# Reinforcement Learning

PhD course @ “Dottorato di ingegneria industriale e dell’informazione”
Trieste, 2024

Simone Silvetti
Research & Development

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# Who am I?

**Simone Silvetti** (silvetti@esteco.com)

- ➔ Studied mathematics in Rome
- ➔ Phd in Computer Science @ Udine
- ➔ Currently working in ESTECO

application of quantitative formal methods and machine learning techniques to Verification and Model-based Testing of Complex Systems

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Numerical Methods Group

multi-objective optimization algorithms, machine learning, object-oriented programming

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process mining, research projects related to technology and domains useful for ESTECO products

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multi-objective optimization algorithms, machine learning, object-oriented programming

I worked on “Inverse Reinforcement Learning” applied to autonomous driving

Research and Development

process mining, research projects related to technology and domains useful for ESTECO products

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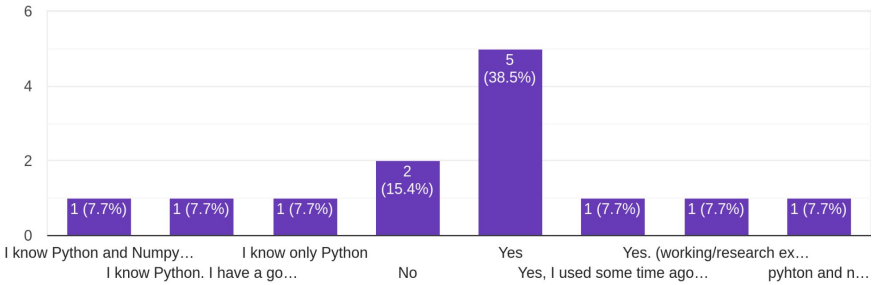
# Who are you?

Do you know Python? Numpy, Scipy?

13 responses



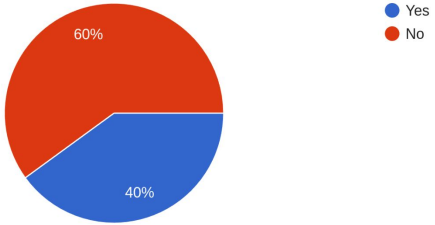

During your studies have you participated in courses of Reinforcement Learning? If yes, which topics have you covered?

13 responses

- No
- Only partially
- I did not partecipate to any course.
- I have never participated in a course about Reinforcement Learning.
- I have never participated at any course of bayesian optimization
- no
- Foundations

Will you follow the "Learning-based Controllers and the Reality Gap" course?

5 responses

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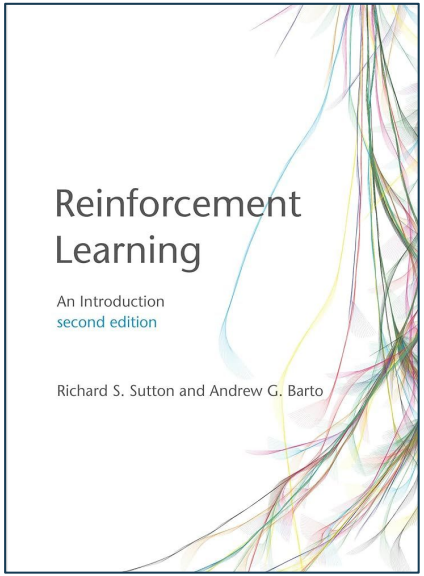


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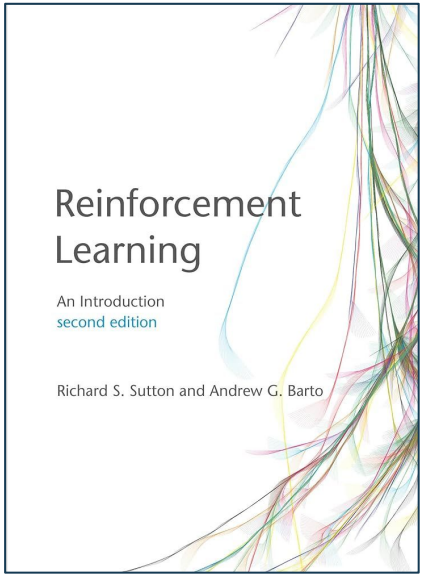


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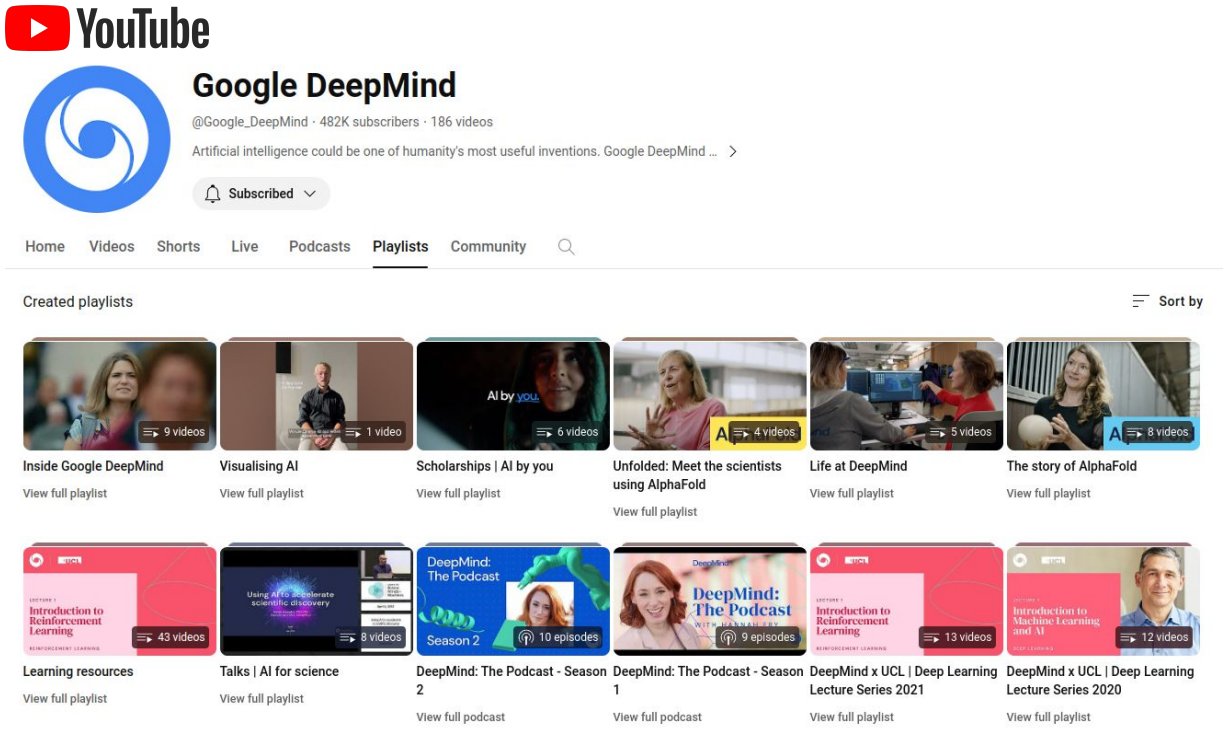


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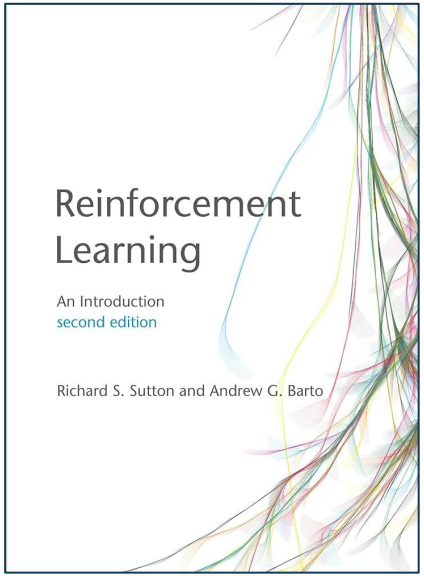


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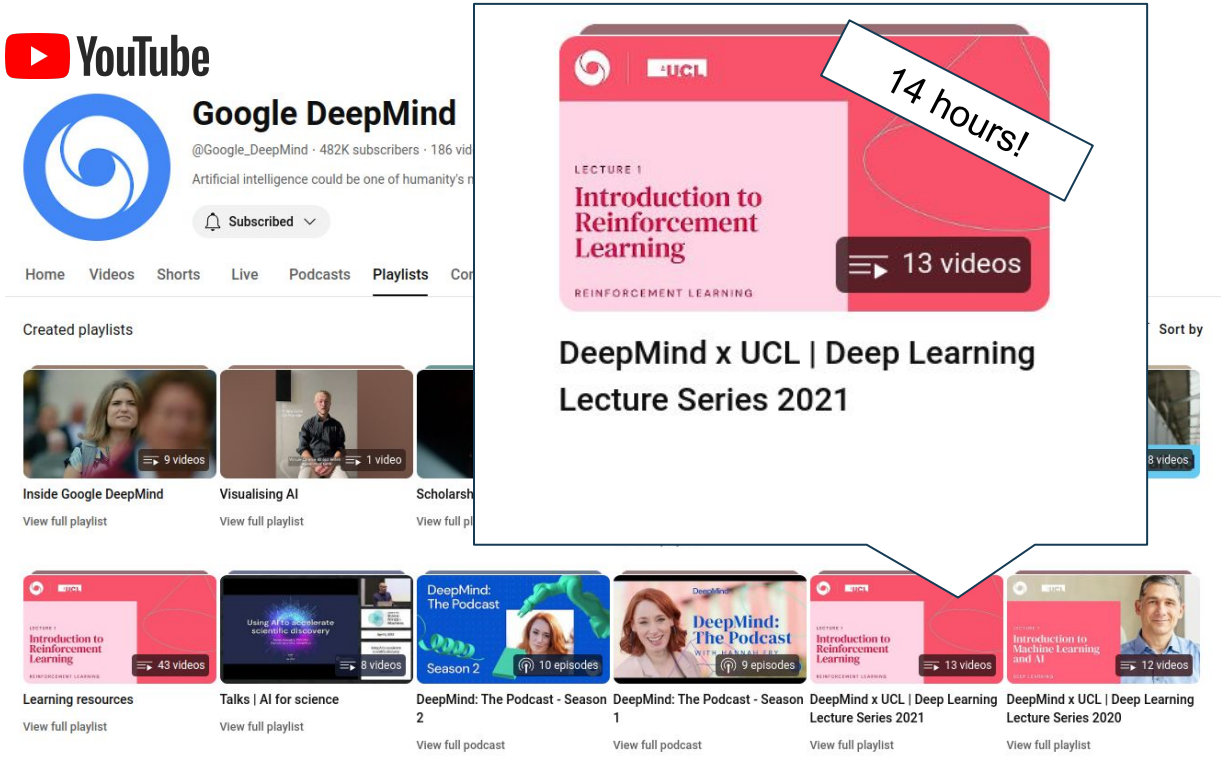


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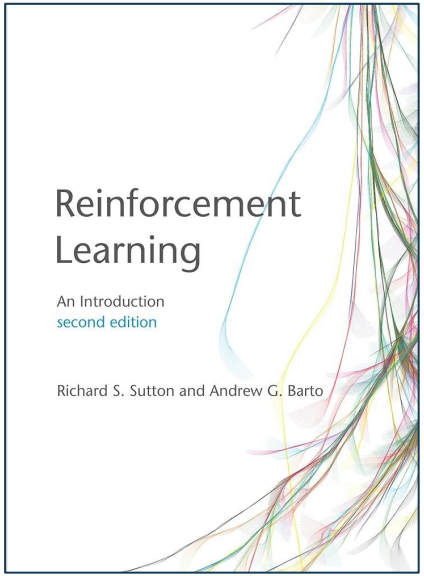


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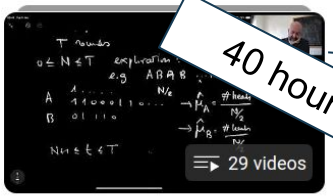


40 hours!

2020-2021 Reinforcement Learning (QLS-RL)

Prof. Antonio Celani

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# Introduction

What is Reinforcement Learning?

