(Higher-order) Total directional variation

with imaging applications

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Plan of this talk

Motivation: theoretical

- Let $u : \Omega \to \mathcal{T}^\ell(\mathbb{R}^d)$ be a function with $\Omega \subset \mathbb{R}^d$ open bounded Lipschitz domain;
- $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{q-1})$ be a collection of positive weights.

Bredies, Holler, Kunisch, Pock (2010-present)

$$\mathrm{TGV}^{q,\ell}_{\boldsymbol{\alpha}}(u) = \sup_{\boldsymbol{\Psi}} \left\{ \int_\Omega u \cdot \mathrm{div}^q\, \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x}, \text{ for all } \boldsymbol{\Psi} \in \mathcal{Y}^{q,\ell}_{\boldsymbol{\alpha}} \right\}.$$

$$\mathcal{Y}^{q,\ell}_{\boldsymbol{\alpha}} := \left\{ \boldsymbol{\Psi} \,:\, \boldsymbol{\Psi} \in \mathrm{C}^q_c(\Omega, \mathrm{Sym}^{q+\ell}(\mathbb{R}^d)),\ \left\| \mathrm{div}^j\, \boldsymbol{\Psi} \right\|_\infty \le \alpha_j,\ j = 0, \dots, q-1 \right\}.$$

Proposed: (higher-order) directional generalization

Let also a collection of weighting fields $\mathcal{M} = (\mathbf{M}_j)_{j=1}^q$ with $\mathbf{M}_j \in \mathrm{C}^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$:

$$\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M}) = \sup_{\boldsymbol{\Psi}} \left\{ \int_\Omega u \cdot \mathrm{div}^q_{\mathcal{M}}\, \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x}, \text{ for all } \boldsymbol{\Psi} \in \mathcal{Y}^{q,\ell}_{\mathcal{M},\boldsymbol{\alpha}} \right\}.$$

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Motivation: theoretical (State of the Art)

Functions of bounded variation [Ambrosio,Fusco & Pallara ('00), etc. . .]:

- TV: total variation [Rudin, Osher & Fatemi, ('92)] for denoising;
- varying fidelity weight [Dong ('09)] for denoising;

Functions of bounded Hessian [Demengel ('84)]:

- combination of first and second TV orders [Schönlieb & Papafitsoros ('14)];

Functions of bounded deformation [Teman & Strang ('80), Bredies ('13)]:

- TGV: total generalized variation [Bredies, Kunisch & Pock ('10), Bredies & Holler ('14)] e.g. for JPEG decompression [Bredies & Holler ('12)] and Wavelet zooming [Bredies & Holler ('13)];

Weighted TV for the adaptation to the local image content:

- by fixed orthogonal matrix [Berkels et al. ('06)];
- by modified test functions:
  $\mathrm{TV}_{a,\theta}$ [Bayram & Kamasak ('12)], $\mathrm{EADTV}_{\alpha,\boldsymbol{\theta}}$ [Zhang & Wang ('13)],
  DTV and $\mathrm{DTGV}^2_{\boldsymbol{\alpha}}$ [Kongskov & Dong ('17)];
- by eigen-values/vectors of structure tensor [Weickert ('98)]:
  [Setzer & Steidl ('09), Grasmair & Lenzen ('10), Lenzen et al. ('13)], STV_p [Lefkimmiatis et al. ('15)];
- projection onto suitable directions dTV for MRI [Ehrhardt & Betcke ('16)];

Motivation: imaging applications

$(q = 1)$ weak formulation

$$\int_\Omega u \cdot \mathrm{div}_{\mathbf{M}} \, \boldsymbol{\Psi} \, \mathrm{d}\mathbf{x}$$

$(q = 1)$ strong formulation

$$\int_\Omega \mathbf{M} \nabla u \cdot \boldsymbol{\Psi} \, \mathrm{d}\mathbf{x}$$

- fix $\mathbf{b} : \Omega \to [0,1]^2$, with $\mathbf{b} = (b_1, b_2)$.
- fix $\boldsymbol{\theta} : \Omega \to [0, 2\pi)$, $\mathbf{v} = (\cos\boldsymbol{\theta}, \sin\boldsymbol{\theta})$;

Interpretation 1: $\mathbf{M} \nabla u = \underbrace{\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} \cos\boldsymbol{\theta} & \sin\boldsymbol{\theta} \\ -\sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} \end{pmatrix}}_{\mathbf{M} = \Lambda_{\mathbf{b}} R_{\boldsymbol{\theta}}^{\mathrm{T}}} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} b_1 \nabla_{\mathbf{v}} u \\ b_2 \nabla_{\mathbf{v}^\perp} u \end{pmatrix}.$

Interpretation 2: $\mathbf{M}$ describes the local metric on the tangent plane of u ($\mathbb{R}^3$-manifold) in x.

Motivation: imaging applications

$(q = 1)$ weak formulation	$(q = 1)$ strong formulation
$\displaystyle\int_\Omega u \cdot \operatorname{div}_{\mathbf{M}} \boldsymbol{\Psi} \, \mathrm{d}\mathbf{x}$	$\displaystyle\int_\Omega \mathbf{M} \nabla u \cdot \boldsymbol{\Psi} \, \mathrm{d}\mathbf{x}$

- fix $\mathbf{b} : \Omega \to [0,1]^2$, with $\mathbf{b} = (b_1, b_2)$.
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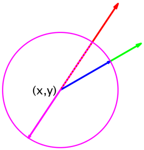
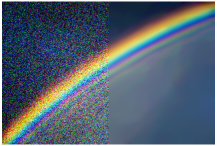
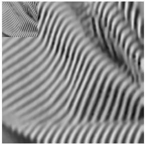


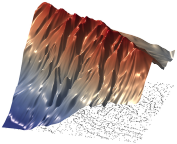
$\mathbf{b} = (1.00, 1.00)$



Denoising (20% G. noise)



4X Wavelet zooming



DEM interp. (7% data)

Adjointness property

$(q=1)$ weak formulation	$(q=1)$ strong formulation
$\int_\Omega u \cdot \mathrm{div}_{\mathbf{M}}\, \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x}$	$\int_\Omega \mathbf{M}\, \nabla\, u \cdot \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x}$

Following [Bredies, Kunisch & Pock, '10], we deal with $(0,q)$-tensors (covariant vectors).

