Computational lower bounds via almost orthonormal polynomials

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co-supervision with Alexandra Carpentier, Christophe Giraud, Nicolas Verzelen

Toy model: planted sub-matrix/sub-graph/random clique

For (λ, k) sample the $n \times n$ matrix:

$$X_{ij} = x_i x_j, \qquad x_i \overset{\text{i.i.d.}}{\sim} \sqrt{\lambda}\mathrm{Ber}\left(\frac{k}{n}\right)$$

and observe the $n \times n$ matrix:

$$Y_{ij} = \begin{cases} 1 & \text{with probability } \frac{1+X_{ij}}{2} \\ -1 & \text{with probability } \frac{1-X_{ij}}{2} \end{cases}.$$

Latent clique of (“strength”, “size”)$= (\lambda, k)$.

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Latent clique of ("strength", "size")$= (\lambda, k)$.

Question: prove when cannot detect perturbations in poly-time

$$H_0 \ : \ Y \text{ structure is } (\lambda, k), \qquad H_1 : Y \text{ structure is } (\lambda + \eta, k).$$

Performance in hypothesis test

Statistical optimality

$$err_{IT}(\lambda, \eta, k) \coloneqq \inf_{t \text{ measurable test}} \mathbb{P}_{H_0}[t(Y) = H_1] + \mathbb{P}_{H_1}[t(Y) = H_0].$$

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Statistical-to-computational gap [KWB19; BPW18]

Less studied **poly-time criterion is the important one in practice**.
If $1-\Omega(1) = err_{IT}(\lambda,\eta,k) < err_{poly}(\lambda,\eta,k) = 1-o(1)$ then η is hard.

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Computational bound, conjecture [Hop18]

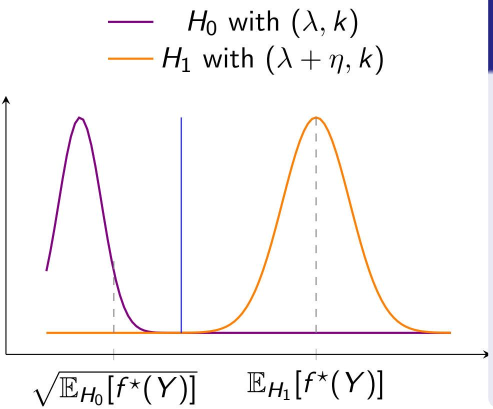
log n degree polynomials surrogate poly-time algorithms, and if:

$$Adv(\lambda, \eta, k) := \sup_{f : \deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}} = 1 + o(1),$$

and $err_{IT}(\lambda, \eta, k) \leq 1 - \Omega(1)$ then **statistical-to-computational gap**.

Intuition on advantage

When the advantage $Adv(\lambda, \eta, k) := \sup_{f:\deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}}$ is large:



If it is small for each polynomial of degree $\leq D$

no polynomial separates distributions! **Extrapolate hardness for polynomials, conjecture it for poly-time algorithms.**

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- captures best known algorithms, but also polynomials and spectral methods [KWB19];
- applied to many problems for hypothesis testing, estimation, optimization [Wei25; EGV25a; EGV25b; EGV24];
- linked to other techniques to claim/conjecture algorithmic hardness [Ban+22; BB20; MW22; Che+25; Wei25; GMZ22].

The orthonormal trick to bound *Adv*

Question: prove when cannot detect perturbations **in poly-time**,

$$H_0 \;:\; Y \text{ structure is } (\lambda, k), \qquad H_1 : Y \text{ structure is } (\lambda + \eta, k).$$

Basic idea: decomposition along basis of $Adv(\lambda, \eta, k)$.

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Imagine the basis is orthonormal in H_0 . . .

$$Adv(\lambda, \eta, k) = \sup_{f:\deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}}$$

Decompose in ψ_G orthonormal basis numerator and denominator.

$$f(Y) = \sum_{G \in \text{basis}} \alpha_G \psi_G, \qquad \alpha_G = \mathbb{E}_{H_0:(\lambda,k)}\left[f(Y)\psi_G\right].$$

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$$= \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{G \in \text{ basis}} \alpha_G \psi_G\right]}{\|\alpha\|_2} = LinAdv(\lambda, \eta, k),$$

Linear: **easy**!

Outside of pure noise, $(\lambda, k), \lambda, k \neq 0$

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In literature implicit recursive solutions [SW22; SW25].

