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Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Random Vectors with Solution 2022-12-08

Problem 1 *Let (X_1, X_2) a real random vector with a joint density $f_{X_1,X_2} : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f_{X_1,X_2}(x_1, x_2) \stackrel{def}{=} 1_{[0,1]\times[0,1]}(x_1, x_2), \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Consider the real random variables $Y \equiv \min(X_1, X_2)$ and $Z \equiv \max(X_1, X_2)$. Determine:

1. *the distribution functions of Y and Z;*
2. *the joint distribution function of Y and Z;*
3. *the marginal distributions functions of Y and Z;*
4. *the expectations of Y and Z.*

Solution.

1. We have
$$1_{[0,1]\times[0,1]}(x_1, x_2) = 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2),$$
for every $(x_1, x_2) \in \mathbb{R}^2$. As a consequence, for the marginal density $f_{X_2} : \mathbb{R} \to \mathbb{R}$ [resp. $f_{X_2} : \mathbb{R} \to \mathbb{R}$] of the entry X_1 [resp. X_2] of the random vector $(X_1, X_2)^\intercal$, we obtain
$$\begin{aligned} f_{X_1}(x_1) &= \int_{\mathbb{R}} 1_{[0,1]\times[0,1]}(x_1, x_2)\, d\mu_L(x_2) = \int_{\mathbb{R}} 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_2) \\ &= 1_{[0,1]}(x_1) \int_{\mathbb{R}} 1_{[0,1]}(x_2)\, d\mu_L(x_2) = 1_{[0,1]}(x_1) \int_{[0,1]} d\mu_L(x_2) = 1_{[0,1]}(x_1)\, \mu_L([0,1]) \\ &= 1_{[0,1]}(x_1), \end{aligned}$$
for every $x_1 \in \mathbb{R}$ [resp.
$$f_{X_2}(x_2) = 1_{[0,1]}(x_2),$$
for every $x_2 \in \mathbb{R}$]. It then follows,
$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2),$$
for every $(x_1, x_2) \in \mathbb{R}^2$. Hence, the entries X_1 and X_2 of the random vector (X_1, X_2) are independent random variables, and both are standard uniformly distributed. Now, we have
$$\{Y \leq y\} = \{X_1 \leq y, X_2 \leq y\} \cup \{X_1 > y, X_2 \leq y\} \cup \{X_1 \leq y, X_2 > y\},$$
for every $y \in \mathbb{R}$, where the three events on the right hand side are pairwise incompatible, and
$$\{Z \leq z\} = \{X_1 \leq z, X_2 \leq z\},$$
for every $z \in \mathbb{R}$. By virtue of the indepencence of X_1 and X_2, it then follows,
$$\begin{aligned} & F_Y(y) \\ &= \mathbf{P}(Y \leq y) = \mathbf{P}(X_1 \leq y, X_2 \leq y) + \mathbf{P}(X_1 > y, X_2 \leq y) + \mathbf{P}(X_1 \leq y, X_2 > y) \\ &= \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) + \mathbf{P}(X_1 > y)\, \mathbf{P}(X_2 \leq y) + \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 > y) \\ &= \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) + (1 - \mathbf{P}(X_1 \leq y))\, \mathbf{P}(X_2 \leq y) + \mathbf{P}(X_1 \leq y)\, (1 - \mathbf{P}(X_2 \leq y)) \\ &= \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) + \mathbf{P}(X_2 \leq y) - \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) + \mathbf{P}(X_1 \leq y) - \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) \\ &= \mathbf{P}(X_1 \leq y) + \mathbf{P}(X_2 \leq y) - \mathbf{P}(X_1 \leq y)\, \mathbf{P}(X_2 \leq y) \\ &= F_{X_1}(y) + F_{X_2}(y) - F_{X_1}(y)\, F_{X_2}(y) \end{aligned}$$

and

$$F_Z(z) = \mathbf{P}(X_1 \leq z, X_2 \leq z) = \mathbf{P}(X_1 \leq z)\,\mathbf{P}(X_2 \leq z) = F_{X_1}(z)\,F_{X_2}(z).$$

Note that instead of the event $\{Y \leq y\}$ we could have considered the event

$$\{Y > y\} = \{X_1 > y, X_2 > y\},$$

for every $y \in \mathbb{R}$, obtaining

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) = 1 - \mathbf{P}(Y > y) = 1 - \mathbf{P}(X_1 > y, X_2 > y) \\ &= 1 - \mathbf{P}(X_1 > y)\,\mathbf{P}(X_2 > y) = 1 - (1 - \mathbf{P}(X_1 \leq y))(1 - \mathbf{P}(X_2 \leq y)) \\ &= 1 - (1 - F_{X_1}(y))((1 - F_{X_2}(y))) \\ &= 1 - (1 - F_{X_2}(y) - F_{X_1}(y) + F_{X_1}(y)\,F_{X_2}(y)) \\ &= F_{X_1}(y) + F_{X_2}(y) - F_{X_1}(y)\,F_{X_2}(y), \end{aligned}$$

for every $y \in \mathbb{R}$, as above. On the other hand, both the random variables X_1 and X_2 are standard uniformly distributed on the interval $[0,1]$. Therefore,

$$F_Y(y) = F_X(y)\,(2 - F_X(y)) \qquad \text{and} \qquad F_Z(z) = F_X(z)^2,$$

for all $y, x \in \mathbb{R}$, where F_X is the distribution function of the random variable $X\tilde{\ }Unif(0,1)$, given by

$$F_X(x) = x \cdot 1_{[0,1]}(x) + 1_{(1,+\infty)}(x),$$

for every $x \in \mathbb{R}$. It then follows

$$\begin{aligned} &F_Y(y) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,+\infty)}(y) - \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\right) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,0)}(y) + 2 \cdot 1_{[0,1]}(y) + 2 \cdot 1_{(1,+\infty)}(y) - \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\right) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,0)}(y) + (2 - y) \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right) \\ &= (2 - y)\,y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y), \end{aligned}$$

for every $y \in \mathbb{R}$, and

$$F_Z(z) = \left(z \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z)\right)^2 = z^2 \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z).$$

for every $z \in \mathbb{R}$. Note that we have

$$F_Y'(y) = 2(1 - y) \cdot 1_{(0,1)}(y) \qquad \text{and} \qquad F_Z'(z) = 2z \cdot 1_{(0,1)}(z),$$

for every $y, z \in \mathbb{R} - \{0,1\}$. These imply

$$\begin{aligned} \int_{(-\infty,y)} F_Y'(u)\,d\mu_L(u) &= \int_{(-\infty,y)} 2(1-u)\,1_{(0,1)}(u)\,d\mu_L(u) \\ &= \begin{cases} 0, & \text{if } y \leq 0, \\ \int_{(0,y)} 2(1-u)\,d\mu_L(u), & \text{if } 0 < y < 1, \\ \int_{(0,1)} 2(1-u)\,d\mu_L(u), & \text{if } 1 \leq y, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_{(-\infty,z)} F_Z'(v)\,d\mu_L(v) &= \int_{(-\infty,z)} 2z \cdot 1_{(0,1)}(z)\,d\mu_L(v) \\ &= \begin{cases} 0, & \text{if } z \leq 0, \\ \int_{(0,z)} 2v d\mu_L(v), & \text{if } 0 < z < 1, \\ \int_{(0,1)} 2v d\mu_L(v), & \text{if } 1 \leq z. \end{cases} \end{aligned}$$

On the other hand,

$$\int_{(0,y)} 2(1-u)\,d\mu_L(u) = \int_0^y 2(1-u)\,du = \left. 2u - u^2 \right|_0^y = y(2-y),$$

for every $0 < y \leq 1$, and

$$\int_{(0,z)} 2v d\mu_L(v) = \int_0^z 2v dv = v^2\Big|_0^z = z^2,$$

for every $0 < z \leq 1$. We can then write

$$\int_{(-\infty,y)} F_Y'(u)\, d\mu_L(u) = y(2-y)\cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y) = F_Y(y),$$

for every $y \in \mathbb{R}$, and

$$\int_{(-\infty,z)} F_Z'(v)\, d\mu_L(v) = z^2 \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z) = F_Z(z),$$

for every $z \in \mathbb{R}$. These imply that Y and Z are absolutely continuous random variables.

2. We have

$$\begin{aligned}
&\{Y \leq y, Z \leq z\} \\
&= (\{X_1 \leq y, X_2 \leq y\} \cup \{X_1 > y, X_2 \leq y\} \cup \{X_1 \leq y, X_2 > y\}) \cap \{X_1 \leq z, X_2 \leq z\} \\
&= (\{X_1 \leq y, X_2 \leq y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&\quad \cup (\{X_1 > y, X_2 \leq y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&\quad \cup (\{X_1 \leq y, X_2 > y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&= \{X_1 \leq \min(y,z), X_2 \leq \min(y,z)\} \\
&\quad \cup \{y < X_1 \leq z, X_2 \leq \min(y,z)\} \\
&\quad \cup \{X_1 \leq \min(y,z), y < X_2 \leq z\}.
\end{aligned}$$

Therefore, considering the joint distribution function $F_{Y,Z} : \mathbb{R}^2 \to \mathbb{R}$ of Y and Z, on account of the independence of X_1 and X_2, we can write

$$\begin{aligned}
F_{Y,Z}(y,z) &= \mathbf{P}(Y \leq y, Z \leq z) \\
&= \mathbf{P}(X_1 \leq \min(y,z), X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(y < X_1 \leq z, X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(X_1 \leq \min(y,z), y < X_2 \leq z) \\
&= \mathbf{P}(X_1 \leq \min(y,z))\,\mathbf{P}(X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(y < X_1 \leq z)\,\mathbf{P}(X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(X_1 \leq \min(y,z))\,\mathbf{P}(y < X_2 \leq z),
\end{aligned}$$

for every $(y,z) \in \mathbb{R}^2$. On the other hand,

$$\begin{array}{lr}
\min(y,z) = y, & \text{if } y \leq z, \\
\mathbf{P}(y < X_1 \leq z) = 0 \quad \text{and} \quad \min(y,z) = z, & \text{if } y > z.
\end{array}$$

Hence, considerning that X_1 and X_2 have the same distribution, we obtain

$$F_{Y,Z}(y,z) = \begin{cases} F_X(y)(2F_X(z) - F_X(y)), & \text{if } y \leq z, \\ F_X(z)^2, & \text{if } y > z. \end{cases}$$

In fact, if $y \leq z$

$$\begin{aligned}
&\mathbf{P}(X_1 \leq \min(y,z))\,\mathbf{P}(X_2 \leq \min(y,z)) + \mathbf{P}(y < X_1 \leq z)\,\mathbf{P}(X_2 \leq \min(y,z)) \\
&+ \mathbf{P}(X_1 \leq \min(y,z))\,\mathbf{P}(y < X_2 \leq z) \\
&= \mathbf{P}(X \leq y)\,\mathbf{P}(X \leq y) + 2\mathbf{P}(X \leq y)\,\mathbf{P}(y < X \leq z) \\
&= F_X(y)^2 + 2F_X(y)(F_X(z) - F_X(y)) \\
&= F_X(y)(2F_X(z) - F_X(y))
\end{aligned}$$

and if $y > z$

$$\begin{aligned}
&\mathbf{P}\left(X_1 \leq \min\left(y,z\right)\right)\mathbf{P}\left(X_2 \leq \min\left(y,z\right)\right) + \mathbf{P}\left(y < X_1 \leq z\right)\mathbf{P}\left(X_2 \leq \min\left(y,z\right)\right) \\
&+ \mathbf{P}\left(X_1 \leq \min\left(y,z\right)\right)\mathbf{P}\left(y < X_2 \leq z\right) \\
&= \mathbf{P}\left(X \leq z\right)\mathbf{P}\left(X \leq z\right) + 2\mathbf{P}\left(X \leq z\right)\mathbf{P}\left(y < X \leq z\right) \\
&= F_X\left(z\right)^2.
\end{aligned}$$

