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LM in Ingegneria dell'Informazione e dell'Automazione
Complementi di Probabilità e Statistica
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Problem 1 *Let Ω be the sample space of a random phenomenon and let $\mathcal{E}_1$, $\mathcal{E}_2$ algebras [resp. σ-algebras] of events of Ω. May we say that the family $\mathcal{E}_1 \cup \mathcal{E}_2$ of events of Ω given by*

$$\mathcal{E}_1 \cup \mathcal{E}_2 \stackrel{def}{=} \{E \in \mathcal{P}(\Omega) : E \in \mathcal{E}_1 \text{ or } E \in \mathcal{E}_2\}$$

is an algebra [resp. σ-algebras]?

Solution. Clearly, since $\mathcal{E}_1$, $\mathcal{E}_2$ algebras [resp. σ-algebras] of events of Ω, the family $\mathcal{E}_1 \cup \mathcal{E}_2$ is not empty. Now, assume that an event E is in $\mathcal{E}_1 \cup \mathcal{E}_2$, then E is in $\mathcal{E}_1$ or E is in $\mathcal{E}_2$. As a consequence, E^c is in $\mathcal{E}_1$ or E^c is in $\mathcal{E}_2$. Hence, $E^c \in \mathcal{E}_1 \cup \mathcal{E}_2$. However, assuming that E and F are in $\mathcal{E}_1 \cup \mathcal{E}_2$, unless they are both in $\mathcal{E}_1$ or $\mathcal{E}_2$, there is no reason why $E \cup F$ should be in $\mathcal{E}_1 \cup \mathcal{E}_2$. This is confirmed by the following example: with reference to the die sample space $\Omega \equiv \{\omega_1, \ldots, \omega_6\}$ choose

$$\mathcal{E}_1 \equiv \{\varnothing, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\}, \Omega\} \quad \text{and} \quad \mathcal{E}_2 \equiv \{\varnothing, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \Omega\}.$$

$\mathcal{E}_1$ and $\mathcal{E}_2$ are algebras of events of Ω, but

$$\mathcal{E}_1 \cup \mathcal{E}_2 = \{\varnothing, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\} \{\omega_4, \omega_5, \omega_6\}, \Omega\}$$

is not. Note that $\mathcal{E}_1 \cup \mathcal{E}_2$ is closed with respect to the complement operator, but not with respect to the union.

Problem 2 *Let Ω be the infinite sample space of a random phenomenon. The family*

$$\mathcal{E}_{count} \equiv \{E \in \mathcal{P}(\Omega) : |E| \leq \aleph_0\}$$

of all countable events of Ω is a σ-algebra of events of Ω if and only if Ω itself is countable. In this case, we have $\mathcal{E}_{count} = \mathcal{P}(\Omega)$. On the other hand, the family $\mathcal{E}_{count\text{-}cocount}$ of all events of Ω that are countable or have countable complement, in symbols

$$\mathcal{E}_{count\text{-}cocount} \equiv \{E \in \mathcal{P}(\Omega) : |E| \leq \aleph_0 \ \vee \ |E^c| \leq \aleph_0\},$$

is a σ-algebra of events of Ω.

Solution. Given any $\omega \in \Omega$ we have

$$|\{\omega\}| = 1 \leq \aleph_0.$$

Hence, $\{\omega\} \in \mathcal{E}_{\text{count}}$. Assume that $\mathcal{E}_{\text{count}}$ is a σ-algebra, then also $\{\omega\}^c \equiv \Omega - \{\omega\}$ is in $\mathcal{E}_{\text{count}}$. By definition, it follows

$$|\Omega - \{\omega\}| \leq \aleph_0,$$

which clearly implies

$$|\Omega| \leq \aleph_0.$$

Conversely, if Ω is countable, then every $E \in \mathcal{P}(\Omega)$ is countable. This implies

$$\mathcal{P}(\Omega) \subseteq \mathcal{E}_{\text{count}},$$

that is

$$\mathcal{E}_{\text{count}} = \mathcal{P}(\Omega).$$

As a trivial consequence, $\mathcal{E}_{\text{count}}$ is σ-algebra of events of Ω. As a consequence of the above argument, if Ω is not countable, that is

$$|\Omega| > \aleph_0$$

or, according the continuum hypothesis,

$$|\Omega| \geq \aleph_1,$$

the family $\mathcal{E}_{\text{count}}$ cannot be a σ-algebra. On the other hand, the family $\mathcal{E}_{\text{count-cocount}}$ is. In fact, clearly $\mathcal{E}_{\text{count-cocount}} \neq \varnothing$. Furthermore, if $E \in \mathcal{E}_{\text{count-cocount}}$, according to the definition, we have two cases:

$$|E| \leq \aleph_0 \quad \text{or} \quad |E^c| \leq \aleph_0.$$

In the first case,

$$|(E^c)^c| = |E| \leq \aleph_0.$$

This implies that $E^c \in \mathcal{E}_{\text{count-cocount}}$. In the second case, we have $E^c \in \mathcal{E}_{\text{count-cocount}}$ straightforwardly. Hence, in either cases $E^c \in \mathcal{E}_{\text{count-cocount}}$. In the end, consider a sequence $(E_n)_{n\geq 1}$ of elements in $\mathcal{E}_{\text{count-cocount}}$. If $|E_n| \leq \aleph_0$ for every $n \in \mathbb{N}$, then

$$\left| \bigcup_{n\geq 1} E_n \right| \leq \aleph_0,$$

which implies $\bigcup_{n\geq 1} E_n \in \mathcal{E}_{\text{count-cocount}}$. Otherwise, there exists at least $n_0 \in \mathbb{N}$ such that $|E_{n_0}| > \aleph_0$. However, in this case, since $E_{n_0} \in \mathcal{E}_{\text{count-cocount}}$, we necessarily have

$$\left|E_{n_0}^c\right| \leq \aleph_0.$$

This implies

$$\left| \left(\bigcup_{n\geq 1} E_n \right)^c \right| = \left| \bigcap_{n\geq 1} E_n^c \right| \leq \left|E_{n_0}^c\right| \leq \aleph_0.$$

Thus, it still follows that $\bigcup_{n\geq 1} E_n \in \mathcal{E}_{\text{count-cocount}}$.

Problem 3 *Let Ω be the sample space of a random phenomenon or experiment, let $\mathcal{E}$ be an algebra of events of Ω and let $\mathbf{P} : \mathcal{E} \to \mathbb{R}_+$ be an additive probability on Ω. Prove that we have*

1. $\mathbf{P}(\varnothing) = 0$;
2. $\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$ *for any* $E \in \mathcal{E}$;
3. $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E \cap F)$ *for all* $E, F \in \mathcal{E}$, *in particular* $\mathbf{P}(F - E) = \mathbf{P}(F) - \mathbf{P}(E)$ *when* $E \subseteq F$;
4. $\mathbf{P}(E) \leq \mathbf{P}(F)$ *for all* $E, F \in \mathcal{E}$ *such that* $E \subseteq F$;
5. $\mathbf{P}(E) \leq 1$ *for any* $E \in \mathcal{E}$;
6. $\mathbf{P}\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mathbf{P}(E_k)$ *for any finite sequence* $(E_k)_{k=1}^n$ *in* $\mathcal{E}$ *of pairwise exclusive events;*

7. $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$ *for all* $E, F \in \mathcal{E}$, *in particular* $\mathbf{P}(E \cup F) \leq \mathbf{P}(E) + \mathbf{P}(F)$;
8. $\mathbf{P}\left(\bigcup_{k=1}^{n} E_k\right) \leq \sum_{k=1}^{n} \mathbf{P}(E_k)$ *for any finite sequence* $(E_k)_{k=1}^{n}$ *in* $\mathcal{E}$.

Solution. See Notes on Probability and Statistics, Proposition 146.

Problem 4 *Let* Ω *be the sample space of a random phenomenon or experiment, let* $\mathcal{E}$ *be a* σ*-algebra of events of* Ω *and let* $\mathbf{P} : \mathcal{E} \to \mathbb{R}_+$ *be a countably additive probability on* Ω. *Prove that we have*

1. $\mathbf{P}\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mathbf{P}(E_k)$ *for any finite sequence* $(E_k)_{k=1}^{n}$ *of pairwise incompatible events in* $\mathcal{E}$;
2. $\mathbf{P}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mathbf{P}(E_n)$ *for any sequence* $(E_n)_{n \geq 1}$ *in* $\mathcal{E}$.

Solution.

1. Let $(E_k)_{k=1}^{n}$ be a finite sequence of pairwise incompatible events in $\mathcal{E}$. Then the sequence $(F_k)_{k \geq 1}$ given by
$$F_k \overset{\text{def}}{=} \begin{cases} E_k & \forall k = 1, \dots, n \\ \varnothing & \forall k \geq n+1 \end{cases},$$
is a denumerable sequence of pairwise incompatible events in $\mathcal{E}$ such that
$$\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} F_k$$
and
$$\mathbf{P}(F_k) = \begin{cases} \mathbf{P}(E_k) & \forall k = 1, \dots, n \\ 0 & \forall k \geq n+1 \end{cases}.$$
As a consequence,
$$\mathbf{P}\left(\bigcup_{k=1}^{n} E_k\right) = \mathbf{P}\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mathbf{P}(F_k) = \sum_{k=1}^{n} \mathbf{P}(F_k) = \sum_{k=1}^{n} \mathbf{P}(E_k),$$
This proves that is an additive probability

