Generalized Image Osmosis Filtering

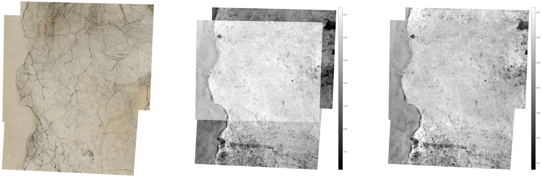
with shadow removal imaging applications

Simone Parisotto[1] joint work with L. Calatroni[2], M. Caliari[3] and C.B. Schönlieb[4]

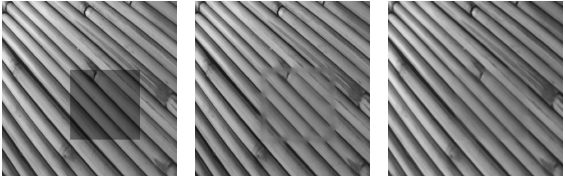
[1]Cambridge Centre for Analysis, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK
[2]CMAP, Ecole Polytechnique, 91128 Palaiseau Cedex, France
[3]University of Verona, Strada Le Grazie 15, 37134, Verona, Italy
[4]DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

SIAM Imaging Science 2018 - Bologna

1. **Efficiency**: standard dimensional splitting on osmosis for large images in CH imaging;



2. **Anisotropic**: directional (non-) linear inpainting for removing constant shadow.



Standard osmosis equation

Weickert, Vogel, Setzer, Hagenburg, Breuß ('11-'13): given $v : \Omega \to \mathbb{R}^3$, find $u : \Omega \to \mathbb{R}^3$:

$$\min_u E^v(u), \quad E^v(u) := \int_\Omega v \left| \nabla \left(\frac{u}{v} \right) \right|^2 \mathrm{d}x.$$

Here, u is the steady-state solution, of the following drift-diffusion PDE:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) = \Delta u - \operatorname{div}(\mathbf{d}u) & \text{on } \Omega \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega, \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T], \end{cases}$$

where $\mathbf{d} = \nabla \log v$ and $f : \Omega \to \mathbb{R}$.

Standard osmosis equation

Weickert, Vogel, Setzer, Hagenburg, Breuß ('11-'13): given $v : \Omega \to \mathbb{R}^3$, find $u : \Omega \to \mathbb{R}^3$:

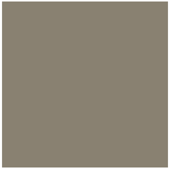
$$\min_u E^v(u), \quad E^v(u) := \int_\Omega v \left| \nabla \left(\frac{u}{v} \right) \right|^2 \mathrm{d}x.$$

Here, u is the steady-state solution, of the following drift-diffusion PDE:

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) = \Delta u - \operatorname{div}(\mathbf{d}u) & \text{on } \Omega \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega, \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T], \end{cases}$$

where $\mathbf{d} = \nabla \log v$ and $f : \Omega \to \mathbb{R}$.

v



u^{t_0}

The shadow removal problem

Good news: we can modify (slightly) the vector field $\mathbf{d}$ for other imaging applications.

Let $u : \Omega \to \mathbb{R}^+$ be a positive image corrupted by a constant shadow, $\Omega = \Omega_{\text{out}} \cup \Omega_{\text{sb}} \cup \Omega_{\text{in}}$.

Decomposition of a shadowed image into Ω_{out}, Ω_{sb} and Ω_{in}.

Shadow removal problem via osmosis: recover the correct light coefficient

$$\begin{cases} \partial_t u = \Delta u - \operatorname{div}(\mathbf{d} u) & \text{on } \Omega_{\text{in}} \cup \Omega_{\text{out}} \times (0, T], \\ \partial_t u = \Delta u & \text{on } \Omega_{\text{sb}} \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega \\ \langle \nabla u - \mathbf{d} u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases} \quad \text{where} \quad \mathbf{d} := \begin{cases} \nabla \log f & \text{on } \Omega_{\text{in}} \cup \Omega_{\text{out}} \\ 0 & \text{on } \Omega_{\text{sb}} \end{cases}$$

Osmosis acts as inpainting on Ω_{sb}: Δu diffuses the structures isotropically and smoothly.

Possible solution: post-processing inpainting step as Arias, Facciolo, Caselles & Sapiro ('11)

Shadowed.

Boundary.

Standard osmosis.

Inpainted 1x1.

Standard osmosis (zoom).

Inpainted (zoom).

Properties of standard osmosis equation

Continuous osmosis model

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) = \Delta u - \operatorname{div}(\mathbf{d}u) & \text{on } \Omega \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega, \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

Continuous model, Proposition 1 in Weickert et al. ('13)

The solution of the osmosis model enjoys the following properties:

- **mass conservation (AVG)**: $\frac{1}{|\Omega|} \int_\Omega u(x,t)\,\mathrm{d}x = \frac{1}{|\Omega|} \int_\Omega f(x)\,\mathrm{d}x$ for all $t > 0$;
- **non-negativity**: $u(x,t) \geq 0$ for all $x \in \Omega$ and $t > 0$ (osmosis may violate max-min);
- **non-constant steady states**: given $v > 0$ and $\mathbf{d} := \nabla \log v$, then energy E^v is minimised by the steady states of the associated PDE. Furthermore, the stationary solution is given by $w(x) := \frac{\mu_f}{\mu_v} v(x)$, where μ_f, μ_v are the AVG of f, v, respectively.

Valid also in the discrete for suitable space discretization $\mathbf{A}$ and Explicit/Implicit-Euler.

Spatial discretisation of osmosis in Vogel et al. ('13)

The FD spatial discretisation at $(x_i, y_j) = ((i-0.5)h, (j-0.5)h)$ is a 5 point stencil:

$$(\partial_t u)_{i,j} = (\mathbf{A}u)_{i,j}, \quad \text{with } \mathbf{A} \in \mathbb{R}^{MN \times MN}$$

$$\begin{aligned}\mathbf{A} :=& \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{1}{h}\left(d_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}+u_{i,j}}{2} - d_{1,i-\frac{1}{2},j}\frac{u_{i,j}+u_{i-1,j}}{2}\right) \\ &+ \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} - \frac{1}{h}\left(d_{2,i,j+\frac{1}{2}}\frac{u_{i,j+1}+u_{i,j}}{2} - d_{2,i,j-\frac{1}{2}}\frac{u_{i,j}+u_{i,j-1}}{2}\right)\end{aligned}$$

$\mathbf{A}$ is irreducible (non-symm.), with zero-column sum and non-neg. off-diagonal entries.

