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Four lectures course: 4x2 hrs

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- Introduction to MOO framework and taxonomy
- Classical scalarization methods

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- Meta-heuristic methods
- Simulated annealing
- Swarm Particle
- Genetic algorithms

Lecture 3:

- Evaluation metrics
- Python packages

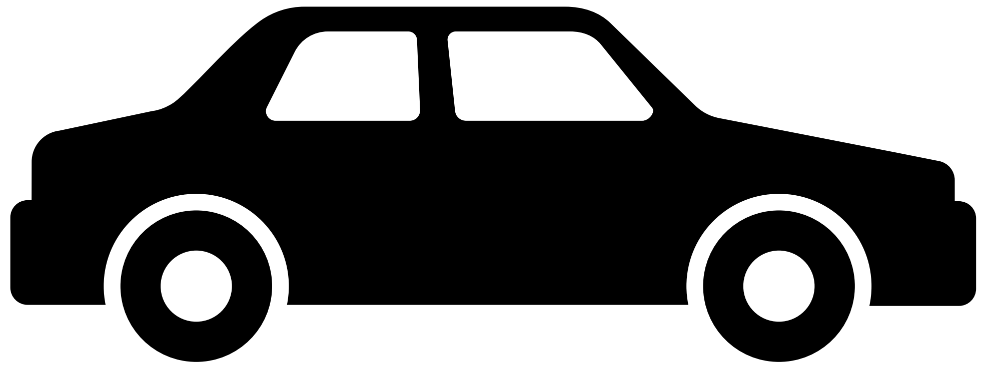
Lecture 4:

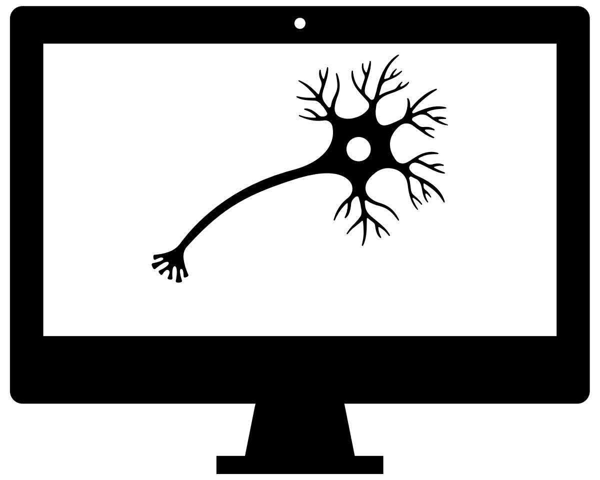
- Symbolic regression
- Multi-objective symbolic regression
- Case studies

Multi-Objective Optimization (MOO)

SCAN ME

MOO in almost every real-world application





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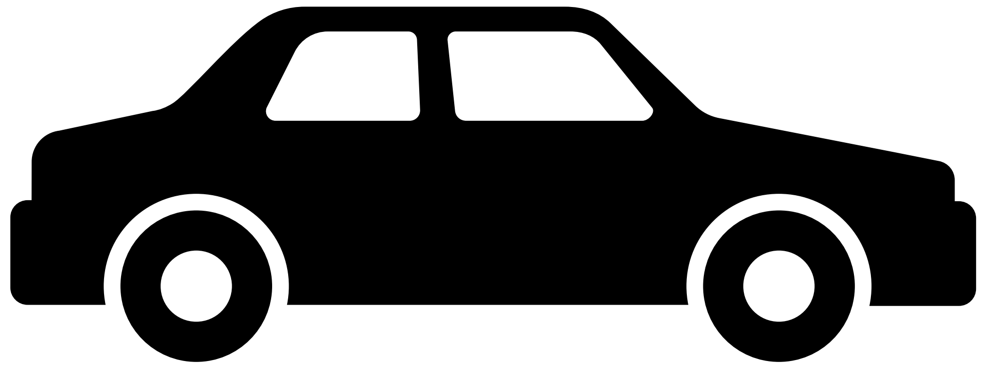

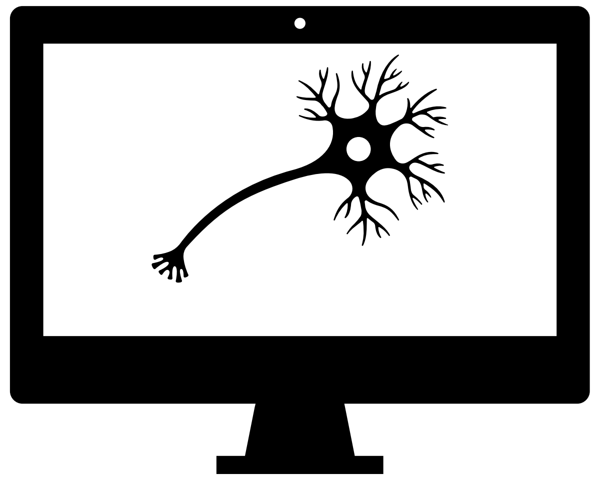

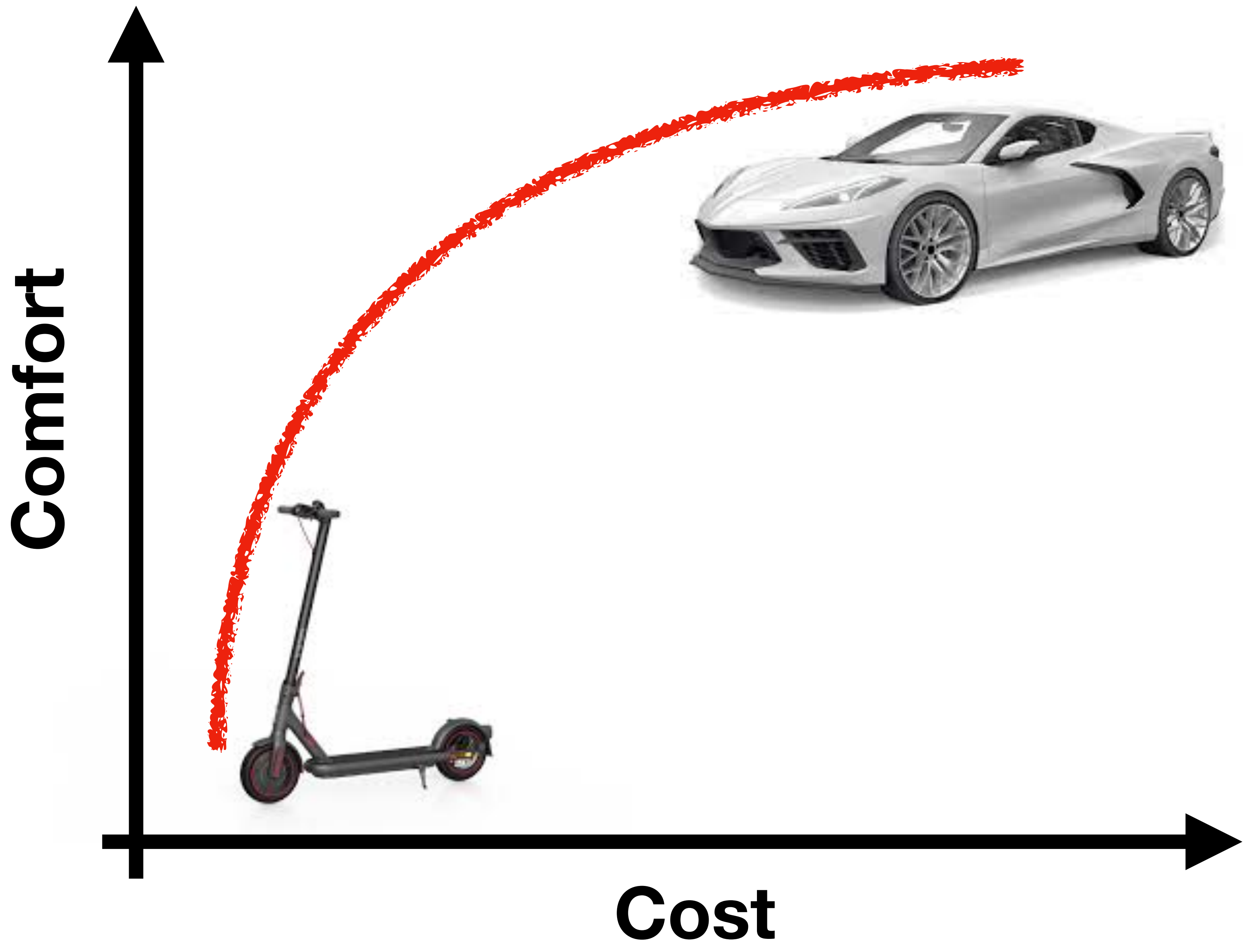

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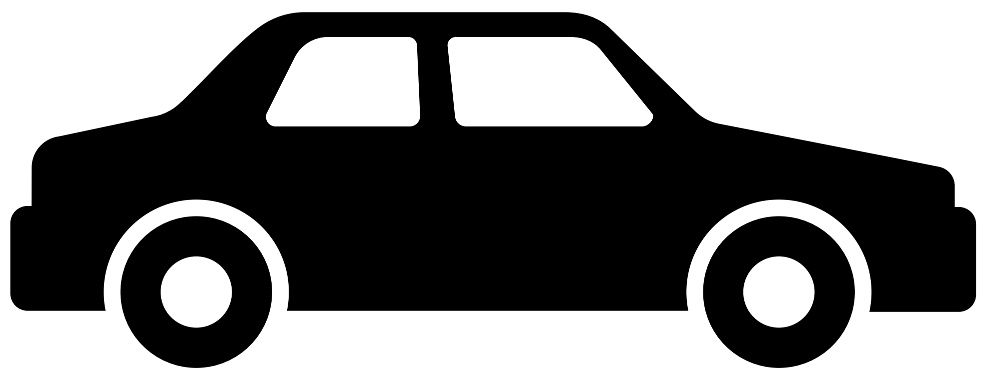

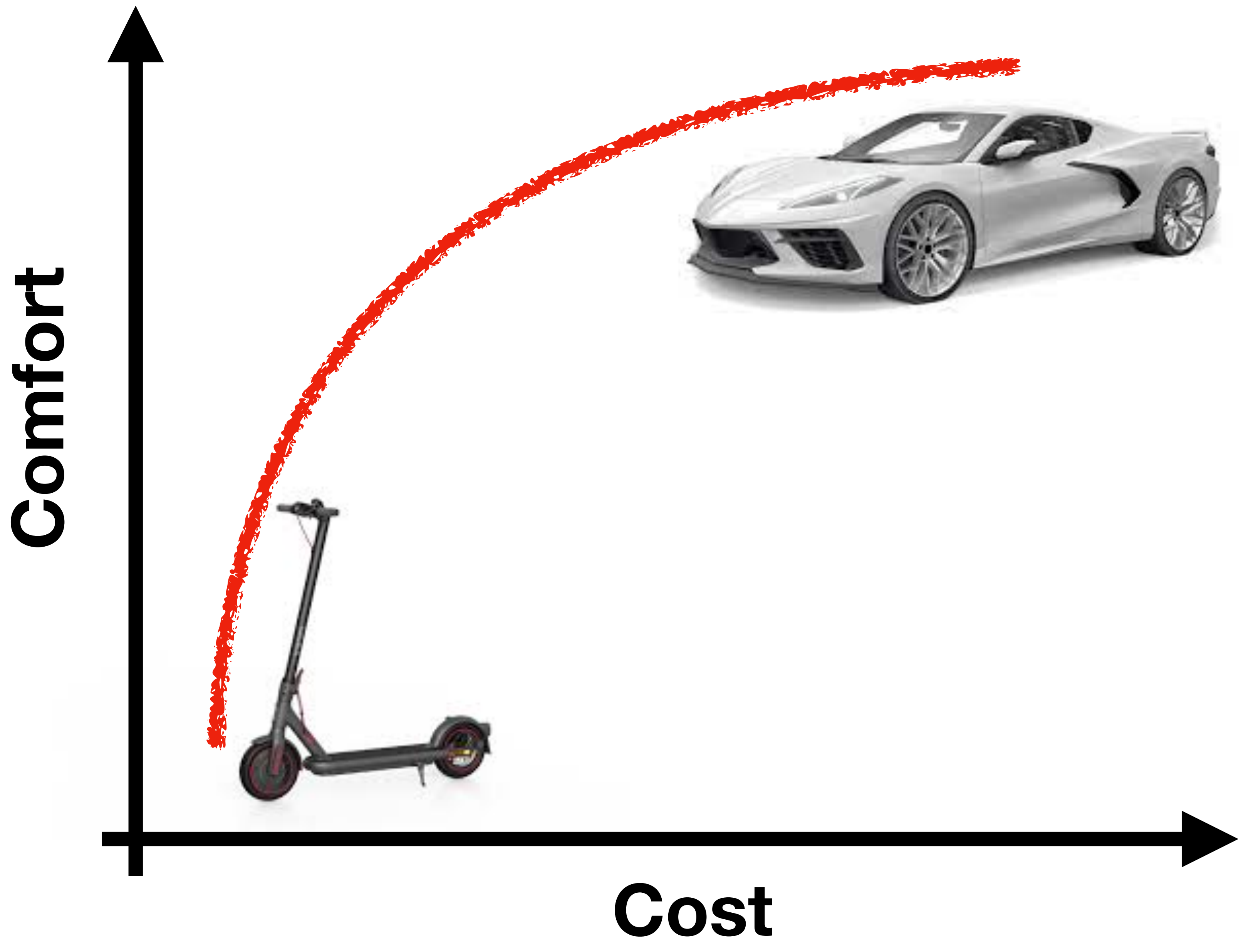

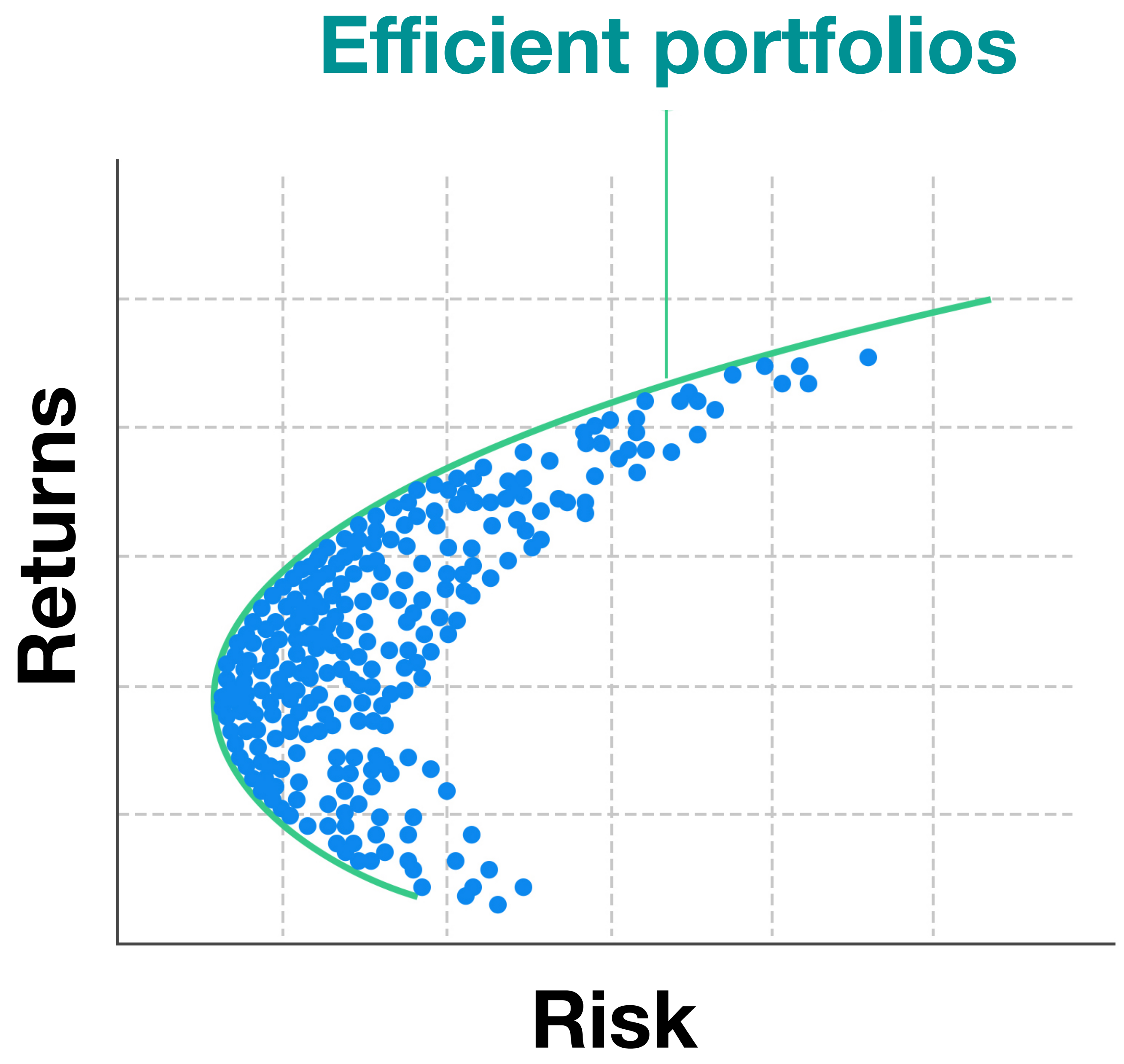

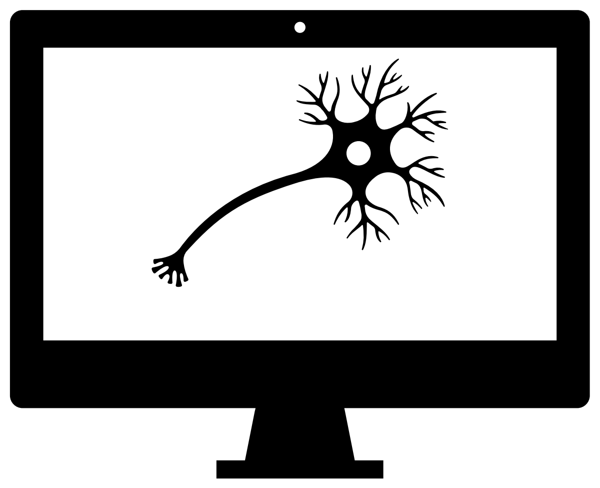

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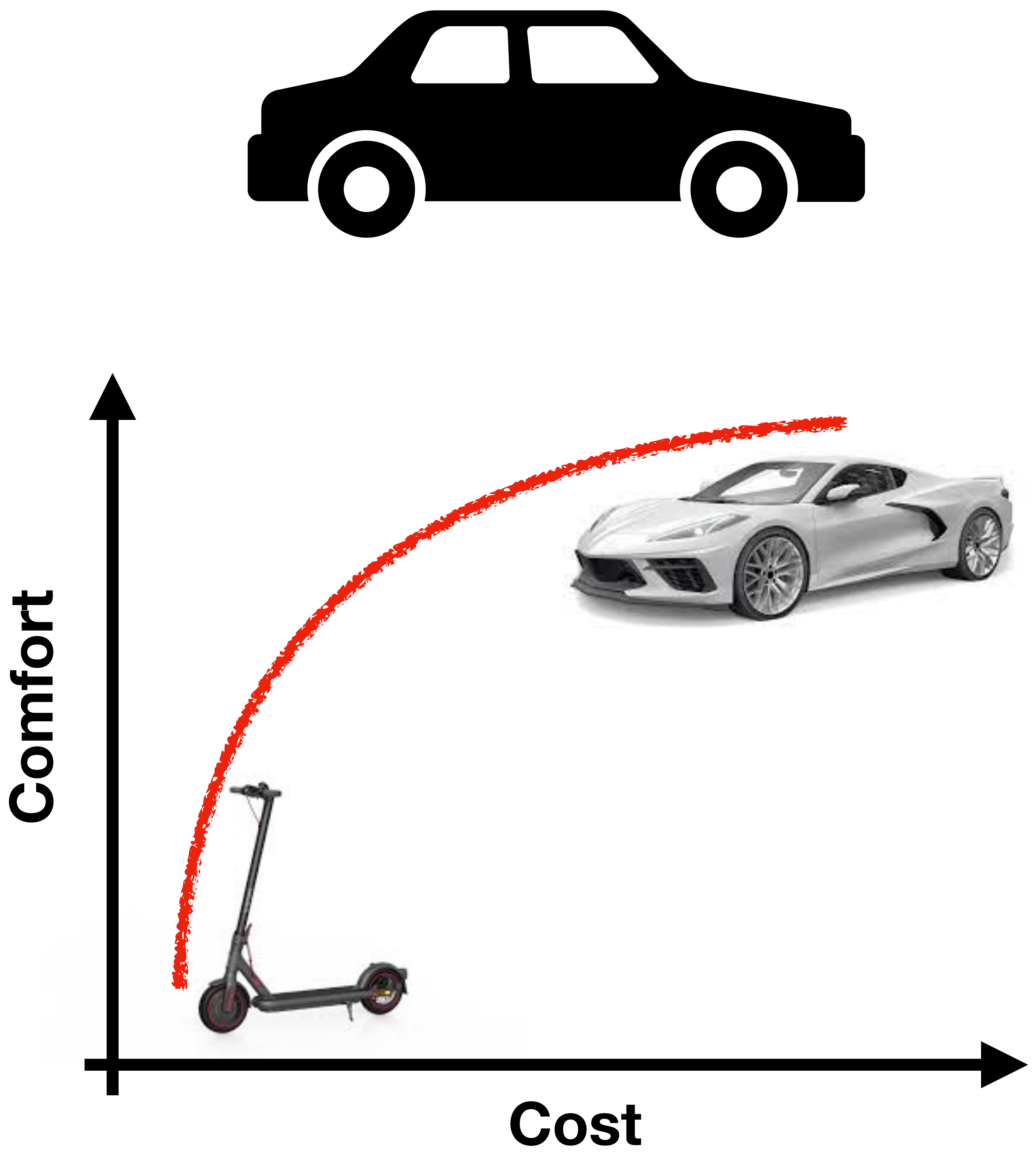
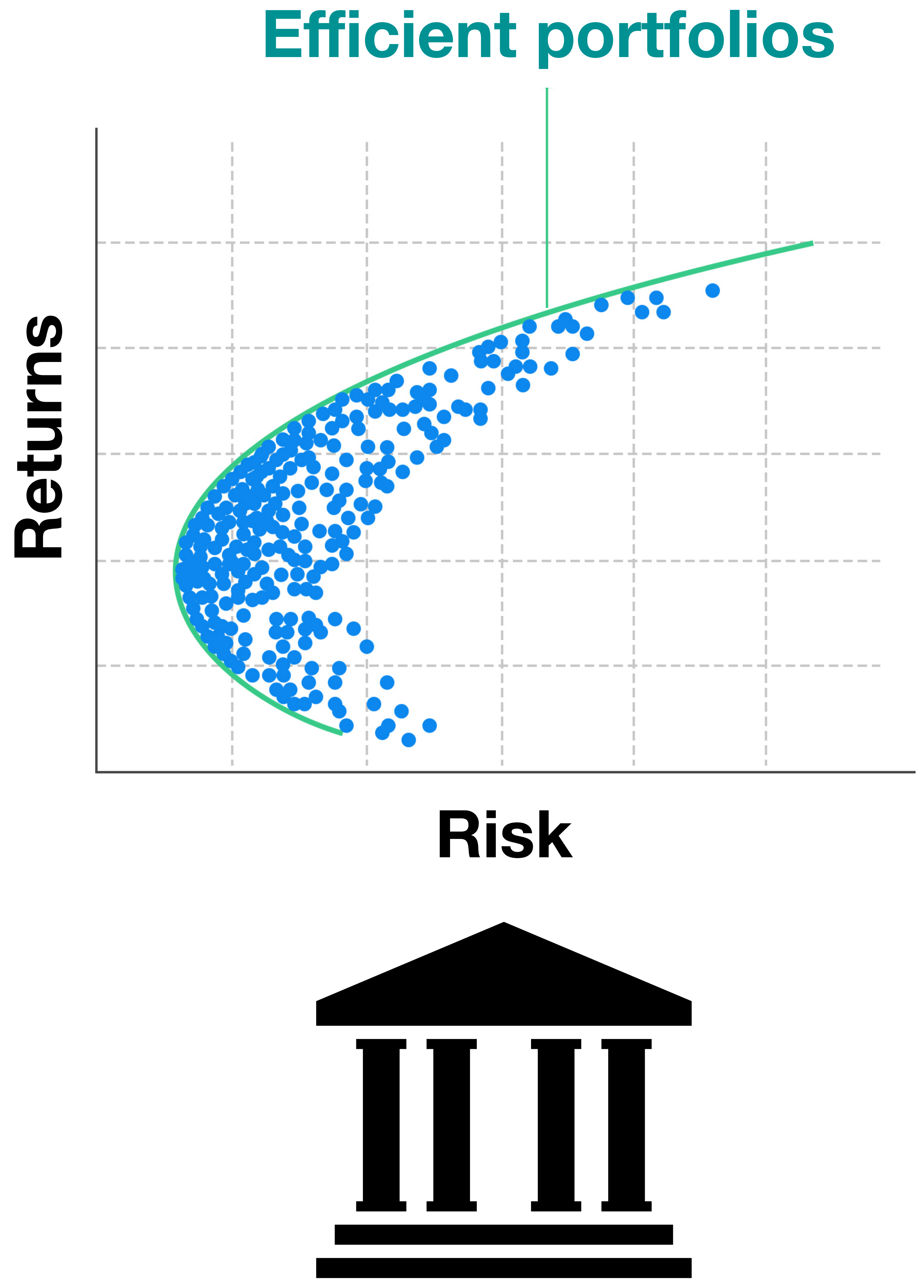
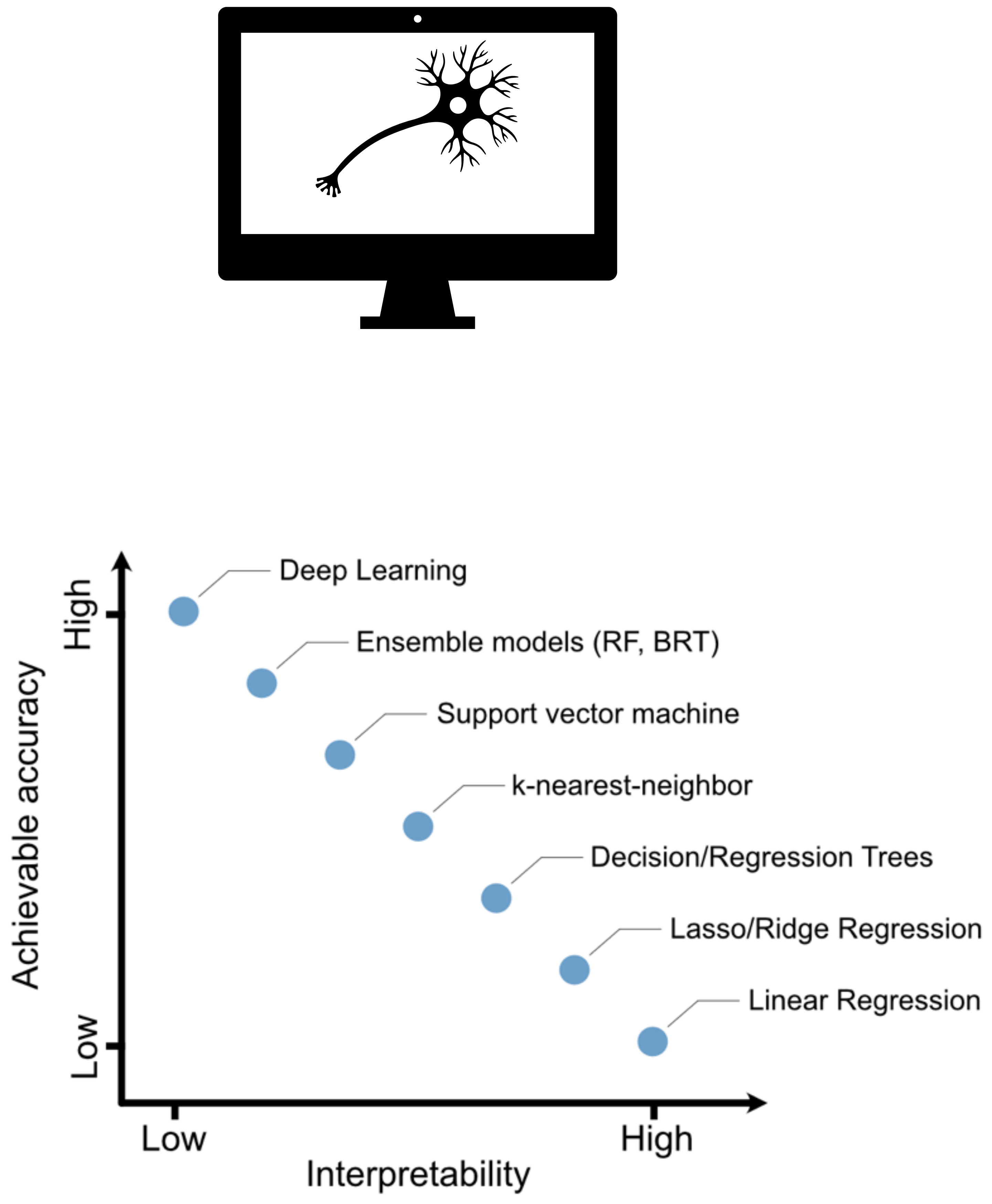
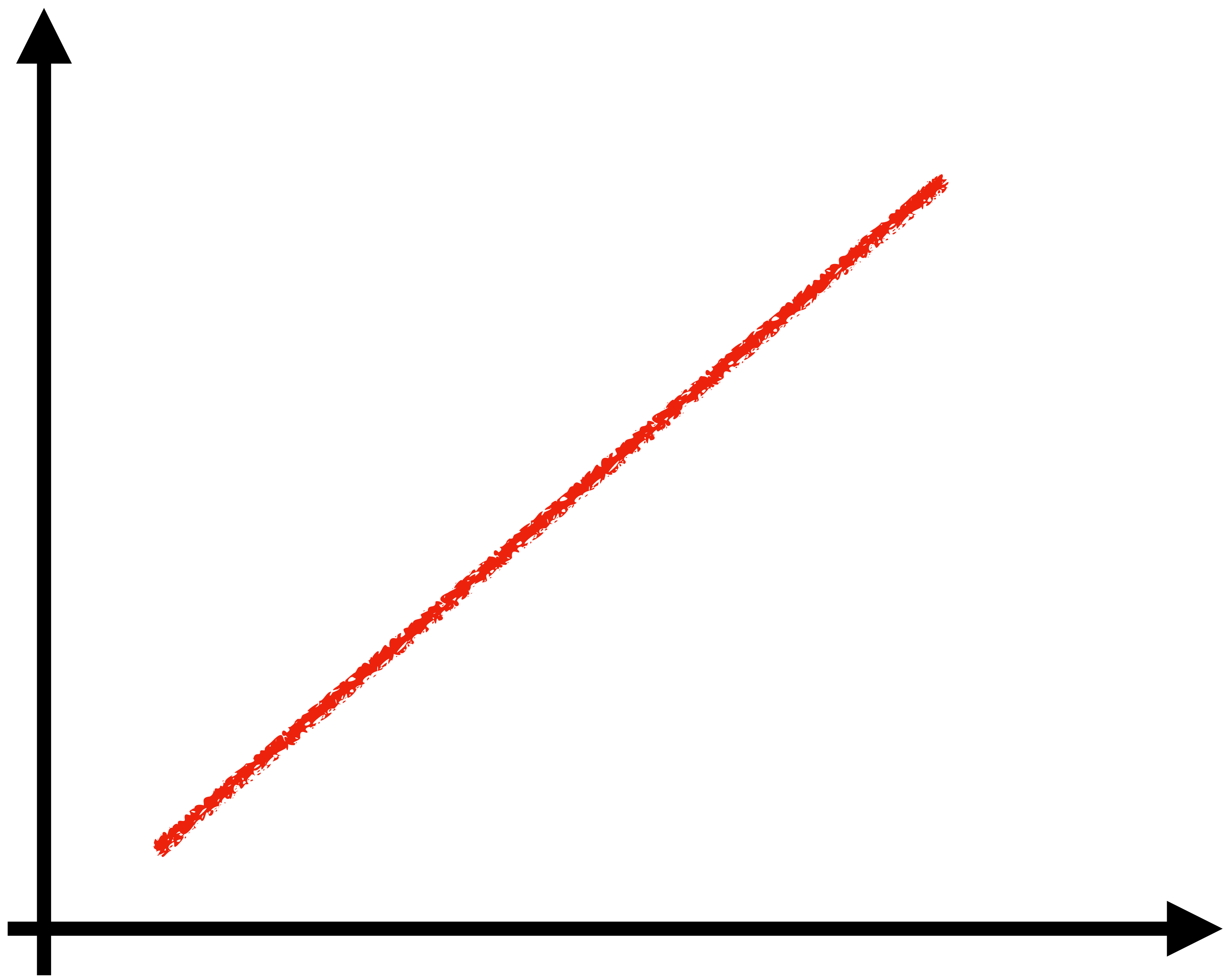