Note that we can write

$$F_{Y,Z}\left(y,z\right) = F_X\left(y\right)\left(2F_X\left(z\right) - F_X\left(y\right)\right)1_{\{(y,z)\in\mathbb{R}^2:y\leq z\}} + F_X\left(z\right)^2 1_{\{(y,z)\in\mathbb{R}^2:y>z\}}.$$

3. To determine the marginal distribution functions $F_Y : \mathbb{R} \to \mathbb{R}$ and $F_Z : \mathbb{R} \to \mathbb{R}$ of the random vector $(Y,Z)^\intercal$, respectively, we can apply the formula

$$\begin{aligned}
F_Y\left(y\right) &= \lim_{z\to+\infty} F_{Y,Z}\left(y,z\right) \\
&= \lim_{z\to+\infty}\left(F_X\left(y\right)\left(2F_X\left(z\right) - F_X\left(y\right)\right)1_{\{(y,z)\in\mathbb{R}^2:y\leq z\}}\left(y,z\right) + F_X\left(z\right)^2 1_{\{(y,z)\in\mathbb{R}^2:y>z\}}\left(y,z\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
F_Z\left(z\right) &= \lim_{y\to+\infty} F_{Y,Z}\left(y,z\right) = \\
&= \lim_{y\to+\infty}\left(F_X\left(y\right)\left(2F_X\left(z\right) - F_X\left(y\right)\right)1_{\{(y,z)\in\mathbb{R}^2:y\leq z\}}\left(y,z\right) + F_X\left(z\right)^2 1_{\{(y,z)\in\mathbb{R}^2:y>z\}}\left(y,z\right)\right).
\end{aligned}$$

as $z \to +\infty$ for every $y \in \mathbb{R}$ we have

$$1_{\{(y,z)\in\mathbb{R}^2:y\leq z\}}\left(y,z\right) = 1 \qquad \text{and} \qquad 1_{\{(y,z)\in\mathbb{R}^2:y>z\}}\left(y,z\right) = 0.$$

Conversely, as $y \to +\infty$ for every $z \in \mathbb{R}$ we have

$$1_{\{(y,z)\in\mathbb{R}^2:y\leq z\}}\left(y,z\right) = 0 \qquad \text{and} \qquad 1_{\{(y,z)\in\mathbb{R}^2:y>z\}}\left(y,z\right) = 1.$$

It then follows

$$F_Y\left(y\right) = F_X\left(y\right)\left(2F_X\left(z\right) - F_X\left(y\right)\right) \qquad \text{and} \qquad F_Z\left(z\right) = F_X\left(z\right)^2,$$

which shows that the marginal distribution functions of the random vector (Y,Z) concide with the distribution functions of the random variables X and Y. As a consequence, the random variables $Y \equiv \min\left(X_1,X_2\right)$ and $Z \equiv \max\left(X_1,X_2\right)$ are indepemdent.

4. In the end, we have

$$\begin{aligned}
\mathbf{E}\left[Y\right] &= \int_{\mathbb{R}} y f_Y\left(y\right)d\mu_L\left(y\right) = \int_{\mathbb{R}} 2y\left(1-y\right)1_{[0,1]}\left(y\right)d\mu_L\left(y\right) = \int_{[0,1]} 2y\left(1-y\right)d\mu_L\left(y\right) \\
&= \int_0^1 2\left(1-y\right)ydy = 2\left(\int_0^1 ydy - \int_0^1 y^2dy\right) = 2\left(\frac{1}{2}y^2\bigg|_0^1 - \frac{1}{3}y^3\bigg|_0^1\right) = \frac{1}{3}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}\left[Z\right] &= \int_{\mathbb{R}} z f_Z\left(z\right)d\mu_L\left(z\right) = \int_{\mathbb{R}} 2z^2 \cdot 1_{[0,1]}\left(z\right)d\mu_L\left(z\right) = \int_{[0,1]} 2z^2 d\mu_L\left(z\right) \\
&= \int_0^1 2z^2dz = 2\int_0^1 z^2dz = 2\,\frac{1}{3}z^3\bigg|_0^1 = \frac{2}{3}.
\end{aligned}$$

Problem 2 *Let $(\Omega,\mathcal{E},\mathbf{P}) \equiv \Omega$ be a probability space and let $\left(\mathbb{R}^2,\mathcal{B}\left(\mathbb{R}^2\right),\mu_L^2\right) \equiv \mathbb{R}^2$ be the Euclidean real plane endowed with the Borel σ-algebra and the Borel-Lebesgue measure $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right)$. Let $f : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f\left(x,y\right) \overset{def}{=} kxye^{-(x^2+y^2)}1_{\mathbb{R}_+^2}\left(x,y\right), \quad \forall\left(x,y\right) \in \mathbb{R}^2$$

where $\mathbb{R}_+^2 \equiv \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. Determine $k \in \mathbb{R}$ such that $f : \mathbb{R}^2 \to \mathbb{R}_+$ is a probability density and let $Z \equiv (X,Y)$ be the random vector of density $f : \mathbb{R}^2 \to \mathbb{R}_+$.

1. *Determine the distribution function* $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *of the vector* Z *and check that*

$$\frac{\partial F^2}{\partial x \partial y}(x, y) = f(x, y), \quad \mu_L^2 - a.e. \text{ on } \mathbb{R}^2.$$

2. *Determine the marginal distribution function* $F_X : \mathbb{R} \to \mathbb{R}_+$ *and* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z.

3. *Determine the densities* $f_X : \mathbb{R} \to \mathbb{R}_+$ *and* $f_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z *and check that*

$$\frac{dF_X}{dx}(x) = f_X(x) \quad \text{and} \quad \frac{dF_Y}{dy}(y) = f_Y(y), \quad \mu_L - a.e. \text{ on } \mathbb{R}.$$

4. *Are* X *and* Y *independent random varables?*

5. *Compute* $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$ *and* $Cov(X, Y)$.

6. *Compute* $\mathbf{E}[(X, Y)]$ *and the covariance matrix of the vector* (X, Y).

Solution. . $\square$

Problem 3 *Determine the value of the parameter* k *such that the function* $f : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f(x_1, x_2, x_3) \stackrel{def}{=} \begin{cases} k\left(x_1 + x_2^2 + x_3^3\right), & \text{if } (x_1, x_2, x_3) \in [0,1] \times [0,1] \times [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

is a probability density. Hence, consider the random vector $(X_1, X_2, X_3)^\intercal$ *with density* $f_{X_1,X_2,X_3} : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) \stackrel{def}{=} f(x_1, x_2, x_3).$$

Compute:

1. *the probability* $\mathbf{P}(X_2 \leq 1/2, X_3 > 1/2)$;

2. *the marginal densities of the random vector* $(X_1, X_2)^\intercal$;

3. *the expectation of* $(X_1, X_2)^\intercal$;

4. *the conditional density* $f_{X_1,X_2|X_3=1/2}(x_1, x_2)$.

Solution. To determine the value of the parameter k such that the function $f : \mathbb{R}^3 \to \mathbb{R}$ is a probability density we have to solve the equation

$$\int_{\mathbb{R}^3} f(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) = 1.$$

We have

$$f(x_1, x_2, x_3) = k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1, x_2, x_3),$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^3} f(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) &= \int_{\mathbb{R}^3} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) \\
&= \int_{[0,1]\times[0,1]\times[0,1]} k\left(x_1 + x_2^2 + x_3^3\right) d\mu_L(x_1, x_2, x_3) \\
&= k \int_{[0,1]\times[0,1]\times[0,1]} \left(x_1 + x_2^2 + x_3^3\right) d\mu_L(x_1, x_2, x_3)
\end{aligned}$$

Now the real function $x_1 + x_2^2 + x_3^3$ is continuous on $[0,1] \times [0,1] \times [0,1]$. Therefore, the Lebesue integral can be computed as a Riemann integral. A as consequence, on account of the additive property of the Riemann integral and the separability of the integrand function on the pluri-interval domain, we can write

$$\begin{aligned}
&\int_{[0,1]\times[0,1]\times[0,1]} \left(x_1 + x_2^2 + x_3^3\right) d\mu_L\left(x_1, x_2, x_3\right) \\
&= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} \left(x_1 + x_2^2 + x_3^3\right) dx_1 dx_2 dx_3 \\
&= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_1 dx_1 dx_2 dx_3 \\
&+ \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_2^2 dx_1 dx_2 dx_3 \\
&+ \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_3^3 dx_1 dx_2 dx_3 \\
&= \int_{x_1=0}^{1} x_1 dx_1 \int_{x_2=0}^{1} dx_2 \int_{x_3=0}^{1} dx_3 \\
&+ \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} x_2^2 dx_2 \int_{x_3=0}^{1} dx_3 \\
&+ \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} dx_2 \int_{x_3=0}^{1} x_3^3 dx_3 \\
&= \frac{1}{2}\, x_1^2\Big|_{x_1=0}^{1} \cdot x_2\Big|_{x_2=0}^{1} \cdot x_3\Big|_{x_3=0}^{1} \\
&+ x_1\Big|_{x_1=0}^{1} \cdot \frac{1}{3}\, x_2^3\Big|_{x_2=0}^{1} \cdot x_3\Big|_{x_3=0}^{1} \\
&+ x_1\Big|_{x_1=0}^{1} \cdot x_2\Big|_{x_2=0}^{1} \frac{1}{4} \cdot x_3^4\Big|_{x_3=0}^{1} \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&= \frac{13}{12}
\end{aligned}$$

It follows

$$k = \frac{12}{13}.$$

With similar computation, we have

$$\begin{aligned}
\mathbf{P}\left(X_2 \leq 1/2, X_3 > 1/2\right) &= \int_{x_1=0}^{1} \int_{x_2=0}^{1/2} \int_{x_3=1/2}^{1} \frac{12}{13} \left(x_1 + x_2^2 + x_3^3\right) dx_1 dx_2 dx_3 \\
&= \frac{12}{13} \left(\frac{1}{2}\, x_1^2\Big|_{x_1=0}^{1} \cdot x_2\Big|_{x_2=0}^{1/2} \cdot x_3\Big|_{x_3=1/2}^{1} \right. \\
&+ x_1\Big|_{x_1=0}^{1} \cdot \frac{1}{3}\, x_2^3\Big|_{x_2=0}^{1/2} \cdot x_3\Big|_{x_3=1/2}^{1} \\
&+ \left. x_1\Big|_{x_1=0}^{1} \cdot x_2\Big|_{x_2=0}^{1/2} \cdot \frac{1}{4}\, x_3^4\Big|_{x_3=1/2}^{1} \right) \\
&= \frac{12}{13} \left(\frac{1}{8} + \frac{1}{48} + \frac{15}{128} \right) \\
&= \frac{101}{416}.
\end{aligned}$$

