Performance Modeling of Computer Systems and Networks

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Generating Discrete Random Variates

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Prerequisite

We assume the knowledge of discrete random variables (sect.6.1).
In particular:

- *Equilakely*(a,b)
- *Geometric*(p)
- *Bernoulli*(p)
- *Binomial*(n,p)
- *Pascal*(n,p)
- *Poisson*(μ)

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sis2 è inventory system, usa equilikely.

sis2.c

```
#include <stdio.h>
#include "rng.h"

#define MINIMUM   20
#define MAXIMUM   80
#define STOP     100     /* 100 weeks = about 2 years*/
#define sqr(x)    ((x) * (x))

long Equilikely(long a, long b)
{  return (a + (long) ((b - a + 1) * Random()));}

long GetDemand(void)
{
        return (Equilikely(10, 50)); }
```

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Noi generiamo un valore tra 0 e 1, e lo trasformiamo a seconda della variabile.

ssq2.c distribution-driven simulation

```
#include <stdio.h>
#include <math.h>
#include "rng.h"
#define LAST          10000L    /* number of jobs processed */
#define START         0.0

double Exponential(double m)          /* ------*
{return (-m * log(1.0 - Random())); }    m > 0.0
                                        ------ */
double Uniform(double a, double b)     /* ------*
{return (a + (b - a) * Random());    }    a < b
                                       * ------*/
          double GetArrival(void)
        {static double arrival = START;
          arrival += Exponential(2.0);
               return (arrival);}

          double GetService(void)
        {return (Uniform(1.0, 2.0));}
```

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X è v.a., F è la cumulativa (funzione di distribuzione).
Esiste F* che sarebbe "funzione inversa", ma formalmente non lo è. Perchè?
Perchè se voglio passare dal continuo al discreto, molti punti continui potrebbero convergere nello stesso valore discreto.

Preliminary Definitions

X random variable, $F(\cdot)$ is the cdf of X

The *inverse distribution function* (idf) of X is the function

$F^* : (0, 1) \rightarrow \chi$, $\forall\, u \in (0, 1)$

$$F^*(u) = \min_x \{x : u < F(x)\}$$

that is, if $F^*(u)=x$, x is the smallest possible value of X for which $F(x)$ is greater than u

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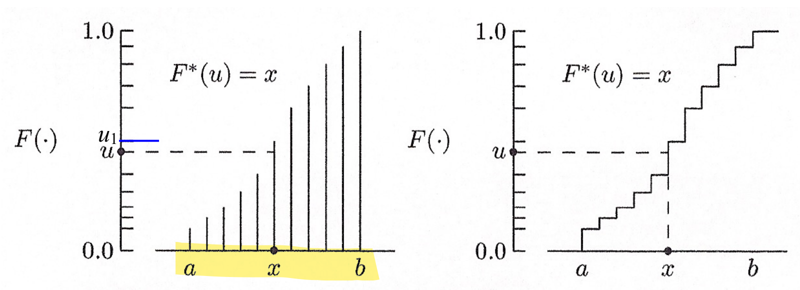
Sull'asse 'x' ho insieme discreto. Prendo un x, faccio F(x) = u1 > u. Ora faccio l'inverso,
F*(u) = "x più piccola tale che F(x) > u ". Ciò è vero se prendo sulle ascisse proprio 'x', perchè F(x) = u1 > u.
In pratica, quando faccio l'inverso di un certo "u", devo prendere la x sulle ascisse più piccola tale che, F(x) < u minore stretto.

- $\chi = \{a, a + 1, \ldots, b\}$, where b may be ∞, $F(\cdot)$ is the cdf of X,
- $F(x)=\text{Prob}\{X \leq x\}=u_1 > u$ $\qquad F^*(u) = \min_x \{x : u < F(x)\}$



Theorem
- if $u < F(a)$, $F^*(u) = a$
- else $F^*(u) = x$ where $x \in \chi$ is the unique possible value of X for which $F(x - 1) \leq u < F(x)$

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Il teorema ci dice che, presa un'ascissa "a" (che non è per forza il primo punto sull'asse x).
Se F(a) > u, allora F*(u) = a.
Altrimenti prendo un'ascissa tale che F(x' -1) <= u < F(x)

L'algoritmo 1 presenta un modo "dummy" per calcolare questa inversa.
Sostanzialmente, partendo dal minimo sulle ascisse 'a', io incremento finchè non trovo il primo valore di x tale che F(x) > u. Allora ritorno questa 'x', ovvero x = F*(u).
L'aspetto negativo è la lentezza delle iterazioni, poichè parto sempre dall'inizio.

