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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Distribution Functions 2021-11-17 with Some Solutions

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$. *In symbols,* $X \sim Unif\,(-1, 1)$. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} \alpha + \beta x, \quad \forall x \in \mathbb{R},$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\beta \neq 0$.

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution.

1. The function $g : \mathbb{R} \to \mathbb{R}$ is a Borel function. Therefore, $Y = g \circ X$ is a random variable.

2. Recall that $X \sim Unif\,(-1, 1)$ is absolutely continuous with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x),$$

   for every $x \in \mathbb{R}$. Hence, writing $F_X : \mathbb{R} \to \mathbb{R}$ for the distribution function of X, we have

$$\begin{aligned} F_X(x) &= \int_{(-\infty,x]} f_X(u)\, d\mu_L(u) = \int_{(-\infty,x]} \frac{1}{2} 1_{[-1,1]}(u)\, d\mu_L(u) \\ &= \frac{1}{2} \int_{(-\infty,x]\cap[-1,1]} d\mu_L(u) = \frac{1}{2} \mu_L((-\infty, x] \cap [-1, 1]). \end{aligned}$$

   On the other hand,

$$(-\infty, x] \cap [-1, 1] = \begin{cases} \varnothing, & \text{if } x < -1, \\ \{-1\}, & \text{if } x = -1, \\ [-1, x], & \text{if } x < -1. \end{cases}$$

   Therefore,

$$F_X(x) = \begin{cases} 0, & \text{if } x < -1, \\ \frac{x+1}{2}, & \text{if } -1 \leq x < 1, \\ 1, & \text{if } 1 \leq x. \end{cases}$$

Now, since g is a continuously differentiable real function on $\mathbb{R}$, in particular a Borel function, then $Y \equiv g(X) = \alpha + \beta X$ is a real random variable. To compute the distribution function F_Y, we apply the definition

$$F_Y(y) \overset{\text{def}}{=} \mathbf{P}(Y \leq y), \quad \forall y \in \mathbb{R}.$$

On the other hand, considering that $\beta \neq 0$, we have

$$\begin{aligned}
\mathbf{P}(Y \leq y) &= \mathbf{P}(\alpha + \beta X \leq y) = \mathbf{P}\left(X \leq \frac{y-\alpha}{\beta}\right) \\
&= F_X\left(\frac{y-\alpha}{\beta}\right) = \begin{cases} 0, & \text{if } \frac{y-\alpha}{\beta} < -1 \Leftrightarrow y < \alpha - \beta, \\ \frac{\frac{y-\alpha}{\beta}+1}{2} = \frac{y+\beta-\alpha}{2\beta}, & \text{if } -1 \leq \frac{y-\alpha}{\beta} < 1 \Leftrightarrow \alpha - \beta \leq y < \alpha + \beta, \\ 1, & \text{if } 1 \leq \frac{y-\alpha}{\beta} \Leftrightarrow \alpha + \beta \leq y. \end{cases}
\end{aligned}$$

Summarizing,

$$F_Y(y) = \begin{cases} 0, & \text{if } y < \alpha - \beta, \\ \frac{y+\beta-\alpha}{2\beta}, & \text{if } \alpha - \beta \leq y \leq \alpha + \beta, \\ 1, & \text{if } \alpha + \beta < y. \end{cases}$$

Therefore, the random variable Y turns out to be a uniformly distributed random variable on the interval $[\alpha - \beta, \alpha + \beta]$. In symbols, $Y \sim Unif(\alpha - \beta, \alpha + \beta)$. It then follows that Y is absolutely continuous with density $f_Y : \mathbb{R} \to \mathbb{R}$ given by

$$f_Y(y) = \frac{1}{2\beta} 1_{[\alpha-\beta,\alpha+\beta]}(y).$$

3. Since X is in the linear space $\mathcal{L}^2(\Omega; \mathbb{R})$, the random variable $Y = \alpha + \beta X$ is also in the linear space $\mathcal{L}^2(\Omega; \mathbb{R})$. Hence, Y has finite moments of order 1 and 2.