Scale-space theory in Vogel et al. ('13)

Let $\mathbf{A}$ as above. For $k \geq 0$, the time-discretisation schemes for $\mathbf{P} \in \mathbb{R}^{MN \times MN}$ is:

$$\begin{cases} u^0 = f & \text{if } k = 0, \\ u^{k+1} = \mathbf{P}u^k & \text{if } k \geq 1. \end{cases}$$

Then $\mathbf{P}$ is the (non-symmetric) irreducible, non-negative matrix with strictly positive diagonal entries and unitary column sum such that

1. the evolution preserves positivity and the average grey value of f;
2. the eigenvector of $\mathbf{P}$ associated to eigenvalue 1 is the unique steady state for $k \to \infty$.

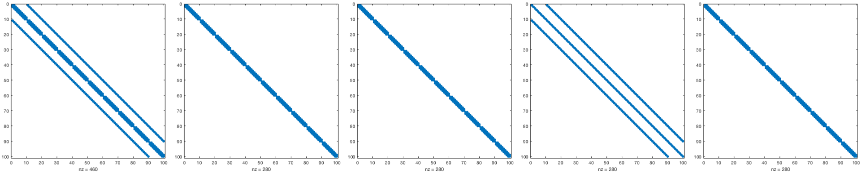
Efficiency: dimensional splitting for osmosis

We focus on dimensional splitting for $\mathbf{A}_{\textsf{full}} := \mathbf{A_1} + \mathbf{A_2}$ preserving the continuous properties.

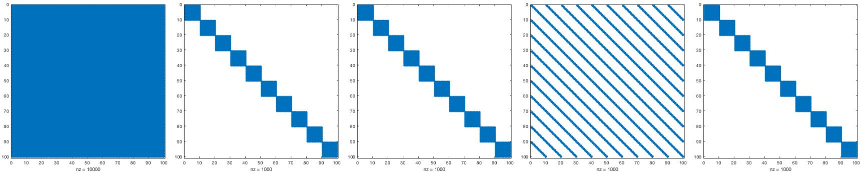
$$\mathbf{A_1}(\mathbf{u}) := \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{1}{h}\left(d_{1,i+\frac{1}{2},j}\frac{u_{i+1,j}+u_{i,j}}{2} - d_{1,i-\frac{1}{2},j}\frac{u_{i,j}+u_{i-1,j}}{2}\right)$$
$$\mathbf{A_2}(\mathbf{u}) := \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} - \frac{1}{h}\left(d_{2,i,j+\frac{1}{2}}\frac{u_{i,j+1}+u_{i,j}}{2} - d_{2,i,j-\frac{1}{2}}\frac{u_{i,j}+u_{i,j-1}}{2}\right)$$

Method	$\mathbf{P}$	$\tau <$ step-size limitation	# diags	bandwidth
Explicit Euler	$\mathbf{P}^{+}_{\textsf{full},\tau} = (\mathbf{I} + \tau\mathbf{A}_{\textsf{full}})$	$\left(\max_i \left\|(\mathbf{A}_{\textsf{full}})_{i,i}\right\|\right)^{-1}$	5	M
Implicit Euler	$\mathbf{P}^{-}_{\textsf{full},\tau} = (\mathbf{I} - \tau\mathbf{A}_{\textsf{full}})^{-1}$	-	5	M
P.R.	$\mathbf{P}^{-}_{\mathbf{1},\frac{\tau}{2}}\mathbf{P}^{+}_{\mathbf{2},\frac{\tau}{2}}\mathbf{P}^{-}_{\mathbf{2},\frac{\tau}{2}}\mathbf{P}^{+}_{\mathbf{1},\frac{\tau}{2}},$	$2\left(\max_{\mathbf{n}}\max_i \left\|(\mathbf{A_n})_{i,i}\right\|\right)^{-1}$	3	1
Douglas	$\mathbf{I} + \tau\mathbf{P}^{-}_{\mathbf{2},\theta\tau}\mathbf{P}^{-}_{\mathbf{1},\theta\tau}\mathbf{A}_{\textsf{full}}$	$\left(\max_i \left\|\mathbf{P}^{-}_{\mathbf{2},\theta\tau}\mathbf{P}^{-}_{\mathbf{1},\theta\tau}\mathbf{A}_{\textsf{full}}\right\|_{i,i}\right)^{-1}$	3	1
AOS	$\frac{1}{2}\sum_{\mathbf{n}=1}^{2}\mathbf{P}^{-}_{\mathbf{n},2\tau}$	no	3	1
MOS	$\frac{1}{2}\prod_{\mathbf{n}=1}^{2}\mathbf{P}^{-}_{\mathbf{n},\tau}$	no	3	1
AMOS	$\frac{1}{2}\sum_{\substack{\mathbf{n}=1\\ \mathbf{i}=\{1,2\}\\ \mathbf{j}=\{2,1\}}}^{2}\mathbf{P}^{-}_{\mathbf{j_n},\tau}\mathbf{P}^{-}_{\mathbf{i_n},\tau}$	no	3	1

- the inverse of a sparse matrix is not necessarily sparse;
- the more the bandwidth of $\mathbf{A}$ is small, the better is (to reduce fill-in when inverting);
- make use of symmetric reverse Cuthill–McKee algorithm to minimize the bandwidth.