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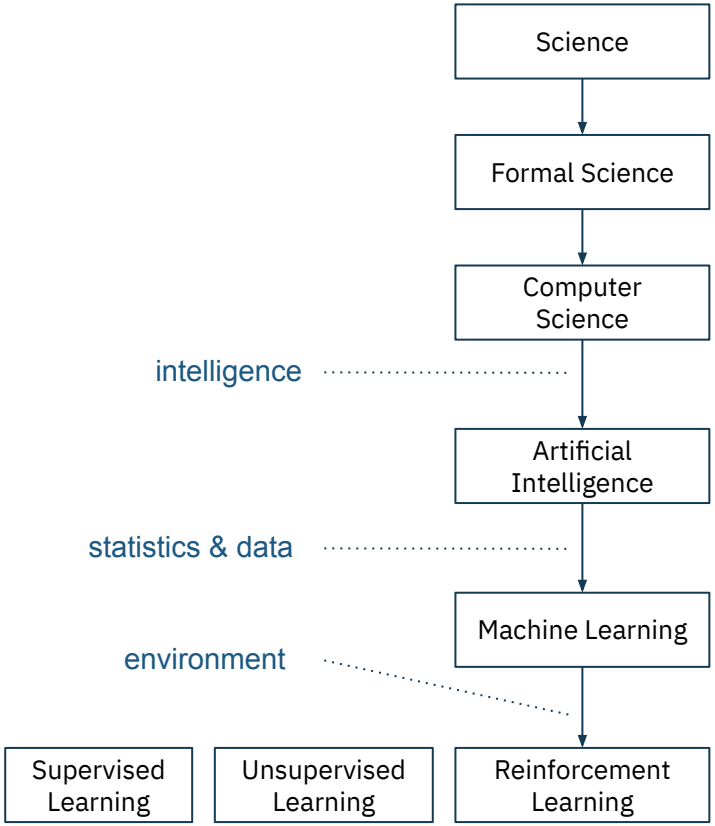
# A map




the systematic study of physical and natural world through observation, experimentation, and the testing of theories against the evidence obtained

uses formal systems to generate knowledge

is the study of computation, information and automation

enabling machines to perceive their environment and uses learning and intelligence to take actions that maximize their chances of achieving defined goals

development and study of **statistical algorithms** that can learn from data and **generalize** to unseen data, and thus perform tasks without explicit instructions.

technique that trains software to make decisions to achieve the most optimal results

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# A definition

| | |
|---|---|
| Reinforcement Learning | technique that trains software to make decisions to achieve the most optimal results |

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# A definition

Reinforcement Learning

technique that trains ~~software~~ to ~~make decisions~~ to ~~achieve the most optimal results~~

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# A definition

Reinforcement Learning

technique that trains **agents** to ~~make decisions~~ to ~~achieve the most optimal results~~

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# A definition

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technique that trains **agents** to **map states into actions** to ~~achieve the most optimal results~~

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# A definition

Reinforcement Learning

technique that trains **agents** to **map states into actions** to **maximize a cumulative reward**

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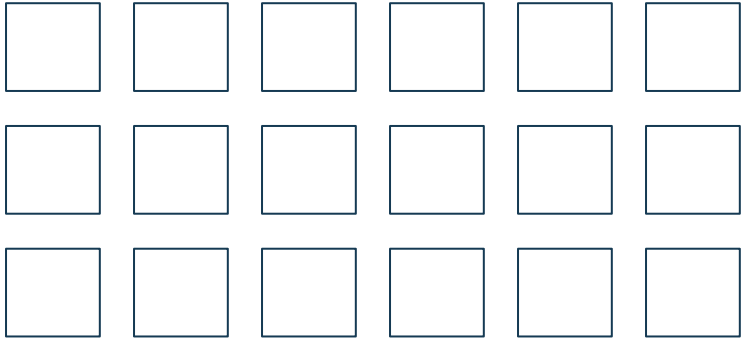
technique that trains **agents** to **map states into actions** to **maximize a cumulative reward**

agent



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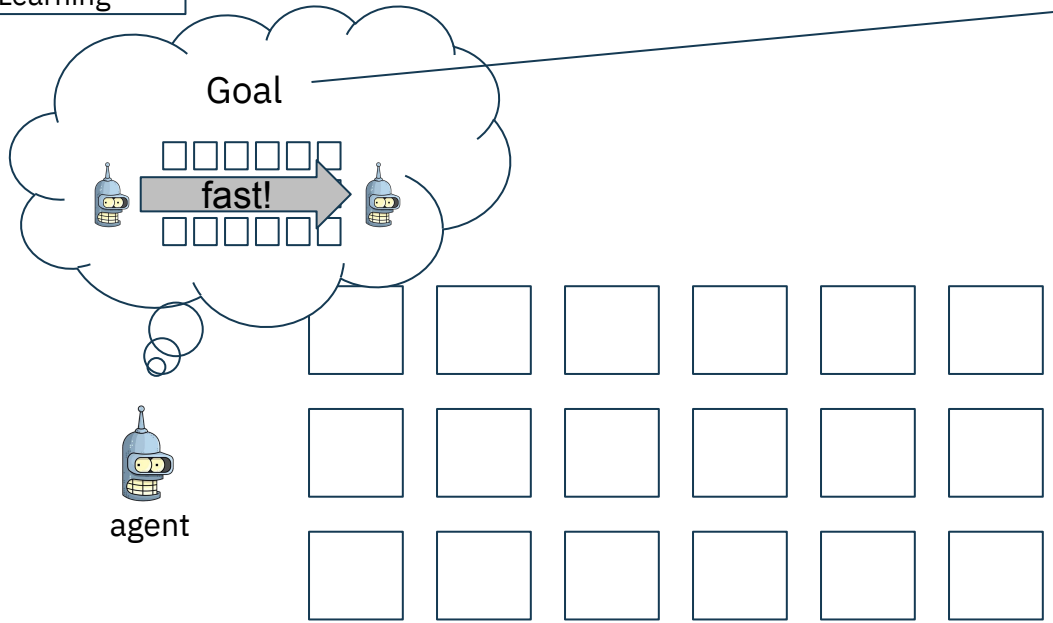


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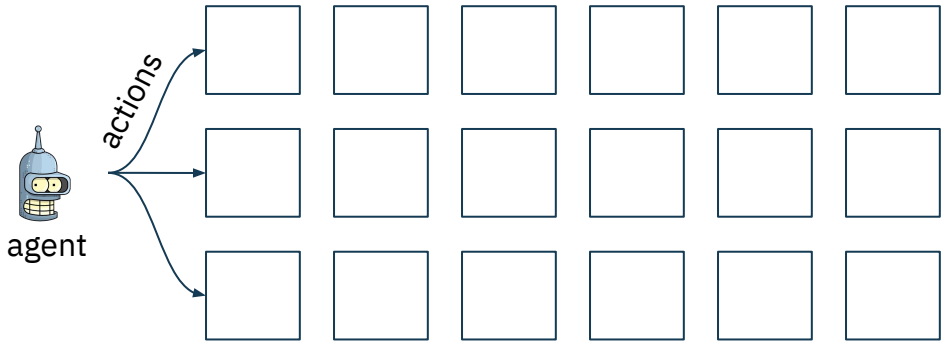


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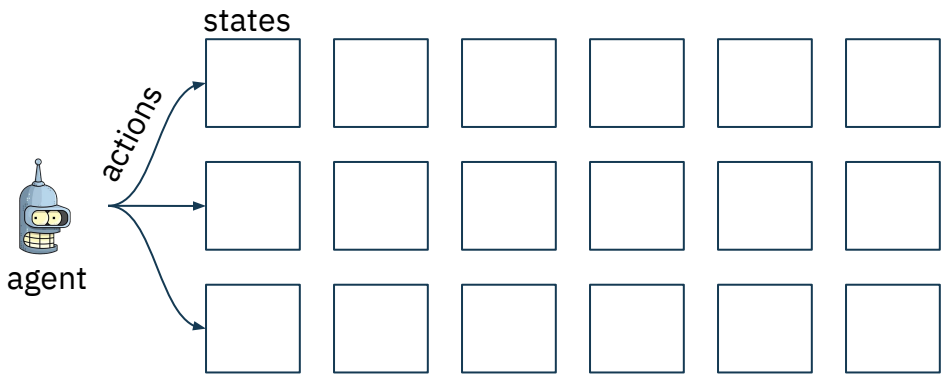


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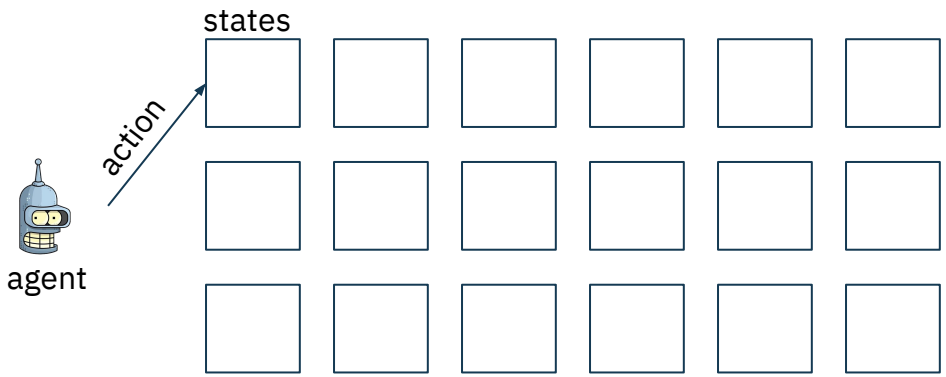


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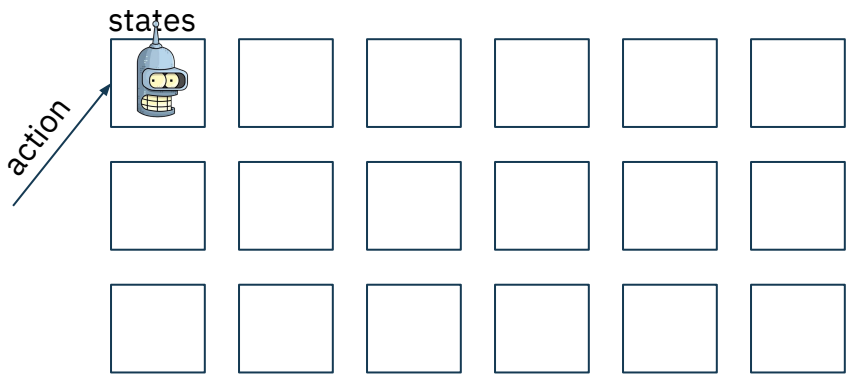


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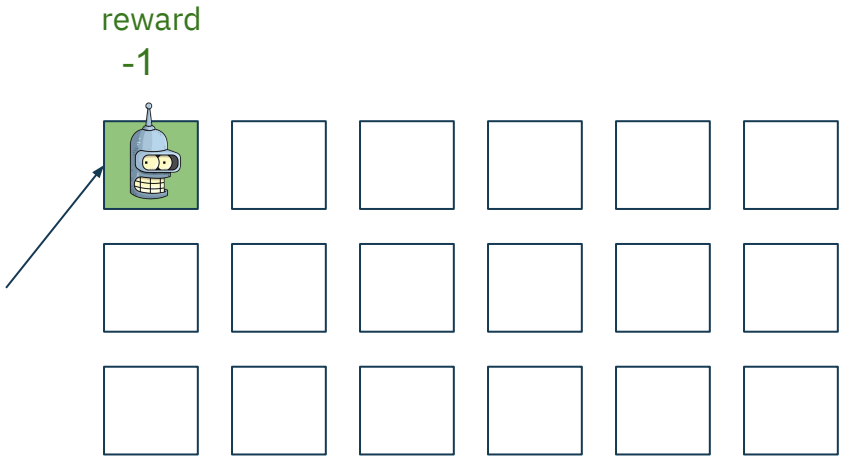


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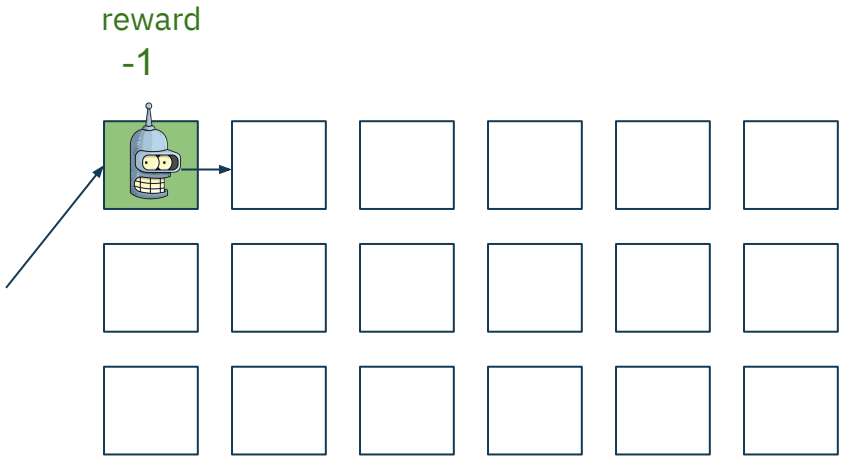


