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Complementi di Probabilità e Statistica - Advanced Statistics
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Problems on Random Vectors with Some Solutions 2021-11-23

Problem 1 *Let (X_1, X_2) a real random vector with a joint density $f_{X_1,X_2} : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f_{X_1,X_2}(x_1, x_2) \overset{def}{=} 1_{[0,1]\times[0,1]}(x_1, x_2), \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Consider the real random variables $Y \equiv \min(X_1, X_2)$ and $Z \equiv \max(X_1, X_2)$. Determine:

1. *the distribution functions of Y and Z;*
2. *the joint distribution function of Y and Z;*
3. *the marginal distributions functions of Y and Z;*
4. *the expectations of Y and Z.*

Solution.

1. We clearly have
$$1_{[0,1]\times[0,1]}(x_1, x_2) = 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2),$$
for every $(x_1, x_2) \in \mathbb{R}^2$. As a consequence, considering the marginal densities of the entries X_1 and X_2 of the random vector (X_1, X_2), we obtain
$$\begin{aligned} f_{X_1}(x_1) &= \int_{\mathbb{R}} 1_{[0,1]\times[0,1]}(x_1, x_2)\, d\mu_L(x_2) = \int_{\mathbb{R}} 1_{[0,1]}(x_1)\, 1_{[0,1]}(x_2)\, d\mu_L(x_2) \\ &= 1_{[0,1]}(x_1) \int_{\mathbb{R}} 1_{[0,1]}(x_2)\, d\mu_L(x_2) = 1_{[0,1]}(x_1) \int_{[0,1]} d\mu_L(x_2) = 1_{[0,1]}(x_1)\, \mu_L([0,1]) \\ &= 1_{[0,1]}(x_1). \end{aligned}$$
Similarly,
$$f_{X_2}(x_2) = 1_{[0,1]}(x_2).$$
Hence, the entries X_1 and X_2 of the random vector (X_1, X_2) are independent random variables and both standard uniformly distributed. We have
$$\{Y \le y\} = \{X_1 \le y, X_2 \le y\} \cup \{X_1 > y, X_2 \le y\} \cup \{X_1 \le y, X_2 > y\},$$
$F_{X_1}(y)\, F_{X_2}(y)$where the three events on the right hand side are pairwise incompatible, and
$$\{Z \le z\} = \{X_1 \le z, X_2 \le z\},$$
for every $z \in \mathbb{R}$. By virtue of the indepencence of X_1 and X_2, it follows,
$$\begin{aligned} &F_Y(y) \\ &= \mathbf{P}(Y \le y) = \mathbf{P}(X_1 \le y, X_2 \le y) + \mathbf{P}(X_1 > y, X_2 \le y) + \mathbf{P}(X_1 \le y, X_2 > y) \\ &= \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) + \mathbf{P}(X_1 > y)\,\mathbf{P}(X_2 \le y) + \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 > y) \\ &= \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) + (1 - \mathbf{P}(X_1 \le y))\,\mathbf{P}(X_2 \le y) + \mathbf{P}(X_1 \le y)\,(1 - \mathbf{P}(X_2 \le y)) \\ &= \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) + \mathbf{P}(X_2 \le y) - \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) + \mathbf{P}(X_1 \le y) - \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) \\ &= \mathbf{P}(X_1 \le y) + \mathbf{P}(X_2 \le y) - \mathbf{P}(X_1 \le y)\,\mathbf{P}(X_2 \le y) \\ &= F_{X_1}(y) + F_{X_2}(y) - F_{X_1}(y)\, F_{X_2}(y) \end{aligned}$$
and
$$F_Z(z) = \mathbf{P}(X_1 \le z, X_2 \le z) = \mathbf{P}(X_1 \le z)\,\mathbf{P}(X_2 \le z) = F_{X_1}(z)\, F_{X_2}(z),$$

Note that instead of the event $\{Y \leq y\}$ we could have considered the event

$$\{Y > y\} = \{X_1 > y, X_2 > y\}$$

for every $y \in \mathbb{R}$, obtaining

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) = 1 - \mathbf{P}(Y > y) = 1 - \mathbf{P}(X_1 > y, X_2 > y) \\ &= 1 - \mathbf{P}(X_1 > y)\,\mathbf{P}(X_2 > y) = 1 - (1 - \mathbf{P}(X_1 \leq y))(1 - \mathbf{P}(X_2 \leq y)) \\ &= 1 - (1 - F_{X_1}(y))((1 - F_{X_2}(y))) \\ &= 1 - (1 - F_{X_2}(y) - F_{X_1}(y) + F_{X_1}(y)\,F_{X_2}(y)) \\ &= F_{X_1}(y) + F_{X_2}(y) - F_{X_1}(y)\,F_{X_2}(y) \end{aligned}$$

as above. On the other hand, both the random variables X_1 and X_2 are standard uniformly distributed on the interval $[0,1]$. Therefore,