4. Thanks to the linearity of the expectation operator, we have

$$\mathbf{E}[Y] = \mathbf{E}[\alpha + \beta X] = \alpha + \beta \mathbf{E}[X],$$

where

$$\mathbf{E}[X] = \int_{\mathbb{R}} \frac{1}{2} x 1_{[-1,1]}(x)\, d\mu_L(x) = \frac{1}{2} \int_{[-1,1]} x\, d\mu_L(x) = \frac{1}{2} \int_{-1}^{1} x\, dx = \frac{1}{4} x^2\Big|_{-1}^{1} = 0.$$

Therefore,

$$\mathbf{E}[Y] = \alpha.$$

Moreover considering the properties of the variance operator, we have

$$\mathbf{D}^2[Y] = \mathbf{D}^2[\alpha + \beta X] = \beta^2 \mathbf{D}^2[X],$$

where

$$\mathbf{D}^2[X] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X] = \mathbf{E}\left[X^2\right]$$

and

$$\mathbf{E}\left[X^2\right] = \int_{\mathbb{R}} \frac{1}{2} x^2 1_{[-1,1]}(x)\, d\mu_L(x) = \frac{1}{2} \int_{[-1,1]} x^2\, d\mu_L(x) = \frac{1}{2} \int_{-1}^{1} x^2\, dx = \frac{1}{6} x^3\Big|_{-1}^{1} = \frac{1}{3}.$$

Therefore,

$$\mathbf{D}^2[Y] = \frac{\beta^2}{3}.$$

Problem 2 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $X : \Omega \to \mathbb{R}$ *be a uniformly distributed random variable with states in the interval* $[-1, 1]$*. In symbols,* $X \sim Unif(-1, 1)$*. Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} |x|, \quad \forall x \in \mathbb{R},$$

where $|x|$ *is the absolute value of* x*.*

1. *Can you show that the function* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

   is a random variable?

2. *Can you compute the distribution function* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the random variable* Y*?*

3. *Is* Y *absolutely continuous?*

4. *Are the first and second order moments of* Y *finite?*

5. *If the first and second order moments of* Y *are finite, can you compute* $\mathbf{E}[Y]$ *and* $\mathbf{D}^2[Y]$*?*

Solution. Recall that, since $X \sim Unif(-1, 1)$, the random variable X is absolutely continuous with density

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x),$$

for every $x \in \mathbb{R}$. Now, we have

$$F_Y(y) \overset{\text{def}}{=} \mathbf{P}(Y \leq y) = \mathbf{P}(g(X) \leq y) = \mathbf{P}(|X| \leq y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(-y \leq X \leq y), & \text{if } y \geq 0. \end{cases}$$

On the othe hand, under the assumption $y \geq 0$, we have

$$\begin{aligned}
\mathbf{P}(-y \leq X \leq y) &= \int_{[-y,y]} f_X(x)\, d\mu_X(x) \\
&= \int_{[-y,y]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\
&= \frac{1}{2} \int_{[-y,y]\cap[-1,1]} d\mu_X(x) \\
&= \frac{1}{2} \mu_X([-y, y] \cap [-1, 1]),
\end{aligned}$$

where

$$\mu_X([-y, y] \cap [-1, 1]) = \begin{cases} \mu_X([-y, y]) = 2y, & \text{if } y \leq 1, \\ \mu_X([-1, 1]) = 2, & \text{if } y > 1. \end{cases}$$

It follows

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ y, & \text{if } 0 \leq y \leq 1, \\ 1, & \text{if } y > 1. \end{cases}$$

