**II Università di Roma, Tor Vergata**
**Dipartimento d'Ingegneria Civile e Ingegneria Informatica**
**LM in Ingegneria dell'Informazione e dell'Automazione**
**Complementi di Probabilità e Statistica**
**Homework - 2021-10-13**

**Problem 1** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $E, F \in \mathcal{E}$. *Show that* $E$ *and* $F$ *are independent if and only if:*

1. $E^c$ *and* $F^c$ *are independent;*
2. $E$ *and* $F^c$ *are independent;*
3. $E^c$ *and* $F$ *are independent.*

**Solution.** .

**Problem 2** *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(E_k)_{k=1}^n$ *a finite sequence of independent events of* $\Omega$.

1. *Show that the probability that none of the events of the sequence* $(E_k)_{k=1}^n$ *occurs is less or equal to* $\exp\left(-\sum_{k=1}^n \mathbf{P}(E_k)\right)$.

   *Hint: consider the elementary properties of the exponential and logarithmic functions.*

2. *Let* $E, F$ *independent events of* $\Omega$ *such that* $\mathbf{P}(F) > 0$. *Prove that*

$$\mathbf{P}(E \cap F \mid E \cup F) \leq \mathbf{P}(E \cap F \mid F).$$

3. *Let* $E, F, G$ *independent events of* $\Omega$ *such that* $\mathbf{P}(E \cap F) > 0$. *Prove that*

$$\mathbf{P}(G \mid E \cap F) = \mathbf{P}(G).$$

**Solution.** .

**Problem 3** *A urn, say Urn A, contains* 5 *white balls and* 10 *black balls. Another urn, say Urn B, contains* 3 *white balls and* 12 *black balls. A fair coin is tossed. If it shows heads [resp. tails] a ball is drawn from Urn A [resp. B]. Suppose that this random experiment has been done and we know that a white ball has been drawn. What is the probability that the ball has been drawn from Urn A? What is the probability that the ball has been drawn from Urn B?*

**Solution.** .

**Problem 4** *An urn contains* $N$ *balls of which* $M$ *are white and the remaining* $N - M$ *are black. From the urn the balls are drawn* with replacement. *We know that the naive probability of drawing the white ball at the nth drawn is* $M/N$. *Show that the naive probability of drawing the white ball at the* $n$*-th draw, given that all previous draws have failed is again* $M/N$. *Assume then that the balls are drawn* without replacement. *Show that the naive probability of drawing a white ball at the* $n$*-th draw is still* $M/N$ *and the naive probability of drawing the white ball at the* $n$*-th draw, given that all previous draws have failed is* $M/(N - n + 1)$, *where* $n \leq N - M + 1$.

**Solution.** .

**Problem 5** *Students are allowed to take a test twice in an examination session. Assume that* 8 *students over* 10 *pass the test the first time. For those who fail or do not show up, only* 6 *students over* 10 *pass the test the second time.*

1. *Find the probability that a randomly selected student (who needs to pass the test) passes the test.*
2. *Assuming that a student passed the test what is the probability she passed on the first try?*
3. *Consider that a part of the first test presented the following problem: two dice are rolled and the number on the upper faces are observed. Is the event "the sum of the observed numbers is* 7*" independent of the event "the number observed on the upper face of a die is* 5*"? Could you give a solution to this problem?*

**Solution.** .

**Problem 6** *The National Health Service (NHS) aims to introduce a new test for the screening of a disease. The pharmaceutical company which produces the test states that:*

- *the test yields a positive result on the* 95% *of people who are affected by the disease (*sensitivity *or* true positive rate *of the test);*
- *the test yields a negative result on the* 99% *of people who are not affected by the disease (*specificity *or* true negative rate *of the test);*

*On the other hand, the NHS knows the the disease is currently affecting the* 10% *of the population. Compute:*

1. *the probability that a randomly chosen individual of the population is affected by the disease given that the test yields a positive result;*
2. *the probability that a randomly chosen individual of the population is not affected by the disease given that the test yields a positive result;*
3. *the probability that a randomly chosen individual of the population is affected by the disease given that the test yields a negative result;*
4. *the probability that a randomly chosen individual of the population is not affected by the disease given that the test yields a negative result;*
5. *the probability that the test yields a positive result on a randomly chosen individual of the population.*
6. *the probability that the test yields a negative result on a randomly chosen individual of the population.*