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Our solution: almost orthonormal basis

Find a collection of functions $(\widetilde{\psi}_G)_G$ forming a basis of H_0, H_1:

$$f(Y) = \sum_{G \in \text{ basis}} \alpha_G \widetilde{\psi}_G,$$

such that for some constants:

$$c \left\|\alpha\right\|_2^2 \leq \mathbb{E}_{H_0:(\lambda,k)} \left[\sum_{G, G' \in \text{basis}} \alpha_G \alpha_{G'} \widetilde{\psi}_G \widetilde{\psi}_{G'} \right] \leq C \left\|\alpha\right\|_2^2 .$$

Key proposition

Assume

The parameter (λ, k) to sample Y in H_0 is such that $\max\left\{\frac{k}{n}, \frac{\lambda k}{n}, \lambda\right\} \leq \mathrm{polylog}(n)$.

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There is an almost orthonormal basis and for (λ, k):

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Decompose in $\widetilde{\psi}_G$ almost orthonormal basis numerator and denominator.

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$$\leq \frac{1}{\sqrt{c}} \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{G \in \text{ basis}} \alpha_G \widetilde{\psi}_G\right]}{\|\alpha\|_2} = \frac{1}{\sqrt{c}} LinAdv(\lambda, \eta, k),$$

Like orthonormal up to constants, explicit, linear: **easy**!

Main theorem

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The parameter (λ, k) to sample Y in H_0 is such that
$\max\left\{\frac{k}{n}, \frac{\lambda k}{n}, \lambda\right\} \leq \mathrm{polylog}(n)$.
The perturbation η for H_1 is such that $\eta\frac{k^2}{n} \leq \mathrm{polylog}(n)$.

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The perturbation η for H_1 is such that $\eta\frac{k^2}{n} \leq \mathrm{polylog}(n)$.

Main Theorem

Under the conditions above for the planted sub-matrix model:

$$Adv(\lambda, \eta, k) \leq \frac{1}{\sqrt{c}} LinAdv(\lambda, \eta, k) = 1 + o(1)$$

so we conjecture $err_{poly}(\lambda, \eta, k)$ is large.

The region of the assumption and the perturbation is hard for poly-time algorithms.

Takeaways and extensions

Statistical-to-computational gap

The region of the assumptions is not hard for all functions so **there is a gap**. Conjecturally $1 - \Omega(1) = err_{IT}(\lambda, \eta, k) \ll err_{poly}(\lambda, \eta, k) = 1 - o(1)$.

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Novelty

- proof technique via almost orthonormal basis;
- more explicit ;
- potentially sharper.

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Novelty

- proof technique via almost orthonormal basis;
- more explicit ;
- potentially sharper.

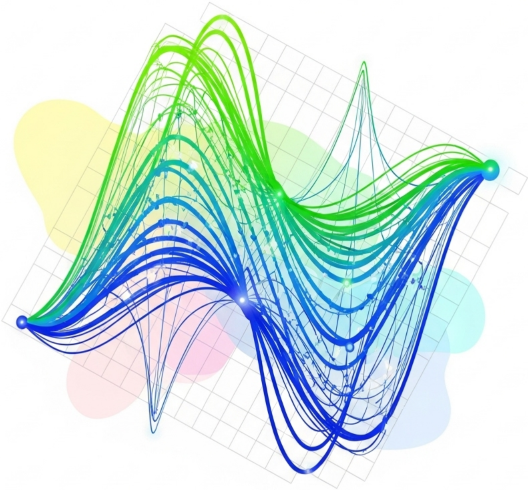
Extensions

Can let D vary, perturb k instead of λ, and nonasympotic result.
Many more models admit an almost orthonormal basis: stochastic block model, allow for fixed latent size, estimation instead of testing.

Concluding

Thank you!

Image credit: Gemini; prompt: *Generate an image with the following prompt "Computational lower bounds via almost orthonormal polynomials". Do not put text, and make the background white.*



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Proof sketch

1. Clarifications on low-degree (if needed);
2. what is the candidate basis;
3. invariance ideas;
4. proving almost orthonormality in practice.

Performance in hypothesis test

Idea: minimize type I and type II errors over a *class of functions*.

Statistical optimality

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Poly-time optimality

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If $1 - \Omega(1) = err_{IT}(\lambda, \eta, k) < err_{poly}(\lambda, \eta, k) = 1 - o(1)$ then η is hard.