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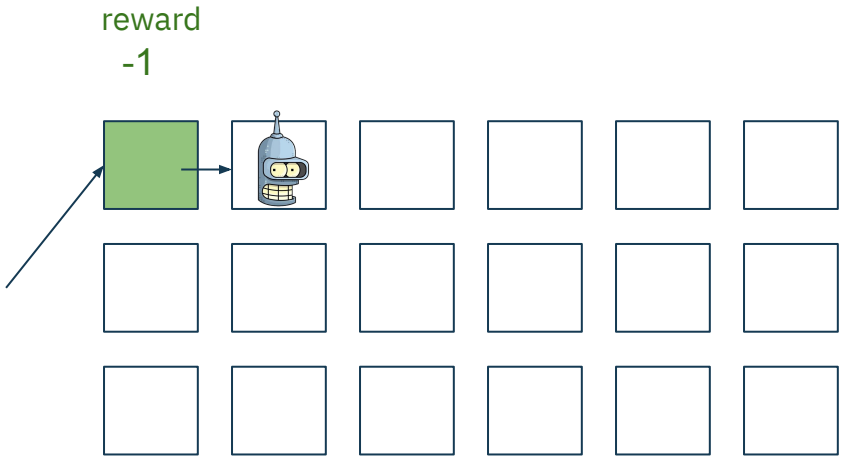


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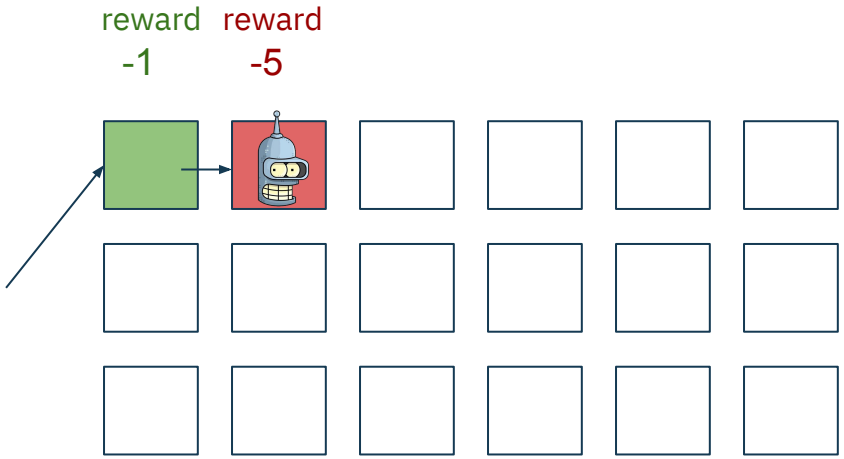
Cumulative Reward

-6

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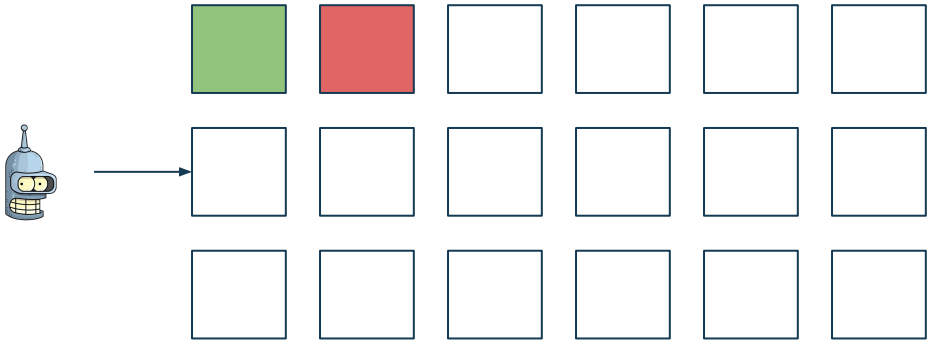

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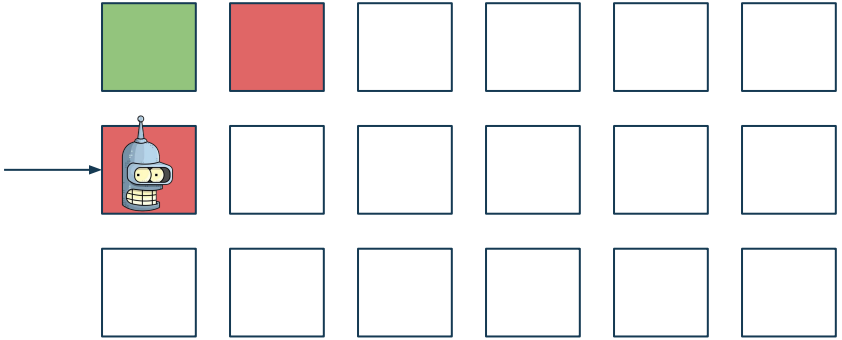
Cumulative Reward

-5

reward

-5



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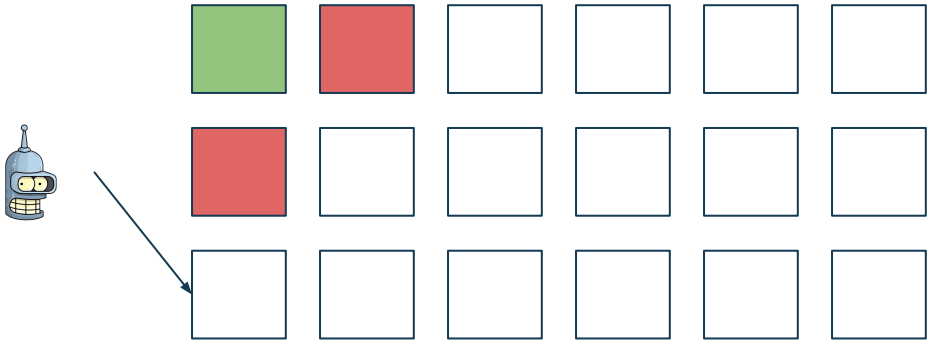

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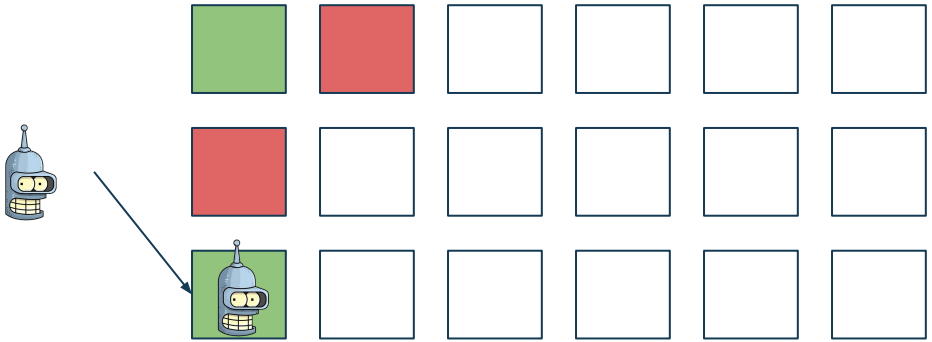
Cumulative Reward

-1



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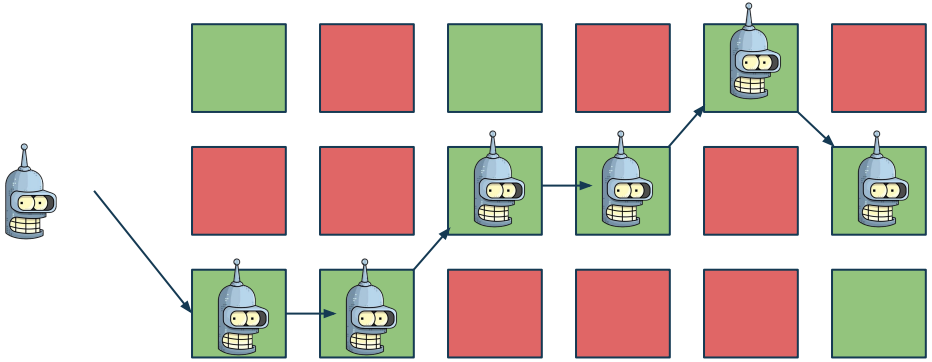
Cumulative Reward

-6



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Reinforcement Learning

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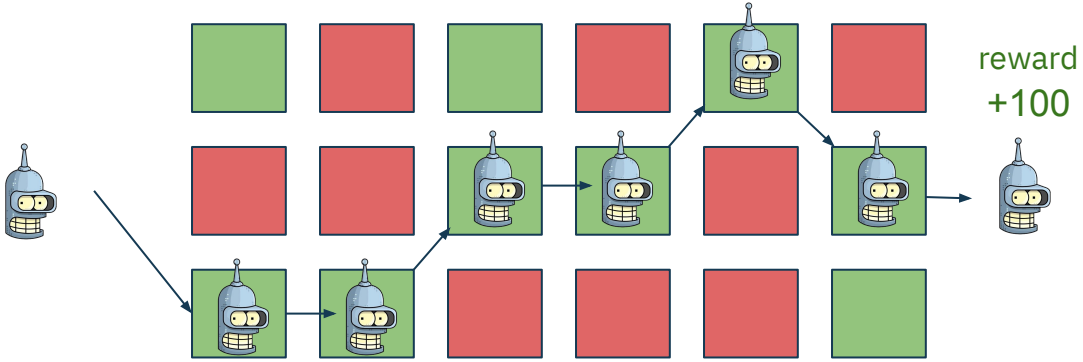
Cumulative Reward

94

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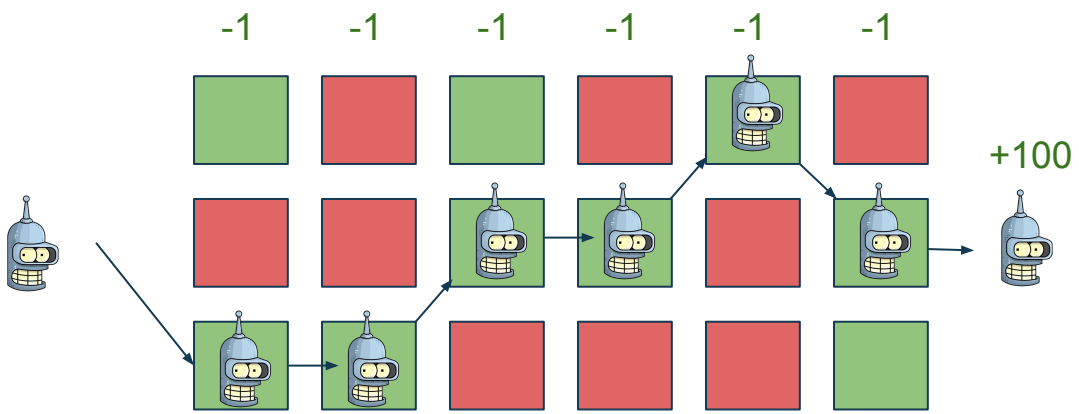
# On reward

Goal and reward coherence

we want the agent goes as fast as possible from A to B. We need to choose an appropriate reward signal!

Cumulative Reward

94

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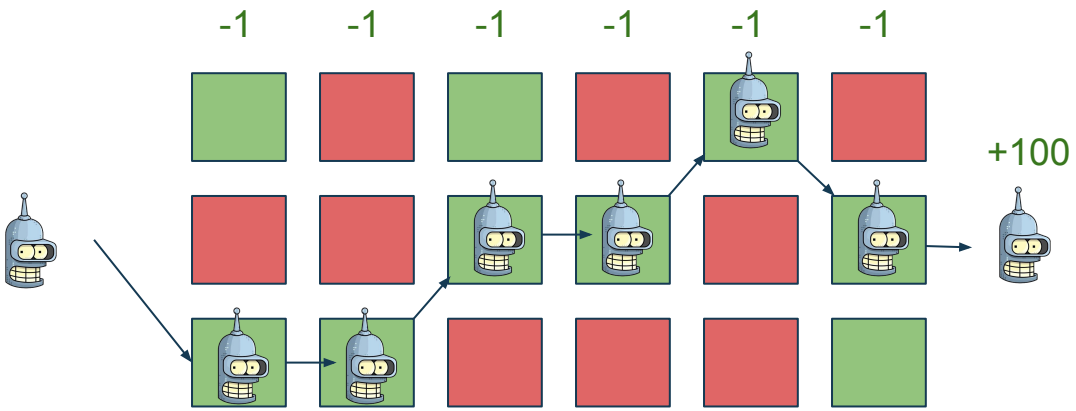
Cumulative Reward

94

Why negative values?

