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Dipartimento d'Ingegneria Civile e Ingegneria Informatica
LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Intermediate Test - 2019-11-22

Problem 1 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let E, F, G events of Ω such that*

1. *E and G are independent;*
2. *F and G are independent;*
3. *E and F are exclusive;*
4. *$\mathbf{P}(E \cup G) \equiv a$, $\mathbf{P}(F^c \cap G^c) \equiv b$, $\mathbf{P}(E \cup F \cup G) \equiv c$, such that $a - b - c \neq 0$.*

Determine $\mathbf{P}(E)$, $\mathbf{P}(F)$, and $\mathbf{P}(G)$ in terms of a, b, c. Hint: think on a "smart" way to solve the system of equation yielding $\mathbf{P}(E)$, $\mathbf{P}(F)$, and $\mathbf{P}(G)$.

Solution. . □

Problem 2 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ be the Euclidean real plane endowed with the Borel σ-algebra, $\mathcal{B}\left(\mathbb{R}^2\right)$, and the Lebesgue measure, $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$. Let $f : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f(x,y) \stackrel{def}{=} kxe^{-(x+y)} 1_{\mathbb{R}^2_+}(x,y), \quad \forall (x,y) \in \mathbb{R}^2$$

where $\mathbb{R}^2_+ \equiv \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. Determine $k \in \mathbb{R}$ such that $f : \mathbb{R}^2 \to \mathbb{R}_+$ is a probability density. Let $Z \equiv (X,Y)$ be the random vector of density $f : \mathbb{R}^2 \to \mathbb{R}_+$.

1. *Determine the distribution function $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ of the vector Z and check that*
$$\frac{\partial F^2}{\partial x \partial y}(x,y) = f(x,y), \quad \mu_L^2\text{-a.e. on } \mathbb{R}^2.$$
2. *Determine the marginal distribution function $F_X : \mathbb{R} \to \mathbb{R}_+$ and $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the entries X and Y of Z.*
3. *Determine the densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ of the entries X and Y of Z and check that*
$$\frac{dF_X}{dx}(x) = f_X(x) \quad \text{and} \quad \frac{dF_Y}{dy}(y) = f_Y(y), \quad \mu_L\text{-a.e. on } \mathbb{R}.$$
4. *Are X and Y independent random varables?*
5. *Compute $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$ and $Cov(X,Y)$.*
6. *Compute $\mathbf{E}[(X,Y)]$ and the covariance matrix of the vector (X,Y).*

Solution. . □

Problem 3 *Let X [resp. B] be a standard Gaussian [Bernoulli] random variable on a probability space Ω. In symbols, $X \sim N(0,1)$ and $B \sim Ber(1/2)$. Assume that X and B are independent and define $Y \equiv B \cdot X$.* ***Specifying carefully the properties used****, answer the following questions:*

1. *Is the random variable Y Gaussian? Is Y absolutely continuous?*
2. *Are the random variables X and Y uncorrelated? Are X and Y independent?*
3. *Are the random variables B and Y uncorrelated? Are B and Y independent?*
4. *Does the random vector $(X, Y)^\intercal$ have a bivariate Gaussian distribution?*
5. *Can you compute $\mathbf{E}[Y \mid X]$? What about $\mathbf{E}[X \mid Y]$?*

Solution. .

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space and let $B \sim Ber(1/2)$ [resp. $R \sim Rad(1/2)$] a standard Bernoulli [resp. Rademacher] random variable on Ω. Assume that B and R are independent and define $X \stackrel{def}{=} B + R$.*

1. *Compute $\mathbf{E}[X \mid B]$, $\mathbf{E}[X \mid R]$, $\mathbf{E}[B \mid X]$, and $\mathbf{E}[R \mid X]$. In addition, **specifying carefully the properties used**, answer the following questions:*
2. *Are the random variables $\mathbf{E}[X \mid B]$, $\mathbf{E}[X \mid R]$ uncorrelated? Are $\mathbf{E}[X \mid B]$, $\mathbf{E}[X \mid R]$ independent?*
3. *Are the random variables $\mathbf{E}[B \mid X]$ and $\mathbf{E}[R \mid X]$ uncorrelated? Are $\mathbf{E}[B \mid X]$ and $\mathbf{E}[R \mid X]$ independent?*
4. *By using the properties of the conditional expectation, on account that you are dealing with a Bernoulli and a Rademacher random variable, can you compute $\mathbf{E}[BR \mid X]$?*

Solution. .

Problem 5 *Let $X \sim Exp(1)$ an exponential random variable of rate parameter $\lambda = 1$ and let $(Y_n)_{n\geq 1}$ be the sequence of independent real random variables such that*

$$Y_n \stackrel{def}{=} \begin{cases} n & \text{if } 0 \leq X < \frac{1}{n}, \\ 0 & \text{if } 1/n \leq X. \end{cases}, \quad \forall n \geq 1$$

1. *Does $(Y_n)_{n\geq 1}$ converges in distribution?*
2. *Does $(Y_n)_{n\geq 1}$ converges in probability?*
3. *Does $(Y_n)_{n\geq 1}$ converges almost surely?*
4. *Does $(Y_n)_{n\geq 1}$ converges in mean?*
5. *Does $(Y_n)_{n\geq 1}$ converges in quadratic mean?*

Solution. .