Objectives conflict to some degree



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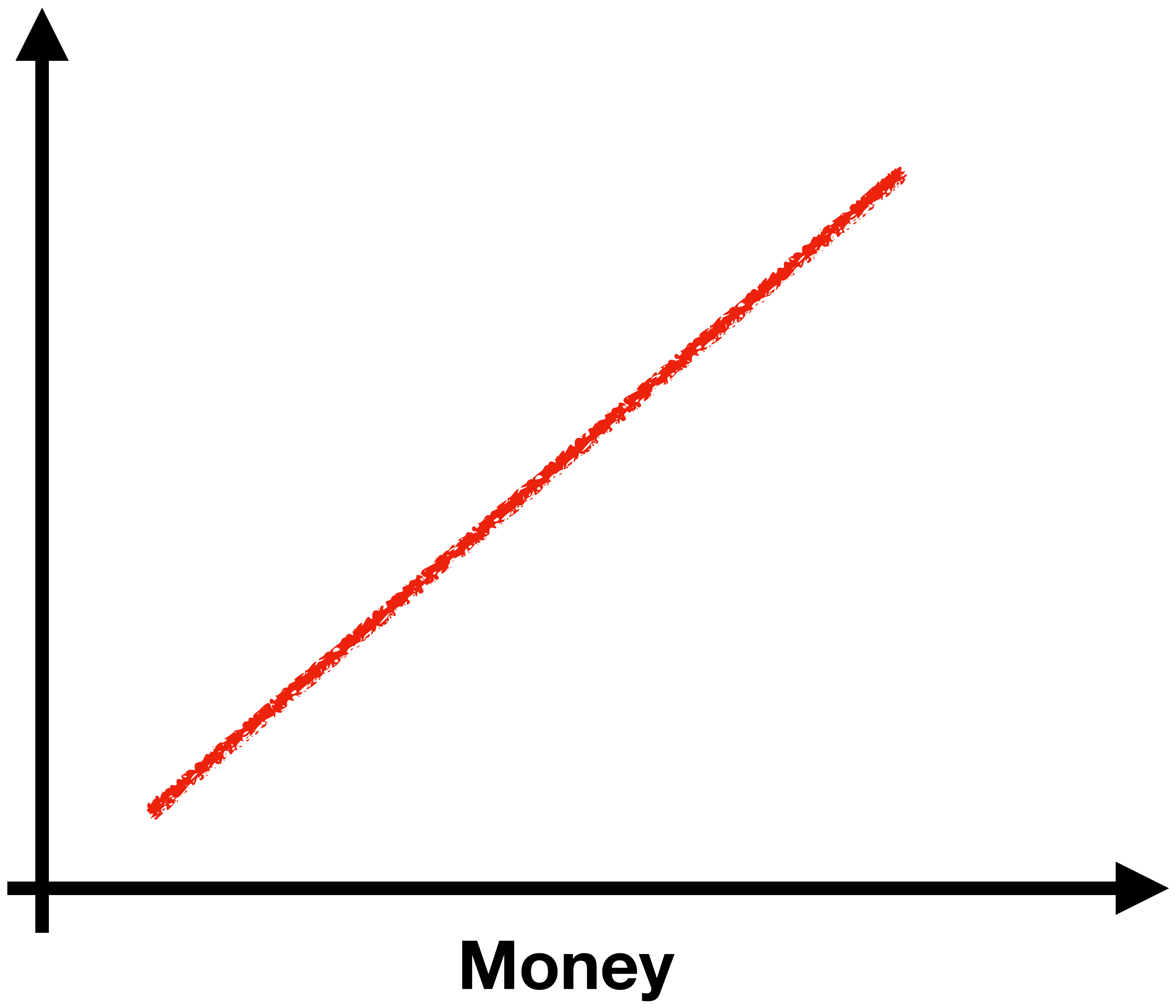

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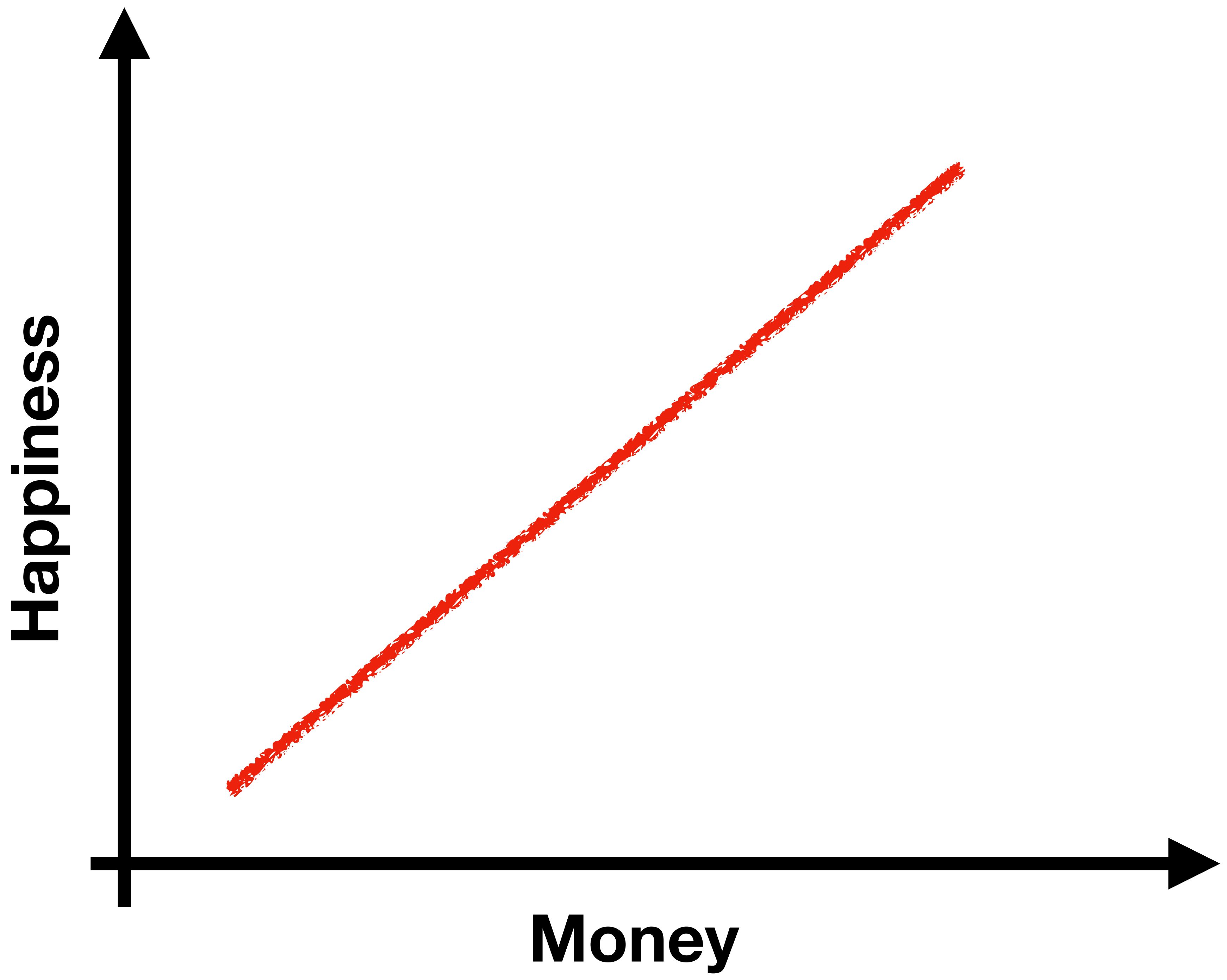

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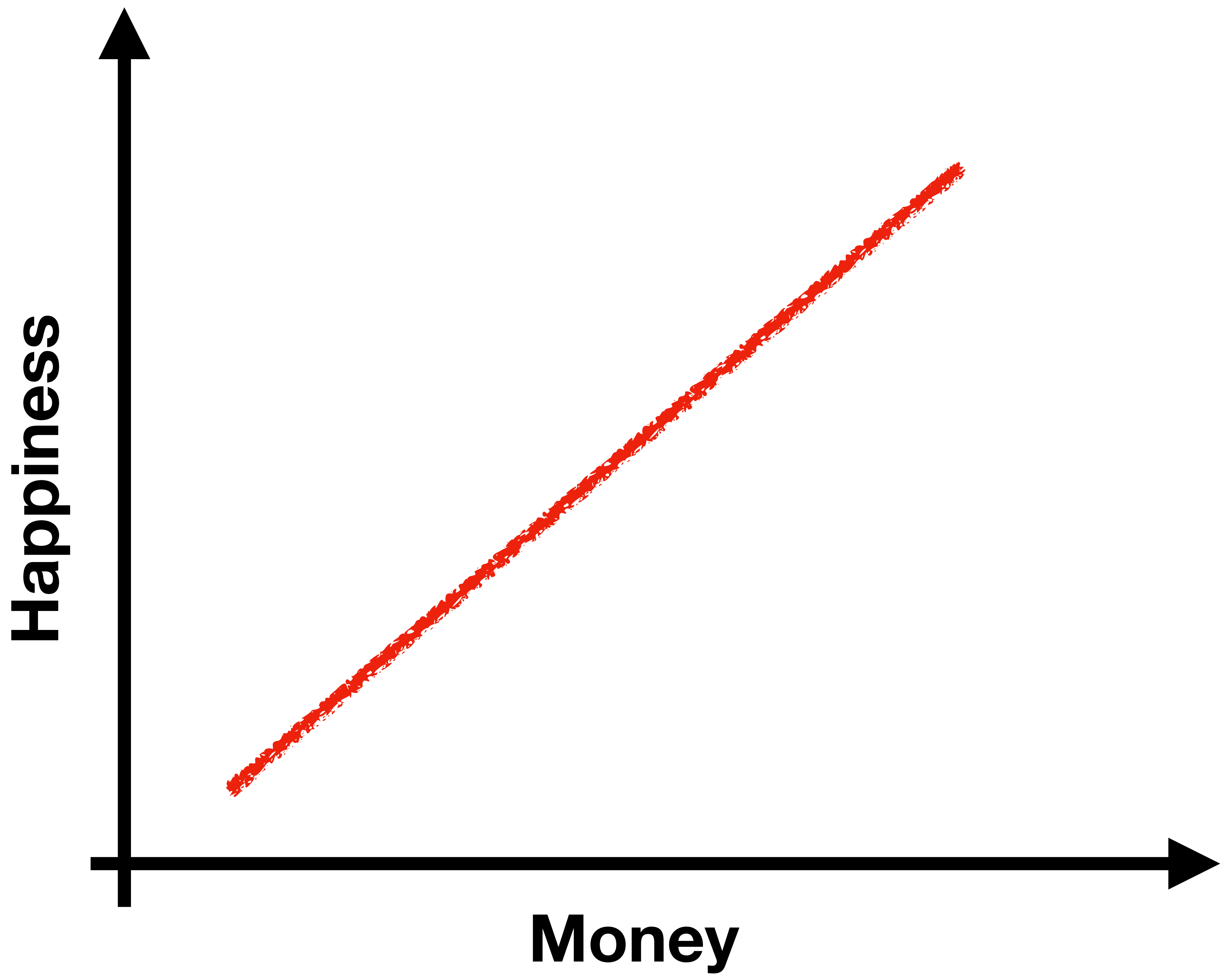

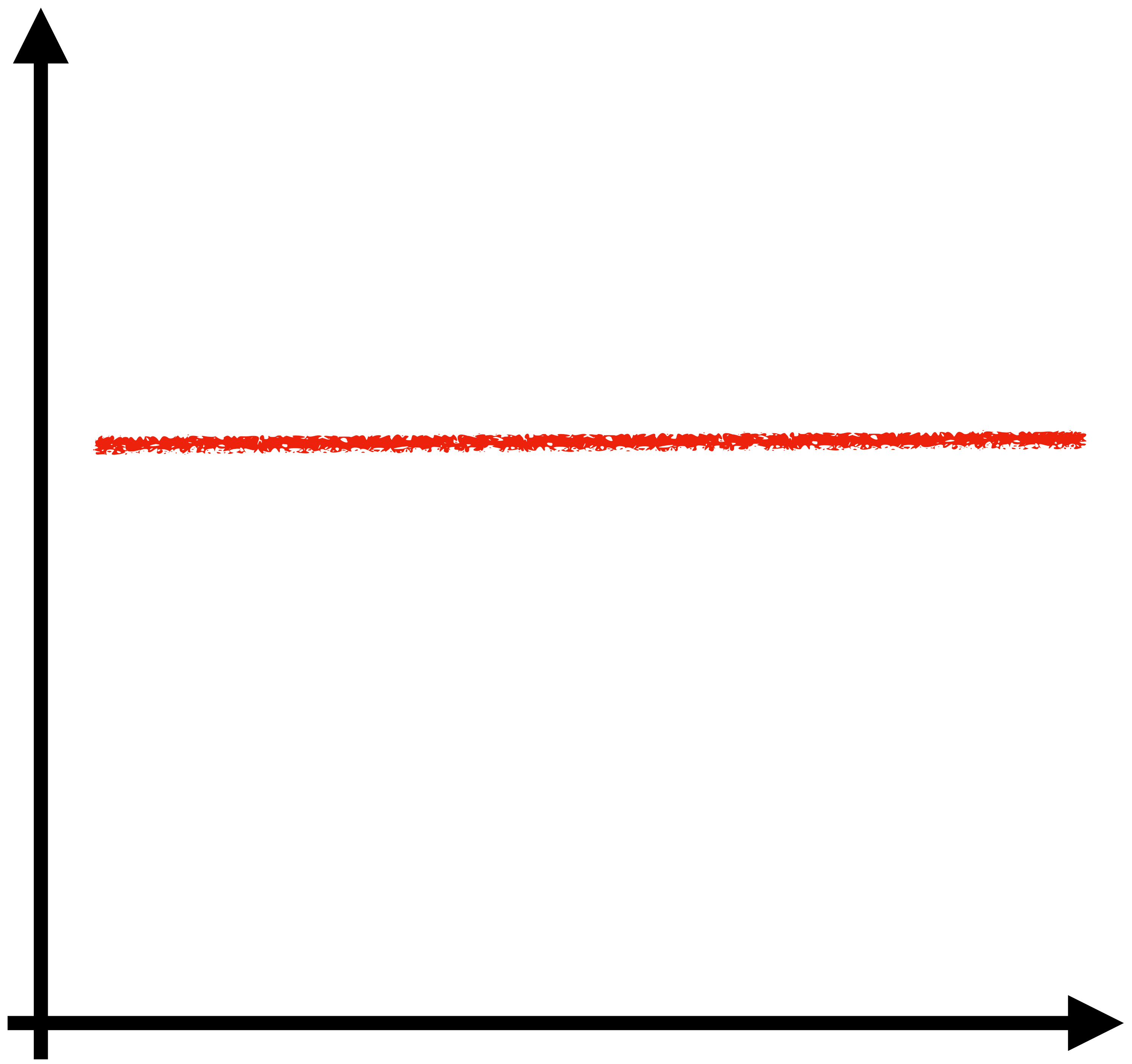

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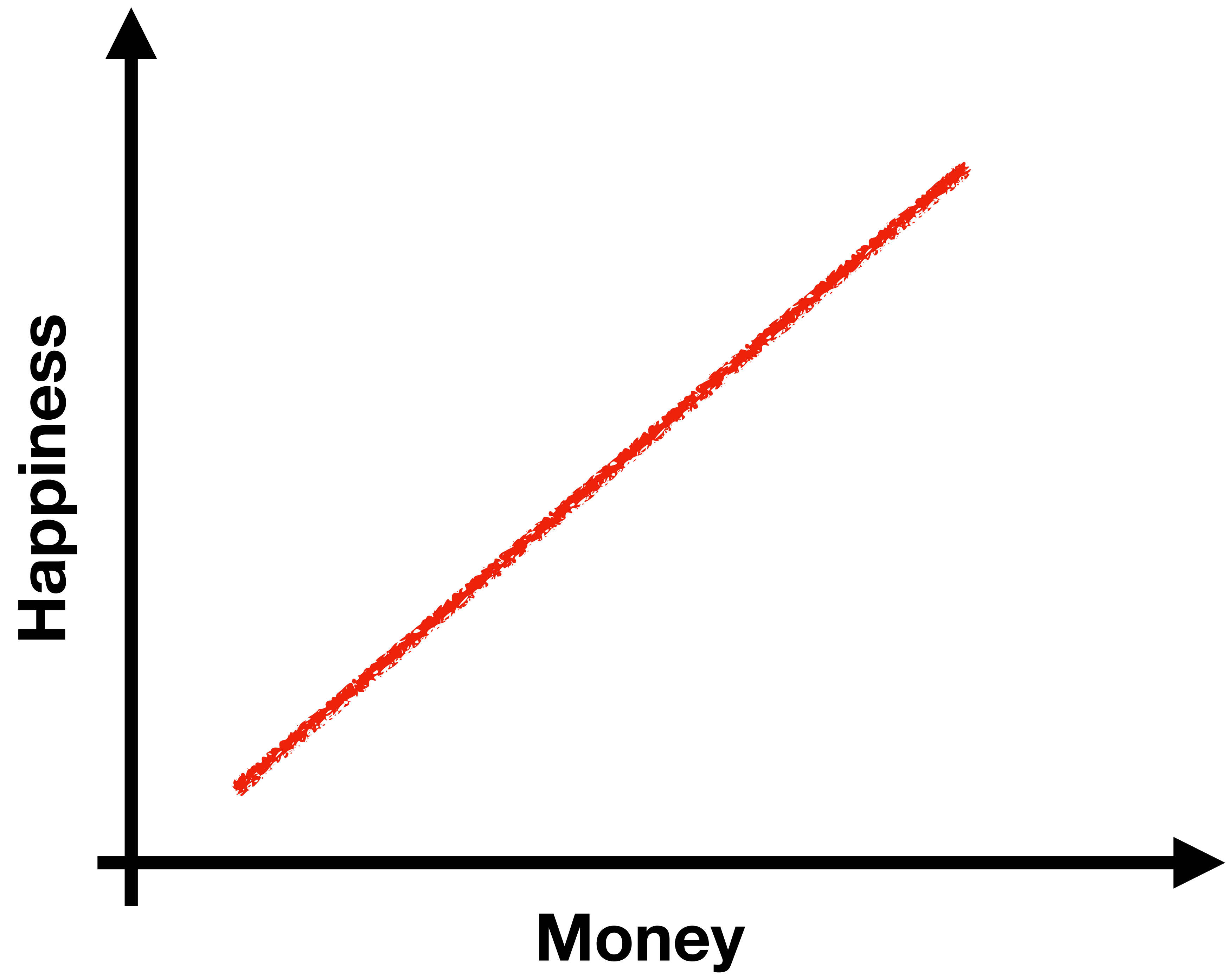

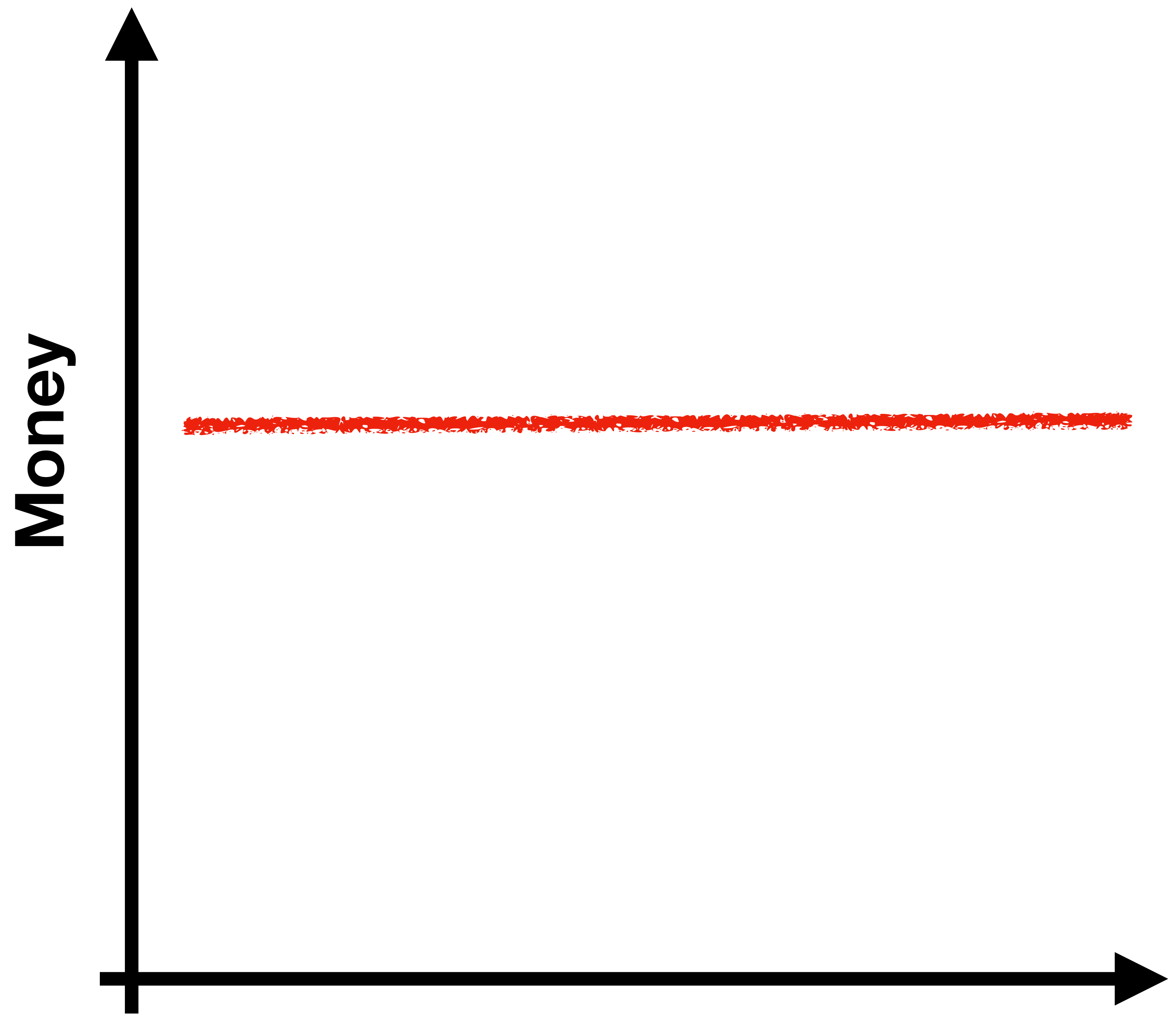

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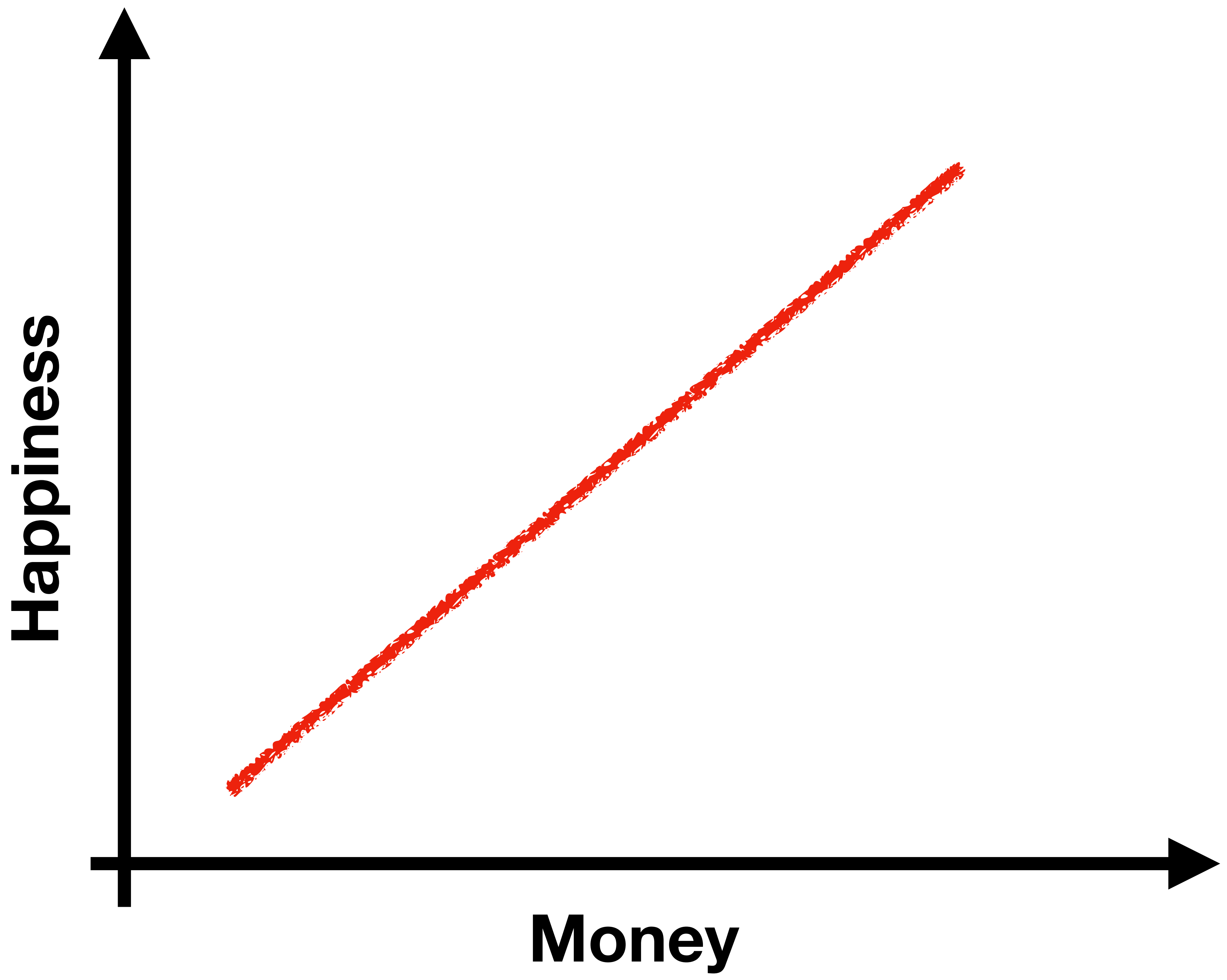

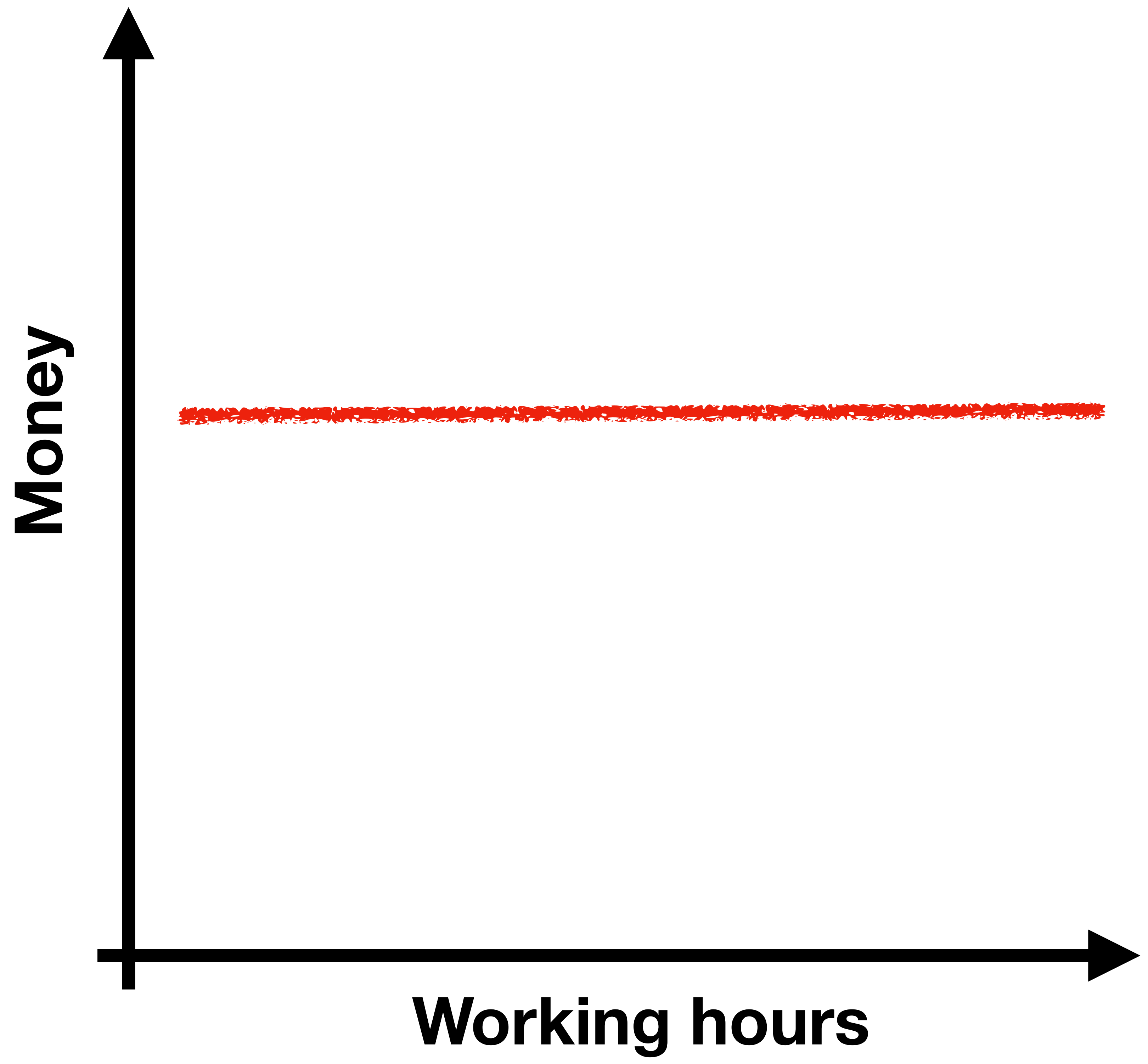

… at least in Academia

MOO getting famous?