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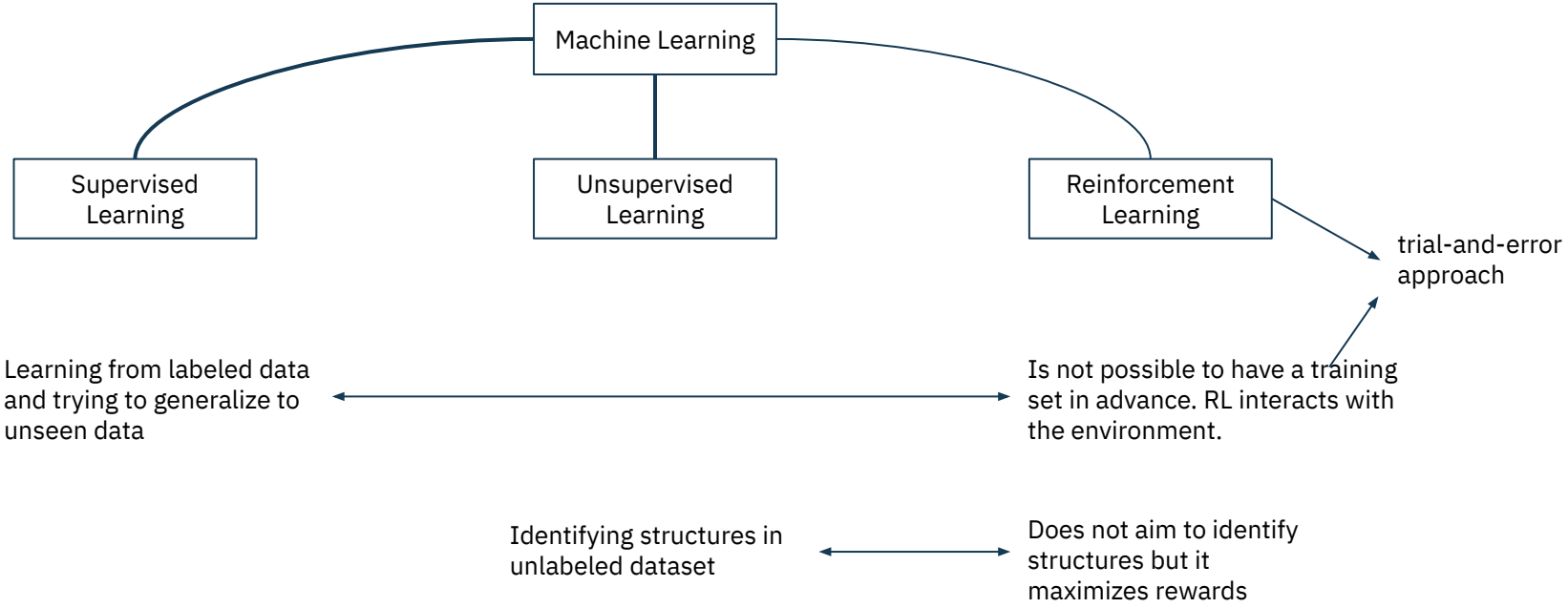


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# Elements of RL

Mathematical definition

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# List of the ingredients

Elements of Reinforcement Learning

- **Environment** — usually exposes uncertainty
- **Agent** — is a decision-maker and has a goal
- **Policy (of the agent)** — defines the agent behaviour
- **Reward signal** — defines the goal of the agent
- **Model of the environment** — exists model-free and model-based RL algorithms
- **Value function (of a policy)** — quantifies how good is for an agent to be in a specific state (if it follows that policy)

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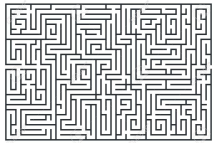
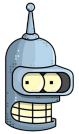
# Observability



environment



agent

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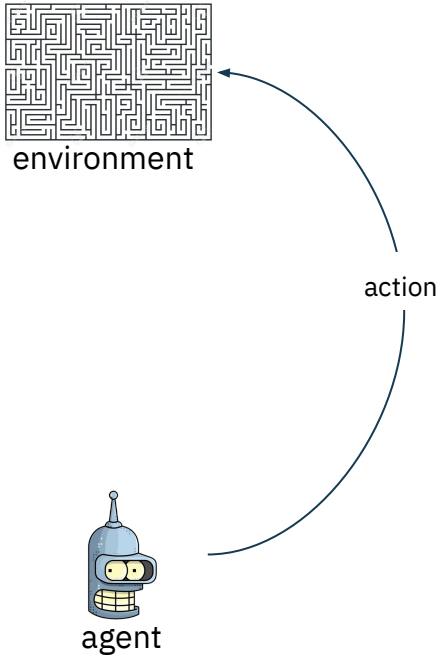


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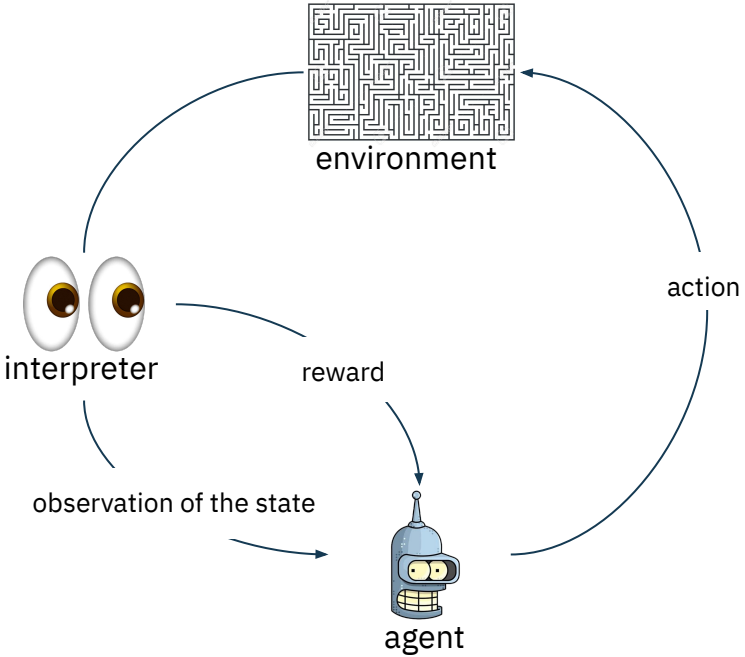


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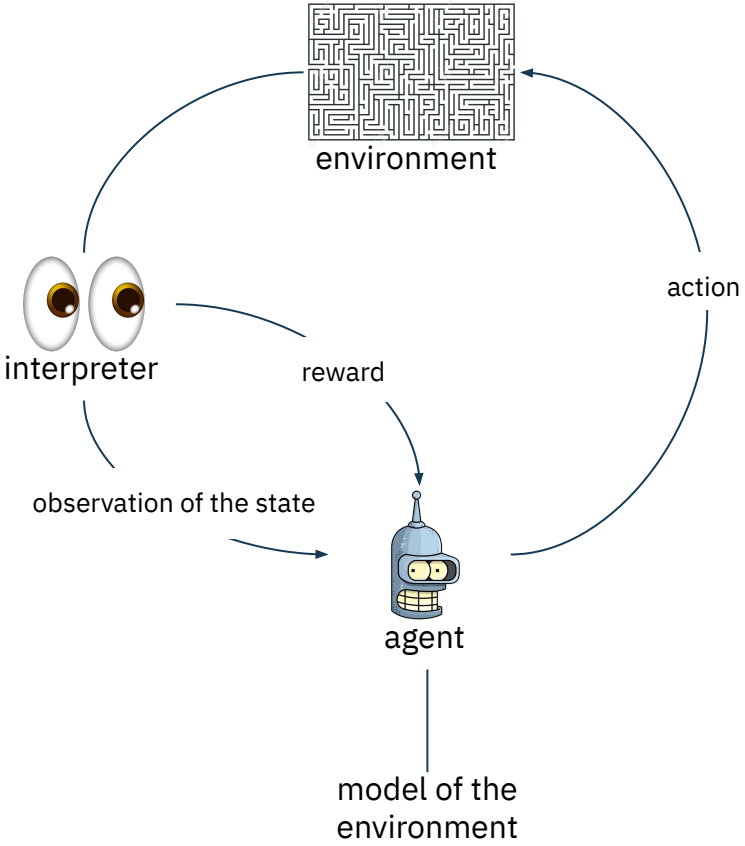


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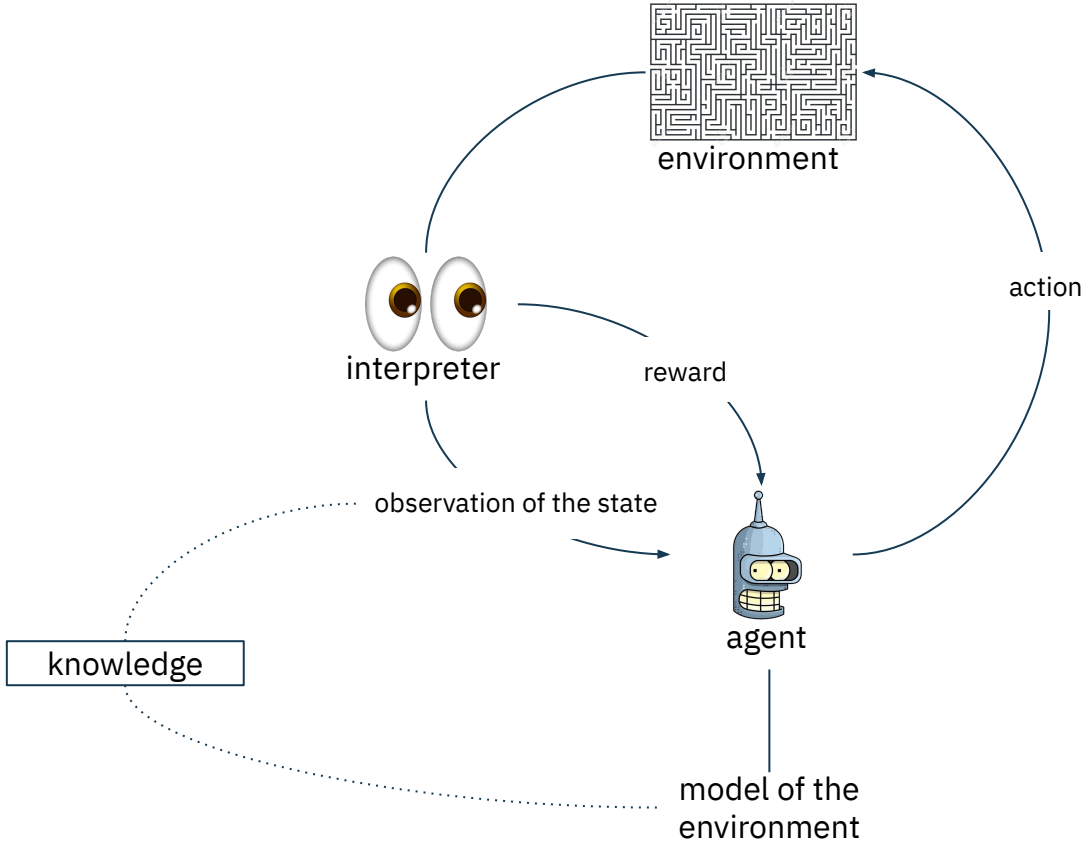


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# Observability

RL = learning + prediction + controlling

Building a model of the environment

Knowing the cumulative reward I'll get following a policy

Discovering the best action

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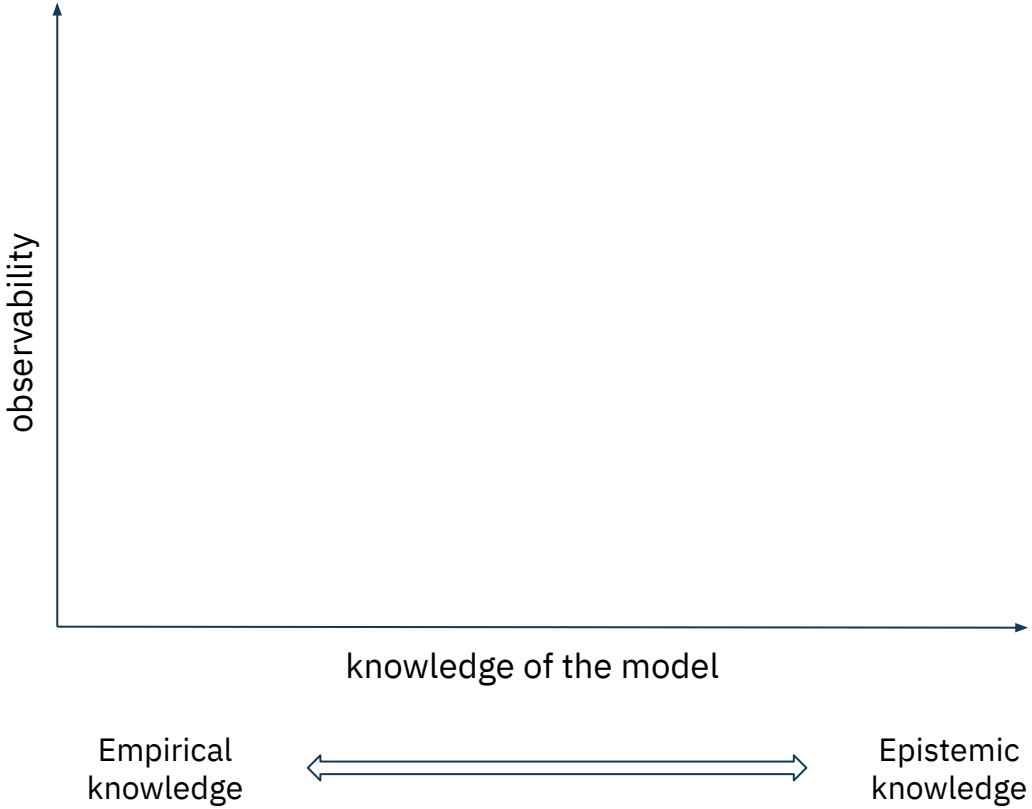
# Knowledge of the environment