$$F_Y(y) = F_X(y)(2 - F_X(y)) \qquad \text{and} \qquad F_Z(z) = F_X(z)^2,$$

for all $y, x \in \mathbb{R}$, where F_X is the distribution function of the standard uniformly distributed random variable X, given by

$$F_X(x) = x \cdot 1_{[0,1]}(x) + 1_{(1,+\infty)}(x),$$

for every $x \in \mathbb{R}$. It then follows

$$\begin{aligned} &F_Y(y) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,+\infty)}(y) - \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\right) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,0)}(y) + 2 \cdot 1_{[0,1]}(y) + 2 \cdot 1_{(1,+\infty)}(y) - \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\right) \\ &= \left(y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right)\left(2 \cdot 1_{(-\infty,0)}(y) + (2 - y) \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y)\right) \\ &= (2 - y)\,y \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y) \end{aligned}$$

and

$$F_Z(z) = \left(z \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z)\right)^2 = z^2 \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z).$$

Note that we have

$$F'_Y(y) = 2(1 - y) \cdot 1_{(0,1)}(y) \qquad \text{and} \qquad F'_Z(z) = 2z \cdot 1_{(0,1)}(z),$$

for every $y, z \in \mathbb{R} - \{0,1\}$. These imply

$$\begin{aligned} \int_{(-\infty,y)} F'_Y(u)\,d\mu_L(u) &= \int_{(-\infty,y)} 2(1 - u)\,1_{(0,1)}(u)\,d\mu_L(u) \\ &= \begin{cases} 0, & \text{if } y \leq 0, \\ \int_{(0,y)} 2(1-u)\,d\mu_L(u), & \text{if } 0 < y < 1, \\ \int_{(0,1)} 2(1-u)\,d\mu_L(u), & \text{if } 1 \leq y, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_{(-\infty,z)} F'_Z(v)\,d\mu_L(v) &= \int_{(-\infty,z)} 2z \cdot 1_{(0,1)}(z)\,d\mu_L(v) \\ &= \begin{cases} 0, & \text{if } z \leq 0, \\ \int_{(0,z)} 2v\,d\mu_L(v), & \text{if } 0 < z < 1, \\ \int_{(0,1)} 2v\,d\mu_L(v), & \text{if } 1 \leq z. \end{cases} \end{aligned}$$

On the other hand,

$$\int_{(0,y)} 2(1-u)\,d\mu_L(u) = \int_0^y 2(1-u)\,du = 2u - u^2\big|_0^y = y(2-y),$$

for every $0 < y \leq 1$, and

$$\int_{(0,z)} 2v\,d\mu_L(v) = \int_0^z 2v\,dv = v^2\big|_0^z = z^2,$$

for every $0 < z \leq 1$. We can then write

$$\int_{(-\infty,y)} F'_Y(u)\, d\mu_L(u) = y(2-y) \cdot 1_{[0,1]}(y) + 1_{(1,+\infty)}(y) = F_Y(y),$$

for every $y \in \mathbb{R}$, and

$$\int_{(-\infty,z)} F'_Z(v)\, d\mu_L(v) = z^2 \cdot 1_{[0,1]}(z) + 1_{(1,+\infty)}(z) = F_Z(z),$$

for every $z \in \mathbb{R}$. These imply that Y and Z are absolutely continuous random variables.

2. We have

$$\begin{aligned}
&\{Y \leq y, Z \leq z\} \\
&= (\{X_1 \leq y, X_2 \leq y\} \cup \{X_1 > y, X_2 \leq y\} \cup \{X_1 \leq y, X_2 > y\}) \cap \{X_1 \leq z, X_2 \leq z\} \\
&= (\{X_1 \leq y, X_2 \leq y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&\quad \cup (\{X_1 > y, X_2 \leq y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&\quad \cup (\{X_1 \leq y, X_2 > y\} \cap \{X_1 \leq z, X_2 \leq z\}) \\
&= \{X_1 \leq \min(y,z), X_2 \leq \min(y,z)\} \\
&\quad \cup \{y < X_1 \leq z, X_2 \leq \min(y,z)\} \\
&\quad \cup \{X_1 \leq \min(y,z), y < X_2 \leq z\}.
\end{aligned}$$

Therefore, considering the joint distribution function $F_{Y,Z} : \mathbb{R}^2 \to \mathbb{R}_+$ of Y and Z, on account of the independence of X_1 and X_2, we can write

$$\begin{aligned}
F_{Y,Z}(y,z) &= \mathbf{P}(Y \leq y, Z \leq z) \\
&= \mathbf{P}(X_1 \leq \min(y,z), X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(y < X_1 \leq z, X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(X_1 \leq \min(y,z), y < X_2 \leq z) \\
&= \mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(y < X_1 \leq z)\, \mathbf{P}(X_2 \leq \min(y,z)) \\
&\quad + \mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(y < X_2 \leq z),
\end{aligned}$$

for every $(y,z) \in \mathbb{R}^2$. On the other hand,

$$\begin{array}{lr}
\min(y,z) = y, & \text{if } y \leq z, \\
\mathbf{P}(y < X_1 \leq z) = 0 \quad \text{and} \quad \min(y,z) = z, & \text{if } y > z.
\end{array}$$