**Solution.** .

**Problem 7** *The scrutiny of group of* $100,000$ *randomly chosen male people in the age* $40-79$ *in UK during* $2013-2015$ *reveals the following table of average lung cancer incidence*

| | *smoker* | *not smoker* | *total* |
|---|---|---|---|
| *lung cancer* | $10,395$ | $7,407$ | $17,802$ |
| *not lung cancer* | $50,078$ | $32,120$ | $82,198$ |
| | $60,473$ | $39,527$ | $100,000$ |

*Write* $\Omega$ *for the sample space consisting of these* $100,000$ *people and write* $S$ *[resp.* $C$*] for the subsets of* $\Omega$ *consisting of the smokers [the people affected by lung canger]. Let* $1_S : \Omega \to \{0,1\}$ *and* $1_C : \Omega \to \{0,1\}$ *the indicator functions of the events* $S$ *and* $C$ *respectively.*

1. *Determine the joint distribution of the random vector* $(1_S, 1_C)$ *and the marginal distributions of the random variables* $1_S$ *and* $1_C$.
2. *Are the random variables* $1_S$ *and* $1_C$ *independent?*
3. *What is the probability that a randomly chosen person in* $\Omega$ *is affected by lung cancer, given that he is a smoker?*
4. *What is the probability that a randomly chosen person in* $\Omega$ *is a smoker, given that he is affected by lung cancer?*
5. *Check the validity of the total probability formula for* $\mathbf{P}(S)$ *and the Bayes Formula for* $\mathbf{P}(C \mid S)$.

**Solution.** .

**Problem 8** *Consider an e-mail server equipped with a spam filter. The spam filter scrutinizes each incoming e-mail for the appearance of some key sentences.which make it likely that the message is spam. Assume that a key sentence is "check this out" in the the subject line of the e-mail. Assume also that*

1. $40\%$ *of incoming e-mails are spam;*
2. $1\%$ *of emails contain the sentence "check this out" in the subject line, given that they are spam;*
3. $0.4\%$ *of emails contain the sentence "check this out" in the subject line, given that they are not spam.*

   *Compute the probability that an e-mail is spam given that it contains the sentence "check this out" in the subject line.*

**Solution.** .

**Problem 9** *In a large town, after a robbery, a thief jumped into a cab and disappeared. An eyewitness on the crime scene told the police that the cab is yellow. Having some doubt on the reliability of the eyewitness, the police consulted a mathematician. Assuming that*

1. $20\%$ *of the cabs in the town are yellow;*
2. *from the past experience police knows that an eyewitness is* $\%80$ *accurate, that an eyewitness identifies correctly the colour of a taxi yellow or not yellow* $8$ *out* $10$ *times.*

*Compute the probability that the information reported by the eyewitness is true. That is the probability that the cab was yellow given that the eyewitness said so. Do you think this information is useful?*

*Hint: consider the events "the cab is yellow", "the cab is not yellow", "the eyewitness says the cab is yellow", and "the eyewitness says the cab is not yellow" and formulate in terms of conditional probability the accuracy of the eyewitness.*

**Solution.** .

**Problem 10** *A box contains five coins labeled by $C_1, \ldots, C_5$. Assume that both the faces of the coin $C_1$ [resp. $C_5$] show head [resp. tail]; assume that both the coins $C_2$ and $C_4$ are rigged in such a way that the probability of getting head [resp. tail] when tossing the coin $C_2$ [resp. $C_4$] is double than the probability of getting tail [resp. head]; in the end assume that $C_3$ is a fair coin. A coin is randomly drawn from the box and is flipped 5 times. It happens that 5 heads show up in a row, what is the probability that we have drawn the coin $C_k$, for $k = 1, \ldots, 5$.*

*Hint: given a probability space $(\Omega, \mathcal{E}, \mathbf{P})$ and three events $E, F, G \in \mathcal{E}$ it may occur that*

$$\mathbf{P}(E \cap F \mid G) = \mathbf{P}(E \mid G) \mathbf{P}(F \mid G).$$

*In this case, the events $E$ and $F$ are said to be* conditionally independent *given $G$.*