Getting much better over the years…

MOO getting famous?

Getting much better over the years…

… clearly not as much as deep learning …

Founders of MOO

Mathematicians and Economists

- **Francis Ysidro Edgeworth (1845-1926)**

  Mathematical Physics: An Essay on the Application of Mathematics to the Moral Sciences, published in 1881

  “It is required to find a point (x, y) such that, in whatever direction we take an infinitely small step, $f1$ and $f2$ do not decrease together, but that, while one increases, the other decreases”

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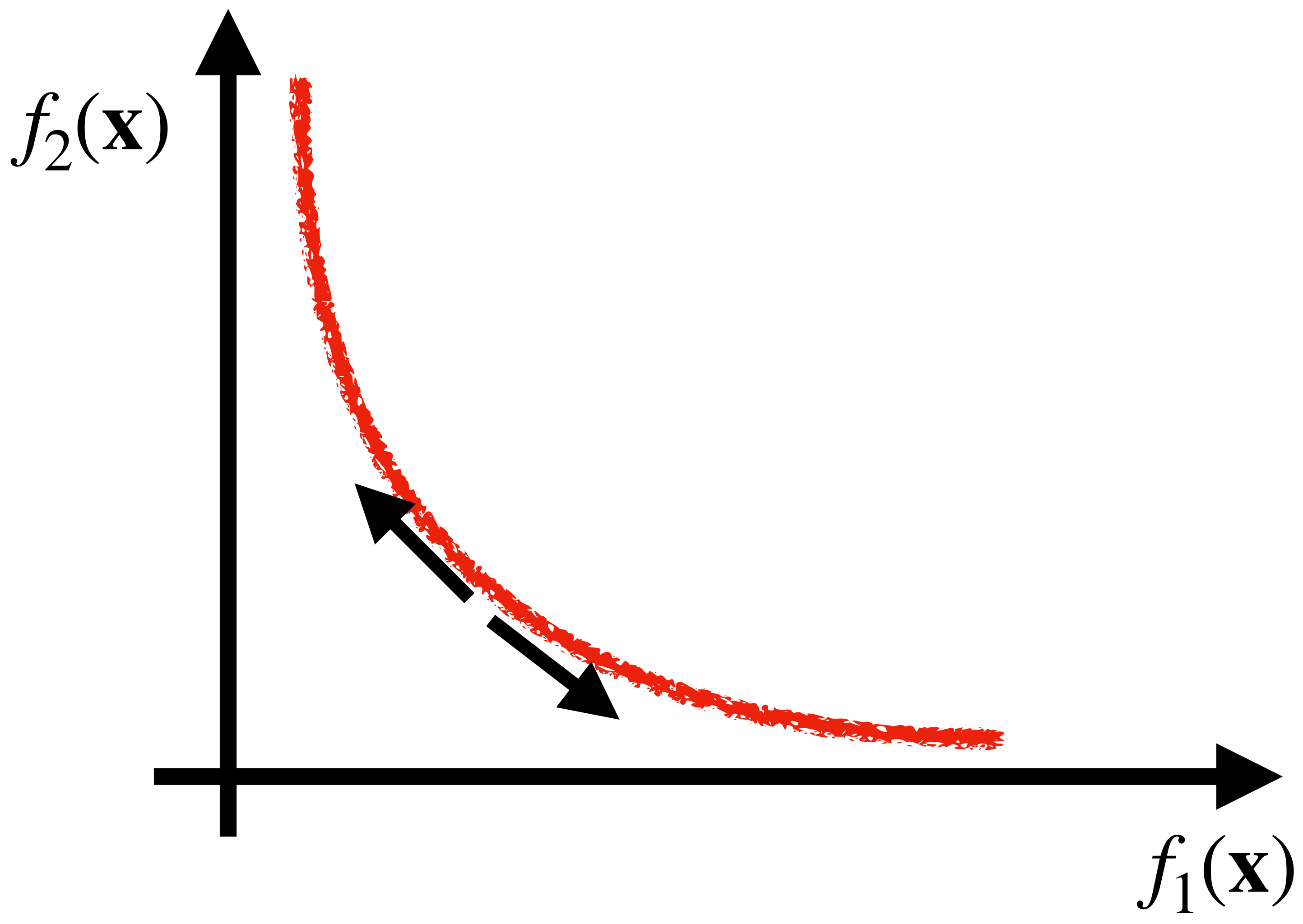

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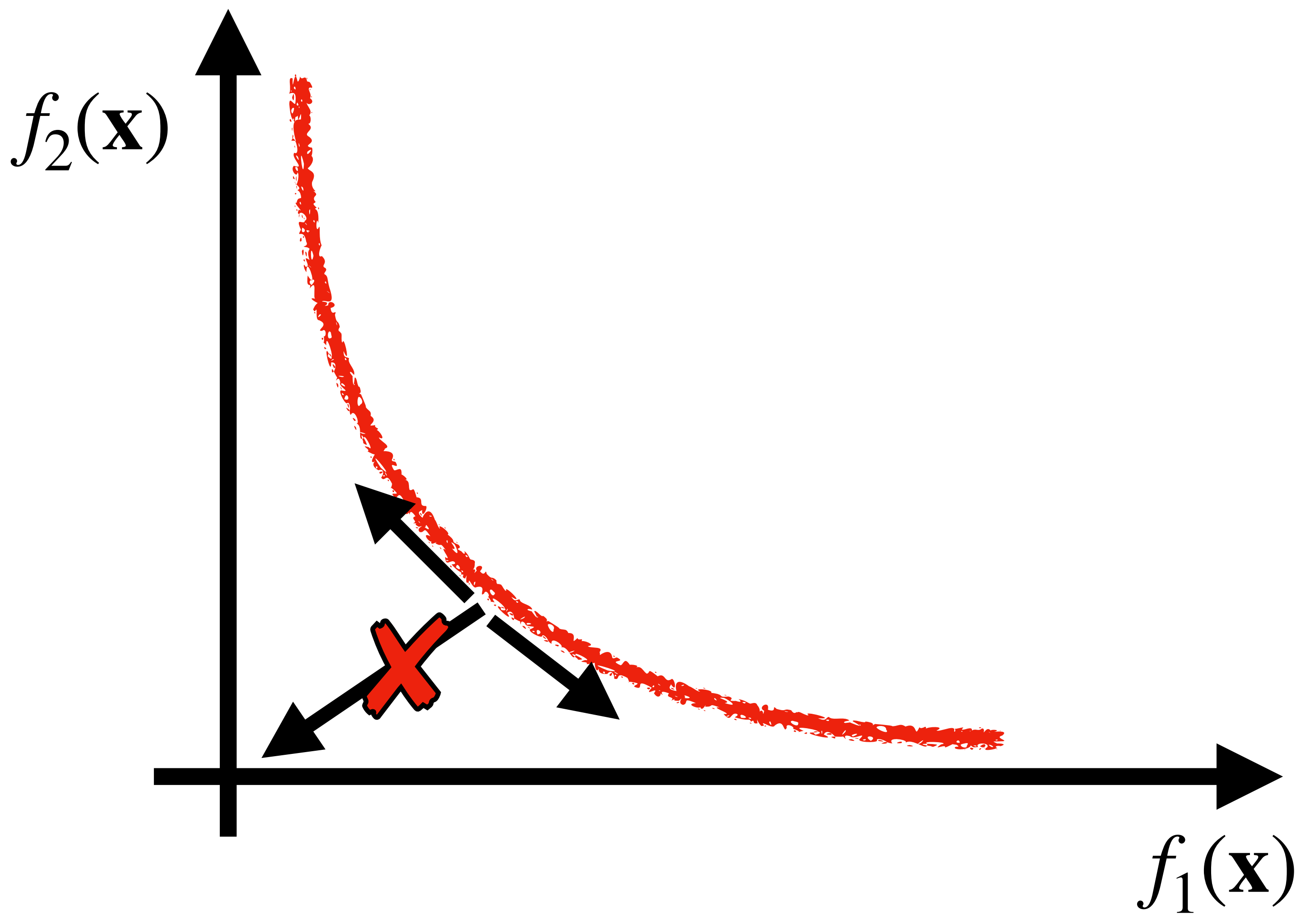

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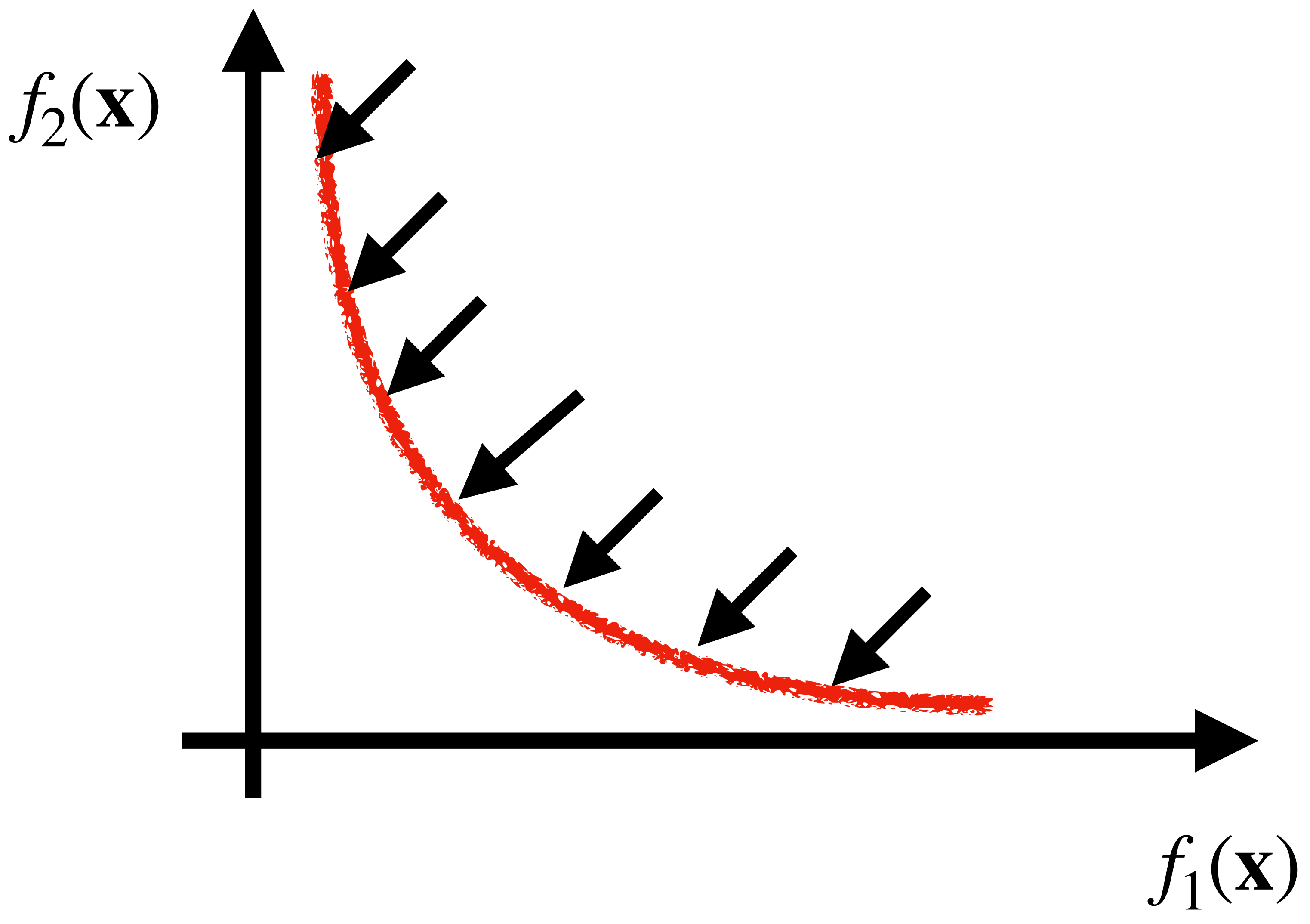

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MOO definition

- Decision variables and domain $\mathbf{x} \in \mathbf{R}^n$ $\mathscr{D} = \left\{\mathbf{x} \in \mathbf{R}^n : x_i^L \leq x_i \leq x_i^U \; ; i = 1,\ldots,n\right\}$
- Objective functions $(f_1, f_2, \ldots, f_M) \in \mathbf{R}^M$ $f_i(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$
- Constraints $g_j(\mathbf{x}),\, h_k(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$
- Problem $\min_{\mathbf{x}} \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x})\right)$

  s.t. $g_j(\mathbf{x}) \geq 0 \;; \quad h_k(\mathbf{x}) = 0 \;; \quad \mathbf{x} \in \mathscr{D}$

- Feasible solutions

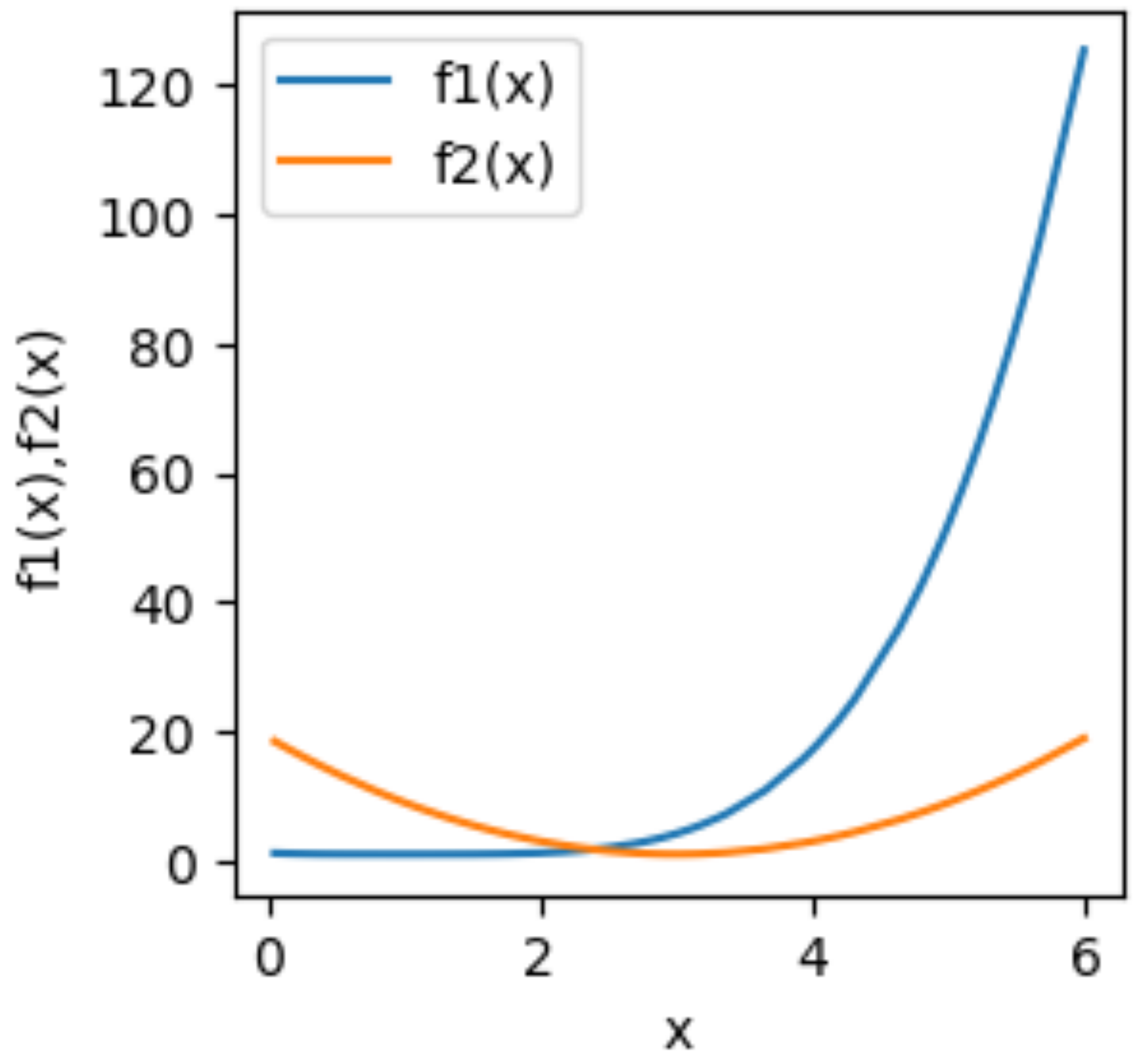
$$S = \left\{\mathbf{x} \in \mathbf{R}^n : x_i^L \leq x_i \leq x_i^U \cap g_j(\mathbf{x}) \geq 0 \; \cap h_k(\mathbf{x}) = 0\right\}$$

Example

$$f_1(x) = \frac{1}{5}(x-1)^4 + 1$$

$$f_2(x) = 2(x-3)^2 + 1$$

$$x \in [0,6] \subset \mathbf{R}$$



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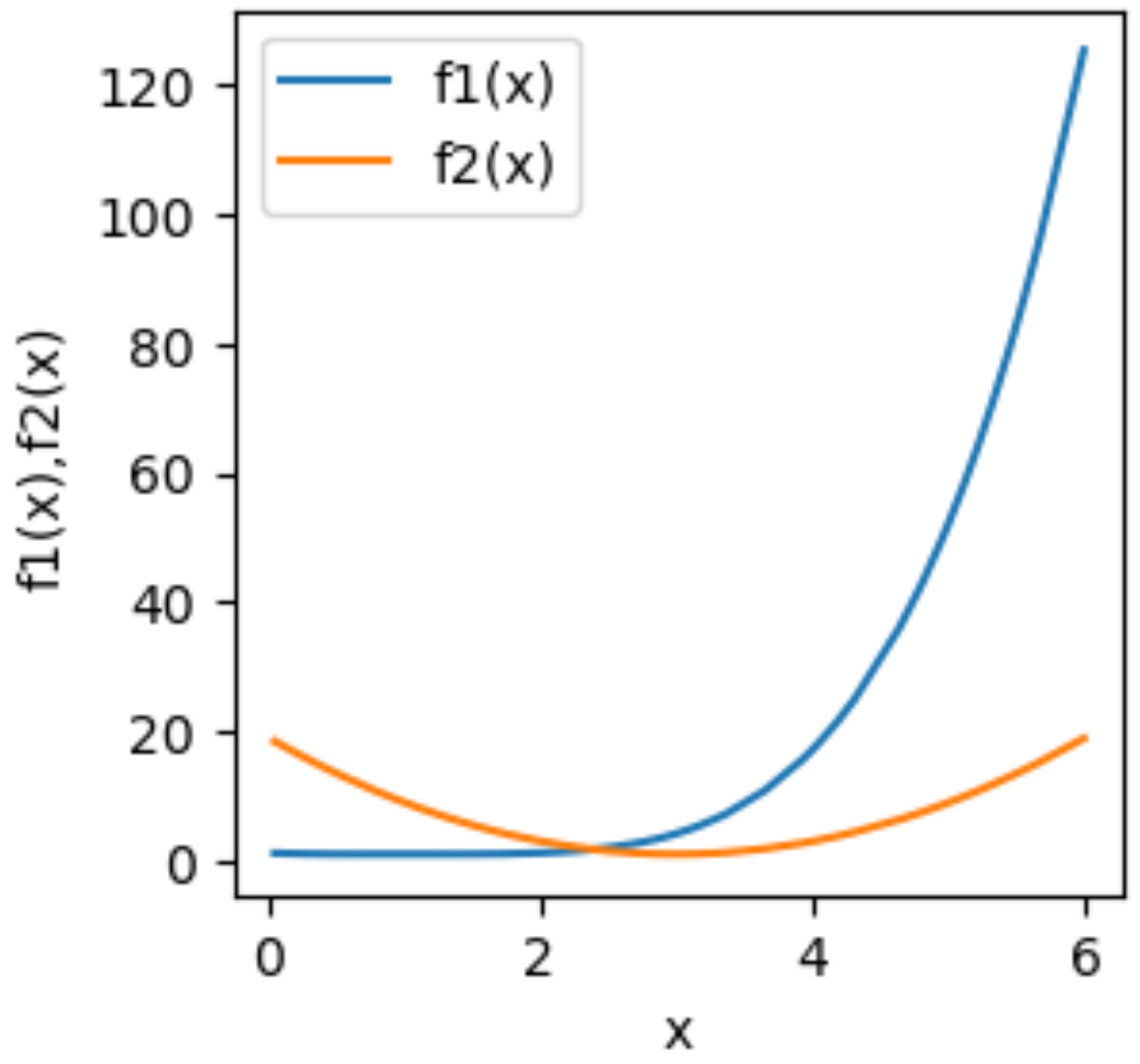
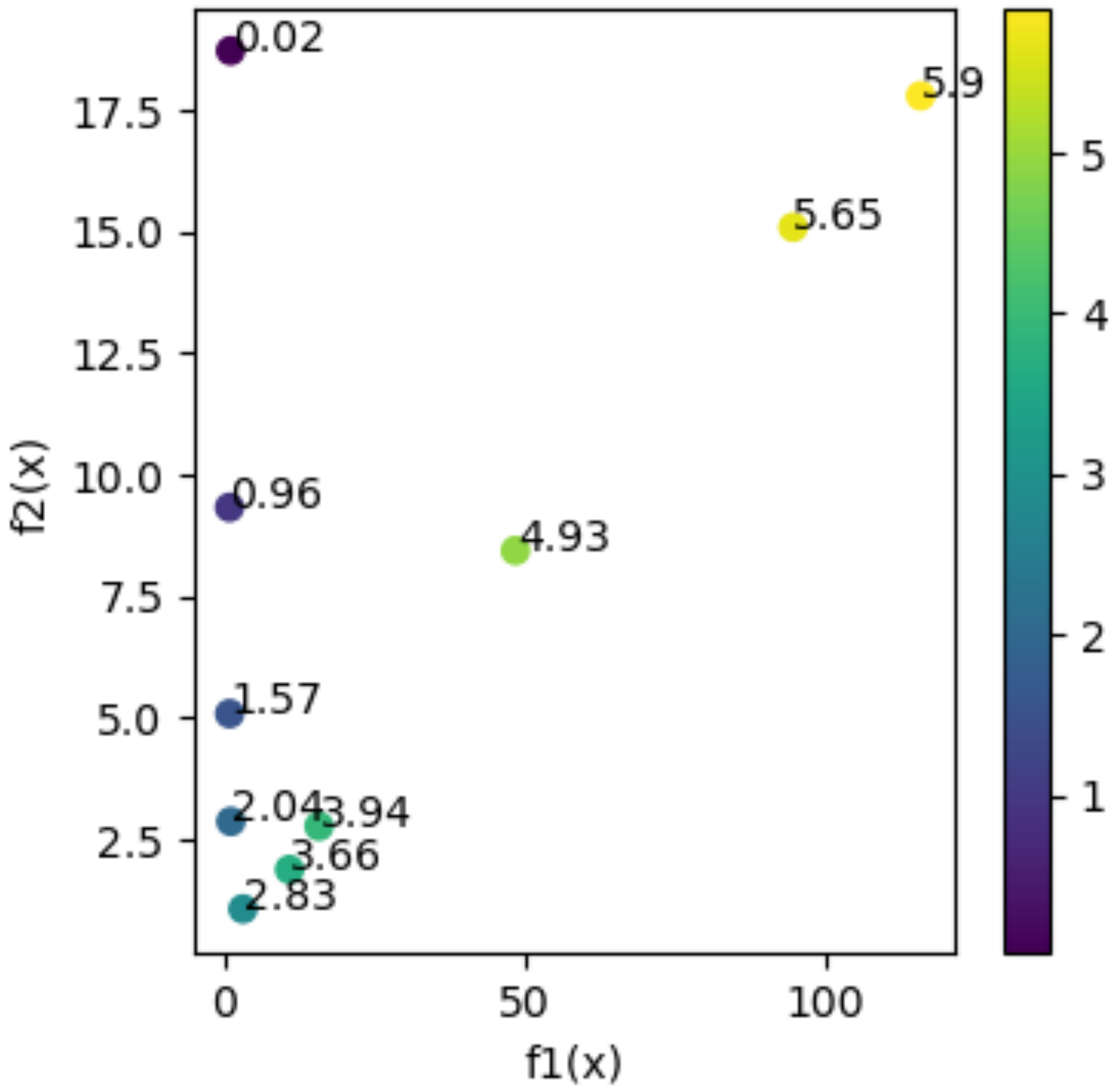


Visualization in low dimensions: from solution to objective space



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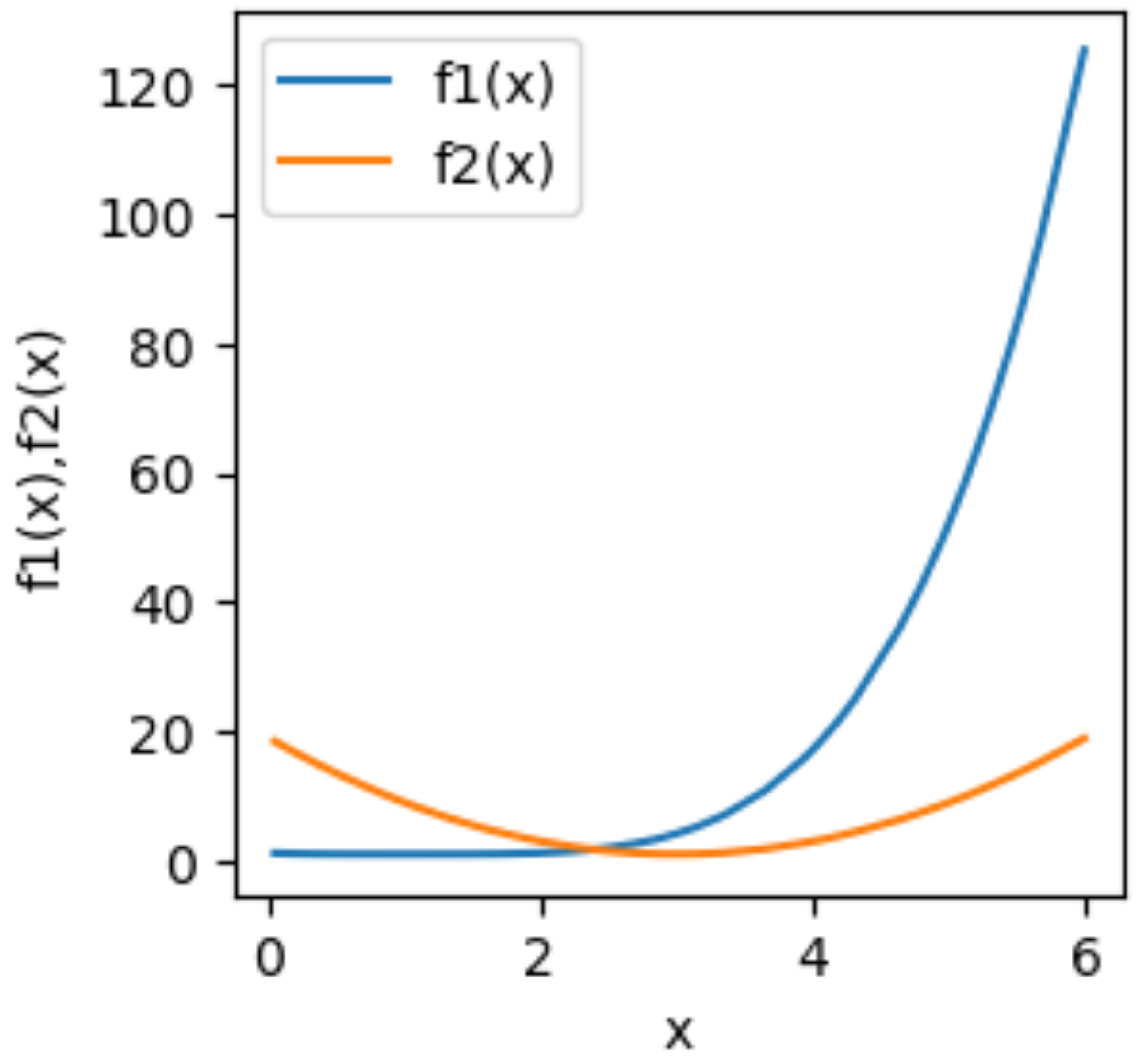