The low-degree method & conjecture

Main issue: no clear idea of how to tackle poly-time tests so:

Conjecture [Hop18]

Poly-time test functions are less powerful than test functions thresholding polynomials of degree $D \lesssim \log n$:

$$err_{IT}(\lambda, \eta, k) \leq err_{LD}(\lambda, \eta, k) \overset{\text{conjecture}}{\leq} err_{poly}(\lambda, \eta, k),$$

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Statistical-to-computational gap

If $err_{IT}(\lambda, \eta, k)$ is small and $err_{LD}(\lambda, \eta, k)$ is **large then so is** $err_{poly}(\lambda, \eta, k)$ and we have a gap.

Last simplification: bounding the advantage

Main issue: Working on $err_{LD}(\lambda, \eta, k)$ is hard, find a surrogate condition.

Advantage bound

If:

$$Adv(\lambda, \eta, k) := \sup_{f:\deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}} = 1 + o(1),$$

then **expect** $err_{LD}(\lambda, \eta, k) \geq 1 - o(1)$.

Statistical-to-computational gap

Show $Adv(\lambda, \eta, k)$ is bounded, **extrapolate** $err_{LD}(\lambda, \eta, k)$ is large, **conjecture** $err_{poly}(\lambda, \eta, k)$ is large.

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In planted sub-matrix: $(\lambda, k) = (0, 0)$ is such that $Y_{ij} \overset{i.i.d.}{\sim} \mathrm{Rad}\left(\frac{1}{2}\right)$.

$$x_i \overset{i.i.d.}{\sim} \sqrt{0}\mathrm{Ber}(0) \text{ for all } i \in [n] \implies Y_{ij} = \begin{cases} 1 & \text{with probability } \frac{1+0}{2} \\ -1 & \text{with probability } \frac{1-0}{2} \end{cases}.$$

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Orthonormal basis for pure noise

$$Y^G = \prod_{(i,j)\in E} Y_{ij},$$

for all $G = (V, E)$ labelled sub-graphs of the $n \times n$ complete graph.

Correlations in canonical basis

$$
\begin{aligned}
\mathbb{E}_{(0,0)}\left[Y^G Y^{G'}\right] &= \mathbb{E}_{(\lambda,k)=(0,0)}\left[\prod_{(i,j)\in E,E'} \underbrace{Y_{ij}^2}_{=1\text{ a.s.}} \prod_{(i,j)\in E \text{ only}} Y_{ij} \prod_{(i,j)\in E' \text{ only}} Y_{ij}\right] \\
&= \mathbb{E}_{(0,0)}\left[\mathbb{E}\left[\prod_{(i,j)\in E \text{ only}} Y_{ij} \prod_{(i,j)\in E' \text{ only}} Y_{ij} \mid (x_i x_j)_{i,j\in[n]}\right]\right] \\
&= \mathbb{E}_{(0,0)}\left[\prod_{(i,j)\in E \text{ only}} X_{ij} \prod_{(i,j)\in E' \text{ only}} X_{ij}\right] \\
&= \delta_{G=G'}.
\end{aligned}
$$

The orthonormal trick to bound *Adv*

If we have an orthonormal basis in H_0 can **decompose functions along such basis**:

$$f(Y) = \sum_{G \in \text{basis}} \alpha_G Y^G, \qquad \alpha_G = \mathbb{E}_{(\lambda,k)=(0,0)}\left[f(Y)Y^G\right].$$

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Rewrite the advantage

$$Adv(\lambda = 0, \eta, k = 0) = \sup_{f: deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}}$$

Decompose in Y^G orthonormal basis numerator and denominator.

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$$= \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{G \in \text{basis}} \alpha_G Y^G\right]}{\|\alpha\|_2} = LinAdv((\eta, k), (\lambda, k))$$

by orthonormality the denominator simplifies and the **advantage is a linear function**.

Outside of pure noise $(\lambda, k), \lambda, k \neq 0$

Question: prove when cannot detect perturbations in poly-time

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Conjecturally hard when decomposition along basis of $Adv(\lambda, \eta, k)$ but at $(\lambda, k), \lambda, k \neq 0$ **not explicit ortho basis**!

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Problem

When $(\lambda, k), \lambda, k \neq 0$ the basis $\{Y^G\}_G$ is **not orthonormal**!

$$\mathbb{E}_{H_0:(\lambda,k)}\left[Y^G Y^{G'}\right] = \lambda^{\#\text{edges in symm. diff.}} \left(\frac{k}{n}\right)^{\#\text{vertices symm. diff}}.$$

No explicit formula. No linearization of advantage. In literature complicated recursive implicit solutions [SW22; SW25].