We can then recognize that $Y \sim Unif(0, 1)$, which implies that Y is absolutely continuous with density given by

$$f_Y(y) = 1_{[0,1]}(y),$$

for every $y \in \mathbb{R}$, and Y has finite first and second order moments. More specifically

$$\begin{aligned}\mathbf{E}[Y] &= \int_{\mathbb{R}} y f_Y(y)\, d\mu_X(y) = \int_{\mathbb{R}} y 1_{[0,1]}(y)\, d\mu_X(y) \\ &= \int_{[0,1]} y d\mu_X(y) = \int_0^1 y dy = \frac{1}{2} y^2 \Big|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left[Y^2\right] &= \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_X(y) = \int_{\mathbb{R}} y^2 1_{[0,1]}(y)\, d\mu_X(y) \\ &= \int_{[0,1]} y^2 d\mu_X(y) = \int_0^1 y^2 dy = \frac{1}{3} y^3 \Big|_0^1 \\ &= \frac{1}{3}.\end{aligned}$$

It follows

$$\mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Note that, since $Y = |X|$ it would be possible to compute $\mathbf{E}[Y]$ and $\mathbf{E}\left[Y^2\right]$ by using the density of X. That is

$$\begin{aligned}\mathbf{E}[Y] = \mathbf{E}[|X|] &= \int_{\mathbb{R}} |x|\, f_X(x)\, d\mu_X(x) \\ &= \int_{\mathbb{R}} |x| \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\ &= \frac{1}{2} \int_{[-1,1]} |x|\, d\mu_X(x) \\ &= \frac{1}{2}\left(\int_{[-1,0]} -x d\mu_X(x) + \int_{[0,1]} x d\mu_X(x) \right) \\ &= \frac{1}{2}\left(\int_{-1}^0 -x dx + \int_0^1 x dx \right) \\ &= \frac{1}{2}\left(-\int_{-1}^0 x dx + \int_0^1 x dx \right) \\ &= \frac{1}{2}\left(-\frac{1}{2} x^2 \Big|_{-1}^0 + \frac{1}{2} x^2 \Big|_0^1 \right) \\ &= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} \right) \\ &= \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[Y^2\right] &= \mathbf{E}\left[|X|^2\right] = \mathbf{E}\left[X^2\right] = \int_{\mathbb{R}} x^2 f_X(x)\, d\mu_X(x) \\
&= \int_{\mathbb{R}} x^2 \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_X(x) \\
&= \frac{1}{2}\int_{[-1,1]} x^2 d\mu_X(x) \\
&= \frac{1}{2}\int_{-1}^{1} x^2 dx \\
&= \frac{1}{2}\frac{1}{3}x^3\Big|_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) \\
&= \frac{1}{3}.
\end{aligned}$$

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Problem 3 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \stackrel{def}{=} x^2, \quad \forall x \in \mathbb{R}.$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \stackrel{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of the random variable Y?*
3. *Is Y absolutely continuous?*
4. *Are the first and second order moments of Y finite?*
5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \stackrel{def}{=} \begin{cases} 0, & \text{if } x \le 0. \\ x^2, & \text{if } x > 0. \end{cases}$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \stackrel{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

   is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the random variable Y?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. Recall that $X \sim Unif(-1,1)$ is absolutely continuous, with density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \frac{1}{2} 1_{[-1,1]}(x).$$

Note also that we can write

$$g(x) = x^2 1_{(0,+\infty)}(x),$$

for every $x \in \mathbb{R}$.

1. The function g is clearly continuous. In particular, g is a Borel function. Therefore, $Y = g \circ X$ is a random variable.

