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Dipartimento d'Ingegneria Civile e Ingegneria Informatica
LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica
Homework - 2019-10-31

Problem 1 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra and the Borel-Lebesgue measure* $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right)$. *Let* $f : \mathbb{R}^2 \to \mathbb{R}_+$ *given by*

$$f(x,y) \stackrel{def}{=} kxye^{-(x^2+y^2)} 1_{\mathbb{R}_+^2}(x,y), \quad \forall (x,y) \in \mathbb{R}^2$$

where $\mathbb{R}_+^2 \equiv \left\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. *Determine* $k \in \mathbb{R}$ *such that* $f : \mathbb{R}^2 \to \mathbb{R}_+$ *is a probability density and let* $Z \equiv (X,Y)$ *be the random vector of density* $f : \mathbb{R}^2 \to \mathbb{R}_+$.

1. *Determine the distribution function* $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *of the vector* Z *and check that*

$$\frac{\partial F^2}{\partial x \partial y}(x,y) = f(x,y), \quad \mu_L^2 - a.e. \text{ on } \mathbb{R}^2.$$

2. *Determine the marginal distribution function* $F_X : \mathbb{R} \to \mathbb{R}_+$ *and* $F_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z.

3. *Determine the densities* $f_X : \mathbb{R} \to \mathbb{R}_+$ *and* $f_Y : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X *and* Y *of* Z *and check that*

$$\frac{dF_X}{dx}(x) = f_X(x) \quad \text{and} \quad \frac{dF_Y}{dy}(y) = f_Y(y), \quad \mu_L - a.e. \text{ on } \mathbb{R}.$$

4. *Are* X *and* Y *independent random varables?*

5. *Compute* $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$ *and* $Cov(X,Y)$.

6. *Compute* $\mathbf{E}[(X,Y)]$ *and the covariance matrix of the vector* (X,Y).

Solution. . $\square$

Problem 2 *Determine the value of the parameter* k *such that the function* $f : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f(x_1,x_2,x_3) \stackrel{def}{=} \begin{cases} k\left(x_1 + x_2^2 + x_3^3\right) & \text{if } (x_1,x_2,x_3) \in [0,1] \times [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density. Hence, consider the random vector $X \equiv (X_1, X_2, X_3)^\intercal$ *with density* $f_X : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f_X(x_1,x_2,x_3) \stackrel{def}{=} f(x_1,x_2,x_3).$$

1. *Determine the distribution function* $F_X : \mathbb{R}^3 \to \mathbb{R}_+$ *and check that*

$$\frac{\partial F_X^2}{\partial x_1 \partial x_2 \partial x_3}(x_1,x_2,x_3) = f_X(x_1,x_2,x_3), \quad \mu_L^3\text{-}a.e. \text{ on } \mathbb{R}^3.$$

2. *Determine the marginal distribution function* $F_{X_1} : \mathbb{R} \to \mathbb{R}_+$, $F_{X_2} : \mathbb{R} \to \mathbb{R}_+$, *and* $F_{X_3} : \mathbb{R} \to \mathbb{R}_+$ *of the entries* X_1, X_2, *and* X_3 *of* X.

3. *Determine the marginal densities* $f_{X_1}: \mathbb{R} \to \mathbb{R}_+$, $f_{X_2}: \mathbb{R} \to \mathbb{R}_+$, *and* $f_{X_3}: \mathbb{R} \to \mathbb{R}_+$ *of the entries* X_1, X_2, *and* X_3 *of* X *and check that*

$$\frac{dF_{X_n}}{dx}(x) = f_{X_n}(x), \text{ for } n = 1, 2, 3, \quad \mu_L\text{-a.e. on } \mathbb{R}.$$

4. *Determine the joint distribution function* $F_{X_1,X_2}: \mathbb{R}^2 \to \mathbb{R}_+$, $F_{X_1,X_3}: \mathbb{R} \to \mathbb{R}_+$, *and* $F_{X_2,X_3}: \mathbb{R} \to \mathbb{R}_+$. *Is it useful to compute the joint distribution function* $F_{X_2,X_1}: \mathbb{R}^2 \to \mathbb{R}_+$, $F_{X_3,X_1}: \mathbb{R} \to \mathbb{R}_+$, *and* $F_{X_3,X_2}: \mathbb{R} \to \mathbb{R}_+$*?*

5. *Determine the joint densities* $f_{X_1,X_2}: \mathbb{R}^2 \to \mathbb{R}_+$, $f_{X_1,X_3}: \mathbb{R} \to \mathbb{R}_+$, *and* $f_{X_2,X_3}: \mathbb{R} \to \mathbb{R}_+$. *What is the relationship between the joint distribution function* $F_{X_m,X_n}: \mathbb{R}^2 \to \mathbb{R}_+$ *and the joint density* $f_{X_m,X_n}: \mathbb{R}^2 \to \mathbb{R}_+$ *for* $m, n = 1, 2, 3$, $m < n$.

