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Dipartimento d'Ingegneria Civile e Ingegneria Informatica
LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica - Advanced Statistics
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Final Test - 2020-02-25 - Probability

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space, let* $(\mathbb{R}, \mathcal{B}(\mathbb{R})) \equiv \mathbb{R}$ *be the Euclidean real line endowed with the Borel* σ*-algebra, and let* X *be a real random variable on* Ω*. Consider the functions* $g : \mathbb{R}_+ \to \mathbb{R}$ *and* $h : \mathbb{R}_{++} \to \mathbb{R}$ *given by*

$$g(x) \overset{def}{=} \sqrt{x}, \quad \forall x \in \mathbb{R}_+ \qquad \text{and} \qquad h(x) \overset{def}{=} \ln(x), \quad \forall x \in \mathbb{R}_{++}.$$

1. *Can you always state that the functions* $Y : \Omega \to \mathbb{R}$ *given by*

$$Y(\omega) \overset{def}{=} g(X(\omega)), \quad \forall \omega \in \Omega \qquad \text{and} \qquad Z(\omega) \overset{def}{=} h(X(\omega)) \quad \forall \omega \in \Omega$$

   are real random variables on Ω*?*

2. *Considering the answer you gave to the above question, can you compute the distribution function,* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of* $Y : \Omega \to \mathbb{R}$ *and* $F_Z : \mathbb{R} \to \mathbb{R}_+$ *of* $Z : \Omega \to \mathbb{R}$*?*

3. *Can you show that fixed any* $\lambda \in (0,1)$*, the function* $\lambda F_Y + (1-\lambda) F_Z$ *is a distribution function?*

4. *What about the functions* F_Y^2*,* F_X^2*, and* $F_Y F_Z$*?*

Solution. .

Problem 2 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra* $\mathcal{B}\left(\mathbb{R}^2\right)$ *and the Lebesgue measure* $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$*. Let*

$$\mathbb{R}_+^2(x > y) \equiv \left\{(x,y) \in \mathbb{R}_+^2 : x > y\right\}, \qquad \mathbb{R}_+^2(x \leq y) \equiv \left\{(x,y) \in \mathbb{R}_+^2 : x \leq y\right\},$$

and let $F : \mathbb{R}^2 \to \mathbb{R}_+$ *given by*

$$F(x,y) \overset{def}{=} \left(1 - e^{-y} - \frac{1}{2} y e^{-x}\right) 1_{\mathbb{R}_+^2(x>y)}(x,y) + \left(1 - e^{-x} - \frac{1}{2} x e^{-y}\right) 1_{\mathbb{R}_+^2(x \leq y)}(x,y), \quad \forall (x,y) \in \mathbb{R}^2.$$

1. *Can you show that the function* $F : \mathbb{R}^2 \to \mathbb{R}_+$ *is a distribution function? Hint: consider carefully the sets* $\mathbb{R}_+^2(x > y)$ *and* $\mathbb{R}_+^2(x \leq y)$ *(draw a graph).*

   Let $Z \equiv (X, Y)$ *be the random vector on* Ω *with distribution function* $F : \mathbb{R}^2 \to \mathbb{R}_+$*.*

2. *Can you determine the marginal distribution of the entries* X *and* Y*?*

3. *Is the random vector* Z *absolutely continuous? Can you determine a density* $f_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *for* Z*?*

4. *If* Z *is absolutely continuous, can you determine the marginal densities of the entries* X *and* Y*? Hint: it may be useful to rewrite the indicator fnctions* $1_{\mathbb{R}_+^2(x>y)}(x,y)$ *and* $1_{\mathbb{R}_+^2(x \leq y)}(x,y)$ *in terms of product of other indicator functions.*

Solution. .

Problem 3 *Let Z_1, Z_2, Z_3 independent random variables on a probability space Ω such that such that $X_k \sim N(0,1)$, fo $k = 1,2,3$. Consider the real random variables*

$$X_1 \stackrel{def}{=} Z_1 + Z_2 + Z_3, \quad X_2 \stackrel{def}{=} Z_1 - Z_2 + Z_3, \quad X_3 \stackrel{def}{=} Z_1 - Z_3.$$

1. *What is the distribution of the vector $X \equiv (X_1, X_2, X_3)^\intercal$?*
2. *Can you compute the distribution function of X?*
3. *Among the pairs (X_1, X_2), (X_1, X_3), and (X_2, X_3) of entries of X what are made by independent random variables?*
4. *Compute the distributions of X_1, X_2, and X_3;*
5. *Think on a quick and smart way to compute $\mathbf{E}\left[X_1 X_2^2\right]$, $\mathbf{E}\left[X_1^2 X_2^2\right]$, $\mathbf{E}\left[X_2 X_3^2\right]$, $\mathbf{E}\left[X_2^2 X_3^2\right]$.*

Solution. .

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space and let R_1 and R_2 be standard Rademacher random variables on Ω. In symbols, $R_k \sim Rad(1/2)$, for $k = 1,2$. Assume that R_1 and R_2 are independent and set*

$$X \stackrel{def}{=} R_1 - R_2, \quad Y \stackrel{def}{=} -R_1 \cdot R_2$$

1. *Compute $\mathbf{E}[R_k \mid X]$ and $\mathbf{E}[R_k \mid Y]$ for $k = 1,2$.*
2. *Are the random variables $\mathbf{E}[R_1 \mid X]$ and $\mathbf{E}[R_2 \mid X]$ uncorrelated? Are they independent?*
3. *Are the random variables $\mathbf{E}[R_1 \mid Y]$ and $\mathbf{E}[R_2 \mid Y]$ uncorrelated? Are they independent?*
4. *Compute $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$.*
5. *Are the random variables $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[Y \mid X]$ uncorrelated? Are they independent?*
6. *Compute $\mathbf{E}\left[X^2 \mid Y\right]$ and $\mathbf{E}\left[Y^2 \mid X\right]$.*

Solution. .

Problem 5 *Consider the probability space $([0,1], \mathcal{B}([0,1]), \mu_L) \equiv \Omega$, where $\mathcal{B}([0,1])$ is the Borel σ-algebra on the interval $[0,1] \subseteq \mathbb{R}$ and $\mu_L : \mathcal{B}([0,1]) \to \mathbb{R}_+$ is the Borel-Lebesgue measure on $[0,1]$. Consider the sequence $(X_n)_{n\geq 1}$ given by*

$$X_n(\omega) = \begin{cases} 1 & \text{if } 0 \leq \omega \leq \frac{n+1}{2n} \\ 0 & \text{otherwise} \end{cases} .$$

1. *Can you show that $(X_n)_{n\geq 1}$ is a sequence of random variables on Ω?*
2. *Can you prove that the sequence $(X_n)_{n\geq 1}$ converges in distribution to a random variable X on Ω?*
3. *Can you prove that the sequence $(X_n)_{n\geq 1}$ converges in probability to X.*
4. *Does the sequence $(X_n)_{n\geq 1}$ converges in mean to X.*
5. *Does the sequence $(X_n)_{n\geq 1}$ converges in square mean to X?*
6. *Can you prove that the sequence $(X_n)_{n\geq 1}$ converges almost surely to X?*

Solution. .