2. The distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of Y is given by

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(g(X) \leq y)$$

for every $y \in \mathbb{R}$. Now, due to the definition of g, we have

$$\{x \in \mathbb{R} : g(x) \leq y\} = \begin{cases} \varnothing, & \text{if } y < 0, \\ \{x \in \mathbb{R} : x \leq \sqrt{y}\}, & \text{if } y \geq 0. \end{cases}$$

Hence,

$$\{g(X) \leq y\} = \begin{cases} \varnothing, & \text{if } y < 0, \\ \{X \leq \sqrt{y}\}, & \text{if } y \geq 0. \end{cases}$$

It follows,

$$\mathbf{P}(g(X) \leq y)) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbf{P}(X \leq \sqrt{y}), & \text{if } y \geq 0. \end{cases}$$

On the other hand, since $X \sim Unif(-1,1)$, we have

$$\begin{aligned} \mathbf{P}(X \leq \sqrt{y}) &= \int_{(-\infty,\sqrt{y}]} f_X(x)\, d\mu_L(x) \\ &= \int_{(-\infty,\sqrt{y}]} \frac{1}{2} 1_{[-1,1]}(x)\, d\mu_L(x) \\ &= \frac{1}{2} \int_{(-\infty,\sqrt{y}]\cap[-1,1]} d\mu_L(x) \\ &= \frac{1}{2} \mu_L((-\infty,\sqrt{y}] \cap [-1,1]), \end{aligned}$$

where

$$(-\infty,\sqrt{y}] \cap [-1,1] = \begin{cases} [-1,\sqrt{y}], & \text{if } 0 \leq y < 1, \\ [-1,1], & \text{if } y \geq 1. \end{cases}$$

Therefore,

$$\mathbf{P}(X \leq \sqrt{y}) = \begin{cases} \frac{1}{2}(\sqrt{y}+1), & \text{if } y < 1, \\ 1, & \text{if } y \geq 1. \end{cases}$$

We can then write,

$$F_Y(y) = \frac{1}{2}(\sqrt{y}+1)\,1_{[0,1]}(y) + 1_{(1,+\infty)}(y).$$

Note that

$$\mathbf{P}(Y<0) = F_Y(0) = 0.$$

Hence, Y is a non negative random variable.

3. Note that $F_Y : \mathbb{R} \to \mathbb{R}$ is not continuous since

$$\lim_{x\to 0^-} F_Y(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} F_Y(x) = \frac{1}{2}.$$

A fortiori it is not absolutely continuous.

4. We have

$$\int_\Omega Y^2 d\mathbf{P} = \int_\Omega g(X)^2\, d\mathbf{P}.$$

Therefore, Y has finite moment of order 2 or not according to whether

$$\int_\Omega g(X)^2\, d\mathbf{P} < \infty.$$

Now, since X is absolutely continuous, we can write

$$\begin{aligned}
\int_\Omega g(X)^2\, d\mathbf{P} &= \int_\mathbb{R} g(x)^2 f_X(x)\, d\mu_L(x) \\
&= \int_\mathbb{R} x^4 1_{(0,+\infty)}(x)\, 1_{(-1,1)}(x)\, d\mu_L(x) \\
&= \frac{1}{2}\int_{(0,1)} x^4 d\mu_L(x) \\
&= \frac{1}{2}\int_0^1 x^4 dx \\
&= \frac{1}{10}\, x^5\Big|_0^1 \\
&= \frac{1}{10}.
\end{aligned}$$

It follows, that Y has finite moment of order 2 and

$$\mathbf{E}\left[Y^2\right] = \int_\Omega Y^2 d\mathbf{P} = \frac{1}{10}.$$

A fortiori Y has finite moment of order 1 and

$$\begin{aligned}
\mathbf{E}[Y] = \mathbf{E}[g(X)] &= \int_\mathbb{R} g(x)\, f_X(x)\ d\mu_L(x) \\
&= \int_\mathbb{R} x^2 1_{(0,+\infty)}(x)\, 1_{(-1,1)}(x)\ d\mu_L(x) \\
&= \frac{1}{2}\int_{(0,1)} x^2\ d\mu_L(x) \\
&= \frac{1}{2}\int_0^1 x^2\ dx \\
&= \frac{1}{6}\, x^3\Big|_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

In the end,

$$\mathbf{D}^2[Y] = \mathbf{E}\left[Y^2\right] - \mathbf{E}[Y]^2 = \frac{1}{10} - \frac{1}{36} = \frac{13}{180}.$$