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Basic idea: decomposition along basis of $Adv(\lambda, \eta, k)$ but at $(\lambda, k), \lambda, k \neq 0$ **not explicit ortho basis**!
In literature implicit recursive solutions [SW22; SW25].

Outside of pure noise, $(\lambda, k), \lambda, k \neq 0$

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Our solution: almost orthonormal basis

Find a collection of functions $(\psi_G)_G$ forming a basis of H_0, H_1:

$$f(Y) = \sum_{G \in \text{ basis}} \alpha_G \psi_G,$$

such that for some constants:

$$c \|\alpha\|_2^2 \leq \mathbb{E}_{(\lambda,k)} \left[\sum_{G, G' \in \text{basis}} \alpha_G \alpha_{G'} \psi_G \psi_{G'} \right] \leq C \|\alpha\|_2^2 .$$

Adjusting the orthonormal basis

As we said the basis:

$$(Y^G)_G, \qquad Y^G = \prod_{(i,j)\in E} Y_{ij},$$

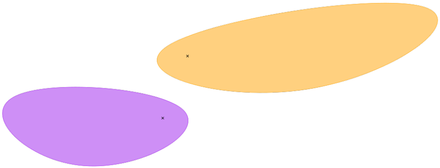
is **not orthonormal** when $(\lambda, k), \lambda, k \neq 0$, and has correlations dep. on symm. diff.:

$$\mathbb{E}_{H_0:(\lambda,k)}\left[Y^G Y^{G'}\right] = \lambda^{|E_{G\triangle G'}|}\left(\frac{k}{n}\right)^{V_{G\triangle G'}}.$$

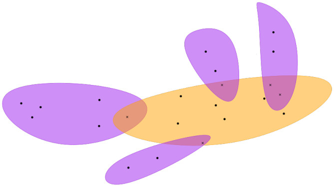
Rough intuition

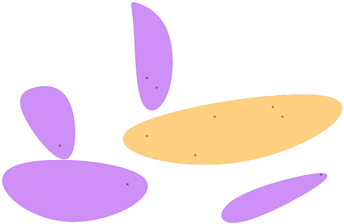
Adjust the Y^G basis to decrease correlations.
Use independence of random variables when G and G' are disconnected (latent Bernoullis $(x_i x_j)^G_{i,j\in[n]}, (x_i x_j)^{G'}_{i,j\in[n]}$ are indep.).

Some visuals with blobs of vertices



The two graphs correlate $\mathbb{E}_{H_0:(\lambda,k)}\left[Y^G Y^{G'}\right] \neq 0$.





The two graphs correlate $\mathbb{E}_{H_0:(\lambda,k)}\left[Y^G Y^{G'}\right] \neq 0$ in different ways.

Partial adjustment

Centered basis

The basis:

$$\widehat{Y}^G := Y^G - \mathbb{E}_{H_0}\left[Y^G\right],$$

correlates less than Y^G.

Indeed:

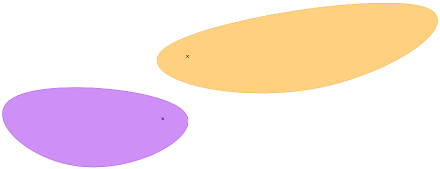
$$\mathbb{E}_{H_0:(\lambda,k)}\left[\widehat{Y}^G \widehat{Y}^{G'}\right] = \mathbb{E}_{H_0:(\lambda,k)}\left[Y^G Y^{G'}\right] - \mathbb{E}_{H_0:(\lambda,k)}\left[Y^G\right]\mathbb{E}_{H_0:(\lambda,k)}\left[Y^{G'}\right]$$

If G, G' are disconnected then $G \triangle G' = G \cup G'$ and the correlation is zero.

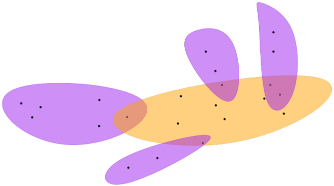
But this is not enough

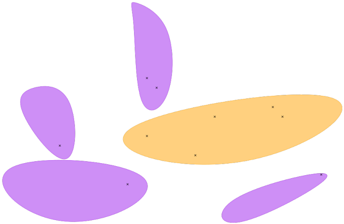
Can use more independence to zero out correlations.

