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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Solved Problems on Hypothesis Tests 2023-01-17

Problem 1 *Let X be a Gaussian random variable with unknown mean μ_X and variance σ_X^2 representing a population. Assume that testing the sample mean $\bar{X}_n$ and the sample standard deviation S_n of a simple random sample $X_1, \ldots, X_n$ of size $n \equiv 9$ drawn from X we obtain the value $\bar{X}_n(\omega) \equiv \bar{x}_n = 251.50cm$ and $S_n(\omega) \equiv s_n = 2.30cm$.*

1. *Considering both the rejection region method and the p-value method, should the null hypothesis $H_0 : \mu_X = 250cm$ be rejected against the alternative $H_a : \mu_X \neq 250cm$ at the significance level $\alpha = 0.1$?*

2. *Considering both the rejection region method and the p-value method, should the null hypothesis $H_0 : \sigma_X^2 = 4$ be rejected against of the alternative $H_a : \sigma_X^2 > 4$ at the significance level $\alpha = 0.05$? Calculate the probability $\beta(5)$ of a II type error.*

Solution.

1. Since X is Gaussian distributed with unknown mean and variance and the size of the sample is small, the statistic to be used is
$$\frac{\bar{X}_n - \mu_X}{S_n/\sqrt{n}}. \tag{1}$$
Consider testing the null hypothesis $H_0 : \mu_X = \mu_0$, where $\mu_0 \equiv 250cm$, against the alternative $H_1 : \mu_X \neq \mu_0$ at the significance level $\alpha = 0.1$. Under the assumption that the null hypothesis is true the statistic (1) with $\mu_X = \mu_0$ has the Student distribution with $n-1$ degree of freedom, that is
$$\frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \sim T_{n-1}.$$
Moreover, the stucture of the alternative hypothesis calls for a rejection region of the form
$$R = \left\{T_{n-1} < t_{n-1,1-\alpha/2}\right\} \cup \left\{T_{n-1} > t_{n-1,\alpha/2}\right\}.$$
where,
$$t_{n-1,\alpha/2} = t_{8,0.05} = 1.860 \quad \text{and} \quad t_{n-1,1-\alpha/2} = -t_{n-1,\alpha/2} = -t_{8,0.05} = -1.860.$$
Hence,
$$R = (-\infty, -1.860) \cup (1.860, +\infty)$$
Computing the realization of the statistic, we have
$$\frac{\bar{X}_n(\omega) - \mu_X}{S_n(\omega)/\sqrt{n}} = \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{9}} = \frac{251.50 - 250}{2.30/3} = 1.96 \in R.$$

This implies a rejection of the null hypothesis in favor of the alternative. Adopting the p-value method, we recall that, on account of the alternative hypothesis, the p-value is the probability that the absolute value of the test statistic under the null assumption yields a value not less than the realization of the statistic. We will reject the null hypothesis when the computed p-value is smaller that the given significance level α. In symbols,

$$p = \mathbf{P}\left(|T_{n-1}| \geq \frac{\bar{x}_n - \mu_0}{s_n/\sqrt{9}} \mid H_0 = T\right) = \mathbf{P}(T_8 \leq -1.96) + \mathbf{P}(T_8 \geq 1.96) = 0.087 < 0.1,$$

which confirms the rejection of the null hypothesis.

2. Since ve are interested in testing a hypothesis on the variance of X, which is normally distributed with unknown variance and the size of the sample is small, the statistic to be used is

$$\frac{(n-1)\,S_n^2}{\sigma_X^2}. \tag{2}$$

Consider testing the null hypothesis $H_0 : \sigma_X^2 = \sigma_0^2$, where $\sigma_0^2 \equiv 4$, against the alternative $H_1 : \sigma_X^2 > \sigma_0^2$ at the significance level $\alpha \equiv 0.05$. Under the assumption that the null hypothesis is true the statistic (2) with $\sigma_X^2 = \sigma_0^2$ has the chi-square distribution with $n-1$ degrees of freedom, that is

$$\frac{(n-1)\,S_n^2}{\sigma_0^2} \sim \chi_{n-1}^2.$$

Hence, the upper tail rejection region is given by

$$R = \left\{\chi_{n-1}^2 > \chi_{n-1,\alpha}^2\right\}$$

where, the upper $\alpha = 0.05$ critical value $\chi_{n-1,\alpha}^2$ of the chi-square distribution with $n-1 = 8$ degrees of freedom is given by

$$\chi_{8,0.05}^2 \simeq 15.51.$$

Hence,

$$R = (15.51, +\infty)$$

Computing the realization of the statistic, we have

$$\frac{(n-1)\,S_n^2(\omega)}{\sigma_X^2} = \frac{(n-1)\,s_n^2}{\sigma_0^2} = \frac{8 \cdot 2.30^2}{4} = 10.58 \notin R$$

Hence, the realization of the statistic does not belong to the rejection region. This implies that H_0 cannot be rejected.