Adjointness property

- $\Omega \subset \mathbb{R}^d$, $\mathbf{M} \in \mathrm{C}^1(\Omega, \mathcal{T}^2(\mathbb{R}^d))$, $\xi \in \mathrm{C}^q(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))$, $\boldsymbol{\Psi} \in \mathrm{C}^1_c(\Omega, \mathcal{T}^{q+1}(\mathbb{R}^d))$;
- $\xi^\sim(a_1, \dots a_\ell) = \xi(a_\ell, a_1, \dots, a_{\ell-1})$.

Moving $\mathbf{M}$: $\quad \int_\Omega (\mathbf{M}\, \nabla \otimes \xi) \cdot \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x} = \int_\Omega (\nabla \otimes \xi) \cdot \mathrm{trace}\, (\mathbf{M} \otimes \boldsymbol{\Psi}^\sim)$, for all $\mathbf{M}$, ξ, $\boldsymbol{\Psi}$.

Adjoint: $\quad \int_\Omega (\mathbf{M}\, \nabla \otimes \xi) \cdot \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x} = -\int_\Omega \xi \cdot \mathrm{div}_{\mathbf{M}}\, \boldsymbol{\Psi}\, \mathrm{d}\mathbf{x}$, for all $\mathbf{M}$, ξ, $\boldsymbol{\Psi}$,

where $\mathrm{div}_{\mathbf{M}}\, \boldsymbol{\Psi} := \mathrm{trace}\, (\nabla \otimes [\mathrm{trace}\, (\mathbf{M} \otimes \boldsymbol{\Psi}^\sim)]^\sim)$.

If $\boldsymbol{\Psi} \in \mathrm{Sym}^\ell(\mathbb{R}^d)$ ($\boldsymbol{\Psi}^\sim = \boldsymbol{\Psi}$), then $\mathrm{div}_{\mathrm{Id}}(\boldsymbol{\Psi}) = \mathrm{trace}(\nabla \otimes \boldsymbol{\Psi})$ [Bredies, Kunisch, Pock '10].

Total directional variation

Definition (Total directional variation of order q, associated with $\mathcal{M}$ and α)

- let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain;
- let $u : \Omega \to \mathcal{T}^\ell(\mathbb{R}^d)$ be a measurable function;
- let $\mathcal{M} := (\mathbf{M}_j)_{j=1}^q$ be a collection of fields in $\mathrm{C}^\infty(\Omega, \mathcal{T}^2(\mathbb{R}^d))$;
- let $\boldsymbol{\alpha} := (\alpha_0, \dots, \alpha_{q-1})$ be a collection of positive real numbers.

$$\mathrm{TDV}_{\boldsymbol{\alpha}}^{q,\ell}(u, \mathcal{M}) := \sup_{\boldsymbol{\Psi}} \left\{ \int_\Omega u \cdot \mathrm{div}_{\mathcal{M}}^q \boldsymbol{\Psi} \, \mathrm{d}\mathbf{x} \,\middle|\, \text{for all } \boldsymbol{\Psi} \in \mathcal{Y}_{\mathcal{M},\boldsymbol{\alpha}}^{q,\ell} \right\};$$

$$\mathcal{Y}_{\mathcal{M},\boldsymbol{\alpha}}^{q,\ell} = \left\{ \boldsymbol{\Psi} \,:\, \boldsymbol{\Psi} \in \mathrm{C}_c^q(\Omega, \mathcal{T}^{q+\ell}(\mathbb{R}^d)),\ \left\| \mathrm{div}_{\mathcal{M}}^j \boldsymbol{\Psi} \right\|_\infty \leq \alpha_j, \forall\, j = 0, \dots q-1 \right\}.$$

Divergence of order q, recursive definition

$$\mathrm{div}_{\mathcal{M}}^0 \boldsymbol{\Psi} := \boldsymbol{\Psi}, \quad \mathrm{div}_{\mathcal{M}}^j(\boldsymbol{\Psi}) := \mathrm{div}_{\mathbf{M}_{q-j+1}} \left(\mathrm{div}_{\mathcal{M}}^{j-1} \boldsymbol{\Psi} \right).$$

When $\mathcal{M} = (\mathrm{Id})_{j=1}^q$ we have $\neg\mathrm{symTGV}_{\boldsymbol{\alpha}}^q(u)$ [Bredies, Kunisch & Pock, '10].

Advanced: Total directional variation as Radon norm

Example: For $q = 1$ then $\mathrm{TDV}^{1,\ell}_{\boldsymbol{\alpha}}(u, \mathbf{M}) = \|\mathbf{M} \nabla \otimes u\|_{\mathscr{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d))}$ and

$$\mathrm{BDV}^1(\Omega, \mathbf{M}, \mathcal{T}^\ell(\mathbb{R}^d)) = \left\{ u \in \mathrm{L}^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \,\middle|\, \mathbf{M} \nabla \otimes u \in \mathscr{M}(\Omega, \mathcal{T}^{\ell+1}(\mathbb{R}^d)) \right\}.$$

TDV as Radon norm for general q

$$\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M}) = \|\mathbf{M}_q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes u\|_{\mathscr{M}(\Omega, \mathcal{T}^{\ell+q}(\mathbb{R}^d))}$$

$$\mathrm{BDV}^q(\Omega, \mathcal{M}, \mathcal{T}^\ell(\mathbb{R}^d)) = \left\{ u \in \mathrm{L}^1(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \,\middle|\, \mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M}) < \infty \right\}.$$

- the spaces are nested: the larger is q, the smaller is the space;
- BDV^q is a Banach space with $\|u\|_1 + \|\mathbf{M}_q \nabla \otimes \cdots \otimes \mathbf{M}_1 \nabla \otimes u\|_{\mathscr{M}}$;
- $\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}$ is convex and lower semi-continuous.