The marginal density of the random vector $(X_1, X_2)^\intercal$ is given by

$$\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \int_{\mathbb{R}} f(x_1,x_2,x_3)\, d\mu_L(x_3) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)\, d\mu_L(x_3) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, 1_{[0,1]}(x_3)\, d\mu_L(x_3) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_3) \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \int_{x_3=0}^{1} \left(x_1 + x_2^2 + x_3^3\right) dx_3 \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \left(\int_{x_3=0}^{1} x_1 d\mu_L(x_3) + \int_{x_3=0}^{1} x_2^2 d\mu_L(x_3) + \int_{x_3=0}^{1} x_3^3 d\mu_L(x_3)\right) \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \left(x_1 \cdot x_3\big|_{x_3=0}^{1} + x_2^2 \cdot x_3\big|_{x_3=0}^{1} + \frac{1}{4}\, x_3^4\big|_{x_3=0}^{1}\right) \\
&= k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \\
&= k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2).
\end{aligned}$$

We have

$$\mathbf{E}\left[(X_1, X_2)^\intercal\right] = \left(\mathbf{E}[X_1], \mathbf{E}[X_2]\right)^\intercal,$$

where

$$\mathbf{E}[X_k] = \int_{\mathbb{R}} x_k f_{X_k}(x_k)\, d\mu_L(x_k), \quad k = 1, 2,$$

and $f_{X_k}(x_k)$ is the marginal density of the random variable X_k, for $k = 1, 2$. Now,

$$\begin{aligned}
f_{X_1}(x_1) &= \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_2) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)\, d\mu_L(x_2) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_2) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, d\mu_L(x_2) \\
&= k 1_{[0,1]}(x_1) \int_{x_2=0}^{1} \left(x_1 + x_2^2 + \frac{1}{4}\right) dx_2 \\
&= k 1_{[0,1]}(x_1) \left(x_1 \cdot x_2\big|_{x_2=0}^{1} + \frac{1}{3} \cdot x_2^3\big|_{x_2=0}^{1} + \frac{1}{4} \cdot x_2\big|_{x_2=0}^{1}\right) \\
&= k 1_{[0,1]}(x_1) \left(x_1 + \frac{1}{3} + \frac{1}{4}\right) \\
&= k\left(x_1 + \frac{7}{12}\right) 1_{[0,1]}(x_1).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
f_{X_2}(x_2) &= \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_1) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)\, d\mu_L(x_1) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x1) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_2)\, d\mu_L(x_1) \\
&= k 1_{[0,1]}(x_2) \int_{x_1=0}^{1} \left(x_1 + x_2^2 + \frac{1}{4}\right) dx_1 \\
&= k 1_{[0,1]}(x_2) \left(\frac{1}{3}\cdot x_1^2\Big|_{x_1=0}^{1} + x_2^2\cdot x_1\Big|_{x_1=0}^{1} + \frac{1}{4}\cdot x_1\Big|_{x_1=0}^{1}\right) \\
&= k 1_{[0,1]}(x_2) \left(\frac{1}{3} + x_2^2 + \frac{1}{4}\right) \\
&= k\left(x_2^2 + \frac{7}{12}\right) 1_{[0,1]}(x_2).
\end{aligned}
$$

It follows

$$
\begin{aligned}
\mathbf{E}[X_1] &= \int_{\mathbb{R}} k\left(x_1 + \frac{7}{12}\right) 1_{[0,1]}(x_1) = k\int_{x_1=0}^{1} \left(x_1 + \frac{7}{12}\right) dx_1 \\
&= k\left(\frac{1}{2}\cdot x_1^2\Big|_{x_1=0}^{1} + \frac{7}{12}\cdot x_1\Big|_{x_1=0}^{1}\right) = \frac{13}{12}k
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}[X_2] &= \int_{\mathbb{R}} k\left(x_2^2 + \frac{7}{12}\right) 1_{[0,1]}(x_2) = k\int_{x_2=0}^{1} \left(x_2^2 + \frac{7}{12}\right) dx_2 \\
&= k\left(\frac{1}{3}\cdot x_1^3\Big|_{x_2=0}^{1} + \frac{7}{12}\cdot x_2\Big|_{x_2=0}^{1}\right) = \frac{11}{12}k.
\end{aligned}
$$

The conditional density $f_{X_1,X_2|X_3=1/2}(x_1,x_2)$ is simply given by

$$
f_{X_1,X_2|X_3=1/2}(x_1,x_2) = \frac{f_{X_1,X_2,X_3}(x_1,x_2,1/2)}{\int_{\mathbb{R}^2} f_{X_1,X_2,X_3}(x_1,x_2,1/2)\, d\mu_L(x_1,x_2)} = \frac{f_{X_1,X_2,X_3}(x_1,x_2,1/2)}{f_{X_3}(1/2)},
$$

for every $(x_1,x_2)\in\mathbb{R}^2$. Now, since

$$
f_{X_1,X_2,X_3}(x_1,x_2,1/2) = k\left(x_1 + x_2^2 + \frac{1}{8}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^2} f_{X_1,X_2,X_3}\left(x_1,x_2,1/2\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{\mathbb{R}^2} k\left(x_1+x_2^2+\frac{1}{8}\right) 1_{[0,1]\times[0,1]}\left(x_1,x_2\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{[0,1]\times[0,1]} k\left(x_1+x_2^2+\frac{1}{8}\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{x_1=0}^{1}\int_{x_2=0}^{1} k\left(x_1+x_2^2+\frac{1}{8}\right) dx_1 dx_2 \\
&= k\left(\int_{x_1=0}^{1}\int_{x_2=0}^{1} x_1 dx_1 dx_2 + \int_{x_1=0}^{1}\int_{x_2=0}^{1} x_2^2 dx_1 dx_2 + \int_{x_1=0}^{1}\int_{x_2=0}^{1} \frac{1}{8} dx_1 dx_2\right) \\
&= k\left(\int_{x_1=0}^{1} x_1 dx_1 \int_{x_2=0}^{1} dx_2 + \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} x_2^2 dx_2 + \frac{1}{8}\int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} dx_2\right) \\
&= k\left(\frac{1}{2}\cdot x_1^2\Big|_{x_1=0}^{1} \cdot x_2\Big|_{x_2=0}^{1} + x_1\Big|_{x_1=0}^{1} \cdot \frac{1}{3}\cdot x_2^3\Big|_{x_2=0}^{1} + \frac{1}{8}\cdot x_1\Big|_{x_1=0}^{1} x_2\Big|_{x_2=0}^{1}\right) \\
&= k\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{8}\right) \\
&= \frac{23}{24}k,
\end{aligned}
$$

we obtain

$$
f_{X_1,X_2|X_3=1/2}\left(x_1,x_2\right) = \frac{24}{23}\left(x_1+x_2^2+\frac{1}{8}\right) 1_{[0,1]\times[0,1]}\left(x_1,x_2\right).
$$

Problem 4 *Determine the value of the parameter k such that the function $f:\mathbb{R}^3\to\mathbb{R}$ given by*

$$
f\left(x_1,x_2,x_3\right) \overset{def}{=} \begin{cases} k\left(x_1+x_2^2+x_3^3\right), & \text{if } \left(x_1,x_2,x_3\right)\in[0,1]\times[0,1]\times[0,1], \\ 0, & \text{otherwise,} \end{cases}
$$

is a probability density. Hence, consider the random vector $X\equiv\left(X_1,X_2,X_3\right)^{\intercal}$ with density $f_X:\mathbb{R}^3\to\mathbb{R}$ given by

$$
f_X\left(x_1,x_2,x_3\right) \overset{def}{=} f\left(x_1,x_2,x_3\right).
$$

1. *Determine the distribution function $F_X:\mathbb{R}^3\to\mathbb{R}_+$ and check that*

$$
\frac{\partial^3 F_X}{\partial x_1\partial x_2\partial x_3}\left(x_1,x_2,x_3\right) = f_X\left(x_1,x_2,x_3\right), \quad \mu_L^3\text{-a.e. on } \mathbb{R}^3.
$$

2. *Determine the marginal distribution functions $F_{X_1}:\mathbb{R}\to\mathbb{R}$, $F_{X_2}:\mathbb{R}\to\mathbb{R}$, and $F_{X_3}:\mathbb{R}\to\mathbb{R}$ of the entries X_1, X_2, and X_3 of X.*

3. *Determine the marginal densities $f_{X_1}:\mathbb{R}\to\mathbb{R}$, $f_{X_2}:\mathbb{R}\to\mathbb{R}$, and $f_{X_3}:\mathbb{R}\to\mathbb{R}$ of the entries X_1, X_2, and X_3 of X and check that*

$$
\frac{dF_{X_n}}{dx}(x) = f_{X_n}(x), \text{ for } n=1,2,3, \quad \mu_L\text{-a.e. on } \mathbb{R}.
$$

4. *Determine the joint distribution function $F_{X_1,X_2}:\mathbb{R}^2\to\mathbb{R}$, $F_{X_1,X_3}:\mathbb{R}\to\mathbb{R}$, and $F_{X_2,X_3}:\mathbb{R}\to\mathbb{R}$.*

5. *Determine the joint densities $f_{X_1,X_2}:\mathbb{R}^2\to\mathbb{R}$, $f_{X_1,X_3}:\mathbb{R}\to\mathbb{R}$, and $f_{X_2,X_3}:\mathbb{R}\to\mathbb{R}$. What is the relationship between the joint distribution function $F_{X_m,X_n}:\mathbb{R}^2\to\mathbb{R}$ and the joint density $f_{X_m,X_n}:\mathbb{R}^2\to\mathbb{R}$ for $m,n=1,2,3$, $m<n$.*

6. *Determine the the expectation of X.*

7. *Determine the variance-covariance matrix of* X.

Solution. .

Problem 5 *Determine the value of the parameter* k *such that the function* $f:\mathbb{R}^3\to\mathbb{R}$ *given by*

$$f(x_1,x_2,x_3)\overset{def}{=}\begin{cases} k\left(x_1^2+x_2^2+x_3^3\right), & \text{if } (x_1,x_2,x_3)\in[0,1]\times[0,1]\times[0,1],\\ 0, & \text{otherwise,}\end{cases}$$

is a probability density. Hence, consider the random vector $(X_1,X_2,X_3)^\intercal$ *with density* $f_{X_1,X_2,X_3}:\mathbb{R}^3\to\mathbb{R}$ *given by*

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3)\overset{def}{=}f(x_1,x_2,x_3).$$

Compute:

1. *the marginal density of the random vector* $(X_1,X_2)^\intercal$*;*
2. *the expectation of the product* $X_1\cdot X_2$*;*
3. *the conditional density* $f_{X_1|X_2=1/2,X_3=3/4}(x_1)$*;*
4. *the probability* $\mathbf{P}(X_1\le 1/2, X_2<1/2, X_3<1/2)$.

Solution. .

Problem 6 *Let* $F:\mathbb{R}^2\to\mathbb{R}_+$*, briefly* F*, given by*

$$F(x_1,x_2)\overset{def}{=}\left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right)1_{\mathbb{R}_+}(x_1)\,1_{\mathbb{R}_+}(x_2),\quad \forall(x_1,x_2)\in\mathbb{R}^2.$$

Show that F *is the distribution function of a real random vector* (X_1,X_2) *and compute the marginal distribution functions of* (X_1,X_2).