Hence, considerning that X_1 and X_2 have the same distribution, we obtain

$$F_{Y,Z}(y,z) = \begin{cases} F_X(y)\,(2F_X(z) - F_X(y)), & \text{if } y \leq z, \\ F_X(z)^2, & \text{if } y > z. \end{cases}$$

In fact, if $y \leq z$

$$\begin{aligned}
&\mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(X_2 \leq \min(y,z)) + \mathbf{P}(y < X_1 \leq z)\, \mathbf{P}(X_2 \leq \min(y,z)) \\
&+ \mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(y < X_2 \leq z) \\
&= \mathbf{P}(X \leq y)\, \mathbf{P}(X \leq y) + 2\mathbf{P}(X \leq y)\, \mathbf{P}(y < X \leq z) \\
&= F_X(y)^2 + 2F_X(y)\,(F_X(z) - F_X(y)) \\
&= F_X(y)\,(2F_X(z) - F_X(y)).
\end{aligned}$$

and if $y > z$

$$\begin{aligned}
&\mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(X_2 \leq \min(y,z)) + \mathbf{P}(y < X_1 \leq z)\, \mathbf{P}(X_2 \leq \min(y,z)) \\
&+ \mathbf{P}(X_1 \leq \min(y,z))\, \mathbf{P}(y < X_2 \leq z) \\
&= \mathbf{P}(X \leq z)\, \mathbf{P}(X \leq z) + 2\mathbf{P}(X \leq z)\, \mathbf{P}(y < X \leq z) \\
&= F_X(z)^2
\end{aligned}$$

Note that we can write

$$F_{Y,Z}(y,z) = F_X(y)(2F_X(z) - F_X(y))\, 1_{\{(y,z)\in\mathbb{R}^2: y\le z\}} + F_X(z)^2\, 1_{\{(y,z)\in\mathbb{R}^2: y>z\}}.$$

3. To determine the marginal distribution functions $F_Y : \mathbb{R} \to \mathbb{R}_+$ and $F_Z : \mathbb{R} \to \mathbb{R}_+$ of the random vector (Y, Z), respectively, we can apply the formula

$$\begin{aligned} F_Y(y) &= \lim_{z\to+\infty} F_{Y,Z}(y,z) \\ &= \lim_{z\to+\infty} \left(F_X(y)(2F_X(z) - F_X(y))\, 1_{\{(y,z)\in\mathbb{R}^2: y\le z\}}(y,z) + F_X(z)^2\, 1_{\{(y,z)\in\mathbb{R}^2: y>z\}}(y,z) \right) \end{aligned}$$

and

$$\begin{aligned} F_Z(z) &= \lim_{y\to+\infty} F_{Y,Z}(y,z) = \\ &= \lim_{y\to+\infty} \left(F_X(y)(2F_X(z) - F_X(y))\, 1_{\{(y,z)\in\mathbb{R}^2: y\le z\}}(y,z) + F_X(z)^2\, 1_{\{(y,z)\in\mathbb{R}^2: y>z\}}(y,z) \right). \end{aligned}$$

as $z \to +\infty$ for every $y \in \mathbb{R}$ we have

$$1_{\{(y,z)\in\mathbb{R}^2: y\le z\}}(y,z) = 1 \qquad \text{and} \qquad 1_{\{(y,z)\in\mathbb{R}^2: y>z\}}(y,z) = 0.$$

Conversely, as $y \to +\infty$ for every $z \in \mathbb{R}$ we have

$$1_{\{(y,z)\in\mathbb{R}^2: y\le z\}}(y,z) = 0 \qquad \text{and} \qquad 1_{\{(y,z)\in\mathbb{R}^2: y>z\}}(y,z) = 1.$$

It then follows

$$F_Y(y) = F_X(y)(2F_X(z) - F_X(y)) \qquad \text{and} \qquad F_Z(z) = F_X(z)^2,$$

which shows that the marginal distribution functions of the random vector (Y, Z) concide with the distribution functions of the random variables X and Y. As a consequence, the random variables $Y \equiv \min(X_1, X_2)$ and $Z \equiv \max(X_1, X_2)$ are indepemdent.