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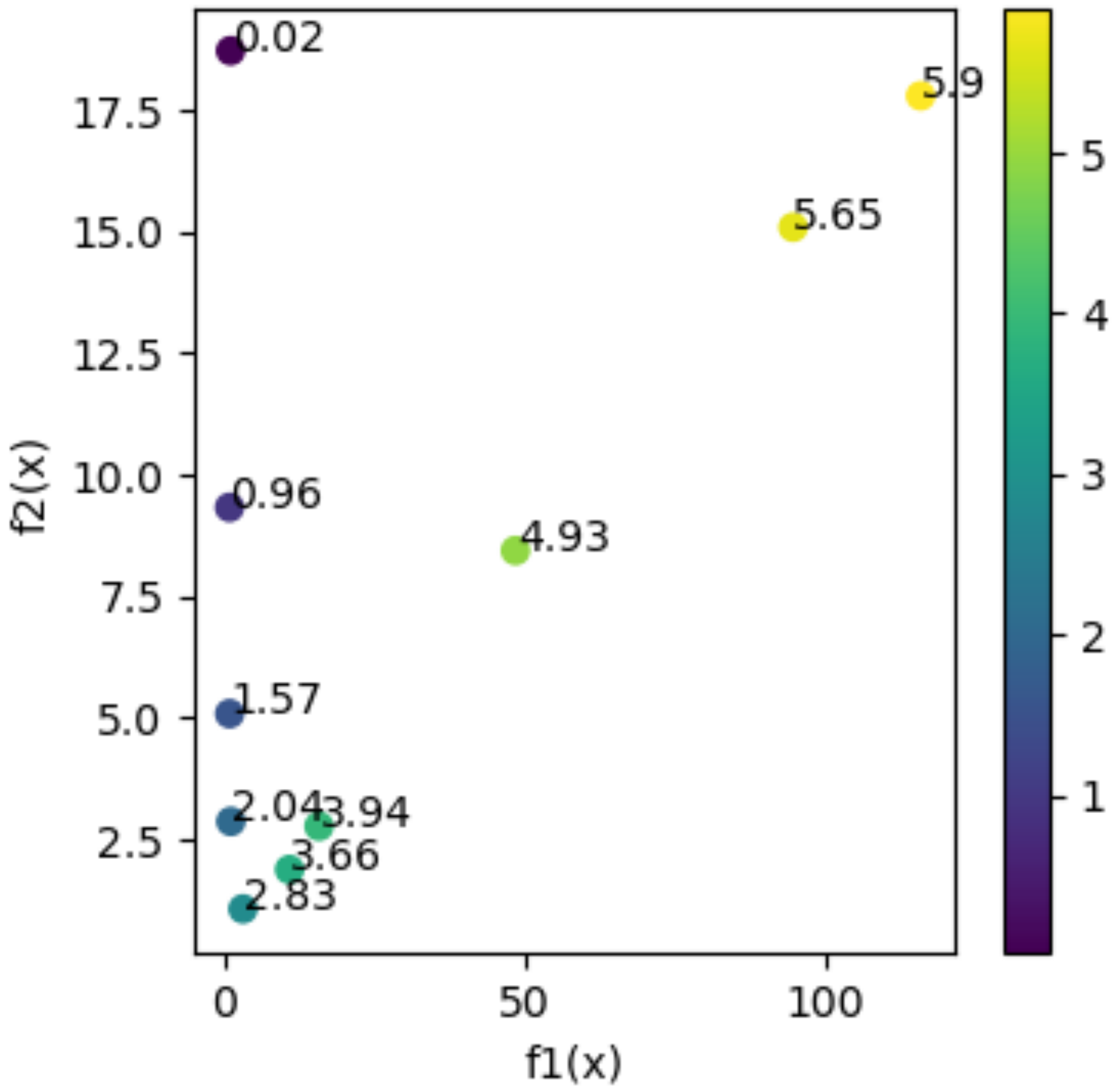
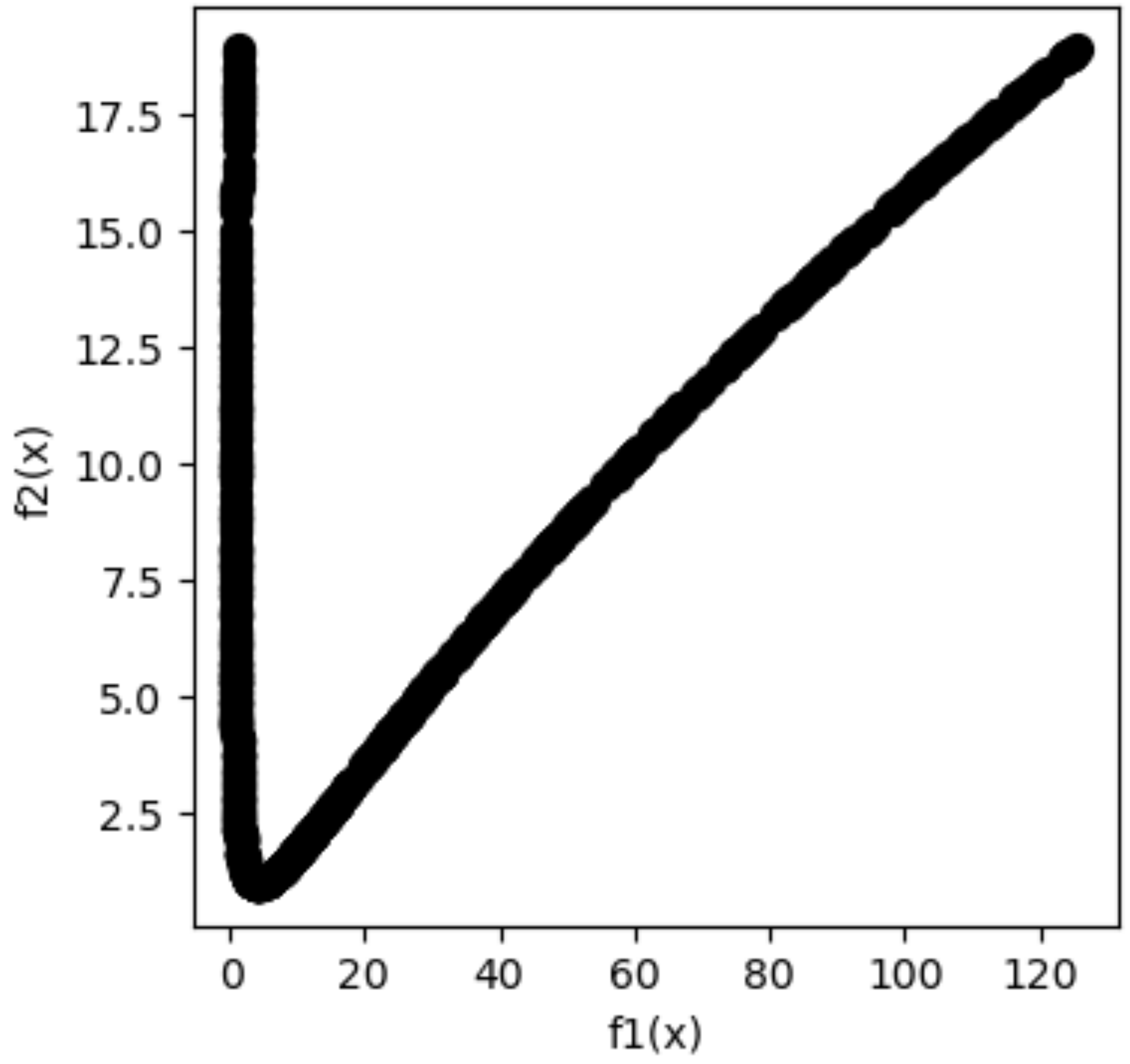




Pareto Dominance

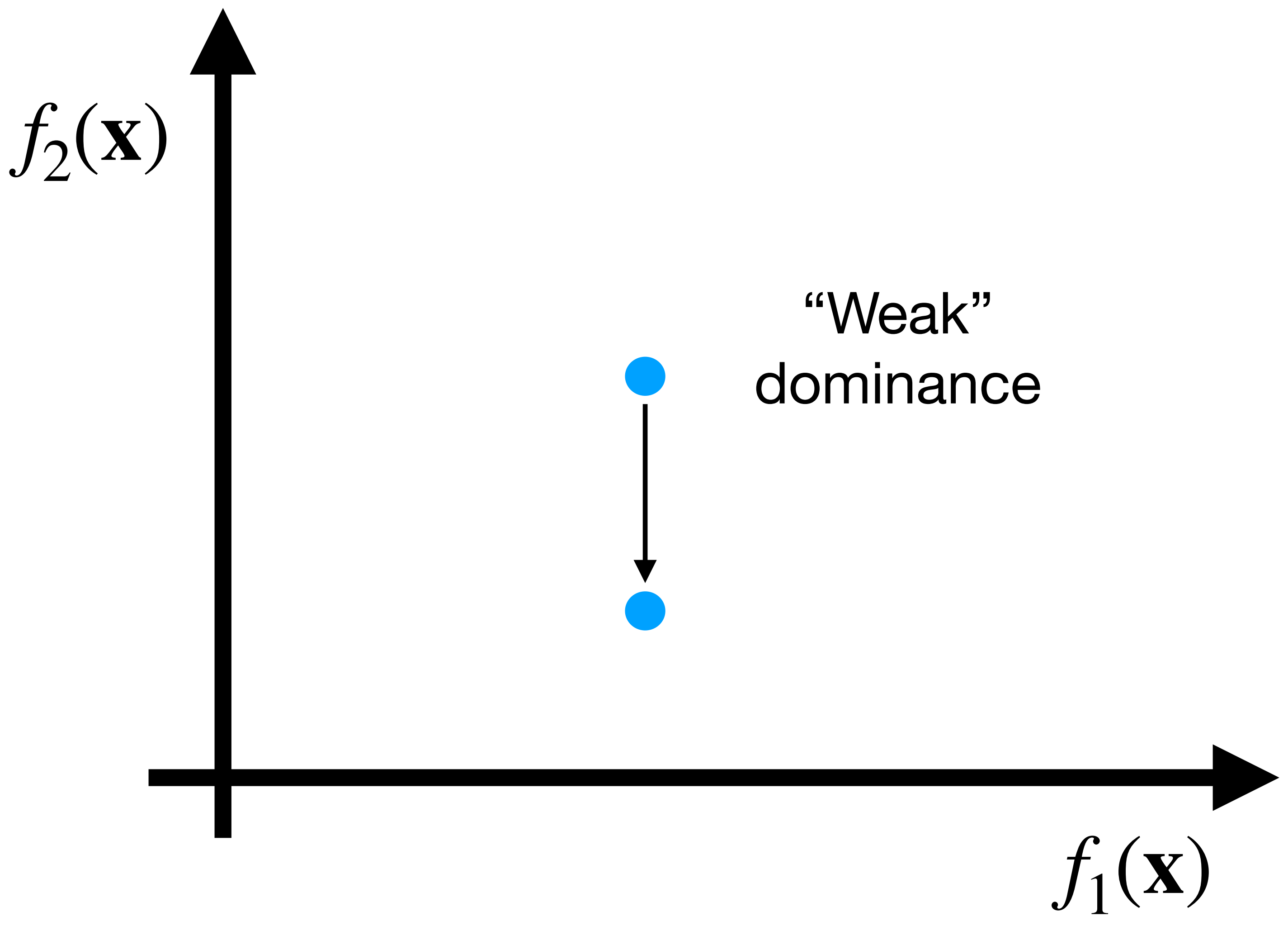
Definition

Pareto dominance: a solution $\mathbf{x}$ is said to dominate a solution $\mathbf{x}^*$ $(\mathbf{x} \preceq \mathbf{x}^*)$ if and only if

$$f_k(\mathbf{x}) \preceq f_k(\mathbf{x}^*) \quad \forall k = 1,\ldots,M$$

and

$$\exists\, j \in 1,\ldots,M \quad s.t. \quad f_j(\mathbf{x}) \prec f_j(\mathbf{x}^*)$$



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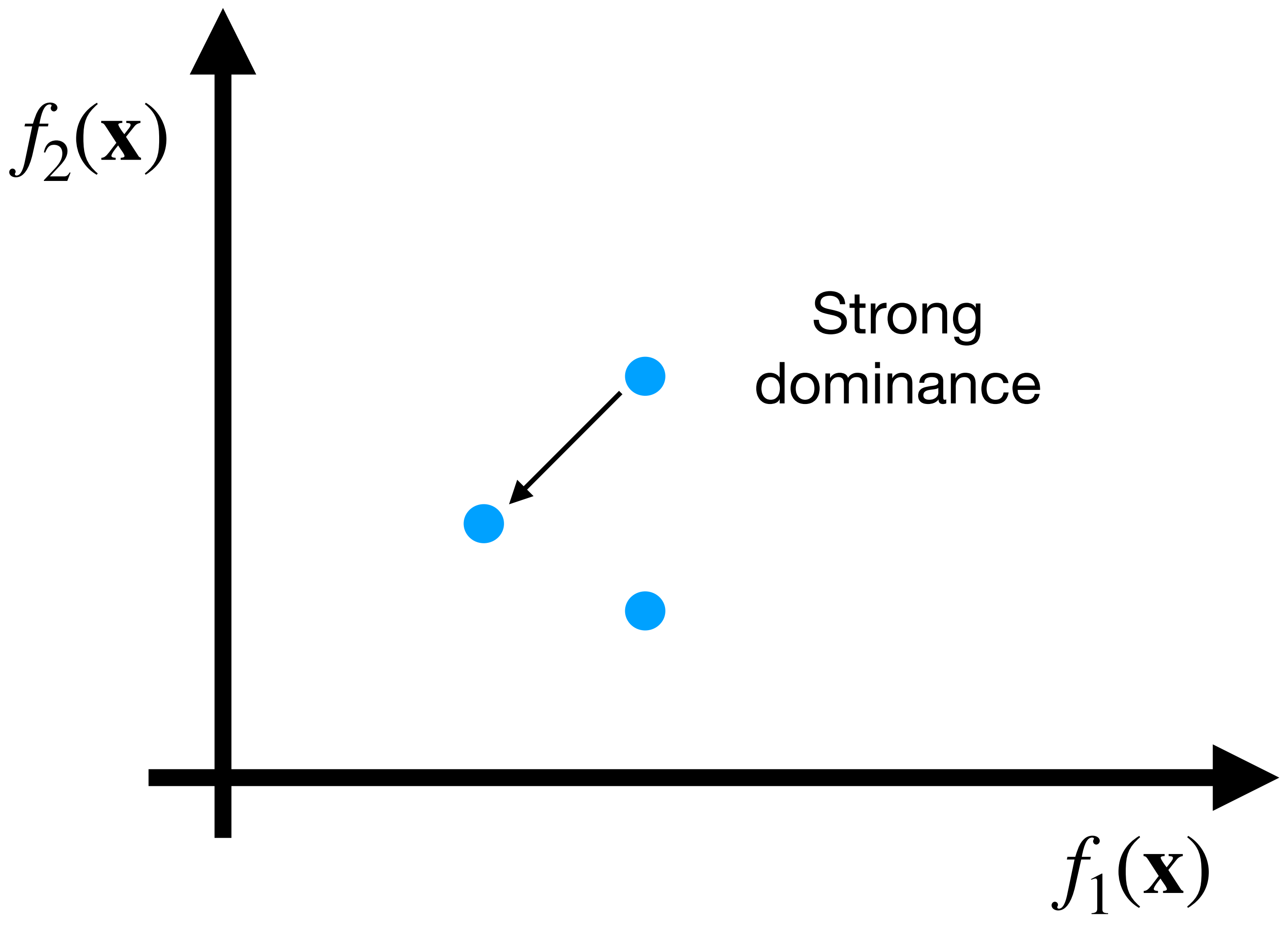

Strong Pareto dominance: a solution $\mathbf{x}$ is said to strongly dominate a solution $\mathbf{x}^*$ $(\mathbf{x} \prec \mathbf{x}^*)$ if it is strictly better than $\mathbf{x}^*$ in all the objectives.

$$f_k(\mathbf{x}) \prec f_k(\mathbf{x}^*) \quad \forall k = 1,\ldots,M$$

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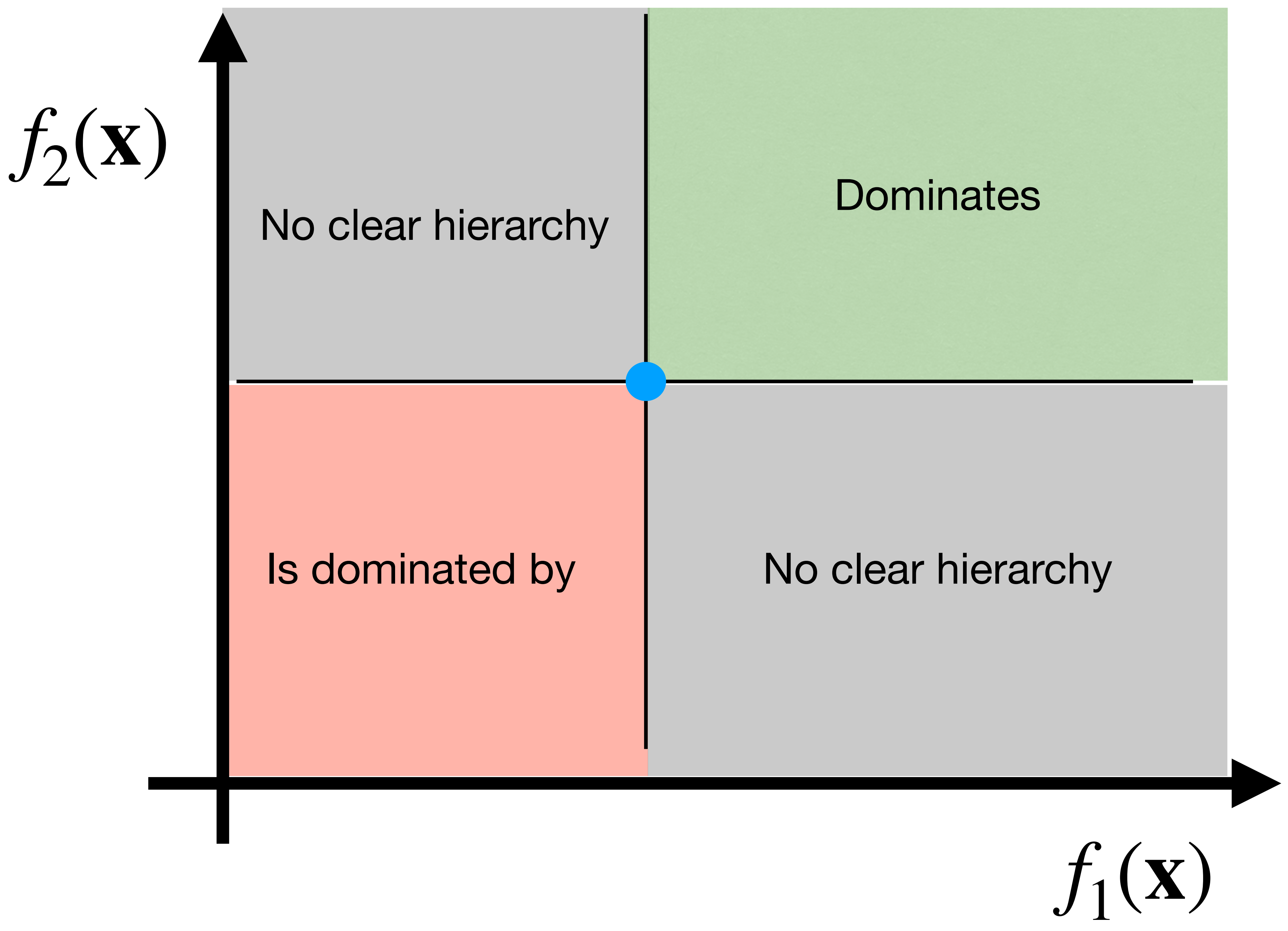

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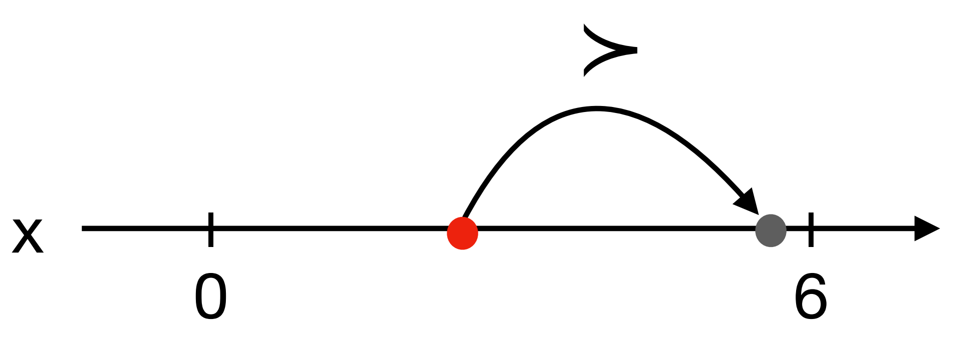
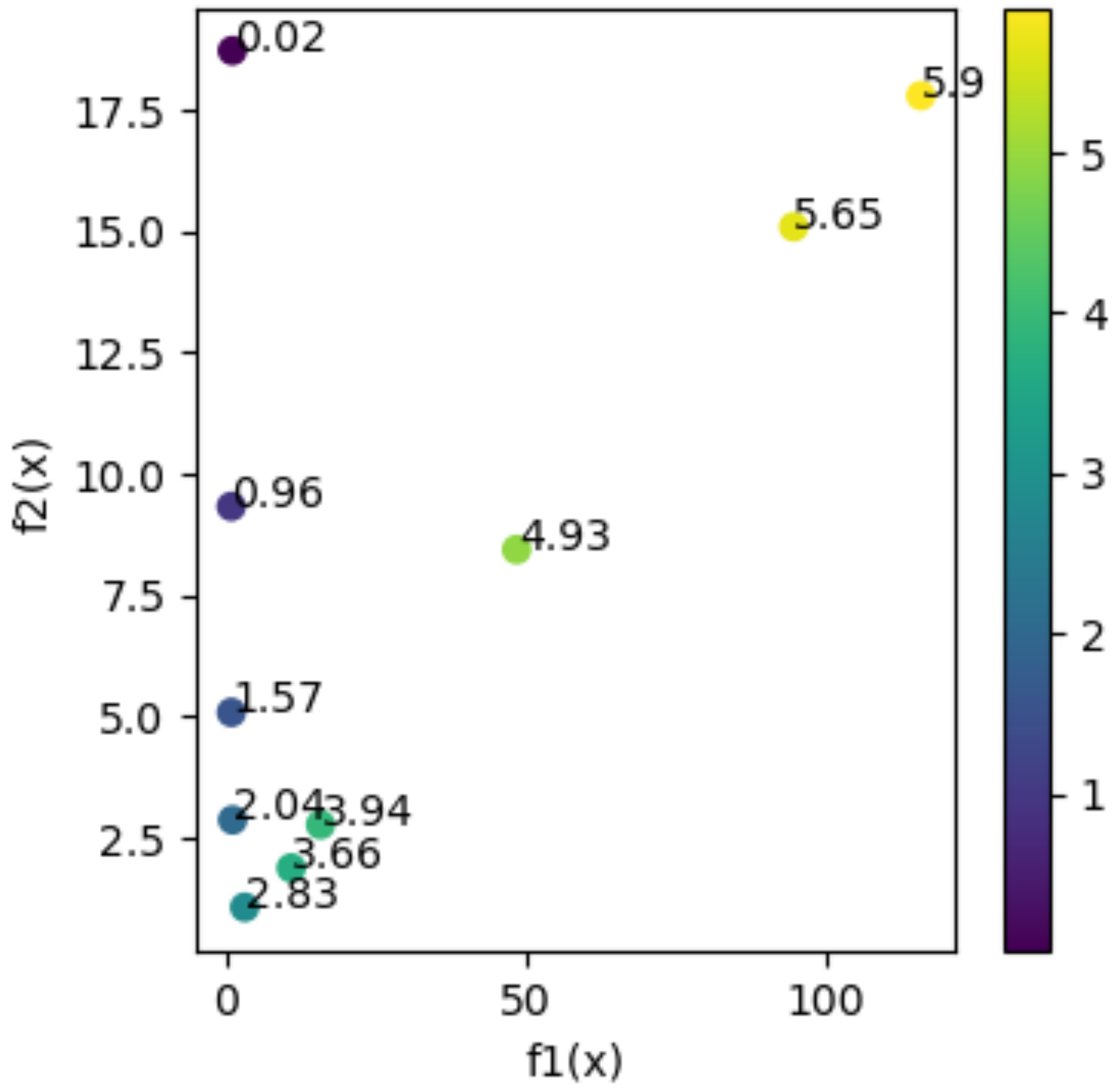
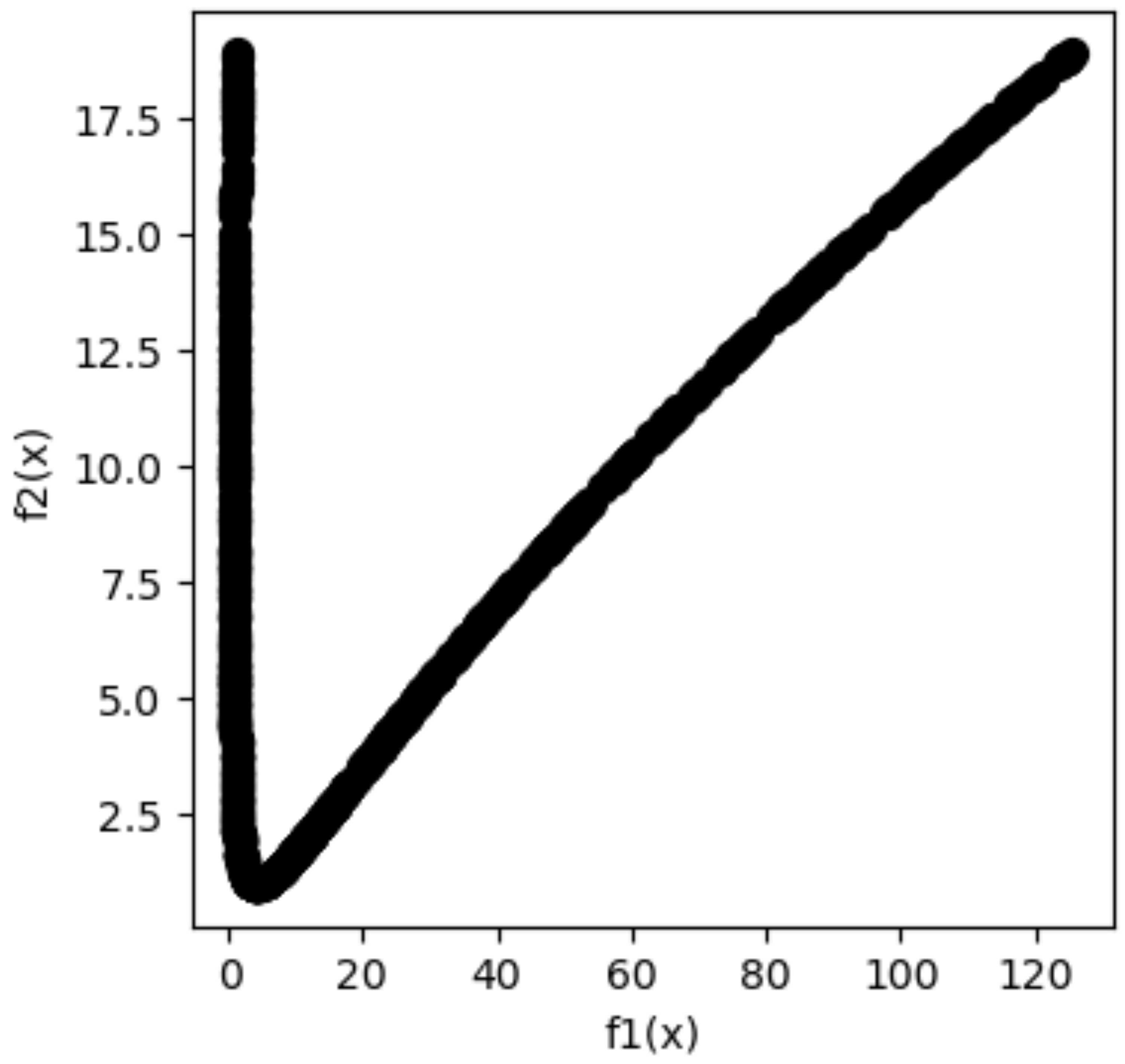


`x1 = 2.8 dominates x2 = 5.7:`

`[f1(x1),f2(x1)]=[(3.1, 1.08)]`

`[f1(x2),f2(x2)]=[(98.59, 15.58)]`





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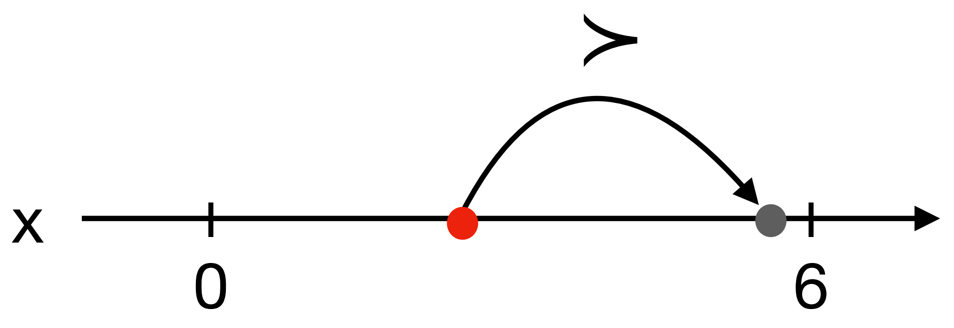