1. *Is the function* F *absolutely continuous?*
2. *Are the entries* X_1 *and* X_2 *of the random vector* (X_1,X_2) *independent random variables?*
3. *Are the entries* X_1 *and* X_2 *of the random vector* (X_1,X_2) *absolutely continuous random variables?*
4. *What is the distribution* $F_Z:\mathbb{R}^2\to\mathbb{R}_+$*, briefly* F_Z*, of the real random variable* $Z=\max\{X_1,X_2\}$.
5. *Is the function* F_Z *absolutely continuous?*

   Hint: it might be useful to rewrite $F(x_1,x_2)$ *in a more convenient form.*

Solution. .

Problem 7 *Let* $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ *be a probability space and let* $\left(\mathbb{R}^2,\mathcal{B}\left(\mathbb{R}^2\right),\mu_L^2\right)\equiv\mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra* $\mathcal{B}\left(\mathbb{R}^2\right)$ *and the Lebesgue measure* $\mu_L^2:\mathcal{B}\left(\mathbb{R}^2\right)\to\mathbb{R}_+$*. Let* $f:\mathbb{R}^2\to\mathbb{R}$ *given by*

$$f(x,y)\overset{def}{=}ke^{-(x^2-xy+y^2/2)},\quad\forall(x,y)\in\mathbb{R}^2,$$

where $k\in\mathbb{R}$ *is a parameter.*

1. *Determine* k *such that* $f:\mathbb{R}^2\to\mathbb{R}$ *is a probability density. Hint: can you compute* $\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(y-x)^2}dy$ *with no computation?*

   Let $Z\equiv(X,Y)$ *be the random vector on* Ω *with density* $f:\mathbb{R}^2\to\mathbb{R}_+$.
2. *Determine the marginal density of the entries* X *and* Y*. Are the random variables* X *and* Y *Gaussian?*
3. *Is the random vector* Z *Gaussian?*

4. *Compute* $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$, *and* $Cov(X,Y)$.

5. *Are* X *and* Y *independent random variables?*

6. *Is the random vector* Z *Gaussian? Hint: consider the answer you gave to 4., what you know from the theory, and try to make a simple guess.*

Solution.

1. We can write
$$\int_{\mathbb{R}^2} f(x,y)\,d\mu_L^2(x,y) = k\int_{\mathbb{R}^2} e^{-(x^2-xy+y^2/2)}d\mu_L^2(x,y).$$
On the other hand, since $e^{-(x^2-xy+y^2/2)}$ is a continuous positive function
$$\begin{aligned}\int_{\mathbb{R}^2} e^{-(x^2-xy+y^2/2)}d\mu_L^2(x,y) &= \int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} e^{-(x^2-xy+y^2/2)}dxdy \\ &= \int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(y^2-2xy+x^2\right)}e^{-\frac{1}{2}x^2}dxdy \\ &= \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\left(\int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2}dy\right)dx.\end{aligned}$$
Now, we have
$$\int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2}dy = \int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2}dz = \sqrt{2\pi},$$
for every $x\in\mathbb{R}$. Therefore,
$$\int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} e^{-(x^2-xy+y^2/2)}dxdy = \sqrt{2\pi}\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2}dx = 2\pi.$$
If follows that
$$\int_{\mathbb{R}^2} f(x,y)\,d\mu_L^2(x,y) = 1 \Rightarrow k = \frac{1}{2\pi}.$$

2. Considering what shown above, we have
$$f_X(x) = \int_{\mathbb{R}} \frac{1}{2\pi} f(x,y)\,d\mu_L(y) = \frac{1}{2\pi}\int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}\left(y^2-2xy+x^2\right)}e^{-\frac{1}{2}x^2}dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},$$
for every $x\in\mathbb{R}$. Similarly, since
$$e^{-(x^2-xy+y^2/2)} = e^{-\frac{1}{2}(2x^2-2xy+y^2)} = e^{-\frac{1}{2}\left(\left(\sqrt{2}x\right)^2-2xy+\left(\frac{y}{\sqrt{2}}\right)^2\right)}e^{-\frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2} = e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2}e^{-\frac{y^2}{4}},$$
we have
$$f_Y(y) = \int_{\mathbb{R}} \frac{1}{2\pi} f(x,y)\,d\mu_L(x) = \frac{1}{2\pi}\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2}e^{-\frac{y^2}{4}}dx = \frac{1}{2\pi}e^{-\frac{y^2}{4}}\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2}dx,$$
for every $y\in\mathbb{R}$. Furthermore,
$$\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2}dx = \frac{1}{\sqrt{2}}\int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2}dz = \sqrt{\pi}.$$
Hence,
$$f_Y(y) = \frac{1}{2\sqrt{\pi}}e^{-\frac{y^2}{4}} = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{1}{2}\left(\frac{y}{\sigma_Y}\right)^2}, \quad \sigma_Y \equiv \sqrt{2}.$$
This shows that the random variables X and Y are Gaussian.

3. We clearly have

$$\mathbf{E}[X] = \mathbf{E}[Y] = 0.$$

Moreover,

$$\mathbf{D}^2[X] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2} dx = 1, \quad \mathbf{D}^2[Y] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} y^2 e^{-\frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2} dy = 2.$$

In addition,

$$\begin{aligned} Cov(X,Y) = \mathbf{E}[XY] &= \int_{\mathbb{R}^2} xyf(x,y)\, d\mu_L^2(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} xye^{-(x^2-xy+y^2/2)} d\mu_L^2(x,y) \\ &= \frac{1}{2\pi}\int_{x=-\infty}^{+\infty} xe^{-\frac{1}{2}x^2}\left(\int_{y=-\infty}^{+\infty} ye^{-\frac{1}{2}(y-x)^2} dy\right) dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{y=-\infty}^{+\infty} ye^{-\frac{1}{2}(y-x)^2} dy &= \int_{y=-\infty}^{+\infty} (y-x)\, e^{-\frac{1}{2}(y-x)^2} dy + \int_{y=-\infty}^{+\infty} xe^{-\frac{1}{2}(y-x)^2} dy \\ &= \int_{z=-\infty}^{+\infty} ze^{-\frac{1}{2}z^2} dz + x\int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz \\ &= \sqrt{2\pi}x. \end{aligned}$$

Hence,

$$Cov(X,Y) = \frac{1}{2\pi}\int_{x=-\infty}^{+\infty} \sqrt{2\pi}x^2 e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}}\int_{x=-\infty}^{+\infty} x^2 e^{-\frac{1}{2}x^2} = 1.$$

4. Since

$$Cov(X,Y) \neq 0,$$

the random variables X and Y are not independent.

5. Since not independent, despite X and Y are Gaussian, we cannot state at present whether the random vector $(X,Y)^{\intercal}$ is Gaussian or not. To solve this doubt, we can try to write

$$\begin{pmatrix} X \\ Z \end{pmatrix} = A\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

for independent standard Gaussian random variables Z_1 and Z_1 and a suitable matrix

$$A \equiv \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

If this is true, we have

$$\Sigma_{X,Y}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = AA^{\intercal}.$$

Thus, we are led to find a matrix A such that

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}^2 + a_{1,2}^2 & a_{1,1}a_{2,1} + a_{1,2}a_{2,2} \\ a_{1,1}a_{2,1} + a_{1,2}a_{2,2} & a_{2,1}^2 + a_{2,2}^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

To this goal, observe that $\Sigma_{X,Y}^2$ has eigenvalues

$$\frac{3}{2} + \frac{1}{2}\sqrt{5} \quad \text{and} \quad \frac{3}{2} - \frac{1}{2}\sqrt{5},$$

with corresponding orthogonal eigenvectors

$$\begin{pmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix}.$$

In fact, we have

$$\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2}\sqrt{5}-\frac{1}{2}\\ 1\end{pmatrix}-\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{2}\sqrt{5}-\frac{1}{2}\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$\left(\frac{3}{2}-\frac{1}{2}\sqrt{5}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-\frac{1}{2}\sqrt{5}-\frac{1}{2}\\ 1\end{pmatrix}-\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}-\frac{1}{2}\sqrt{5}-\frac{1}{2}\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

and

$$\begin{pmatrix}\frac{1}{2}\sqrt{5}-\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}-\frac{1}{2}\sqrt{5}-\frac{1}{2}\\ 1\end{pmatrix}=0.$$

Therefore, normalizing the eigenvectors, we have that

$$B\equiv\left\{\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\end{pmatrix},\begin{pmatrix}-\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix}\right\}$$

is a basis of orthonormal eigenvectors in $\mathbb{R}^2$. We then have

$$M_E^B(id)\,\Lambda M_B^E(id)=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix},$$

where

$$E\equiv\left\{\begin{pmatrix}1\\ 0\end{pmatrix},\begin{pmatrix}0\\ 1\end{pmatrix}\right\}$$

is the standard orthonormal basia in $\mathbb{R}^2$,

$$M_E^B(id)=\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix},\quad \Lambda\equiv\begin{pmatrix}\frac{3}{2}+\frac{1}{2}\sqrt{5} & 0\\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{5}\end{pmatrix},$$

and

$$M_B^E(id)=M_E^B(id)^{-1}=M_E^B(id)^{\intercal}=\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}.$$

Hence,

$$\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix}\begin{pmatrix}\frac{3}{2}+\frac{1}{2}\sqrt{5} & 0\\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{5}\end{pmatrix}\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}.$$

In addition, we can write

$$\begin{aligned}
&\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix}\begin{pmatrix}\frac{3}{2}+\frac{1}{2}\sqrt{5} & 0\\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{5}\end{pmatrix}\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}\\
&=\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{3}{2}+\frac{1}{2}\sqrt{5}} & 0\\ 0 & \sqrt{\frac{3}{2}-\frac{1}{2}\sqrt{5}}\end{pmatrix}\\
&\cdot\begin{pmatrix}\sqrt{\frac{3}{2}+\frac{1}{2}\sqrt{5}} & 0\\ 0 & \sqrt{\frac{3}{2}-\frac{1}{2}\sqrt{5}}\end{pmatrix}\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}\\
&=\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}+\frac{1}{2}\sqrt{5} & 0\\ 0 & \frac{1}{2}\sqrt{5}-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2}+\frac{1}{2}\sqrt{5} & 0\\ 0 & \frac{1}{2}\sqrt{5}-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\\ -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\end{pmatrix}.
\end{aligned}$$

Therefore, we obtain

$$\left(\begin{array}{cc} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{array}\right)\left(\begin{array}{cc} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{array}\right)=\left(\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right).$$

Setting

$$A=\left(\begin{array}{cc} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{array}\right),$$

it then follows

$$a_{1,1}^2+a_{1,2}^2=\left(\frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\right)^2+\left(-\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\right)^2=1,$$

$$a_{2,1}^2+a_{2,2}^2=\left(\frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\right)^2+\left(\frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\right)^2=2,$$

$$a_{1,1}a_{2,1}+a_{1,2}a_{2,2}=\frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}-\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}=1.$$

This proves that $(X,Y)^\intercal$ is Gaussian. Note that, from

$$\left(\begin{array}{c} X \\ Y \end{array}\right)=A\left(\begin{array}{c} Z_1 \\ Z_2 \end{array}\right),$$

it follows

$$\left(\begin{array}{c} X \\ Y \end{array}\right)\left(\begin{array}{cc} X & Y \end{array}\right)=A\left(\begin{array}{c} Z_1 \\ Z_2 \end{array}\right)\left(\begin{array}{cc} Z_1 & Z_2 \end{array}\right)A^\intercal,$$

that is to say

$$\left(\begin{array}{cc} X^2 & XY \\ XY & Y^2 \end{array}\right)=A\left(\begin{array}{cc} Z_1^2 & Z_1Z_2 \\ Z_1Z_2 & Z_2^2 \end{array}\right)A^\intercal.$$