Theorem: existence of solutions

Let $p \in [1, \infty]$ with $p \leq d/(d-1)$ and assume that $F : \mathrm{L}^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d)) \to [-\infty, \infty]$ is proper, convex, lower semi-continuous and coercive. Then there exist a solution of

$$\min_{u \in \mathrm{L}^p(\Omega, \mathcal{T}^\ell(\mathbb{R}^d))} \mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M}) + F(u).$$

The continuous and discrete imaging problem

- $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain;
- let $\mathcal{M} = (\mathcal{M}_1, \dots, \mathcal{M}_Q)$, with $\mathcal{M}_q = (\mathbf{M}_{q,1}, \dots, \mathbf{M}_{q,q})$;
- let $\boldsymbol{\alpha}_q = (\alpha_{q,0}, \dots, \alpha_{q,q-1})$ be positive weights;

- let $\mathcal{S}$ be the operator (e.g. Id, projection,...);
- let $u^\diamond$ be the input.

Single continuous problem ($q \in \mathbb{N}$ fixed)

$$u^\star = \arg\min_{u \in \Omega} \mathrm{TDV}^{q,0}_{\boldsymbol{\alpha}_q}(u, \mathcal{M}_q) + \frac{\eta}{2} \left\| \mathcal{S}u - u^\diamond \right\|_2^2 ,$$

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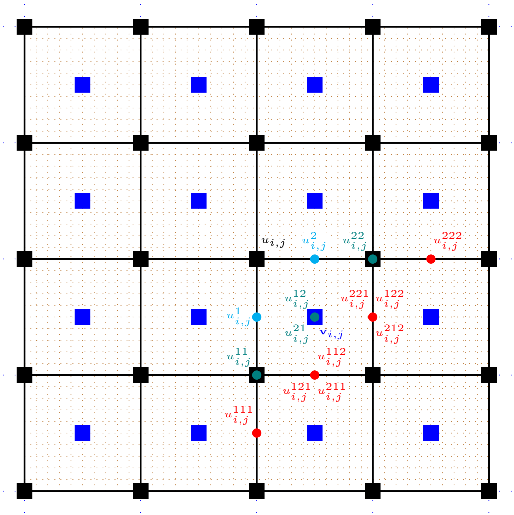
- superscript h for the discrete quantities;
- Images: $u : \Omega^h \to [0, 255]$ (greyscale), $u : \Omega^h \to [0, 255]^3$ (RGB-scale, channel-wise).

Single discrete problem ($q \in \mathbb{N}$ fixed)

$$u^\star = \underset{u^h \in \Omega^h}{\arg\min}\, \mathrm{TDV}^{q,h}_{\boldsymbol{\alpha}_q}(u^h, \mathcal{M}^h_q) + \frac{\eta}{2} \left\|\mathcal{S}u^h - u^{h,\diamond}\right\|_2^2 ,$$



Staggered Grids. $u^h \in \Omega^h$ (■), $\mathbf{v}^h \in \Gamma^h$ (■), $\nabla^h u$ (•), $\nabla^{2,h} u$ (•), $\nabla^{3,h} u$ (•).

We need transfer operators $\mathcal{W}$ to bring (•, •, •) to (■) and vice-versa.

Discrete weighted adjointness property

For every $u^h \in \Omega^h, \mathbf{p}^h \in (\Gamma^h \times \cdots \times \Gamma^h)$ and a fixed $q \in \mathbb{N}$:

$$\langle \nabla^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} u^h, \mathbf{p}^h \rangle = (-1)^q \sum_{(i,j)\in\Omega^h} u^h_{i,j} (\mathrm{div}^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} \mathbf{p}^h)_{i,j},$$

with $\nabla^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} u^h : \Omega^h \to (\Gamma^h \times \cdots \times \Gamma^h)$ and $\mathrm{div}^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} : (\Gamma^h \times \cdots \times \Gamma^h) \to \Omega^h$.

In view of the primal-dual algorithm, a bound for the weighted gradient is:

$$L_q^2 = \left\| \nabla^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} \right\|^2 = \left\| \mathrm{div}^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} \right\|^2 = \sup_{\mathbf{\Psi}^h \in \mathcal{Y}^{q,h}_{\mathcal{M}^h_q,\boldsymbol{\alpha}}} \left\| \mathrm{div}^{q,h}_{\mathcal{M}^h_q,\mathcal{W}} \mathbf{\Psi}^h \right\|^2 \left\| \mathbf{\Psi}^h \right\|^{-2}.$$

With finite differences scheme:

- $L_1^2 = \left\| \mathrm{div}^{1,h}_{\mathcal{M}^h_1,\mathcal{W}} \mathbf{\Psi}^h \right\|^2 \leq \frac{8}{h^2} \left\| \mathrm{M}^h_{1,1} \mathcal{W}^1 \right\|_F^2 \left\| \mathbf{\Psi}^h \right\|^2 \leq 8h^{-2}$
- $L_q^2 \leq (8h^{-2})^q \prod_{j=1}^{q} \left\| (\mathcal{W}^j)^{\mathrm{T}} \mathrm{M}^h_{q,j} \mathcal{W}^j \right\|_2^2 \leq (8h^{-2})^q$.

This agrees with the classic results: $L^2 \leq 8h^{-2}$ for TV and $L^2 \leq 64h^{-4}$ for $\mathrm{TGV}^2_{\boldsymbol{\alpha}}$.

Total directional variation decomposition

Continuous setting

Similar to TGV [Bredies & Holler (2014)], now with $\mathbf{M}_{q,j}\nabla$ instead of $\mathcal{E}$ (symm. gradient):

$$\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}_q}(u,\mathcal{M}_q) = \inf_{\substack{u_j\in \mathrm{BDV}^{q-j}(\Omega,\mathbf{M}_{q,j+1}\mathcal{T}^{\ell+j}(\mathbb{R}^d))\\ j=1,\dots,q-1,\, u_0=u,\, u_q=0}} \left(\sum_{j=1}^{q} \alpha_{q,q-j}\,\|\mathbf{M}_{q,j}\nabla u_{j-1}-u_j\|_1\right).$$