Algorithm 1

```
x = a;
while (F(x) <= u)
        x++;
return x;            /*x is F*(u)*/
```

Average case analysis:
- let Y be the number of `while loop` passes
- $Y = X - a$
- $\mathrm{E}[Y] = \mathrm{E}[X{-}a] = \mathrm{E}[X] - a = \mu - a$

Linear search algorithm !

qui partiamo dalla "moda" (un valore più centrale), muovendoci con ricerca binaria.

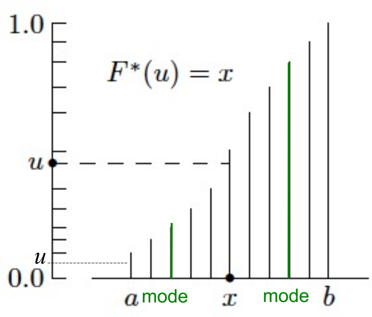
Idea: start at a more likely point

For $\chi=\{a, a+1, \ldots, b\}$, a more efficient linear search algorithm defines $F^*(u)$

Algorithm 2

```
x = mode;           /*initialize with the mode of X */
if (F(x) <= u)
    while (F(x) <= u)
        x++;    avanti
else if (F(a) <= u)
        while (F(x-1) > u)
          x--;  indietro
    else
        x = a;
return x;           /*x is F*(u)*/
```

For large χ, consider binary search



alcuni esempi di inversi calcolate.

Idf Examples

- In some cases $F^*(u)$ can be determined explicitly
- If X is *Bernoulli*(*p*) and $F(x) = u$,
  then x=0 iff $0 < u < 1$-p

$$F^*(u) = \begin{cases} 0 & 0 < u < 1 - p \\ 1 & 1 - p \le u < 1 \end{cases}$$

Random Variate Generation By Inversion

- X is a discrete random variable with idf $F^*(\cdot)$ deve essere esplicita
- continuous random variable U is *Uniform*(0,1)
- Z is the discrete random variable defined by $Z = F^*(U)$

Theorem
Z and X are identically distributed

this Theorem allows any discrete random variable (with known idf) to be generated with one call to `Random()`

Algorithm 3

```
u = Random();
return F*(u);
```

Ho questa funzione densità f(x).

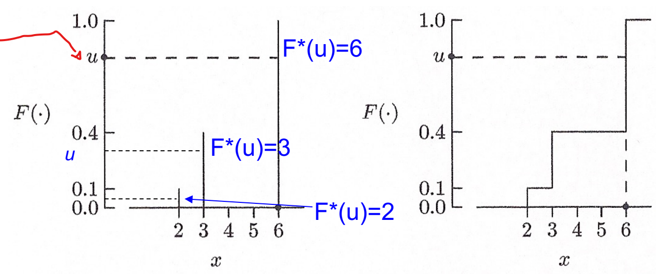
Inversion Examples

- Consider X with pdf

$$f(x) = \begin{cases} 0.1 & x = 2 \\ 0.3 & x = 3 \\ 0.6 & x = 6 \end{cases}$$

- The cdf for X is plotted using two formats

Nel grafico a sinistra, parto dall'asse y e ritorna il valore sulle x che si interseca con la 'u'. Ad esempio, per "u" vicino a 1.0 interseca F*(u) = 6



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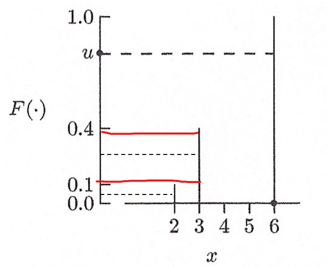
```
if (u < 0.1)
    return 2;
else if (u < 0.4)
        return 3;
    else
        return 6;
```

I valori tra 0.1 e 0.4 vanno tutti in x = 3



This algorithm returns
2 with probability 0.1,
3 with probability 0.3 and
6 with probability 0.6.
This corresponds to the pdf of X.

more efficiency: check the ranges for *u* associated with $x = 6$ first (the mode), then $x = 3$, then $x = 2$

- problems may arise when $|\mathcal{X}|$ is large or infinite

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More inversion examples

Generating a *Bernoulli*(*p*) random variate

```
u = Random();
if (u < 1-p)
        return 0;
else
        return 1;
```

Generating an *Equilikely*(*a*,*b*) random variate

```
u = Random();
return a + (long) (u * (b - a + 1));
```