6. *Determine the the expectation of* X.

7. *Determine the variance-covariance matrix of* X.

Solution. To determine the value of the parameter k such that the function $f : \mathbb{R}^3 \to \mathbb{R}$ is a probability density we have to solve the equation

$$\int_{\mathbb{R}^3} f(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) = 1.$$

We have

$$f(x_1, x_2, x_3) = k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1, x_2, x_3),$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^3} f(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) &= \int_{\mathbb{R}^3} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1, x_2, x_3)\, d\mu_L(x_1, x_2, x_3) \\
&= \int_{[0,1]\times[0,1]\times[0,1]} k\left(x_1 + x_2^2 + x_3^3\right) d\mu_L(x_1, x_2, x_3) \\
&= k \int_{[0,1]\times[0,1]\times[0,1]} \left(x_1 + x_2^2 + x_3^3\right) d\mu_L(x_1, x_2, x_3)
\end{aligned}$$

Now the real function $x_1 + x_2^2 + x_3^3$ is continuous on $[0,1]\times[0,1]\times[0,1]$. Therefore, the Lebesue integral can be computed as a Riemann integral. A as consequence, on account of the additive property of the Riemann integral and the separability of the integrand function on the pluri-interval domain, we can write

$$
\begin{aligned}
&\int_{[0,1]\times[0,1]\times[0,1]} \left(x_1 + x_2^2 + x_3^3\right) d\mu_L \left(x_1, x_2, x_3\right) \\
&= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} \left(x_1 + x_2^2 + x_3^3\right) dx_1 dx_2 dx_3 \\
&= \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_1 dx_1 dx_2 dx_3 \\
&+ \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_2^2 dx_1 dx_2 dx_3 \\
&+ \int_{x_1=0}^{1} \int_{x_2=0}^{1} \int_{x_3=0}^{1} x_3^3 dx_1 dx_2 dx_3 \\
&= \int_{x_1=0}^{1} x_1 dx_1 \int_{x_2=0}^{1} dx_2 \int_{x_3=0}^{1} dx_3 \\
&+ \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} x_2^2 dx_2 \int_{x_3=0}^{1} dx_3 \\
&+ \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} dx_2 \int_{x_3=0}^{1} x_3^3 dx_3 \\
&= \frac{1}{2} \left. x_1^2 \right|_{x_1=0}^{1} \cdot \left. x_2 \right|_{x_2=0}^{1} \cdot \left. x_3 \right|_{x_3=0}^{1} \\
&+ \left. x_1 \right|_{x_1=0}^{1} \cdot \frac{1}{3} \left. x_2^3 \right|_{x_2=0}^{1} \cdot \left. x_3 \right|_{x_3=0}^{1} \\
&+ \left. x_1 \right|_{x_1=0}^{1} \cdot \left. x_2 \right|_{x_2=0}^{1} \frac{1}{4} \cdot \left. x_3^4 \right|_{x_3=0}^{1} \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&= \frac{13}{12}
\end{aligned}
$$

It follows

$$
k = \frac{12}{13}.
$$

With similar computation, we have

$$
\begin{aligned}
\mathbf{P}\left(X_2 \leq 1/2, X_3 > 1/2\right) &= \int_{x_1=0}^{1} \int_{x_2=0}^{1/2} \int_{x_3=1/2}^{1} \frac{12}{13} \left(x_1 + x_2^2 + x_3^3\right) dx_1 dx_2 dx_3 \\
&= \frac{12}{13} \left(\frac{1}{2} \left. x_1^2 \right|_{x_1=0}^{1} \cdot \left. x_2 \right|_{x_2=0}^{1/2} \cdot \left. x_3 \right|_{x_3=1/2}^{1} \right. \\
&+ \left. x_1 \right|_{x_1=0}^{1} \cdot \frac{1}{3} \left. x_2^3 \right|_{x_2=0}^{1/2} \cdot \left. x_3 \right|_{x_3=1/2}^{1} \\
&+ \left. \left. x_1 \right|_{x_1=0}^{1} \cdot \left. x_2 \right|_{x_2=0}^{1/2} \cdot \frac{1}{4} \left. x_3^4 \right|_{x_3=1/2}^{1} \right) \\
&= \frac{12}{13} \left(\frac{1}{8} + \frac{1}{48} + \frac{15}{128} \right) \\
&= \frac{101}{416}.
\end{aligned}
$$

