.32} = 0.5.$$

This means that the information provided by the eyewitness is not that useful.

Problem 12 *The National Health Service (NHS) aims to introduce a new test for the screening of a disease. The pharmaceutical company which produces the test states that:*

- *the test yields a positive result on the* 95% *of people who are affected by the disease (*sensitivity *or* true positive rate *of the test);*
- *the test yields a negative result on the* 99% *of people who are not affected by the disease (*specificity *or* true negative rate *of the test);*

On the other hand, the NHS knows the the disease is currently affecting the 10% *of the population. Compute:*

1. *the probability that a randomly chosen individual of the population is affected by the disease given that the test yields a positive result;*
2. *the probability that a randomly chosen individual of the population is not affected by the disease given that the test yields a positive result;*
3. *the probability that a randomly chosen individual of the population is affected by the disease given that the test yields a negative result;*
4. *the probability that a randomly chosen individual of the population is not affected by the disease given that the test yields a negative result;*
5. *the probability that the test yields a positive result on a randomly chosen individual of the population.*
6. *the probability that the test yields a negative result on a randomly chosen individual of the population.*

Solution. Write D [resp. H] for the event "a randomly chosen individual of the population is affected [resp. not affected] by the disease". We have

$$D \cup H = \Omega \quad \text{and} \quad D \cap H = \varnothing.$$

It follows

$$\mathbf{P}(D) + \mathbf{P}(H) = 1. \tag{13}$$

Write T_+ [resp. T_-] for the event "the test yields a positive [resp. negative] result on a randomly chosen individual of the population". We have

$$T_+ \cup T_- = \Omega \quad \text{and} \quad T_+ \cap T_- = \varnothing.$$

It follows

$$\mathbf{P}(T_+) + \mathbf{P}(T_-) = 1. \tag{14}$$

In terms of the above notations, the information provided by the pharmaceutical company means

$$\mathbf{P}(T_+ \mid D) = 0.95, \qquad \text{and} \qquad \mathbf{P}(T_- \mid H) = 0.99. \tag{15}$$

The information provided by NHS is

$$\mathbf{P}(D) = 0.10, \tag{16}$$

which clearly implies

$$\mathbf{P}(H) = 0.90. \tag{17}$$

To answer Questions **??**-**??**, we need to compute the following probabilities

$$\mathbf{P}(D \mid T_+), \quad \mathbf{P}(H \mid T_+), \quad \mathbf{P}(D \mid T_-), \quad \mathbf{P}(H \mid T_-), \quad \mathbf{P}(T_+), \quad \mathbf{P}(T_-),$$

respectively. Now, from the symmetry formula of conditional probabilities and on account of (15)-(17), we know that

$$\mathbf{P}(D \mid T_+) = \frac{\mathbf{P}(T_+ \mid D)\,\mathbf{P}(D)}{\mathbf{P}(T_+)} = \frac{0.95 * 0.10}{\mathbf{P}(T_+)} = \frac{0.095}{\mathbf{P}(T_+)}, \tag{18}$$

$$\mathbf{P}(H \mid T_+) = \frac{\mathbf{P}(T_+ \mid H)\,\mathbf{P}(H)}{\mathbf{P}(T_+)} = \frac{0.90 * \mathbf{P}(T_+ \mid H)}{\mathbf{P}(T_+)}, \tag{19}$$

$$\mathbf{P}(D \mid T_-) = \frac{\mathbf{P}(T_- \mid D)\,\mathbf{P}(D)}{\mathbf{P}(T_-)} = \frac{0.10 * \mathbf{P}(T_- \mid D)}{\mathbf{P}(T_-)}, \tag{20}$$

$$\mathbf{P}(H \mid T_-) = \frac{\mathbf{P}(T_- \mid H)\,\mathbf{P}(H)}{\mathbf{P}(T_-)} = \frac{0.99 * 0.90}{\mathbf{P}(T_-)} = \frac{0.891}{\mathbf{P}(T_-)}, \tag{21}$$

On the other hand, we know that the conditional probability is a probability concentrated on the conditioning event. Hence, we have

$$\mathbf{P}(T_+ \mid H) = 1 - \mathbf{P}(T_- \mid H) = 1 - 0.99 = 0.01. \tag{22}$$

and

$$\mathbf{P}(T_- \mid D) = 1 - \mathbf{P}(T_+ \mid D) = 1 - 0.95 = 0.05 \tag{23}$$

