1)

$$P(Y_n \le y) = P(X \le n^\alpha y) = \int_{(-\infty, n^\alpha y]} 3x^{-4} \, 1_{[1,+\infty)}(x) \, d\mu_L(x) =$$

$$= \begin{cases} 0 & \text{se } y < \frac{1}{n^\alpha} \\ \int_1^{n^\alpha y} 3x^{-4} dx = -x^{-3}\Big|_1^{n^\alpha y} = 1 - \frac{1}{n^{3\alpha} y^3} & \text{se } y > \frac{1}{n^\alpha} \end{cases}$$

$$\lim_{n \to +\infty} F_{Y_n}(y) = \begin{cases} 0 & \text{se } y < 0 \\ 1 & \text{se } y \ge 0 \end{cases} \Rightarrow Y_n \xrightarrow{w} Dir(0)$$

Supponiamo $Y_n \xrightarrow{P} Dir(0)$

$$P(|Y_n - Dir(0)| \ge \varepsilon) \to 0 \quad n \to +\infty$$

$$F'_{Y_n}(y) = \begin{cases} 0 & \text{se } y < 0 \\ \frac{3}{n^{3\alpha} y^4} & \text{se } y \ge 0 \end{cases}$$

$$P(Y_n \ge \varepsilon) = \int_{[\varepsilon, +\infty)} \frac{3}{n^{3\alpha}} y^{-4} \, 1_{[0,+\infty)} \, d\mu_L(y) = \frac{3}{n^{3\alpha}} \int_\varepsilon^{+\infty} y^{-4} dy =$$

$$= \frac{-3}{n^{3\alpha}} y^{-3} \Big|_\varepsilon^{+\infty} = \frac{3}{n^{3\alpha} \varepsilon^3} \longrightarrow 0 \quad \text{per } n \to +\infty$$

$$E[|Y_n - Dir(0)|^p]^{1/p} = E[Y_n^p]^{1/p}$$

$$E[Y_n^p] = E\left[\frac{X^p}{n^{\alpha p}}\right] = \frac{1}{n^{\alpha p}} E[X^p] = \frac{1}{n^\alpha} \int_{\mathbb{R}} x^p 3x^{-4} \, 1_{[1,+\infty)}(x) \, d\mu_L(x) =$$

$$= \frac{3}{n^\alpha} \int_1^{+\infty} x^{p-4} dx = \frac{3}{n^\alpha} \left(\frac{x^{p-3}}{p-3} \right) \Big|_1^{+\infty} < +\infty \iff p < 3$$

$$\text{Se } p < 3 \Rightarrow E[Y_n^p] = \frac{-3}{n^\alpha (p-3)} \longrightarrow 0 \quad n \to +\infty$$

$$\lim_{n \to \infty} Y_n(\omega) = \lim_{n \to \infty} \frac{X(\omega)}{n^\alpha} = 0 = Dir(\omega) \quad \forall \omega \quad \forall \alpha > 0$$

$(X_n)_{n\geq 1}$ converge q. c. a X se e solo se:

$$\lim_{m\to\infty} P\left(\bigcap_{n\geq m} \{|X_n - X| < \varepsilon\}\right) = 1$$

o se:

$$\lim_{m\to\infty} P\left(\bigcup_{n\geq m} \{|X_n - X| \geq \varepsilon\}\right) = 0$$

COROLLARIO

Se $\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < +\infty$, allora $X_n \xrightarrow{q.c.} X$

Se $(Z_n)_{n\geq 1}$ sono indipendenti, allora

$$Z_n \xrightarrow{q.c.} Dir(0) \iff \sum_{n=1}^{\infty} P(|Z_n| \geq \varepsilon) < +\infty \quad \forall \varepsilon > 0$$

Applicando l'ultima:

$$P(|Z_n| \geq \varepsilon) = \frac{1}{n^{3\alpha}\varepsilon^3}$$

$$\sum_{n=1}^{\infty} P(|Z_n| \geq \varepsilon) = \frac{1}{\varepsilon^3} \sum_{n=1}^{\infty} \frac{1}{n^{3\alpha}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3\alpha}}$ serie armonica generalizzata, converge per $3\alpha > 1$