4. In the end, we have

$$\begin{aligned} \mathbf{E}[Y] &= \int_{\mathbb{R}} y f_Y(y)\, d\mu_L(y) = \int_{\mathbb{R}} 2y(1-y)\, 1_{[0,1]}(y)\, d\mu_L(y) = \int_{[0,1]} 2y(1-y)\, d\mu_L(y) \\ &= \int_0^1 2(1-y)\, y dy = 2\left(\int_0^1 y dy - \int_0^1 y^2 dy \right) = 2\left(\frac{1}{2} y^2 \Big|_0^1 - \frac{1}{3} y^3 \Big|_0^1 \right) = \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}[Z] &= \int_{\mathbb{R}} z f_Z(z)\, d\mu_L(z) = \int_{\mathbb{R}} 2z^2 \cdot 1_{[0,1]}(z)\, d\mu_L(z) = \int_{[0,1]} 2z^2 d\mu_L(z) \\ &= \int_0^1 2z^2 dz = 2\int_0^1 z^2 dz = 2\, \frac{1}{3} z^3 \Big|_0^1 = \frac{2}{3}. \end{aligned}$$

Problem 2 *Let $F : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$F(x_1, x_2) \stackrel{def}{=} \left(1 - e^{-x_1} - e^{-x_2} + e^{-(x_1+x_2)} \right) 1_{\mathbb{R}_+}(x_1)\, 1_{\mathbb{R}_+}(x_2), \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

Show that $F : \mathbb{R}^2 \to \mathbb{R}_+$ is the distribution function of a real random vector (X_1, X_2) and compute the marginal distribution functions of (X_1, X_2).

1. *Is the function $F : \mathbb{R}^2 \to \mathbb{R}_+$ absolutely continuous?*
2. *Are the entries X_1 and X_2 of the random vector (X_1, X_2) independent random variables?*
3. *Are the entries X_1 and X_2 of the random vector (X_1, X_2) absolutely continuous random variables?*

4. *What is the distribution* $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *of the real random variable* $Z = \max\{X_1, X_2\}$.

5. *Is the function* $F_Z : \mathbb{R}^2 \to \mathbb{R}_+$ *absolutely continuous?*

   Hint: it might be useful to rewrite $F(x_1, x_2)$ *in a more convenient form.*

Solution. .

Problem 3 *Let* $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ *be a probability space and let* $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ *be the Euclidean real plane endowed with the Borel* σ*-algebra* $\mathcal{B}\left(\mathbb{R}^2\right)$ *and the Lebesgue measure* $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(x, y) \stackrel{def}{=} k e^{-(x^2 - xy + y^2/2)}, \quad \forall (x, y) \in \mathbb{R}^2,$$

where $k \in \mathbb{R}$ *is a parameter.*

1. *Determine* k *such that* $f : \mathbb{R}^2 \to \mathbb{R}$ *is a probability density. Hint: can you compute* $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2} dy$ *with no computation?*

   Let $Z \equiv (X, Y)$ *be the random vector on* Ω *with density* $f : \mathbb{R}^2 \to \mathbb{R}_+$.

2. *Determine the marginal density of the entries* X *and* Y. *Are the random variables* X *and* Y *Gaussian?*

3. *Is the random vector* Z *Gaussian?*

4. *Compute* $\mathbf{E}[X]$, $\mathbf{E}[Y]$, $\mathbf{D}^2[X]$, $\mathbf{D}^2[Y]$, *and* $Cov(X, Y)$.

5. *Are* X *and* Y *independent random variables?*

6. *Is the random vector* Z *Gaussian? Hint: consider the answer you gave to 4., what you know from the theory, and try to make a simple guess.*

Solution.

1. We can write

$$\int_{\mathbb{R}^2} f(x, y)\, d\mu_L^2(x, y) = k \int_{\mathbb{R}^2} e^{-(x^2 - xy + y^2/2)} d\mu_L^2(x, y).$$

   On the other hand, since $e^{-(x^2 - xy + y^2/2)}$ is a continuous positive function

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{-(x^2 - xy + y^2/2)} d\mu_L^2(x, y) &= \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} e^{-(x^2 - xy + y^2/2)} dx dy \\
&= \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(y^2 - 2xy + x^2\right)} e^{-\frac{1}{2}x^2} dx dy \\
&= \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} \left(\int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2} dy \right) dx.
\end{aligned}$$

   Now, we have

$$\int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}(y-x)^2} dy = \int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{2\pi},$$

   for every $x \in \mathbb{R}$. Therefore,

$$\int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{+\infty} e^{-(x^2 - xy + y^2/2)} dx dy = \sqrt{2\pi} \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = 2\pi.$$

   If follows that

$$\int_{\mathbb{R}^2} f(x, y)\, d\mu_L^2(x, y) = 1 \Rightarrow k = \frac{1}{2\pi}.$$

2. Considering what shown above, we have

$$f_X(x) = \int_{\mathbb{R}} \frac{1}{2\pi} f(x,y)\, d\mu_L(y) = \frac{1}{2\pi} \int_{y=-\infty}^{+\infty} e^{-\frac{1}{2}\left(y^2-2xy+x^2\right)} e^{-\frac{1}{2}x^2} dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

for every $x \in \mathbb{R}$. Similarly, since

$$e^{-\left(x^2-xy+y^2/2\right)} = e^{-\frac{1}{2}\left(2x^2-2xy+y^2\right)} = e^{-\frac{1}{2}\left(\left(\sqrt{2}x\right)^2-2xy+\left(\frac{y}{\sqrt{2}}\right)^2\right)} e^{-\frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2} = e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2} e^{-\frac{y^2}{4}},$$

we have

$$f_Y(y) = \int_{\mathbb{R}} \frac{1}{2\pi} f(x,y)\, d\mu_L(x) = \frac{1}{2\pi} \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2} e^{-\frac{y^2}{4}} dx = \frac{1}{2\pi} e^{-\frac{y^2}{4}} \int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2} dx,$$

for every $y \in \mathbb{R}$. Furthermore,

$$\int_{x=-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{y}{\sqrt{2}}\right)^2} dx = \frac{1}{\sqrt{2}} \int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{\pi}.$$