Therefore, on account of (14), to answer Questions **??**-**??**, we are left with computing $\mathbf{P}(T_+)$. To this, by the Total Probability Formula, we can write

$$\begin{aligned}\mathbf{P}(T_+) &= \mathbf{P}(T_+ \mid H)\,\mathbf{P}(H) + \mathbf{P}(T_+ \mid D)\,\mathbf{P}(D)\\ &= 0.01 * 0.90 + 0.95 * 0.10 = 0.104.\end{aligned} \tag{24}$$

As a consequence,

$$\mathbf{P}(T_-) = 1 - 0.104 = 0.896. \tag{25}$$

In the end, replacing (22)-(25) into (18)-(21), we obtain

$$\begin{aligned}\mathbf{P}(D \mid T_+) &= \frac{0.095}{0.104} = 0.913,\\ \mathbf{P}(H \mid T_+) &= \frac{0.90 * 0.01}{0.104} = 8.654 \times 10^{-2},\\ \mathbf{P}(D \mid T_-) &= \frac{0.10 * 0.05}{0.896} = 5.58 \times 10^{-3},\\ \mathbf{P}(H \mid T_-) &= \frac{0.891}{0.896} = 0.994,\end{aligned}$$

which complete the answers.

Problem 13 *Machineries* 1, 2 *and* 3 *contribute to the total production of an industry with the percentages of* 50%, 30%, *and* 20% *respectively. The percentages of defective items produced by the machineries are* 2%, 4%, *and* 5% *respectively. Compute the naive probability that a randomly chosen item is defective. Compute also the naive probability that a randomly chosen item has been produced by the machinery* 1, *provided it is defective.*

Solution. Denote by D the event "the randomly chosen item is defective". Denote by M_k the event "the randomly chosen item is produced by the kth machinery", where $k = 1, 2, 3$. We are interested in computing $\mathbf{P}(D)$ and $\mathbf{P}(M_1 \mid D)$. Clearly we have

$$\mathbf{P}(M_1) = 0.50, \quad \mathbf{P}(M_2) = 0.30, \quad \mathbf{P}(M_3) = 0.20$$

and

$$\mathbf{P}(D \mid M_1) = 0.02, \quad \mathbf{P}(D \mid M_2) = 0.04, \quad \mathbf{P}(D \mid M_3) = 0.05.$$

Therefore, applying (**??**), we obtain

$$\begin{aligned}\mathbf{P}(D) &= \mathbf{P}(D \mid M_1)\mathbf{P}(M_1) + \mathbf{P}(D \mid M_2)\mathbf{P}(M_2) + \mathbf{P}(D \mid M_3)\mathbf{P}(M_3)\\ &= 0.02 \cdot 0.50 + 0.04 \cdot 0.30 + 0.05 \cdot 0.20\\ &= 0.032.\end{aligned}$$

Hence, applying the Bayes formula,

$$\mathbf{P}(M_1 \mid D) = \frac{\mathbf{P}(D \mid M_1)\mathbf{P}(M_1)}{\mathbf{P}(D)} = \frac{0.02 \cdot 0.50}{0.032} = 0.312\,5.$$

Problem 14 *Guns A and B are shooting the same target. Gun A shoots on the average* 9 *shots during the same time that gun B shoots* 10 *shots. On the average, out of* 10 *shots from gun A only* 8 *hit the target, and from gun B, only* 7. *During the shooting the target has been hit by a bullet. Compute the naive probability that the target was hit by gun A or B. Compute also the probability of hitting the target.*

Solution. Denote by G_k the events "the bullet is shot by gun k", where $k = A, B$, and denote by T the event "a bullet hits the target". We are interested in $\mathbf{P}(G_A \mid T)$, $\mathbf{P}(G_A \mid T)$, and $\mathbf{P}(T)$. We know that the target has been hit by a bullet. Hence the events G_A and G_B constitute a partition of the sure event. Then, we have

$$\mathbf{P}(G_A) + \mathbf{P}(G_B) = 1. \tag{26}$$

In addition, the assumption on the speed of firing yields

$$\mathbf{P}(G_A) = 0.9 \cdot \mathbf{P}(G_B) \tag{27}$$

Combining (26) and (27) we obtain

$$\mathbf{P}(G_A) = 0.475 \qquad \text{and} \qquad \mathbf{P}(G_B) = 0.525.$$

In addition, the assumption on the precision of firing yields

$$\mathbf{P}(T \mid G_A) = 0.8 \quad \text{and} \quad \mathbf{P}(T \mid G_B) = 0.7.$$