Problem 5 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $[-1, 1]$. In symbols, $X \sim Unif(-1, 1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \stackrel{def}{=} x^2 - 2x, \quad \forall x \in \mathbb{R},$$

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \stackrel{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the random variable Y?*
3. *Is Y absolutely continuous?*
4. *Are the first and second order moments of Y finite?*
5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 6 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be an exponentially distributed random variable with rate parameter $\lambda = 1$. In symbols, $X \sim Exp(1)$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) \stackrel{def}{=} 1 - \exp(-x), \quad \forall x \in \mathbb{R},$$

where $\exp : \mathbb{R} \to \mathbb{R}$ *is the Neper exponential function.*

1. *Can you show that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \stackrel{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega,$$

is a random variable?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}_+$ of the random variable Y?*
3. *Is Y absolutely continuous?*
4. *Are the first and second order moments of Y finite?*
5. *If the first and second order moments are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

Solution. .

Problem 7 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a uniformly distributed random variable with states in the interval $(0, 1)$. In symbols, $X \sim Unif(0, 1)$. Consider the function $g : \mathbb{R}_{++} \to \mathbb{R}$ given by*

$$g(y) \stackrel{def}{=} -\frac{1}{\lambda} \ln(y), \quad \forall \in \mathbb{R}_{++},$$

where $\ln : \mathbb{R}_{++} \to \mathbb{R}$ *is the natural logarithm function and* $\lambda > 0$.

1. *Can you state that the function $Y : \Omega \to \mathbb{R}$ given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega.$$

is a real random variable on Ω?

2. *Can you compute the distribution function $F_Y : \mathbb{R} \to \mathbb{R}$ of $Y : \Omega \to \mathbb{R}$?*

3. *Is Y absolutely continuous?*

4. *Are the first and second order moments of Y finite?*

5. *If the first and second order moments of Y are finite, can you compute $\mathbf{E}[Y]$ and $\mathbf{D}^2[Y]$?*

   *Hint: recall the properties of the **logarithm** and **exponential** function.*

Solution.

1. Note that, since $X \sim Unif(0,1)$, that is X has density $f_X : \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) \overset{\text{def}}{=} 1_{[0,1]}(x), \quad \forall x \in \mathbb{R},$$

we have

$$\mathbf{P}(X \leq 0) = \int_{(-\infty,0]} f_X(x)\, d\mu_L(x) = \int_{(-\infty,0]} 1_{[0,1]}(x)\, d\mu_L(x) = \int_{(-\infty,0]\cap[0,1]} d\mu_L(x) = \int_{\{0\}} d\mu_L(x) = \mu_L(0) =$$

Therefore, since $\ln : \mathbb{R}_{++} \to \mathbb{R}$ is a Borel function on $\mathbb{R}_{++}$, the function $Y : \Omega \to \mathbb{R}$ is well defined and it is a real random variable on Ω.

2. Considering that $\lambda > 0$ and the exponential function is the inverse of the logarithm function, we have

$$\{Y \leq y\} = \left\{-\frac{1}{\lambda}\ln(X) \leq y\right\} = \{\ln(X) \geq -\lambda y\} = \left\{X \geq e^{-\lambda y}\right\}$$

for every $y \in \mathbb{R}$. As a consequence,

$$\mathbf{P}(Y \leq y) = \mathbf{P}\left(X \geq e^{-\lambda y}\right) = \int_{[e^{-\lambda y},+\infty)} 1_{[0,1]}(x)\, d\mu_L(x) = \int_{[e^{-\lambda y},+\infty)\cap[0,1]} d\mu_L(x).$$

On the other hand,

$$\left[e^{-\lambda y}, +\infty\right) \cap [0,1] = \begin{cases} \left[e^{-\lambda y}, 1\right], & \text{if } y \geq 0, \\ \varnothing, & \text{if } y < 0. \end{cases}$$