**Solution.** .

**Problem 11** *An empathic professor aims to help his students to pass the hard final exam of his course in Probability and Statistics. To this goal he splits the course propram in two parts and gives his students an intermediate written test on the first part of the course. Assume that*

- *the 18% of the students who pass the final exam on their first try got a mark not lower than 25 in the intermediate test;*
- *the 24% of the students who pass the final exam on their first try got a mark in the range $20 - 24$ in the intermediate test;*
- *the 30% of the students who pass the final exam on their first try got a mark not higher than 19 in the intermediate test;*
- *the 4% of the students who pass the final exam on their first try did not take the intermediate test.*

  *Assume also that*

- *the 20% of the students attending the course get a mark not lower than 25 in the intermediate test;*
- *the 30% of the students attending the course get a mark in the range $20 - 24$ in the intermediate test;*
- *the 10% of the students attending the course do not take the intermediate test.*

**Exercise 12** *Compute:*

1. *the probability that a student passes the final exam on her first try, given that she gets a mark not lower than 25 in the intermediate test;*

2. *the probability that a student passes the final exam on her first try, given that she gets a mark not higer than* 19 *in the intermediate test;*

3. *the probability that a student passes the final exam on her first try, given that she does not take the intermediate test;*

4. *the probability that a student passes the final exam on her first try;*

5. *the probability that a student gets a mark not lower than* 25 *in the intermediate test, given that she passes the final exam on her first try;*

6. *the probability that a student gets a mark not lower than* 25 *in the intermediate test, given that she does not pass the final exam on her first try;*

7. *the probability that a student does not pass the final exam on her first try, given that she does not take the intermediate test.*

**Solution.** Write $T_{\geq 25}$ [resp. $T_{20-24}$, $T_{\leq 19}$] for the event "a randomly choosen student attending the course got a mark not lower than 25 [resp. in the range $20-24$, not higher than 19] in the intermediate test". Write also $T_0$ for the event "a randomly choosen student did not take the intermediate test". In the end, write $S$ for the event "a randomly choosen student attending the course passed the final exam on her first try".

We clearly have that $\{T_0, T_{\leq 19}, T_{20-24}, T_{\geq 25}\}$ is a partition of the sure event $\Omega$. In addition, we know that

$$\mathbf{P}\left(T_{\geq 25}\right) = 0.20, \quad \mathbf{P}\left(T_{20-24}\right) = 0.30, \quad \mathbf{P}\left(T_0\right) = 0.10.$$

It follows

$$\mathbf{P}\left(T_{\leq 19}\right) = 1 - \left(\mathbf{P}\left(T_{\geq 25}\right) + \mathbf{P}\left(T_{20-24}\right) + \mathbf{P}\left(T_0\right)\right) = 1 - (0.20 + 0.30 + 0.10) = 0.40.$$

We also know the probabilities

$$\mathbf{P}\left(S \cap T_{\geq 25}\right) = 0.18, \quad \mathbf{P}\left(S \cap T_{20-24}\right) = 0.24, \quad \mathbf{P}\left(S \cap T_{\leq 19}\right) = 0.30, \quad \mathbf{P}\left(S \cap T_0\right) = 0.04.$$

.

**Example 13 (Monty's Hall Strikes Back)** *Consider Monty's Hall problem. Still assume that the quiz master knows what box contains the prize and the quiz master never shows a box containing the prize. However, in this case assume that after watching the game many times you notice that when a quiz participant chooses box A the quiz master shows empty box B [resp. C] the* 60% *[resp.* 40%*] of the time. May this information improve the quiz participant's strategy for winning the prize? What about if you notice that when a quiz participant chooses box A the quiz master shows empty box B [resp. C] the* 75% *[resp.* 25%*] of the time?*