2. Let $(E_n)_{n \geq 1}$ be any sequence of events in $\mathcal{E}$. Then the sequence $(F_n)_{n \geq 1}$ given by
$$F_n \overset{\text{def}}{=} \begin{cases} E_1 & \text{if } n = 1 \\ E_n - \bigcup\limits_{k=1}^{n-1} E_k & \text{if } n > 1 \end{cases},$$
is a sequence of pairwise incompatible events in $\mathcal{E}$ such that
$$\bigcup_{n \geq 1} E_n = \bigcup_{n \geq 1} F_n.$$
In fact, clearly $F_n \subseteq E_n$, for every $n \geq 1$. Hence,
$$\bigcup_{n \geq 1} F_n \subseteq \bigcup_{n \geq 1} E_n.$$

Conversely, if $x \in \bigcup_{n\geq 1} E_n$, then the set $N_x \equiv \{n \in \mathbb{N} : x \in E_n\} \neq \varnothing$. Write $\hat{n}_x \equiv \min(N_x)$. We have $x \in E_{\hat{n}_x}$. In case $\hat{n}_x = 1$, by definition we have $x \in F_1$. In case $\hat{n}_x > 1$, we have $x \notin E_k$ for every $k = 1, \ldots, \hat{n}_x - 1$. Hence, $x \notin \bigcup\limits_{k=1}^{\hat{n}_x - 1} E_k$ and, again by definition, $x \in F_{\hat{n}_x}$. Therefore, in any case, we obtain $x \in F_{\hat{n}_x}$. This implies that $x \in \bigcup_{n\geq 1} F_n$.

Now, given $n_1, n_2 \in \mathbb{N}$ such that $n_1 \neq n_2$, we have

$$F_{n_1} \cap F_{n_2} = \varnothing.$$

In fact, assuming for instance $n_1 < n_2$, we have

$$F_{n_1} \subseteq E_{n_1} \quad \text{and} \quad F_{n_2} \cap E_{n_1} = \left(E_{n_2} - \bigcup_{k=1}^{n_2 - 1} E_k \right) \cap E_{n_1} = \varnothing.$$

This implies that the events of the sequence $(F_n)_{n\geq 1}$ are pairwise incompatible. As a consequence of the above arguments, it follows

$$\mathbf{P}\left(\bigcup_{n\geq 1} E_n \right) = \mathbf{P}\left(\bigcup_{n\geq 1} F_n \right) = \sum_{n\geq 1} \mathbf{P}(F_n) \leq \sum_{n\geq 1} \mathbf{P}(E_n),$$

which is the desired result.

Problem 5 *Let Ω be the sample space of a random phenomenon or experiment, let $\mathcal{E}$ be a σ-algebra of events of Ω and let $\mathbf{P} : \mathcal{E} \to \mathbb{R}_+$ be a countably additive probability on Ω. A sequence $(E_n)_{n\geq 1}$ of events in $\mathcal{E}$ is said to be increasing [resp. decreasing] if*

$$E_n \subseteq E_{n+1} \quad \textit{[resp. } E_n \supseteq E_{n+1}\textit{]}, \quad \forall n \in \mathbb{N}.$$

Let $(E_n)_{n\geq 1}$ be an increasing sequence of events belonging to $\mathcal{E}$. Prove that

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mathbf{P}(E_n).$$

Use this result to show that for a decreasing sequence $(E_n)_{n\geq 1}$ of events belonging to $\mathcal{E}$ we have

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mathbf{P}(E_n).$$

Solution. .

Problem 6 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ a probability space and let $E, F \in \mathcal{E}$ such that*

$$\mathbf{P}(E) + \mathbf{P}(F) \geq 1. \tag{1}$$

Prove that

$$\mathbf{P}(E) + \mathbf{P}(F) - 1 \leq \mathbf{P}(E \cap F) \leq \min\{\mathbf{P}(E), \mathbf{P}(F)\} \tag{2}$$

Determine a similar lower and upper bound for $\mathbf{P}(E \cap F)$ under the assumption

$$\mathbf{P}(E) + \mathbf{P}(F) < 1. \tag{3}$$

Solution. .