$(\mathbf{I} - \tau \mathbf{A}_{\textbf{full}})$ | $(\mathbf{I} - \tau \mathbf{A}_1)$ | $(\mathbf{I} - \tau \mathbf{A}_1)$ with symrcm | $(\mathbf{I} - \tau \mathbf{A}_2)$ | $(\mathbf{I} - \tau \mathbf{A}_2)$ with symrcm



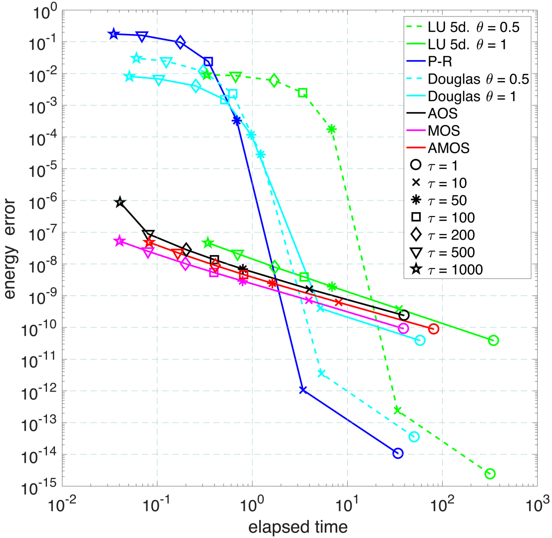
$(\mathbf{I} - \tau \mathbf{A}_{\textbf{full}})^{-1}$ | $(\mathbf{I} - \tau \mathbf{A}_1)^{-1}$ | $(\mathbf{I} - \tau \mathbf{A}_1)^{-1}$ with symrcm | $(\mathbf{I} - \tau \mathbf{A}_2)^{-1}$ | $(\mathbf{I} - \tau \mathbf{A}_2)^{-1}$ with symrcm

P.R. | Douglas, $\theta = 1$.

AOS. | MOS.

AMOS. | LU, $\theta = 1$.

Parameters: $\tau = 1000$, $T = 100000$



x-axis: elapsed time (cputime) to reach final T for different τ.

Application: Light balance in TQR

We apply dimensional splitting for osmosis on light unbalanced images from CH imaging.

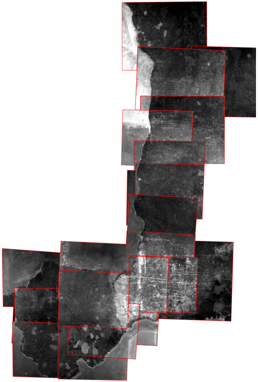
Thermal Quasi-Reflectography (Mid-infrared imaging), Daffara, SP et al. ('12-'18)

Monocromo, Leonardo da Vinci, Castello Sforzesco (Milan)

We apply dimensional splitting for osmosis on light unbalanced images from CH imaging.

Thermal Quasi-Reflectography (Mid-infrared imaging), Daffara, SP et al. ('12-'18)

Monocromo, L. da Vinci

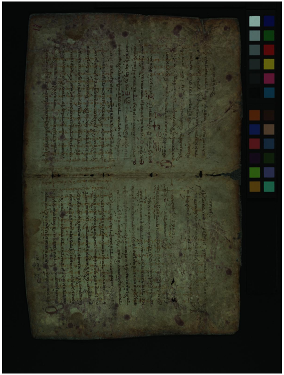


Shadowed with mask
33 TQR tiles (28 Megapixels)

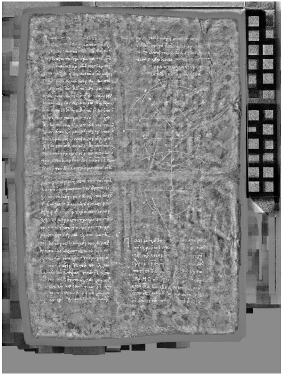
Result $\tau = 1000$, $T = 100000$
(MOS 629 s., `BiCGStab` 4H)

Application: Data fusion on Archimedes palimpsest

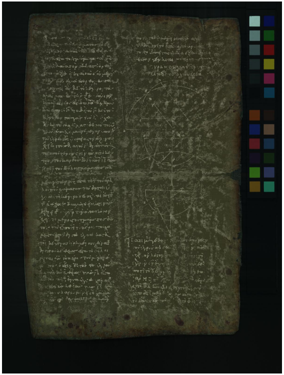
We apply dimensional splitting for osmosis on data integration in Archimedes Palimpsest (X century copy of the works of Archimedes, overwritten in XIII century).



Original parchment (2MB)



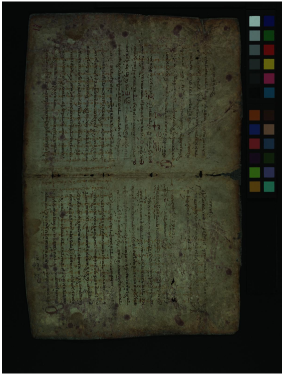
v provided



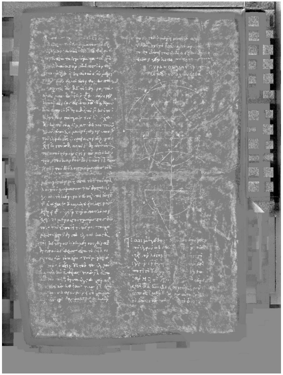
Result (MOS 137 s.)

Application: Data fusion on Archimedes palimpsest

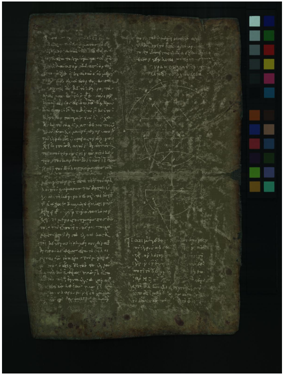
We apply dimensional splitting for osmosis on data integration in Archimedes Palimpsest (X century copy of the works of Archimedes, overwritten in XIII century).



Original parchment (2MB)

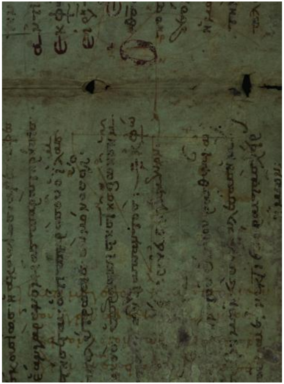


v with local Otsu

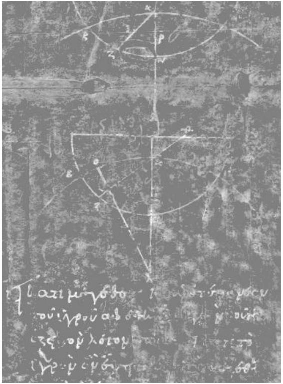


Result (MOS 137 s.)