## The two axes of knowledge

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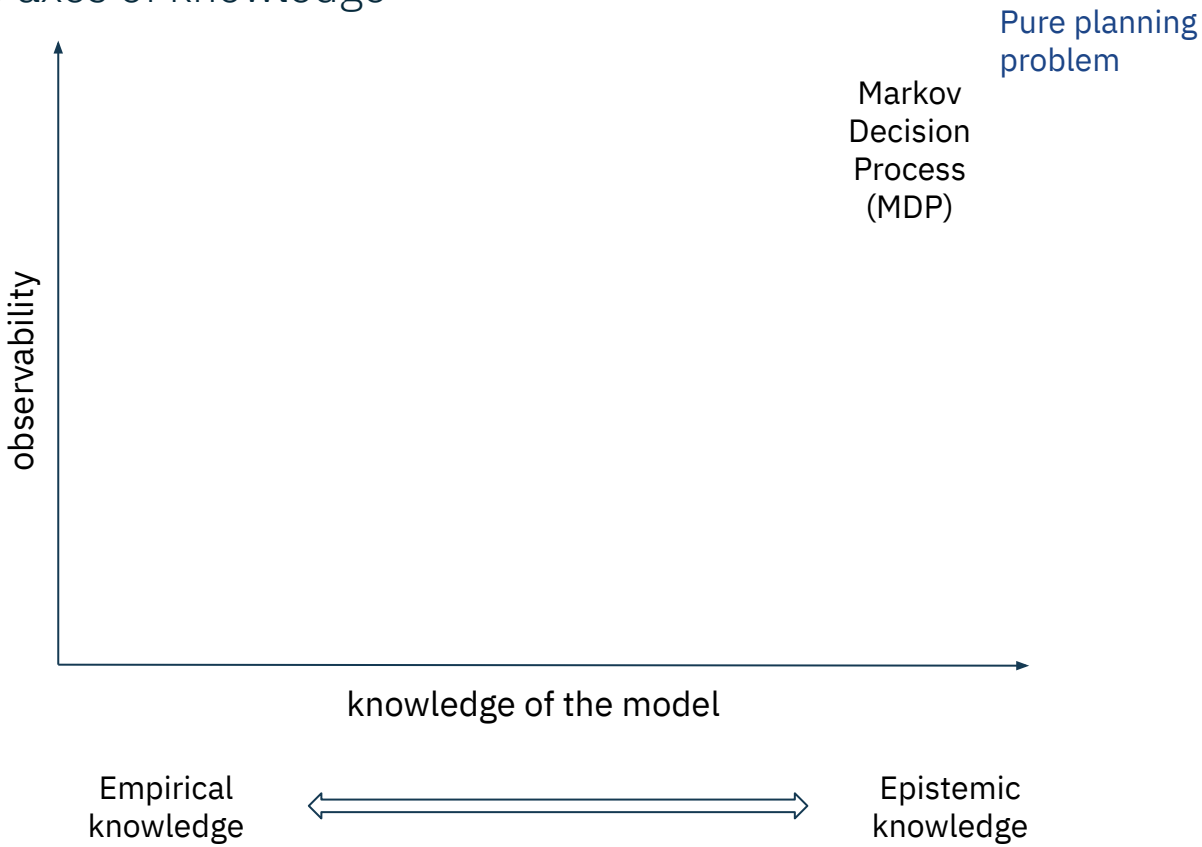
# Knowledge of the environment

## The two axes of knowledge

<u>Markovian process</u>
only matter knowledge of the actual state

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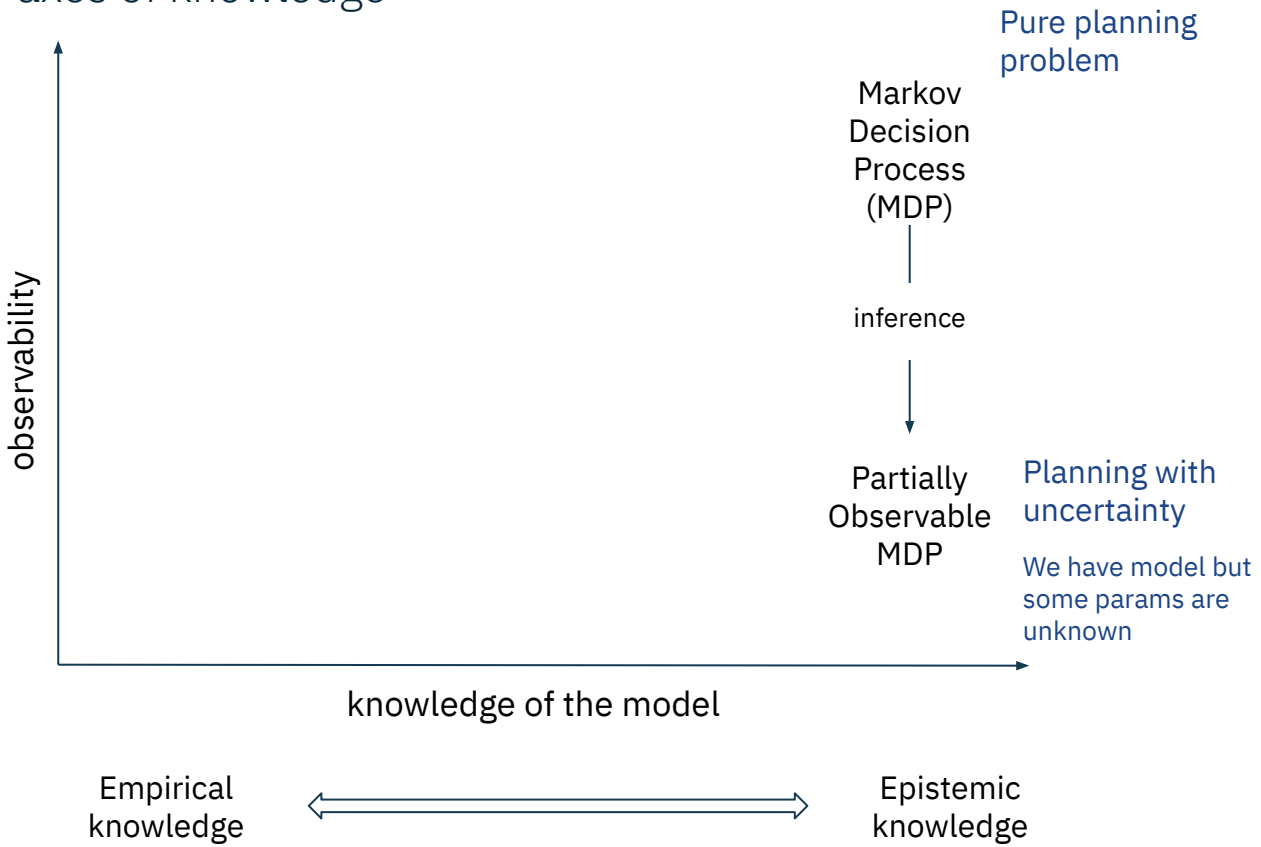


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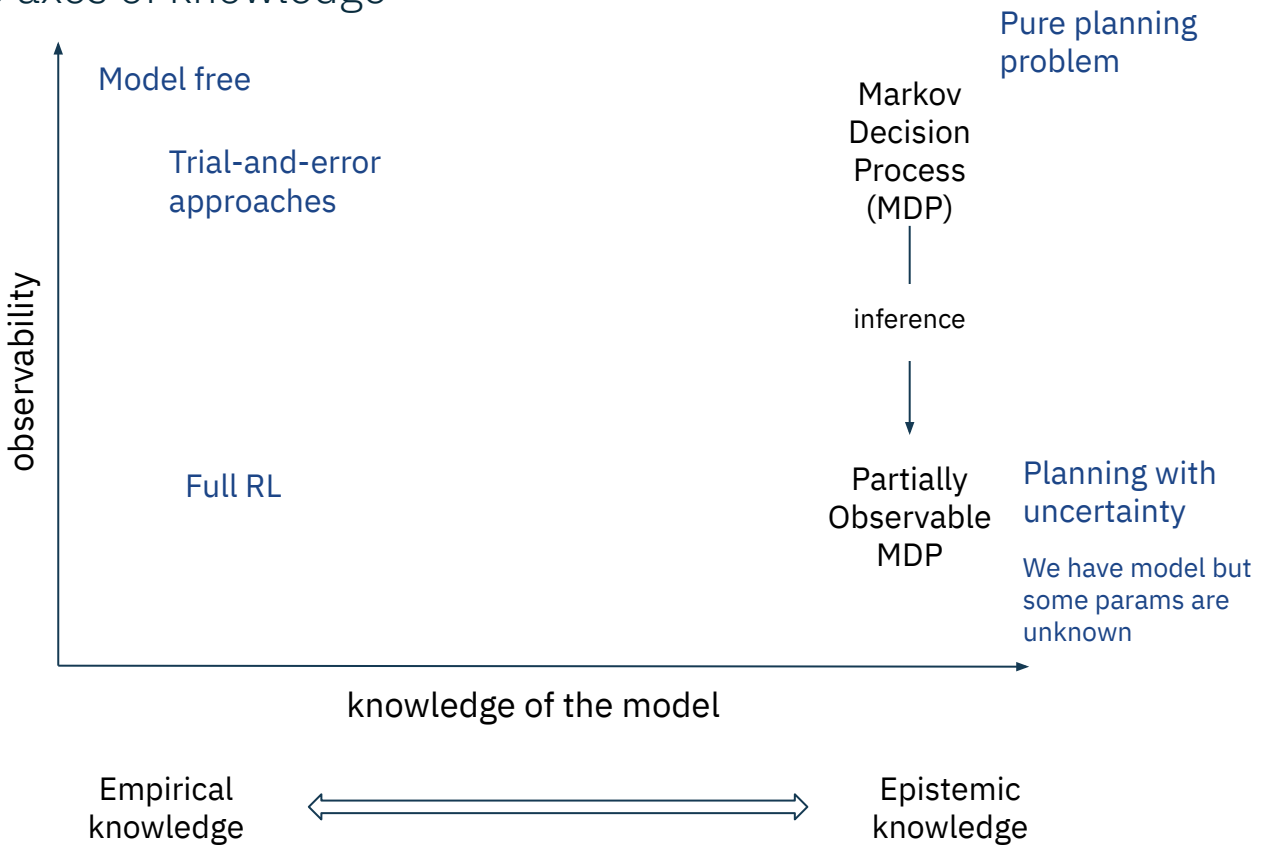


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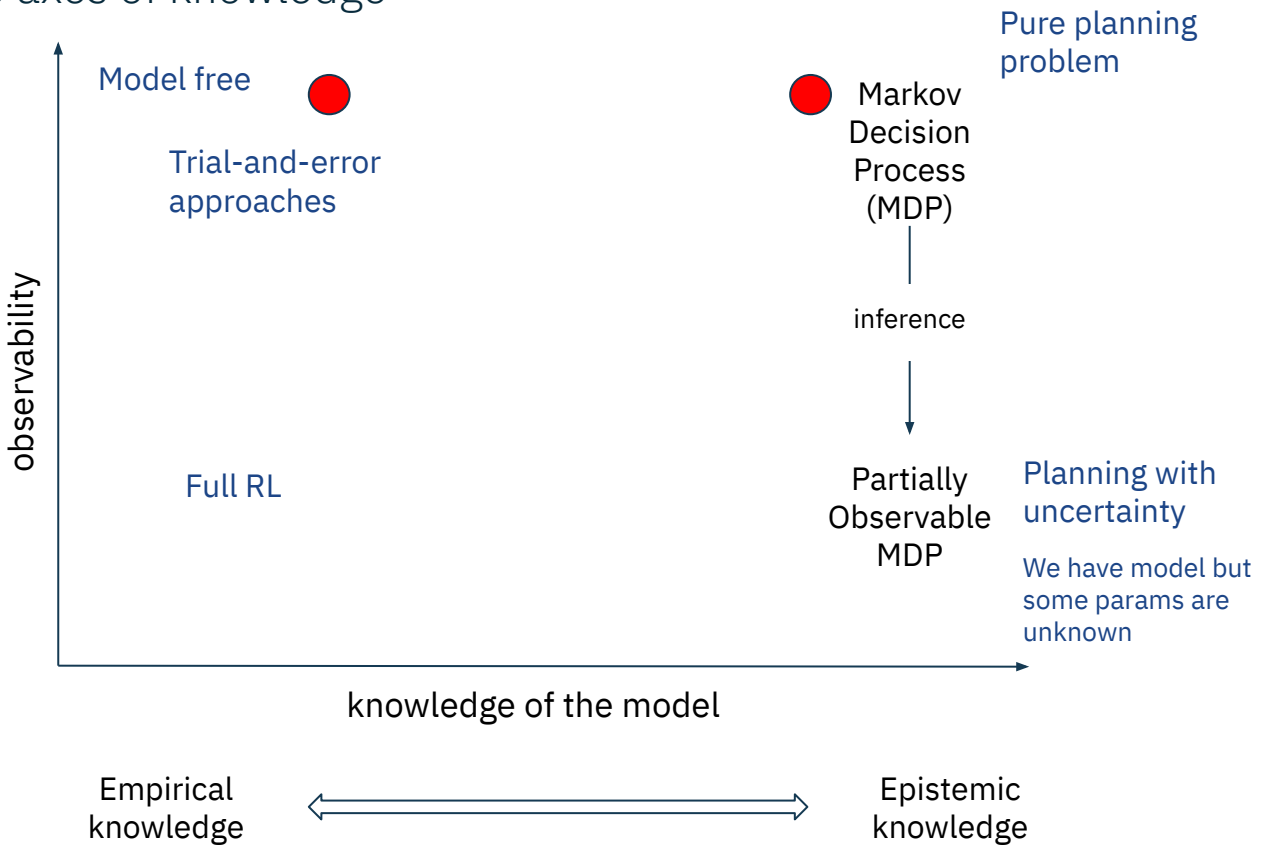
## The two axes of knowledge

