A sample paper

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In this paper we

1 Introduction

We consider the equation

$$(I - \Delta)^s u = a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $a \in L^\infty(\mathbb{R}^N)$ and $2 < p < 2^\star = 2N/(N-2)$ if $N \geq 3$, and $p > 2$ if $N = 1$ or $N = 2$.

2 The variational setting

Lemma 2.1. *Let $u \in L^\infty(\mathbb{R}^N)$. There results*

$$\operatorname{ess\,sup} u = \sup \left\{ \int_{\mathbb{R}^N} u\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\}. \tag{2.1}$$

Proof. Whenever $\varphi \in L^1(\mathbb{R}^N)$, $\varphi \geq 0$, $\int_{\mathbb{R}^N} \varphi = 1$, we compute

$$\int_{\mathbb{R}^N} u\varphi \leq \operatorname{ess\,sup} u \int_{\mathbb{R}^N} \varphi = \operatorname{ess\,sup} u.$$

Hence

$$\operatorname{ess\,sup} u \geq \sup \left\{ \int_{\mathbb{R}^N} u\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\}. \tag{2.2}$$

On the other hand, if we set

$$\sup \left\{ \int_{\mathbb{R}^N} u\varphi \mid \varphi \in L^1(\mathbb{R}^N),\ \varphi \geq 0,\ \int_{\mathbb{R}^N} \varphi = 1 \right\} = b$$

and we assume that $\operatorname{ess\,sup} u > b$, then for some $\delta > 0$ we can say that the set $\Omega = \left\{ x \in \mathbb{R}^N \mid u(x) \geq b + \delta \right\}$ has positive measure. Let us define $\varphi = \chi_\Omega / \mathcal{L}^N(\Omega)$, so that

$$\int_{\mathbb{R}^N} u\varphi = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u \geq b + \delta,$$

contrary to (2.2). This completes the proof. □

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References

[1] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math., **113** (2000), 401–410.

[2] M. Struwe, Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, *Springer Verlag*, Berlin–Heidelberg (1990).

[3] M. Willem, Minimax Theorems, Birkhäuser, Basel (1999).

[4] L. S. Yu, *Nonlinear p–Laplacian problems on unbounded domains*, Proc. Amer. Math. Soc. **115** (1992), 1037–1045.