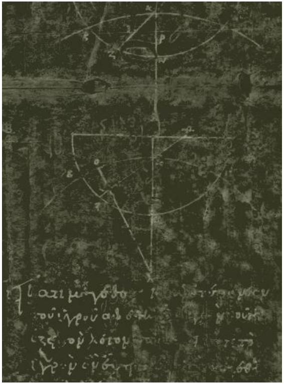
We apply dimensional splitting for osmosis on data integration in Archimedes Palimpsest (X century copy of the works of Archimedes, overwritten in XIII century).



Original parchment (2MB) (zoom)



v with local Otsu (zoom)



Result (MOS 137 s.) (zoom)

Generalization: anisotropic osmosis

Aim: do shadow removal and inpainting jointly to avoid the blurring artefact.

- fix $\mathbf{b} : \Omega \to [0,1]^2$, with $\mathbf{b} = (b_1, b_2)$.
- fix $\boldsymbol{\theta} : \Omega \to [0, 2\pi)$, $\mathbf{z} = (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$;

Interpretation 1: $\mathbf{M}\,\nabla\, u = \underbrace{\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} \cos\boldsymbol{\theta} & \sin\boldsymbol{\theta} \\ -\sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} \end{pmatrix}}_{\mathbf{M}=\Lambda_{\mathbf{b}} R_{\boldsymbol{\theta}}^{\mathrm{T}}} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} b_1\,\nabla_{\mathbf{z}}\, u \\ b_2\,\nabla_{\mathbf{z}^\perp}\, u \end{pmatrix}.$

Interpretation 2: $\mathbf{M}$ describes the local metric on the tangent plane of u ($\mathbb{R}^3$-manifold) in x.

Generalization: anisotropic osmosis

Aim: do shadow removal and inpainting jointly to avoid the blurring artefact.

- fix $\mathbf{b} : \Omega \to [0,1]^2$, with $\mathbf{b} = (b_1, b_2)$.
- fix $\boldsymbol{\theta} : \Omega \to [0, 2\pi)$, $\mathbf{z} = (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$;

Interpretation 1: $\mathbf{M}\,\nabla\, u = \underbrace{\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} \cos\boldsymbol{\theta} & \sin\boldsymbol{\theta} \\ -\sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} \end{pmatrix}}_{\mathbf{M}=\Lambda_{\mathbf{b}} R_{\boldsymbol{\theta}}^{\mathrm{T}}} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} b_1\, \nabla_{\mathbf{z}}\, u \\ b_2\, \nabla_{\mathbf{z}^\perp}\, u \end{pmatrix}.$

Interpretation 2: $\mathbf{M}$ describes the local metric on the tangent plane of u ($\mathbb{R}^3$-manifold) in x.

from *standard osmosis energy* to *anisotropic osmosis energy*

from $\quad E^v(u) := \int_\Omega v \left| \nabla \left(\frac{u}{v} \right) \right|^2 \mathrm{d}x \quad$ to $\quad E^v_{\mathbf{b},\theta}(u) := \int_\Omega v \left| \mathbf{M}_{\mathbf{b},\theta}\, \nabla \left(\frac{u}{v} \right) \right|^2 \mathrm{d}x.$

or with the anisotropic diffusivity tensor $\mathbf{W}_{\mathbf{b},\theta} := \mathbf{M}_{\mathbf{b},\theta}^{\mathrm{T}} \mathbf{M}_{\mathbf{b},\theta}$:

$$E_{\mathbf{b},\theta}(u) := \int_\Omega v \left\langle \mathbf{W}_{\mathbf{b},\theta}\, \nabla \left(\frac{u}{v} \right), \nabla \left(\frac{u}{v} \right) \right\rangle \mathrm{d}x, \quad \text{where } \mathbf{W}_{\mathbf{b},\theta} = b_1^2 (\mathbf{z} \otimes \mathbf{z}) + b_2^2 (\mathbf{z}^\perp \otimes \mathbf{z}^\perp).$$

from *standard osmosis PDE* to *anisotropic osmosis PDE*

from $\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) \\ u(x,0) = f(x) \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 \end{cases}$ to $\begin{cases} \partial_t u = \operatorname{div}\big(\mathbf{W}_{\mathbf{b},\theta}(\nabla u - \mathbf{d}u)\big) & \text{on } \Omega \times (0,T], \\ u(x,0) = f(x) & \text{on } \Omega \\ \langle \mathbf{W}_{\mathbf{b},\theta}(\nabla u - \mathbf{d}u), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$

- for $\mathbf{b} = (1,1)$ then $\mathbf{W}_{\mathbf{b},\theta} = \mathbf{I}$ and the models are the same.

Theorem: properties of the continuous model

- **mass conservation (AVG)**;
- **non-negativity**;
- **non-constant steady states**: given $v > 0$ and $\mathbf{d} := \nabla \log v$, then energy E^v is minimised by the steady states of the associated PDE. Furthermore, the stationary solution is given by $w(x) := \frac{\mu_f}{\mu_v} v(x)$, where μ_f, μ_v are the AVG of f, v, respectively.

- $\operatorname{div}\big(\mathbf{W}_{\mathbf{b},\mathbf{z}}(\nabla u - \mathbf{d}u)\big) = \mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u + \big(\operatorname{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}) - \mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d}\big) \cdot \nabla u - \operatorname{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d})u.$

We now focus on $\mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u$.

Connections with anisotropic PDE inpainting models

Definition (Directional Hessian)

Let $u : \Omega \to \mathbb{R}$, $\mathbf{r}$, $\mathbf{s} : \Omega \to \mathbb{R}^2$ be unitary vectors and $\mathrm{A} := \mathrm{D}^2 u \in \mathbb{R}^{2\times 2}$ be the Hessian of u. We define the *directional Hessian* of u in the directions $\mathbf{r}$ and $\mathbf{s}$ as:

$$\mathrm{D}^2 u(\mathbf{r}, \mathbf{s}) := \sum_{i,j} \mathrm{A}_{i,j} \mathbf{r}_i \mathbf{s}_j .$$

normalized gradient $\mathbf{z} := \dfrac{\nabla u}{|\nabla u|}$, normalized tanget to isolines $\mathbf{z}^\perp := \dfrac{\nabla^\perp u}{|\nabla u|}$.

