II Università di Roma, Tor Vergata
Dipartimento d'Ingegneria Civile e Ingegneria Informatica
LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
Instructors: Roberto Monte & Massimo Regoli
Intermediate Test - 2020-01-27 - Probability

Problem 1 *Students are allowed to take a test twice in an examination session. Assume that* 7 *students over* 10 *pass the test on the first try. For those who fail, only* 4 *students over* 10 *pass the test on the second try.*

1. *Find the probability that a randomly selected student passes the test.*
2. *Assuming that a student passed the test what is the probability she passed on the first try?*
3. *The first test presents the following problem: two dice are rolled and the number on the upper faces are observed. Is the event "the sum of the observed numbers is* 8*" independent of the event "the number observed on the upper face of a die is* 4*"? Can you give a solution to this problem?*

   Hint. it might be useful to consider the event E_k "a randomly selected student passes the test on the kth try" for $k = 1, 2$.

Solution. .

Problem 2 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ be the Euclidean real plane endowed with the Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^2\right)$ and the Lebesgue measure $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$. Let $f : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f(x, y) \stackrel{def}{=} k e^{-(x^2 - xy + y^2/2)}, \quad \forall (x, y) \in \mathbb{R}^2.$$

1. *Determine $k \in \mathbb{R}$ such that $f : \mathbb{R}^2 \to \mathbb{R}_+$ is a probability density. Hint: can you compute $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2}\, dy$ with no computation?*

   Let $Z \equiv (X, Y)$ be the random vector on Ω with density $f : \mathbb{R}^2 \to \mathbb{R}_+$.

2. *Determine the marginal density of the entries X and Y. Are the random variables X and Y Gaussian?*
3. *Compute $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$, and $Cov(X, Y)$.*
4. *Are X and Y independent random varables?*
5. *Is the random vector Z Gaussian? Hint: consider the answer you gave to* 4.*, what you know from the theory, and try to make a simple guess.*

Solution. .

Problem 3 *Let B [resp. R] be a standard Bernoulli [resp. Rademacher] random variable on a probability space Ω. In symbols, $B \sim Ber(1/2)$ [resp. $R \sim Rad(1/2)$. Assume that B and R are independent and define $X \equiv B \cdot R$.*

1. *Compute $\mathbf{E}[X \mid B]$ and $\mathbf{E}[X \mid R]$.*
2. *Are the random variables $\mathbf{E}[X \mid B]$ and $\mathbf{E}[X \mid R]$ uncorrelated? Are they independent?*

3. *Compute* $\mathbf{E}[B \mid X]$ *and* $\mathbf{E}[R \mid X]$.

4. *Are the random variables* $\mathbf{E}[B \mid X]$ *and* $\mathbf{E}[R \mid X]$ *uncorrelated? Are they independent?*

5. *Are the random variables* B *and* X *uncorrelated? Are* B *and* X *independent?*

6. *Are the random variables* R *and* X *uncorrelated? Are* R *and* X *independent?*

7. *Compute* $\mathbf{E}[B+R \mid X]$ *and* $\mathbf{E}\left[(B+R)^2 \mid X\right]$.

Solution. .

Problem 4 *Let* B *be a binomial random variable with number of trials paramenter* n *and success probability* p*, which models the number of successes in* n *independent trials, and let* $(X_k)_{k=1}^n$ *be a finite sequence of independent and exponentially distributed random variables with rate parameter* λ*, which are also independent of* B*. Study the conditional expectation*

$$\mathbf{E}\left[\textstyle\sum_{k=1}^{B} X_k \mid B\right].$$

Use the properties of the conditional expectation to compute the expectation (and the variance) of the random sum

$$S_B \stackrel{def}{=} \textstyle\sum_{k=1}^{B} X_k.$$

Solution. .

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Problem 5 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(X_n)_{n\geq 1}$ *be a sequence of independent real random variables on* Ω *such that*

$$X_n \sim Ber\,(1/n^\alpha)\,,$$

for some $\alpha > 0$*. Consider the sequence* $(Y_n)_{n\geq 1}$ *of real random variables on* Ω *given by*

$$Y_n \stackrel{def}{=} \min\{X_1, \ldots, X_n\}\,.$$

1. *After characterizing* Y_n*, study the convergence in distribution, in probability, in mean, and in quadratic mean of* $(X_n)_{n\geq 1}$ *and* $(Y_n)_{n\geq 1}$ *on varying of* $\alpha > 0$.

2. *Study the almost sure convergence of* $(X_n)_{n\geq 1}$ *on varying of* $\alpha > 0$*. Hint: it might be useful to recall when the series* $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ *or equivalently the integral* $\int_1^{+\infty} \frac{1}{x^\alpha} dx$ *converges.*

3. *Can you study the almost sure convergence of* $(Y_n)_{n\geq 1}$ *on varying of* $\alpha > 0$*?*

Solution. .