In terms of p-value we have to compute

$$\mathbf{P}\left(\chi_{n-1}^2 \geq \frac{(n-1)\,s_n^2}{\sigma_0^2} \mid H_0 \text{ true}\right) = \mathbf{P}\left(\chi_8^2 \geq 10.58\right) = 1 - \mathbf{P}\left(\chi_8^2 \leq 10.58\right) = 0.227 > 0.05.$$

The p-value method confirms that H_0 cannot be rejected.

With regard to the evaluation of $\beta(5)$, setting $\sigma_1^2 \equiv 5$, we have

$$
\begin{aligned}
\beta(5) &= \mathbf{P}\left(\text{accept } H_0 \mid \sigma_X^2 = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2}\frac{\sigma_1^2}{\sigma_1^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_1^2}\frac{\sigma_1^2}{\sigma_0^2} \notin R \mid \sigma_X = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_X^2}\frac{\sigma_1^2}{\sigma_0^2} \notin R\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2\frac{\sigma_1^2}{\sigma_0^2} \notin R\right) \\
&= \mathbf{P}\left(\chi_8^2\frac{\sigma_1^2}{\sigma_0^2} \leq 15.51\right) \\
&= \mathbf{P}\left(\chi_8^2 \leq 15.51 \cdot \frac{\sigma_0^2}{\sigma_1^2}\right) \\
&= \mathbf{P}\left(\chi_8^2 \leq 8.46\right) \\
&= 0.61.
\end{aligned}
$$

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Solution. .

Problem 2 *Let X be a normal random variable with unknown mean μ and variance σ^2, which represents a certain characteristic of a population and let $X_1, \ldots, X_n$ be a simple random sample of size n drawn from X. Assume that $n = 25$ and the realizations $x_1, \ldots, x_{25}$ of the sample give the information summarized by*

$$\sum_{k=1}^{25} x_k = 100 \quad \textit{and} \quad \sum_{k=1}^{25} x_k^2 = 550$$

1. *Considering both the rejection region method and the P-value method, should the null hypothesis $H_0 : \sigma^2 = 4$ be rejected against of the alternative $H_1 : \sigma^2 > 4$ with a significance level $\alpha = 0.05$? Calculate the probability $\beta(5)$ of a II type error. What should have been the size n of the sample to achieve $\beta(5) = 0.5$?*

2. *Considering both the rejection region method and the P-value method, should the null hypothesis $H_0 : \sigma^2 = 4$ be rejected against of the alternative $H_1 : \sigma^2 \neq 4$ with a significance level $\alpha = 0.05$? Calculate the probability $\beta(5)$ of a II type error. What should have been the size n of the sample to achieve $\beta(5) = 0.5$?*

Solution. Note that the given information allows the knowledge of the realization of sample variance, which is given by

$$
\begin{aligned}
s_n^2 &= \frac{1}{n-1}\sum_{k=1}^n \left(x_k - \frac{1}{n}\sum_{\ell=1}^n x_\ell\right)^2 \\
&= \frac{1}{n-1}\sum_{k=1}^n \left(x_k^2 - \frac{2}{n}x_k\sum_{\ell=1}^n x_\ell + \frac{1}{n^2}\left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{2}{n}\left(\sum_{k=1}^n x_k\right)\left(\sum_{\ell=1}^n x_\ell\right) + \frac{1}{n^2}\sum_{k=1}^n \left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{2}{n}\left(\sum_{k=1}^n x_k\right)^2 + \frac{1}{n^2}n\left(\sum_{\ell=1}^n x_\ell\right)^2\right) \\
&= \frac{1}{n-1}\left(\sum_{k=1}^n x_k^2 - \frac{1}{n}\left(\sum_{k=1}^n x_k\right)^2\right).
\end{aligned}
$$

Thus, in the present case, we have

$$
s_n^2 = \frac{1}{24}\left(550 - \frac{1}{25}100^2\right) = \frac{25}{4} = 6.25.
$$

Now, since ve are interested in testing a hypothesis on the variance of X, which is normally distributed, the standard statistic to be considered is

$$
\frac{(n-1)\,S_n^2}{\sigma^2} \sim \chi^2_{n-1}.
$$

1. By virtue of the above considerations, the upper tail rejection region is given by

$$
R = \left\{\frac{(n-1)\,S_n^2}{\sigma^2} > \chi^2_{n-1,\alpha}\right\}
$$

where, the upper $\alpha = 0.05$ critical value of the chi-square distribution with $n-1 = 24$ degrees of freedom is given by

$$
\chi^2_{24,0.05} = 36.4150.
$$

Now, we have

$$
\frac{(n-1)\,s_n^2}{\sigma^2} = \frac{24}{4}\cdot\frac{25}{4} = 37.50 \in R \tag{3}
$$

Hence, the realization of the statistic belongs to the rejection region. This implies that the null hypotesis H_0 has to be rejected in favor of H_1.