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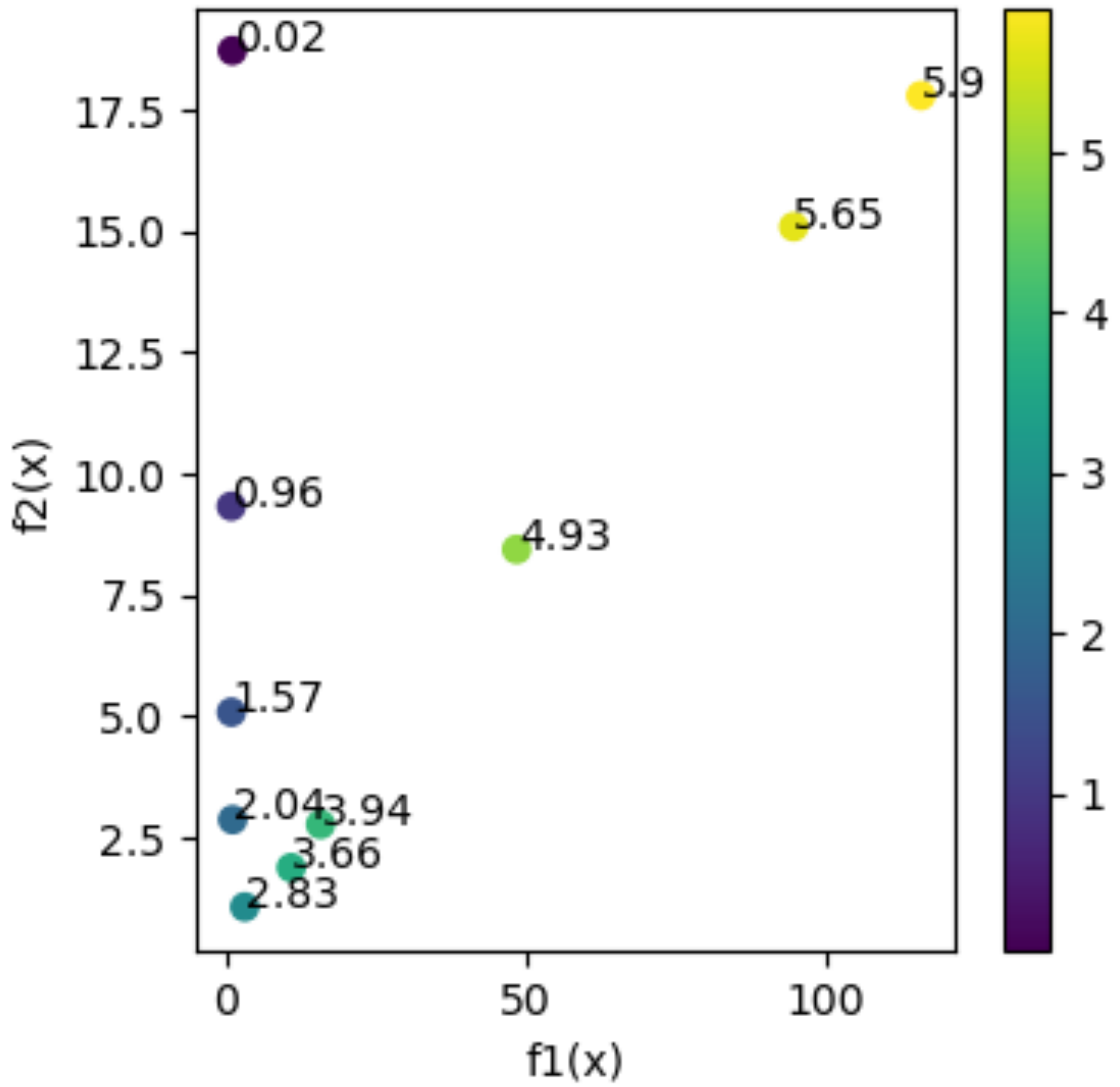
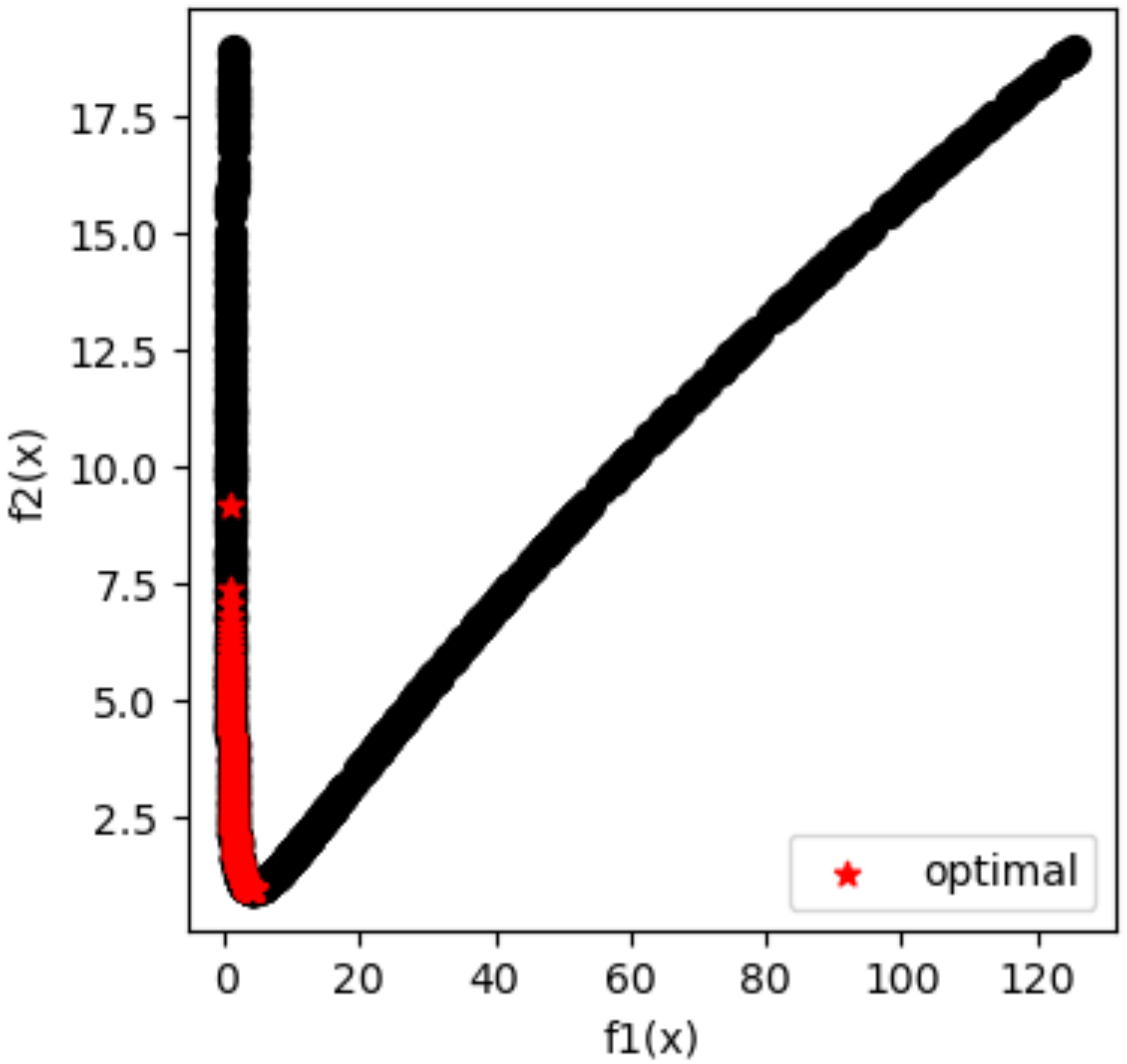




Pareto Dominance

Properties

PD induces a **strong partial ordering**:

Partial ordering

- Reflexivity

  $$\mathbf{x} \preceq \mathbf{x} \quad \forall \mathbf{x} \in \mathcal{D}$$

- Antisymmetry

  If $\mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{x}$

  $$\implies \mathbf{x} = \mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathcal{D}$$

- Transitivity

  if $\mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{z}$

  $$\implies \mathbf{x} \preceq \mathbf{z} \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{D}$$

Pareto Dominance

Properties

PD induces a **strong partial ordering**:

- Not reflexive

$$\mathbf{x} \not\preceq \mathbf{x} \quad \forall \mathbf{x} \in \mathscr{D}$$

- Asymmetry

$$\nexists \mathbf{x}, \mathbf{y} \quad s.t. \quad \mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{x}$$

- Transitivity

$$\text{if } \mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{z}$$

$$\implies \mathbf{x} \preceq \mathbf{z} \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathscr{D}$$

Partial ordering

- Reflexivity

$$\mathbf{x} \preceq \mathbf{x} \quad \forall \mathbf{x} \in \mathscr{D}$$

- Antisymmetry

$$\text{If } \mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{x}$$

$$\implies \mathbf{x} = \mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathscr{D}$$

- Transitivity

$$\text{if } \mathbf{x} \preceq \mathbf{y} \cap \mathbf{y} \preceq \mathbf{z}$$

$$\implies \mathbf{x} \preceq \mathbf{z} \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathscr{D}$$

Pareto Optimality

Nomenclature

- **Non-dominated (Pareto optimal) solutions**

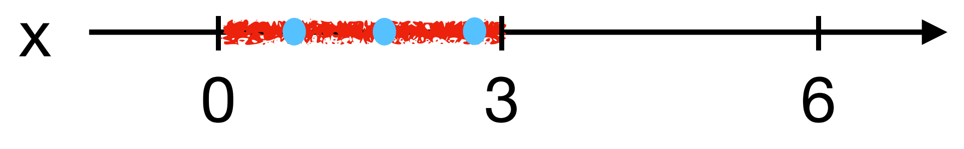
  $\mathbf{x} \in \mathscr{D} \quad s.t. \quad \vec{f}(\mathbf{x})$ is not dominated by any other solution. Blue dots.

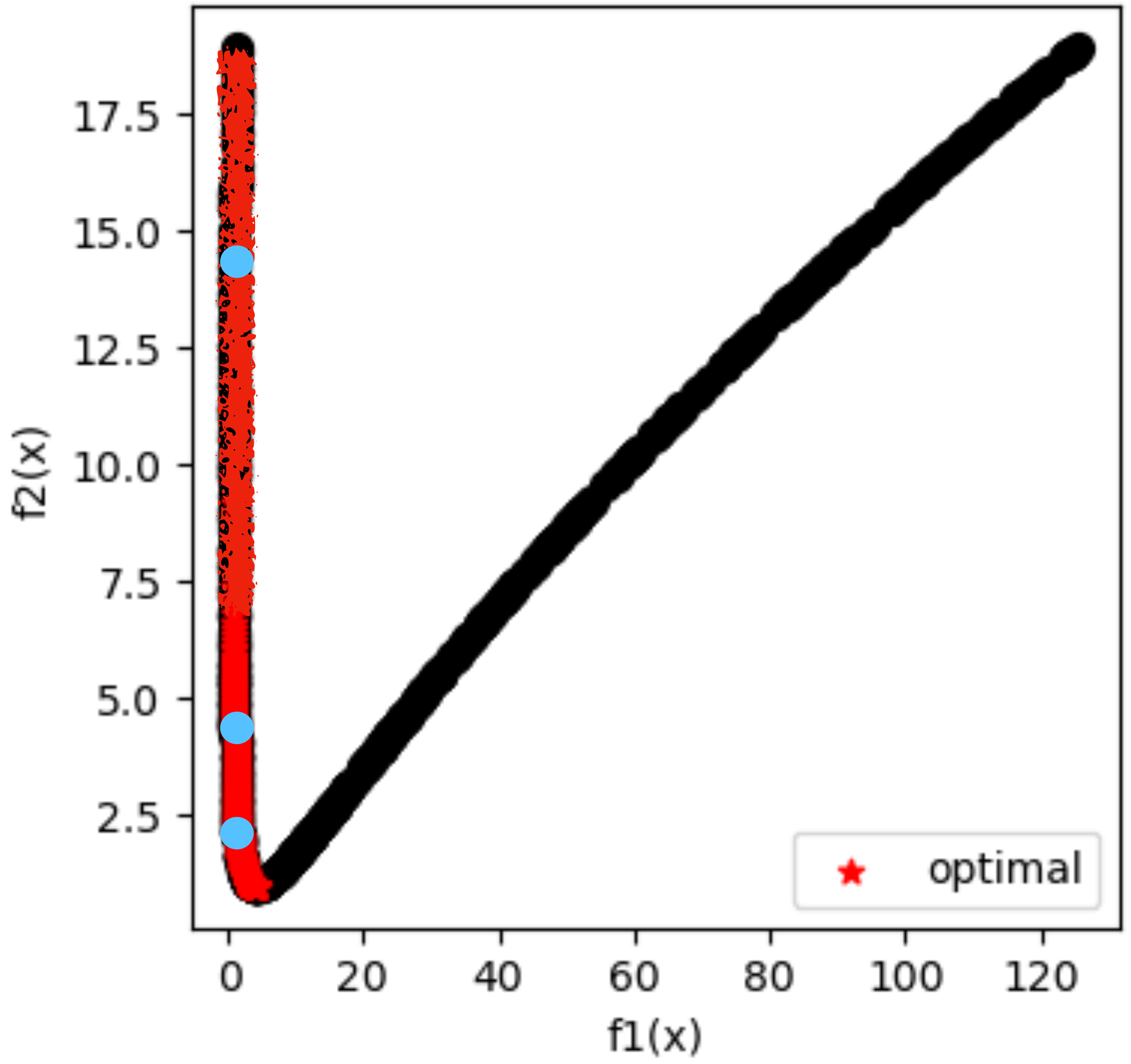
- **Pareto front** (in objective space)

  All $\mathbf{x} \in \mathscr{D} \quad s.t. \quad \vec{f}(\mathbf{x})$ belongs to the red curve in the objective space

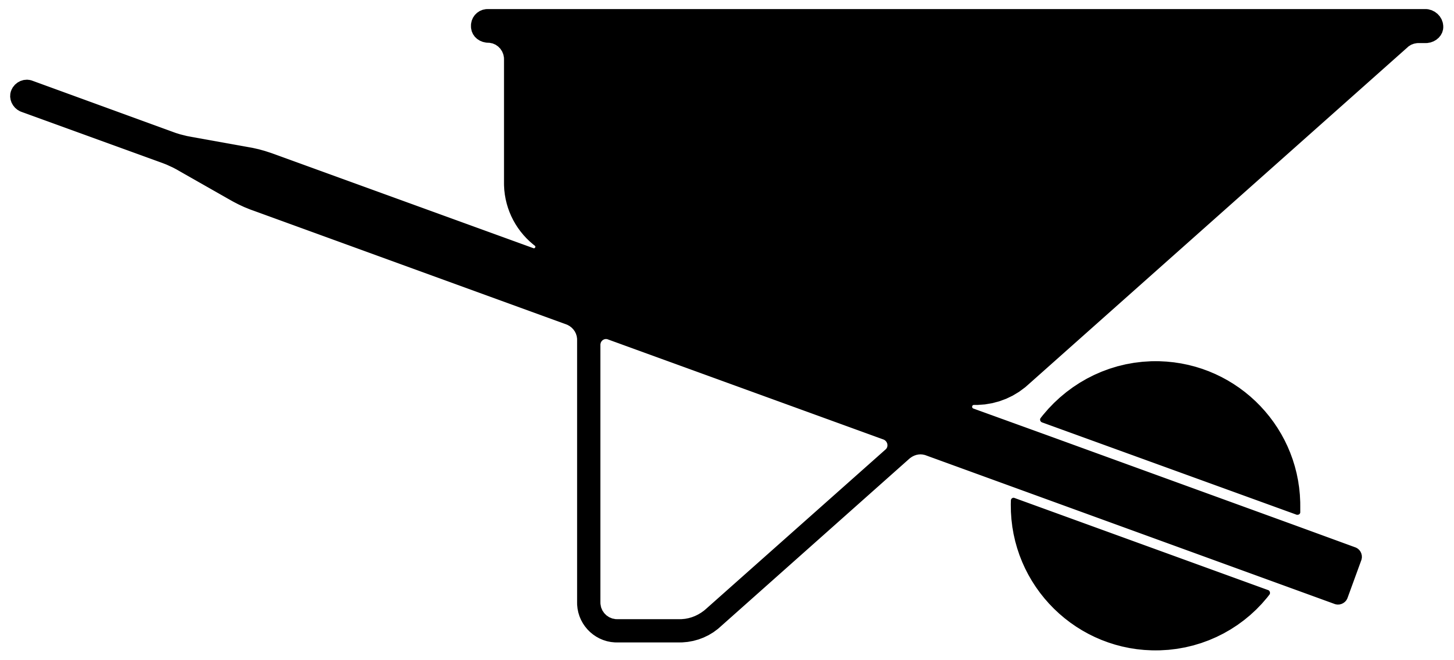
- **Pareto optimal set** (in variables space)

  All $\mathbf{x} \in \mathscr{D}$ belonging to the red curve in the decision variable space





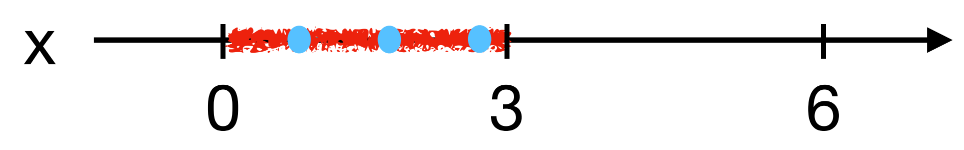
Hands on

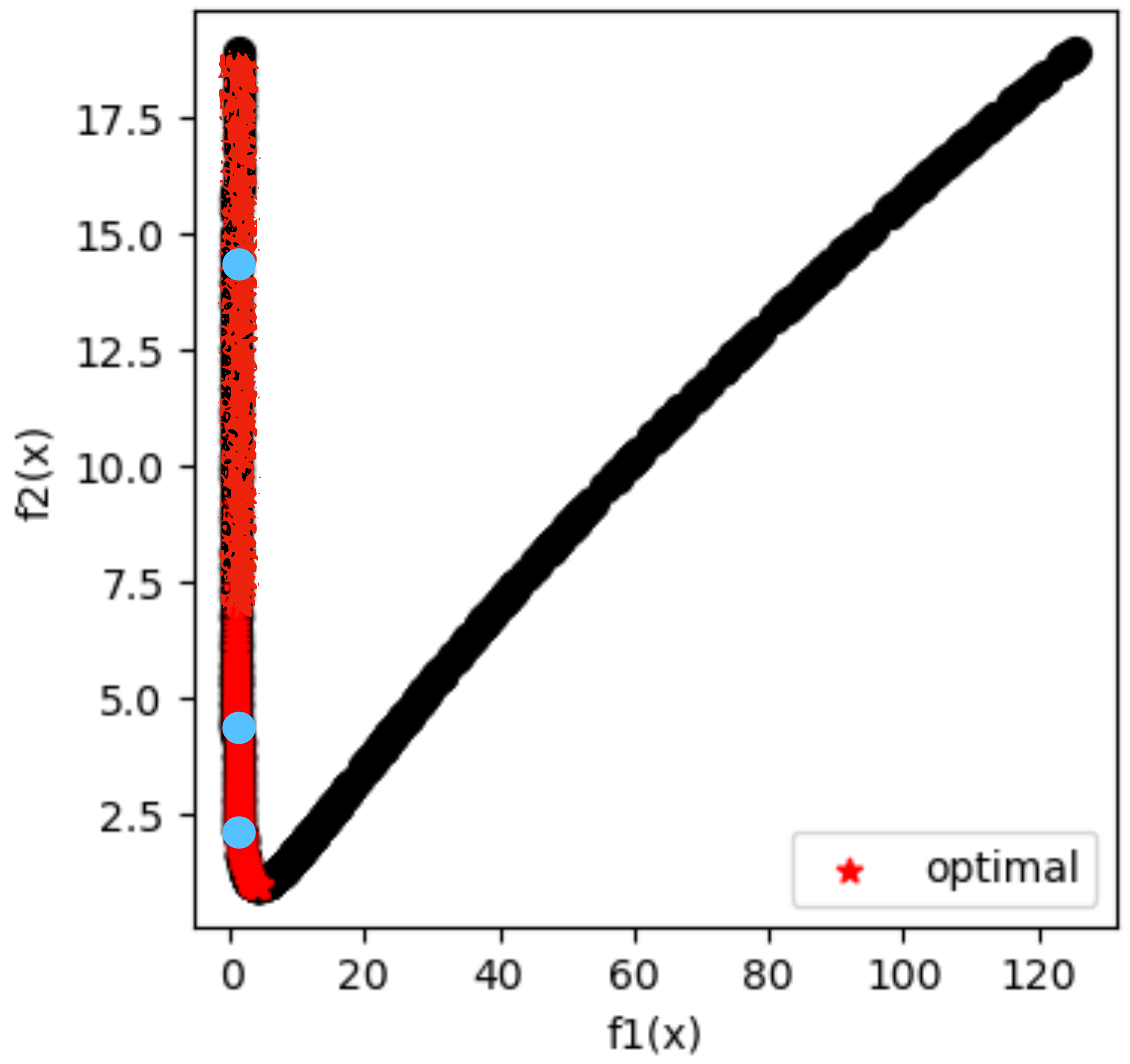


Approaches to MOO

Goals

- Find a non-dominated solution
- Find all non-dominated solution (Pareto set)





Approaches to MOO

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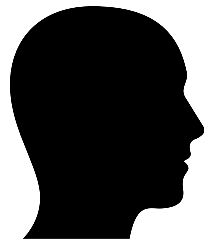
Decision making  ***<- decision maker***

- A priori: Define  —> MOO, No Pareto set —> Single solution

Approaches to MOO

Goals

- Find a non-dominated solution
- Find all non-dominated solution (Pareto set)

Decision making

 ***<- decision maker***

- A priori: Define  —> MOO, No Pareto set —> Single solution
- A posteriori: Define —> MOO, Pareto set —> Single solution 

Approaches to MOO

Goals

- Find a non-dominated solution
- Find all non-dominated solution (Pareto set)

Decision making  ***<- decision maker***

- A priori: Define  —> MOO, No Pareto set —> Single solution
- A posteriori: Define —> MOO, Pareto set —> Single solution 
- Interactive: Define —> MOO, interactive ( HITL) —> Single solution

Approaches to MOO

Goals

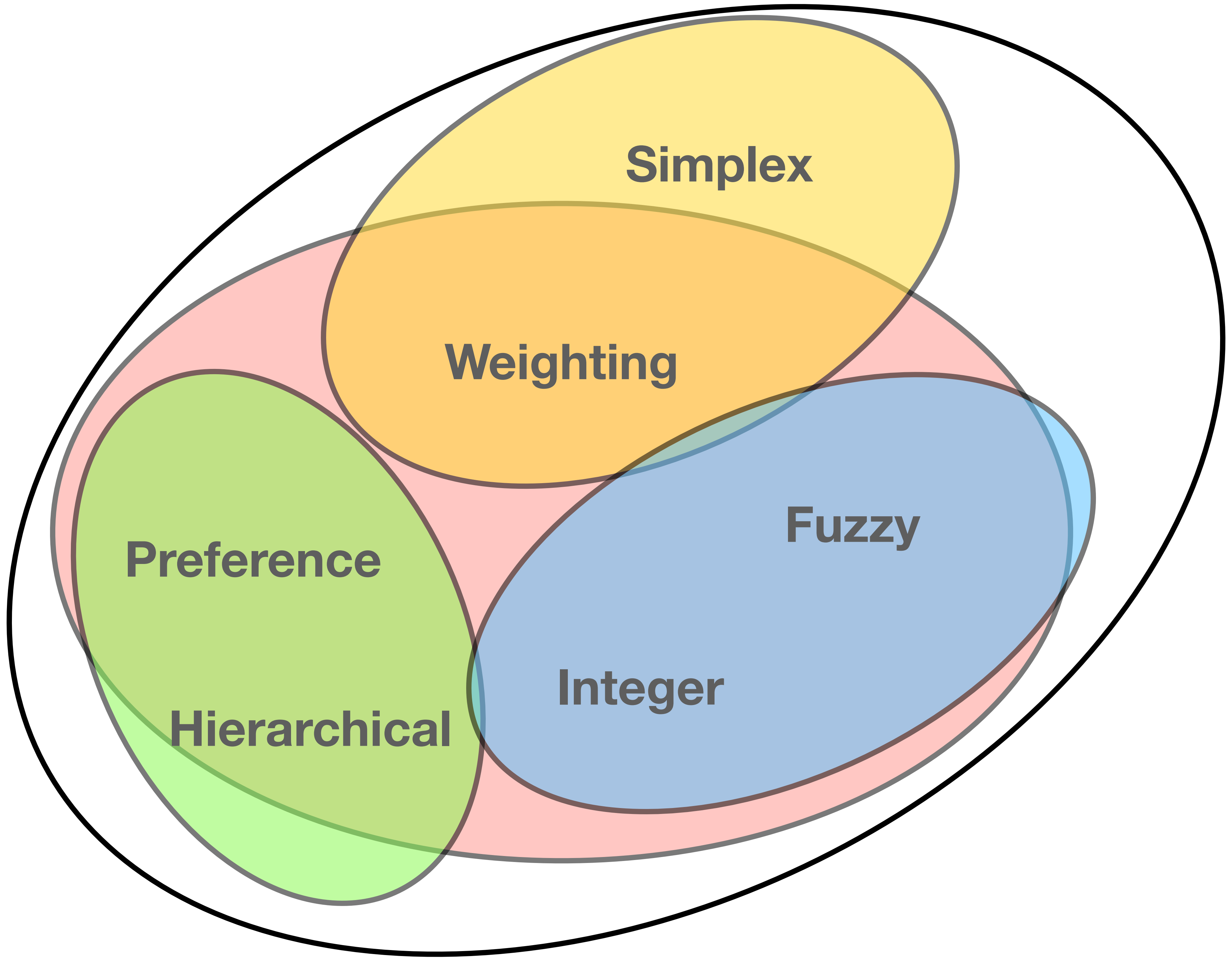
- Find a non-dominated solution
- Find all non-dominated solution (Pareto set)

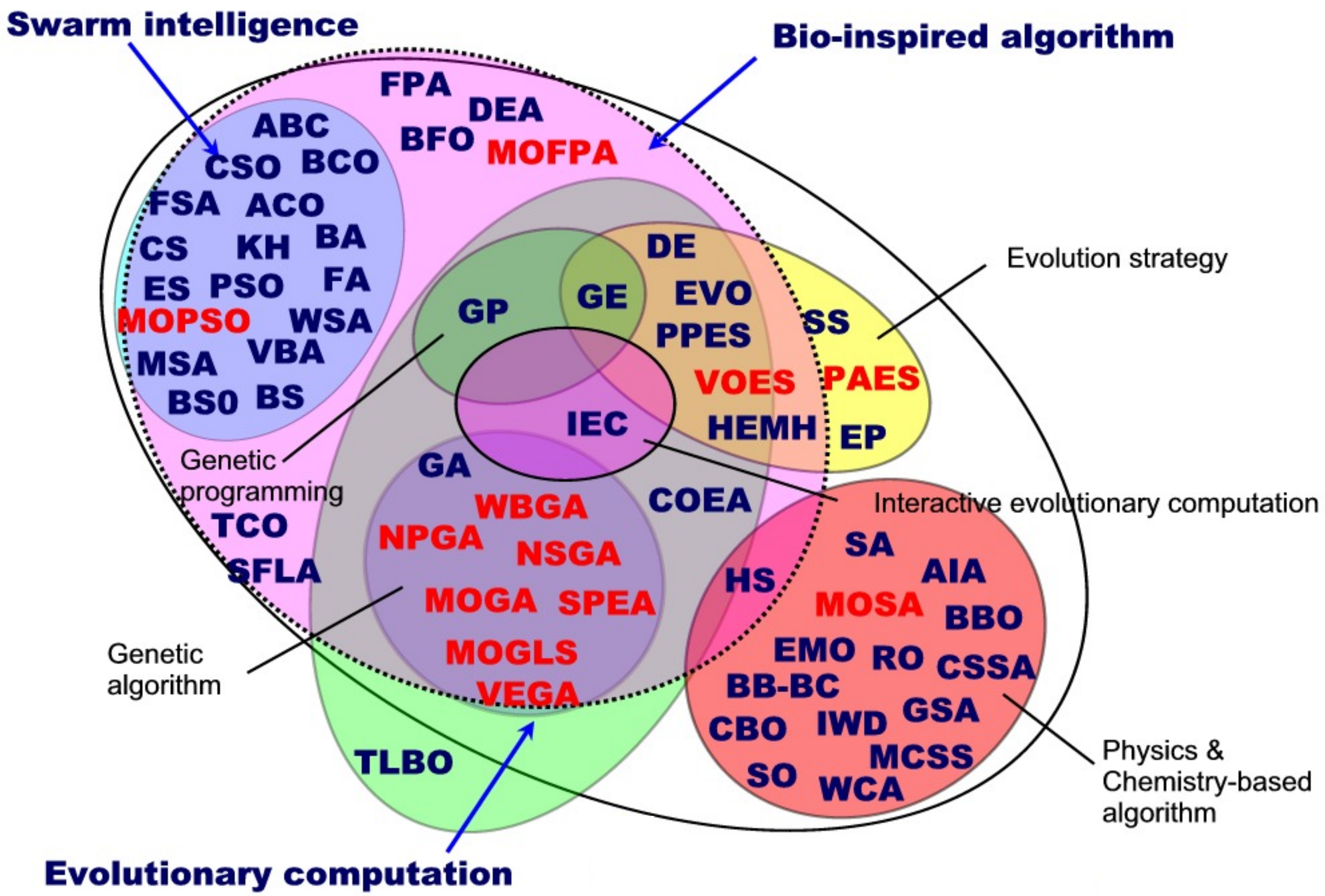
Decision making  ***<- decision maker***

- A priori: Define  —> MOO, No Pareto set —> Single solution
- A posteriori: Define —> MOO, Pareto set —> Single solution 
- Interactive: Define —> MOO, interactive ( HITL) —> Single solution
- No DM / preference: Define —> Run —> Pareto set

Taxonomy

Classical and meta-heuristic methods





Taken from (Keller, 2017)

Scalarization Algorithms

Basic idea

- Combine multiple objectives into a single utility function
- Solve the single objective problem
- Find one Pareto optimal solution
- Repeat to find the whole Pareto optimal set