The marginal density of the random vector $(X_1, X_2)^\intercal$ is given by

$$\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \int_{\mathbb{R}} f(x_1,x_2,x_3)\, d\mu_L(x_3) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]\times[0,1]\times[0,1]}(x_1,x_2,x_3)\, d\mu_L(x_3) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, 1_{[0,1]}(x_3)\, d\mu_L(x_3) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + x_3^3\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_3) \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \int_{x_3=0}^{1} \left(x_1 + x_2^2 + x_3^3\right) dx_3 \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \left(\int_{x_3=0}^{1} x_1 d\mu_L(x_3) + \int_{x_3=0}^{1} x_2^2 d\mu_L(x_3) + \int_{x_3=0}^{1} x_3^3 d\mu_L(x_3)\right) \\
&= k 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \left(x_1 \cdot x_3\big|_{x_3=0}^{1} + x_2^2 \cdot x_3\big|_{x_3=0}^{1} + \frac{1}{4}\, x_3^4\big|_{x_3=0}^{1}\right) \\
&= k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2) \\
&= k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2).
\end{aligned}$$

We have

$$\mathbf{E}\left[(X_1, X_2)^\intercal\right] = \left(\mathbf{E}[X_1], \mathbf{E}[X_2]\right)^\intercal,$$

where

$$\mathbf{E}[X_k] = \int_{\mathbb{R}} x_k f_{X_k}(x_k)\, d\mu_L(x_k), \quad k = 1, 2,$$

and $f_{X_k}(x_k)$ is the marginal density of the random variable X_k, for $k = 1, 2$. Now,

$$\begin{aligned}
f_{X_1}(x_1) &= \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_2) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)\, d\mu_L(x_2) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_2) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, d\mu_L(x_2) \\
&= k 1_{[0,1]}(x_1) \int_{x_2=0}^{1} \left(x_1 + x_2^2 + \frac{1}{4}\right) dx_2 \\
&= k 1_{[0,1]}(x_1) \left(x_1 \cdot x_2\big|_{x_2=0}^{1} + \frac{1}{3} \cdot x_2^3\big|_{x_2=0}^{1} + \frac{1}{4} \cdot x_2\big|_{x_2=0}^{1}\right) \\
&= k 1_{[0,1]}(x_1) \left(x_1 + \frac{1}{3} + \frac{1}{4}\right) \\
&= k\left(x_1 + \frac{7}{12}\right) 1_{[0,1]}(x_1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
f_{X_2}(x_2) &= \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, d\mu_L(x_1) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)\, d\mu_L(x_1) \\
&= \int_{\mathbb{R}} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x1) \\
&= \int_{[0,1]} k\left(x_1 + x_2^2 + \frac{1}{4}\right) 1_{[0,1]}(x_2)\, d\mu_L(x_1) \\
&= k1_{[0,1]}(x_2) \int_{x_1=0}^{1} \left(x_1 + x_2^2 + \frac{1}{4}\right) dx_1 \\
&= k1_{[0,1]}(x_2) \left(\frac{1}{3}\cdot x_1^2\Big|_{x_1=0}^{1} + x_2^2\cdot x_1\Big|_{x_1=0}^{1} + \frac{1}{4}\cdot x_1\Big|_{x_1=0}^{1}\right) \\
&= k1_{[0,1]}(x_2) \left(\frac{1}{3} + x_2^2 + \frac{1}{4}\right) \\
&= k\left(x_2^2 + \frac{7}{12}\right) 1_{[0,1]}(x_2).
\end{aligned}$$