Markovian process
only matter knowledge of the actual state

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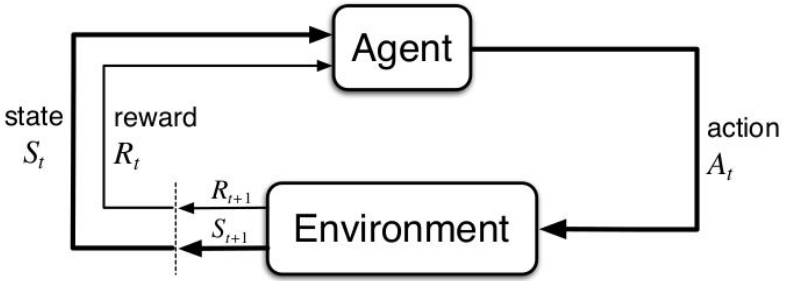
# (finite) Markov Decision Process




trajectory

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, \ldots$$

dynamics

$$p(s', r \mid s, a) \doteq \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\}$$

$$\sum_{s' \in \mathcal{S}} \sum_{r \in \mathcal{R}} p(s', r \mid s, a) = 1, \text{ for all } s \in \mathcal{S}, a \in \mathcal{A}(s)$$

Perfect knowledge
of the model

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# (finite) Markov Decision Process

trajectory

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, \ldots$$

dynamics

$$p(s', r \mid s, a) \doteq \Pr\{S_t = s', R_t = r \mid S_{t-1} = s, A_{t-1} = a\}$$

state-transition probability

$$p(s' \mid s, a) \doteq \Pr\{S_t = s' \mid S_{t-1} = s, A_{t-1} = a\} = \sum_{r \in \mathcal{R}} p(s', r \mid s, a)$$

expected reward (I)

$$r(s, a) \doteq \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a] = \sum_{r \in \mathcal{R}} r \sum_{s' \in \mathcal{S}} p(s', r \mid s, a)$$

expected reward (II)

$$r(s, a, s') \doteq \mathbb{E}[R_t \mid S_{t-1} = s, A_{t-1} = a, S_t = s'] = \sum_{r \in \mathcal{R}} r \frac{p(s', r \mid s, a)}{p(s' \mid s, a)}$$

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# Reward signal

**Reward hypothesis:** that all of what we mean by goals and purposes can be well thought of as the maximization of the expected value of the cumulative sum of a received scalar signal (called reward).

Reward $R_{t+1}$

Return $G_t \doteq R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$

Discounted Return $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$

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# Reward signal

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Short-term view

Long-term view

| | |
|---|---|
| Reward | $R_{t+1}$ |
| Return | $G_t \doteq R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$ |
| Discounted Return | $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$ |

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$$\boxed{G_t = R_{t+1} + \gamma G_{t+1}}$$

**RECURSIVE DEFINITION**

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# Policy

Policy

is a mapping from states to probabilities of selecting each possible action

$$\pi : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$$

If we are at time t, $\pi(a|s)$ is the probability of having $A_t = a \wedge S_t = s$

can be *deterministic*

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# Value function

Value Function

is a function that quantify how good is to be on a state and follows a specific policy

$$v_\pi : \mathcal{S} \to \mathbb{R}$$

state-value function

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right], \text{ for all } s \in \mathcal{S}$$

action-value function

$$q_\pi(s,a) \doteq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a\right]$$

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# Solving a RL problem

find a policy that achieves the maximum reward over the long run

optimal policy

$\pi_* \succeq \pi \quad \forall \pi \in$ policies

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$$\pi' \succeq \pi \iff \forall s \in \mathcal{S},\ v_{\pi'}(s) \geq v_{\pi}(s)$$

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optimal state-value function

$$v_*(s) \doteq \max_\pi v_\pi(s)$$

optimal action-value function

$$q_*(s,a) \doteq \max_\pi q_\pi(s,a)$$

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$$q_*(s,a) \doteq \max_\pi q_\pi(s,a)$$

$$q_*(s,a) = \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a]$$

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# Dynamic Programming

How to solve MDP problems

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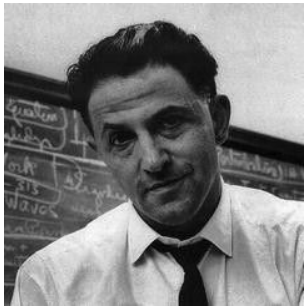
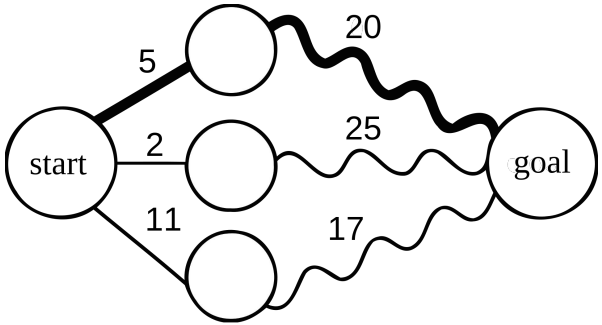
# Dynamic Programming

## Mr. Richard Ernest Bellman



Bellman, 1950s

Algorithm paradigm useful to solve a specific class of problems that can be decomposed in sub-problems in recursive way

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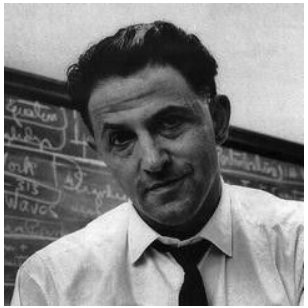
# Dynamic Programming

## In the RL context

Collection of algorithms that can be used to compute optimal policies given a perfect model of the environment as a MDP.



Bellman, 1950s

**Key idea:** use value function to organize and structure the search of optimal policies

Consistency relation of state-value function

$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r \mid s, a) \Big[ r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s'] \Big] \\
&= \sum_a \pi(a|s) \sum_{s', r} p(s', r \mid s, a) \Big[ r + \gamma v_\pi(s') \Big], \quad \text{for all } s \in \mathcal{S}.
\end{aligned}
$$

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# Dynamic Programming

## Towards the Bellman Equation

Consistency relation of state-value function

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\,[r + \gamma v_\pi(s')]$$

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# Dynamic Programming

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Consistency relation of state-value function

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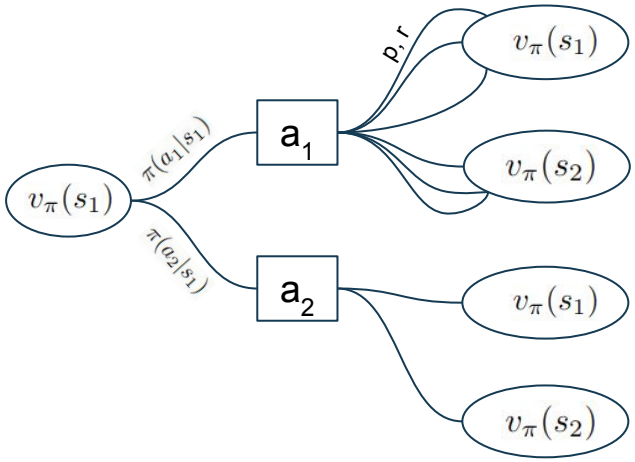


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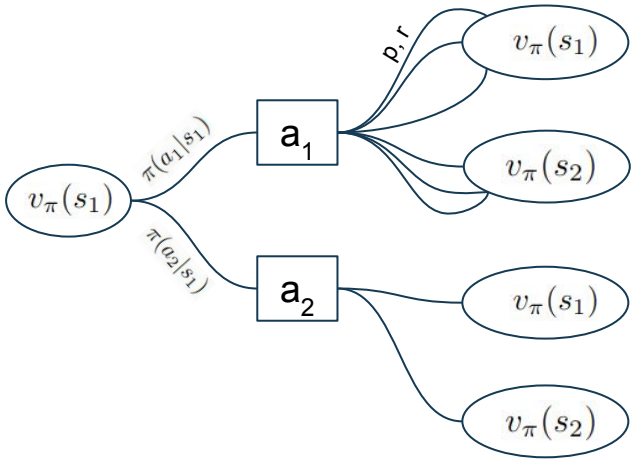


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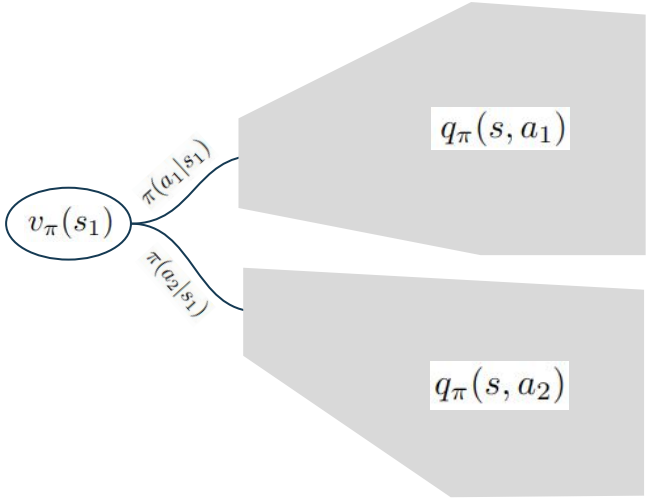


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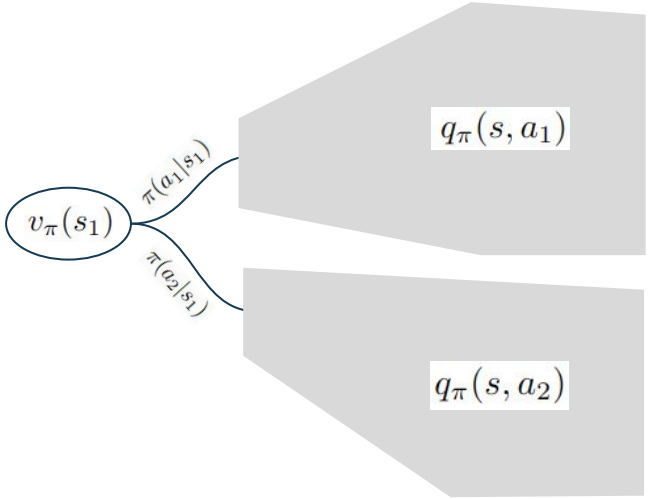
# Dynamic Programming

## Towards the Bellman Equation

Consistency relation of state-value function

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$$

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# Dynamic Programming

## Towards the Bellman Equation

What about the optimal policy and the optimal state-value function?