Therefore,

$$\mathbf{P}(Y \leq y) = \begin{cases} \mu_L\left(e^{-\lambda y}, 1\right) = 1 - e^{-\lambda y}, & \text{if } y \geq 0, \\ \mu_L(\varnothing) = 0, & \text{if } y < 0. \end{cases}$$

That is

$$F_Y(y) = \left(1 - e^{-\lambda y}\right) 1_{\mathbb{R}_+}(y).$$

It then follows that Y is an exponentially distributed random variable with rate parameter λ, in symbols $Y \sim Exp(\lambda)$.

3. Since $Y \sim Exp(\lambda)$ it is well known that Y is absolutely continuous with density $f_Y : \mathbb{R} \to \mathbb{R}$ given by

$$f_Y(y) \overset{\text{def}}{=} \lambda e^{-\lambda y} 1_{\mathbb{R}_+}(y), \quad \forall y \in \mathbb{R}.$$

If we are not aware of this, we can observe that

$$F'_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ \lambda e^{-\lambda y}, & \text{if } y > 0. \end{cases}$$

That is

$$F'_Y(y) = f_Y(y),$$

for every $y \in \mathbb{R} - \{0\}$. On the other hand, F_Y is not differentiable at $y = 0$. Nevertheless, we have

$$\int_{(-\infty,y]} f_Y(v)\, d\mu_L(v) = \int_{(-\infty,y]} \lambda e^{-\lambda v} 1_{\mathbb{R}_+}(v)\, d\mu_L(v) = \int_{(-\infty,y]\cap\mathbb{R}_+} \lambda e^{-\lambda v}\, d\mu_L(v),$$

where

$$(-\infty, y] \cap \mathbb{R}_+ = \begin{cases} \varnothing, & \text{if } y < 0, \\ 0 & \text{if } y = 0, \\ [0, y], & \text{if } 0 < y. \end{cases}$$

Hence,

$$\int_{(-\infty,y]\cap\mathbb{R}_+} \lambda e^{-\lambda y}\, d\mu_L(v) = \begin{cases} 0, & \text{if } y \leq 0, \\ \int_{[0,y]} \lambda e^{-\lambda v}\, d\mu_L(v), & \text{if } y > 0. \end{cases}$$

Now, we have

$$\int_{[0,y]} \lambda e^{-\lambda v}\, d\mu_L(v) = \int_0^y \lambda e^{-\lambda v}\, dv = -\int_0^y de^{-\lambda v} = -\left. e^{-\lambda v} \right|_0^y = 1 - e^{-\lambda y}.$$

It then follows

$$\int_{(-\infty,y]} f_Y(v)\, d\mu_L(v) = \left(1 - e^{-\lambda y}\right) 1_{\mathbb{R}_+}(y) = F_Y(y),$$

which shows that Y is absolutely continuous with density $f_Y : \mathbb{R} \to \mathbb{R}$.

4. Since $Y \sim Exp(\lambda)$ it is well known that Y ha finite moments of order 1 and 2. If we are not aware of this, we can observe that

$$\int_\Omega Y^2 d\mathbf{P} = \int_{\mathbb{R}} y^2 f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y^2 \lambda e^{-\lambda y} 1_{\mathbb{R}_+}(y)\, d\mu_L(y) = \int_{\mathbb{R}_+} \lambda y^2 e^{-\lambda y}\, d\mu_L(y) = \int_0^{+\infty} \lambda y^2 e^{-\lambda y}\, dy.$$

