Performance Modeling of Computer Systems and Networks

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Generating Discrete Random Variates

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Prerequisite

We assume the knowledge of discrete random variables (sect.6.1).
In particular:

- *Equilakely*(a,b)
- *Geometric*(p)
- *Bernoulli*(p)
- *Binomial*(n,p)
- *Pascal*(n,p)
- *Poisson*(μ)

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in sis2.c equilikely fa chiamata random e poi calcola qualcosa.

sis2.c

```
#include <stdio.h>
#include "rng.h"

#define MINIMUM   20
#define MAXIMUM   80
#define STOP     100     /* 100 weeks = about 2 years*/
#define sqr(x)    ((x) * (x))

long Equilikely(long a, long b)
{  return (a + (long) ((b - a + 1) * Random()));}

long GetDemand(void)
{
        return (Equilikely(10, 50)); }
```

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anche esponenziale fa qualcosa, manipolando un random.
uguale anche la uniform (il continuo dell'equilikely)

ssq2.c distribution-driven simulation

```
#include <stdio.h>
#include <math.h>
#include "rng.h"
#define LAST          10000L    /* number of jobs processed */
#define START         0.0

double Exponential(double m)                /* ------*
{return (-m * log(1.0 - Random())); }          m > 0.0
                                             ------- */
double Uniform(double a, double b)          /* ------*
{return (a + (b - a) * Random());    }         a < b
                                            * -------*/

          double GetArrival(void)
        {static double arrival = START;
          arrival += Exponential(2.0);
               return (arrival);}

          double GetService(void)
        {return (Uniform(1.0, 2.0));}
```

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cdf = cumulativa.
uno F* perchè nel caso var. discrete non è una vera inversa, prende numero (0,1) che è probabilità e ci dà x che corrisponde alla cumulativa in x.
la cumulativa in x dovrebbe avere quel valore di probabilità.

Preliminary Definitions

X random variable, $F(\cdot)$ is the cdf of X

The *inverse distribution function* (idf) of X is the function

$F^* : (0, 1) \rightarrow \chi$, $\forall\, u \in (0, 1)$

$$F^*(u) = \min_x \{x : u < F(x)\}$$

that is, if $F^*(u)=x$, x is the smallest possible value of X for which $F(x)$ is greater than u

non è esattamente quell'x, ma il più piccolo u tale che F(x) sia minore di u.

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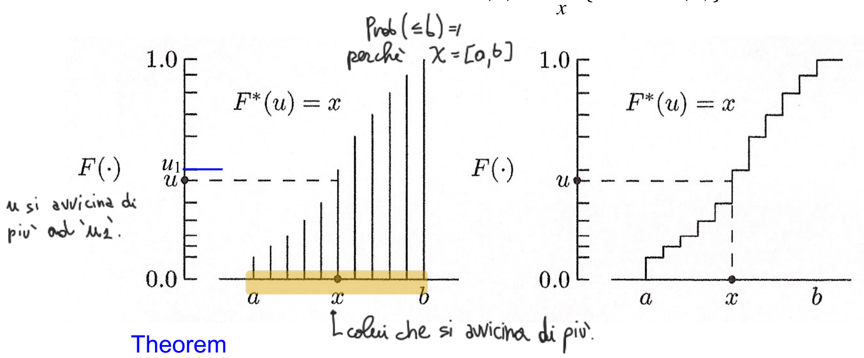
il supporto χ assume valori discreti da "a" a "b". definisco anche prob X<=x = u1

l'inversa è definita come prima

- $\chi = \{a, a+1, \ldots, b\}$, where b may be ∞, $F(\cdot)$ is the cdf of X,
- $F(x)=\text{Prob}\{X \le x\}=u_1 > u$ $\qquad F^*(u) = \min_x \{x : u < F(x)\}$



Theorem

- if $u < F(a)$, $F^*(u) = a$
- else $F^*(u) = x$ where $x \in \chi$ is the unique possible value of X for which $F(x-1) \le u < F(x)$

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come lo trovo? modo 1, parto dal minimo, incremento finchè rispetto la condizione. con molti valori è ricerca lineare lentissima.

Algorithm 1

```
x = a;
while (F(x) <= u)
        x++;
return x;            /*x is F*(u)*/
```

Average case analysis:
- let Y be the number of `while loop` passes
- $Y = X - a$
- $E[Y] = E[X - a] = E[X] - a = \mu - a$

Linear search algorithm !

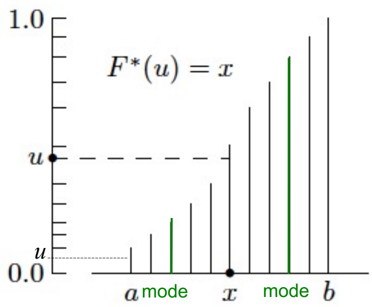
parto dalla moda (il piu probabile). da li mi muovo avanti o indietro.