In terms of P-value we have to compute

$$
\begin{aligned}
\mathbf{P}\left(\frac{(n-1)\,S_n^2}{\sigma^2} \geq \frac{(n-1)\,s_n^2}{\sigma^2} \mid H_0 = T\right) &= \mathbf{P}\left(\frac{(n-1)\,S_n^2}{\sigma^2} \geq \frac{(n-1)\,s_n^2}{\sigma^2} \mid \sigma = \sigma_0\right) \\
&= \mathbf{P}\left(\chi^2_{24} \geq 37.50\right) \\
&= 1 - \mathbf{P}\left(\chi^2_{24} \leq 37.50\right) \\
&= 0.039 < 0.050.
\end{aligned}
$$

As expected, the P-value method confirms the rejection of H_0 in favor of H_1.

With regard to the evaluation of $\beta(5)$, $\sigma_0^2 \equiv 4$, $\sigma_1^2 \equiv 5$, and $n = 25$, we have that

$$\begin{aligned}
\beta(5) &= \mathbf{P}(\text{II type error}) = \mathbf{P}(\text{accept } H_0 \mid H_0 = F) = \mathbf{P}\left(\text{accept } H_0 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_0^2} \frac{\sigma_1^2}{\sigma_1^2} < 36.4150 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma_1^2} < 36.4150 \frac{\sigma_0^2}{\sigma_1^2} \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma^2} < 36.4150 \frac{4}{5}\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2 \leq 29.132\right) \\
&= 0.7848.
\end{aligned}$$

In the end, to achieve $\beta(5) = 0.5$ we have to solve the equation

$$0.5 = \mathbf{P}\left(\chi_{n-1}^2 \leq 29.132\right)$$

in terms of the smallest n. A computer aided computation yields

$$\mathbf{P}\left(\chi_{n-1}^2 \leq 29.132\right) = \begin{cases} 0.54 & \text{if } n = 29 \\ 0.49 & \text{if } n = 30 \end{cases},$$

which implies that $n = 30$ is the desired n.

2. In this case the rejection region is given by

$$R = \left\{\frac{(n-1)\, S_n^2}{\sigma^2} < \chi_{n-1,1-\alpha/2}^2\right\} \cup \left\{\frac{(n-1)\, S_n^2}{\sigma^2} > \chi_{n-1,\alpha/2}^2\right\}$$

where, the lower $1 - \alpha/2 = 0.975$ and the upper $\alpha/2 = 0.025$ critical values of the chi-square distribution with $n - 1 = 24$ degrees of freedom are given by

$$\chi_{24,0.975}^2 = 12.40 \quad \text{and} \quad \chi_{24,0.025}^2 = 39.36,$$

respectively. In this case the values 37.50 of the statistics (see 3) does not belong to the rejection region. Therefore, H_0 cannnot be rejected against H_1. In terms of P-value we have to compute

$$\begin{aligned}
2\mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma^2} \geq \frac{(n-1)\, s_n^2}{\sigma^2} \mid H_0 = T\right) &= 2\mathbf{P}\left(\frac{(n-1)\, S_n^2}{\sigma^2} \geq \frac{(n-1)\, s_n^2}{\sigma^2} \mid \sigma = \sigma_0^2\right) \\
&= 2\mathbf{P}\left(\chi_{24}^2 \geq 37.50\right) \\
&= 2\left(1 - \mathbf{P}\left(\chi_{24}^2 < 37.50\right)\right) \\
&= 0.078 > 0.050.
\end{aligned}$$

Hence, the P-value method confirms that H_0 cannnot be rejected against H_1.

With regard to the evaluation of $\beta(5)$, setting $\sigma_0^2 \equiv 4$, $\sigma_1^2 \equiv 5$, and $n = 25$, we have that

$$
\begin{aligned}
\beta(5) &= \mathbf{P}(\text{II type error}) = \mathbf{P}(\text{accept } H_0 \mid H_0 = F) = \mathbf{P}\left(\text{accept } H_0 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(\frac{(n-1)\,S_n^2}{\sigma_0^2} \notin R \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.40 \leq \frac{(n-1)\,S_n^2}{\sigma_0^2}\frac{\sigma_1^2}{\sigma_1^2} \leq 39.36 \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.40\frac{\sigma_0^2}{\sigma_1^2} \leq \frac{(n-1)\,S_n^2}{\sigma_1^2} \leq 39.36\frac{\sigma_0^2}{\sigma_1^2} \mid \sigma = \sigma_1^2\right) \\
&= \mathbf{P}\left(12.40\frac{4}{5} \leq \frac{(n-1)\,S_n^2}{\sigma^2} \leq 39.36\frac{4}{5}\right) \\
&= \mathbf{P}\left(9.92 \leq \chi_{n-1}^2 \leq 31.49\right) \\
&= \mathbf{P}\left(\chi_{n-1}^2 \leq 31.49\right) - \mathbf{P}\left(\chi_{n-1}^2 \leq 9.92\right) \\
&= 0.860 - 0.005 = 0.855.
\end{aligned}
$$