Performance Modeling of Computer Systems and Networks

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Continuous Random Variates: applications

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1

Noi modelliamo processo arrivi e processo servizi.

For our application framework, we will look at:

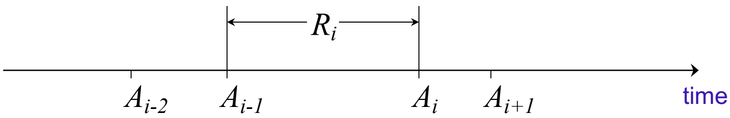
- arrival process model
- service process model

2

In particolare processi di interrarrivo, siamo partiti da una sequenza random R1, R2, R3
e considerando poi i tempi di arrivo come di seguito:

Arrival process model

- Model *interarrival* times as RV sequence $R_1, R_2, R_3, \ldots$
- Construct corresponding arrival times $A_1, A_2, A_3, \ldots$ defined by
  $A_0 = 0$ and $A_i = A_{i-1} + R_i \quad i=1, 2, \ldots$
- by induction, $A_i = R_1 + R_2 + \ldots + R_i \quad i=1, 2, \ldots$
- since $R_i > 0$, $0 = A_0 < A_1 < A_2 < A_3 < \ldots$



3

Gli arrivi sono ordinati in ordine crescente per costruzione (sommo valori positivi). Questo è alla base della costruzione di ssq2 ed ssq3

Example

(descritti come sopra)

- programs `ssq2` and `ssq3` generate job arrivals in this way, where $R_1, R_2, R_3, \ldots$ are *Exponential*$(1/\lambda)$. $E[R_i] = \frac{1}{\lambda}$ $\longrightarrow$ PROCESSO STAZIONARIO DI TASSO "λ".
  In both programs, the arrival rate is equal to $\lambda = 0.5$ jobs per unit time

```
double GetArrival()
{ static double arrival = START;
  SelectStream(0);
  arrival += Exponential(2.0);
  return (arrival);}
```


4

Nel caso dell'inventory system, invece di generare un numero di richieste di arrivo, e spalmarlo flat in un tempo, generavo istanze di domande, generando tempo interrarivo come esponenziale e sommando.

Example

- programs `sis3` and `sis4` generate demand instances in this way, with *Exponential*($1/\lambda$) interdemand times.
  The demand rate corresponds to an average of
  - $\lambda = 30.00$ actual demands per time interval in `sis3`
  - $\lambda = 120.00$ potential demands per time interval in `sis4`

```
double GetDemand(long *amount)
/* ------------------------------------------
 * generate a demand instance with rate 120
 * and generate a corresponding demand amount
 * ------------------------------------------
 */
{
  static double time = START;

  SelectStream(0);
  time += Exponential(1.0 / 120.0);      /* demand instance */
  SelectStream(2);
  *amount = Geometric(0.2);              /* demand amount   */
  return (time);
}
```

```
double GetDemand(void)
/* ---------------------------------------------------------------
 * generate the next demand instance (time) with rate 30 per time
 * interval and exactly one unit of demand per demand instance
 * ---------------------------------------------------------------
 */
{
  static double time = START;

  SelectStream(0);
  time += Exponential(1.0 / 30.0);
  return (time);
}
```


In generale, indipendentemente dal caso particolare, un processo di arrivi stazionario (media arrivo 1/lambda mantenuto anche se vado a infinito), lo posso costruire come già fatto: prendo sequenza interrarivi positivi, ma sono variabili random indipendenti e identicamente distribuite.

Def stationary arrival process

If $R_1, R_2, R_3, \ldots$ is an *iid* sequence of positive interarrival times with $E[R_i] = 1/\lambda > 0$, then the corresponding sequence of arrival times $A_1, A_2, A_3, \ldots$ is a *stationary arrival process* with rate λ

- also known as
  - *Renewal* processes
  - *Homogeneous* arrival processes
- arrival rate λ has units "arrivals per unit time"
  - If average interarrival time is 0.1 minutes,
  - then the arrival rate is 10 arrivals per minute

" inversi tra loro "

- stationary arrival processes are "convenient fiction"
  important theoretically
  good approximation
  understand for nonstationary
- If the arrival rate λ varies with time, the arrival process is *nonstationary*

Nella realtà non è detto che il processo arrivo sia stazionario, è più reale un non stazionario, nel tempo varia (aerei che volano di meno nella notte). Noi ci limitiamo a vedere gli stazionari, sopratutto in termini di difficoltà.