Idea: start at a more likely point

For $\mathcal{X}=\{a, a+1, \ldots, b\}$, a more efficient linear search algorithm defines $F^*(u)$

Algorithm 2

```
x = mode;          /*initialize with the mode of X */
if (F(x) <= u)
    while (F(x) <= u)
        x++; (avanti)
else if (F(a) <= u)
        while (F(x-1) > u)
          x--; (indietro)
    else
        x = a;
return x;          /*x is F*(u)*/
```



For large $\mathcal{X}$, consider binary search con funzioni semplice si possono scrivere inverse esplicite.

Idf Examples

- In some cases $F^*(u)$ can be determined explicitly
- If X is *Bernoulli(p)* and $F(x) = u$, then x=0 iff $0 < u < 1$-p

$\rightarrow \begin{cases} 1 & u > p \\ 0 & 0 < u < 1-p \end{cases}$

$$F^*(u) = \begin{cases} 0 & 0 < u < 1-p \\ 1 & 1-p \leq u < 1 \end{cases}$$

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posso fare questa cosa perchè:

Random Variate Generation By Inversion

- X is a discrete random variable with idf $F^*(\cdot)$ (esplicita)
- continuous random variable U is *Uniform*(0,1)
- Z is the discrete random variable defined by $Z = F^*(U)$

Theorem
Z and X are identically distributed

this Theorem allows any discrete random variable (with known idf) to be generated with one call to `Random()`

Algorithm 3

```
u = Random();
return F*(u);
```
→ l'inversa nel punto 'u'.

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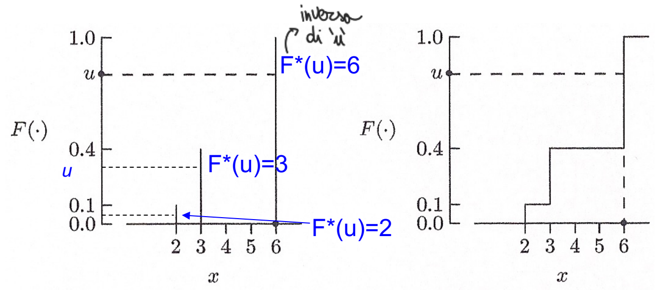
Inversion Examples

- Consider X with pdf

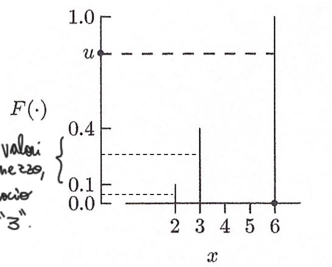
$$f(x) = \begin{cases} 0.1 & x = 2 \\ 0.3 & x = 3 \\ 0.6 & x = 6 \end{cases}$$

- The cdf for X is plotted using two formats




```
if (u < 0.1)
    return 2;
else if (u < 0.4)
        return 3;
    else
        return 6;
```

∃ ∞ valori in mezzo, a cui associo sempre "3".



This algorithm returns
2 with probability 0.1,
3 with probability 0.3 and
6 with probability 0.6.
This corresponds to the pdf of X.

more efficiency: check the ranges for *u* associated with $x = 6$ first (the mode), then $x = 3$, then $x = 2$

- problems may arise when $|\mathcal{X}|$ is large or infinite

More inversion examples

Generating a *Bernoulli*(*p*) random variate

```
u = Random();
if (u < 1-p)
        return 0;
else
        return 1;
```

Generating an *Equilikely*(*a*,*b*) random variate

```
u = Random();
return a + (long) (u * (b - a + 1));
```


Library `rvgs`

pseudo-rand by lehmer

- Includes 6 discrete random variate generators (as below) and 7 continuous random variate generators
  - `long Bernoulli(double p)`
  - `long Binomial(long n, double p)`
  - `long Equilikely(long a, long b)`
  - `long Geometric(double p)`
  - `long Pascal(long n, double p)`
  - `long Poisson(double μ)`
- Functions Bernoulli, Equilikely, Geometric use inversion; essentially ideal
- Functions Binomial, Pascal, Poisson do not use inversion

Library `rvms`

- Provides accurate pdf, cdf, idf functions for many random variates
- Idfs can be used to generate random variates by inversion
- Functions `idfBinomial`, `idfPascal`, `idfPoisson` may have high marginal execution times
- Not recommended when many observations are needed due to time inefficiency
- Array of cdf values with inversion may be preferred

a volte è meglio usare un sottoinsieme, sia per realismo che per efficienza.