Lemma: Anisotropic Hessian

For any $\mathbf{b} : \Omega \to \mathbb{R}^2$ and $\mathbf{z}(x) := (\cos\theta(x), \sin\theta(x))$, and for any $x \in \Omega$ it holds:

$$\mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u = b_1^2 \mathrm{D}^2 u(\mathbf{z}, \mathbf{z}) + b_2^2 \mathrm{D}^2 u(\mathbf{z}^\perp, \mathbf{z}^\perp).$$

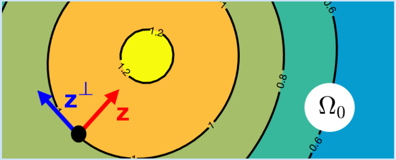
For osmosis:

- $\mathrm{div}\left(\mathbf{W}_{\mathbf{b},\mathbf{z}}(\nabla u - \mathbf{d}u)\right) = \mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u + \left(\mathrm{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}) - \mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d}\right) \cdot \nabla u - \mathrm{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d})u.$

Let Ω_0 be the inpainting domain:

$$\begin{cases} \alpha \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) + 2\beta \mathrm{D}^2 u(\mathbf{z},\mathbf{z}^\perp) + \gamma \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp) & \text{on } \Omega_0 \\ u = \varphi & \text{on } \partial\Omega_0 \end{cases}$$

Alvarez et al. ('93) and Caselles, Morel & Sbert ('98).



Example (Harmonic Inpainting)

$$\begin{cases} \Delta u = 0 & \text{on } \Omega_0 \\ u = \varphi & \text{on } \partial\Omega_0, \end{cases} \quad \text{where} \quad \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) + \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp) = \Delta u.$$

Example (AMLE Inpainting, see Aronsson ('67), Jensen ('93))

$$\begin{cases} \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) = 0 & \text{on } \Omega_0 \\ u = \varphi & \text{on } \partial\Omega_0, \end{cases} \quad \text{where} \quad \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) = \left(\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) \cdot \mathrm{D}^2 u.$$

Example (TV Inpainting, mean flow curvature)

$$\begin{cases} \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp) = 0 & \text{on } \Omega_0 \\ u = \varphi & \text{on } \partial\Omega_0, \end{cases} \quad \text{where} \quad \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp) = |\nabla u| \cdot \operatorname{div}\left(\frac{\nabla u}{|\nabla u|} \right).$$

Anisotropic shadow removal with (non-)linear inpainting

Given shadowed $f > 0$ and $\mathbf{d}(x) = \begin{cases} \nabla \log f(x), & \text{if } x \in \Omega \setminus \Omega_{\mathsf{sb}}, \\ 0, & \text{if } x \in \Omega_{\mathsf{sb}}. \end{cases}$

Standard shadow removal

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) & \text{on } (\Omega \setminus \Omega_{\mathsf{sb}}) \times (0, T], \\ \partial_t u = \Delta u & \text{on } \Omega_{\mathsf{sb}} \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega, \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

Reinterpretation of standard shadow removal

Recall $\mathbf{W}_{(1,1),\mathbf{z}} = \mathbf{I}$ for any unitary $\mathbf{z}$.

$$\begin{cases} \partial_t u = \operatorname{div}(\mathbf{W}_{(1,1),\mathbf{z}}(\nabla u - \mathbf{d}u)) & \text{on } (\Omega \setminus \Omega_{\mathsf{sb}}) \times (0, T], \\ \partial_t u = \operatorname{div}(\mathbf{W}_{(1,1),\mathbf{z}} \nabla u) & \text{on } \Omega_{\mathsf{sb}} \times (0, T], \\ u(x, 0) = f(x) & \text{on } \Omega, \\ \langle \mathbf{W}_{(1,1),\mathbf{z}}(\nabla u - \mathbf{d}u), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

To avoid over-smoothing on Ω_{sb}...

Anisotropic shadow removal with (non-)linear inpainting

Given shadowed $f > 0$ and $\mathbf{d}(x) = \begin{cases} \nabla \log f(x), & \text{if } x \in \Omega \setminus \Omega_{\mathsf{sb}}, \\ 0, & \text{if } x \in \Omega_{\mathsf{sb}}. \end{cases}$

Standard shadow removal

$$\begin{cases} \partial_t u = \operatorname{div}(\nabla u - \mathbf{d}u) & \text{on } (\Omega \setminus \Omega_{\mathsf{sb}}) \times (0,T], \\ \partial_t u = \Delta u & \text{on } \Omega_{\mathsf{sb}} \times (0,T], \\ u(x,0) = f(x) & \text{on } \Omega, \\ \langle \nabla u - \mathbf{d}u, \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

Reinterpretation of standard shadow removal as anisotropic shadow removal!

Recall $\mathbf{W}_{(1,1),\mathbf{z}} = \mathbf{I}$ for any unitary $\mathbf{z}$.

$$\begin{cases} \partial_t u = \operatorname{div}(\mathbf{W}_{(1,1),\mathbf{z}}(\nabla u - \mathbf{d}u)) & \text{on } (\Omega \setminus \Omega_{\mathsf{sb}}) \times (0,T], \\ \partial_t u = \operatorname{div}(\mathbf{W}_{(0,1),\mathbf{z}} \nabla u) & \text{on } \Omega_{\mathsf{sb}} \times (0,T], \\ u(x,0) = f(x) & \text{on } \Omega, \\ \langle \mathbf{W}_{(1,1),\mathbf{z}}(\nabla u - \mathbf{d}u), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

To avoid over-smoothing on Ω_{sb}... introduce anisotropic (TV-type) diffusion.

The anisotropic shadow removal in practice

$$\begin{cases} \partial_t u = \operatorname{div}\left(\mathbf{W}_{\mathbf{b},\mathbf{z}}(\nabla u - \mathbf{d}u)\right) & \text{on } \Omega \times (0,T], \\ u(x,0) = f(x) & \text{on } \Omega, \\ \langle \mathbf{W}_{\mathbf{b},\mathbf{z}}(\nabla u - \mathbf{d}u), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \times (0,T], \end{cases}$$

$$\mathbf{d}(x) = \begin{cases} \nabla \log f, & \text{if} \quad x \in \Omega \setminus \Omega_{\mathsf{sb}}, \\ 0, & \text{if} \quad x \in \Omega_{\mathsf{sb}}. \end{cases} \quad \text{and} \quad \mathbf{b}(x) = \begin{cases} (1,1), & \text{if} \quad x \in \Omega \setminus \Omega_{\mathsf{sb}}, \\ (0,1), & \text{if} \quad x \in \Omega_{\mathsf{sb}}. \end{cases}$$

Recall $\mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u = b_1^2 \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) + b_2^2 \mathrm{D}^2 u(\mathbf{z}^\perp, \mathbf{z}^\perp)$.