All'inizio sono partito con arrivi random, e assunto poi fossero esponenziali.
In realtà ciò che succede è che, partendo da cose random con interrarrivi esponenziali, considero arrivi di Poisson, con istanti di arrivi di Poisson che sono delle Erlang!

stationary Poisson arrival process

random events!

- As in `ssq2`, `ssq3`, `sis3`, `sis4`, with lack of information it is usually most appropriate to assume that the interarrival times are *Exponential*($1/\lambda$)

- If $R_1, R_2, R_3, \ldots$ is an *iid* sequence of *Exponential*($1/\lambda$) interarrival times, the corresponding sequence $A_1, A_2, A_3, \ldots$ of arrival times is a stationary *Poisson* arrival process with rate λ
  Equivalently, for i=1, 2, 3, …, the arrival time A_i is an *Erlang*(i,1/λ) random variable

sequenza "Poisson"

singolo ARRIVO

$$A_i = R_1 + R_2 + \ldots + R_i$$

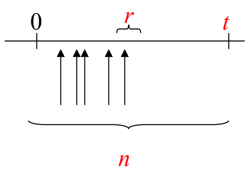
7

che relazione c'è tra arrivi random(=uniformi), esponenziali e poisson? considero intervallo fisso di lunghezza 't', con 'n' arrivi, in questo tempo prendo random un sottointervallo di lunghezza 'r'

Random Arrivals

We now demonstrate the interrelation between Uniform, Exponential and Poisson random variables.

Consider:
- $t > 0$, defines a fixed time interval $(0, t)$
- n rappresents the number of arrivals in the interval $(0, t)$
- $r > 0$, is the length of a small subinterval located at random interior to $(0, t)$

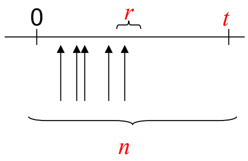


(se $r=1$, $t=10 \longrightarrow p = \frac{1}{10} = 10\%$)

Correspondingly:
- $\lambda = n / t$ is the arrival rate
- $p = r / t$ is the probability that a particular arrival will be in the subinterval
- $np = nr / t = \lambda r$ is the expected number of arrivals in the subinterval

8

random arrivals → Poisson



<u>Theorem 1</u>

Let:

- $A_1, A_2, A_3, \ldots$ be an iid sequence of *Uniform*(0, t) random variables ("unsorted" arrivals).

- the discrete random variable X be the number of A_i that fall in a fixed subinterval of length $r = pt$ interior to $(0, t)$

If n is large and r / t small, X is indistinguishable from a *Poisson*(λr) random variable with $\lambda = n / t$

'X' mi conta gli Ai che cadono in quell'intervallino.

9

Conclusions on random arrivals

- if *many* arrivals occur *at random* with a rate of λ, the number of arrivals X that will occurr in an interval of length r is *Poisson*(λr)
- The probability of x arrivals in an interval with length r is

$$Pr(X = x) = \frac{e^{-\lambda r}(\lambda r)^x}{x!} \qquad x = 0,1,2,..$$

- The probability of <u>no arrivals</u> is: $Pr(X = 0) = e^{-\lambda r}$
- The probability of <u>at least one arrival</u> is

$$Pr(X > 0) = 1 - Pr(X = 0) = 1 - e^{-\lambda r}$$ (complemento ad 1)

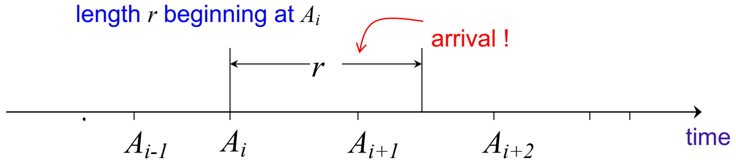
For a fixed λ, the probability of at least one arrival increases with increasing interval length r

(se $r \nearrow$ allora ho intervallo maggiore e "più probabilità")