Some visuals with blobs of vertices



The two graphs **do not** correlate
$\mathbb{E}_{H_0:(\lambda,k)}\left[\widehat{Y}^G\widehat{Y}^{G'}\right] = 0.$





Above graphs correlate
$\mathbb{E}_{H_0:(\lambda,k)}\left[\widehat{Y}^G\widehat{Y}^{G'}\right] \neq 0$ below do not.

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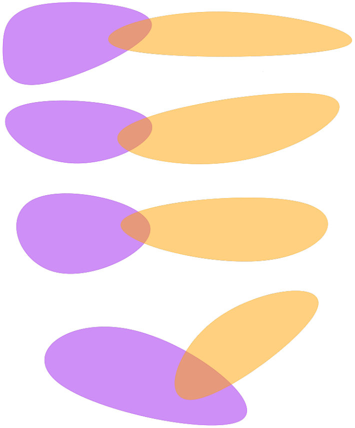

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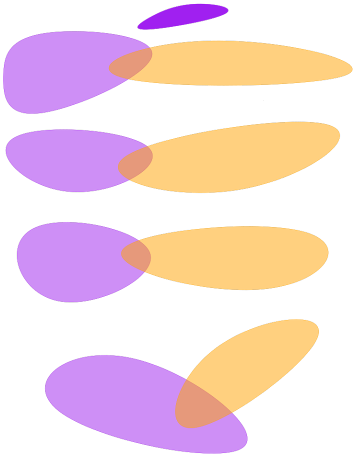


The two graphs correlate
$\mathbb{E}_{H_0:(\lambda,k)}\left[\widehat{Y}^{G}\widehat{Y}^{G'}\right] \neq 0$

Final fix

Basis proposal

The basis:

$$\overline{Y}^G := \prod_{\ell=1}^{m} Y^{G_\ell} - \mathbb{E}_{H_0:(\lambda,k)}\left[Y^{G_\ell}\right], \qquad G = (G_\ell)_{\ell=1}^m \text{ conn. comp.}$$

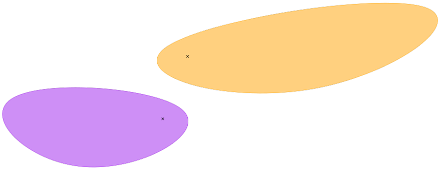
Correlates less than Y^G, $\widehat{Y}^G$.

Imagine G, G' have shared edges/vertices (so $\mathbb{E}_{H_0:(\lambda,k)}\left[\widehat{Y}^G \widehat{Y}^{G'}\right] \neq 0$), but one conn. comp. in G is isolated from all of G', then:

$$\mathbb{E}_{H_0:(\lambda,k)}\left[\overline{Y}^G \overline{Y}^{G'}\right] = \mathbb{E}_{H_0:(\lambda,k)}\left[\overline{Y}^{G\setminus G_{\ell^\star}} \overline{Y}^{G'}\right] \mathbb{E}_{H_0:(\lambda,k)}\left[\overline{Y}^{G_{\ell^\star}}\right] = 0,$$

since in the basis we center connected components.

Some visuals with blobs of vertices



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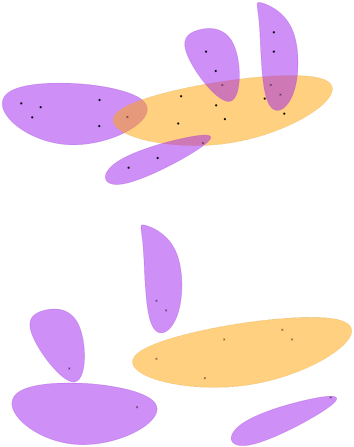

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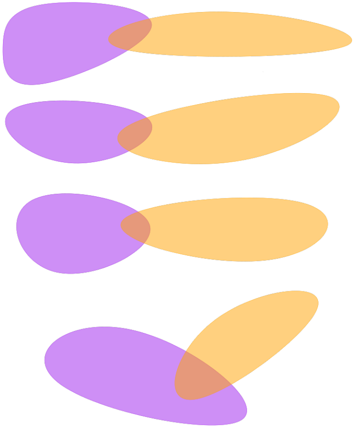