Discrete setting

$$\mathrm{TDV}^{q,h}_{\boldsymbol{\alpha}_q}(u^h,\mathcal{M}^h_q) = \inf_{\substack{\mathbf{z}^h_j\in \mathrm{X}^{j,h}_{\iota}\\ j=0,\dots,q,\\ \mathbf{z}_0=u^h,\, \mathbf{z}^h_q=0}} \sum_{j=1}^{q} \alpha_{q,q-j}\left\|(\mathcal{K}^h_q)_{j,j}\mathbf{z}^h_{j-1}-\mathbf{z}^h_j\right\|_{2,1},$$

with $(\mathcal{K}^h_q)_{j,j} = \begin{cases} (\mathcal{W}^j)^{\mathrm{T}}\mathbf{M}^h_{q,j}\mathcal{W}^j\,\nabla^h & \text{if } j=1,\dots q-1,\\ \mathbf{M}^h_{q,q}\mathcal{W}^q\,\nabla^h & \text{if } j=q.\end{cases}$

The joint problem with TDV decomposed

$$u^\star = \operatorname*{arg\,min}_{u \in \mathbb{R}} \sum_{q=1}^{\mathrm{Q}} \mathrm{TDV}^{q,0}_{\boldsymbol{\alpha}_q}(u, \mathcal{M}_q) + \frac{\eta}{2} \left\| \mathcal{S}u - u^\diamond \right\|_2^2 ,$$

Joint discrete problem ($\mathrm{Q} \in \mathbb{N}$ fixed)

Let $\mathbf{z}^h = (\mathbf{z}^h_q)_{q=0}^{\mathrm{Q}-1}$ and $\mathbf{w}^h = ((\mathbf{w}^h_{q,j})_{j=1}^{q})_{q=1}^{\mathrm{Q}}$ be primal and dual variables. Then

$$u^\star = \operatorname*{arg\,min}_{u^h \in \Omega^h} \inf_{\mathbf{z}^h} \sum_{q=1}^{\mathrm{Q}} \underbrace{\sum_{j=1}^{q} \mathbf{A}_{j,q} \left\| \sum_{\ell=1}^{q} (\mathcal{K}^h_q)_{j,\ell} \mathbf{z}^h_{\ell-1} \right\|_{2,1}}_{\mathrm{TDV}^{q,h}_{\boldsymbol{\alpha}_q}(u^h, \mathcal{M}^h_q)} + \frac{\eta}{2} \left\| \mathcal{S}u^h - u^{\diamond,h} \right\|_2^2 .$$

$$\mathbf{w}^h_q = \mathcal{K}^h_q \mathbf{z}^h = \begin{pmatrix} \mathbf{M}^h_{q,1} & -\mathrm{Id} & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{M}^h_{q,q-1} & -\mathrm{Id} \\ 0 & \dots & 0 & \mathbf{M}^h_{q,q} \end{pmatrix} \begin{pmatrix} \mathbf{z}^h_1 \\ \vdots \\ \vdots \\ \mathbf{z}^h_q \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \alpha_{1,0} & \alpha_{2,1} & \dots & \alpha_{\mathrm{Q},\mathrm{Q}-1} \\ 0 & \alpha_{2,0} & \ddots & \alpha_{\mathrm{Q},\mathrm{Q}-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \alpha_{\mathrm{Q},0} \end{pmatrix}$$

Primal-dual algorithm

Joint saddle-point minimization problem (by duality of $\|\cdot\|_{2,1}$)

$$\min_{\mathbf{z}^h}\max_{\mathbf{w}^h}\sum_{q=1}^{Q}\langle\sum_{\ell=1}^{q}(\mathcal{K}_q^h)_{j,\ell}\mathbf{z}_{\ell-1}^h,\mathbf{w}_{q,j}^h\rangle-\underbrace{\sum_{q=1}^{Q}\sum_{j=1}^{q}\delta_{\{\|\cdot\|_{2,\infty}\le\mathbf{A}_{j,q}\}}(\mathbf{w}_{q,j}^h)}_{F^*(\mathbf{w}^h)}+\underbrace{\frac{\eta}{2}\left\|\mathcal{S}\mathbf{z}_0^h-\mathbf{z}_0^{\diamond,h}\right\|_2^2}_{G(\mathbf{z}_0^h)}.$$

- $\mathbf{prox}_{\sigma F^*}(\mathbf{w}^h)=\sum_{q=1}^{Q}\sum_{j=1}^{q}\mathbf{prox}_{\sigma F^*}(\mathbf{w}_{q,j}^h)=\sum_{q=1}^{Q}\sum_{j=1}^{q}\frac{\mathbf{w}_{q,j}^h}{\max\left(1,\mathcal{A}_{j,q}^{-1}\left\|\mathbf{w}_{q,j}^h\right\|_2\right)};$
- $\mathbf{prox}_{\tau G}(\widehat{u}^h)=\widehat{u}^h+(\mathrm{Id}+\tau\eta\mathcal{S}^*\mathcal{S})^{-1}\tau\eta\mathcal{S}^*(u^{\diamond,h}-\mathcal{S}\widehat{u}^h).$

Primal-dual algorithm (Chambolle & Pock, 2011). Iterate on $n\ge 0$:

$$\begin{aligned}
\mathbf{w}^{n+1,h}&=\mathbf{prox}_{\sigma_n F^*}\left(\mathbf{w}^{n,h}+\sigma_n\mathcal{K}^h\overline{\mathbf{z}}^{n,h}\right);\\
\mathbf{z}^{n+1,h}&=\mathbf{prox}_{\tau_n G}\left(\mathbf{z}^{n,h}-\tau_n(\mathcal{K}^h)^*\mathbf{w}^{n+1,h}\right);\\
\theta_n&=(1+2\gamma\tau_n)^{-0.5};\quad \tau_{n+1}=\theta_n\tau_n;\quad \sigma_{n+1}=\sigma_n\theta_n^{-1};\quad //\text{acceleration};\\
\overline{\mathbf{z}}^{n+1,h}&=\mathbf{z}^{n+1,h}+\theta_n(\mathbf{z}^{n+1,h}-\mathbf{z}^{n,h}).
\end{aligned}$$

Application: image denoising

For the problem:

- Denoising operator $\mathcal{S} = \mathrm{Id}$;
- fidelity weight η is estimated from noise level γ: $\eta \approx \gamma^{-1} \| u_{\mathsf{GT}} - u^{\diamond} \|_2^2$;

