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Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Sequences of Random Variables with Some Solutions 2021-12-07

Problem 1 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X be a uniformly distributed real random variable on the interval $[0,1]$. In symbols, $X \sim U(0,1)$. Consider the sequence $(Y_n)_{n\geq 1}$ of real random variables given by*

$$Y_n \overset{def}{=} \begin{cases} n, & \text{if } 0 \leq X < \frac{1}{n}, \\ 0, & \text{if } 1/n \leq X \leq 1, \end{cases} \qquad \forall n \geq 1.$$

Check whether the sequence $(Y_n)_{n\geq 1}$ converges in distribution, converges in probability, converges in mean, converges almost surely, in the order indicated.

Hint: to deal with the almost sure convergence consider the event $E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}$ and the complement E_0^c.

Solution. Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}$ for the distribution function of Y_n. We have

$$F_{Y_n}(y) = \mathbf{P}(Y_n \leq y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(1/n \leq X \leq 1) = 1 - 1/n, & \text{if } 0 \leq y < n, \\ 1, & \text{if } n \leq y. \end{cases}$$

On the other hand, for every $y \geq 0$ there exists $n_y \in \mathbb{N}$, (e.g. $n_y = \lceil y \rceil$, where $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{R}$, is the ceiling function), such that $y < n$ for every $n > n_y$. Therefore, definitively,

$$\mathbf{P}(Y_n \leq y) = 1 - 1/n.$$

It then follows

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \lim_{n\to\infty} 1/n = 1, & \text{if } 0 \leq y. \end{cases}$$

Considering the Heavside function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } 0 \leq y, \end{cases}$$

we clearly have

$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R}$. Hence, the sequence $(Y_n)_{n\geq 1}$ converges in distribution to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0 \in \mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. By direct approach, according to the definition, we have

$$\mathbf{P}(Y_n = n) = \mathbf{P}\left(0 \leq X < \frac{1}{n}\right) = \frac{1}{n} \quad \text{and} \quad \mathbf{P}(Y_n = 0) = \mathbf{P}\left(\frac{1}{n} \leq X \leq 1\right) = 1 - \frac{1}{n}.$$

Therefore, definitively,

$$\mathbf{P}(|Y_n| \leq \varepsilon) \geq \mathbf{P}(Y_n = 0) = 1 - \frac{1}{n},$$

for every $\varepsilon > 0$. It follows

$$\lim_{n\to\infty} \mathbf{P}(|Y_n| \leq \varepsilon) \geq 1 - \lim_{n\to\infty} \frac{1}{n} = 1,$$

for every $\varepsilon > 0$, which is the convergence in probability of $(Y_n)_{n\geq 1}$ to $Dir\,(0)$. Now, to check the convergence in mean, since $Y_n(\Omega) = \{0, 1\}$, we consider

$$\mathbf{E}[Y_n] = n\mathbf{P}(Y_n = n) = n\frac{1}{n} = 1.$$

It follows that

$$\lim_{n\to\infty} \mathbf{E}[|Y_n|] = \lim_{n\to\infty} \mathbf{E}[Y_n] = 1 \neq 0.$$

Hence, $(Y_n)_{n\geq 1}$ does not converge in mean to $Dir\,(0)$, which implies that $(Y_n)_{n\geq 1}$ does not converge in mean at all (recall that convergence in mean at some random variable implies convergence in probability at the same random variable). In the end, consider the event

$$E_0 \equiv \{\omega \in \Omega : X(\omega) = 0\}.$$

Since $X \sim U(0,1)$, we have $\mathbf{P}(E_0) = 0$. In addition, for every $\omega \in E_0^c$ we have $X(\omega) > 0$ and it is possible to find n_ω such that

$$\frac{1}{n} < X(\omega),$$

for every $n > n_\omega$. It then follows that

$$Y_n(\omega) = 0,$$

for every $n > n_\omega$. This implies

$$\lim_{n\to\infty} Y_n(\omega) = 0,$$

for every $\omega \in E_0^c$, which is the almost sure convergence of the sequence $(Y_n)_{n\geq 1}$ to $Dir\,(0)$. □

Problem 2 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ be the Euclidean real line endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) \stackrel{def}{=} \frac{\alpha - 1}{x^\alpha} 1_{[1,+\infty)}, \quad \forall x \in \mathbb{R},$$

where $\alpha > 1$, is a density. Then, consider a random variable X with density $f_X = f$ and the sequence $(Y_n)_{n\geq 1}$ of random variables given by

$$Y_n \stackrel{def}{=} \frac{X}{n}, \quad \forall n \in \mathbb{N}.$$

Study the convergence in distribution, in probability, and in p-th mean of the sequence $(Y_n)_{n\geq 1}$, on varying of $\alpha > 1$, in the order indicated.