10

Random Arrivals → Exponential Interarrivals

- If R represents the time between consecutive arrivals, the possible values of R are $r > 0$
- Consider arrival time A_i selected at random and an interval of length r beginning at A_i



- $R = A_{i+1} - A_i$ will be less than r iff there is at least one arrival in this interval
- the cdf of R is

$$Pr(R \le r) = Pr\text{(at least one arrival in } r) = 1 - e^{-\lambda r}$$

- R is an *Exponential(1/λ)* random variable

Theorem 2

If arrivals occur at random with rate λ, the corresponding interarrival times form an iid sequence of *Exponential(1/λ)* RVs.

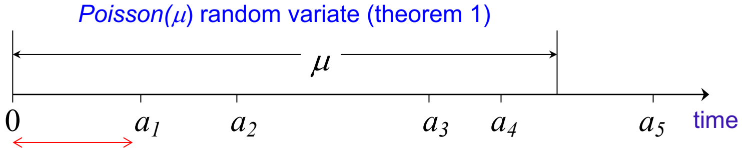
This result justifies the use of *Exponential* interarrival times in programs `ssq2`, `ssq3`, `sis3`, `sis4`

- If we know only that arrivals occur at random with a constant rate λ, the function `GetArrival` in `ssq2` and `ssq3` is appropriate
- If we know only that demand instances occur at random with a constant rate λ, the function `GetDemand` in `sis3` and `sis4` is appropriate

Generating Poisson random *variates*

Observation:

- If arrivals occur at random with rate $\lambda=1$
- the number of arrivals X in an interval of length μ will be a *Poisson(μ)* random variate (theorem 1)



exponential

```
a0 = 0.0;
x = 0;
while (a < μ) {
    a += Exponential(1.0);
    x++;
}
return x-1;
```

$a_5 > \mu$

13

Posso generare Poisson cosi:
Generando variabili uniformi in (0,t) e ordinarle OPPURE, come nell'algoritmo 1, contando gli a(i) (nell'esempio ne ho 4)

Summary of Poisson arrival processes

Given a fixed time interval $(0, t)$, there are two ways of generating a realization of a stationary Poisson arrival process with rate λ

1. Generate the number of arrivals: n = *Poisson(λt)*
   Generate a *Uniform*(0,t) random variate sample of size n and sort to form $0 < a_1 < a_2 < a_3 < \ldots < a_n$
2. use algorithm 1 with *Exponential*($1/\lambda t$)

- Statistically, the two approaches are equivalent
- The first approach is computationally more expensive, especially for large n
- The second approach is always preferred

14

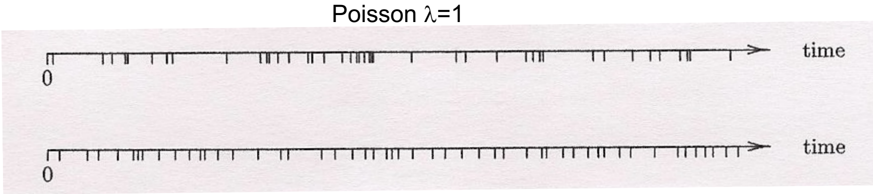
La moda è valore più probabile, nell'esponenziale è 0, che è anche un tempo di interrarivo.
Allora dovrebbero essere molto probabili arrivi simultanei (non visti). Cioè se vedo interrarivi di Poisson ho comportamenti a cluster, mentre se vedo i singoli interrarivi, l'aspetto 'cluster' si vede molto meno.

Discrete Simulation
Continuous RV applications

Summary of Poisson arrival processes

- The *mode* of the exponential distribution is 0
  $\rightarrow$ A stationary Poisson arrival process exhibits "clustering"

Poisson λ=1



Interarrival times *Erlang*(4,1/4)

The stationary Poisson arrival process generalizes to

- a stationary arrival process when exponential interarrival times are replaced by any continuous RV with positive support
- a nonstationary Poisson arrival process when λ varies over time

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15

Nel confronto hanno stessa media.Ovviamente devo considerare la varianza (che è diversa), questo viene dal fatto che la moda dell'esponenziale è 0.