Connection with directional interpolators

on $\Omega \setminus \Omega_{\mathsf{sb}} \times (0,T]$:

$$\begin{aligned} \partial_t u &= \operatorname{div}\left(\mathbf{W}_{\mathbf{b},\mathbf{z}}(\nabla u - \mathbf{d}u)\right) \\ &= \mathbf{W}_{\mathbf{b},\mathbf{z}} \cdot \mathrm{D}^2 u + \left(\operatorname{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}) - \mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d}\right) \cdot \nabla u - \operatorname{div}(\mathbf{W}_{\mathbf{b},\mathbf{z}}\mathbf{d})u. \end{aligned}$$

TV on $\Omega_{\mathsf{sb}} \times (0,T]$: $\quad \partial_t u = \operatorname{div}(\mathbf{W}_{(0,1),\mathbf{z}} \nabla u) = \mathrm{D}^2 u(\mathbf{z}^\perp, \mathbf{z}^\perp)$.

Numerical details: space and time discretisation

(a selection of) Tested space discretisation for $\mathbf{A}$

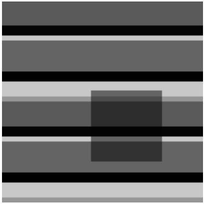
- FD: finite differences (not ideal), negative for strong anisotropy;
- WS (coherence enanching) Weickert & Scharr ('02): (25d), checkerboard, negative;
- AD-LBR, Fehrenbach & Mirebeau ('13): sparse (6d) non-neg., oriented as $\mathbf{W}_{\mathbf{b},\theta}$.

Time solver: exponential integrators for exact-in-time solution at T

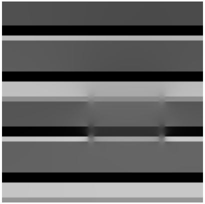
$$u^T = \exp(\tau \mathbf{A}) u^0, \quad \text{with } \tau := T - t_0.$$

- we compute $\exp(\tau A)u^0$ for $\tau = T - t_0$ since $\exp(\tau \mathbf{A})$ is too expensive;
- we approximate $\exp(\tau A)u^0$ via Leja interpolation as in Caliari et al. ('16);
- **scale-space properties are preserved** by exponential integrators provided good $\mathbf{A}$;

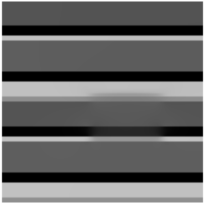
Application: synthetic images with known directions



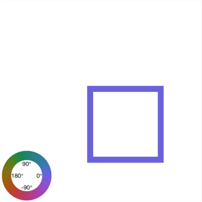
Shadow, $\theta = 000^\circ$



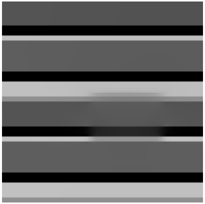
Harmonic



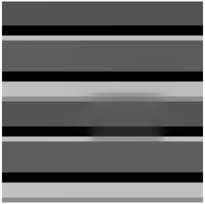
TV with FD



$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^\perp$



TV with WS



TV with sparse AD-LBR

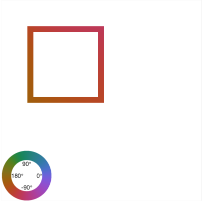
Comparison: Harmonic osmosis vs Total Variation Osmosis (TV) with different spatial discretisations.

Application: synthetic images with known directions

Shadow

Harmonic

TV with FD



$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^{\perp}$

TV with WS

TV with sparse AD-LBR

Comparison: Harmonic osmosis vs. Total Variation Osmosis (TV) with different spatial discretisations.

Estimation of directions in shadowed image

Let $\mathbf{A}$ be a 2-tensor on $\mathbb{R}^2$, decomposed as

$$\mathbf{A} = \underbrace{(\lambda_1 - \lambda_2)}_{s=\text{saliency}}(\mathbf{e}_1 \otimes \mathbf{e}_1) + \underbrace{\lambda_2}_{\text{ballness}}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2)$$

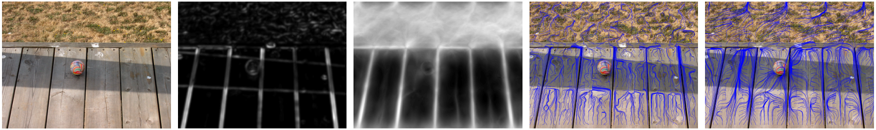
- θ is the *orientation* (angle) formed by $\mathbf{e}_1$.

Tensor voting framework (TVF): Guy and Medioni ('96)

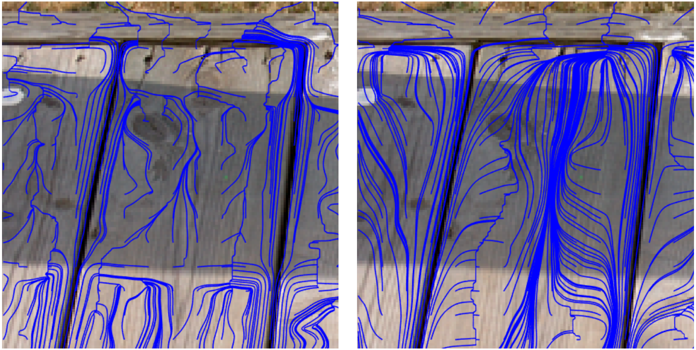
- neighbourhoods vote for the most likely structure (scale-depending);
- efficiently implemented by steerable filters in Franken et al. ('06);
- improves the structure tensor $K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma)$, see Moreno et al. ('12).