It follows

$$\begin{aligned}
\mathbf{E}[X_1] &= \int_{\mathbb{R}} k\left(x_1 + \frac{7}{12}\right) 1_{[0,1]}(x_1) = k\int_{x_1=0}^{1} \left(x_1 + \frac{7}{12}\right) dx_1 \\
&= k\left(\frac{1}{2}\cdot x_1^2\Big|_{x_1=0}^{1} + \frac{7}{12}\cdot x_1\Big|_{x_1=0}^{1}\right) = \frac{13}{12}k
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}[X_2] &= \int_{\mathbb{R}} k\left(x_2^2 + \frac{7}{12}\right) 1_{[0,1]}(x_2) = k\int_{x_2=0}^{1} \left(x_2^2 + \frac{7}{12}\right) dx_2 \\
&= k\left(\frac{1}{3}\cdot x_1^3\Big|_{x_2=0}^{1} + \frac{7}{12}\cdot x_2\Big|_{x_2=0}^{1}\right) = \frac{11}{12}k.
\end{aligned}$$

The conditional density $f_{X_1,X_2|X_3=1/2}(x_1,x_2)$ is simply given by

$$f_{X_1,X_2|X_3=1/2}(x_1,x_2) = \frac{f_{X_1,X_2,X_3}(x_1,x_2,1/2)}{\int_{\mathbb{R}^2} f_{X_1,X_2,X_3}(x_1,x_2,1/2)\, d\mu_L(x_1,x_2)} = \frac{f_{X_1,X_2,X_3}(x_1,x_2,1/2)}{f_{X_3}(1/2)},$$

for every $(x_1,x_2)\in\mathbb{R}^2$. Now, since

$$f_{X_1,X_2,X_3}(x_1,x_2,1/2) = k\left(x_1 + x_2^2 + \frac{1}{8}\right) 1_{[0,1]\times[0,1]}(x_1,x_2)$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^2} f_{X_1,X_2,X_3}\left(x_1,x_2,1/2\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{\mathbb{R}^2} k\left(x_1+x_2^2+\frac{1}{8}\right) 1_{[0,1]\times[0,1]}\left(x_1,x_2\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{[0,1]\times[0,1]} k\left(x_1+x_2^2+\frac{1}{8}\right) d\mu_L\left(x_1,x_2\right) \\
&= \int_{x_1=0}^{1}\int_{x_2=0}^{1} k\left(x_1+x_2^2+\frac{1}{8}\right) dx_1 dx_2 \\
&= k\left(\int_{x_1=0}^{1}\int_{x_2=0}^{1} x_1 dx_1 dx_2 + \int_{x_1=0}^{1}\int_{x_2=0}^{1} x_2^2 dx_1 dx_2 + \int_{x_1=0}^{1}\int_{x_2=0}^{1} \frac{1}{8} dx_1 dx_2\right) \\
&= k\left(\int_{x_1=0}^{1} x_1 dx_1 \int_{x_2=0}^{1} dx_2 + \int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} x_2^2 dx_2 + \frac{1}{8}\int_{x_1=0}^{1} dx_1 \int_{x_2=0}^{1} dx_2\right) \\
&= k\left(\frac{1}{2}\cdot x_1^2\Big|_{x_1=0}^{1}\cdot x_2\Big|_{x_2=0}^{1} + x_1\Big|_{x_1=0}^{1}\cdot\frac{1}{3}\cdot x_2^3\Big|_{x_2=0}^{1} + \frac{1}{8}\cdot x_1\Big|_{x_1=0}^{1} x_2\Big|_{x_2=0}^{1}\right) \\
&= k\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{8}\right) \\
&= \frac{23}{24}k,
\end{aligned}
$$

we obtain

$$
f_{X_1,X_2|X_3=1/2}\left(x_1,x_2\right) = \frac{24}{23}\left(x_1+x_2^2+\frac{1}{8}\right) 1_{[0,1]\times[0,1]}\left(x_1,x_2\right).
$$

Problem 3 *Let $F:\mathbb{R}^2\to\mathbb{R}_+$ given by*

$$
F\left(x_1,x_2\right) \overset{def}{=} \left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right) 1_{\mathbb{R}_+}\left(x_1\right) 1_{\mathbb{R}_+}\left(x_2\right), \quad \forall\left(x_1,x_2\right)\in\mathbb{R}^2.
$$

Show that $F:\mathbb{R}^2\to\mathbb{R}_+$ is the distribution function of a real random vector (X_1,X_2) and compute the marginal distribution functions of the entries X_1 and X_2 of (X_1,X_2).