Consistency relation of state-value function

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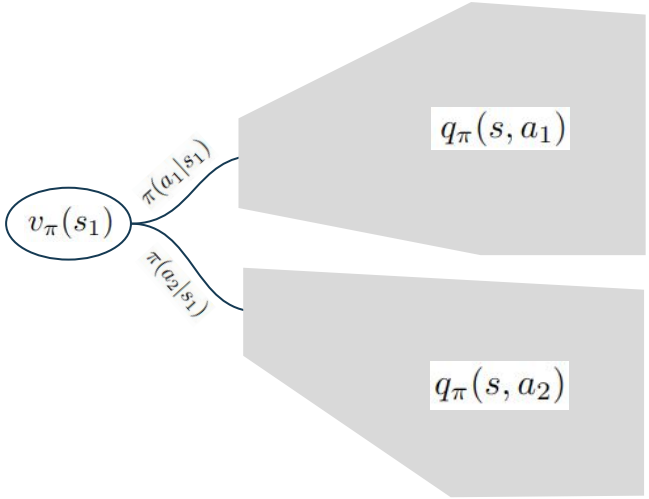


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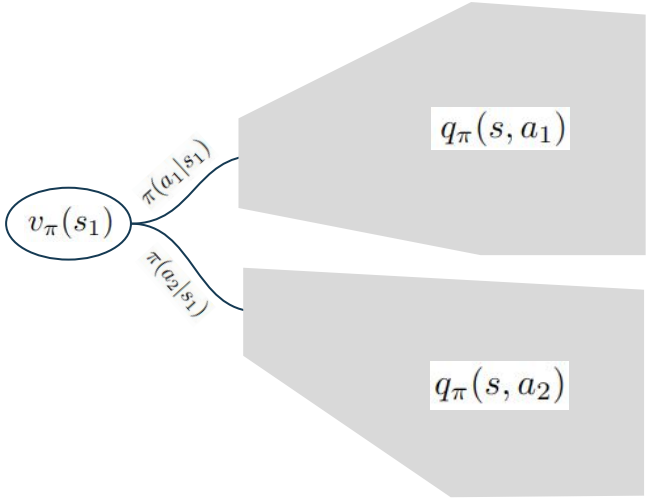
It's an average

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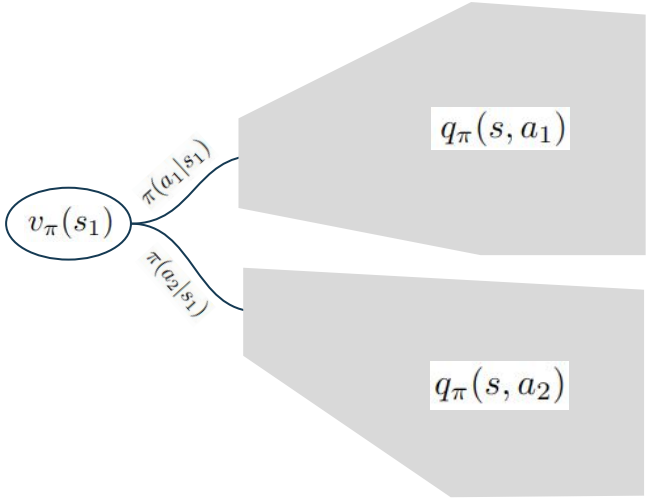
It's an average

The optimal policy is a policy so it should satisfy the consistency relation

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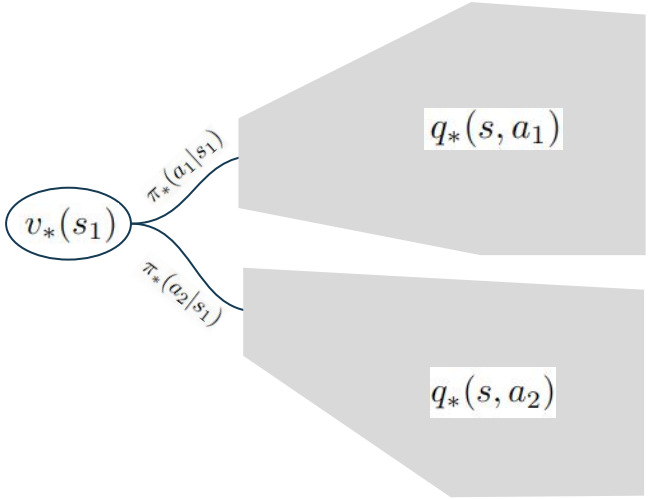


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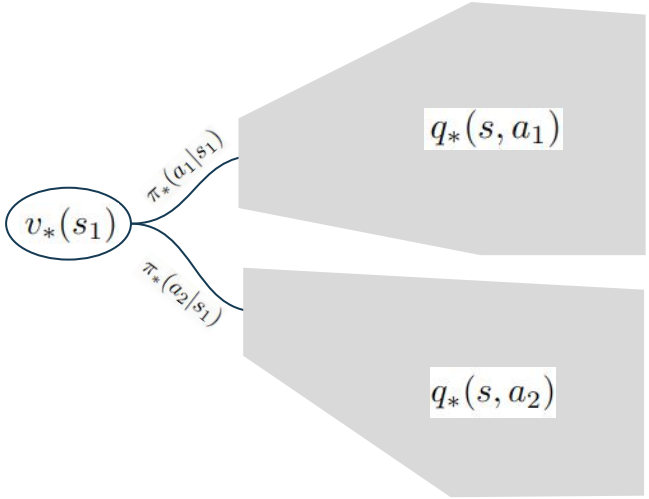


The optimal policy is *optimal*

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# Dynamic Programming

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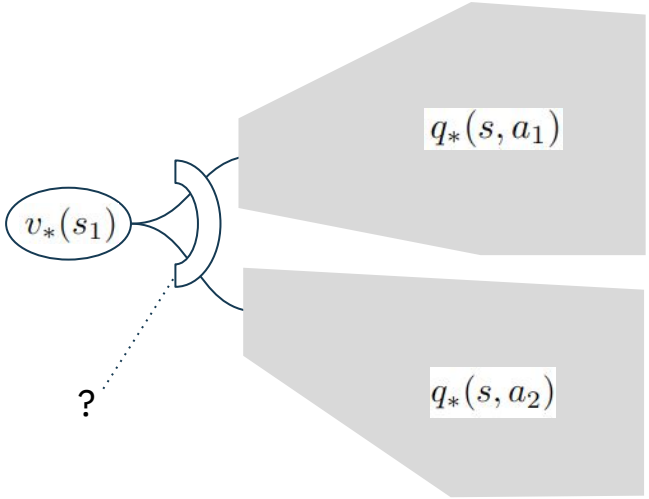


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The optimal policy is *optimal*

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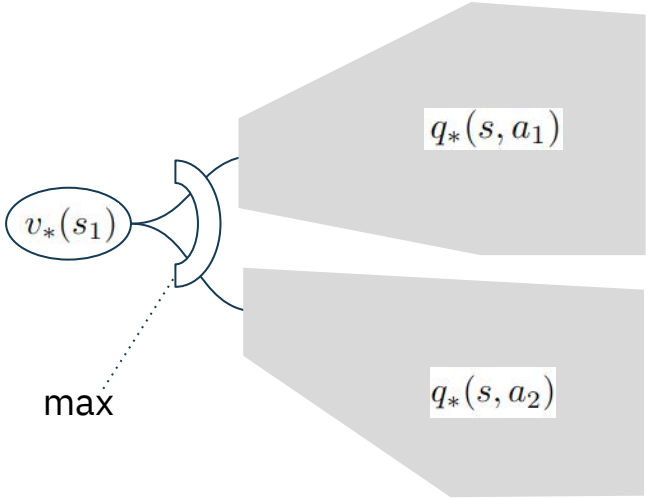
# Dynamic Programming

## Bellman Equation

Bellman equation

$$v_*(s) = \max_a q_*(s, a)$$

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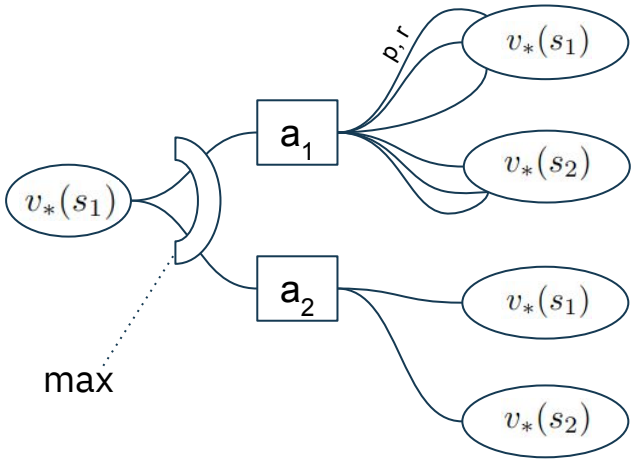
# Dynamic Programming

## Bellman Equation

Bellman equation

$$v_*(s) = \max_a q_*(s,a)$$

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma v_*(s')]$$

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# Dynamic Programming

## Bellman Equation

Bellman equation

$$v_*(s) = \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma v_*(s')]$$

$$q_*(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma \max_a q_*(s',a')]$$

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# Dynamic Programming

## How to find the optimal policy?

$$\pi \xrightarrow{\text{Iterative procedure}} \pi^*$$

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# Dynamic Programming

## How to find the optimal policy?

Consistency relation of the state-value function

$$\pi \xrightarrow{\text{Policy evaluation}} v_\pi \longrightarrow \pi'$$

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# Dynamic Programming

## How to find the optimal policy?

Consistency relation of the state-value function

Bellman intuition

$$\pi \xrightarrow{\text{Policy evaluation}} v_\pi \xrightarrow{\text{Policy improvement}} \pi'$$

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# Dynamic Programming

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Policy Iteration

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# Dynamic Programming

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Consistency relation of the state-value function

Bellman intuition

$$\pi \xrightarrow{\text{Policy evaluation}} v_\pi \xrightarrow{\text{Policy improvement}} \pi'$$

Policy Iteration

$$\pi_0 \xrightarrow{\mathrm{E}} v_{\pi_0} \xrightarrow{\mathrm{I}} \pi_1 \xrightarrow{\mathrm{E}} v_{\pi_1} \xrightarrow{\mathrm{I}} \pi_2 \xrightarrow{\mathrm{E}} \cdots \xrightarrow{\mathrm{I}} \pi_* \xrightarrow{\mathrm{E}} v_*$$

Does it converge? Yes

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# Dynamic Programming

## Policy evaluation

$$\pi \xrightarrow{\text{Policy evaluation}} v_\pi$$

Consistency relation of state-value function

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\,[r + \gamma v_\pi(s')]$$

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Iterative policy evaluation

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_k(s')]$$

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# Dynamic Programming

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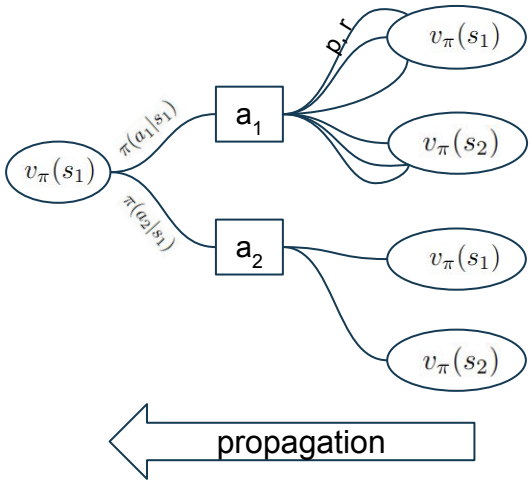
<u>2 ways of updating</u>: *in-place* vs *two arrays version*

Faster, depends on ordering of update

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# Policy Evaluation

## Algorithm

**Iterative Policy Evaluation, for estimating $V \approx v_\pi$**

Input $\pi$, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) \big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

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    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ Stability of state-value function

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# Policy Evaluation

## Example

actions

Uniform Policy

| | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | |

$$R_t = -1$$

on all transitions

non-terminal state

terminal state

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# Policy Evaluation

## Example - 1st iteration

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | | | |
| | | | |
| | | | |
| | | | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - 1st iteration

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | | | |
| | | | |
| | | | |
| | | | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[\underbrace{r}_{-1} + \gamma \underbrace{v_\pi(s')}_{0}]$$

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# Policy Evaluation

## Example - 1st iteration

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - 1st iteration

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | ? | | |
| ? | | | |
| | | | ? |
| | | ? | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - 2nd iteration

← -1/3 +

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

→

| | | | |
|---|---|---|---|
| 0.0 | ? | | |
| ? | | | |
| | | | ? |
| | | ? | 0.0 |

$$v_\pi(s) = \sum_a \underbrace{\pi(a|s)}_{1/3} \sum_{s',r} \underbrace{p(s',r|s,a)}_{1}[\underbrace{r}_{-1} + \gamma \underbrace{v_\pi(s')}_{0}]$$

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# Policy Evaluation

## Example - 2nd iteration

← -1/3 ✚

↓ -2/3 ✚

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

⇒

| | | | |
|---|---|---|---|
| 0.0 | ? | | |
| ? | | | |
| | | | ? |
| | | ? | 0.0 |

$$v_\pi(s) = \sum_a \underbrace{\pi(a|s)}_{1/3} \sum_{s',r} \underbrace{p(s',r|s,a)}_{1}[\underbrace{r}_{-1} + \gamma \underbrace{v_\pi(s')}_{-1}]$$

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# Policy Evaluation

## Example - 2nd iteration

← -1/3 +

↓ -2/3 +

→ -2/3 +

$\boxed{-1.7}$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | ? | | |
| ? | | | |
| | | | ? |
| | | ? | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - 2nd iteration

← -1/3 +

↓ -2/3 +

→ -2/3 +

$\boxed{-1.7}$

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

⇒

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | | |
| -1.7 | | | |
| | | | -1.7 |
| | | -1.7 | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - 2nd iteration