**Solution.** Retaining the notation of Example **??**, assume the quiz participant chooses box $A$ and set

$$\mathbf{P}\left(EB\right) = x, \qquad \mathbf{P}\left(EC\right) = 1 - x \tag{1}$$

for the probability that the quiz master shows empty box $B$ and $C$, respectively. Applying the Total Probability Formula we have

$$\mathbf{P}\left(EB\right) = \mathbf{P}\left(EB \mid PA\right)\mathbf{P}\left(PA\right) + \mathbf{P}\left(EB \mid PB\right)\mathbf{P}\left(PB\right) + \mathbf{P}\left(EB \mid PC\right)\mathbf{P}\left(PC\right) \tag{2}$$

and

$$\mathbf{P}\left(EC\right) = \mathbf{P}\left(EC \mid PA\right)\mathbf{P}\left(PA\right) + \mathbf{P}\left(EC \mid PB\right)\mathbf{P}\left(PB\right) + \mathbf{P}\left(EC \mid PC\right)\mathbf{P}\left(PC\right). \tag{3}$$

On the other hand, under the assumptions of this version of Monty's Hall problem, we still have

$$\mathbf{P}(PA) = \mathbf{P}(PB) = \mathbf{P}(PC) = \frac{1}{3}. \tag{4}$$

and

$$\begin{aligned} &\mathbf{P}(EB \mid PB) = 0, \quad \mathbf{P}(EB \mid PC) = 1, \\ &\mathbf{P}(EC \mid PB) = 1, \quad \mathbf{P}(EC \mid PC) = 0. \end{aligned} \tag{5}$$

Combining (1)-(5) it then follows

$$\mathbf{P}(EB \mid PA) = 3x - 1 \qquad \text{and} \qquad \mathbf{P}(EC \mid PA) = 2 - 3x.$$

Now, thanks to symmetry formula (**??**) we obtain

$$\mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{1}{3}\frac{3x-1}{x}, \qquad \mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC \mid PA)\,\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{1}{3}\frac{2-3x}{1-x}, \tag{6}$$

and

$$\mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB \mid PC)\,\mathbf{P}(PC)}{\mathbf{P}(EB)} = \frac{1}{3}\frac{1}{x}, \qquad \mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC \mid PB)\,\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{1}{3}\frac{1}{1-x}. \tag{7}$$

In the end, note that since we are dealing with probabilities the choice of $x$ in the interval $(0,1)$ cannot be free but subject to the further constraints

$$0 < 3x - 1 < 1 \qquad \text{and} \qquad 0 < 2 - 3x < 1.$$

These yields

$$\frac{1}{3} < x < \frac{2}{3}, \tag{8}$$

which excludes the possibility that we notice that when a quiz participant chooses box $A$ the quiz master shows empty box $B$ [resp. $C$] the 75% [resp. 25%] of the time. However, when $x = 0.6 = \frac{3}{5}$ which fulfills the constraint (8) we obtain

$$\mathbf{P}(PA \mid EB) = \frac{4}{9} = 0.4\bar{4}, \qquad \mathbf{P}(PA \mid EC) = \frac{1}{6} = 0.167$$

and

$$\mathbf{P}(PB \mid EC) = \frac{5}{6} = 0.8\bar{3}, \qquad \mathbf{P}(PC \mid EB) = \frac{5}{9} = 0.5\bar{5}.$$

These imply that the participant should again exchange chosen box $A$ with closed box $B$ or $C$. Note that comparing Equations (6) and (7) it turns out that the condition who could suggest the quiz participant to stick to her first choice would be

$$3x - 1 > 1 \qquad \text{or} \qquad 2 - 3x > 1.$$

Bothe these conditions do not respect the constraint (8). At most, an indifference condition can be realized for

$$x = \frac{1}{3} = 0.3\bar{3} \qquad \text{or} \qquad x = 0.6\bar{6}.$$

in thsi cases we have

$$\begin{aligned} &\mathbf{P}(PA \mid EB) = 0, \quad \mathbf{P}(PA \mid EC) = \tfrac{1}{2}, \\ &\mathbf{P}(PC \mid EB) = 1, \quad \mathbf{P}(PB \mid EC) = \tfrac{1}{2}, \end{aligned}$$

or

$$\begin{aligned} &\mathbf{P}(PA \mid EB) = \tfrac{1}{2}, \quad \mathbf{P}(PA \mid EC) = 0, \\ &\mathbf{P}(PC \mid EB) = \tfrac{1}{2}, \quad \mathbf{P}(PB \mid EC) = 1. \end{aligned}$$

Hence, in the first [resp. second] case the quiz participant could either stick to is first choice or exchange with the rejected box, the odds being fifty-fifty, if the quiz master shows the empty box $C$ [resp. $B$].