Truncation

Sometimes, the realistic values of a variable are restricted to a subset

X random variable with possible values $\chi=\{0, 1, 2, \ldots\}$ and cdf $F(x)=\Pr(X \le x)$

voglio • want to restrict X to the finite range $0 \le a \le x \le b < \infty$

- if $a > 0$, taglio $\alpha = \Pr(X < a)$, $\beta = \Pr(X > b)$

coda sx: $\alpha = \Pr(X < a) = \Pr(X \le a\text{-}1) = F(a\text{-}1)$ (cumulativa)

coda dx: $\beta = \Pr(X > b) = 1\text{-}\Pr(X \le b) = 1\text{-}F(b)$ (1 - cumulativa)

$$\Pr(a \le X \le b) = \Pr(X \le b) - \Pr(X < a) = F(b) - F(a\text{-}1)$$

essentially, always true iff $F(b) \cong 1.0$ and $F(a\text{-}1) \cong 0.0$

Specifying truncation points

- if a and b are specified (punti limite)

  Left-tail, right-tail probabilities α and β obtained using cdf

  $$\alpha = \Pr(X < a) = F(a\text{-}1) \quad \text{and} \quad \beta = \Pr(X > b) = 1\text{-}F(b)$$

  transformation is exact

- if α and β are specified

  idf can be used to obtain a and b

  $$a = F^*(\alpha) \quad \text{and} \quad b = F^*(1\text{-}\beta)$$

  transformation is not exact because X is discrete

  $$\Pr(X < a) \leq \alpha \quad \text{and} \quad \Pr(X > b) < \beta$$

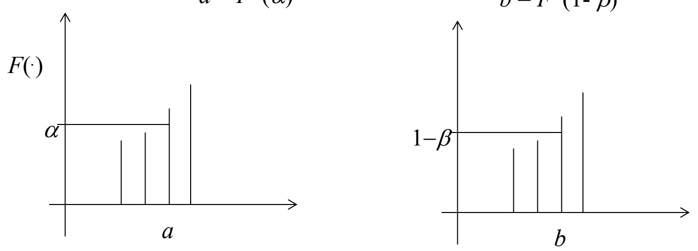
F* è approssimazione inversa, non è "esatta", perchè passo
17 dal continuo ad alcuni valori discreti.

$F(x-1) \leq u < F(x)$

Specifying truncation points

- if α and β are specified

$a = F^*(\alpha)$ $\qquad$ $b = F^*(1\text{-}\beta)$



$$F(a) > \alpha$$
$$\Pr(X < a) \leq \alpha$$

$$F(b) > 1\text{-}\beta$$
$$\Pr(X \leq b) > 1\text{-}\beta$$
$$\text{-}\Pr(X \leq b) < \beta\text{-}1$$
$$1\text{-}\Pr(X \leq b) < \beta$$
$$\Pr(X > b) < \beta$$

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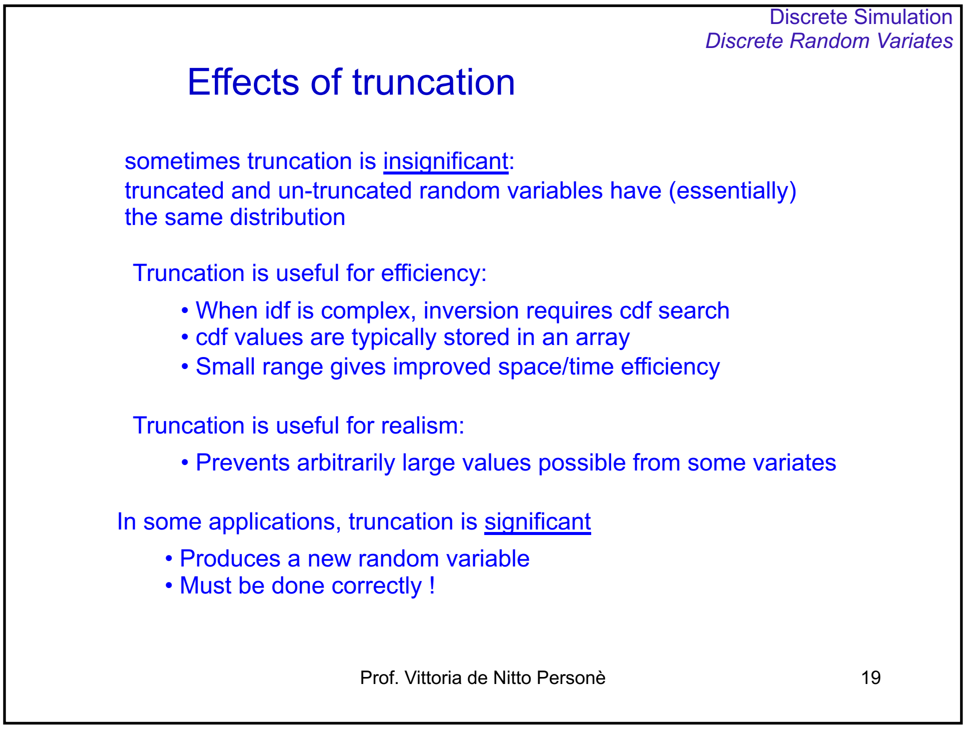
qui vediamo come calcolare i vari passaggi.
ho una disuguaglianza, con un <=, mentre beta è <. quindi non sono "esatti".

gli effetti sono insignificanti se tolgo poco, ma può avere senso per efficienza, se la funzione è complessa, per realismo. Se la troncata è molto diversa, ho nuova var.random!



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