Discrete Simulation
Continuous RV applications

For our application framework, we will look at:

- arrival process model
- service process model

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16

Per i servizi ho solo linee guida, non posso dire che siano random semplicemente come nel caso degli arrivi.

Service Process Models

differently from the case of arrival processes, there are no well-defined "default", only application-dependent guidelines:

- *Uniform*(a, b) service times are usually inappropriate since they rarely "cut off" at a maximum value b
- Service times are positive, so they cannot be *Normal*(μ,σ) unless truncated to positive values
- Positive probability models "with tails", such as the *Lognormal*(a, b) distribution, are candidates
  - jobs UNIX
  - web file size
  - Internet topology
  - IP packet flow
  - …
- If service times are the sum of n iid *Exponential*(b) sub-task times, then the *Erlang*(n, b) model is appropriate

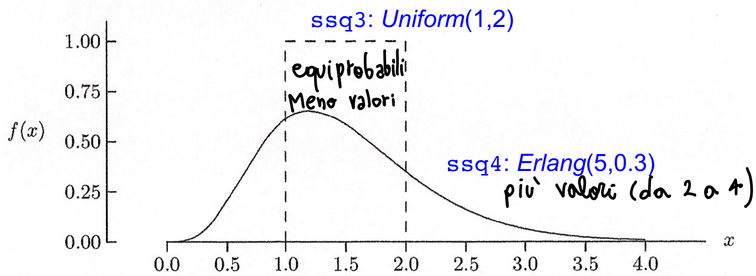
Program `ssq4`

- `ssq4` is based on program `ssq3`, but with a more realistic *Erlang*(5, 0.3) service time model
  The corresponding service rate is 2/3
- As in program `ssq3`, `ssq4` uses *Exponential*(2) random variate interarrivals.
  The corresponding arrival rate is 1/2

inverso Media

Example

- For both `ssq3` and `ssq4`, the arrival rate is λ=0.5 and the service rate is ν= 2/3 ≈ 0.667
- The distribution of service times for two programs is very different



first conjecture how the steady-state statistics will differ for these two programs and then investigate the correctness of your conjecture via discrete-event simulation

Example

- suppose using a *Normal*(1.5,2.0) random variable to model service times
- Truncate distribution so that
  - Service times are non-negative (a=0)
  - Service times are less than 4 (b=4)

```
α = cdfNormal(1.5, 2.0, a);          /* a is 0.0 */
β = 1.0 - cdfNormal(1.5, 2.0, b);    /* b is 4.0 */
```

- the result: $\alpha = 0.2266$ and $\beta = 0.1056$

perchè ho Troncato (togliendo <0)

NB
the *truncated Normal*(1.5,2.0) random variable has a mean of 1.85 (not 1.5) and a standard deviation of 1.07 (not 2.0)

Why is the mean increased?????

Constrained Inversion

once α and β are determined, the corresponding truncated random variate can be generated by using constrained inversion

```
u = Uniform(α, 1.0 - β);
return F⁻¹(u);
```

Exercise:
- generate n truncated random variates;
- compute sample mean and standard deviation (by Welford)
- verify for which n the statistics converge to the theoretical values

Example (cont.)

- The idf capability in `rvms` can be used to generate the *truncated Normal*(1.5,2.0) RV

```
α = 0.2266274;
β = 0.1056498;
u = Uniform(α, 1.0 - β);
return idfNormal(1.5, 2.0, u);
```

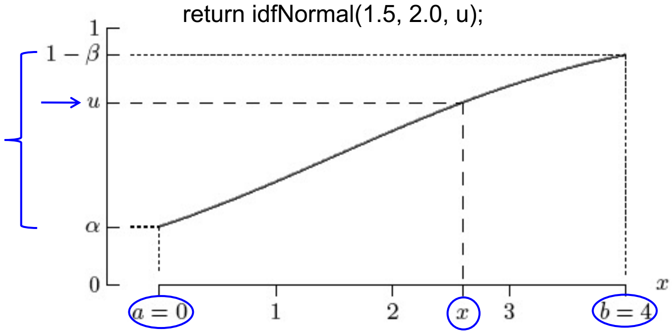


Figure shows u=0.7090 and x=2.601

Exercises

- Exercise 7.3.1