**Example 14 (The Return of Monty's Hall)** *Consider Monty's Hall problem. Still assume that the quiz master knows what box contains the prize, the quiz master never shows a box containing the prize, and the quiz master chooses an empty box between two with uniform probability. However, in this case assume that after watching the game many times you notice that the prize turns out to be in box A [resp. B] for 45% [resp. 30%] of the time and in box C the rest of the time. What is the quiz participant's best strategy?*

**Solution.** Retaining the notation of Example **??**, in this third episode of Monty's Hall saga ve have

$$\mathbf{P}(PA) = 0.50 = \frac{1}{2}, \qquad \mathbf{P}(PB) = 0.30 = \frac{3}{10}, \qquad \mathbf{P}(PC) = 0.20 = \frac{1}{5}.$$

However, in this case, we have to determine the quiz participant's best strategy which is made by a first and a second choice. To this, assume the quiz participant's first choice is $A$ box. We can clearly replicate the argument of Example **??** and end up with evaluating

$$\mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}, \quad \mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2},$$

and

$$\mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB|PC)\mathbf{P}(PC)}{\mathbf{P}(EB)} = \frac{1\cdot\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}, \quad \mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{1\cdot\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}.$$

Now, assume the quiz participant first choice is $B$ box. In this case, with a similar argument, we end up with evaluating

$$\mathbf{P}(PB \mid EA) = \frac{\mathbf{P}(EA|PB)\mathbf{P}(PB)}{\mathbf{P}(EA)} = \frac{\frac{1}{2}\cdot\frac{3}{10}}{\frac{1}{2}} = \frac{3}{10}, \quad \mathbf{P}(PB \mid EC) = \frac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{\frac{1}{2}\cdot\frac{3}{10}}{\frac{1}{2}} = \frac{3}{10},$$

and

$$\mathbf{P}(PC \mid EA) = \frac{\mathbf{P}(EA|PC)\mathbf{P}(PC)}{\mathbf{P}(EA)} = \frac{1\cdot\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}, \quad \mathbf{P}(PA \mid EC) = \frac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \frac{1\cdot\frac{1}{2}}{\frac{1}{2}} = 1.$$

In the end, assume the quiz participant first choice is $C$ box. We end up with evaluating

$$\mathbf{P}(PC \mid EA) = \frac{\mathbf{P}(EA|PC)\mathbf{P}(PC)}{\mathbf{P}(EA)} = \frac{\frac{1}{2}\cdot\frac{1}{5}}{\frac{1}{2}} = \frac{1}{5}, \quad \mathbf{P}(PC \mid EB) = \frac{\mathbf{P}(EB|PC)\mathbf{P}(PB)}{\mathbf{P}(EB)} = \frac{\frac{1}{2}\cdot\frac{3}{10}}{\frac{1}{2}} = \frac{3}{10},$$

and

$$\mathbf{P}(PB \mid EA) = \frac{\mathbf{P}(EA|PB)\mathbf{P}(PB)}{\mathbf{P}(EA)} = \frac{1\cdot\frac{3}{10}}{\frac{1}{2}} = \frac{3}{5}, \quad \mathbf{P}(PA \mid EB) = \frac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \frac{1\cdot\frac{1}{2}}{\frac{1}{2}} = 1.$$

In light of what shown above, the best strategy is to choose box $C$ and change with the other rejected box as the quiz master gives the opportunity.

**Example 15 (Monty's Hall Awakens)** *Consider Monty's Hall problem. Still assume that the quiz master knows what box contains the prize, the quiz master never shows a box containing the prize. Assume also that after watching the game many times you notice that the prize turns out to be in box A [resp. B] for 45% [resp. 30%] of the time and in box C the rest of the time. Moreover, assume that after a quiz participant chooses a box the quiz master chooses an empty box to show between two by fliping a rigged coin with success probability p. What is the quiz participant's best strategy?*