Hence,

$$f_Y(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{y^2}{4}} = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{y}{\sigma_Y}\right)^2}, \quad \sigma_Y \equiv \sqrt{2}.$$

This shows that the random variables X and Y are Gaussian.

3. We clearly have

$$\mathbf{E}[X] = \mathbf{E}[Y] = 0.$$

Moreover,

$$\mathbf{D}^2[X] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2} dx = 1, \quad \mathbf{D}^2[Y] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} y^2 e^{-\frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2} dy = 2.$$

In addition,

$$\begin{aligned} Cov(X,Y) = \mathbf{E}[XY] &= \int_{\mathbb{R}^2} xy f(x,y)\, d\mu_L^2(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} xy e^{-\left(x^2-xy+y^2/2\right)} d\mu_L^2(x,y) \\ &= \frac{1}{2\pi} \int_{x=-\infty}^{+\infty} x e^{-\frac{1}{2}x^2} \left(\int_{y=-\infty}^{+\infty} y e^{-\frac{1}{2}(y-x)^2} dy \right) dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{y=-\infty}^{+\infty} y e^{-\frac{1}{2}(y-x)^2} dy &= \int_{y=-\infty}^{+\infty} (y-x) e^{-\frac{1}{2}(y-x)^2} dy + \int_{y=-\infty}^{+\infty} x e^{-\frac{1}{2}(y-x)^2} dy \\ &= \int_{z=-\infty}^{+\infty} z e^{-\frac{1}{2}z^2} dz + x \int_{z=-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz \\ &= \sqrt{2\pi} x. \end{aligned}$$

Hence,

$$Cov(X,Y) = \frac{1}{2\pi} \int_{x=-\infty}^{+\infty} \sqrt{2\pi} x^2 e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{+\infty} x^2 e^{-\frac{1}{2}x^2} = 1.$$

4. Since

$$Cov(X,Y) \neq 0,$$

the random variables X and Y are not independent.

5. Since not independent, despite X and Y are Gaussian, we cannot state at present whether the random vector $(X,Y)^\intercal$ is Gaussian or not. To solve this doubt, we can try to write

$$\begin{pmatrix} X \\ Z \end{pmatrix} = A \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

for independent standard Gaussian random variables Z_1 and Z_1 and a suitable matrix

$$A \equiv \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

If this is true, we have

$$\Sigma^2_{X,Y} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = AA^{\intercal}.$$

Thus, we are led to find a matrix A such that

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{pmatrix} = \begin{pmatrix} a^2_{1,1} + a^2_{1,2} & a_{1,1}a_{2,1} + a_{1,2}a_{2,2} \\ a_{1,1}a_{2,1} + a_{1,2}a_{2,2} & a^2_{2,1} + a^2_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

To this goal, observe that $\Sigma^2_{X,Y}$ has eigenvalues

$$\frac{3}{2} + \frac{1}{2}\sqrt{5} \quad \text{and} \quad \frac{3}{2} - \frac{1}{2}\sqrt{5},$$

with corresponding orthogonal eigenvectors

$$\begin{pmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix}.$$

In fact, we have

$$\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{pmatrix} = 0.$$

Therefore, normalizing the eigenvectors, we have that

$$B \equiv \left\{ \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5} - \frac{1}{2}}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} \end{pmatrix}, \begin{pmatrix} -\frac{\frac{1}{2}\sqrt{5} + \frac{1}{2}}{\sqrt{\frac{5}{2} + \frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2} + \frac{1}{2}\sqrt{5}}} \end{pmatrix} \right\}$$

is a basis of orthonormal eigenvectors in $\mathbb{R}^2$. We then have

$$M^B_E\,(id)\,\Lambda M^E_B\,(id) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

where

$$E \equiv \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is the standard orthonormal basia in $\mathbb{R}^2$,

$$M^B_E\,(id) = \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5} - \frac{1}{2}}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5} + \frac{1}{2}}{\sqrt{\frac{5}{2} + \frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2} + \frac{1}{2}\sqrt{5}}} \end{pmatrix}, \quad \Lambda \equiv \begin{pmatrix} \frac{3}{2} + \frac{1}{2}\sqrt{5} & 0 \\ 0 & \frac{3}{2} - \frac{1}{2}\sqrt{5} \end{pmatrix},$$