Solution. Since $\alpha > 1$, that is $\alpha - 1 > 0$ and $1 - \alpha < 0$, we have clearly

$$f(x) \geq 0,$$

for every $x \in \mathbb{R}$, and

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\, d\mu_L(x) &= \int_{\mathbb{R}} \frac{\alpha - 1}{x^\alpha} 1_{[1,+\infty)}(x)\, d\mu_L(x) = \int_{[1,+\infty)} \frac{\alpha - 1}{x^\alpha} d\mu_L(x) \\
&= \int_1^{+\infty} \frac{\alpha - 1}{x^\alpha} dx = \lim_{x\to+\infty} \int_1^x \frac{\alpha - 1}{u^\alpha} du = \lim_{x\to+\infty} -\int_1^x du^{1-\alpha} \\
&= \lim_{x\to+\infty} -\left. u^{1-\alpha}\right|_1^x = 1 - \lim_{x\to+\infty} x^{1-\alpha} = 1.
\end{aligned}$$

This proves that $f : \mathbb{R} \to \mathbb{R}$ is a density.

Write $F_{Y_n} : \mathbb{R} \to \mathbb{R}$ for the distribution function of Y_n, for every $n \geq 1$. We have

$$F_{Y_n}(y) = \mathbf{P}(Y_n \leq y) = \mathbf{P}(X/n \leq y) = \mathbf{P}(X \leq ny) = \int_{(-\infty,ny]} f(x)\, d\mu_L(x).$$

On the other hand,

$$\begin{aligned}
\int_{(-\infty,ny]} f(x)\, d\mu_L(x) &= \int_{(-\infty,ny]} \frac{\alpha-1}{x^{\alpha}} 1_{[1,+\infty)}(x)\, d\mu_L(x) \\
&= \int_{(-\infty,ny]\cap[1,+\infty)} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x) \\
&= \begin{cases} \int_{\varnothing} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x), & \text{if } ny < 1, \\ \int_{\{ny\}} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x), & \text{if } ny = 1, \\ \int_{[1,ny]} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x), & \text{if } 1 < ny, \end{cases}
\end{aligned}$$

where

$$\int_{\varnothing} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x) = \int_{\{ny\}} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x) = 0$$

and

$$\int_{[1,ny]} \frac{\alpha-1}{x^{\alpha}} d\mu_L(x) = \int_1^{ny} \frac{\alpha-1}{x^{\alpha}} dx = -\int_1^{ny} dx^{1-\alpha} = -\left. x^{1-\alpha}\right|_1^{ny} = 1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}.$$

Therefore,

$$F_{Y_n}(y) = \begin{cases} 0, & \text{if } y \leq \frac{1}{n}, \\ 1 - \frac{1}{n^{\alpha-1}y^{\alpha-1}}, & \text{if } \frac{1}{n} < y. \end{cases}$$

As a consequence,

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ 1 - \lim_{n\to\infty} \frac{1}{n^{\alpha-1}y^{\alpha-1}} = 1, & \text{if } 0 < y. \end{cases}$$

Considering the Heavside function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } 0 \leq y, \end{cases}$$

we then have

$$\lim_{n\to\infty} F_{Y_n}(y) = H(y),$$

at any point $y \in \mathbb{R} - \{0\}$, where the Heavside function is continuous. Hence, the sequence $(Y_n)_{n\geq 1}$ converges in distribution to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0 \in \mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. By direct approach, since

$$F_{Y_n}(y) = \int_{(-\infty,y]} \frac{1-\alpha}{n^{\alpha-1}u^{\alpha}} 1_{(1/n,+\infty)}(u)\, d\mu_L(u)$$

for every $y \in \mathbb{R}$, we have that the random variables of the sequence $(Y_n)_{n\geq 1}$ are absolutely continuous with density $f_{Y_n} : \mathbb{R} \to \mathbb{R}$ given by

$$f_{Y_n}(y) = \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y), \quad \forall y \in \mathbb{R}.$$