Therefore, thanks to the total probability formula,

$$\mathbf{P}(T) = \mathbf{P}(T \mid G_A)\mathbf{P}(G_A) + \mathbf{P}(T \mid G_B)\mathbf{P}(G_B) = 0.8 \cdot 0.9 \cdot 0.525 + 0.7 \cdot 0.525 = 0.745.$$

In the end, applying the simmetry formula, we obtain

$$\mathbf{P}(G_A \mid T) = \frac{\mathbf{P}(T \mid G_A)\mathbf{P}(G_A)}{\mathbf{P}(T)} = \frac{0.8 \cdot 0.475}{0.745} = 0.510$$

and

$$\mathbf{P}(G_B \mid T) = \frac{\mathbf{P}(T \mid G_B)\mathbf{P}(G_B)}{\mathbf{P}(T)} = \frac{0.7 \cdot 0.525}{0.745} = 0.493.$$

Problem 15 (Monty's hall) *A quiz master shows a participant three closed boxes labeled by A, B, C. One of the boxes, chosen by the quiz organisers with uniform probability, contains a prize of* \$1,000. *The remaining two are empty. The quiz master asks the participant to choose a box. Once the participant makes own choice the quiz master opens one of the two rejected boxes and shows that it is empty. Thereafter, the quiz master gives the participant the opportunity either to stick to the first choice or to exchange the chosen box with the one of the rejected boxes which is still closed. Assume that the quiz master knows what box contains the prize, the quiz master never shows a box containing the prize, and the quiz master chooses an empty box between two with uniform probability. What should the participant do? To stick to her first choice, to accept the exchange or it does not matter at all because the odds are now fifty-fifty?*

Solution. Assume the quiz participant chooses box A and thereafter the quiz master shows the empty box B or C. Denote by PX the event "the prize is in box X" for $X = A, B, C$ and EY the event "the quiz master shows empty box Y", for $Y = B, C$. Under the assumptions of the problem, we have

$$\mathbf{P}(PA) = \mathbf{P}(PB) = \mathbf{P}(PC) = \frac{1}{3}.$$

Now, since the family of events $\{PA, PB, PC\}$ is a partition of the sure event Ω, by virtue of the Total Probability Formula, we have

$$\mathbf{P}(EB) = \mathbf{P}(EB \mid PA)\,\mathbf{P}(PA) + \mathbf{P}(EB \mid PB)\,\mathbf{P}(PB) + \mathbf{P}(EB \mid PC)\,\mathbf{P}(PC) \tag{28}$$

and

$$\mathbf{P}(EC) = \mathbf{P}(EC \mid PA)\,\mathbf{P}(PA) + \mathbf{P}(EC \mid PB)\,\mathbf{P}(PB) + \mathbf{P}(EC \mid PC)\,\mathbf{P}(PC) \tag{29}$$

In addition, still the assumptions of the problem yield

$$\begin{array}{lll} \mathbf{P}(EB \mid PA) = \frac{1}{2}, & \mathbf{P}(EB \mid PB) = 0, & \mathbf{P}(EB \mid PC) = 1, \\ \mathbf{P}(EC \mid PA) = \frac{1}{2}, & \mathbf{P}(EC \mid PB) = 1, & \mathbf{P}(EC \mid PC) = 0. \end{array} \tag{30}$$

Combining (28), (29) and (30), we obtain the somewhat obvious result

$$\mathbf{P}(EB) = \mathbf{P}(EC) = \frac{1}{2}. \tag{31}$$

By (**??**), it Then, follows

$$\mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}, \quad \mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}, \tag{32}$$

and

$$\mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB|PC)\mathbf{P}(PC)}{\mathbf{P}(EB)} = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}, \quad \mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}. \tag{33}$$

Hence, whether the quiz master shows empty box B or C the conditional probability that the prize is in the other box C or B is higher than the conditional probability that the prize is contained in the initially chosen box A. We can conclude that the quiz participant should exchange the initially chosen box A with closed box C or B.