$\Rightarrow \alpha > \frac{1}{3}$

Non si ha convergenza quasi certa per $0 \leq \alpha \leq \frac{1}{3}$

2)

1. $Z_n \sim Bin(n, p)$ $\qquad \bar{X}_n = \frac{1}{n} Z_n$

$E[Z_n] = np \;,\quad D^2[Z_n] = np(1-p)$

TEOREMA DEL LIMITE CENTRALE

Se la v.a. X ha momento finito di ordine 2 $(E[X] = \mu_X$, $D^2[X] = \sigma^2_X)$, allora la v.a.

Se $Z_n = \sum_{k=1}^{n} X_k$, allora $\tilde{Z}_n = \frac{Z_n - n\mu_X}{\sqrt{n}\,\sigma_X} \xrightarrow{w} N(0,1)$

$$\tilde{Z}_n = \frac{Z_n - np}{\sqrt{np(1-p)}} \xrightarrow{w} N(0,1) \quad \text{per } n \to +\infty$$

$$\sqrt{\frac{p(1-p)}{n}}\;\tilde{Z}_n = \frac{\sqrt{p(1-p)}}{\sqrt{n}}\left(\frac{Z_n}{\sqrt{n}\cdot\sqrt{p(1-p)}} - \frac{\sqrt{n}\,p}{\sqrt{p(1-p)}}\right) =$$

$$= \frac{1}{n} Z_n - p = \bar{X}_n - p$$

$$\bar{X}_n = \sqrt{\frac{p(1-p)}{n}}\;\tilde{Z}_n + p \Rightarrow E[\bar{X}_n] = p \,,\; D^2[\bar{X}_n] = \frac{p(1-p)}{n}$$

2. $P(49\,500 \le Z_n \le 50\,500) =$ $\qquad n = 100000 \,,\; p = 0.5$

$$= P\left(\frac{49500 - np}{\sqrt{np(1-p)}} \le \frac{Z_n - np}{\sqrt{np(1-p)}} \le \frac{50500 - np}{\sqrt{np(1-p)}}\right) =$$

$$= P\left(\frac{-500}{\sqrt{25000}} \le \tilde{Z}_n \le \frac{500}{\sqrt{25000}}\right) = P(-\sqrt{10} \le \tilde{Z}_n \le \sqrt{10}) =$$

$$= \Phi(\sqrt{10}) - \Phi(-\sqrt{10}) = 2\,\Phi(\sqrt{10}) - 1$$

$$P(-\sqrt{10} \le \tilde{Z}_n \le \sqrt{10}) = P(|\tilde{Z}_n| \le \sqrt{10}) =$$

$$= 1 - P(|\tilde{Z}_n| > \sqrt{10})$$

$$P(|\tilde{Z}_n| > \sqrt{10}) = P(|\tilde{Z}_n - E[\tilde{Z}_n]| > \sqrt{10}) \le \frac{D^2[\tilde{Z}_n]}{10} = \frac{1}{10}$$

$$1 - P(|\tilde{Z}_n| > \sqrt{10}) \ge 1 - \frac{1}{10} = 0.9$$

3. $Z_n \sim \text{Poisson}(n\lambda)$

Applicando il teorema del limite centrale:

$$\tilde{Z}_n : \frac{Z_n - n\lambda}{\sqrt{n\lambda}} \xrightarrow{w} N(0,1)$$

$$\sqrt{\frac{\lambda}{n}}\, \tilde{Z}_n = \frac{\sqrt{\lambda}}{\sqrt{n}} \left(\frac{Z_n}{\sqrt{n\lambda}} - \sqrt{n\lambda} \right) = \frac{Z_n}{n} - \lambda = \bar{X}_n - \lambda$$

$$\bar{X}_n = \sqrt{\frac{\lambda}{n}}\, \tilde{Z}_n + \lambda \qquad E[\bar{X}_n] = \lambda, \quad D^2[\bar{X}_n] = \frac{\lambda}{n}$$