Proposed estimation: shadow boundary directions are biased:

1. estimate s and θ from shadowed u;
2. $s = 0$ and θ randomized on shadow boundaries Ω_{sb};
3. multi-scale inspection, tensor voting is performed;

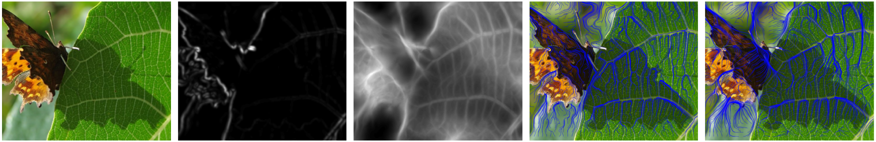


Input — STF, λ_1 — TVF, λ_1 — STF, $\mathbf{e}_1 = \mathbf{z}^\perp$ — TVF, $\mathbf{e}_1 = \mathbf{z}^\perp$

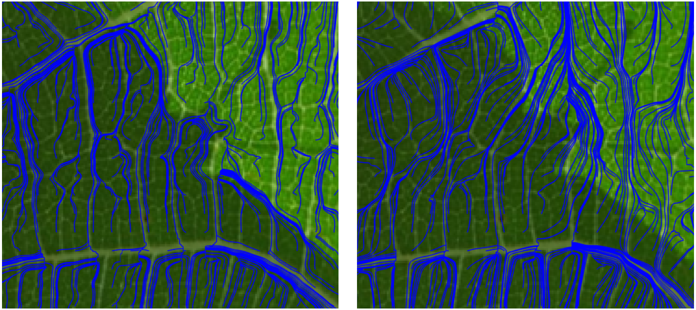


STF, $\mathbf{e}_1 = \mathbf{z}^\perp$ (zoom) — TVF, $\mathbf{e}_1 = \mathbf{z}^\perp$ (zoom)

Comparison: structure tensor (STF) with $(\sigma, \rho) = (2, 2)$ and tensor voting (TVF) with multi-scale $(11, 21, 31)$.

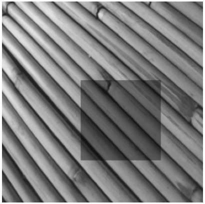


Input | STF, λ_1 | TVF, λ_1 | STF, $\mathbf{e}_1 = \mathbf{z}^\perp$ | TVF, $\mathbf{e}_1 = \mathbf{z}^\perp$

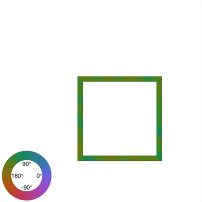


STF, $\mathbf{e}_1 = \mathbf{z}^\perp$ (zoom) | TVF, $\mathbf{e}_1 = \mathbf{z}^\perp$ (zoom)

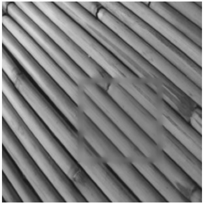
Comparison: structure tensor (STF) with $(\sigma, \rho) = (2, 2)$ and tensor voting (TVF) with multi-scale $(11, 21, 31)$.



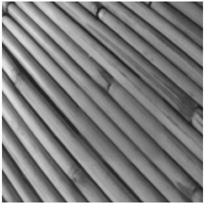
Shadowed



$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^{\perp}$



Harmonic



TV with AD-LBR stencil

Shadowed

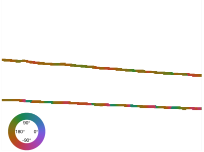
$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^{\perp}$

Harmonic

TV with AD-LBR stencil

Parameters: $\mathbf{b} = (0.05, 1.00)$, $T = 1\text{e}5$, $\tau = 1\text{e}2$.

Shadowed

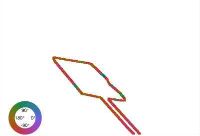


$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^{\perp}$

Harmonic

TV with AD-LBR stencil

Shadowed



$\Omega_{\mathbf{sb}}$, θ for $\mathbf{z}^{\perp}$

Harmonic

TV with AD-LBR stencil

Parameters: $\mathbf{b} = (0.05, 1.00)$, $T = 1e5$, $\tau = 1e2$.

Conclusions

Messages

- we can plug in anisotropic directions into osmosis;
- the shadow removal problem is a mix of osmosis and (non-)linear inpainting;
- discretisation of anisotropic diffusion is not a free lunch!

Contributions

- ADI splitting for isotropic osmosis with applications to real CH imaging;
- anisotropic model for the shadow removal problem;
- estimation of directional structures via tensor voting, even for light jumps.

Thank you for your attention!

(selected) Bibliography

Arias, Facciolo, Caselles and Sapiro. A Variational Framework for Exemplar-Based Image Inpainting. Int. J. Comput. Vision Vol. 93 (3), (2011),

Caliari, Kandolf, Ostermann and Rainer. The Leja method revisited: backward error analysis for the matrix exponential. SIAM Journal on Scientific Computation, 38 (3), (2016)

Caselles, Morel and Sbert. An axiomatic approach to image interpolation. IEEE Trans on Image Processing, Vol. 7 (3), (1998).

Daffara, Parisotto and Ambrosini. Multipurpose, dual-mode imaging in the 3 μm-5 μm range (MWIR) for artwork diagnostics: A systematic approach. Optics and Laser in Engineering, Vol. 104 (2018)

Fehrenbach, Mirabeau, Sparse Non-Negative Stencils for Anisotropic Diffusion. JMIV Vol. 49(1), (2014)

Parisotto, Calatroni, Caliari and Schönlieb. Dimensional splitting and anisotropic modelling of a drift-diffusion osmosis equation for imaging applications, (2018) (In preparation).

Parisotto, Calatroni and Daffara. Digital Cultural Heritage imaging via osmosis filtering. ICISP (accepted), (2018)

Vogel, Hagenburg, Weickert, and Setzer. A Fully Discrete Theory for Linear Osmosis Filtering. SSVM, LNCS Vol. 7893, (2013).

Weickert, Hagenburg, Breußand Vogel. Linear Osmosis Models for Visual Computing. EMMCVPR, LNCS Vol. 8081, Springer-Verlag (2013)

Weickert and Scharr. A Scheme for Coherence-Enhancing Diffusion Filtering with Optimized Rotation Invariance, J Vis Commun Image Represent, Vol. 13, (2002)

Addendum 1: Directional Interpolators

Axiomatic description of a 2nd-order PDE inpainting interpolator in Alvarez et al. ('93).

