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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Sequences of Random Variables with Solutions 2021-11-23

Problem 1 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X be a uniformly distributed real random variable on the interval $[0,1]$. In symbols, $X \sim U(0,1)$. Consider the sequence $(Y_n)_{n\geq 1}$ of real random variables given by*

$$Y_n \stackrel{def}{=} \begin{cases} n, & \text{if } 0 \leq X < \frac{1}{n}, \\ 0, & \text{if } 1/n \leq X \leq 1, \end{cases} \qquad \forall n \geq 1.$$

Check whether the sequence $(Y_n)_{n\geq 1}$ converges in distribution, converges in probability, converges in mean, converges almost surely, in the assigned order.

Exercise 2 *Hint: to deal with the almost sure convergence consider the event $E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}$ and the complement E_0^c.*

Solution. . □

Problem 3 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ be the Euclidean real line endowed with the Borel σ-algebra. Prove that the function $f : \mathbb{R} \to \mathbb{R}_+$ given by*

$$f(x) \stackrel{def}{=} \frac{\alpha - 1}{x^\alpha} 1_{[1,+\infty)}, \quad \forall x \in \mathbb{R},$$

where $\alpha > 1$, is a density. Then, consider a random variable X with density $f_X = f$ and the sequence $(Y_n)_{n\geq 1}$ of random variables given by

$$Y_n \stackrel{def}{=} \frac{X}{n}, \quad \forall n \in \mathbb{N}.$$

Exercise 4 *Study the convergence in distribution, in probability and in p-th mean of the sequence $(Y_n)_{n\geq 1}$ on varying of $\alpha > 1$.*

Solution. .

Problem 5 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a complete probability space and let $(X_n)_{n\geq 1}$ be a sequence of independent real random variables such that $X_n \sim Ber(1/n^\alpha)$ for some $\alpha > 0$. Consider the sequence $(Y_n)_{n\geq 1}$ of real random variables on Ω given by*

$$Y_n \stackrel{def}{=} \min\{X_1, \dots, X_n\}.$$

1. *study the convergence in distribution, in probability and in $L^p(\Omega; \mathbb{R})$ of $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ on varying of $\alpha > 0$;*
2. *study the almost sure convergence of $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ on varying of $\alpha > 0$.*

Solution. .

Exercise 6 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(X_n)_{\geq n}$ *be a sequence of real random variables on* Ω. *Assume that* $(X_n)_{\geq n}$ *are identically distributed and let* $f_X : \mathbb{R} \to \mathbb{R}_+$ *their common density function given by*

$$f_X(x) \stackrel{def}{=} \frac{2}{x^3} 1_{(1,+\infty)}(x), \quad \forall x \in \mathbb{R}.$$

Set

$$Y_n \equiv \frac{X_n}{n^\alpha}, \quad \forall n \geq 1,$$

where $\alpha > 0$.

1. *Study the convergence in distribution, probability, and* L^p *of the sequence* $(Y_n)_{n\geq 1}$ *on varying of* $\alpha > 0$.
2. *Under the additional assumption of independence of the random variables of the sequence* $(X_n)_{\geq n}$, *compute* $\limsup_{n\to\infty} Y_n$ *and* $\liminf_{n\to\infty} Y_n$ *on varying of* $\alpha > 0$. *Does the sequence* $(Y_n)_{n\geq 1}$ *converge almost surely?*

Solution. . $\square$