and

$$M^E_B\,(id) = M^B_E\,(id)^{-1} = M^B_E\,(id)^{\intercal} = \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5} - \frac{1}{2}}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2} - \frac{1}{2}\sqrt{5}}} \\ -\frac{\frac{1}{2}\sqrt{5} + \frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5} + \frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5} + \frac{5}{2}}} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \frac{3}{2}+\frac{1}{2}\sqrt{5} & 0 \\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In addition, we can write

$$\begin{aligned} &\begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \frac{3}{2}+\frac{1}{2}\sqrt{5} & 0 \\ 0 & \frac{3}{2}-\frac{1}{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{3}{2}+\frac{1}{2}\sqrt{5}} & 0 \\ 0 & \sqrt{\frac{3}{2}-\frac{1}{2}\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{3}{2}+\frac{1}{2}\sqrt{5}} & 0 \\ 0 & \sqrt{\frac{3}{2}-\frac{1}{2}\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \\ \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}+\frac{1}{2}\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\sqrt{5} & 0 \\ 0 & \frac{1}{2}\sqrt{5}-\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\sqrt{5} & 0 \\ 0 & \frac{1}{2}\sqrt{5}-\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}-\frac{1}{2}}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix}. \end{aligned}$$

Therefore, we obtain

$$\begin{pmatrix} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} \\ -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Setting

$$A = \begin{pmatrix} \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & -\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} & \frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix},$$

it then follows

$$a_{1,1}^2 + a_{1,2}^2 = \left(\frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\right)^2 + \left(-\frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\right)^2 = 1,$$

$$a_{2,1}^2 + a_{2,2}^2 = \left(\frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\right)^2 + \left(\frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\right)^2 = 2,$$

$$a_{1,1}a_{2,1} + a_{1,2}a_{2,2} = \frac{1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}}\frac{1}{2}\frac{\sqrt{5}+1}{\sqrt{\frac{5}{2}-\frac{1}{2}\sqrt{5}}} - \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}}\frac{1}{2}\frac{\sqrt{5}-1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} = 1.$$

This proves that $(X,Y)^\intercal$ is Gaussian. Note that, from

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

it follows

$$\begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} = A \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \begin{pmatrix} Z_1 & Z_2 \end{pmatrix} A^\intercal,$$

that is to say

$$\begin{pmatrix} X^2 & XY \\ XY & Y^2 \end{pmatrix} = A \begin{pmatrix} Z_1^2 & Z_1Z_2 \\ Z_1Z_2 & Z_2^2 \end{pmatrix} A^\intercal.$$

It follows,

$$\Sigma^2_{X,Y} = \begin{pmatrix} \mathbf{D}^2[X] & Cov(X,Y) \\ Cov(X,Y) & \mathbf{D}^2[X] \end{pmatrix} = \begin{pmatrix} \mathbf{E}\left[X^2\right] & \mathbf{E}[XY] \\ \mathbf{E}[XY] & \mathbf{E}\left[Y^2\right] \end{pmatrix}$$
$$= A \begin{pmatrix} \mathbf{E}\left[Z_1^2\right] & \mathbf{E}[Z_1 Z_2] \\ \mathbf{E}[Z_1 Z_2] & \mathbf{E}\left[Z_2^2\right] \end{pmatrix} A^\intercal = A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^\intercal = AA^\intercal.$$

Problem 4 *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right), \mu_L^2\right) \equiv \mathbb{R}^2$ be the Euclidean real plane endowed with the Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^2\right)$ and the Lebesgue measure $\mu_L^2 : \mathcal{B}\left(\mathbb{R}^2\right) \to \mathbb{R}_+$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x,y) \overset{def}{=} k e^{-\frac{x^2 - xy + y^2}{2}}, \quad \forall (x,y) \in \mathbb{R}^2,$$

where $k \in \mathbb{R}$ is a parameter.

1. *Determine k such that $f : \mathbb{R}^2 \to \mathbb{R}$ is a probability density. Hint: It may be useful to recall that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = 1$.*

2. *Determine the marginal density functions of the entries X and Y. Are X and Y independent?*

3. *Compute $\mathbf{P}(X = Y)$ and $\mathbf{P}(X \geq Y)$.*

Solution. .

Exercise 5 (Sheldon M. Ross - 4.11) *Let $(\Omega, \mathcal{E}, \mathbf{P}) \equiv \Omega$ be a probability space and let X and Y be real random variables on Ω such that the random vector (X,Y) is absolutely continuous with a density $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ given by*

$$f_{X,Y}(x,y) \overset{def}{=} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \cdot 1_{(0,1)\times(0,2)}(x,y), \quad \forall (x,y) \in \mathbb{R}^2.$$

1. *Check that $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ is a density function.*

2. *Are the random variables X and Y absolutely continuous? In case of affirmative answer determine the marginal densities $f_X : \mathbb{R} \to \mathbb{R}_+$ and $f_Y : \mathbb{R} \to \mathbb{R}_+$ of X and Y, respectively.*

3. *Check whether the random variables X and Y are independent.*

4. *Compute $\mathbf{P}(X > Y)$.*

Solution.