**Solution.** Retaining the notation of Example **??**, the only difference between this episode of Monty's Hall saga and Episode 14 is that we have

$$\begin{aligned} \mathbf{P}(EB) &= p, \quad \mathbf{P}(EC) = 1-p, \\ \mathbf{P}(EA) &= p, \quad \mathbf{P}(EC) = 1-p, \\ \mathbf{P}(EA) &= p, \quad \mathbf{P}(EB) = 1-p, \end{aligned}$$

according to whether the quiz participant's first choice is $A$ or $B$ or $C$ box In particular, we still have

$$\mathbf{P}(PA) = 0.50 = \frac{1}{2}, \qquad \mathbf{P}(PB) = 0.30 = \frac{3}{10}, \qquad \mathbf{P}(PC) = 0.20 = \frac{1}{5}.$$

As a consequence, assume the quiz participant's first choice is $A$ box. We end up with evaluating

$$\mathbf{P}(PA \mid EB) = \tfrac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \tfrac{\frac{1}{2}\cdot\frac{1}{2}}{p} = \tfrac{1}{4p}, \quad \mathbf{P}(PA \mid EC) = \tfrac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \tfrac{\frac{1}{2}\cdot\frac{1}{2}}{1-p} = \tfrac{1}{4(1-p)},$$

and

$$\mathbf{P}(PC \mid EB) = \tfrac{\mathbf{P}(EB|PC)\mathbf{P}(PC)}{\mathbf{P}(EB)} = \tfrac{1\cdot\frac{1}{5}}{p} = \tfrac{1}{5p}, \quad \mathbf{P}(PB \mid EC) = \tfrac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \tfrac{1\cdot\frac{3}{10}}{1-p} = \tfrac{3}{10(1-p)}.$$

Assume the quiz participant's first choice is $B$ box. We obtain

$$\mathbf{P}(PB \mid EA) = \tfrac{\mathbf{P}(EA|PB)\mathbf{P}(PB)}{\mathbf{P}(EA)} = \tfrac{\frac{1}{2}\cdot\frac{3}{10}}{p} = \tfrac{3}{20p}, \quad \mathbf{P}(PB \mid EC) = \tfrac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \tfrac{\frac{1}{2}\cdot\frac{3}{10}}{1-p} = \tfrac{3}{20(1-p)},$$

and

$$\mathbf{P}(PC \mid EA) = \tfrac{\mathbf{P}(EA|PC)\mathbf{P}(PC)}{\mathbf{P}(EA)} = \tfrac{1\cdot\frac{1}{5}}{p} = \tfrac{1}{5p}, \quad \mathbf{P}(PA \mid EC) = \tfrac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \tfrac{1\cdot\frac{1}{2}}{1-p} = \tfrac{1}{2(1-p)}.$$

In the end, assume the quiz participant's first choice is $C$ box. We obtain

$$\mathbf{P}(PC \mid EA) = \tfrac{\mathbf{P}(EA|PC)\mathbf{P}(PC)}{\mathbf{P}(EA)} = \tfrac{\frac{1}{2}\cdot\frac{1}{5}}{p} = \tfrac{1}{10p}, \quad \mathbf{P}(PC \mid EB) = \tfrac{\mathbf{P}(EB|PC)\mathbf{P}(PB)}{\mathbf{P}(EB)} = \tfrac{\frac{1}{2}\cdot\frac{3}{10}}{1-p} = \tfrac{3}{20(1-p)},$$

and

$$\mathbf{P}(PB \mid EA) = \tfrac{\mathbf{P}(EA|PB)\mathbf{P}(PB)}{\mathbf{P}(EA)} = \tfrac{1\cdot\frac{3}{10}}{p} = \tfrac{3}{10p}, \quad \mathbf{P}(PA \mid EB) = \tfrac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \tfrac{1\cdot\frac{1}{2}}{1-p} = \tfrac{1}{2(1-p)}.$$

Hence, also in this case, no matter of the value of the success probability $p$, the best strategy is to choose box $C$ and change with the other rejected box as the quiz master gives the opportunity.