Weighted sum methods

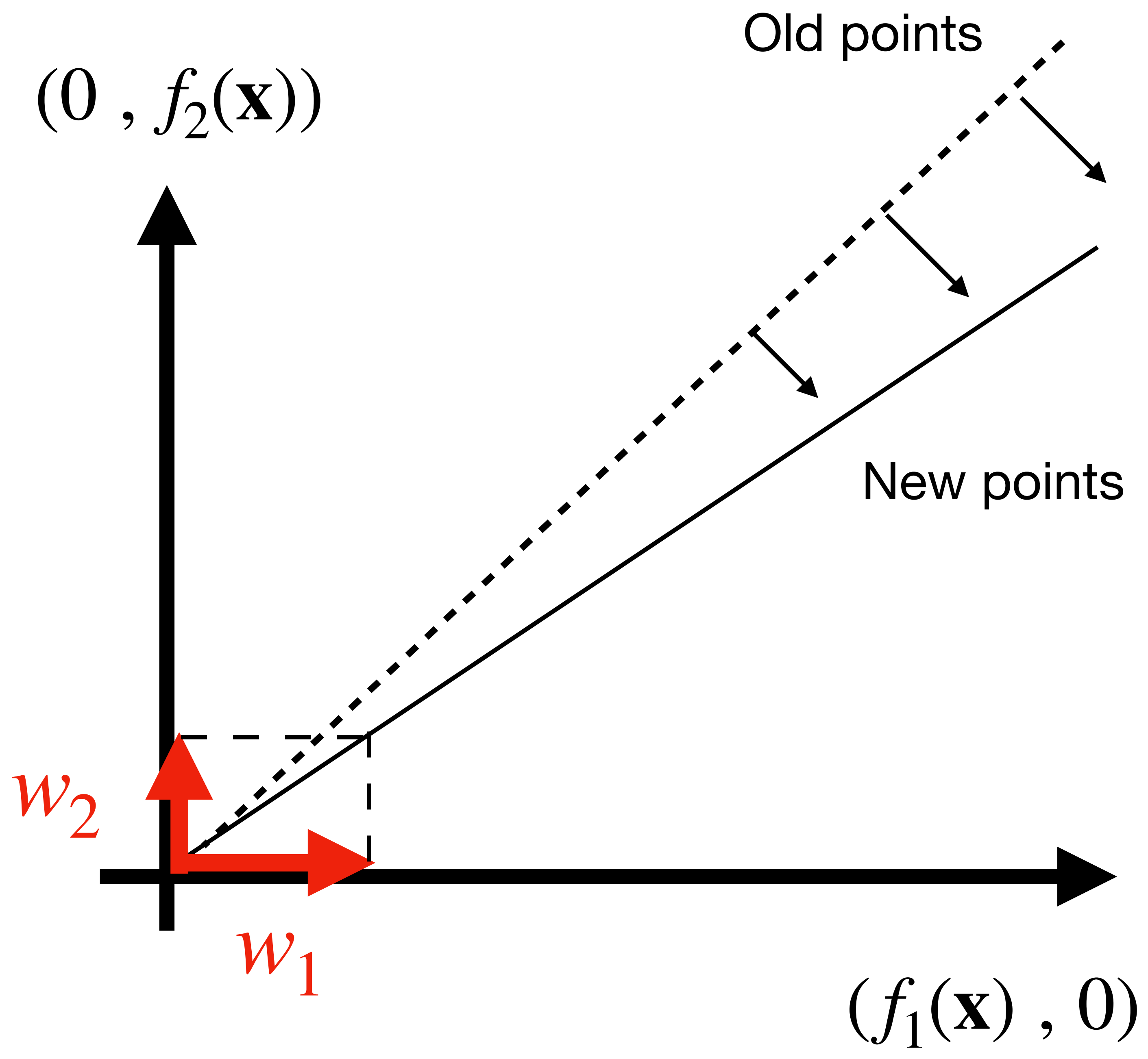
Definition

- Identify a weight vector in the objective space

$$\min_{\mathbf{x}} \sum_{i=1}^{M} w_i f_i(\mathbf{x})$$

$$\mathbf{x} \in \mathscr{D} \qquad w_i > 0 \quad \forall i \in 1,\ldots,M$$

$$\sum_{i=1}^{M} w_i = 1$$



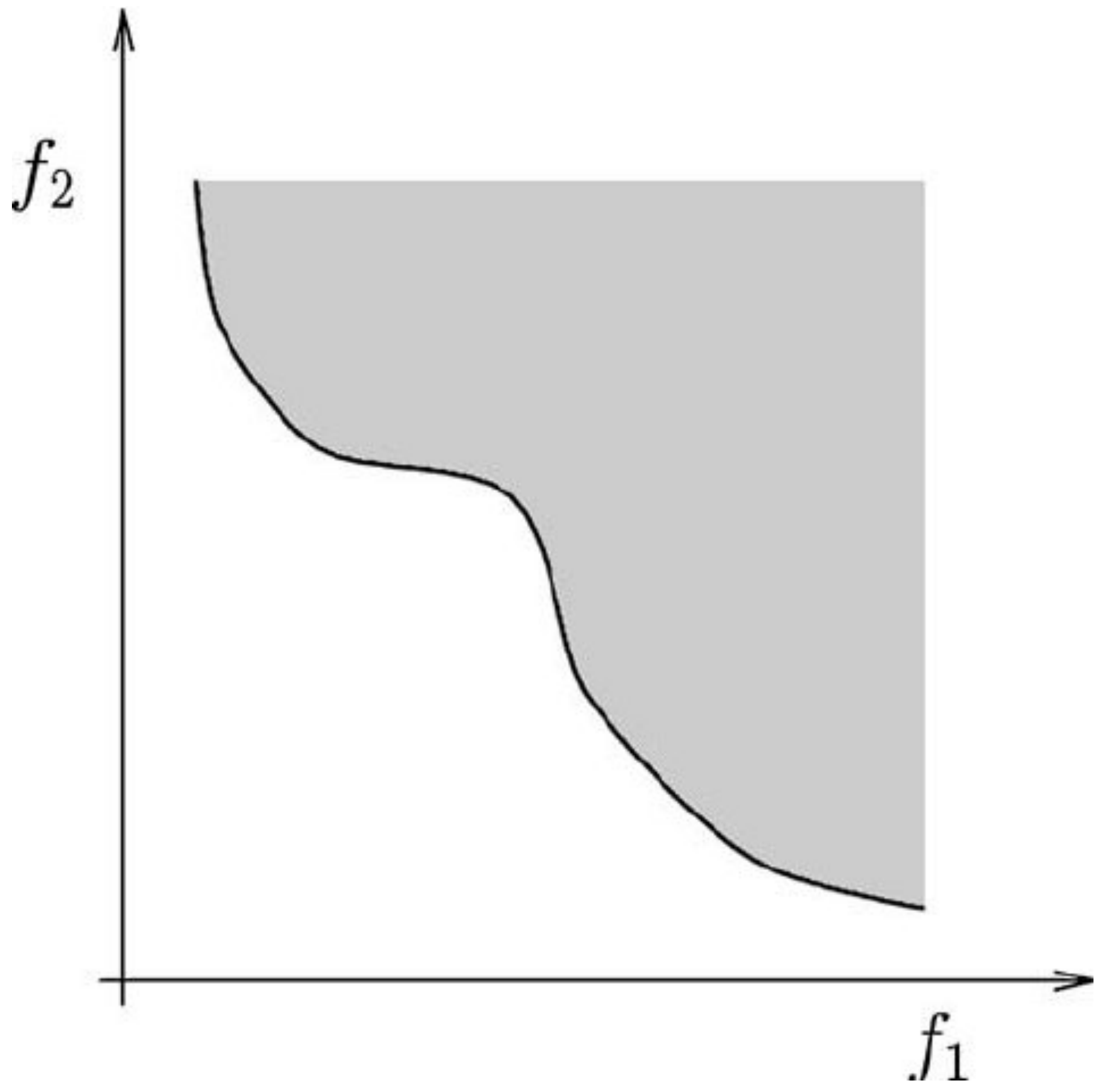
- Normalized weights ensure Pareto optimality

Weighted sum methods
Definition

Can we get all optimal points?

It depends on:

- Convexity of the domain
- Convexity of objectives



Censor, Y. (1977). Pareto optimality in multiobjective problems. Applied Mathematics and Optimization, 4(1), 41-59.

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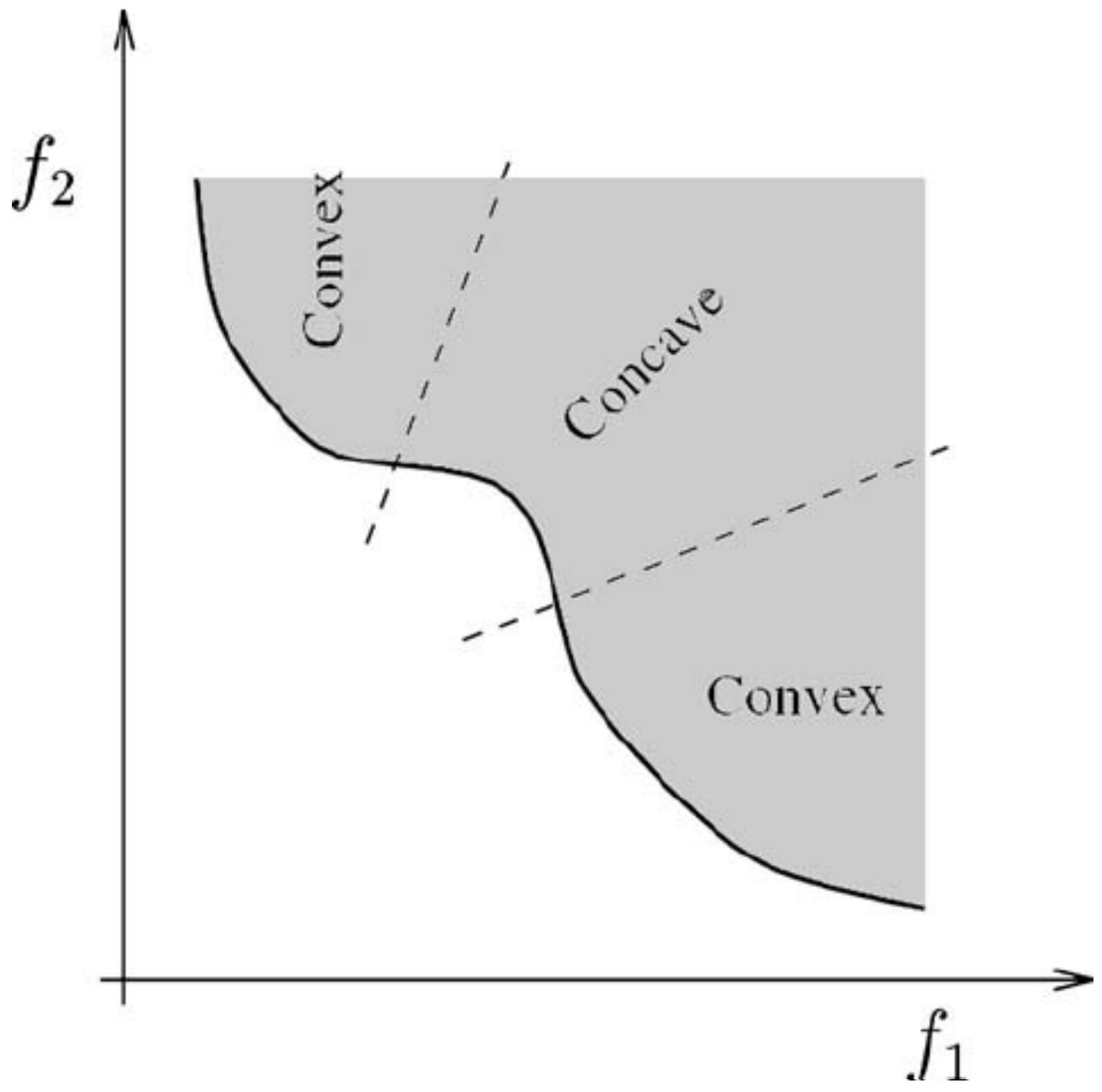

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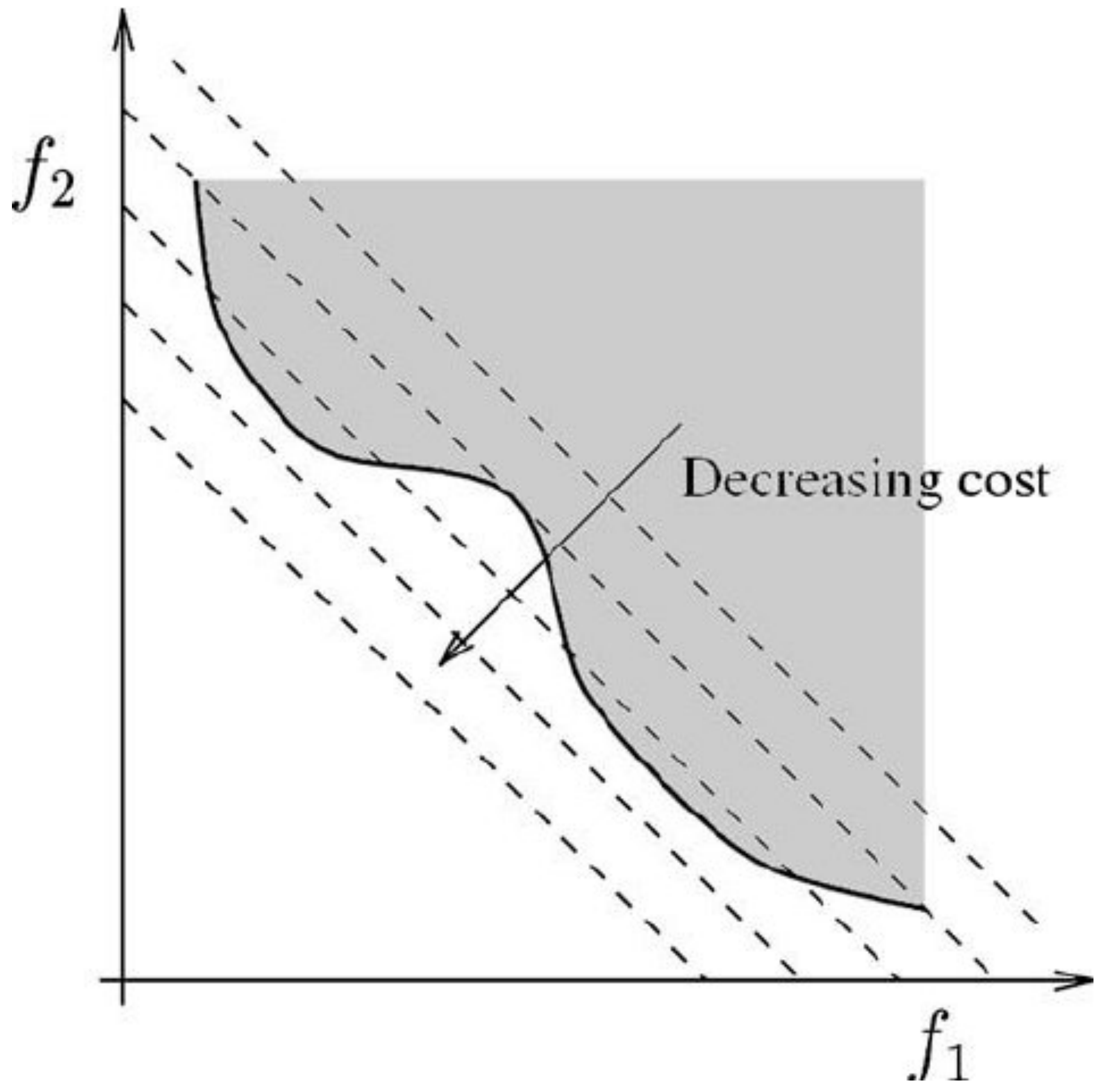

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Example

Convex to non-convex

$$f_1(x_1, x_2) = x_1$$

$$f_2(x_1, x_2) = 1 - x_1 - \alpha \sin(\beta \pi x_1) + x_2^2$$

Solve

$$\min_{\mathbf{x}} \left(w_1 f_1(x_1, x_2) + w_2 f_2(x_1, x_2) \right)$$

$$w_1 + w_2 = 1 \ , \qquad w_1, w_2 > 0$$

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$$\boxed{\begin{aligned} & x_1^{opt} = \frac{1}{\beta\pi} \cos^{-1} \left[\frac{1}{\alpha\beta\pi} \left(\frac{w_1}{w_2} - 1 \right) \right] \\ & x_2^{opt} = 0 \\ & 1 - \alpha\beta\pi \leq \frac{w1}{w_2} \leq 1 + \alpha\beta\pi \end{aligned}}$$

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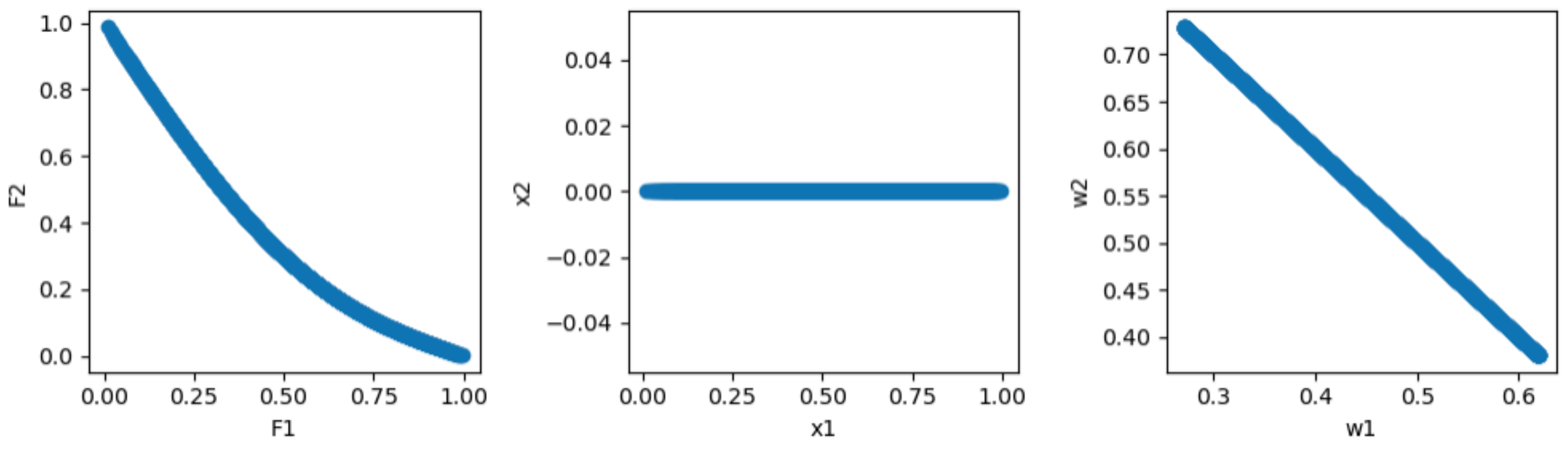
$$x_2^{opt} = 0$$

$$1 - \alpha\beta\pi \le \frac{w1}{w_2} \le 1 + \alpha\beta\pi$$

$$\alpha = 1/5 \qquad \beta = 1$$



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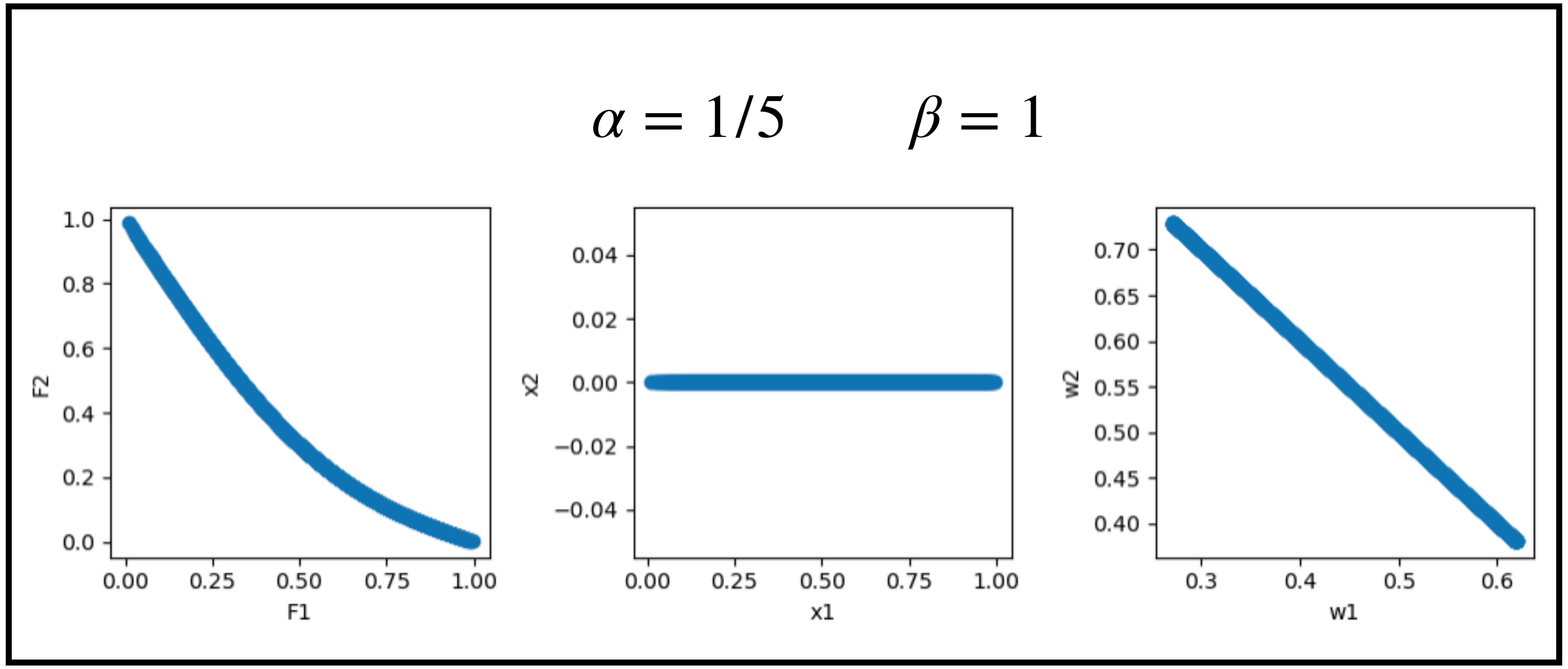

Solve

$$\min_{\mathbf{x}} \left(w_1 f_1(x_1, x_2) + w_2 f_2(x_1, x_2) \right)$$

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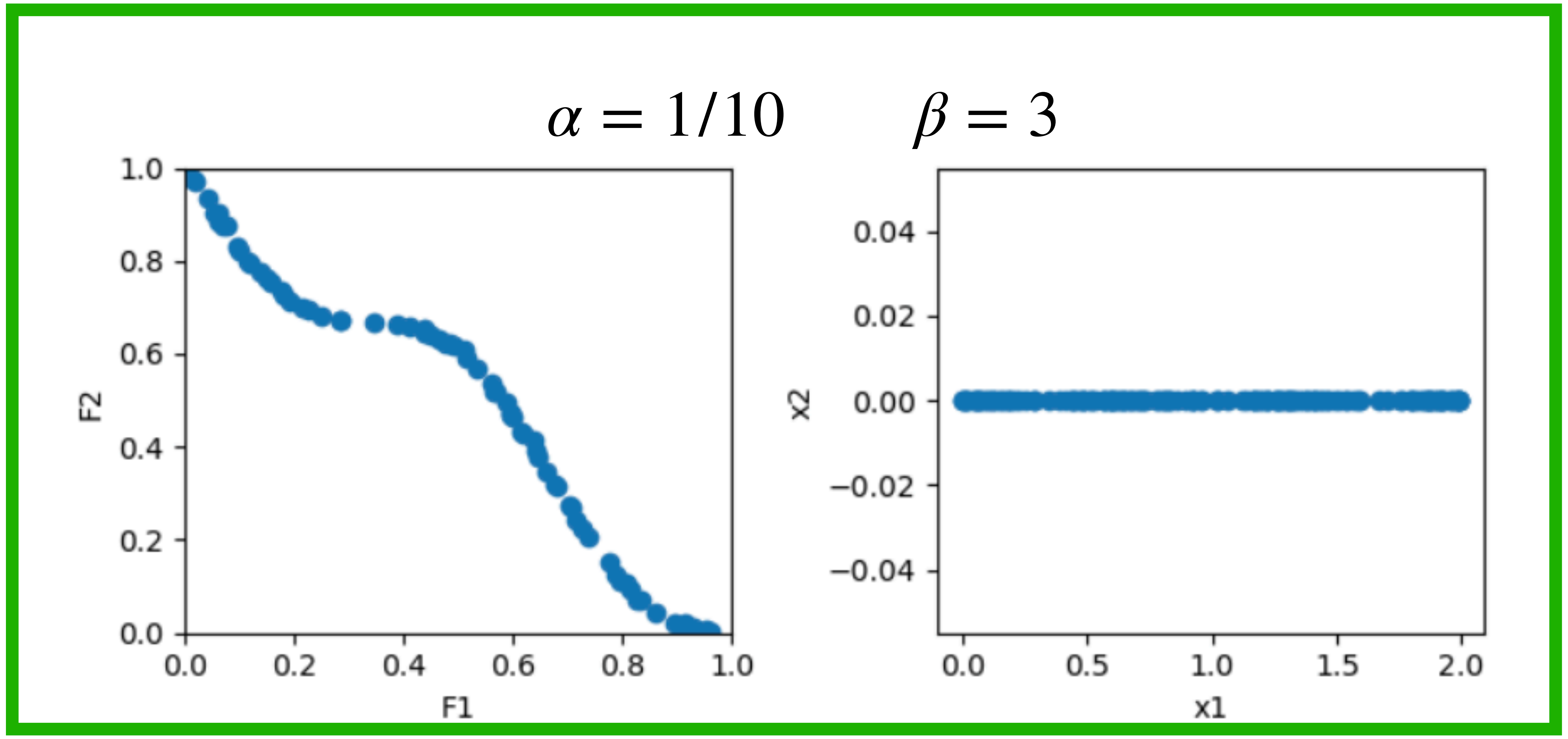

Weighted sum methods

We can now answer

Can we get all optimal points?

… not guaranteed …

A sufficient condition is:

- The domain $\mathscr{D}, S$ is convex
- Each objective $f_k(\mathbf{x})$ is convex

Censor, Y. (1977). Pareto optimality in multiobjective problems. Applied Mathematics and Optimization, 4(1), 41-59.

Weighted sum methods

We can now answer

Can we get all optimal points?

… not guaranteed …

Other methods can solve this limitation

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Weighted exponential *(P. L. Yu, 1973)*

The scalar objective to optimize becomes:

$$f_s = \sum_{i=1}^{M} w_i [f_i(\mathbf{x})]^p$$

where

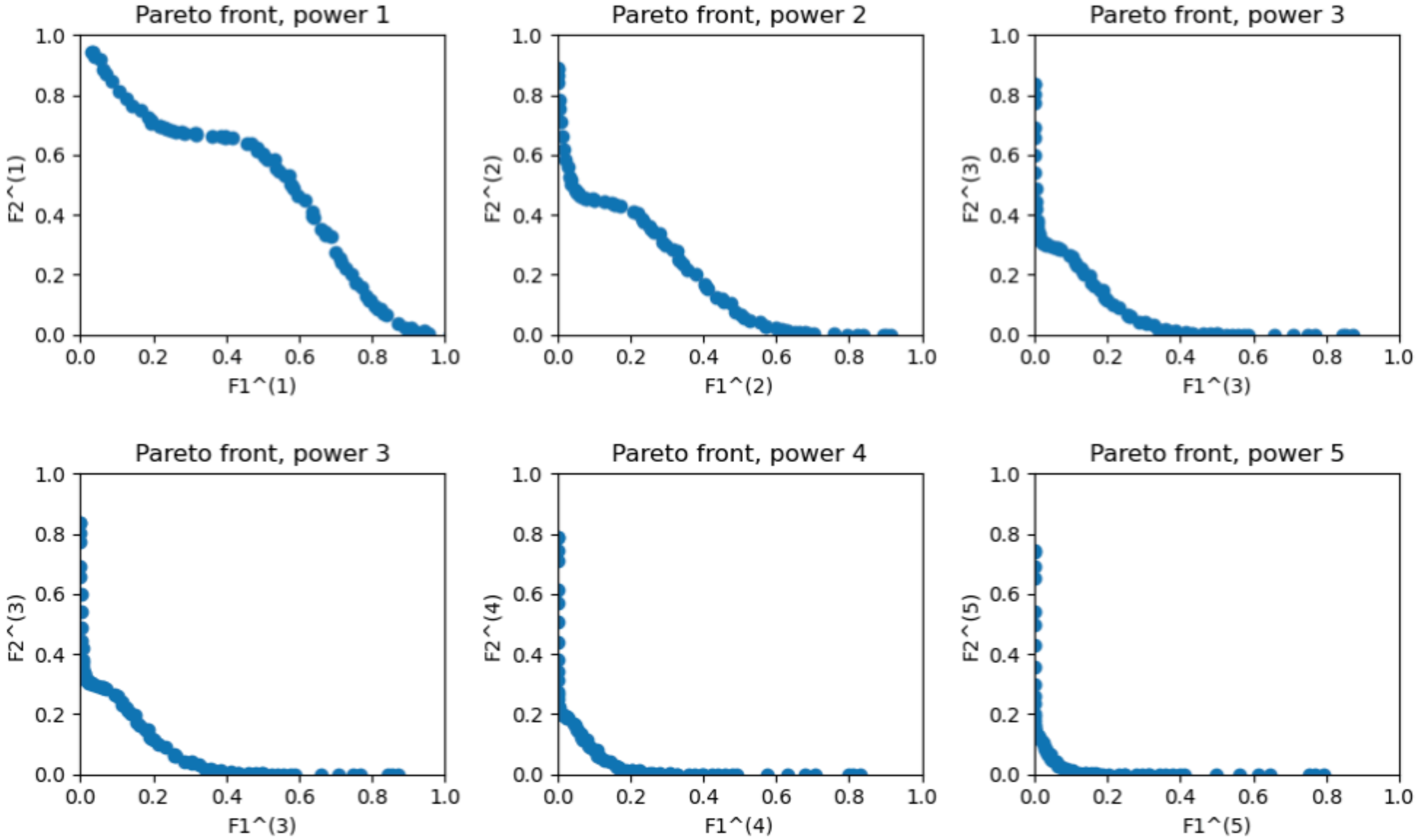
$$f_i(\mathbf{x}) > 0 \quad \forall i = 1, \ldots, M \qquad 1 \leq p < \infty$$

$$\mathbf{x} \in \mathscr{D} \qquad w_i > 0 \quad \forall i \in 1, \ldots, M \qquad \sum_{i=1}^{M} w_i = 1$$

- The condition on weights ensures optimality
- Bigger p, bigger effectiveness
- Bigger p, may give non-Pareto solutions

Weighted exponential *(P. L. Yu, 1973)*

Pictorial view of why it works:



Weighted metric *(P. L. Yu and G. Leitmann, 1974)*

Define the ideal point for each objective $f^* = (f_1{}^*, \ldots, f_M{}^*)$

1) Utopian point - data-driven - min value of each objective

2) Goal point - decision maker 

$$f_s = \left[\sum_{i=1}^{M} w_i^p \, | f_i(\mathbf{x}) - f_i{}^* |^p \right]^{1/p}$$

where

$$\mathbf{x} \in \mathscr{D} \qquad w_i > 0 \quad \forall i \in 1, \ldots, M$$

$$\sum_{i=1}^{M} w_i = 1, \quad 1 \le p < \infty$$

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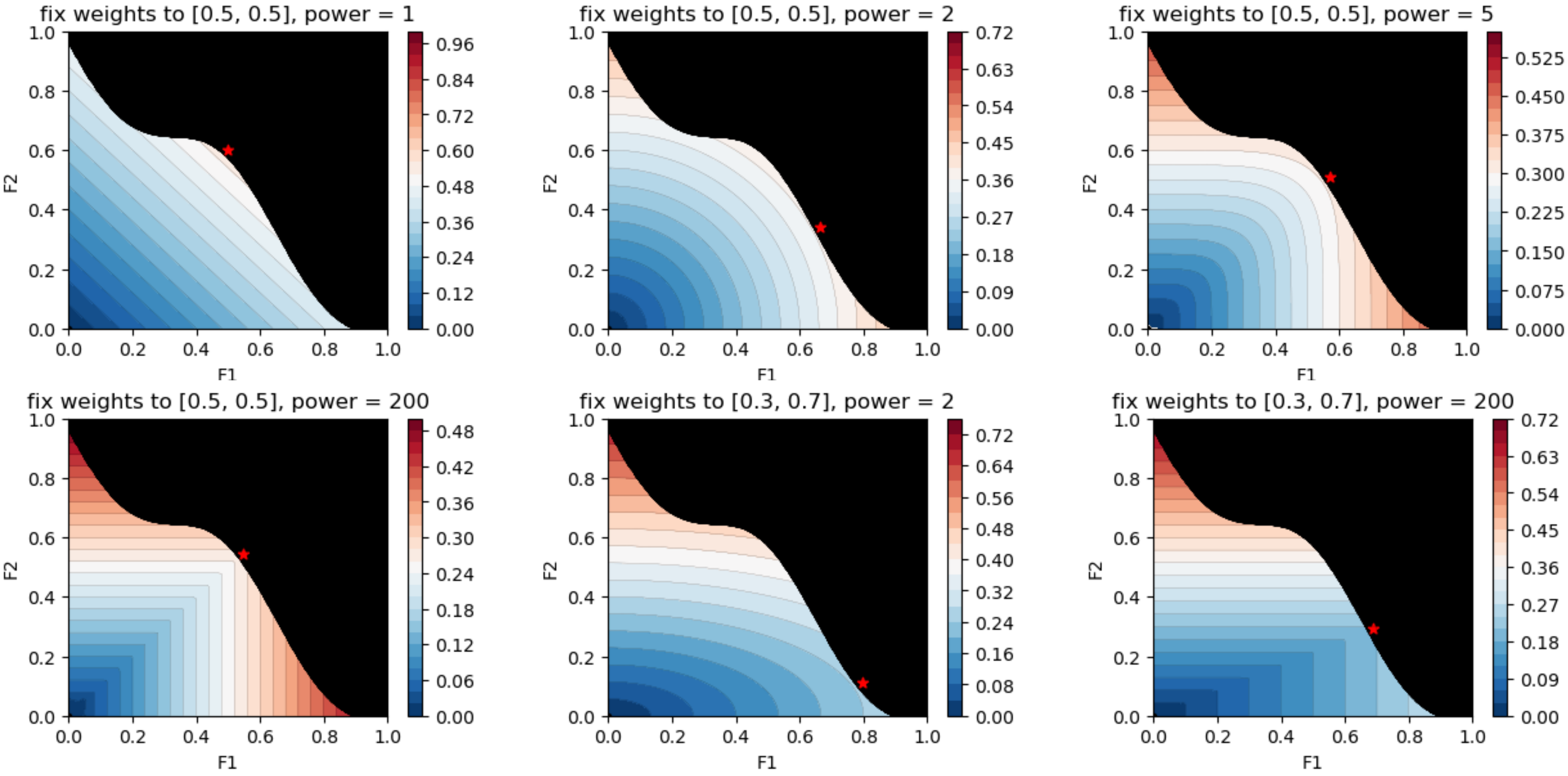
- Bigger p, bigger effectiveness but may be non Pareto
- Can play with ideal point

Weighted metric *(P. L. Yu and G. Leitmann, 1974)*



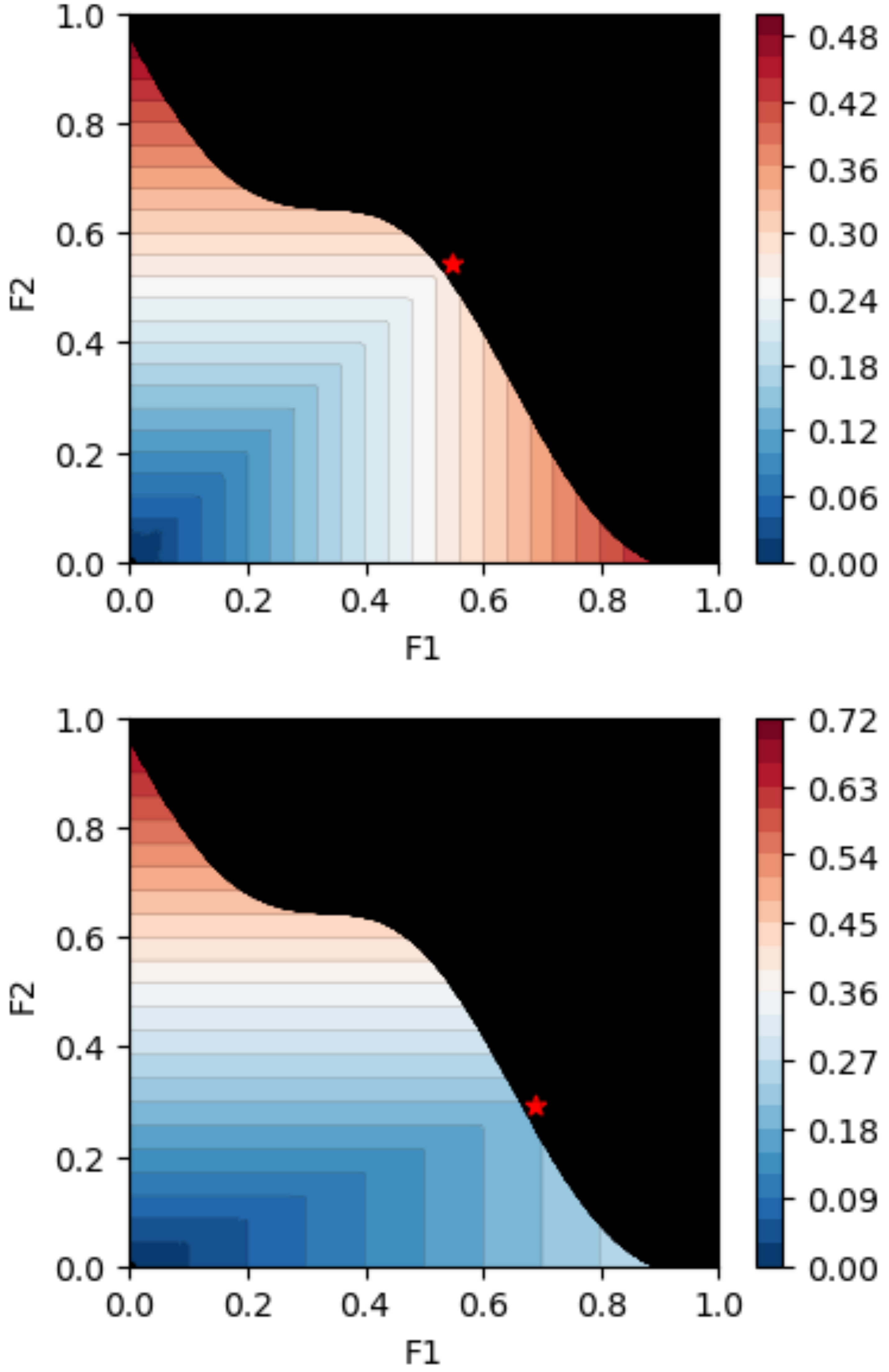
Weighted Chebyshev method *(Lightner, 1981; Messac, 2000)*

$$f_s = \max_{i \in \{1,\ldots,M\}} w_i \,|\, f_i(\mathbf{x}) - f_i^* \,|$$

$$\mathbf{x} \in \mathscr{D} \qquad w_i > 0 \quad \forall i \in 1,\ldots,M$$

$$\sum_{i=1}^{M} w_i = 1, \quad 1 \le p < \infty$$

- Equivalent to taking $p \rightarrow \infty$
- Can overcome lack of convexity conditions
- May get non-Pareto solutions



Other weighted methods

Exponential weighted criterion *(Athan, Papalambros, 1996)*

$$f_s = \sum_{i=1}^{M} \left(e^{pw_i} - 1\right) e^{pf_i(\mathbf{x})} \qquad p > 1$$

- Can overcome lack of convexity conditions
- Problems with numerical stability

Weighted product method *(Gerasimov, 1979)*

$$f_s = \prod_{i=1}^{M} |f_i(\mathbf{x})|^{w_i}$$

- Deal with objectives with different magnitude

ε-constraint method *(Haimes, Lasdon, Wismer, 1971)*

Algorithm:

- Create a grid of possible upper bounds for each objective $g_{f_k} = \{v_0^k, \ldots, v_G^k\}$
- For each objective, consider the (d-1)-dimensional grid related to the other objectives. A single point of the grid is given by $\overrightarrow{\epsilon}_k = \{v^1, \ldots, v^{k-1}, v^{k+1}, \ldots, v^M\}$; $v^i \in g_{f_i}$.
- For each point of the grid solve:

$$f_s(\overrightarrow{\epsilon}_k) = \min f_k(\mathbf{x})$$

$$f_i(\mathbf{x}) < v^i \quad \forall i \neq k$$

Pros/Cons:

- No convex requirement
- Solutions may not be Pareto optimal

ε-constraint method *(Haimes, Lasdon, Wismer, 1971)*

Visual inspection

Solutions are PO if unique

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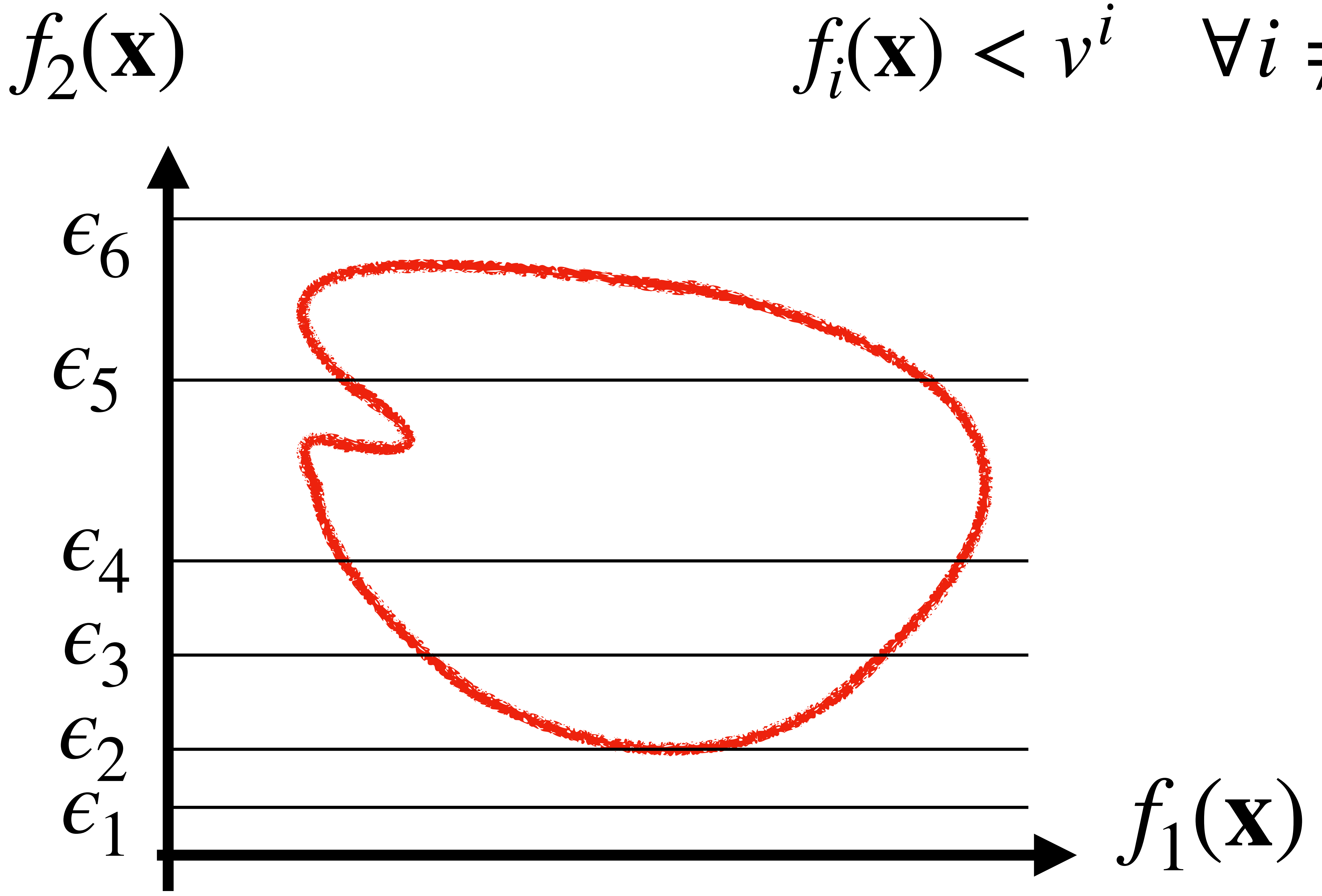

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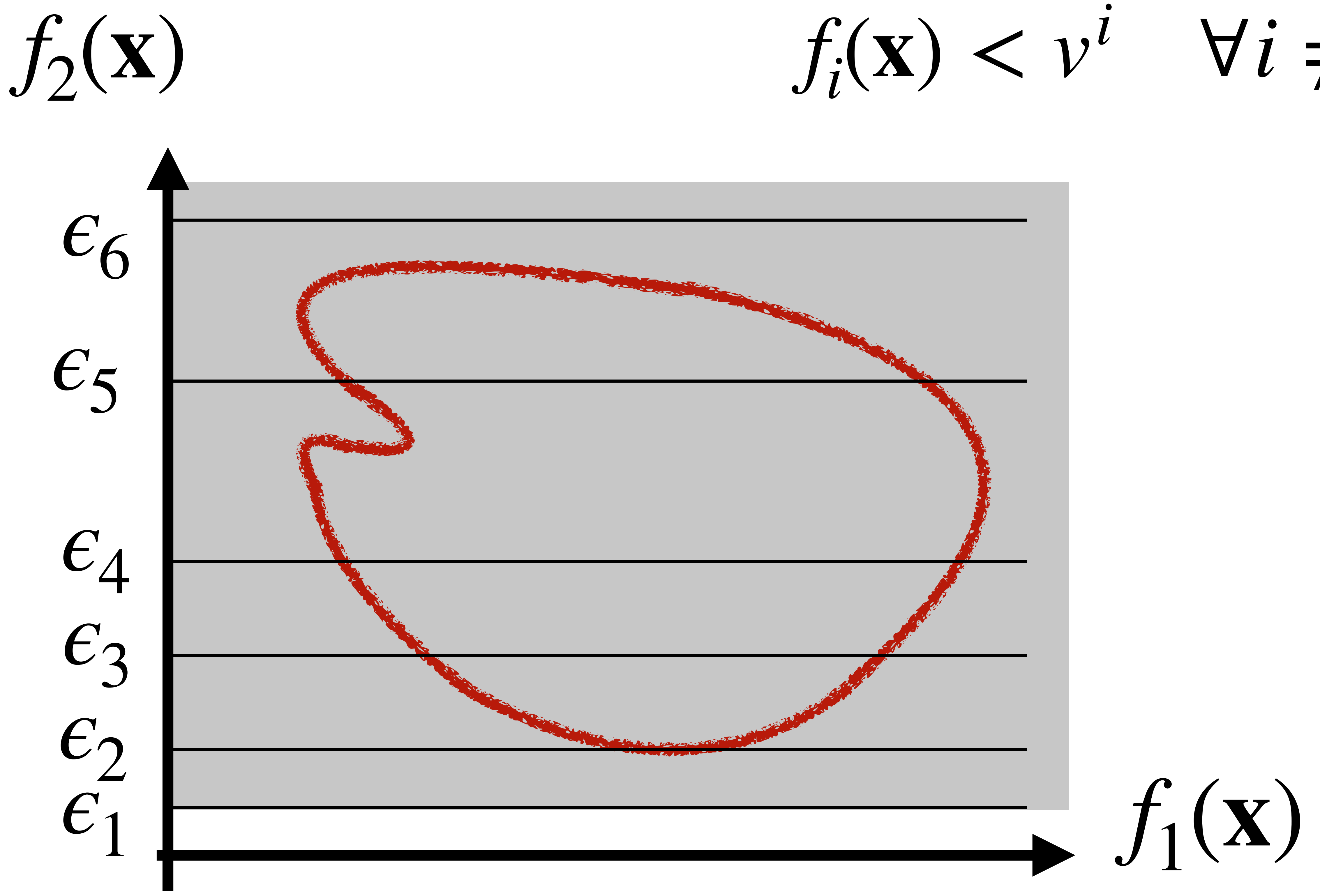
ϵ_1 no solution (PO?)



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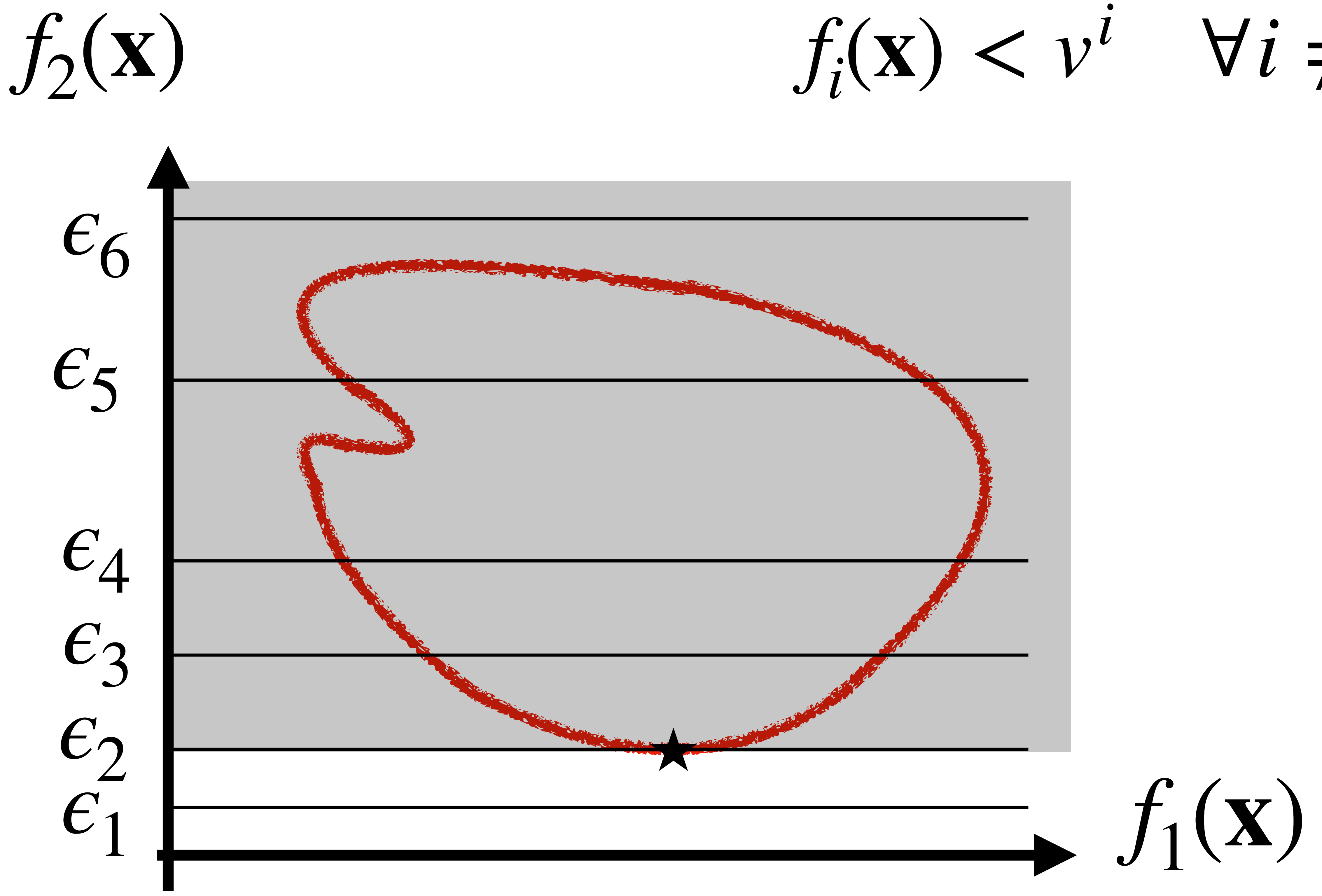

ϵ_1 no solution (PO?)

ϵ_2 1 solution PO

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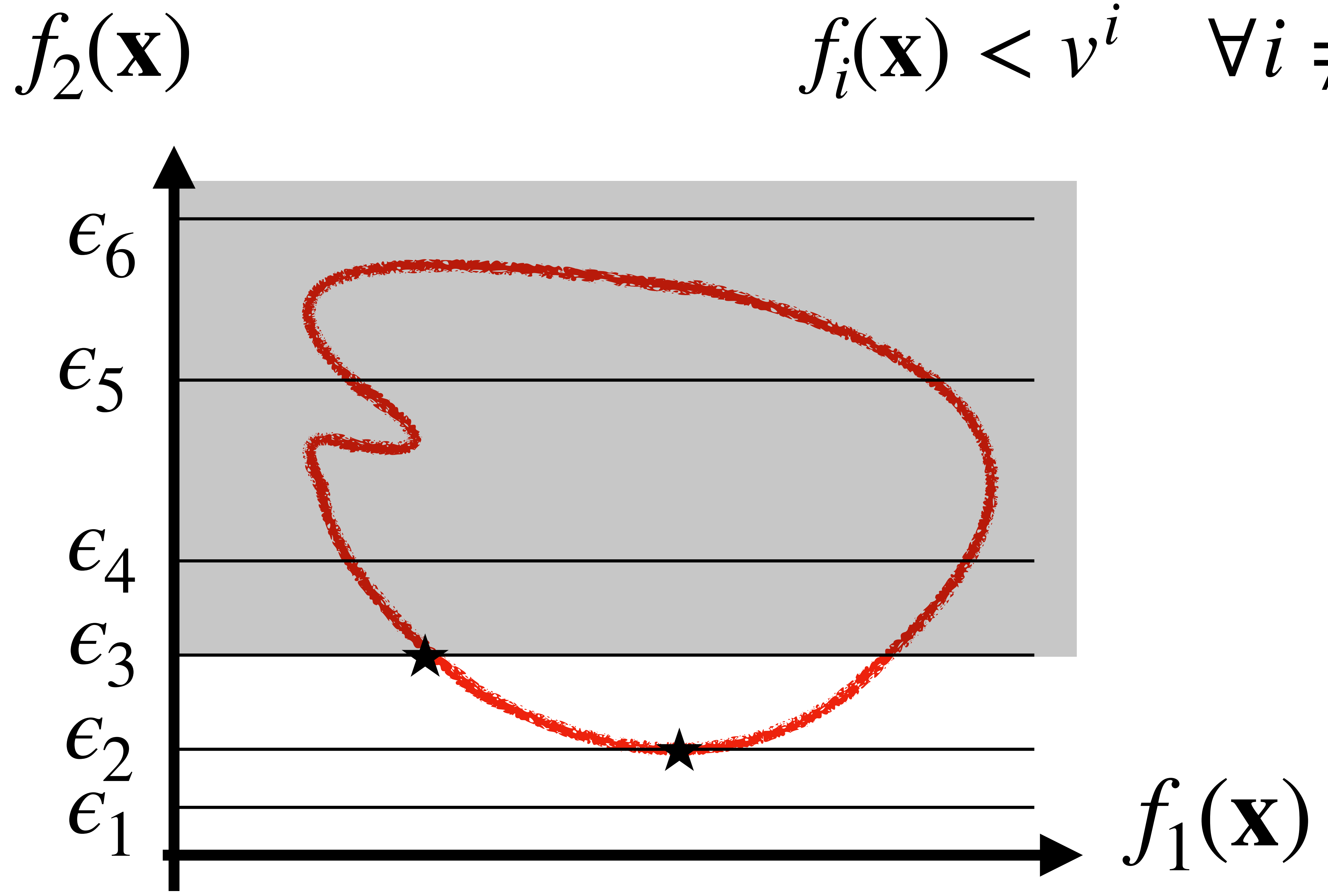
ϵ_1 no solution (PO?)

ϵ_2, ϵ_3 1 solution PO



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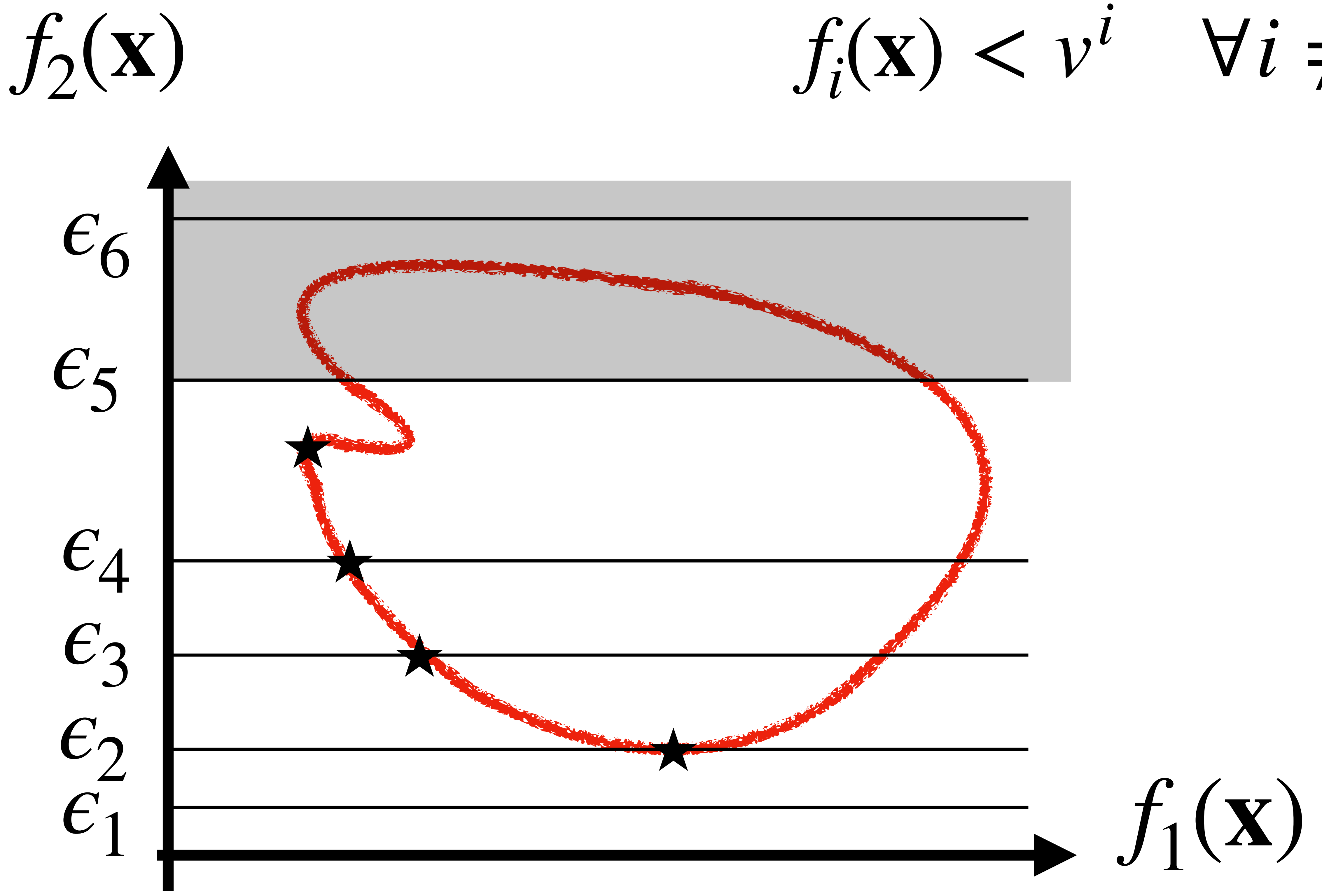
$\epsilon_2, \ldots, \epsilon_5$ 1 solution PO



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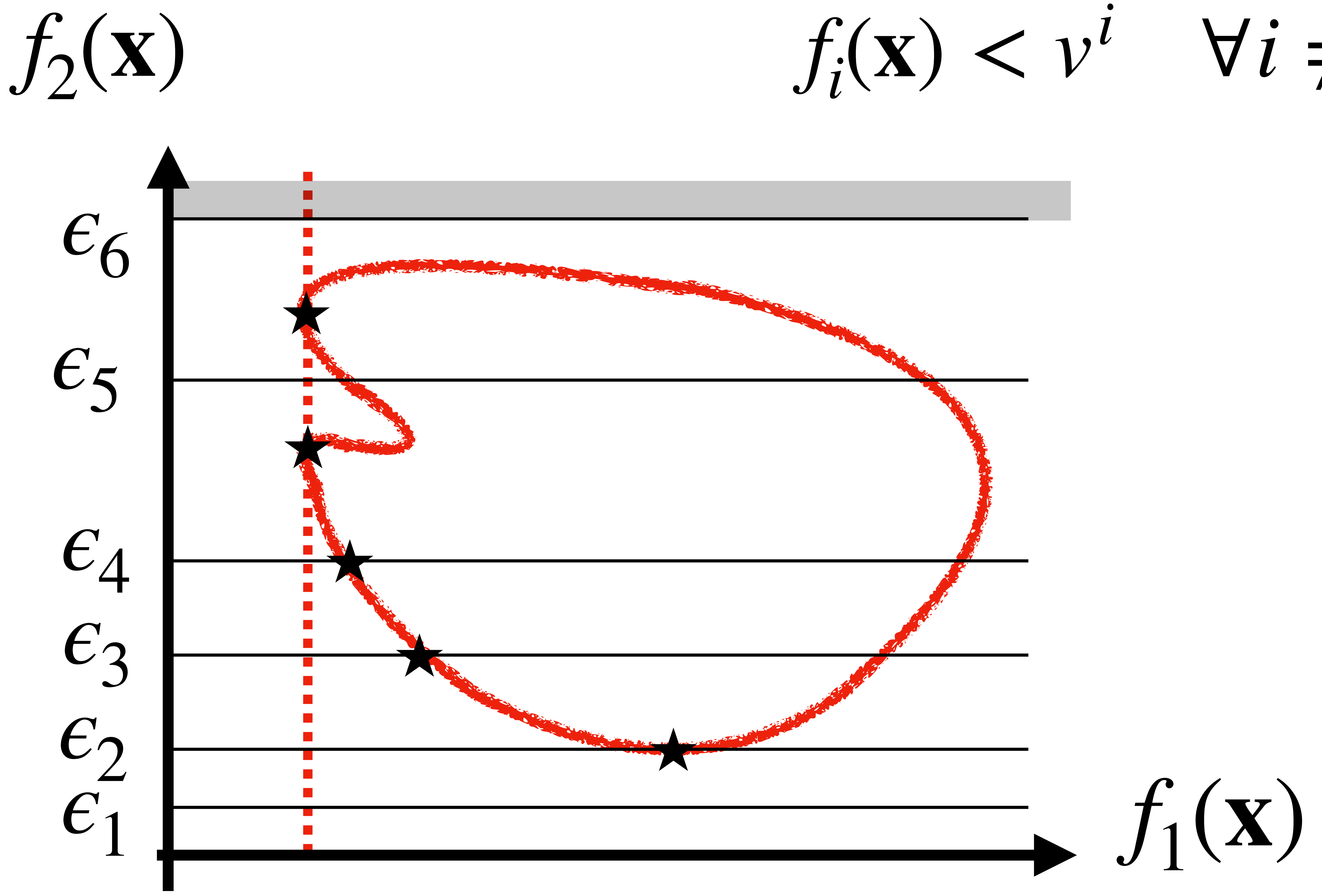
Solutions are PO if unique

ϵ_1	no solution (PO?)
$\epsilon_2, \ldots, \epsilon_5$	1 solution PO
ϵ_6	2 solutions (PO?)