Exercise 4

1. *May we say that $F:\mathbb{R}^2\to\mathbb{R}_+$ is absolutely continuous?*
2. *May we say that the entries X_1 and X_2 of the random vector (X_1,X_2) are independent random variables?*
3. *May we say that the entries X_1 and X_2 of the random vector (X_1,X_2) are absolutely continuous random variables?*
4. *Consider the real random variable $Z=\max\{X_1,X_2\}$. Determine the distribution function $F_Z:\mathbb{R}^2\to\mathbb{R}_+$ of Z is $F_Z:\mathbb{R}^2\to\mathbb{R}_+$ absolutely continuous?*

Solution. We have

$$
\begin{aligned}
\left(1-e^{-x_1}-e^{-x_2}+e^{-(x_1+x_2)}\right) 1_{\mathbb{R}_+}\left(x_1\right) 1_{\mathbb{R}_+}\left(x_2\right) &= (\left(1-e^{-x_1}\right)-\left(1-e^{-x_1}\right)e^{-x_2}) 1_{\mathbb{R}_+}\left(x_1\right) 1_{\mathbb{R}_+}\left(x_2\right) \\
&= \left(1-e^{-x_1}\right) 1_{\mathbb{R}_+}\left(x_1\right) \left(1-e^{-x_2}\right) 1_{\mathbb{R}_+}\left(x_2\right),
\end{aligned}
$$

for every $(x_1,x_2)\in\mathbb{R}^2$. Therefore,

$$
\lim_{x_1\to-\infty}\lim_{x_2\to-\infty} F\left(x_1,x_2\right) = 0 \quad \text{and} \quad \lim_{x_1\to+\infty}\lim_{x_2\to+\infty} F\left(x_1,x_2\right) = 1.
$$

In addition, $F(x_1,x_2)$ is non decreasing and right-hand continuous in each variable. These properties imply that $F:\mathbb{R}^2\rightarrow\mathbb{R}_+$ the distribution function of a real random vector (X_1,X_2). The marginal distributions $F_{X_1}:\mathbb{R}\rightarrow\mathbb{R}_+$ and $F_{X_2}:\mathbb{R}\rightarrow\mathbb{R}_+$ are given by

$$F_{X_1}(x_1)=\lim_{x_2\rightarrow-\infty}F(x_1,x_2),\quad\forall x_1\in\mathbb{R}\quad\text{and}\quad F_{X_2}(x_2)=\lim_{x_1\rightarrow+\infty}F(x_1,x_2),\quad\forall x_2\in\mathbb{R},$$

respectively. Hence,

$$F_{X_1}(x_1)=\lim_{x_2\rightarrow-\infty}\left(1-e^{-x_1}\right)1_{\mathbb{R}_+}(x_1)\left(1-e^{-x_2}\right)1_{\mathbb{R}_+}(x_2)=\left(1-e^{-x_1}\right)1_{\mathbb{R}_+}(x_1)$$

and

$$F_{X_2}(x_2)=\lim_{x_1\rightarrow-\infty}\left(1-e^{-x_1}\right)1_{\mathbb{R}_+}(x_1)\left(1-e^{-x_2}\right)1_{\mathbb{R}_+}(x_2)=\left(1-e^{-x_2}\right)1_{\mathbb{R}_+}(x_2).$$

Note that both $F_{X_1}:\mathbb{R}\rightarrow\mathbb{R}_+$ and $F_{X_2}:\mathbb{R}\rightarrow\mathbb{R}_+$ are absolutely continuous functions with density $f_{X_1}:\mathbb{R}\rightarrow\mathbb{R}_+$ and $f_{X_2}:\mathbb{R}\rightarrow\mathbb{R}_+$ given by

$$f_{X_1}(x_1)=e^{-x_1}1_{\mathbb{R}_+}(x_1)\quad\text{and}\quad f_{X_2}(x_2)=e^{-x_2}1_{\mathbb{R}_+}(x_2),$$

respectively. This proves that the entries X_1 and X_2 of the random vector (X_1,X_2) are absolutely continuous random variables. As a consequence $F:\mathbb{R}^2\rightarrow\mathbb{R}_+$ is itself absolutely continuous. In fact

$$\begin{aligned}F(x_1,x_2)&=\int_{(-\infty,x_1]\times(-\infty,x_2]}f_{X_1}(u_1)f_{X_2}(u_2)\ d\mu_L(u_1,u_2)\\&=\int_{(-\infty,x_1]}f_{X_1}(u_1)\ d\mu_L(u_1)\int_{(-\infty,x_2]}f_{X_2}(u_2)\ d\mu_L(u_2)\\&=\int_{(-\infty,x_1]}e^{-u_1}1_{\mathbb{R}_+}(u_1)\ d\mu_L(u_1)\int_{(-\infty,x_2]}e^{-u_2}1_{\mathbb{R}_+}(u_2)\ d\mu_L(u_2)\\&=\int_{(-\infty,x_1]\cap\mathbb{R}_+}e^{-u_1}\ d\mu_L(u_1)\int_{(-\infty,x_2]\cap\mathbb{R}_+}e^{-u_2}\ d\mu_L(u_2).\end{aligned}$$