Note that

$$F'_{Y_n}(y) = f_{Y_n}(y),$$

for every $y \neq 1/n$. As a consequence, provided n is sufficiently large,

$$\begin{aligned}
\mathbf{P}(|Y_n| > \varepsilon) = \mathbf{P}(Y_n > \varepsilon) &= \int_{(\varepsilon,+\infty)} f_{Y_n}(y)\, d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} 1_{(1/n,+\infty)}(y)\, d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)\cap(1/n,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{(\varepsilon,+\infty)} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} d\mu_L(y) \\
&= \int_{\varepsilon}^{+\infty} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha}} dy \\
&= \frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} \frac{\alpha-1}{u^{\alpha}} du \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{\varepsilon}^{y} du^{\alpha-1} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{1-\alpha}\big|_{\varepsilon}^{y} \\
&= -\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \left(\frac{1}{y^{\alpha-1}} - \frac{1}{\varepsilon^{\alpha-1}}\right) \\
&= \frac{1}{n^{\alpha-1}\varepsilon^{\alpha-1}}.
\end{aligned}$$

It follows

$$\lim_{n\to\infty} \mathbf{P}(|Y_n| > \varepsilon) = \lim_{n\to\infty} \frac{1}{n^{\alpha-1}\varepsilon^{\alpha-1}} = 0,$$

for every $\varepsilon > 0$. This proves directly that $(Y_n)_{n\geq 1}$ converges in probability to $Dir(0)$.

By virtue of what shown above, to study the convergence in p-th mean of the sequence $(Y_n)_{n\geq 1}$ it is sufficient to consider

$$\begin{aligned}
\mathbf{E}[Y_n^p] &= \int_{\mathbb{R}} y^p f_{Y_n}(y)\, d\mu_L(u) = \int_{\mathbb{R}} \frac{1-\alpha}{n^{\alpha-1}y^{\alpha-p}} 1_{(1/n,+\infty)}(y)\, d\mu_L(u) = \frac{1-\alpha}{n^{\alpha-1}} \int_{(1/n,+\infty)} \frac{1}{y^{\alpha-p}} d\mu_L(u) \\
&= \frac{1-\alpha}{n^{\alpha-1}} \int_{1/n}^{+\infty} \frac{1}{y^{\alpha-p}} dy = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} du^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}} \lim_{y\to+\infty} u^{p-\alpha+1}\big|_{1/n}^{y}, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \int_{1/n}^{y} d\ln(u) = \frac{\alpha-1}{n^{\alpha-1}} \lim_{y\to+\infty} \ln(u)\big|_{1/n}^{y}, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mathbf{E}[Y_n^p] = \mathbf{E}\left[\left(\frac{X}{n}\right)^p\right] &= \int_{\mathbb{R}} \frac{x^p}{n^p} f_X(x)\, d\mu_L(x) = \int_{\mathbb{R}} \frac{x^p}{n^p}\frac{\alpha-1}{x^{\alpha}} 1_{[1,+\infty)}(x)\, d\mu_L(x) \\
&= \frac{\alpha-1}{n^p} \int_{[1,+\infty)} \frac{1}{x^{\alpha-p}} d\mu_L(x) = \frac{\alpha-1}{n^p} \int_{1}^{+\infty} \frac{1}{x^{\alpha-p}} dx = \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} \frac{1}{u^{\alpha-p}} du \\
&= \begin{cases} \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} \frac{1}{u^{\alpha-p}} u^{p-\alpha+1} = \frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} \lim_{x\to+\infty} u^{p-\alpha+1}\big|_{1}^{x}, & \text{if } p \neq \alpha-1, \\ \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \int_{1}^{x} d\ln(u) = \frac{\alpha-1}{n^p} \lim_{x\to+\infty} \ln(u)\big|_{1}^{x}, & \text{if } p = \alpha-1. \end{cases}
\end{aligned}$$

Now, if $p \geq \alpha - 1$ we have that $\mathbf{E}[Y_n^p]$ is not finite. The sequence $(Y_n)_{n\geq 1}$ cannot converge in p-th mean. If $1 \leq p < \alpha - 1$, we have

$$\mathbf{E}[Y_n^p] = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^{\alpha-1}}\frac{1}{n^{p-\alpha+1}} = -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p}.$$

Hence,

$$\lim_{n\to\infty} \mathbf{E}\left[Y_n^p\right] = \lim_{n\to\infty} -\frac{\alpha-1}{p-\alpha+1}\frac{1}{n^p} = 0.$$

The sequence converges in p-th mean to the standard Dirac random variable.