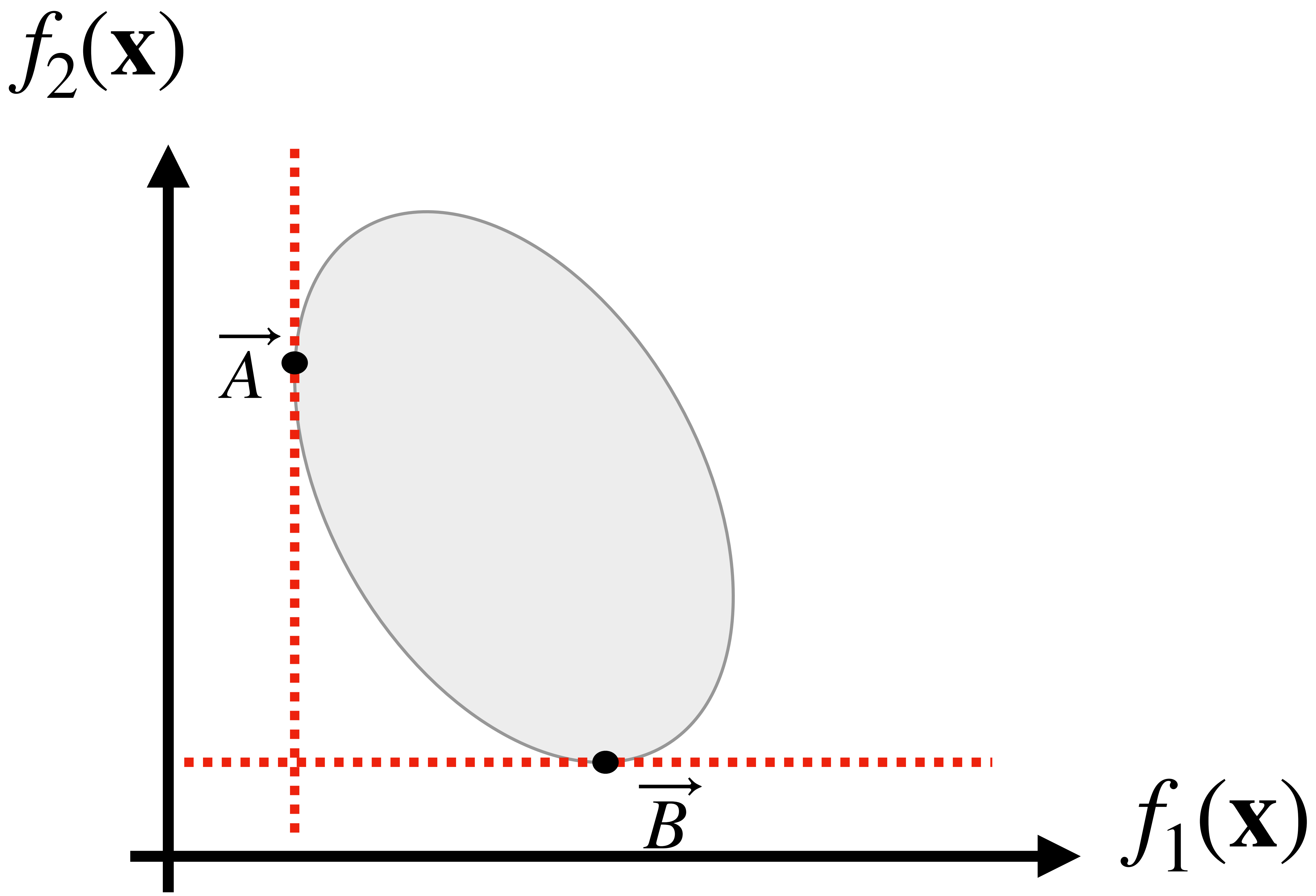


Normal boundary intersection *(Dennis, 1998; Messac, 2003)*

Algorithm:

- Find anchor points

$$\overrightarrow{A} = \{\mathbf{x}_1{}^* = \min_{x \in \mathscr{D}} f_1(\mathbf{x})\} \; ; \qquad \overrightarrow{B} = \{\mathbf{x}_2{}^* = \min_{x \in \mathscr{D}} f_2(\mathbf{x})\}$$



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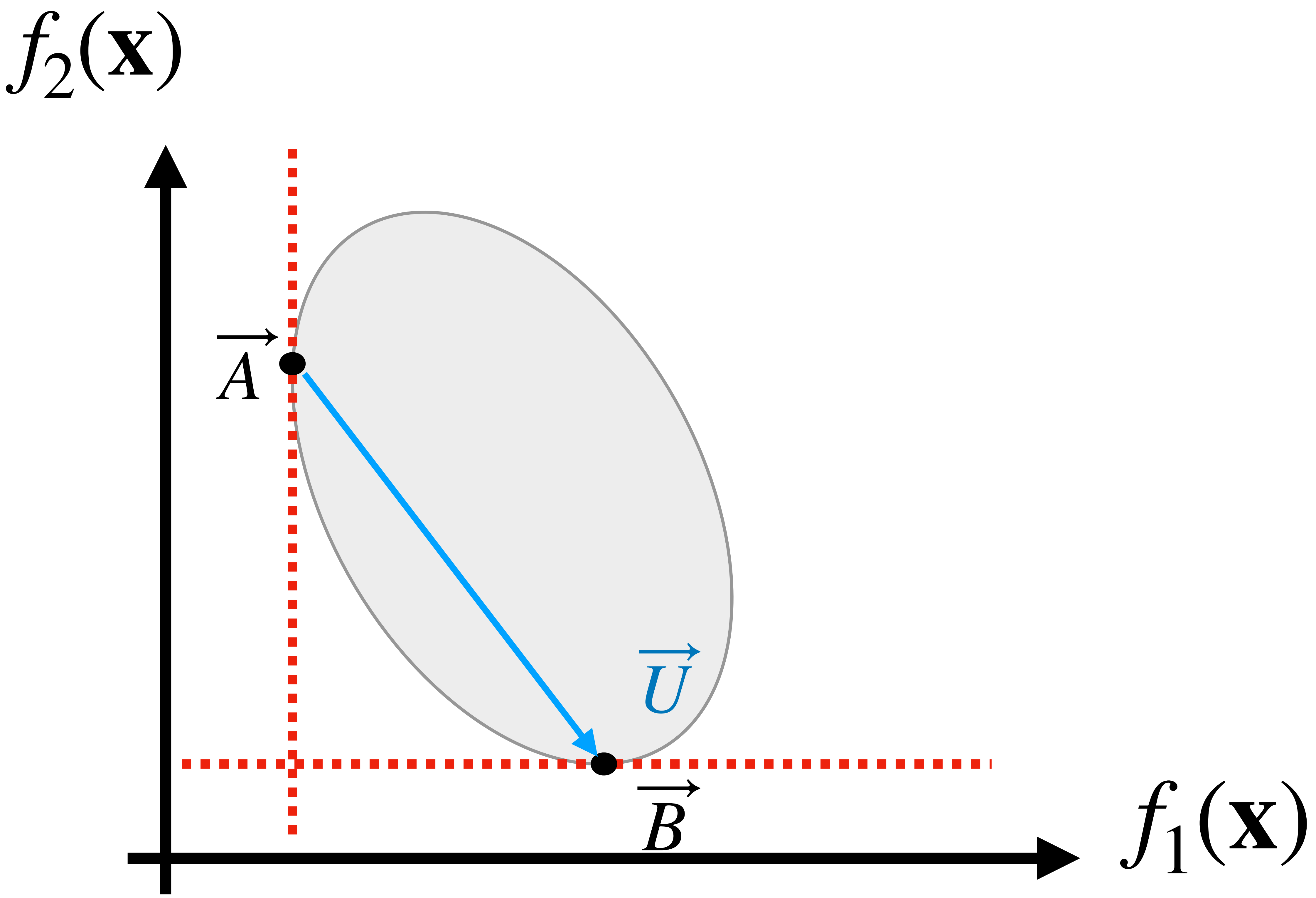
- Define utopian hyperplane through anchors

$$\overrightarrow{U} = \overrightarrow{A} - \overrightarrow{B}$$



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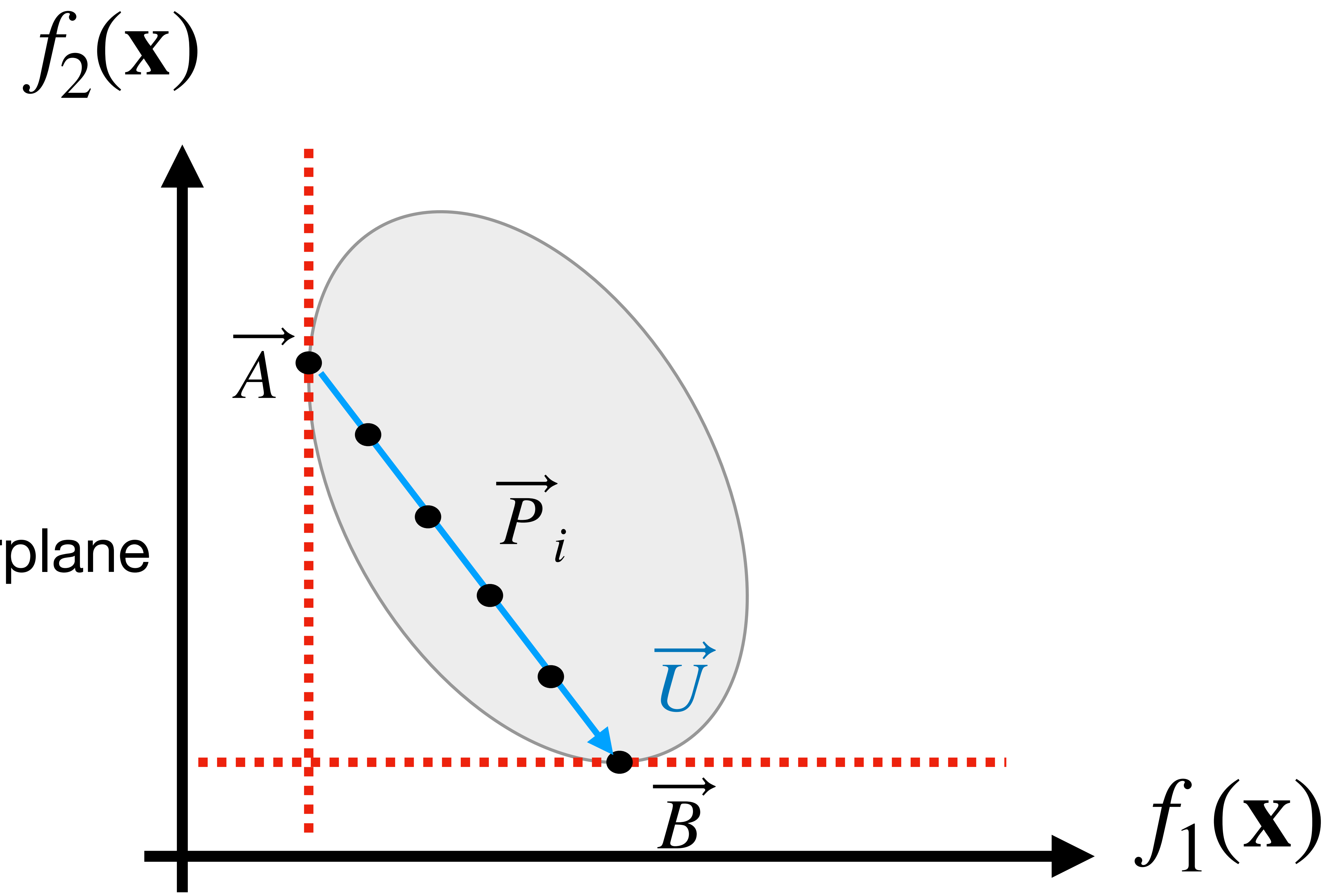
- Define utopian hyperplane through anchors

$$\vec{U} = \vec{A} - \vec{B}$$

- Define an evenly distributed grid on the utopia hyperplane

$$W = \{\vec{w}_i\} = \{(w_{1i}, w_{2i})\} = \{(\delta i, 1 - \delta i)\}_{i=0,\ldots,1/\delta}$$
$$P = \{\vec{P}_i\} = \{w_{1i}\vec{A} + w_{2i}\vec{B}\}_{i=0,\ldots,1/\delta}$$



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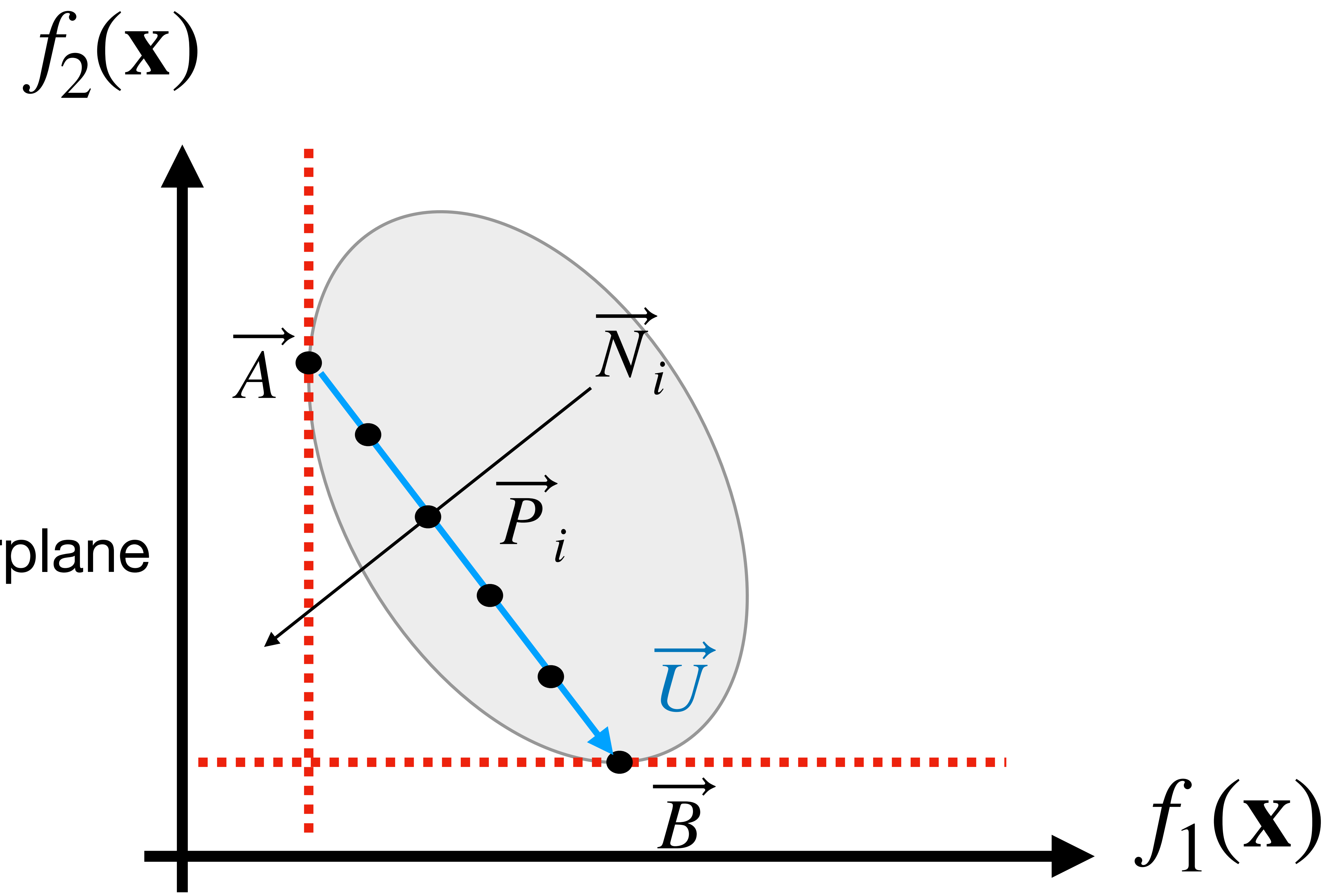
$$P = \{\vec{P}_i\} = \{w_{1i}\vec{A} + w_{2i}\vec{B}\}_{i=0,\ldots,1/\delta}$$

- Define the normal vector at each grid point

$$N = \{\vec{N}_i\} \quad s.t. \quad \vec{P}_i \cdot \vec{N}_i = 0$$



Normal boundary intersection *(Dennis, 1998; Messac, 2003)*

Optimize the objectives along each normal vector

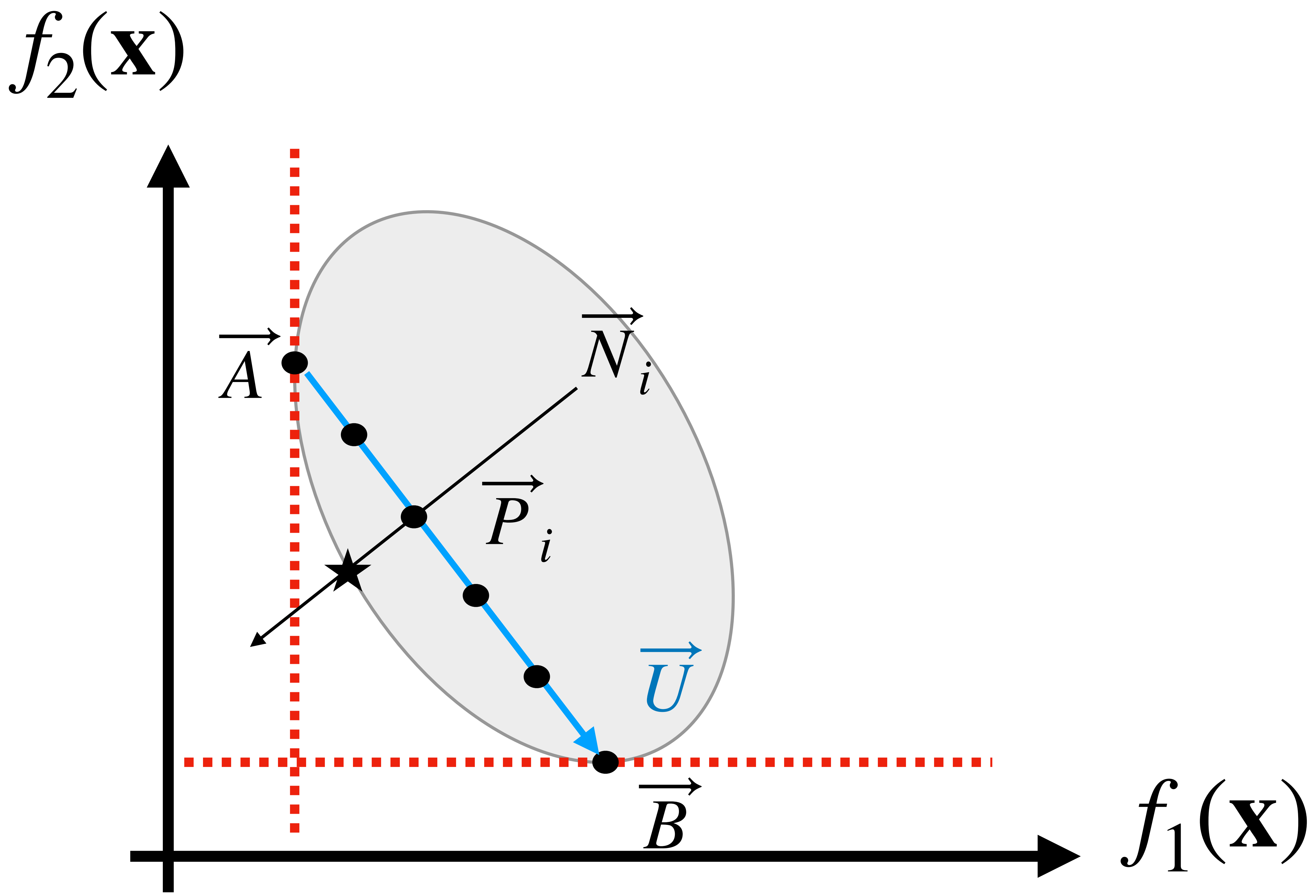
$$f_s = \min_{\mathbf{x} \in \mathscr{D}} f_1(\mathbf{x})$$

$$g_s = \overrightarrow{U} \cdot (\overrightarrow{F}(\mathbf{x}) - P_i) = 0$$

$$\overrightarrow{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$$

Pros/Cons:

- No convex requirement
- Solutions may not be Pareto optimal



Other classical methods

- Goal programming
- Simplex method
- Physical programming
- Lexicographic methods
- ….

References:

A. *Keller, André A. Multi-objective optimization in theory and practice i: classical methods. Bentham Science Publishers, 2017.*
B. *Miettinen, Kaisa. Nonlinear multiobjective optimization. Vol. 12. Springer Science & Business Media, 1999*
C. *Blank, J., & Deb, K. (2020). Pymoo: Multi-objective optimization in python. Ieee access, 8, 89497-89509.*
D. *Deb, K., Sindhya, K., & Hakanen, J. (2016). Multi-objective optimization. In Decision sciences (pp. 161-200). CRC Press.*

Take home message

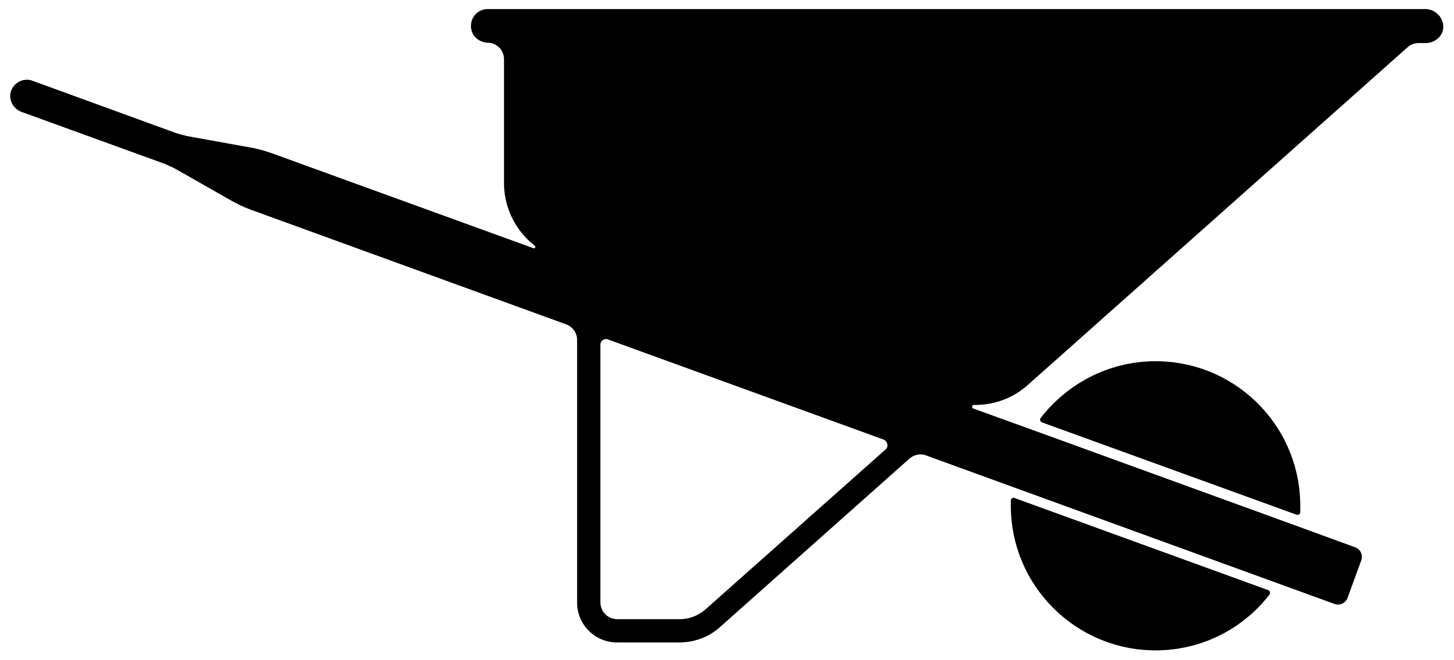
- Classical MOO algorithms usually exploit scalarization methods

- Whenever possible, **rescale objectives**. The trade-off between conflicting objectives reflects the rate of change in one objective resulting from a unit increment in another objective.

- Always check domain and objective convexity

- Scalarization methods robust to non-convexity need Pareto optimality check

Hands on



Case study: Robust Optimization

Robustness: technological insensitivity to variability in industrial and environmental factors.

What about optimization?

What if we have some **noise** in the decision variable space or in the numerical coefficients?

Case study: Robust Optimization

The expected value of the objective becomes

$$F(f, \mathbf{x}, \alpha, \epsilon) = \int \ldots \int f(\mathbf{x} + \epsilon_x, \alpha + \epsilon_\alpha) p(\epsilon_x) p(\epsilon_\alpha) \prod_i d\epsilon_x^i \prod_j d\epsilon_\alpha^j$$

In 1D with only noise in x

$$F(f, x, \epsilon) = \int_{-\infty}^{\infty} f(x + \epsilon, \alpha)\, p(\epsilon)\, d\epsilon$$

Case study: Robust Optimization

So, the following constrained minimization problem

$$\min_{\mathbf{x}\in\mathscr{D}} f(\mathbf{x},\alpha), \qquad \text{satisfying} \qquad g_j(\mathbf{x},\alpha) \leq 0 \qquad j = 1,\ldots,J$$

Is rephrased as follows in the presence of uncertainties

$$\min_{\mathbf{x}\,,\,\alpha} F\left[f(\mathbf{x}+\epsilon_x,\alpha+\epsilon_\alpha)\right] \qquad \text{satisfying} \qquad G_j\left[g_j(\mathbf{x}+\epsilon_x,\alpha+\epsilon_\alpha)\right] \leq 0 \qquad j = 1,\ldots,J$$

It incorporates uncertainties!

Case study: Robust Optimization

Let us assume for simplicity that random variables are independent:

$$p(\overrightarrow{\epsilon_x}) = \prod_{i=1}^{n} p_i(\epsilon_{xi}) \qquad\qquad p(\overrightarrow{\epsilon_\alpha}) = \prod_{\rho=1}^{P} p_\rho(\epsilon_{\alpha\rho})$$

In general, we assume p to be Gaussian PDFs. In practice, F, G use the mean and variance of f and g_j

$$F = w_1\mu_f + w_2\sigma_f \qquad\qquad G_j = \mu_{g_j} + k_j\sigma_{g_j}^2$$

Case study: Robust Optimization

$$F = w_1\mu_f + w_2\sigma_f \qquad G_j = \mu_{g_j} + k_j\sigma^2_{g_j}$$

Where (same for G_j)

$$\mu_f = \mathbf{E}[f(\mathbf{x},\alpha)] = \int \dots \int f(\mathbf{x}+\epsilon_x, \alpha+\epsilon_\alpha)\prod_i p_i(\epsilon_x^i)d\epsilon_x^i \prod_\rho p_\rho(\epsilon_\alpha^\rho)d\epsilon_\alpha^\rho$$

$$\sigma_f = \mathbf{E}[(f(\mathbf{x},\alpha)-\mu_f)^2] = \int \dots \int \Big(f(\mathbf{x}+\epsilon_x, \alpha+\epsilon_\alpha)-\mu_f\Big)^2\prod_i p_i(\epsilon_x^i)d\epsilon_x^i \prod_\rho p_\rho(\epsilon_\alpha^\rho)d\epsilon_\alpha^\rho$$

We can simplify these formulas by expanding them around they mean value using Taylor series.

Case study: Robust Optimization

$$F = w_1\mu_f + w_2\sigma_f \qquad G_j = \mu_{g_j} + k_j\sigma_{g_j}^2$$

In practice:

$$\mu_f \simeq f(\mu_x, \mu_\alpha) \qquad \mu_{g_j} \simeq g_j(\mu_x, \mu_\alpha)$$

$$\sigma_f^2 \simeq \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \sum_{\rho=1}^{K}\left(\frac{\partial f}{\partial \alpha_\rho}\right)^2 \sigma_{\alpha_\rho}^2 \qquad \sigma_{g_j} \simeq \sum_{i=1}^{n}\left(\frac{\partial g_j}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \sum_{\rho=1}^{K}\left(\frac{\partial g_j}{\partial \alpha_\rho}\right)^2 \sigma_{\alpha_\rho}^2$$

If **tolerance/bounded uncertainties** are specified $|\epsilon_{x_i}|, |\epsilon_{\alpha_\rho}|$, the constraints are usually reformulated as

$$G_j = \mu_{g_j} + k_{x_j}\sum_{i=1}^{n}\left|\frac{\partial g_j}{\partial x_i}\right| |\epsilon_{x_i}| + k_{\alpha_j}\sum_{\rho=1}^{K}\left|\frac{\partial g_j}{\partial \alpha_\rho}\right| |\epsilon_{\alpha_\rho}|$$

Case study: Robust Optimization

Summarising, the minimization problem is reformulated as:

$$F = \min_{\mathbf{x} \in \mathscr{D}} \left[w_1 f(\mu_x, \mu_\alpha) + w_2 \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{\rho=1}^{K} \left(\frac{\partial f}{\partial \alpha_\rho} \right)^2 \sigma_{\alpha_\rho}^2} \right]$$

subject to

$$G_j = \mu_{g_j} + k_{x_j} \sum_{i=1}^{n} \left| \frac{\partial g_j}{\partial x_i} \right| |\epsilon_{x_i}| + k_{\alpha_j} \sum_{\rho=1}^{K} \left| \frac{\partial g_j}{\partial \alpha_\rho} \right| |\epsilon_{\alpha_\rho}| \leq 0 \qquad j = 1, \ldots K$$

or

$$G_j = \min_{\mathbf{x} \in \mathscr{D}} \left[w_1' g_j(\mu_x, \mu_\alpha) + w_2' \sqrt{\sum_{i=1}^{n} \left(\frac{\partial g_j}{\partial x_i} \right)^2 \sigma_{x_i}^2 + \sum_{\rho=1}^{K} \left(\frac{\partial g_j}{\partial \alpha_\rho} \right)^2 \sigma_{\alpha_\rho}^2} \right] \leq 0 \qquad j = 1 \ldots, K$$

Hands on