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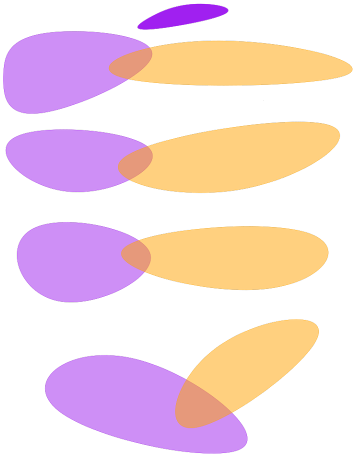


The two graphs **do not** correlate
$\mathbb{E}_{H_0:(\lambda,k)}\left[\overline{Y}^{G}\overline{Y}^{G'}\right] = 0$

Making counting easier

The basis $(\overline{Y}^G)_G$ runs over **all labelled sub-graphs** of the complete n-graph that have less than $\lesssim \log n$ edges (degree constraint).

Counting matters

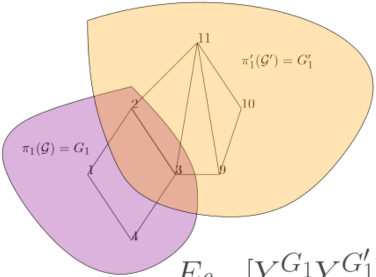
Enumerating such graphs is tedious, plus, the correlations depend on the symmetric difference:

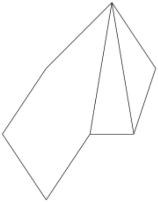
If two different labelled pairs (G_1, G_1'), (G_2, G_2') are such that $G_1 \simeq G_2$, $G_1' \simeq G_2'$ and they have the same symmetric difference then we count them twice.

Rough intuition

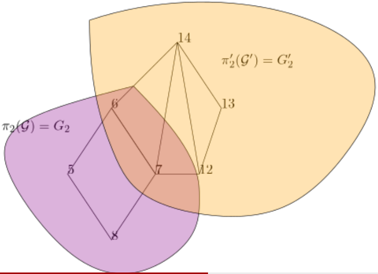
Group isomorphic graphs together and see how the equivalence classes generate different symmetric differences.

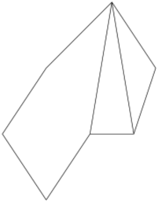
Visualization of symmetric difference





$$E_{\theta_{H_0}}[Y^{G_1}Y^{G'_1}] = E_{\theta_{H_0}}[Y^{G_2}Y^{G'_2}] = \lambda^9 \left(\frac{k}{n}\right)^7$$





Formalizing

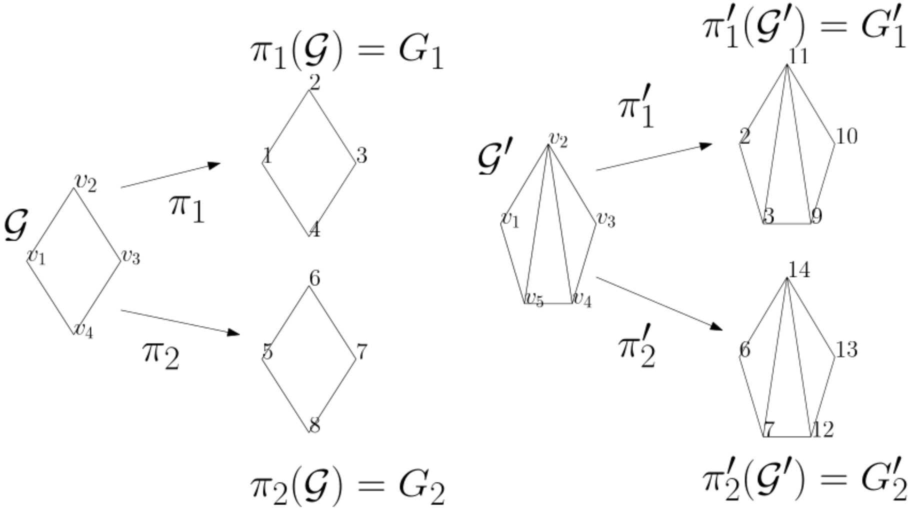
Collect G into labellings of graphs from an abstract space, i.e. $G = \pi(\mathcal{G})$ for some abstract $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and $\pi : \mathcal{V} \mapsto [n]$ a **labelling**.
For two labellings $\pi(\mathcal{G}), \pi'(\mathcal{G})$ we have two graphs that come from the same "shape" in the abstract space.
Form the basis:

$$\overline{Y}^{\mathcal{G}} := \sum_{\pi \text{ labellings}} \overline{Y}^{\pi(\mathcal{G})}, \qquad \text{for all abstract graphs } \mathcal{G}.$$