On the other hand,

$$\int_0^{+\infty} y^2 \lambda e^{-\lambda y}\, dy = \lim_{y\to+\infty} \int_0^y \lambda v^2 e^{-\lambda v}\, dv,$$

where integrating by parts

$$
\begin{aligned}
\int_0^y \lambda v^2 e^{-\lambda v} dv &= -\int_0^y v^2 de^{-\lambda v} \\
&= -\left. v^2 e^{-\lambda v}\right|_0^y + 2\int_0^y v e^{-\lambda v} dv \\
&= -y^2 e^{-\lambda y} + \frac{2}{\lambda}\int_0^y \lambda v e^{-\lambda v} dv \\
&= -y^2 e^{-\lambda y} - \frac{2}{\lambda}\int_0^y v de^{-\lambda v} \\
&= -y^2 e^{-\lambda y} - \frac{2}{\lambda}\left. v e^{-\lambda v}\right|_0^y + \frac{2}{\lambda}\int_0^y e^{-\lambda v} dv \\
&= -y^2 e^{-\lambda y} - \frac{2}{\lambda} y e^{-\lambda y} - \frac{2}{\lambda^2}\int_0^y de^{-\lambda v} \\
&= -y^2 e^{-\lambda y} - \frac{2}{\lambda} y e^{-\lambda y} - \frac{2}{\lambda^2}\left. e^{-\lambda v}\right|_0^y \\
&= -y^2 e^{-\lambda y} - \frac{2}{\lambda} y e^{-\lambda y} - \frac{2}{\lambda^2} e^{-\lambda v} + \frac{2}{\lambda^2}.
\end{aligned}
$$

It follows

$$
\lim_{y\to+\infty}\int_0^y \lambda v^2 e^{-\lambda v} dv = \lim_{y\to+\infty}\left(-y^2 e^{-\lambda y} - \frac{2}{\lambda} y e^{-\lambda y} - \frac{2}{\lambda^2} e^{-\lambda v} + \frac{2}{\lambda^2}\right) = \frac{2}{\lambda^2}.
$$

Therefore, Y ha finite moments of order 2 and we have

$$
\mathbf{E}\left[Y^2\right] = \frac{2}{\lambda^2}.
$$

This implies also that Y has finite moment of order 1.

5. We have

$$
\mathbf{E}[Y] = \int_\Omega Y\, d\mathbf{P} = \int_{\mathbb{R}} y f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} y\lambda e^{-\lambda y} 1_{\mathbb{R}_+}(y)\, d\mu_L(y) = \int_{\mathbb{R}_+} \lambda y e^{-\lambda y} d\mu_L(y) = \int_0^{+\infty} \lambda y e^{-\lambda y} dy.
$$

On the other hand,

$$
\int_0^{+\infty} y\lambda e^{-\lambda y} dy = \lim_{y\to+\infty}\int_0^y \lambda v e^{-\lambda v} dv,
$$

where integrating by parts

$$
\begin{aligned}
\int_0^y \lambda v e^{-\lambda v} dv &= -\int_0^y v de^{-\lambda v} \\
&= -\left. v e^{-\lambda v}\right|_0^y + \int_0^y e^{-\lambda v} dv \\
&= -y e^{-\lambda y} + \frac{1}{\lambda}\int_0^y \lambda e^{-\lambda v} dv \\
&= -y e^{-\lambda y} - \frac{1}{\lambda}\int_0^y de^{-\lambda v} \\
&= -y e^{-\lambda y} - \frac{1}{\lambda}\left. e^{-\lambda v}\right|_0^y \\
&= -y e^{-\lambda y} - \frac{1}{\lambda} e^{-\lambda y} + \frac{1}{\lambda}
\end{aligned}
$$

It follows

$$\lim_{y\to+\infty}\int_0^y \lambda v e^{-\lambda v}dv = \lim_{y\to+\infty}\left(-ye^{-\lambda y}-\frac{1}{\lambda}e^{-\lambda y}+\frac{1}{\lambda}\right)=\frac{1}{\lambda}.$$

Thus,

$$\mathbf{E}\left[Y\right]=\frac{1}{\lambda}.$$

In the end,

$$\mathbf{D}^2\left[Y\right]=\mathbf{E}\left[Y^2\right]-\mathbf{E}\left[Y\right]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2},$$

as it is well known.