1. We will have proven that $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}_+$ is a density function if we can show that

$$\int_{\mathbb{R}^2} f_{X,Y}(x,y)\, d\mu_L^2(x,y) = 1.$$

On the other hand, considering the properties of the Lebesgue integral, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} f_{X,Y}(x,y)\, d\mu_L(x,y) &= \int_{\mathbb{R}^2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\ \ d\mu_L^2(x,y) \\
&= \int_{(0,1)\times(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right) d\mu_L^2(x,y) \\
&= \int_{(0,1)\times(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right) dxdy \\
&= \int_{y=0}^{2}\int_{x=0}^{1} \frac{6}{7}\left(x^2+\frac{xy}{2}\right) dxdy \\
&= \frac{6}{7}\int_{y=0}^{2}\left(\int_{x=0}^{1}\left(x^2+\frac{xy}{2}\right) dx\right) dy \\
&= \frac{6}{7}\int_{y=0}^{2}\left(\frac{x^3}{3}+\frac{x^2y}{4}\Big|_0^1\right) dy \\
&= \frac{6}{7}\int_{y=0}^{2}\left(\frac{1}{3}+\frac{y}{4}\right) dy \\
&= \frac{6}{7}\left(\frac{y}{3}+\frac{y^2}{8}\Big|_0^2\right) \\
&= \frac{6}{7}\left(\frac{2}{3}+\frac{1}{2}\right) \\
&= 1.
\end{aligned}$$

2. Since the random vector is absolutely continuous the entries X and Y are absolutely continuous random variables with densities $f_X:\mathbb{R}\to\mathbb{R}_+$ and $f_Y:\mathbb{R}\to\mathbb{R}_+$ given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(y) \qquad \text{and} \qquad f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(x),$$

μ_L-a.e. on $\mathbb{R}$, respectively. Now, we have

$$\begin{aligned}
\int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(y) &= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\, d\mu_L(y) \\
&= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\, 1_{(0,2)}(y)\, d\mu_L(y) \\
&= \int_{(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\, d\mu_L(y) \\
&= \frac{6}{7}\left(\int_0^2\left(x^2+\frac{xy}{2}\right) dy\right)\cdot 1_{(0,1)}(x) \\
&= \frac{6}{7}\left(x^2y+\frac{xy^2}{4}\Big|_{y=0}^2\right)\cdot 1_{(0,1)}(x) \\
&= \frac{6}{7}\left(2x^2+x\right)\cdot 1_{(0,1)}(x).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\int_{\mathbb{R}} f_{X,Y}(x,y)\, d\mu_L(x) &= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\, d\mu_L(x) \\
&= \int_{\mathbb{R}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)}(x)\, 1_{(0,2)}(y)\, d\mu_L(x) \\
&= \int_{(0,1)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,2)}(y)\, d\mu_L(y) \\
&= \frac{6}{7}\left(\int_0^1 \left(x^2+\frac{xy}{2}\right) dx\right)\cdot 1_{(0,2)}(y) \\
&= \frac{6}{7}\left(\frac{x^3}{3}+\frac{x^2 y}{4}\bigg|_{x=0}^{1}\right)\cdot 1_{(0,2)}(y) \\
&= \frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y).
\end{aligned}
$$

Therefore, we can write

$$
f_X(x) = \frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x) \qquad \text{and} \qquad f_Y(y) = \frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y),
$$

μ_L-a.e. on $\mathbb{R}$, respectively.

3. The random variables X and Y are independent if and only if

$$
f_X(x)\, f_Y(y) = f_{X,Y}(x,y),
$$

μ_L^2-a.e. on $\mathbb{R}^2$. On the other hand,

$$
\begin{aligned}
f_X(x)\, f_Y(x) &= \left(\frac{6}{7}\left(x+2x^2\right)\cdot 1_{(0,1)}(x)\right)\left(\frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right)\cdot 1_{(0,2)}(y)\right) \\
&= \frac{36}{49}\left(\frac{x}{3}+\frac{xy}{4}+\frac{2x^2}{3}+\frac{x^2 y}{2}\right)\cdot 1_{(0,1)}(x)\, 1_{(0,2)}(y) \\
&= \frac{36}{49}\left(\frac{x}{3}+\frac{xy}{4}+\frac{2x^2}{3}+\frac{x^2 y}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y) \\
&\neq \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)
\end{aligned}
$$

for almost all points $(x,y)\in(0,1)\times(0,2)$. Therefore, X and Y are not independent.