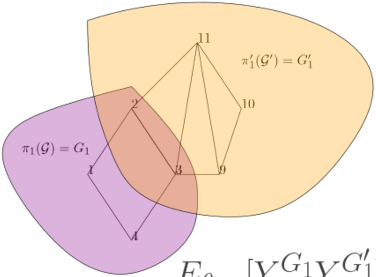
Double result

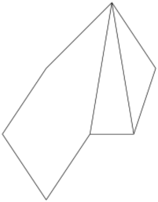
$\{\overline{Y}^{\mathcal{G}}\}_{\mathcal{G}}$ is a basis of **perm. invariant** polynomials, but the advantage $Adv(\lambda, \eta, k)$ is attained by a perm. invariant polynomial since H_0, H_1 are perm. invariant distributions.
No loss by working on this basis, plus simplified counting!

Visualization of labellings



Visualization of symmetric difference





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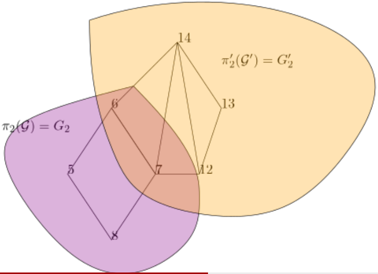

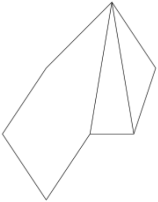

Almost orthonormality exemplified

Recall that we want to write:

$$Adv(\lambda, \eta, k) = \sup_{f : deg(f) \lesssim \log n} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}}$$

Use that Adv is attained by invariant polynomial.

Almost orthonormality exemplified

Recall that we want to write:

$$Adv(\lambda, \eta, k) = \sup_{\substack{f: deg(f) \lesssim \log n \\ f\ perm.invariant}} \frac{\mathbb{E}_{H_1}[f(Y)]}{\sqrt{\mathbb{E}_{H_0}[f(Y)^2]}}$$

Decompose along perm. invariant basis $\{\overline{Y}^{\mathcal{G}}\}$.

Almost orthonormality exemplified

Recall that we want to write:

$$Adv(\lambda, \eta, k) = \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{\mathcal{G}} \alpha_G \overline{Y}^{\mathcal{G}}\right]}{\sqrt{\mathbb{E}_{H_0}\left[\sum_{\mathcal{G},\mathcal{G}'} \alpha_{\mathcal{G}} \alpha_{\mathcal{G}'} \overline{Y}^{\mathcal{G}} \overline{Y}^{\mathcal{G}'}\right]}}$$

Use almost orthonormality to linearize.

Almost orthonormality exemplified

Recall that we want to write:

$$
\begin{aligned}
Adv(\lambda,\eta,k) &= \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{\mathcal{G}} \alpha_G \overline{Y}^{\mathcal{G}}\right]}{\sqrt{\mathbb{E}_{H_0}\left[\sum_{\mathcal{G},\mathcal{G}'} \alpha_{\mathcal{G}}\alpha_{\mathcal{G}'} \overline{Y}^{\mathcal{G}}\overline{Y}^{\mathcal{G}'}\right]}} \\
&\leq \frac{1}{\sqrt{c}} \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{\mathcal{G}} \alpha_{\mathcal{G}} \overline{Y}^{\mathcal{G}}\right]}{\|\alpha\|} \\
&= \frac{1}{\sqrt{c}} LinAdv(\lambda,\eta,k).
\end{aligned}
$$

How to establish

$$
c\,\|\alpha\| \leq \mathbb{E}_{H_0:(\lambda,k)}\left[\sum_{\mathcal{G},\mathcal{G}'} \alpha_{\mathcal{G}}\alpha_{\mathcal{G}'} \overline{Y}^{\mathcal{G}}\overline{Y}^{\mathcal{G}'}\right] \leq C\,\|\alpha\|?
$$

Preliminary

The basis $\{\overline{Y}^{\mathcal{G}}\}$ is **not normalized** because we sum over many graphs, indeed:

$$\overline{Y}^{\mathcal{G}} = \underbrace{\sum_{\pi \text{ labellings}}}_{\textbf{exploding number of labellings}} \underbrace{\overline{Y}^{\pi(\mathcal{G})}}_{\text{norm order one}},$$

so we **need to normalize it**.