4. $P(49.500 \le Z_n \le 50\,500) =$

$$= P\left(\frac{49500 - n\lambda}{\sqrt{n\lambda}} \le \frac{Z_n - n\lambda}{\sqrt{n\lambda}} \le \frac{50500 - n\lambda}{\sqrt{n\lambda}} \right) \overset{\substack{n = 100000 \\ \lambda = 1}}{=}$$

$$= P\left(\frac{-50500}{\sqrt{100000}} \le \tilde{Z}_n \le \frac{-49500}{\sqrt{100000}} \right) =$$

$$= P(-159,7 \le \tilde{Z}_n \le -156,5) = \Phi(-156.5) - \Phi(-159,7)$$

3) OK

4) Ho usato:

Proposition 1172 *Assume the random variables X and Y have finite moment of order 2. In addition, assume that the samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are independent. In the end, assume that the size of both the samples is large[4]. Then, given any $\alpha \in (0,1)$, an approximate $100(1-\alpha)\%$ confidence interval for $\mu_X - \mu_Y$ is given by the statistics*

$$\bar{X}_m - \bar{Y}_n - z_{\frac{\alpha}{2}} \sqrt{\frac{S^2_{X,m}}{m} + \frac{S^2_{Y,n}}{n}} \quad \text{and} \quad \bar{X}_m - \bar{Y}_n + z_{\frac{\alpha}{2}} \sqrt{\frac{S^2_{X,m}}{m} + \frac{S^2_{Y,n}}{n}}, \tag{15.44}$$

where $z_{\frac{\alpha}{2}} \equiv z^+_{\frac{\alpha}{2}}$ is the upper tail critical value of level $\alpha/2$ of the standard Gaussian distribution. The realizations of such a confidence interval are of the form

$$\left(\bar{x}_m - \bar{y}_n - z_{\frac{\alpha}{2}} \sqrt{\frac{s^2_{X,m}}{m} + \frac{s^2_{Y,n}}{n}}, \bar{x}_m - \bar{y}_n + z_{\frac{\alpha}{2},\hat{\nu}} \sqrt{\frac{s^2_{X,m}}{m} + \frac{s^2_{Y,n}}{n}}\right), \tag{15.45}$$

where $\bar{x}_m$ [resp. $\bar{y}_n$] is the value taken by $\bar{X}_m$ [resp. $\bar{Y}_n$] on any of the available realization $x_1, \ldots, x_m$ [resp. $y_1, \ldots, y_n$] of the sample $X_1, \ldots, X_m$ [resp. $Y_1, \ldots, Y_n$] and $s^2_{X,m}$ [resp. $s^2_{Y,n}$] is the value taken by $S^2_{X,m}$ [resp. $S^2_{Y,n}$] on the data set $x_1, \ldots, x_m$ [resp. $y_1, \ldots, y_n$] used to evaluate $\bar{x}_m$ [resp. $\bar{y}_n$].

es. $S^2_{X,50} = \frac{1}{49} \sum_{k=1}^{n} (X_K - \bar{X}_{50})^2 = \frac{1}{49} \sum_{k=1}^{50} (X_K^2 + \bar{X}_{50}^2 - 2 X_K \bar{X}_{50}) =$

$= \frac{1}{49} \left(\sum_{k=1}^{50} X_K^2 - 2 \bar{X}_{50} \sum_{k=1}^{50} X_K + 50 \bar{X}_{50}^2 \right) =$

$= \frac{1}{49} \left(45 - \frac{90}{50} \cdot 45 + 50 \frac{45^2}{50^2} \right) = 0,092$

$H_0 : \mu_0 = 1\,\text{Kg}$ $\quad H_a : \mu_X \neq \mu_0$

$\quad\quad\quad\quad$ // $\quad H_a : \mu_X > \mu_0$

$\quad\quad\quad\quad\quad\quad H_a : \mu_X < \mu_0$

$$Z = \frac{X - \mu}{\sigma_{X,m}/\sqrt{n}} \sim \cancel{T_{m-1}}\ N(0,1)$$

$$Z = \frac{(n+1) S^2_{X,n}}{\sigma_0^2} \sim \chi^2_{n-1}$$