Problem 3 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $(X_n)_{\geq n}$ be a sequence of real random variables on Ω. Assume that $(X_n)_{\geq n}$ are identically distributed and let $f_X : \mathbb{R} \to \mathbb{R}$ their common density function given by*

$$f_X(x) \overset{def}{=} \frac{2}{x^3} 1_{(1,+\infty)}(x), \quad \forall x \in \mathbb{R}.$$

Set

$$Y_n \equiv \frac{X_n}{n^\alpha}, \quad \forall n \geq 1,$$

where $\alpha > 0$.

1. *Study the convergence in distribution, probability, and L^p of the sequence $(Y_n)_{n\geq 1}$, on varying of $\alpha > 0$, in the order indicated.*
2. *Under the additional assumption of independence of the random variables of the sequence $(X_n)_{\geq n}$, compute $\limsup_{n\to\infty} Y_n$ and $\liminf_{n\to\infty} Y_n$ on varying of $\alpha > 0$. Does the sequence $(Y_n)_{n\geq 1}$ converge almost surely?*

Solution. . $\square$

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a complete probability space and let $(X_n)_{n\geq 1}$ be a sequence of independent real random variables such that $X_n \sim Ber(1/n^\alpha)$ for some $\alpha > 0$. Consider the sequence $(Y_n)_{n\geq 1}$ of real random variables on Ω given by*

$$Y_n \overset{def}{=} \min\{X_1, \ldots, X_n\}.$$

1. *study the convergence in distribution, in probability and in $L^p(\Omega; \mathbb{R})$ of $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ on varying of $\alpha > 0$;*
2. *study the almost sure convergence of $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ on varying of $\alpha > 0$.*

Solution.

1. We clearly have

$$Y_n(\omega) = \begin{cases} 1 & \Leftrightarrow X_1(\omega) = \cdots = X_n(\omega) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, by virtue of the independence of the random variables of the sequence $(X_n)_{n\geq 1}$, we have

$$\mathbf{P}(Y_n = 1) = \mathbf{P}(X_1 = 1, \ldots, X_n = 1) = \mathbf{P}(X_1 = 1) \cdots \mathbf{P}(X_n = 1) = \prod_{k=1}^n \frac{1}{k^\alpha} = \frac{1}{n!^\alpha}$$

and

$$\mathbf{P}(Y_n = 0) = 1 - \mathbf{P}(Y_n = 1) = 1 - \frac{1}{n!^\alpha}.$$

In other words, $(Y_n)_{n\geq 1}$ is a sequence of standard Bernoulli random variables with succes probability $\frac{1}{n!^{\alpha}}$. Considering the distribution functions $F_{X_n}:\mathbb{R}\to\mathbb{R}_+$ and $F_{Y_n}:\mathbb{R}\to\mathbb{R}_+$ of X_n and Y_n, respectively, we have

$$F_{X_n}(x)\overset{\text{def}}{=}\mathbf{P}(X_n\leq x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n^{\alpha}}, & \text{if } 0\leq x<1,\\ 1, & \text{if } 1\leq x,\end{cases}$$

and

$$F_{Y_n}(x)\overset{\text{def}}{=}\mathbf{P}(Y_n\leq x)=\begin{cases}0, & \text{if } x<0,\\ 1-\frac{1}{n!^{\alpha}}, & \text{if } 0\leq x<1,\\ 1, & \text{if } 1\leq x.\end{cases}$$