For the model, take $b_1 \equiv 1$ and parameters for the following situation for any $q \in [1, 3]$:

$$\mathbf{M}_{q,q} \, \nabla(\nabla^{q-1} u) = \begin{pmatrix} 1 \cdot \nabla_{\mathbf{v}}(\nabla^{q-1} u) \\ b_2(\mathbf{x}) \cdot \nabla_{\mathbf{v}^\perp}(\nabla^{q-1} u) \end{pmatrix}$$

The denoising problem: $\mathcal{S} = \mathrm{Id}$

$$u^\star = \arg\min_u \sum_{q=1}^{Q} \mathrm{TDV}^q_{\boldsymbol{\alpha}_q}(u, \mathcal{M}_q) + \frac{\eta}{2} \| u - u^{\diamond} \|_2^2 \,,$$

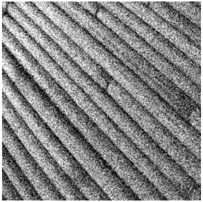
We need a strategy to estimate directions $\mathbf{v}$.

Estimation of the vector field $\mathbf{v}$

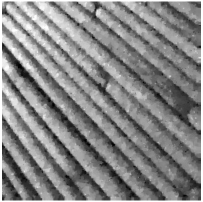
- choose $\sigma, \rho > 0$ pre-smoothing and post-smoothing parameters;
- structure tensor: $J_\rho(u) := K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma) = \lambda_1(\mathbf{e}_1 \otimes \mathbf{e}_1) + \lambda_2(\mathbf{e}_2 \otimes \mathbf{e}_2)$;
- take $\widetilde{\mathbf{v}} = \mathbf{e}_1$;
- estimate the anisotropy weight $w(\mathbf{x}) = \dfrac{\lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x})}{\lambda_1(\mathbf{x}) + \lambda_2(\mathbf{x})}$;
- regularization step (the higher the anisotropy w, the lower the regularization):

$$\mathbf{v} = \arg\min_{\mathbf{z}} \frac{1}{2} \int_\Omega w(\mathbf{x}) \, \|\mathbf{z} - \widetilde{\mathbf{v}}\|_2^2 \, \mathrm{d}\mathbf{x} + \frac{\gamma}{2} \int_\Omega \|\nabla \widetilde{\mathbf{v}}\|_2^2 \, \mathrm{d}\mathbf{x};$$

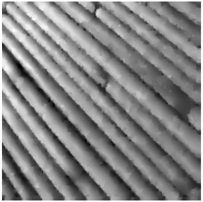
- for $b_2(x)$: take $b_2(\mathbf{x}) = 1 - w(\mathbf{x})$ and rescale it in $[0, 1]$;



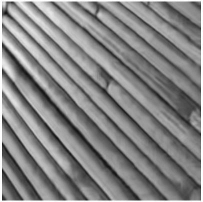
Noisy u (20% Gaussian)



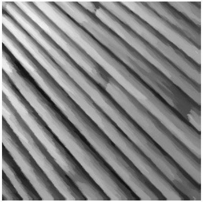
$\mathrm{TV}(u)$
PSNR = 23.8



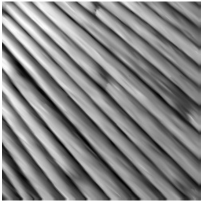
$\mathrm{TGV}(u)$
PSNR = 24.7



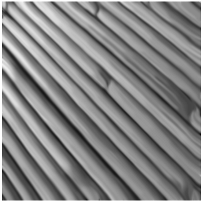
BM3D
PSNR = 29.10



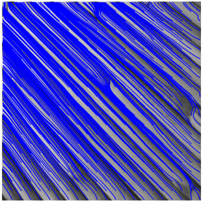
$\mathrm{DTV}(u)$, $\boldsymbol{\theta} = 45^\circ$
PSNR = 26.8



$\mathrm{DTGV}^2(u), \boldsymbol{\theta} = 45^\circ$
PSNR = 28.2



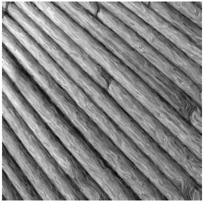
Our, joint 1st and 3rd
$\eta = 3.5$, $\mathbf{v}$, $\mathbf{b} = (1, 0.02)$,
PSNR = 29.20



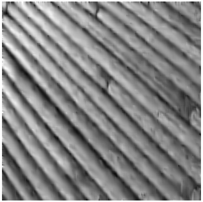
$\mathbf{v}$
$(\sigma_1, \rho_1) = (1.8, 2.8)$
$(\sigma_2, \rho_2) = (1.0, 1.0)$

Bamboo image (253x253 pixels).

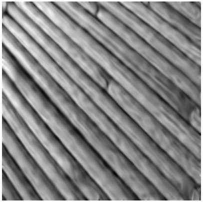
$\mathrm{TV}(u)$,$\mathrm{TGV}(u)$, $\mathrm{DTV}(u)$ **and** $\mathrm{DTGV}^2(u)$ **from Kongskov and Dong (2017).**



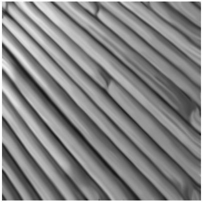
Our, single 1st.
PSNR = 21.40



Our, single 2nd.
PSNR = 25.29

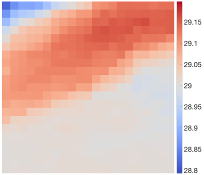


Our, single 3rd.
PSNR = 27.16

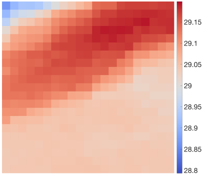


Our, joint 1st and 3rd
PSNR = 29.20

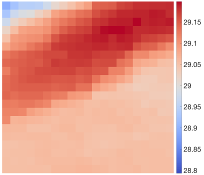
Comparison of different orders of TDV and fixed $\eta = 3.5$, $\mathbf{v}$, $\mathbf{b} = (1, 0.02)$.