Hence,

$$F(x_1,x_2)=\begin{cases}\int_{[0,x_1]}e^{-u_1}\ d\mu_L(u_1)\int_{[0,x_2]}e^{-u_2}\ d\mu_L(u_2)=\int_1^{x_1}e^{-u_1}\ du_1\int_1^{x_1}e^{-x_2}\ du_2 & \text{if } x_1,x_2>0\\ 0 & \text{otherwise}\end{cases}.$$

It follows

$$F(x_1,x_2)=(1-e^{x_1})(1-e^{x_2})1_{\mathbb{R}_{++}}(x_1)1_{\mathbb{R}_{++}}(x_2)=(1-e^{x_1})(1-e^{x_2})1_{\mathbb{R}_+}(x_1)1_{\mathbb{R}_+}(x_2)$$

almost everywhere in $\mathbb{R}^2$. Moreover, we have

$$F(x_1,x_2)=\int_{(-\infty,x_1]}f_{X_1}(u_1)\ d\mu_L(u_1)\int_{(-\infty,x_2]}f_{X_2}(u_2)\ d\mu_L(u_2)=F_{X_1}(x_1)F_{X_2}(x_2),$$

almost everywhere in $\mathbb{R}^2$. This proves that the entries X_1 and X_2 of the random vector (X_1,X_2) are independent random variables.

In the end, to determine the distribution function $F_Z:\mathbb{R}^2\rightarrow\mathbb{R}_+$ of Z, we consider the event $\{Z\leq z\}$. We have

$$\{Z\leq z\}=\{\max\{X_1,X_2\}\leq z\}=\{X_1\leq z,X_2\leq z\}.$$

Hence, on accont of the independence of X_1 and X_2, we can write

$$\begin{aligned}F_Z(z)&=\mathbf{P}(Z\leq z)=\mathbf{P}(X_1\leq z,X_2\leq z)=\mathbf{P}(X_1\leq z)\mathbf{P}(X_2\leq z)\\&=F_{X_1}(z)F_{X_2}(z)=\left(1-e^{-z}\right)1_{\mathbb{R}_+}(z)\left(1-e^{-z}\right)1_{\mathbb{R}_+}(z)\\&=\left(1-e^{-z}\right)^2 1_{\mathbb{R}_+}(z)=\left(1-2e^{-z}+e^{-2z}\right)1_{\mathbb{R}_+}(z).\end{aligned}$$

Now, since

$$\int_0^z e^{-v}\ dv = 1 - e^{-z} \quad \text{and} \quad \int_0^z e^{-2v}\ dv = \frac{1}{2}\left(1 - e^{-2z}\right)$$

for every $z > 0$, we have

$$2\int_0^z \left(e^{-v} - e^{-2v}\right)\ dv = 2\left(1 - e^{-z}\right) - 2\frac{1}{2}\left(1 - e^{-2z}\right) = 1 - 2e^{-z} + e^{-2z},$$

for every $z > 0$. It clearly follows

$$\begin{aligned} F_Z(z) &= \int_{(-\infty,z]} 2\left(e^{-v} - e^{-2v}\right) 1_{\mathbb{R}_+}(v)\ d\mu_L(v) \\ &= \int_{(-\infty,z]\cap\mathbb{R}_+} 2\left(e^{-v} - e^{-2v}\right)\ d\mu_L(v) \\ &= \begin{cases} \int_{[0,z]} 2\left(e^{-v} - e^{-2v}\right)\ d\mu_L(v) = \int_0^z 2\left(e^{-v} - e^{-2v}\right)\ dv & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}, \end{aligned}$$

that is

$$F_Z(z) = \left(1 - 2e^{-z} + e^{-2z}\right) 1_{\mathbb{R}_+}(z)$$

for every $z \in \mathbb{R}$. This proves that $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ is absolutely continuous. □