Problem 7 *Five Italian players are playing poker. The deck of poker cards contains* 36 *cards of the usual ranks* ($6, 7, 8, 9, 10, J, Q, K, A$) *and of the usual suites (hearts* $\heartsuit$*, clubs* $\spadesuit$*, diamonds* $\diamondsuit$*, flowers* $\clubsuit$*).*

1. *How many hands are possible by a random deal?*
2. *How many hands give a straight flush by a random deal?*
3. *How many hands give a four of a kind by a random deal?*
4. *How many hands give a flush by a random deal?*
5. *How many hands give a full house by a random deal?*
6. *How many hands give a straight by a random deal?*
7. *How many hands give a three of a kind by a random deal?*
8. *How many hands give two pair by a random deal?*
9. *How many hands give one pair by a random deal?*
10. *How many hands give no pair by a random deal?*
11. *How many hands fail to give any of the above combinations by a random deal?*
12. *What about if the players are Americans? In this case the deck of poker card contains* 56 *cards of the usual ranks* ($1, \ldots, 10, J, Q, K, A$) *and of the usual suites.*

Solution.

1. Since a poker hand is indifferent to the order in which is arranged by the deal, the number of all possible hands is just the number of all possible subsets of 5 elements that can be selected from a set of 36 elements. Hence,
$$\binom{36}{5}$$
is the number of all possible hands.

2. According to the (Italian) poker rules, there are 6 possibilities for choosing the rank of the first card of a straight. The ranks of the other cards are then consequently determined. That is the ony possible straights in a deck of 36 cards are
$$A, 6, 7, 8, 9; \quad 6, 7, 8, 9, 10; \quad ; \ldots ; \quad 10, J, Q, K, A.$$
We have a straight flush when the cards of the straight have all the same suits. We can choose the suit for the straight flush in 4 differeent ways. Hence, we have
$$4 \cdot 6$$
possible hands giving a straight flush by a random deal.

3. There are 9 possibility for choosing the rank of card for the four of a kind, once the card has been chosen there is no room for the choice of the suites. Then, there are 8 possibilities for choosing the rank of fifth card and for each rank there are $\binom{4}{1} = 4$ possibilities for choosing the suits. As a consequence, we have
$$9 \cdot 8 \cdot 4$$
possible hands giving a four of a kind by a random deal.

4. To be continued.

Problem 8 *An urn contains n distinguishable balls of which r are red, with $1 \le r < n$, and $n - r$ are white. The urn is shaken and the balls are drawn from the urn one after the other without replacement. How many of the possible drawn sequences show the first red ball at the kth draw?*

Ans.
$$\binom{n-r}{k-1}(k-1)!r\,(n-k)!$$

Solution. .

Problem 9 *An urn contains n balls of which r are red, with $1 \le r < n$, and $n - r$ are white. The urn is shaken and the balls are drawn one after the other without replacement. Suppose that both the red balls and the white ones are undistinguishable among them. How many of the possible drawn sequences show the first red ball at the kth draw?*

Ans.
$$\binom{n-k}{r-1}$$

Solution. .

Problem 10 *An urn contains n distinguishable balls of which r are red, with $1 \le r < n$, and $n - r$ are white. The urn is shaken and k balls are drawn without replacement. If $k \le r$, how many of the possible unordered samples contains $s \le k$ red balls and $k - s$ white ones*

Ans.
$$C_{r,s} \cdot C_{n-r,k-s}.$$

Solution. .

Problem 11 *The key that opens a room is in a box containing $n \ge 1$ keys. How many ordered samples of keys containing the right key at the kth place $(1 \le k \le n)$ we can draw from?*

Ans.
$$(n-1)!$$

Solution. .

Problem 12 *An urn contains N different balls numbered from 1 to N. A sample of $n \leq N$ balls is drawn from the urn without replacement. Show that the probability to obtain a given combination of $k \leq n$ elements is*

$$p_{N,n,k} = \frac{(n-k+1) \cdot \ldots \cdot (n-1)\, n}{(N-k+1) \cdot \ldots \cdot (N-1)\, N}.$$

Note that when $n = k$ we obtain

$$p_{N,n,n} = \frac{n!}{N\,(N-1) \cdot \ldots \cdot (N-n+1)}.$$

Solution. .

Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac statistics

Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac Statistics consider the number of ways in which m objects can be placed into n cells, according to whether the objects are distinguishable from each other and the cells can contain more than one or only one object.

Problem 13 (Maxwell-Boltzman Statistics) *Suppose we have m distinguishable balls and n boxes. How many ways are there to distribute the balls in the boxes if each box can contain more than one ball? How many ways are there to distribute the balls in the boxes if a specific box has to contain $\ell \leq m$ balls?*

Solution. .

Problem 14 (Bose-Einstein Statistics) *Suppose we have m indistinguishable balls and n boxes. How many ways are there in which the balls can be distributed in the boxes if every box can contain more than one ball?*

Solution. .

Problem 15 (Fermi-Dirac Statistics) *Suppose we have m indistinguishable balls and n boxes. How many ways are there in which the balls can be distributed in the boxes if every box cannot contain more than one ball?*

Solution. .