Technical

There exist a way to normalize the basis by rescaling $\overline{Y}^{\mathcal{G}}$ into $\widetilde{Y}^{\mathcal{G}} = \frac{\overline{Y}^{\mathcal{G}}}{\sqrt{\nu(\mathcal{G})}}$ such that:

$$\mathbb{E}_{H_0:(\lambda,k)}\left[(\widetilde{Y}^{\mathcal{G}})^2\right] = \mathbb{E}_{H_0:(\lambda,k)}\left[\frac{(\overline{Y}^{\mathcal{G}})^2}{\nu(\mathcal{G})}\right] \approx \text{constant order.}$$

Now all the variances are of the same size.

For the rescaled basis $\widetilde{Y}^{\mathcal{G}} = \frac{\overline{Y}^{\mathcal{G}}}{\sqrt{\nu(\mathcal{G})}}$ we then rewrite the denominator as a quadratic form:

$$\mathbb{E}_{H_0:(\lambda,k)}\left[\sum_{\mathcal{G},\mathcal{G}'} \alpha_{\mathcal{G}}\alpha_{\mathcal{G}'}\widetilde{Y}^{\mathcal{G}}\widetilde{Y}^{\mathcal{G}'}\right] = \alpha^{\top}\mathbb{E}_{H_0:(\lambda,k)}\left[\widetilde{Y}\widetilde{Y}^{\top}\right]\alpha,$$

where:

$$\mathbb{E}_{H_0:(\lambda,k)}\left[\widetilde{Y}\widetilde{Y}^{\top}\right] = \text{Gram matrix of correlations for } \{\widetilde{Y}^{\mathcal{G}}\}_{\mathcal{G}} \text{ basis.}$$

Aim

Show the eigenvalues of the Gram matrix are all constant.

Gershgorin criterion to the rescue

By Gershgorin circle theorem the eigenvalues of a Gram matrix are within the circles

$$\sup_i \left\{ G_{ii} \pm \sum_{j \neq i} |G_{ij}| \right\} = \sup_{\mathcal{G}} \left\{ \mathbb{E}_{H_0:(\lambda,k)} \left[\widetilde{Y}^{\mathcal{G}} \right] \pm \sum_{\mathcal{G}' \neq \mathcal{G}} \left| \mathbb{E}_{H_0:(\lambda,k)} \left[\widetilde{Y}^{\mathcal{G}} \widetilde{Y}^{\mathcal{G}'} \right] \right| \right\}.$$

So **show that it is a constant**.

Advantage of this view

Can go step-by-step, from correlations of labelled graphs $\pi(\mathcal{G})$, to correlations of abstract graphs $\sum_{\pi \in \text{ labellings}}$ and so on.

Steps for almost orthonormality via Gershgorin

1. the basis Y^G correlation is a symmetric difference, the candidate basis $\widetilde{Y}^{\pi(\mathcal{G})}$ correlations **approximate** symmetric differences;

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$$\mathbb{E}_{H_0:(\lambda,k)}\left[\widetilde{Y}^{\mathcal{G}}\widetilde{Y}^{\mathcal{G}'}\right] \lesssim (\log n)^{d(\mathcal{G},\mathcal{G}')},$$

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for some proper distance d between graphs;
3. summing over abstract graphs, the control by a distance is enough to show that the eigenvalues of the Gram matrix $\mathbb{E}_{H_0:(\lambda,k)}\left[\widetilde{Y}\widetilde{Y}^\top\right]$ are constant, and we have the almost orthonormality:

$$c\,\|\alpha\| \leq \mathbb{E}_{H_0:(\lambda,k)}\left[\sum_{\mathcal{G},\mathcal{G}'}\alpha_{\mathcal{G}}\alpha_{\mathcal{G}'}\widetilde{Y}^{\mathcal{G}}\widetilde{Y}^{\mathcal{G}'}\right] \leq C\,\|\alpha\|.$$

Bounding the advantage?

By almost orthonormality $\implies$ advantage takes linear form:

$$Adv(\lambda, \eta, k) \leq LinAdv(\lambda, \eta, k) = \frac{1}{\sqrt{c}} \sup_{\alpha} \frac{\mathbb{E}_{H_1}\left[\sum_{\mathcal{G}} \alpha_{\mathcal{G}} \widetilde{Y}^{\mathcal{G}}\right]}{\|\alpha\|},$$

which is easy to upper bound.