Therefore, considering the Heaviside function $H:\mathbb{R}\to\mathbb{R}_+$ given by

$$H(x)\overset{\text{def}}{=}\begin{cases}0, & \text{if } x<0,\\ 1, & \text{if } 0\leq x,\end{cases}$$

we have

$$\lim_{n\to\infty}F_{X_n}(x)=\lim_{n\to\infty}F_{Y_n}(x)=H(x),$$

for every $x\in\mathbb{R}$. Thus, both the sequences $(F_{X_n})_{n\geq 1}$ and $(F_{Y_n})_{n\geq 1}$ converge pointwise to H. It follows that both the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ converge to the standard Dirac real random variable $Dir(0)$. With regard to the convergence in probability, we know that the convergence in distribution to a Dirac random variables $Dir(y_0)$, concentrated at some $y_0\in\mathbb{R}$, implies also the convergence in probability to $Dir(y_0)$. However, according to the definition, we have definitively

$$\mathbf{P}(|X_n-Dir(0)|<\varepsilon)=\mathbf{P}(X_n<\varepsilon)=\mathbf{P}(X_n=0)=1-\frac{1}{n^{\alpha}}$$

and

$$\mathbf{P}(|Y_n-Dir(0)|<\varepsilon)=\mathbf{P}(Y_n<\varepsilon)=\mathbf{P}(Y_n=0)=1-\frac{1}{n!^{\alpha}},$$

for every $\varepsilon>0$. Therefore,

$$\lim_{n\to\infty}\mathbf{P}(|X_n-Dir(0)|<\varepsilon)=\lim_{n\to\infty}\left(1-\frac{1}{n^{\alpha}}\right)=1$$

and

$$\lim_{n\to\infty}\mathbf{P}(|Y_n-Dir(0)|<\varepsilon)=\lim_{n\to\infty}\left(1-\frac{1}{n!^{\alpha}}\right)=1,$$

which is the convergence in probabiliyty of both $(X_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ to $Dir(0)$. To check the convergence in $L^p(\Omega;\mathbb{R})$, we need to consider $\|X_n-Dir(0)\|_p$ and $\|Y_n-Dir(0)\|_p$, because in case of convergence the limit has to be $Dir(0)$. We then have

$$\|X_n-Dir(0)\|_p=\left(\int_{\Omega}|X_n-Dir(0)|^p\ d\mathbf{P}\right)^{1/p}=\left(\int_{\Omega}X_n^p\ d\mathbf{P}\right)^{1/p}=\mathbf{P}(X_n=1)^{1/p}=\frac{1}{n^{\frac{\alpha}{p}}}$$

and

$$\|Y_n-Dir(0)\|_p=\left(\int_{\Omega}|Y_n-Dir(0)|^p\ d\mathbf{P}\right)^{1/p}=\left(\int_{\Omega}Y_n^p\ d\mathbf{P}\right)^{1/p}=\mathbf{P}(Y_n=1)^{1/p}=\frac{1}{n!^{\frac{\alpha}{p}}},$$

for every $\alpha>0$. Hence, we obtain

$$\lim_{n\to\infty}\|X_n-X\|_p=\lim_{n\to\infty}\|Y_n-X\|_p=0,$$

which proves the convergence of both $(X_n)_{n\geq 1}$ and $(X_n)_{n\geq 1}$ to $Dir(0)$ in $L^p(\Omega;\mathbb{R})$.

2. With regard to the almost sure convergence, note that also in this case, if the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ converge almost surely, the limit has to be $Dir\,(0)$. Hence, for any fixed $\varepsilon > 0$ consider the events $E_0 \equiv \{|X_n - Dir\,(0)| \geq \varepsilon\}$ and $F_0 \equiv \{|Y_n - Dir\,(0)| \geq \varepsilon\}$ we have

$$\{|X_n - Dir\,(0)| \geq \varepsilon\} = \{X_n \geq \varepsilon\} \quad \text{and} \quad \{|Y_n - Dir\,(0)| \geq \varepsilon\} = \{Y_n \geq \varepsilon\}.$$

Hence,

$$\mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \mathbf{P}\left(X_n = 1\right) = \frac{1}{n^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1, \end{cases}$$

and

$$\mathbf{P}\left(|Y_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \mathbf{P}\left(Y_n = 1\right) = \frac{1}{n!^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1. \end{cases}$$