4. To compute $\mathbf{P}(X>Y)$ we apply the formula

$$
\mathbf{P}((X,Y)\in B) = \int_B f_{X,Y}(x,y)\;\; d\mu_L^2(x,y),
$$

which holds true for every $B\in\mathcal{B}\left(\mathbb{R}^2\right)$, by suitably choosing B to represent the event $\{X>Y\}$ in terms of the event $\{(X,Y)\in B\}$. Eventually, setting

$$
B \equiv \left\{(x,y)\in\mathbb{R}^2 : x>y\right\},
$$

it turns out that we can write

$$
\{X>Y\} = \{(X,Y)\in B\}.
$$

In fact, assume that $\omega\in\{X>Y\}\equiv\{\omega\in\Omega : X(\omega)>Y(\omega)\}$, then we have $X(\omega)>Y(\omega)$ so that $(X(\omega),Y(\omega))\in B$ and $\omega\in\{(X,Y)\in B\}\equiv\{\omega\in\Omega : (X(\omega),Y(\omega))\in B\}$. Conversely, assume that $\omega\in\{(X,Y)\in B\}$, then $(X(\omega),Y(\omega))\in B$, which implies $X(\omega)>Y(\omega)$ and consequently $\omega\in\{X>Y\}$.

As a consequence, we have

$$
\begin{aligned}
\mathbf{P}(X>Y) &= \int_{\{(x,y)\in\mathbb{R}^2:x>y\}} f_{X,Y}(x,y)\,d\mu_L^2(x,y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2:x>y\}} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)\cdot 1_{(0,1)\times(0,2)}(x,y)\,d\mu_L^2(x,y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2:x>y\}\cap(0,1)\times(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)d\mu_L^2(x,y) \\
&= \int_{\{(x,y)\in\mathbb{R}^2:x>y\}\cap(0,1)\times(0,2)} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dxdy \\
&= \frac{6}{7}\int_{x=0}^{1}\left(\int_{y=0}^{x}\left(x^2+\frac{xy}{2}\right)dy\right)dx \\
&= \frac{6}{7}\int_{x=0}^{1}\left(x^2y+\frac{xy^2}{4}\Big|_0^x\right)dx \\
&= \frac{6}{7}\int_{x=0}^{1}\frac{5x^3}{4}dx \\
&= \frac{6}{7}\frac{5x^4}{16}\Big|_0^1 \\
&= \frac{6}{7}\frac{5}{16} \\
&= \frac{15}{56}\approx 0.26786
\end{aligned}
$$

.

Problem 6 *Let $f:\mathbb{R}^2\to\mathbb{R}_+$ given by*

$$f(x,y)\overset{def}{=}\frac{4x+2y}{3}1_{[0,1]}(x)\,1_{[0,1]}(y),\quad \forall (x,y)\in\mathbb{R}^2.$$

1. *Show that $f:\mathbb{R}^2\to\mathbb{R}_+$ is the density function of a real random vector (X,Y).*
2. *Compute the marginal densities of (X,Y) and check that the computed marginal densities are actually probability densities.*
3. *May we say that the entries X and Y of the random vector (X,Y) are independent random variables?*
4. *Compute the conditional density function $f_{X|Y}(x,y)$ of X given that $Y=y$ and check the computed density is actually a probability density.*
5. *Compute the function $\mathbf{E}[X\mid Y=y]$ and the conditional expectation $\mathbf{E}[X\mid Y]$.*

Solution. .

Problem 7 *Determine the value of the parameter k such that the function $f:\mathbb{R}^3\to\mathbb{R}$ given by*

$$f(x_1,x_2,x_3)\overset{def}{=}\begin{cases} k\left(x_1+x_2^2+x_3^3\right) & \text{if } (x_1,x_2,x_3)\in[0,1]\times[0,1]\times[0,1] \\ 0 & \text{otherwise}\end{cases}$$

is a probability density. Hence, consider the random vector $(X_1,X_2,X_3)^\intercal$ with density $f_{X_1,X_2,X_3}:\mathbb{R}^3\to\mathbb{R}$ given by

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3)\overset{def}{=}f(x_1,x_2,x_3).$$

Compute:

1. *the probability $\mathbf{P}(X_2\le 1/2, X_3>1/2)$;*

2. *the marginal density of the random vector* $(X_1, X_2)^\intercal$*;*

3. *the expectation of* $(X_1, X_2)^\intercal$*;*

4. *the conditional density* $f_{X_1,X_2|X_3=1/2}(x_1, x_2)$*.*

Solution. .

Problem 8 *Determine the value of the parameter* k *such that the function* $f : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f(x_1, x_2, x_3) \overset{def}{=} \begin{cases} k\left(x_1^2 + x_2^2 + x_3^3\right) & \textit{if } (x_1, x_2, x_3) \in [0,1] \times [0,1] \times [0,1] \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density. Hence, consider the random vector $(X_1, X_2, X_3)^\intercal$ *with density* $f_{X_1,X_2,X_3} : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) \overset{def}{=} f(x_1, x_2, x_3).$$

Compute:

1. *the marginal density of the random vector* $(X_1, X_2)^\intercal$*;*

2. *the expectation of the product* $X_1 \cdot X_2$*;*

3. *the conditional density* $f_{X_1|X_2=1/2,X_3=3/4}(x_1)$*;*

4. *the probability* $\mathbf{P}(X_1 \leq 1/2, X_2 < 1/2, X_3 < 1/2)$*.*

Solution. .