Another argument is the following. Assume the participant chooses box A. Denote by V the event "the participant wins", by PA the event "the prize is in box A" and by PA^c the event "the prize is not in box A". We have clearly

$$\mathbf{P}(PA) = \frac{1}{3}, \quad \mathbf{P}(PA^c) = \frac{2}{3}.$$

The total probability formula yields

$$\mathbf{P}(V) = \mathbf{P}(V \mid PA)\,\mathbf{P}(PA) + \mathbf{P}(V \mid PA^c)\,\mathbf{P}(PA^c) = \frac{1}{3}\mathbf{P}(V \mid PA) + \frac{2}{3}\mathbf{P}(V \mid PA^c). \tag{34}$$

Now, if the participant sticks to the choice of box A we have

$$\mathbf{P}(V \mid PA) = 1, \quad \mathbf{P}(V \mid PA^c) = 0,$$

which implies

$$\mathbf{P}(V) = \frac{1}{3}.$$

In contrast, if the participant exchanges the chosen box with the still-closed rejected box we have

$$\mathbf{P}(V \mid PA) = 0, \quad \mathbf{P}(V \mid PA^c) = 1,$$

which implies

$$\mathbf{P}(V) = \frac{2}{3}.$$

This confirms that the participant should exchange chosen box A with closed box B or C.

Problem 16 (Monty's Hall Strikes Back) *Consider Monty's Hall problem. Still assume that the quiz master knows what box contains the prize and the quiz master never shows a box containing the prize. However, in this case assume that after watching the game many times you notice that when a quiz participant chooses box A the quiz master shows empty box B [resp. C] the* 60% *[resp.* 40%*] of the times. May this information improve the quiz participant's strategy for winning the prize? What about if you notice that when a quiz participant chooses box A the quiz master shows empty box B [resp. C] the* 75% *[resp.* 25%*] of the time?*

Solution. .

Problem 17 (The Return of Monty's Hall) *Consider Monty's Hall problem. Still assume that the quiz master knows what box contains the prize, the quiz master never shows a box containing the prize, and the quiz master chooses an empty box between two with uniform probability. However, in this case assume that after watching the game many times you notice that the prize turns out to be in box A [resp. B] for* 45% *[resp.* 30%*] of the time and in box C the rest of the time. What is the quiz participant's best strategy?*

Solution. .

Problem 18 (Monty's Hall Awakens) *Consider Monty's Hall problem again. Still, assume that the quiz master knows what box contains the prize and that the quiz master never shows a box containing the prize. Assume also that after watching the game many times, you notice that the prize is in box A [resp. B] for* 50% *[resp.* 30%*] of the time and in box C the rest of the time. Moreover, assume that after a quiz participant chooses a box, the quiz master chooses a box to show between two empty boxes, by flipping a rigged coin with success probability p (the participant does not see the flip of the coins). What is the quiz participant's best strategy?*

Solution. Retaining the notation of Example 15, there are only two differences between this episode of Monty's Hall saga and Episode 17. The differences are that, from your observations, the organizers do not select the box in which to put the price by the uniform distribution, but according to the distribution

$$\mathbf{P}(PA) = 0.50 = \frac{1}{2}, \qquad \mathbf{P}(PB) = 0.30 = \frac{3}{10}, \qquad \mathbf{P}(PC) = 0.20 = \frac{1}{5},$$

and that the quiz master chooses a box between two which are empty, not by the uniform distribution, but by tossing a rigged coin. Assuming that in the favorable (heads) result of the flip, the quiz master chooses the box which comes first in the lexicografic ordere, the latter implies

$$\begin{array}{ll} \mathbf{P}(EB \mid PA) = p, & \mathbf{P}(EC \mid PA) = 1-p, \\ \mathbf{P}(EA \mid PB) = p, & \mathbf{P}(EC \mid PB) = 1-p, \\ \mathbf{P}(EA \mid PC) = p, & \mathbf{P}(EB \mid PC) = 1-p, \end{array}$$

Clearly, we still have

$$\mathbf{P}(EA \mid PA) = \mathbf{P}(EB \mid PB) = \mathbf{P}(EC \mid PC) = 0$$

As a consequence, assuming that the quiz participant's first choice is A box. Then, we also have

$$\mathbf{P}(EB \mid PC) = \mathbf{P}(EC \mid PB) = 1$$

Thus, we end up with evaluating

$$\mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{p \cdot \frac{1}{2}}{\mathbf{P}(EB)}, \quad \mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB \mid PC)\,\mathbf{P}(PC)}{\mathbf{P}(EB)} = \frac{(1-p) \cdot \frac{1}{5}}{\mathbf{P}(EB)},$$

and

$$\mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{(1-p) \cdot \frac{1}{2}}{\mathbf{P}(EC)}, \quad \mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC \mid PB)\,\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{(1-p) \cdot \frac{3}{10}}{\mathbf{P}(EC)}.$$

Therefore, if the quiz master shows the empty box C, the participant should stick to her first choice, A. But, if the quiz master shows the empty box B, the participant should stick to her first choice, A for $p > 2/7$, exchange her first choice with C for $p < 2/7$, and the exchange would be irrelevant for $p = 2/7$.