As a consequence,

$$\textstyle\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1, \end{cases}$$

and

$$\textstyle\sum_{n=1}^{\infty} \mathbf{P}\left(|Y_n - Dir\,(0)| \geq \varepsilon\right) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n!^\alpha}, & \text{if } 0 < \varepsilon \leq 1, \\ 0, & \text{if } \varepsilon > 1. \end{cases}$$

It then follows that $\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right)$ converges for every $\alpha > 1$ and $\sum_{n=1}^{\infty} \mathbf{P}\left(|X_n - Dir\,(0)| \geq \varepsilon\right)$ converges for every $\alpha > 0$. This yields the almost sure convergence of $(X_n)_{n\geq 1}$ to $Dir\,(0)$ for every $\alpha > 1$ and the almost sure convergence of $(Y_n)_{n\geq 1}$ to $Dir\,(0)$ for every $\alpha > 0$. In fact, the convergence of the series implies that

$$\lim_{m\to\infty} \mathbf{P}\left(\bigcup_{n\geq m} \{|Z_n - Z| \geq \varepsilon\}\right) \leq \textstyle\sum_{n=m}^{\infty} \mathbf{P}\left(|Z_n - Z| \geq \varepsilon\right) = 0.$$

To check the almost sure convergence of the sequence $(X_n)_{n\geq 1}$ to $Dir\,(0)$ when $0 < \alpha \leq 1$, let us start by considering the case $\alpha = 1$. Choosing any $\varepsilon < 1$, on account of the independence of the random variables of the sequence $(X_n)_{n\geq 1}$, we estimate

$$\begin{aligned} \mathbf{P}\left(\textstyle\bigcap_{n\geq m} \{|X_n| \leq \varepsilon\}\right) &\leq \mathbf{P}\left(\textstyle\bigcap_{n=m}^{2m} \{|X_n| \leq \varepsilon\}\right) = \textstyle\prod_{n=m}^{2m} \mathbf{P}\left(|X_n| \leq \varepsilon\right) \\ &= \textstyle\prod_{n=m}^{2m} \mathbf{P}\left(X_n = 0\right) = \textstyle\prod_{n=m}^{2m} \left(1 - \dfrac{1}{n}\right) \\ &\leq \textstyle\prod_{n=m}^{2m} \left(1 - \dfrac{1}{2m}\right) = \left(1 - \dfrac{1}{2m}\right)^m. \end{aligned}$$

As a consequence,

$$\lim_{m\to\infty} \mathbf{P}\left(\textstyle\bigcap_{n\geq m} \{|X_n| \leq \varepsilon\}\right) \leq \lim_{m\to\infty} \left(1 - \dfrac{1}{2m}\right)^m = e^{-1/2} < 1.$$

This prevents that

$$\lim_{m\to\infty} \mathbf{P}\left(\textstyle\bigcap_{n\geq m} \{|X_n| \leq \varepsilon\}\right) = 1,$$

so that $X_n \overset{\mathbf{a.s.}}{\not\to} 0$. With regard to the case $0 < \alpha < 1$, it is then sufficient to observe that we have

$$\left(1 - \frac{1}{n^\alpha}\right) < \left(1 - \frac{1}{n}\right),$$

for every $n \in \mathbb{N}$. By an analogous computation as above, it then follows

$$\lim_{m\to\infty} \mathbf{P}\left(\bigcap_{n\geq m} \{|X_n| \leq \varepsilon\}\right) \leq \lim_{n\to\infty} \left(1 - \frac{1}{2m^\alpha}\right)^m \leq \lim_{n\to\infty} \left(1 - \frac{1}{2m}\right)^m = e^{-1/2} < 1.$$

Alternatively, considering that

$$\log(1-x) = -x + o(x),$$

we can directly compute

$$\lim_{n\to\infty} \left(1 - \frac{1}{2m^\alpha}\right)^m = \lim_{n\to\infty} \exp\left(m \log\left(1 - \frac{1}{2m^\alpha}\right)\right) = \lim_{n\to\infty} \exp\left(-\frac{m}{2m^\alpha}\right) = \lim_{n\to\infty} \exp\left(-\frac{m^{1-\alpha}}{2}\right) = 0.$$

As a consequence, we still have $X_n \overset{\textbf{a.s.}}{\not\to} 0$.