**Example 16 (The Last Monty's Hall)** *Consider Monty's Hall problem. However, in this case assume that the quiz master does not know what box contains the prize, the quiz master chooses a box between two with uniform probability, and the quiz master shows an empty box by chance. In this episode of the Monty's Hall saga, what should the participant do? To stick to her first choice, to accept the exchange or it does not matter at all because the odds are now fifty-fifty?*

**Solution.** Retaining the notation of Example **??**, still Equations (**??**) and (**??**) hold true, but since the quiz master does not know what box contains the prize and he shows an empty box by chance then the is a substantial change in Equation (**??**) which becomes

$$\begin{array}{lll} \mathbf{P}(EB \mid PA) = \frac{1}{2}, & \mathbf{P}(EB \mid PB) = 0, & \mathbf{P}(EB \mid PC) = \frac{1}{2}, \\ \mathbf{P}(EC \mid PA) = \frac{1}{2}, & \mathbf{P}(EC \mid PB) = \frac{1}{2}, & \mathbf{P}(EC \mid PC) = 0. \end{array} \tag{9}$$

Combining (**??**), (**??**) and (9), we obtain

$$\mathbf{P}(EB) = \mathbf{P}(EC) = \frac{1}{3}. \tag{10}$$

It then follows

$$\mathbf{P}(PA \mid EB) = \tfrac{\mathbf{P}(EB|PA)\mathbf{P}(PA)}{\mathbf{P}(EB)} = \tfrac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{3}} = \tfrac{1}{2}, \quad \mathbf{P}(PA \mid EC) = \tfrac{\mathbf{P}(EC|PA)\mathbf{P}(PA)}{\mathbf{P}(EC)} = \tfrac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{3}} = \tfrac{1}{2}, \tag{11}$$

and

$$\mathbf{P}\left(PC \mid EB\right) = \frac{\mathbf{P}(EB|PC)\mathbf{P}(PC)}{\mathbf{P}(EB)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}} = \frac{1}{3}, \quad \mathbf{P}\left(PB \mid EC\right) = \frac{\mathbf{P}(EC|PB)\mathbf{P}(PB)}{\mathbf{P}(EC)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}} = \frac{1}{3}. \tag{12}$$

Hence, whether the quiz master shows empty box $B$ or $C$ the conditional probability that the prize is in the other box $C$ or $B$ is the same of the conditional probability that the prize is contained in the firstly chosen box $A$. We can conclude that the quiz participant should be indifferent between sticking to her first choice or accepting the exchange. Note that in this episode of the Monty's Hall saga, the assumption that the quiz master ignores what box contains the prize, clearly makes the of couples of events $\{PA, EB\}$, $\{PA, EB\}$, $\{PC, EB\}$, and $\{PB, EC\}$ independent. Therefore, we could write straightforwardly

$$\mathbf{P}\left(PA \mid EB\right) = \mathbf{P}\left(PA\right) = \tfrac{1}{3}, \quad \mathbf{P}\left(PA \mid EC\right) = \mathbf{P}\left(PA\right) = \tfrac{1}{3}, \tag{13}$$

and

$$\mathbf{P}\left(PC \mid EB\right) = \mathbf{P}\left(PC\right) = \tfrac{1}{3}, \quad \mathbf{P}\left(PB \mid EC\right) = \mathbf{P}\left(PB\right) = \tfrac{1}{3}. \tag{14}$$

It is also interesting to note that the second argument applied in the solution of Example **??** for a not very evident reason. What a reasony?

**Theorem 17 (Bayes Formula)** *Let $N \subseteq \mathbb{N}$ and let $(F_n)_{n \in N}$ be a countable partition of $\Omega$. Then for any $n \in N$, we have*

$$\mathbf{P}\left(F_n \mid E\right) = \frac{\mathbf{P}\left(E \mid F_n\right)\mathbf{P}\left(F_n\right)}{\sum_{m \in N} \mathbf{P}\left(E \mid F_m\right)\mathbf{P}\left(F_m\right)}, \quad \forall E \in \mathcal{E} : \mathbf{P}\left(E\right) > 0.$$

**Proof.** From the symmetry formula (**??**), we can write

$$\mathbf{P}\left(F_n \mid E\right) = \frac{\mathbf{P}\left(E \mid F_n\right)\mathbf{P}\left(F_n\right)}{\mathbf{P}\left(E\right)},$$

for any $n \in N$. Hence, applying Equation (**??**) to evaluate $\mathbf{P}\left(E\right)$, we obtain the desired result. □