It follows,

$$\begin{aligned}\Sigma_{X,Y}^2&=\left(\begin{array}{cc} \mathbf{D}^2[X] & Cov(X,Y) \\ Cov(X,Y) & \mathbf{D}^2[X] \end{array}\right)=\left(\begin{array}{cc} \mathbf{E}\left[X^2\right] & \mathbf{E}[XY] \\ \mathbf{E}[XY] & \mathbf{E}\left[Y^2\right] \end{array}\right)\\ &=A\left(\begin{array}{cc} \mathbf{E}\left[Z_1^2\right] & \mathbf{E}[Z_1Z_2] \\ \mathbf{E}[Z_1Z_2] & \mathbf{E}\left[Z_2^2\right] \end{array}\right)A^\intercal=A\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)A^\intercal=AA^\intercal.\end{aligned}$$

Problem 8 *Let* $(\Omega,\mathcal{E},\mathbf{P})\equiv\Omega$ *be a probability space and let* $\left(\mathbb{R}^2,\mathcal{B}\left(\mathbb{R}^2\right),\mu_L^2\right)\equiv\mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra* $\mathcal{B}\left(\mathbb{R}^2\right)$ *and the Lebesgue measure* $\mu_L^2:\mathcal{B}\left(\mathbb{R}^2\right)\to\mathbb{R}_+$*. Let* $f:\mathbb{R}^2\to\mathbb{R}$ *given by*

$$f(x,y)\overset{def}{=}ke^{-\frac{x^2-xy+y^2}{2}},\quad\forall(x,y)\in\mathbb{R}^2,$$

where $k\in\mathbb{R}$ *is a parameter.*

1. *Determine* k *such that* $f:\mathbb{R}^2\to\mathbb{R}$ *is a probability density. Hint: It may be useful to recall that* $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{2}}\,dx=1$.
2. *Determine the marginal density functions of the entries* X *and* Y*. Are* X *and* Y *independent?*
3. *Compute* $\mathbf{P}(X=Y)$ *and* $\mathbf{P}(X\geq Y)$*.*

Solution. .

Exercise 9 (Sheldon M. Ross - 4.11) *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X and Y be real random variables on Ω such that the random vector (X, Y) is absolutely continuous with a density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f_{X,Y}(x, y) \overset{def}{=} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \cdot 1_{(0,1)\times(0,2)}(x, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

1. *Check that $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ is a density function.*

2. *Are the random variables X and Y absolutely continuous? In case of affirmative answer determine the marginal densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ of X and Y, respectively.*

3. *Check whether the random variables X and Y are independent.*

4. *Compute $\mathbf{P}(X > Y)$.*

Solution.

1. We will have proven that $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ is a density function if we can show that

$$\int_{\mathbb{R}^2} f_{X,Y}(x, y)\, d\mu_L^2(x, y) = 1.$$

On the other hand, considering the properties of the Lebesgue integral, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} f_{X,Y}(x, y)\, d\mu_L(x, y) &= \int_{\mathbb{R}^2} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \cdot 1_{(0,1)\times(0,2)}(x, y)\ \ d\mu_L^2(x, y) \\
&= \int_{(0,1)\times(0,2)} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) d\mu_L^2(x, y) \\
&= \int_{(0,1)\times(0,2)} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx dy \\
&= \int_{y=0}^{2} \int_{x=0}^{1} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx dy \\
&= \frac{6}{7} \int_{y=0}^{2} \left(\int_{x=0}^{1} \left(x^2 + \frac{xy}{2}\right) dx \right) dy \\
&= \frac{6}{7} \int_{y=0}^{2} \left(\frac{x^3}{3} + \frac{x^2 y}{4}\bigg|_0^1 \right) dy \\
&= \frac{6}{7} \int_{y=0}^{2} \left(\frac{1}{3} + \frac{y}{4} \right) dy \\
&= \frac{6}{7} \left(\frac{y}{3} + \frac{y^2}{8}\bigg|_0^2 \right) \\
&= \frac{6}{7} \left(\frac{2}{3} + \frac{1}{2} \right) \\
&= 1.
\end{aligned}$$

2. Since the random vector is absolutely continuous the entries X and Y are absolutely continuous random variables with densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, d\mu_L(y) \qquad \text{and} \qquad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)\, d\mu_L(x),$$

μ_L-a.e. on $\mathbb{R}$, respectively. Now, we have

$$\begin{aligned}
\int_{\mathbb{R}} f_{X,Y}(x,y)\,d\mu_L(y) &= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\,d\mu_L(y)\\
&= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\,1_{(0,2)}(y)\,d\mu_L(y)\\
&= \int_{(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\,d\mu_L(y)\\
&= \frac{6}{7}\left(\int_0^2\left(x^2+\frac{xy}{2}\right)dy\right)\cdot 1_{(0,1)}(x)\\
&= \frac{6}{7}\left(x^2y+\left.\frac{xy^2}{4}\right|_{y=0}^{2}\right)\cdot 1_{(0,1)}(x)\\
&= \frac{6}{7}\left(2x^2+x\right)\cdot 1_{(0,1)}(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}} f_{X,Y}(x,y)\,d\mu_L(x) &= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\,d\mu_L(x)\\
&= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\,1_{(0,2)}(y)\,d\mu_L(x)\\
&= \int_{(0,1)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,2)}(y)\,d\mu_L(y)\\
&= \frac{6}{7}\left(\int_0^1\left(x^2+\frac{xy}{2}\right)dx\right)\cdot 1_{(0,2)}(y)\\
&= \frac{6}{7}\left(\frac{x^3}{3}+\left.\frac{x^2y}{4}\right|_{x=0}^{1}\right)\cdot 1_{(0,2)}(y)\\
&= \frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y).
\end{aligned}$$

Therefore, we can write

$$f_X(x)=\frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x) \qquad\text{and}\qquad f_Y(y)=\frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y),$$

μ_L-a.e. on $\mathbb{R}$, respectively.

3. The random variables X and Y are independent if and only if

$$f_X(x)\,f_Y(y)=f_{X,Y}(x,y),$$

μ_L^2-a.e. on $\mathbb{R}^2$. On the other hand,

$$\begin{aligned}
f_X(x)\,f_Y(x) &= \left(\frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x)\right)\left(\frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y)\right)\\
&= \frac{36}{49}\left(\frac{x}{3}+\frac{xy}{4}+\frac{2x^2}{3}+\frac{x^2y}{2}\right)\cdot 1_{(0,1)}(x)\,1_{(0,2)}(y)\\
&= \frac{36}{49}\left(\frac{x}{3}+\frac{xy}{4}+\frac{2x^2}{3}+\frac{x^2y}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\\
&\neq \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)
\end{aligned}$$

for almost all points $(x,y)\in(0,1)\times(0,2)$. Therefore, X and Y are not independent.

4. To compute $\mathbf{P}(X>Y)$ we apply the formula

$$\mathbf{P}((X,Y)\in B)=\int_B f_{X,Y}(x,y)\;d\mu_L^2(x,y),$$

which holds true for every $B \in \mathcal{B}\left(\mathbb{R}^2\right)$, by suitably choosing B to represent the event $\{X > Y\}$ in terms of the event $\{(X, Y) \in B\}$. Eventually, setting

$$B \equiv \left\{(x, y) \in \mathbb{R}^2 : x > y\right\},$$

it turns out that we can write

$$\{X > Y\} = \{(X, Y) \in B\}.$$

In fact, assume that $\omega \in \{X > Y\} \equiv \{\omega \in \Omega : X(\omega) > Y(\omega)\}$, then we have $X(\omega) > Y(\omega)$ so that $(X(\omega), Y(\omega)) \in B$ and $\omega \in \{(X, Y) \in B\} \equiv \{\omega \in \Omega : (X(\omega), Y(\omega)) \in B\}$. Conversely, assume that $\omega \in \{(X, Y) \in B\}$, then $(X(\omega), Y(\omega)) \in B$, which implies $X(\omega) > Y(\omega)$ and consequently $\omega \in \{X > Y\}$. As a consequence, we have

$$\begin{aligned}
\mathbf{P}(X > Y) &= \int_{\{(x,y)\in\mathbb{R}^2 : x>y\}} f_{X,Y}(x, y)\, d\mu_L^2(x, y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x>y\}} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \cdot 1_{(0,1)\times(0,2)}(x, y)\, d\mu_L^2(x, y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x>y\}\cap(0,1)\times(0,2)} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) d\mu_L^2(x, y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x>y\}\cap(0,1)\times(0,2)} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dxdy \\
&= \frac{6}{7}\int_{x=0}^{1}\left(\int_{y=0}^{x}\left(x^2 + \frac{xy}{2}\right) dy\right) dx \\
&= \frac{6}{7}\int_{x=0}^{1}\left(x^2 y + \frac{xy^2}{4}\bigg|_0^x\right) dx \\
&= \frac{6}{7}\int_{x=0}^{1} \frac{5x^3}{4} dx \\
&= \frac{6}{7} \frac{5x^4}{16}\bigg|_0^1 \\
&= \frac{6}{7}\frac{5}{16} \\
&= \frac{15}{56} \approx 0.26786
\end{aligned}$$

.

Problem 10 *Let $f : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f(x, y) \overset{def}{=} \frac{4x + 2y}{3} 1_{[0,1]}(x)\, 1_{[0,1]}(y), \quad \forall (x, y) \in \mathbb{R}^2.$$

1. *Show that $f : \mathbb{R}^2 \to \mathbb{R}_+$ is the density function of a real random vector (X, Y).*
2. *Compute the marginal densities of (X, Y) and check that the computed marginal densities are actually probability densities.*
3. *May we say that the entries X and Y of the random vector (X, Y) are independent random variables?*
4. *Compute the conditional density function $f_{X|Y}(x, y)$ of X given that $Y = y$ and check the computed density is actually a probability density.*
5. *Compute the function $\mathbf{E}[X \mid Y = y]$ and the conditional expectation $\mathbf{E}[X \mid Y]$.*

Solution. .

Problem 11 (Sheldon M. Ross - 4.17) *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X and Y be absolutely continuous real random variables on Ω with densities $f_X : \mathbb{R} \to \mathbb{R}$ and $f_Y : \mathbb{R} \to \mathbb{R}$, respectively. Assume that the densities of X and Y have at most a finite number of discontinuity points and that X and Y are independent.*

1. *Prove that we have*
$$\mathbf{P}(X + Y < a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\ dy,$$
for every $a \in \mathbb{R}$, and
$$\mathbf{P}(X \leq Y) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y)\ dy,$$
where $F_X : \mathbb{R} \to \mathbb{R}_+$ is the distribution function of X.

2. *Show that the latter equation does not hold true if we drop the assumption of independence.*

Solution.