- **Requirement on architecture and stability**: comparison principles, regularity.
- **Requirement on morphology**: translation, rotation, shift, zoom invariance.

Theorem (Characterization I, Caselles, Morel & Sbert ('98), Theorem 1)

Let O be the inpainting domain, $\Gamma = \partial O$ and $\varphi : \Gamma \to \mathbb{R}$ be a continuous function.
Let E be a "good" interpolation operator and $u := E(\varphi, \Gamma)$ be the interpolant operator.
Then u is a viscosity solution of the equation

$$\begin{cases} G(\mathrm{D}^2 u(\mathbf{z},\mathbf{z}), \mathrm{D}^2 u(\mathbf{z},\mathbf{z}^\perp), \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp)) = 0 & \textit{in } O, \\ u = \varphi & \textit{on } \partial O, \end{cases}$$

with derivatives intended in the weak sense.

Theorem (Characterization II, Caselles, Morel & Sbert ('98), Proposition 1(ii))

If G is differentiable at $(0,0,0)$, then the above is re-written for α, $\gamma \geq 0$ and $\alpha\gamma - \beta^2 \geq 0$:

$$\begin{cases} \alpha \mathrm{D}^2 u(\mathbf{z},\mathbf{z}) + 2\beta \mathrm{D}^2 u(\mathbf{z},\mathbf{z}^\perp) + \gamma \mathrm{D}^2 u(\mathbf{z}^\perp,\mathbf{z}^\perp) & \textit{on } O \\ u = \varphi & \textit{on } \partial O \end{cases}$$

Addendum 2: more details on AD-LBR

- $E(u) = \int_\Omega \| \nabla u(x)\|^2_{\mathbf{W}_{\mathbf{b},\theta}} \,\mathrm{d}x \approx h^{d-2} \sum_{x\in\Omega_h} \sum_{e\in V(x)} \gamma_x(e)\, |u(x+he) - u(x)|^2$;
- for each $x \in \mathbb{R}^d$ (use linearisation) $h^d \| \nabla u\|^2_{\mathbf{W}_{\mathbf{b},\theta}} = h^{d-2} \sum_{e\in V(x)} \gamma_x(e) \langle \nabla u(x), he\rangle^2$,
- $V(x) := \{e_0, e_1, e_2, -e_0, -e_1, -e_2\}$, $\gamma_x(\pm e_i) := -0.5\langle e^{\perp}_{i+1}, \mathbf{W}_{\mathbf{b}(x),\theta(x)} e^{\perp}_{i+2}\rangle$;

$$\mathbf{W}_{\mathbf{z},\theta} = R_\theta \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} R_{-\theta}$$

with $\theta = \pi/6 + \pi/2$

$\varepsilon = 1$			$\varepsilon = 0.5$			$\varepsilon = 0.02$		
0.00	1.00	0.00	0.00	0.41	0.22	0.00	0.11	0.16
1.00	$\mathbf{-4.00}$	1.00	0.66	$\mathbf{-2.57}$	0.66	0.01	$\mathbf{-0.55}$	0.01
0.00	1.00	0.00	0.22	0.41	0.00	0.16	0.11	0.00

Addendum 3: Exponential Time Integrators

For stiff ordinary differential equations:

$$\begin{cases} u'(t) &= \mathbf{A}u(t) + g(t, u(t)), \text{ for } t > 0 \\ u(t_0) &= u^0, \end{cases} \Rightarrow u(t) = \mathrm{e}^{(t-t_0)\mathbf{A}} u^0 + \int_{t_0}^{t} \mathrm{e}^{(t-\tau)\mathbf{A}} g(\tau, u(\tau))\,\mathrm{d}\tau$$

- approximation of the integral: $u(t) \approx \mathrm{e}^{(t-t_0)\mathbf{A}} u^0 + \sum_{\ell=1}^{q} \varphi_\ell\left((t-t_0)\mathbf{A}\right)(t-t_0)^\ell u_\ell$;
- for $p = 1$, $g \approx$ piecewise constant (exact if g constant) iterate for $k = 0, \ldots, K$:

$$u^{k+1} = \exp(\tau_k \mathbf{A}) u^k + \left(\frac{\exp(\tau_k \mathbf{A}) - 1}{\mathbf{A}}\right) g^k = u^k + \tau_k \varphi_1(\tau_k \mathbf{A})(\mathbf{A}u^k + g^k).$$

Polynomial approximation of matrix exponential

- uses Newton method at Leja points falling into the spectrum of $\mathbf{A}$
- Leja points are non-equispaced and do not change with the order (unlike Chebyshev);
- approximate $\exp(\tau\mathbf{A})u^0$ by iterating $u^{k+1} = p_{m_k}\left(\frac{\tau\mathbf{A}}{s}\right) u^k$ for p_{m_k} of order m_k;
- $\left(p_m(\tau\mathbf{A}/K)\right)^K u^0 = \exp(\tau\mathbf{A} + \Delta\tau\mathbf{A})u^0$, with $\|\Delta\tau\mathbf{A}\| \leq \mathrm{tol} \cdot \|\tau\mathbf{A}\|$.

References: Certaine ('60), Pope ('63), Cox & Matthew ('02), Al-Mohy & Higham ('09), Caliari et al. ('16), . . .

Addendum 4: Reasons for Exponential Integrators

Reasons for exponential integrators:

- for strong anisotropy: large stencils (AD-LBR ok): high computational costs;
- implicit methods may use preconditioning: inefficient (bandwith and fill-in issue);
- ADI splitting methods not appealing due to mixed-terms in $\mathbf{W}_{\mathbf{b},\theta}$;
- exponential integrators are exact in time (homogeneous PDE): high precision.

Strategy for preserving exactness in time and reducing computational cost

We discretise the time interval $[t_0, T]$ with $t_0, \ldots, t_k, \ldots, T$ and iterate

$$\begin{cases} u^0 = f & \text{if } k = 0, \\ u^{k+1} = \mathbf{P} u^k & \text{if } k \geq 1, \end{cases} \quad \text{with } \mathbf{P} = \exp(\tau_k \mathbf{A}) \text{ and } \tau_k = t_{k+1} - t_k.$$