Library `rvgs`

Parliamo di "variate", non "variabili", perchè le generiamo noi. (Pseudo random by Lehmer)

- Includes 6 discrete random variate generators (as below) and 7 continuous random variate generators
  - `long Bernoulli(double p)`
  - `long Binomial(long n, double p)`
  - `long Equilikely(long a, long b)`
  - `long Geometric(double p)`
  - `long Pascal(long n, double p)`
  - `long Poisson(double μ)`
- Functions Bernoulli, Equilikely, Geometric use inversion; essentially ideal
- Functions Binomial, Pascal, Poisson do not use inversion

Library `rvms`

densità, cumulativa, inversa

- Provides accurate pdf, cdf, idf functions for many random variates
- Idfs can be used to generate random variates by inversion
- Functions `idfBinomial`, `idfPascal`, `idfPoisson` may have high marginal execution times
- Not recommended when many observations are needed due to time inefficiency
- Array of cdf values with inversion may be preferred

Truncation

Sometimes, the realistic values of a variable are restricted to a subset

fino a infinito

X random variable with possible values $\chi=\{0, 1, 2, \ldots\}$ and cdf $F(x)=\Pr(X \leq x)$

- want to restrict X to the finite range $0 \leq a \leq x \leq b < \infty$
- if $a > 0$, $\alpha = \Pr(X < a), \quad \beta = \Pr(X > b)$ che cosa sto tagliando fuori? alfa coda sinistra, beta destra.

$$\alpha = \Pr(X < a) = \Pr(X \leq a\text{-}1) = F(a\text{-}1)$$

$$\beta = \Pr(X > b) = 1\text{-}\Pr(X \leq b) = 1\text{-}F(b)$$

$$\Pr(a \leq X \leq b) = \Pr(X \leq b) - \Pr(X < a) = F(b) - F(a\text{-}1)$$

essentially, always true iff $F(b) \cong 1.0$ and $F(a\text{-}1) \cong 0.0$

la differenza tra variabile teorica e troncata potrebbe essere minima, se "non taglio nulla".

Specifying truncation points

- if a and b are specified

  Ho l'argomento, devo calcolare la cumulativa per quell'argomento.

  Left-tail, right-tail probabilities α and β obtained using cdf

  $\alpha = \Pr(X < a) = F(a\text{-}1)$ and $\beta = \Pr(X > b) = 1\text{-}F(b)$

  transformation is exact

- if α and β are specified

  Ho i valori della probabilità, devo trovare l'argomento però.

  idf can be used to obtain a and b

  $a = F^*(\alpha)$ and $b = F^*(1\text{-}\beta)$

  transformation is not exact because X is discrete

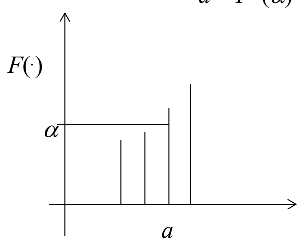
  $\Pr(X < a) \leq \alpha$ and $\Pr(X > b) < \beta$

$F(x-1) \leq u < F(x)$

Specifying truncation points

- if α and β are specified

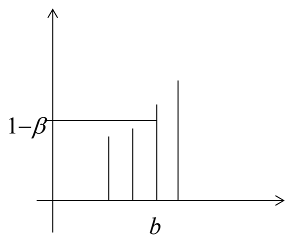
$a = F^*(\alpha)$



$F(a) > \alpha$

$\Pr(X < a) \leq \alpha$

$b = F^*(1\text{-}\beta)$



$F(b) > 1\text{-}\beta$

$\Pr(X \leq b) > 1\text{-}\beta$

$-\Pr(X \leq b) < \beta - 1$

$1\text{-}\Pr(X \leq b) < \beta$

$\Pr(X > b) < \beta$

io conosco 'alfa' (valore cumulativa) asse y.
Su asse x, incremento 'a', calcolo F(a) e il primo
a che mi eccede alfa è l'argomento cercato.

Effects of truncation

sometimes truncation is insignificant:
truncated and un-truncated random variables have (essentially) the same distribution

Truncation is useful for efficiency:

- When idf is complex, inversion requires cdf search
- cdf values are typically stored in an array
- Small range gives improved space/time efficiency

Truncation is useful for realism:

- Prevents arbitrarily large values possible from some variates

In some applications, truncation is significant

- Produces a new random variable
- Must be done correctly !

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Se "taglio poco", il troncamento non porta effetti significativi.
Se "taglio molto", potrei avere una nuova variabile random!
Devo sempre ponderare queste scelte.