1. Since the random variables X and Y are absolutely continuous and independent, the random vector (X, Y) is absolutely continuous with a density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ given by
$$f_{X,Y}(x, y) = f_X(x) f_Y(y),$$
for every $(x, y) \in \mathbb{R}^2$. To compute $\mathbf{P}(X + Y < a)$ we apply the formula
$$\mathbf{P}((X, Y) \in B) = \int_B f_{X,Y}(x, y)\ d\mu_L^2(x, y),$$
which holds true for every $B \in \mathcal{B}(\mathbb{R}^2)$, by suitably choosing B to represent the event $\{X + Y < a\}$ in terms of the event $\{(X, Y) \in B\}$. Eventually, setting
$$B \equiv \left\{(x, y) \in \mathbb{R}^2 : x + y < a\right\},$$
it turns out that we can write
$$\{X + Y < a\} = \{(X, Y) \in B\}.$$
Hence, on account of the continuity property of the densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$, we have
$$\begin{aligned}
\mathbf{P}(X + Y < a) &= \int_{\{(x,y)\in\mathbb{R}^2 : x+y<a\}} f_{X,Y}(x, y)\ d\mu_L^2(x, y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x+y<a\}} f_X(x) f_Y(y)\ d\mu_L^2(x, y) \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{a-y} f_X(x) f_Y(y)\ dxdy \\
&= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{a-y} f_X(x) f_Y(y)\ dxdy \\
&= \int_{y=-\infty}^{\infty} f_Y(y) \left(\int_{x=-\infty}^{a-y} f_X(x)\ dx \right) dy \\
&= \int_{-\infty}^{\infty} f_Y(y) F_X(a - y)\ dy.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\mathbf{P}(X \leq Y) &= \int_{\{(x,y)\in\mathbb{R}^2 : x\leq y\}} f_{X,Y}(x,y)\ d\mu_L^2(x,y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x\leq y\}} f_X(x) f_Y(y)\ d\mu_L^2(x,y) \\
&= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{y} f_X(x) f_Y(y)\ dxdy \\
&= \int_{y=-\infty}^{\infty} f_Y(y)\left(\int_{x=-\infty}^{y} f_X(x)\ dx\right)\ dy \\
&= \int_{-\infty}^{\infty} f_Y(y) F_X(y)\ dy.
\end{aligned}
$$

2. To show that

$$
\mathbf{P}(X \leq Y) = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\ dy,
$$

does not hold true if we drop the assumption of independence, consider the random variables X and Y with densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ given by

$$
f_X(x) \stackrel{\text{def}}{=} \frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x), \quad \forall x \in \mathbb{R} \qquad \text{and} \qquad f_Y(y) \stackrel{\text{def}}{=} \frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y), \quad \forall y \in \mathbb{R}
$$

and a joint density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$
f_{X,Y}(x,y) \stackrel{\text{def}}{=} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y), \quad \forall (x,y) \in \mathbb{R}^2.
$$

We know that

$$
\mathbf{P}(X > Y) = \frac{15}{56}.
$$

Therefore,

$$
\mathbf{P}(X \leq Y) = 1 - \mathbf{P}(X > Y) = 1 - \frac{15}{56} = \frac{41}{56} \approx 0.73214
$$

On the other hand, the distribution function $F_X : \mathbb{R} \to \mathbb{R}_+$ is given by

$$
F_X(x) = \frac{6}{7}\left(\frac{1}{2}x^2+\frac{2}{3}x^3\right)\cdot 1_{(0,1]}(x) + 1_{(1,+\infty)}(x)
$$

for every $x \in \mathbb{R}$. In fact, we have

$$
\begin{aligned}
F_X(x) &= \int_{(-\infty,x)} f_X(u)\ d\mu_L(u) \\
&= \int_{(-\infty,x)} \frac{6}{7}\left(u+2u^2\right)\cdot 1_{(0,1)}(u)\ d\mu_L(u) \\
&= \int_{(-\infty,x)\cap(0,1)} \frac{6}{7}\left(u+2u^2\right)\ d\mu_L(u) \\
&= \begin{cases} 0 & \text{if } x \leq 0 \\ \int_{(0,x)} \frac{6}{7}\left(u+2u^2\right)\ d\mu_L(u) & \text{if } 0 < x < 1 \\ \int_{(0,1)} \frac{6}{7}\left(u+2u^2\right)\ d\mu_L(u) & \text{if } 1 \leq x \end{cases},
\end{aligned}
$$

where

$$
\int_{(0,x)} \frac{6}{7}\left(u+2u^2\right)\ d\mu_L(u) = \frac{6}{7}\int_0^x \left(u+2u^2\right)\ du = \frac{6}{7}\ \frac{1}{2}u^2+\frac{2}{3}u^3\Big|_0^x = \frac{6}{7}\left(\frac{1}{2}x^2+\frac{2}{3}x^3\right),
$$

for every $x \in (0,1]$. As a consequence,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f_Y(y) F_X(y)\ dy \\
&\int_{-\infty}^{\infty} \left(\frac{6}{7}\left(\frac{1}{2}y^2+\frac{2}{3}y^3\right)\cdot 1_{(0,1]}(y)+1_{(1,+\infty)}(y)\right)\left(\frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(0,2)}(y)\right)\ dy \\
&=\int_{-\infty}^{\infty} \left(\frac{36}{49}\left(\frac{1}{2}y^2+\frac{2}{3}y^3\right)\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(0,1]}(y)+\frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(1,2)}(y)\right)\ dy \\
&=\frac{36}{49}\int_0^1\left(\frac{1}{6}y^2+\frac{25}{72}y^3+\frac{1}{6}y^4\right)\ dy+\frac{6}{7}\int_1^2\left(\frac{1}{3}+\frac{1}{4}y\right)\ dy \\
&=\frac{36}{49}\left(\frac{1}{18}y^3+\frac{25}{288}y^4+\frac{1}{30}y^5\Big|_0^1\right)+\frac{6}{7}\left(\frac{1}{3}y+\frac{1}{8}y^2\Big|_1^2\right) \\
&=\frac{36}{49}\left(\frac{1}{18}+\frac{25}{288}+\frac{1}{30}\right)+\frac{6}{7}\left(\left(\frac{2}{3}+\frac{4}{8}\right)-\left(\frac{1}{3}+\frac{1}{8}\right)\right) \\
&=\frac{1443}{1960}\approx 0.73622.
\end{aligned}
$$

It the follows

$$\mathbf{P}(X \le Y) \ne \int_{-\infty}^{\infty} f_Y(y) F_X(y)\ dy.$$

It may be interesting to observe that if we assume that X and Y are independent, then a joint density $f_{X,Y}:\mathbb{R}^2 \to \mathbb{R}_+$ is given by

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\
&=\left(\frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x)\right)\left(\frac{6}{7}\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(0,2)}(y)\right) \\
&=\frac{36}{49}\left(x+2x^2\right)\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(0,1)\times(0,2)}(x,y),
\end{aligned}
$$

for every $(x,y) \in \mathbb{R}^2$. In this case,

$$
\begin{aligned}
\mathbf{P}(X \le Y) &= \int_{\{(x,y)\in\mathbb{R}^2 : x \le y\}} \frac{36}{49}\left(x+2x^2\right)\left(\frac{1}{3}+\frac{1}{4}y\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\ d\mu_L^2(x,y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : x \le y\}\cap(0,1)\times(0,2)} \frac{36}{49}\left(x+2x^2\right)\left(\frac{1}{3}+\frac{1}{4}y\right)\ d\mu_L^2(x,y) \\
&=\frac{36}{49}\int_{x=0}^1\left(x+2x^2\right)\left(\int_{y=x}^2\left(\frac{1}{3}+\frac{1}{4}y\right)\ dy\right)\ dx \\
&=\frac{36}{49}\int_{x=0}^1\left(x+2x^2\right)\left(\frac{1}{3}y+\frac{1}{8}y^2\Big|_x^2\right)\ dx \\
&=\frac{36}{49}\int_{x=0}^1\left(x+2x^2\right)\left(\frac{7}{6}-\left(\frac{1}{3}x+\frac{1}{8}x^2\right)\right)\ dx \\
&=\frac{36}{49}\int_{x=0}^1\left(\frac{7}{6}x+2x^2-\frac{19}{24}x^3-\frac{1}{4}x^4\right)\ dx \\
&=\frac{36}{49}\,\frac{7}{12}x^2+\frac{2}{3}x^3-\frac{19}{96}x^4-\frac{1}{20}x^5\Big|_0^1 \\
&=\frac{36}{49}\left(\frac{7}{12}+\frac{2}{3}-\frac{19}{96}-\frac{1}{20}\right) \\
&=\frac{1443}{1960} \\
&=\int_{-\infty}^{\infty} f_Y(y) F_X(y)\ dy
\end{aligned}
$$

Exercise 12 (Sheldon M. Ross - 4.18) *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X and Z be absolutely continuous real random variables on Ω with densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Z : \mathbb{R} \to \mathbb{R}_+$ given by*

$$f_X(x) \overset{def}{=} 6x(1-x) \cdot 1_{(0,1)}(x), \quad \forall x \in \mathbb{R} \qquad \text{and} \qquad f_Z(y) \overset{def}{=} 2z \cdot 1_{(0,1)}(z), \quad \forall z \in \mathbb{R}.$$

Assume that X and Z are independent and show that the random variable $W = X^2 Z$ is absolutely continuous.

Solution. .

Problem 13 *Let U, V real random variables on a probability space Ω such that such that $U \sim V \sim N(0,1)$, the vector (U, V) is Gaussian, and $Corr(U, V) \equiv \rho < 1$. Consider the real random variables*

$$X \overset{def}{=} U - \rho V \quad \text{and} \quad Y \overset{def}{=} \sqrt{1-\rho} V.$$

1. *Can you prove that the vector (X, Y) Gaussian?*
2. *Are the random variables X and Y independent?*
3. *Compute the distributions of X and Y;*
4. *Compute $\mathbf{E}\left[X^2 Y^2\right]$, $\mathbf{E}\left[XY^3\right]$, $\mathbf{E}\left[Y^4\right]$.*
5. *Compute $\mathbf{E}\left[U^2 V^2\right]$.*

Solution. .

Problem 14 *Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x,y) \overset{def}{=} \begin{cases} \frac{y-x}{2}, & \text{if } (x,y) \in [-1,0] \times [0,1], \\ \frac{x-y}{2}, & \text{if } (x,y) \in [0,1] \times [-1,0], \\ 0, & \text{otherwise,} \end{cases}$$

is a probability density. Hence, consider the random vector $(X, Y)^\intercal$ with density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_{X,Y}(x,y) \overset{def}{=} f(x,y).$$

Determine the marginal densities of entries X and Y of $(X,Y)^\intercal$. Are X and Y correlated? Are X and Y independent? Compute

$$\mathbf{P}(X + Y \geq 0).$$

Solution. .

Problem 15 *Determine the value of the parameter k such that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x,y) \overset{def}{=} \begin{cases} ke^{-(x+y)} & \text{if } 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

is a probability density. Hence, consider the random vector $(X, Y)^\intercal$ with density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_{X,Y}(x,y) \overset{def}{=} f(x,y).$$

Determine the marginal densities of the entries of $(X,Y)^\intercal$. Are X and Y correlated? Are X and Y independent? What is the distribution of Y?

Solution. .

Problem 16 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \equiv \mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra. Let* $f : \mathbb{R}^2 \to \mathbb{R}_+$ *given by*

$$f(x,y) \overset{def}{=} kxe^{-(x+y)} 1_{\mathbb{R}^2_+}(x,y), \quad \forall (x,y) \in \mathbb{R}^2$$

where $\mathbb{R}^2_+ \equiv \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$*. Determine* $k \in \mathbb{R}$ *such that* $f : \mathbb{R}^2 \to \mathbb{R}_+$ *is a probability density. Let* $Z \equiv (X,Y)$ *be the random vector of density* $f : \mathbb{R}^2 \to \mathbb{R}_+$.