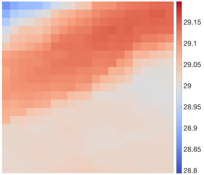
$\mathbf{b} = (1, 0.00)$



$\mathbf{b} = (1, 0.01)$



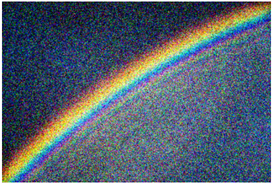
$\mathbf{b} = (1, 0.02)$



$\mathbf{b} = (1, 0.03)$

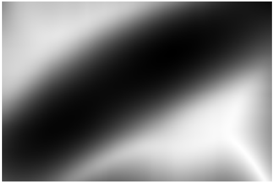
PSNR, joint 1st and 3rd, $\eta = 3.5$.
ρ_1 (x-axis) and σ_1 (y-axis), with $\sigma_1, \rho_1 \in [1.5, 3.5]$ and fixed $(\sigma_2, \rho_2) = (1, 1)$.

Original u

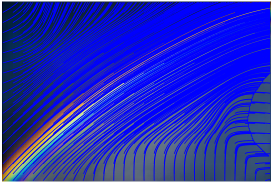


Noisy u (20% Gaussian)

BM3D
PSNR = 34.53



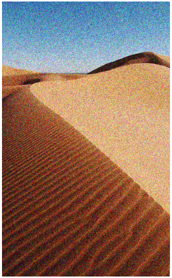
$b_2(\mathbf{x}), (\sigma, \rho) = (2, 30)$



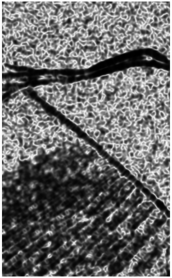
$\mathbf{v}, (\sigma, \rho) = (2, 30)$

Our, joint 1st and 3rd.
$\eta = 3.5$, $\mathbf{v}$, $\mathbf{b} = (1, b_2(\mathbf{x}))$
PSNR = 35.91

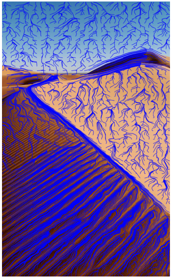
Rainbow image (320×214 pixels, photo by M.P. Markkanen, CC-BY-SA-4.0 license).



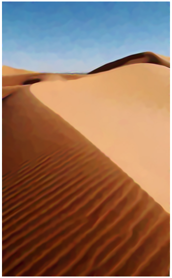
Noisy u
(20% Gaussian)



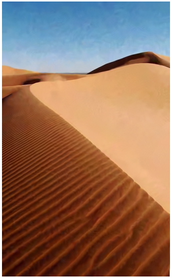
$b_2(x)$
$(\sigma, \rho) = (3.0, 1.5)$



$\mathbf{v}$
$(\sigma, \rho) = (3.0, 1.5)$

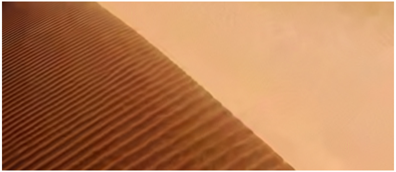


Our, joint 1st-3rd.
$\eta = 3.5$
$(\sigma, \rho) = (3.0, 1.5)$
PSNR = 30.09



BM3D
PSNR = 32.45

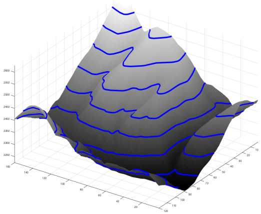
Our, joint 1st-3rd (zoom), PSNR = 30.09

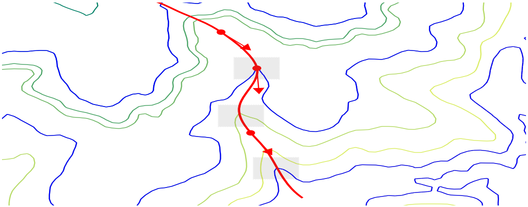


BM3D (zoom), PSNR = 32.45

Desert image (468x768 pixels, photo by Rosino, CC-BY-SA-2.0 license).

Interpolation: Digital Elevation Maps





Find a naturally looking dense surface u that fits contour lines (partially given)!

- u should coincide with the given sparse height values u_0;
- u should preserve the geometry (cusps, kinks) of the given level lines;
- u should be *smooth only* across level lines.

Interpolate the missing data.

In [Lellmann, Schönlieb & Morel (2013)], alternatingly solved (CVX, variable size limit):

$$u^* = \arg\min_u \int_\Omega \left\| \nabla_{\mathbf{v}} (\nabla^2 u) \right\|_2 \mathrm{d}\mathbf{x}$$

$$\widetilde{\mathbf{v}} = \arg\min_{\substack{\mathbf{y} \\ \|\mathbf{y}\|_2 = 1}} \left\| K_\sigma * \nabla \left(\frac{\nabla u}{|\nabla u|} \right) (\mathbf{x})\, \mathbf{y} \right\|_2 + \text{ find } \mathbf{v} \text{ by a further regularization step;}$$

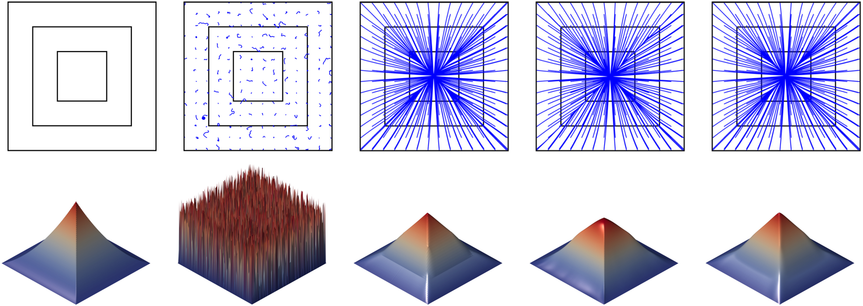
Surface interpolation

Find $u : \Omega \to \mathbb{R}$, $\mathbf{v} : \Omega \to \mathbb{R}^2$ by solving alternatingly via primal-dual algorithm:

$$u^\star = \arg\min_u \sum_{q=1}^{Q} \mathrm{TDV}^q_{\boldsymbol{\alpha}_q}(u, \mathcal{M}_q(\mathbf{v})) + \frac{\eta}{2} \left\| \mathcal{S} u - u^\diamond \right\|_2^2 ,$$

