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Dipartimento d'Ingegneria Civile e Ingegneria Informatica
LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Distribution Functions 2021-10-28

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$*. In symbols,* $X \sim Unif(-1, 1)$*. Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} \alpha + \beta x, \quad \forall x \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\beta \neq 0$.

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 2 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$*. In symbols,* $X \sim Unif(-1, 1)$*. Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} |x|, \quad \forall x \in \mathbb{R},$$

where $|x|$ *is the absolute value of* x.

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 3 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif\,(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} x^2, \quad \forall x \in \mathbb{R}.$$

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

   $$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

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5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 4 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif\,(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} \begin{cases} 0, & \text{if } x \le 0. \\ x^2, & \text{if } x > 0. \end{cases}$$

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

   $$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. .

Problem 5

1. *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif\,(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

   $$g(x) \overset{def}{=} x^2 - 2x, \quad \forall x \in \mathbb{R},$$

   Can you show that the function $Y : \Omega \to \mathbb{R}$ *given by*

   $$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 6 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be an exponentially distributed random variable with rate parameter $\lambda = 1$. In symbols, $X \sim Exp(1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \overset{def}{=} 1 - \exp(-x), \quad \forall x \in \mathbb{R},$$

where $\exp : \mathbb{R} \to \mathbb{R}$ is the Neper exponential function.

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the random variable Y?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 7 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $(-1, 1)$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R}_{++} \to \mathbb{R}$ given by*

$$g(y) \overset{def}{=} -\frac{1}{\lambda} \ln(y), \quad \forall \in \mathbb{R}_{++},$$

where $\ln : \mathbb{R}_{++} \to \mathbb{R}$ is the natural logarithm function and $\lambda > 0$.

1. *Can you state that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a real random variable on Ω?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of $Y : \Omega \to \mathbb{R}$?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

   *Hint: recall the properties of the **logarithm** and **exponential** function.*

Solution. .