1. *Determine the distribution function* $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *of the vector* Z *and check that*

$$\frac{\partial F^2}{\partial x \partial y}(x,y) = f(x,y), \quad \mu^2_L\text{-a.e. on } \mathbb{R}^2.$$

2. *Determine the marginal distribution function* $F_X : \mathbb{R} \to \mathbb{R}_+$ *and* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z.

3. *Determine the densities* $f_X : \mathbb{R} \to \mathbb{R}_+$ *and* $f_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z *and check that*

$$\frac{dF_X}{dx}(x) = f_X(x) \quad \text{and} \quad \frac{dF_Y}{dy}(y) = f_Y(y), \quad \mu_L\text{-a.e. on } \mathbb{R}.$$

4. *Are* X *and* Y *independent random varables?*

5. *Compute* $\mathbf{E}[X]$*,* $\mathbf{E}[Y]$*,* $\mathbf{D}^2[X]$*,* $\mathbf{D}^2[Y]$ *and* $Cov(X,Y)$.

6. *Compute* $\mathbf{E}[(X,Y)]$ *and the covariance matrix of the vector* (X,Y).

Solution. . □

Exercise 17 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* X, Z *be real random variables on* Ω *such that* $X \sim N(0,1)$*,* $Z \sim Rad\left(\frac{1}{2}\right)$ *and* X *and* Z *are independent. Set*

$$Y \overset{def}{=} XZ.$$

Prove that $Y \sim N(0,1)$*, but the random vector* (X,Y) *is not Gaussian.*

Solution. Since X and Z are independent and X is symmetric about zero, we have

$$\begin{aligned}
\mathbf{P}(Y \leq y) &= \mathbf{P}(XZ \leq y) = \mathbf{P}(XZ \leq y, Z = -1) + \mathbf{P}(XZ \leq y, Z = 1) \\
&= \mathbf{P}(XZ \leq y \mid Z = -1)\mathbf{P}(Z = -1) + \mathbf{P}(XZ \leq y \mid Z = 1)\mathbf{P}(Z = 1) \\
&= \frac{1}{2}(\mathbf{P}(-X \leq y \mid Z = -1) + \mathbf{P}(X \leq y \mid Z = 1)) \\
&= \frac{1}{2}(\mathbf{P}(-X \leq y) + \mathbf{P}(X \leq y)) \\
&= \frac{1}{2}(\mathbf{P}(X \geq -y) + \mathbf{P}(X \leq y)) \\
&= \mathbf{P}(X \leq y),
\end{aligned}$$

for every $y \in \mathbb{R}$. This proves that $Y \sim N(0,1)$. Now, consider $Cov(X,Y)$. We have $\mathbf{E}[X] = \mathbf{E}[Y] = 0$ and, thanks again to the independence of X and Z,

$$Cov(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] = \mathbf{E}[XY] = \mathbf{E}[X^2 Z] = \mathbf{E}[X^2]\mathbf{E}[Z] = 0.$$

That is to say that the random variables X and Y are uncorrelated. Therefore, if random vector (X,Y) were Gaussian, the random variables X and Y would be independent. In particular, we would have

$$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X \leq x)\mathbf{P}(Y \leq y),$$

for all $x, y \in \mathbb{R}$. On the other hand, still on account of the independence of X and Z, we have

$$\begin{aligned}
\mathbf{P}(X \leq x, Y \leq y) &= \mathbf{P}(X \leq x, XZ \leq y) = \mathbf{P}(X \leq x, XZ \leq y, Z = -1) + \mathbf{P}(X \leq x, XZ \leq y, Z = 1) \\
&= \mathbf{P}(X \leq x, XZ \leq y \mid Z = -1)\, \mathbf{P}(Z = -1) + \mathbf{P}(X \leq x, XZ \leq y \mid Z = 1)\, \mathbf{P}(Z = 1) \\
&= \mathbf{P}(X \leq x, -X \leq y \mid Z = -1)\, \mathbf{P}(Z = -1) + \mathbf{P}(X \leq x, X \leq y \mid Z = 1)\, \mathbf{P}(Z = 1) \\
&= \mathbf{P}(X \leq x, -X \leq y)\, \mathbf{P}(Z = -1) + \mathbf{P}(X \leq x, X \leq y)\, \mathbf{P}(Z = 1) \\
&= \frac{1}{2}\left(\mathbf{P}(X \leq x, X \geq -y) + \mathbf{P}(X \leq x, X \leq y)\right).
\end{aligned}$$

for all $x, y \in \mathbb{R}$. As a consequence, we would obtain

$$\mathbf{P}(X \leq x)\, \mathbf{P}(Y \leq y) = \frac{1}{2}\left(\mathbf{P}(X \leq x, X \geq -y) + \mathbf{P}(X \leq x, X \leq y)\right)$$

for all $x, y \in \mathbb{R}$, which is clearly false if we consider, for instance, $x = y = 0$. $\square$

Problem 18 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ be the Euclidean real plane endowed with the Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^2\right)$ and the Lebesgue measure $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$. Let*

$$\mathbb{R}_+^2(x > y) \equiv \left\{(x, y) \in \mathbb{R}_+^2 : x > y\right\}, \qquad \mathbb{R}_+^2(x \leq y) \equiv \left\{(x, y) \in \mathbb{R}_+^2 : x \leq y\right\},$$

and let $F : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$F(x, y) \stackrel{def}{=} \left(1 - e^{-y} - \frac{1}{2} y e^{-x}\right) 1_{\mathbb{R}_+^2(x>y)}(x, y) + \left(1 - e^{-x} - \frac{1}{2} x e^{-y}\right) 1_{\mathbb{R}_+^2(x \leq y)}(x, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

1. *Can you show that the function $F : \mathbb{R}^2 \to \mathbb{R}_+$ is a distribution function? Hint: consider carefully the sets $\mathbb{R}_+^2(x > y)$ and $\mathbb{R}_+^2(x \leq y)$ (draw a graph).*

   Let $Z \equiv (X, Y)^\intercal$ be the random vector on Ω with distribution function $F : \mathbb{R}^2 \to \mathbb{R}$.

2. *Can you determine the marginal distribution of the entries X and Y?*

3. *Is the random vector Z absolutely continuous? Can you determine a density $f_Z : \mathbb{R}^2 \to \mathbb{R}_+$ for Z? Hint: it may be useful to rewrite the indicator fnctions $1_{\mathbb{R}_+^2(x>y)}(x, y)$ and $1_{\mathbb{R}_+^2(x \leq y)}(x, y)$ in terms of product of other indicator functions.*

4. *If Z is not absolutely continuous, can you determine the marginal densities of the entries X and Y?*

Solution.

1. We clearly have

   $$\lim_{x \to -\infty} F(x, y) = \lim_{y \to -\infty} F(x, y) = 0$$

   and

   $$\lim_{y \to +\infty} \lim_{x \to +\infty} F(x, y) = \lim_{x \to +\infty} \lim_{y \to +\infty} F(x, y) = 1.$$

   We also have

   $$\frac{\partial F}{\partial x}(x, y) = \frac{1}{2} y e^{-x} 1_{\mathbb{R}_+^2(x>y)}(x, y) + \left(e^{-x} - \frac{1}{2} e^{-y}\right) 1_{\mathbb{R}_+^2(x<y)}(x, y)$$

   and

   $$\frac{\partial F}{\partial y}(x, y) = \frac{1}{2}\left(e^{-y} - \frac{1}{2} e^{-x}\right) 1_{\mathbb{R}_+^2(x>y)}(x, y) + \frac{1}{2} x e^{-y} 1_{\mathbb{R}_+^2(x<y)}(x, y).$$

   Therefore,

   $$\frac{\partial F}{\partial x}(x, y) \geq 0 \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) \geq 0,$$

   for every $(x, y) \in \mathbb{R}_+^2 - \left\{(x, y) \in \mathbb{R}_+^2 : x = y\right\}$. Note also that

   $$F(0, y) = 0,$$

for every $y \in \mathbb{R}^2_+$. In fact, this is clearly true if $y < 0$ and it is also true for $y \geq 0$ since

$$1_{\mathbb{R}^2_+(x>y)}(0,y) = 0 \quad \text{and} \quad \left(1 - e^{-x} - \frac{1}{2}xe^{-y}\right)_{x=0} = 0.$$

Similarly

$$F(x,0) = 0,$$

for every $x \in \mathbb{R}^2_+$. In the end, for every $(x,y) \in \mathbb{R}^2_+$ $(x \leq y)$ such that $x = y$ we have

$$F(x,x) = 0,$$

if $x \leq 0$ and

$$F(x,x) = 1 - e^{-x} - \frac{1}{2}xe^{-x}$$

Hence, for every $(x,y) \in \mathbb{R}^2_+$ $(x \leq y)$ such that $x = y$ we have

$$\lim_{u \to x^\cdot} F(u,y) = \lim_{u \to x^\cdot} F(u,x) = \lim_{u \to x^\cdot} \left(1 - e^{-u} - \frac{1}{2}ue^{-x}\right) = 1 - e^{-x} - \frac{1}{2}xe^{-x} = F(x,x)$$

and

$$\lim_{v \to y^\cdot} F(x,v) = \lim_{v \to x^\cdot} \left(1 - e^{-v} - \frac{1}{2}ve^{-x}\right) = 1 - e^{-x} - \frac{1}{2}xe^{-x} = F(x,x).$$

This is enough to show that F is a distribution function.

2. With regard to the marginal distributions, we have

$$F_X(x) = \lim_{y \to +\infty} F(x,y) = \begin{cases} 0, & \text{if } x \leq 0, \\ \lim_{y \to +\infty} \left(1 - e^{-x} - \frac{1}{2}xe^{-y}\right) = 1 - e^{-x}, & \text{if } x > 0. \end{cases}$$

That is,

$$F_X(x) = \left(1 - e^{-x}\right) 1_{\mathbb{R}_+}(x).$$

Similarly,

$$F_Y(y) = \lim_{x \to +\infty} F(x,y) = \begin{cases} 0, & \text{if } y \leq 0, \\ \lim_{x \to +\infty} \left(1 - e^{-y} - \frac{1}{2}ye^{-x}\right) = 1 - e^{-y}, & \text{if } y > 0. \end{cases}$$

That is

$$F_Y(y) = \left(1 - e^{-y}\right) 1_{\mathbb{R}_+}(y).$$

Note that X and Y are exponential random variables with rate parameter $\lambda = 1$.