Note that the explicit computation of $\mathbf{P}(EB)$ and $\mathbf{P}(EC)$ is not necessary to determine the quiz participant's best strategy. However, by the total probability formula,we have

$$\mathbf{P}(EB) = \mathbf{P}(EB \mid PA)\,\mathbf{P}(PA) + \mathbf{P}(EB \mid PB)\,\mathbf{P}(PB) + \mathbf{P}(EB \mid PC)\,\mathbf{P}(PC) = p \cdot \frac{1}{2} + \frac{1}{5},$$

and

$$\mathbf{P}(EC) = \mathbf{P}(EC \mid PA)\,\mathbf{P}(PA) + \mathbf{P}(EC \mid PB)\,\mathbf{P}(PB) + \mathbf{P}(EC \mid PC)\,\mathbf{P}(PC) = (1-p) \cdot \frac{1}{2} + \frac{3}{10}.$$

It follows,

$$\mathbf{P}(PA \mid EB) = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{5}}, \qquad \mathbf{P}(PC \mid EB) = \frac{\frac{1}{5}(1-p)}{\frac{1}{2}p + \frac{1}{5}},$$

and

$$\mathbf{P}(PA \mid EC) = \frac{\frac{1}{2}(1-p)}{\frac{1}{2}(1-p) + \frac{3}{10}}, \qquad \mathbf{P}(PB \mid EC) = \frac{\frac{3}{10}(1-p)}{\frac{1}{2}(1-p) + \frac{3}{10}},$$

which quantify the probability of finding the prize in the different boxes corresponding to the quiz master's behavior.

Assume the quiz participant's first choice is B box. Then, we also have

$$\mathbf{P}(EA \mid PC) = \mathbf{P}(EC \mid PA) = 1.$$

In this case, we end up with evaluating

$$\mathbf{P}(PB \mid EA) = \frac{\mathbf{P}(EA \mid PB)\,\mathbf{P}(PB)}{\mathbf{P}(EA)} = \frac{p \cdot \frac{3}{10}}{\mathbf{P}(EA)}, \quad \mathbf{P}(PC \mid EA) = \frac{\mathbf{P}(EA \mid PC)\,\mathbf{P}(PC)}{\mathbf{P}(EA)} = \frac{p \cdot \frac{1}{5}}{\mathbf{P}(EA)},$$

and

$$\mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC \mid PB)\,\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{(1-p)\cdot\frac{3}{10}}{\mathbf{P}(EC)}, \quad \mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{(1-p)\cdot\frac{1}{2}}{\mathbf{P}(EC)}.$$

Therefore, if the quiz master shows the empty box A, the participant should always stick to her first choice B, but if the quiz master shows the empty box C, the participant should always exchange her first choice B with the box A.

In the end, assume the quiz participant's first choice is C box. Then, we also have

$$\mathbf{P}(EA \mid PB) = \mathbf{P}(EB \mid PA) = 1.$$

In this case, we end up with evaluating

$$\mathbf{P}(PC \mid EA) = \frac{\mathbf{P}(EA \mid PC)\,\mathbf{P}(PC)}{\mathbf{P}(EA)} = \frac{p\cdot\frac{1}{5}}{\mathbf{P}(EA)}, \quad \mathbf{P}(PB \mid EA) = \frac{\mathbf{P}(EA \mid PB)\,\mathbf{P}(PB)}{\mathbf{P}(EA)} = \frac{p\cdot\frac{3}{10}}{\mathbf{P}(EA)},$$

and

$$\mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB \mid PC)\,\mathbf{P}(PB)}{\mathbf{P}(EB)} = \frac{(1-p)\cdot\frac{3}{10}}{\mathbf{P}(EB)}, \quad \mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{p\cdot\frac{1}{2}}{\mathbf{P}(EB)}.$$

Hence, if the quiz master shows the empty box A, the participant should always exchange her first choice C with box B. But, if the quiz master shows the empty box B, then the quiz participant should stick to her first choice C for $p < 3/8$, exchange the box C with A for $p > 3/8$, and the exchange would be irrelevant for $p = 3/8$.

Problem 19 (The Last Monty's Hall) *Consider Monty's Hall problem. However, in this case assume that the quiz master does not know what box contains the prize, the quiz master chooses a box between two with uniform probability, and the quiz master shows an empty box by chance. In this episode of the Monty's Hall saga, what should the participant do? To stick to her first choice, to accept the exchange or it does not matter at all because the odds are now fifty-fifty?*

Solution. .