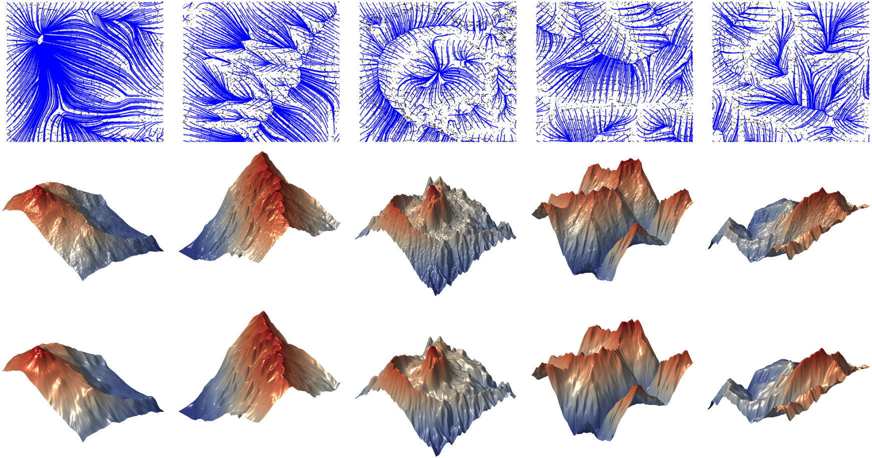
$$\mathbf{v}^\star = \arg\min_{\mathbf{v}} \mu \mathrm{TV}(\mathbf{v}) + \zeta \int_\Omega \left(1 - \mathbf{v} \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \mathrm{d}\mathbf{x}$$

- $\mathcal{M}_q(\mathbf{v}) = (\mathrm{Id}, \ldots, \mathrm{Id}, \mathbf{M}(\mathbf{v}))$, with $\mathbf{M}(\mathbf{v}) := \Lambda_{\mathbf{b}} R^{\mathrm{T}}_{\boldsymbol{\theta}}$;
- $\mathcal{S}$ is a projection operator (leading to non accelerated primal-dual);
- the fidelity term in the second problem echoes the one in [Ballester et al. (2001)].



Ground Truth | Input | Our, single 2nd | Our, single 3rd | Our, joint 2nd-3rd

Pyramid results: 10 iterations. Parameters: $\eta = 100$, $\mu = \zeta = 1$.

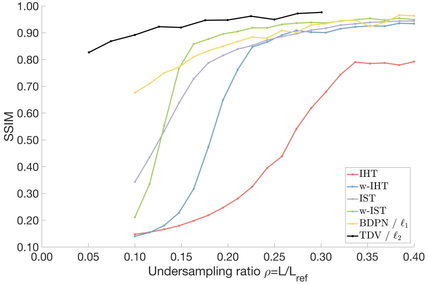


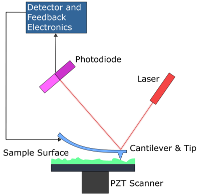
Etna | Monte Baldo | Vesuvio | Vipiteno | Gran Sasso

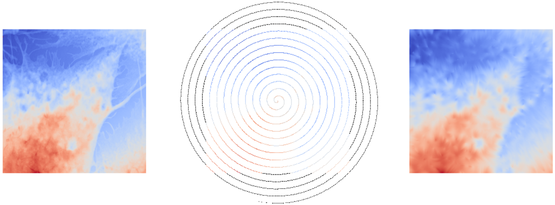
SRTM dataset. Streamlines (top), ground truth (middle), results (bottom).
Parameters: joint 2nd and 3rd orders, $\alpha_{2,0} = 0.1, \alpha_{3,0} = 1$, $\mu = \zeta = 1$, $\eta = 1000$.

Interpolation: Atomic Force Microscopy

Comparison with [Oxvig et al. ('17)]







Ground Truth (left), Sampling (middle), Result (right). Percentage: 05.06%

Conclusions

Messages

- we can control the isotropy/anisotropy locally via the weighted gradients $\mathbf{M}\,\nabla$;
- joint minimization performs better than single minimization in imaging applications;

Contributions

- $\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M})$ as generalization of $\mathrm{TGV}^{q,\ell}_{\boldsymbol{\alpha}}(u)$ and $\mathrm{DTGV}^{q}_{\boldsymbol{\alpha}}(u)$;
- equivalent decomposition of $\mathrm{TDV}^{q,\ell}_{\boldsymbol{\alpha}}(u, \mathcal{M})$ in continuum and discrete case;
- a primal-dual algorithm;
- directional regularizer is effective on a variety of imaging applications;

Thank you for your attention!

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- Bredies, K. and Holler, M., Regularization of linear inverse problems with total generalized variation. J Inverse Ill Pose P 22 (2014).
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- Lellmann J., Morel JM., Schönlieb CB. (2013) Anisotropic Third-Order Regularization for Sparse Digital Elevation Models. SSVM (2013).
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- Parisotto, S., Masnou, S. and Schönlieb, C.B., (Higher-order) Total directional variation. Part I: imaging applications. (in preparation, 2018)
- Parisotto, S., Masnou, S. and Schönlieb, C.B., (Higher-order) Total directional variation. Part II: theory. (in preparation, 2018)

Addendum 1: Wavelet-based zooming

As in [Bredies & Holler, '13], the fidelity term is a wavelet transformation operator.

- $\phi, \psi \in \mathrm{L}^2(\mathbb{R})$ (scaling and mother wavelets);
- $R \in \mathbb{Z}$ is the resolution level;
- u_0 is a low resolution version of $u \in \mathrm{L}^2(\mathbb{R})$;

$$\text{Decompose: } u = \sum_{\mathbf{k} \in M_R} (u, \phi_{R,\mathbf{k}})_2 \widetilde{\phi}_{R,\mathbf{k}} + \sum_{j \leq R, \mathbf{k} \in L_j} (u, \psi_{j,\mathbf{k}})_2 \widetilde{\psi}_{j,\mathbf{k}}.$$

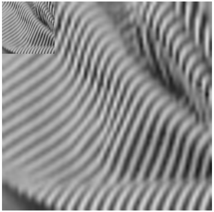
Constraints:

- scaling: $(u, \phi_{R,\mathbf{k}})_2 = (u_0, \phi_{R,\mathbf{k}})_2,$ for all $\mathbf{k} \in M_R$,
- feasibility: $u \in \mathrm{L}^2(\Omega) = \operatorname{span}\left(\left\{ \widetilde{\phi}_{R,\mathbf{k}} \,|\, \mathbf{k} \in M_R \right\} \cup \left\{ \widetilde{\psi}_{j,\mathbf{k}} \,|\, j \leq R, \mathbf{k} \in L_j \right\} \right)$.