3. We check whether Z is absolutely continuous. To this goal, observe that we have

$$\frac{\partial^2 F_Z}{\partial y \partial x}(x,y) = \frac{1}{2}\left(e^{-x}1_{\mathbb{R}^2_+(x>y)}(x,y) + e^{-y}1_{\mathbb{R}^2_+(x<y)}(x,y)\right) = \frac{\partial^2 F_Z}{\partial x \partial y}(x,y)$$

for every $(x,y) \in \mathbb{R}^2 - \left\{(x,y) \in \mathbb{R}^2_+ : x = y\right\}$. However,

$$\mu^2_L\left(\left\{(x,y) \in \mathbb{R}^2_+ : x = y\right\}\right) = 0.$$

Hence, we check whether

$$\begin{aligned}
F_Z(x,y) &= \int_{(-\infty,x]\times(-\infty,y]} \frac{\partial^2 F_Z}{\partial y \partial x}(u,v)\, d\mu^2_L(u,v) \\
&= \frac{1}{2}\int_{(-\infty,x]\times(-\infty,y]} \left(e^{-u}1_{\mathbb{R}^2_+(x>y)}(u,v) + e^{-v}1_{\mathbb{R}^2_+(x<y)}(u,v)\right) d\mu^2_L(u,v) \\
&= \frac{1}{2}\left(\int_{(-\infty,x]\times(-\infty,y]} e^{-u}1_{\mathbb{R}^2_+(x>y)}(u,v)\, d\mu^2_L(u,v) + \int_{(-\infty,x]\times(-\infty,y]} e^{-v}1_{\mathbb{R}^2_+(x<y)}(u,v)\, d\mu^2_L(u,v)\right).
\end{aligned}$$

Note that the above equality is trivially true if $x \leq 0$ or $y \leq 0$. In fact, in this case, we have

$$\frac{\partial^2 F_Z}{\partial y \partial x}(u,v) = 0,$$

for every $(u,v) \in$ is identically $(-\infty, x] \times (-\infty, y]$. Therefore, we consider only the case $x, y \in \mathbb{R}_{++}$ and distinguish two subcases $x > y$ and $x \leq y$. Observe that we can write

$$1_{\mathbb{R}^2_+(x>y)}(x,y) = 1_{[0,x)}(y)\, 1_{\mathbb{R}_+}(x) = 1_{(y,+\infty)}(x)\, 1_{\mathbb{R}_+}(y) \quad \text{and} \quad 1_{\mathbb{R}^2_+(x\leq y)}(x,y) = 1_{\mathbb{R}_+}(x)\, 1_{[x,+\infty)}(y) = 1_{[0,y)}(x)\, 1_{\mathbb{R}_+}(y),$$

for every $(x,y) \in \mathbb{R}^2$. Therefore, for every $x, y \in \mathbb{R}_{++}$ such that $x > y$, applying the Fubini therorem, we can write

$$\begin{aligned}
\int_{(-\infty,x]\times(-\infty,y]} e^{-u} 1_{\mathbb{R}^2_+(x>y)}(u,v)\, d\mu_L^2(u,v) &= \int_{(-\infty,x]\times(-\infty,y]} e^{-u} 1_{[0,u)}(v)\, 1_{\mathbb{R}_+}(u)\, d\mu_L^2(u,v) \\
&= \int_{(-\infty,x]} e^{-u} 1_{\mathbb{R}_+}(u)\, d\mu_L(u) \int_{(-\infty,y]} 1_{[0,u)}(v)\, d\mu_L(v) \\
&= \int_{(-\infty,x]\cap\mathbb{R}_+} e^{-u} d\mu_L(u) \int_{(-\infty,y]\cap[0,u)} d\mu_L(v).
\end{aligned}$$

Now,

$$\int_{(-\infty,y]\cap[0,u)} d\mu_L(v) = \mu_L((-\infty,y]\cap[0,u)) = \begin{cases} \mu_L([0,y)) = y, & \text{if } y < u, \\ \mu_L([0,u)) = u, & \text{if } y \geq u, \end{cases} = u 1_{(-\infty,y]}(u) + y 1_{(y,+\infty)}(u).$$

It follows,

$$\begin{aligned}
\int_{(-\infty,x]\times(-\infty,y]} e^{-u} 1_{\mathbb{R}^2_+(x>y)}(u,v)\, d\mu_L^2(u,v) &= \int_{[0,x]} e^{-u} \left(u 1_{(-\infty,y]}(u) + y 1_{(y,+\infty)}(u)\right) d\mu_L(u) \\
&= \int_{[0,x]} u e^{-u} 1_{(-\infty,y]}(u)\, d\mu_L(u) + \int_{[0,x]} y e^{-u} 1_{(y,+\infty)}(u)\, d\mu_L(u) \\
&= \int_{[0,x]\cap(-\infty,y]} u e^{-u} d\mu_L(u) + y \int_{[0,x]\cap(y,+\infty)} e^{-u} d\mu_L(u)
\end{aligned}$$

where, since $x > y$,

$$\begin{aligned}
\int_{[0,x]\cap(-\infty,y]} u e^{-u} d\mu_L(u) + y \int_{[0,x]\cap(y,+\infty)} e^{-u} d\mu_L(u) &= \int_{[0,y]} u e^{-u} d\mu_L(u) + y \int_{(y,x]} e^{-u} d\mu_L(u) \\
&= \int_0^y u e^{-u} du + y \int_y^x e^{-u} du \\
&= 1 - y e^{-y} - e^{-y} + y\left(e^{-y} - e^{-x}\right) \\
&= 1 - e^{-y} - y e^{-x}.
\end{aligned}$$

On the other hand, if $y \geq x$, we have

$$\begin{aligned}
\int_{[0,x]\cap(-\infty,y]} u e^{-u} d\mu_L(u) + y \int_{[0,x]\cap(y,+\infty)} e^{-u} d\mu_L(u) &= \int_{[0,x]} u e^{-u} d\mu_L(u) + y \int_{\varnothing} e^{-u} d\mu_L(u) \\
&= \int_0^x u e^{-u} du \\
&= 1 - x e^{-x} - e^{-x}.
\end{aligned}$$

We can then write

$$\frac{1}{2}\int_{(-\infty,x]\times(-\infty,y]} e^{-u} 1_{\mathbb{R}^2_+(x>y)}(u,v)\, d\mu_L^2(u,v) = \frac{1}{2}\left(1 - e^{-y} - y e^{-x}\right) 1_{\mathbb{R}^2_+(x>y)}(x,y) + \frac{1}{2}\left(1 - e^{-x} - x e^{-x}\right) 1_{\mathbb{R}^2_+(x\leq y)}(x,y).$$

Similarly,

$$\begin{aligned}\int_{(-\infty,x]\times(-\infty,y]} e^{-v}1_{\mathbb{R}^2_+(x<y)}(u,v)\,d\mu_L^2(u,v) &= \int_{(-\infty,x]\times(-\infty,y]} e^{-v}1_{[0,v)}(u)\,1_{\mathbb{R}_+}(v)\,d\mu_L^2(u,v)\\ &= \int_{(-\infty,y]} e^{-v}1_{\mathbb{R}_+}(v)\,d\mu_L(v)\int_{(-\infty,x]} 1_{[0,v)}(u)\,d\mu_L(v)\\ &= \int_{(-\infty,y]\cap\mathbb{R}_+} e^{-v}d\mu_L(v)\int_{(-\infty,x]\cap[0,v)} d\mu_L(v).\end{aligned}$$

Now,

$$\int_{(-\infty,x]\cap[0,v)} d\mu_L(v) = \mu_L((-\infty,x]\cap[0,v)) = \begin{cases}\mu_L([0,x)) = x, & \text{if } x<v,\\ \mu_L([0,v)) = v, & \text{if } x\geq v,\end{cases} = v1_{(-\infty,x]}(u)+x1_{(x,+\infty)}(u).$$

It follows,

$$\begin{aligned}\int_{(-\infty,x]\times(-\infty,y]} e^{-v}1_{\mathbb{R}^2_+(x<y)}(u,v)\,d\mu_L^2(u,v) &= \int_{[0,y]} e^{-v}\left(v1_{(-\infty,x]}(u)+x1_{(x,+\infty)}(u)\right)d\mu_L(v)\\ &\int_{[0,y]} ve^{-v}1_{(-\infty,x]}(u)\,d\mu_L(v)+\int_{[0,y]} xe^{-v}1_{(x,+\infty)}(u)\,d\mu_L(v)\\ &\int_{[0,y]\cap(-\infty,x]} ve^{-v}d\mu_L(v)+x\int_{[0,y]\cap(x,+\infty)} e^{-v}d\mu_L(v)\end{aligned}$$

where, in case $x>y$,

$$\begin{aligned}\int_{[0,y]\cap(-\infty,x]} ve^{-v}d\mu_L(v)+x\int_{[0,y]\cap(x,+\infty)} e^{-u}d\mu_L(u) &= \int_{[0,y]} ve^{-v}d\mu_L(v)+x\int_{\varnothing} e^{-u}d\mu_L(u)\\ &= \int_0^y ve^{-v}dv\\ &= 1-e^{-y}-ye^{-y}\end{aligned}$$

and, in case $x\leq y$,

$$\begin{aligned}\int_{[0,y]\cap(-\infty,x]} ve^{-v}d\mu_L(v)+x\int_{[0,y]\cap(x,+\infty)} e^{-u}d\mu_L(u) &= \int_{[0,x]} ve^{-v}d\mu_L(v)+x\int_{[x,y]} e^{-u}d\mu_L(u)\\ &= \int_0^x ve^{-v}dv+x\int_x^y e^{-u}du\\ &= 1-e^{-x}-xe^{-x}+x\left(e^{-x}-e^{-y}\right)\\ &= 1-e^{-x}-xe^{-y}.\end{aligned}$$

We then have

$$\frac{1}{2}\int_{(-\infty,x]\times(-\infty,y]} e^{-v}1_{\mathbb{R}^2_+(x<y)}(u,v)\,d\mu_L^2(u,v) = \frac{1}{2}\left(1-e^{-y}-ye^{-y}\right)1_{\mathbb{R}^2_+(x>y)}(x,y)+\frac{1}{2}\left(1-e^{-x}-xe^{-y}\right)1_{\mathbb{R}^2_+(x\leq y)}(x,y).$$

Summarizing,

$$\int_{(-\infty,x]\times(-\infty,y]} \frac{\partial^2 F_Z}{\partial y\partial x}(u,v)\,d\mu_L^2(u,v) = \left(1-e^{-y}-\frac{1}{2}ye^{-x}-\frac{1}{2}ye^{-y}\right)1_{\mathbb{R}^2_+(x>y)}(x,y)+\left(1-e^{-x}-\frac{1}{2}xe^{-y}-\frac{1}{2}xe^{-x}\right)1_{\mathbb{R}^2_+}$$

As a consequence,

$$F(x,y)\neq\int_{(-\infty,x]\times(-\infty,y]} \frac{\partial^2 F_Z}{\partial y\partial x}(u,v)\,d\mu_L^2(u,v),$$

which implies that Z is not absolutely continuous.

4. Despite Z is not absolutely continuous, X and Y, which are exponential random variables with rate parameter $\lambda=1$, are absolutely continuous with the same density $f:\mathbb{R}\to\mathbb{R}$, given by

$$f(z) = e^{-z}1_{\mathbb{R}_+}(z),$$

for every $z\in\mathbb{R}$.

Problem 19 *Let Z_1, Z_2, Z_3 independent random variables on a probability space Ω such that such that $X_k \sim N(0,1)$, fo $k = 1, 2, 3$. Consider the real random variables*

$$X_1 \stackrel{def}{=} Z_1 + Z_2 + Z_3, \quad X_2 \stackrel{def}{=} Z_1 - Z_2 + Z_3, \quad X_3 \stackrel{def}{=} Z_1 - Z_3.$$

1. *What is the distribution of the vector $X \equiv (X_1, X_2, X_3)^\intercal$?*
2. *Can you compute the distribution function of X?*
3. *Among the pairs (X_1, X_2), (X_1, X_3), and (X_2, X_3) of entries of X what are made by independent random variables?*
4. *Compute the distributions of X_1, X_2, and X_3;*
5. *Think on a quick and smart way to compute $\mathbf{E}\left[X_1 X_2^2\right]$, $\mathbf{E}\left[X_1^2 X_2^2\right]$, $\mathbf{E}\left[X_2 X_3^2\right]$, $\mathbf{E}\left[X_2^2 X_3^2\right]$.*

Solution. .