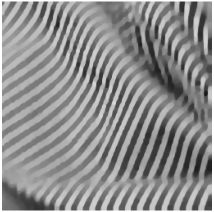
The wavelet-based zooming problem

$$u^\star = \arg\min_u \sum_{q=1}^{Q} \mathrm{TDV}^q_{\boldsymbol{\alpha}_q}(u, \mathcal{M}_q) + \mathcal{I}_{U_D}(u),$$

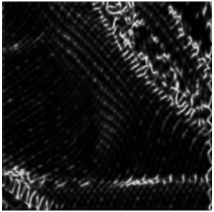
where $U_D = \left\{ u \in \mathrm{L}^2(\Omega) \,|\, (u, \phi_{R,\mathbf{k}})_2 = (u_0, \phi_{R,\mathbf{k}})_2, \text{ for all } \mathbf{k} \in M_R \right\}$.



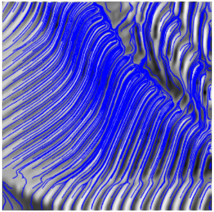
4X Lanczos 2 filter



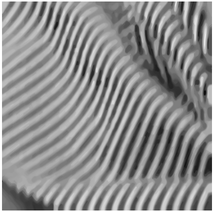
$\mathrm{TGV}^2_{\boldsymbol{\alpha}}$, CDF 9/7
Bredies and Holler (2013)



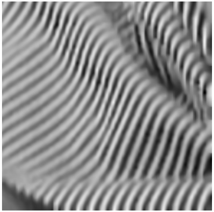
$b_2(\mathbf{x})$
$(\sigma, \rho) = (1.0, 1.0)$



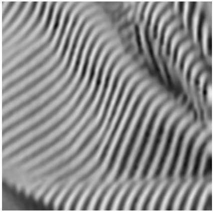
$\mathbf{v}(\mathbf{x})$
$(\sigma, \rho) = (1.0, 1.0)$



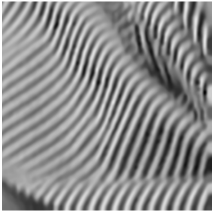
Our, single 1st
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



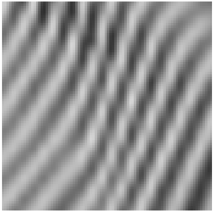
Our, single 2nd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



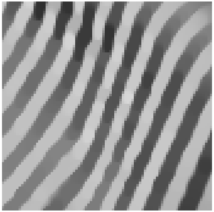
Our, single 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



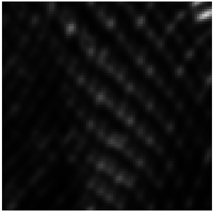
Our, joint 1st and 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



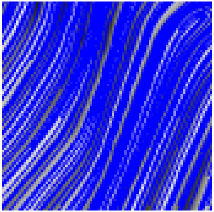
4X Lanczos 2 filter



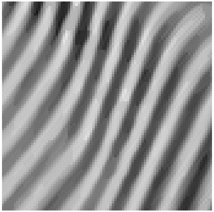
$\mathrm{TGV}^2_{\boldsymbol{\alpha}}$, CDF 9/7
Bredies and Holler (2013)



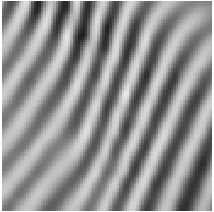
$b_2(\mathbf{x})$
$(\sigma, \rho) = (1.0, 1.0)$



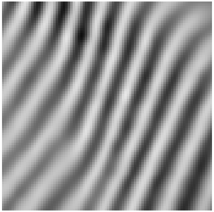
$\mathbf{v}(\mathbf{x})$
$(\sigma, \rho) = (1.0, 1.0)$



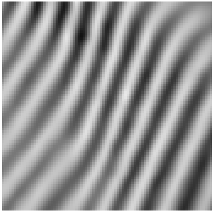
Our, single 1st
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



Our, single 2nd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



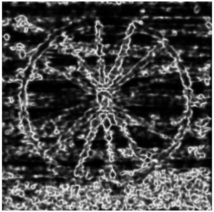
Our, single 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$



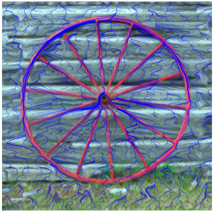
Our, joint 1st and 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

4X Bicubic interpolation

4X Lanczos 2 filter



$b_2(\mathbf{x})$
$(\sigma, \rho) = (1.2, 2.5)$



$\mathbf{v}(\mathbf{x})$
$(\sigma, \rho) = (1.2, 2.5)$

Our, single 1st,
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

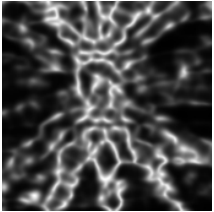
Our, single 2nd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

Our, single 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

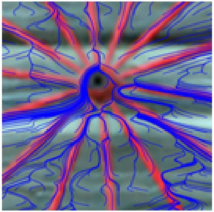
Our, joint 1st and 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

4X Bicubic interpolation

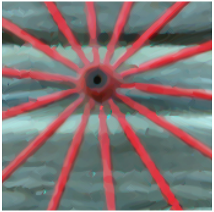
4X Lanczos 2 filter



$b_2(\mathbf{x})$
$(\sigma, \rho) = (1.2, 2.5)$



$\mathbf{v}(\mathbf{x})$
$(\sigma, \rho) = (1.2, 2.5)$



Our, single 1st,
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

Our, single 2nd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

Our, single 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$

Our, joint 1st and 3rd
CDF 9/7, $b_2(\mathbf{x}), \mathbf{v}(\mathbf{x})$