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')]$$

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# Policy Evaluation

## Example - until the end

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

1 →

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

2 →

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

3 →

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

∞ ←

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

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# Policy Evaluation

## Example - until the end

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

1 →

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

2 →

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

3 →

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

∞ →

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

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# Policy Evaluation

## Example - until the end

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

1 →

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

2 →

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

3 →

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

∞ ←

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

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# Policy Improvement

## How to find better policies

$$v_\pi \xrightarrow{\text{Policy improvement}} \pi'$$

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Policy improvement theorem

$$q_\pi(s, \pi'(s)) \geq v_\pi(s),\ \forall s \in \mathcal{S} \Rightarrow v_{\pi'}(s) \geq v_\pi(s),\ \forall s \in \mathcal{S}$$

Greedy policy approach

$$\begin{aligned}
\pi'(s) &\doteq \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}$$

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# Policy Iteration

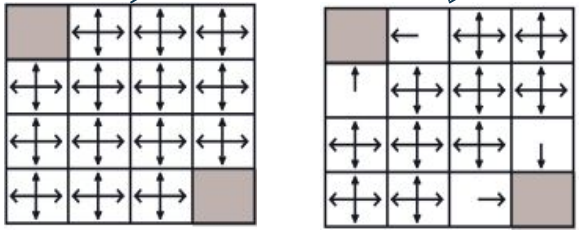
## Example

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

E

| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

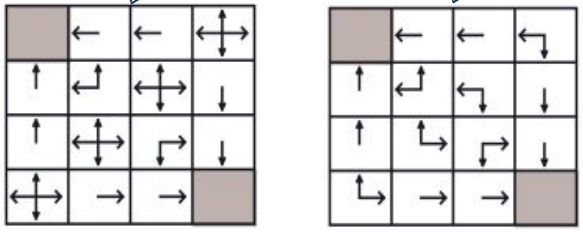
I



E

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

I



E

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

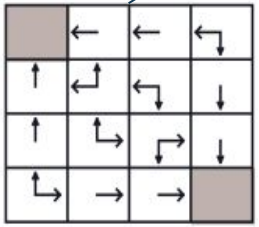
I



E

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

I



E

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# Policy Iteration

## Example

propagation effect

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

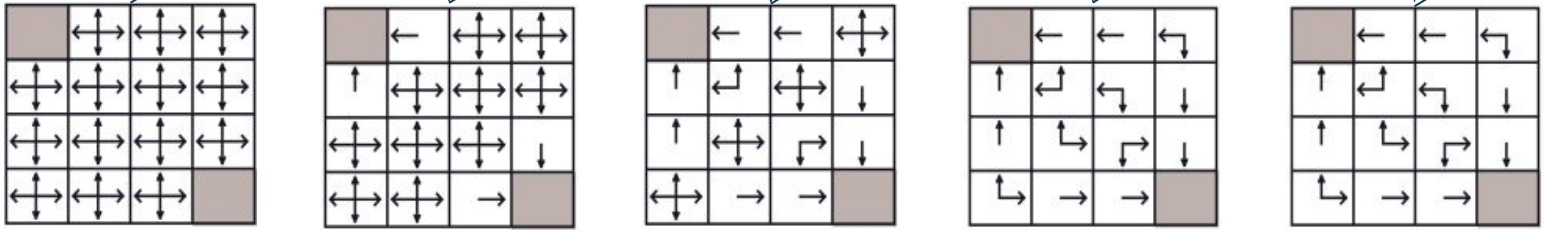
| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

E I E I E I E I E



policy convergence

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# Policy Iteration

## Example

propagation effect

| | | | |
|---|---|---|---|
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 |

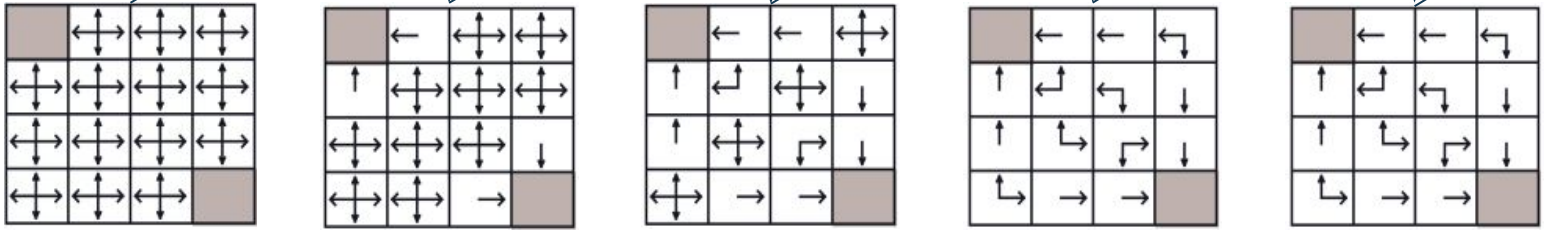
| | | | |
|---|---|---|---|
| 0.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | -1.0 |
| -1.0 | -1.0 | -1.0 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -1.7 | -2.0 | -2.0 |
| -1.7 | -2.0 | -2.0 | -2.0 |
| -2.0 | -2.0 | -2.0 | -1.7 |
| -2.0 | -2.0 | -1.7 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -2.4 | -2.9 | -3.0 |
| -2.4 | -2.9 | -3.0 | -2.9 |
| -2.9 | -3.0 | -2.9 | -2.4 |
| -3.0 | -2.9 | -2.4 | 0.0 |

| | | | |
|---|---|---|---|
| 0.0 | -6.1 | -8.4 | -9.0 |
| -6.1 | -7.7 | -8.4 | -8.4 |
| -8.4 | -8.4 | -7.7 | -6.1 |
| -9.0 | -8.4 | -6.1 | 0.0 |

E I E I E I E I E



policy convergence

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# Policy Iteration

**Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$**

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\quad \Delta \leftarrow 0$
   $\quad$ Loop for each $s \in \mathcal{S}$:
   $\quad\quad v \leftarrow V(s)$
   $\quad\quad V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \big[r + \gamma V(s')\big]$
   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad$ *old-action* $\leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \big[r + \gamma V(s')\big]$
   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

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# Policy Iteration

**Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$**

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\quad \Delta \leftarrow 0$
   $\quad$ Loop for each $s \in \mathcal{S}$:
   $\quad\quad v \leftarrow V(s)$
   $\quad\quad V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \big[r + \gamma V(s')\big]$
   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   *policy-stable* $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   $\quad$ *old-action* $\leftarrow \pi(s)$
   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \big[r + \gamma V(s')\big]$
   $\quad$ If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

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# Value Iteration

## Solving efficiently the Policy Iteration

**Value Iteration, for estimating $\pi \approx \pi_*$**

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $\quad v \leftarrow V(s)$
| $\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
| $\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$

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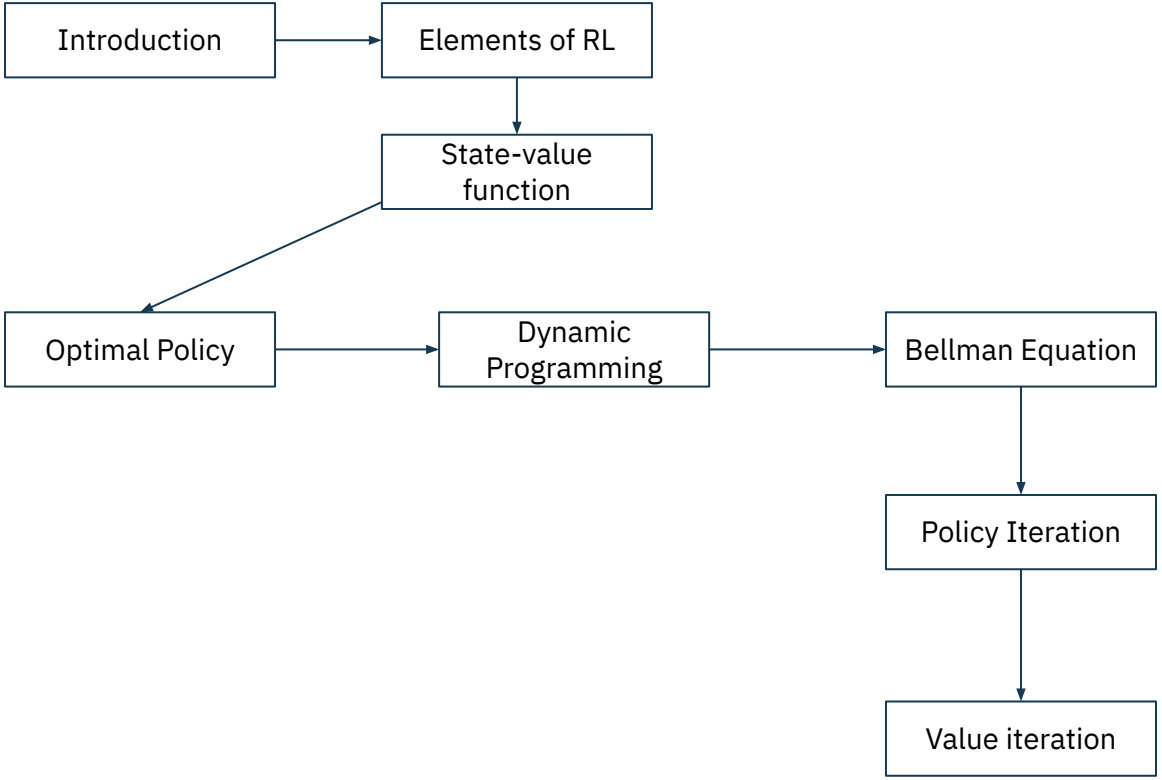
# Recap

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# Monte Carlo Methods

We are ignorant, we need to learn

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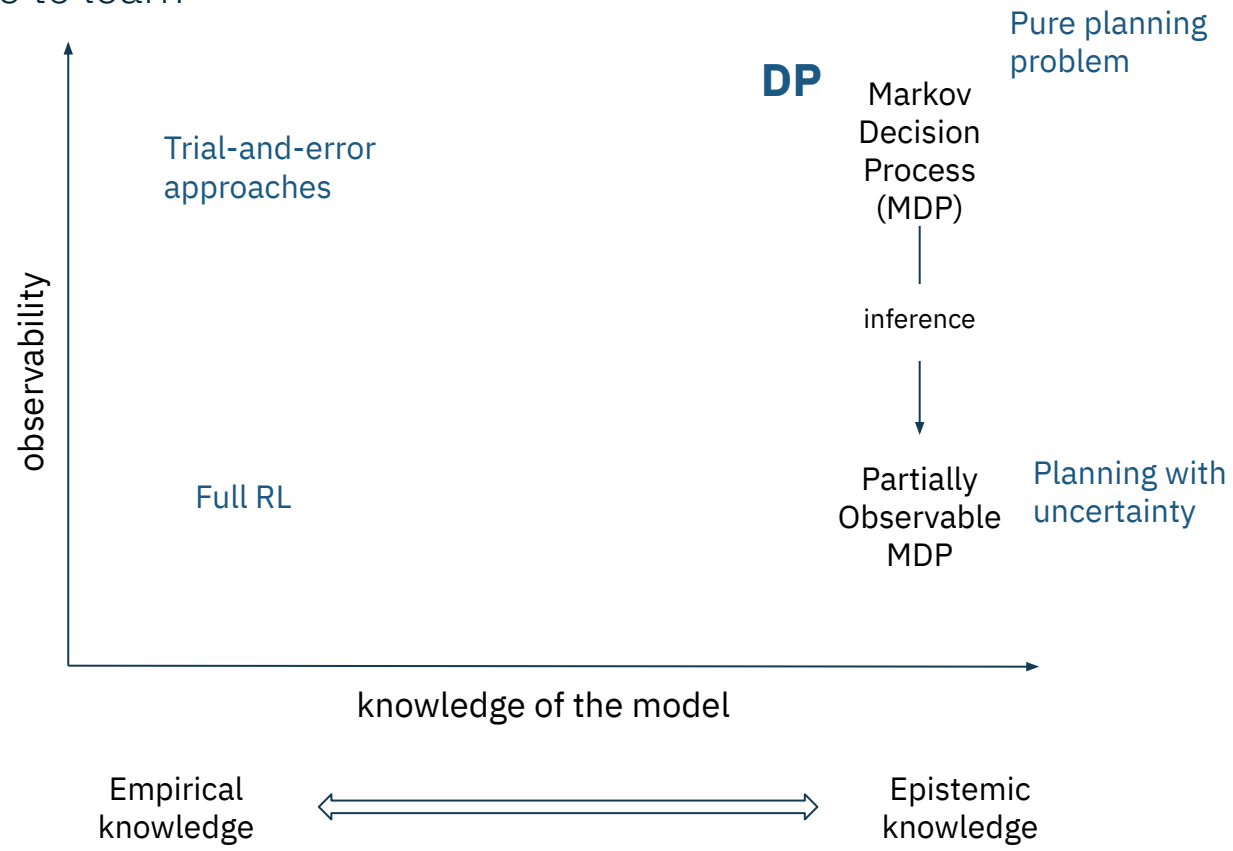
# Monte Carlo Methods

It’s time to learn

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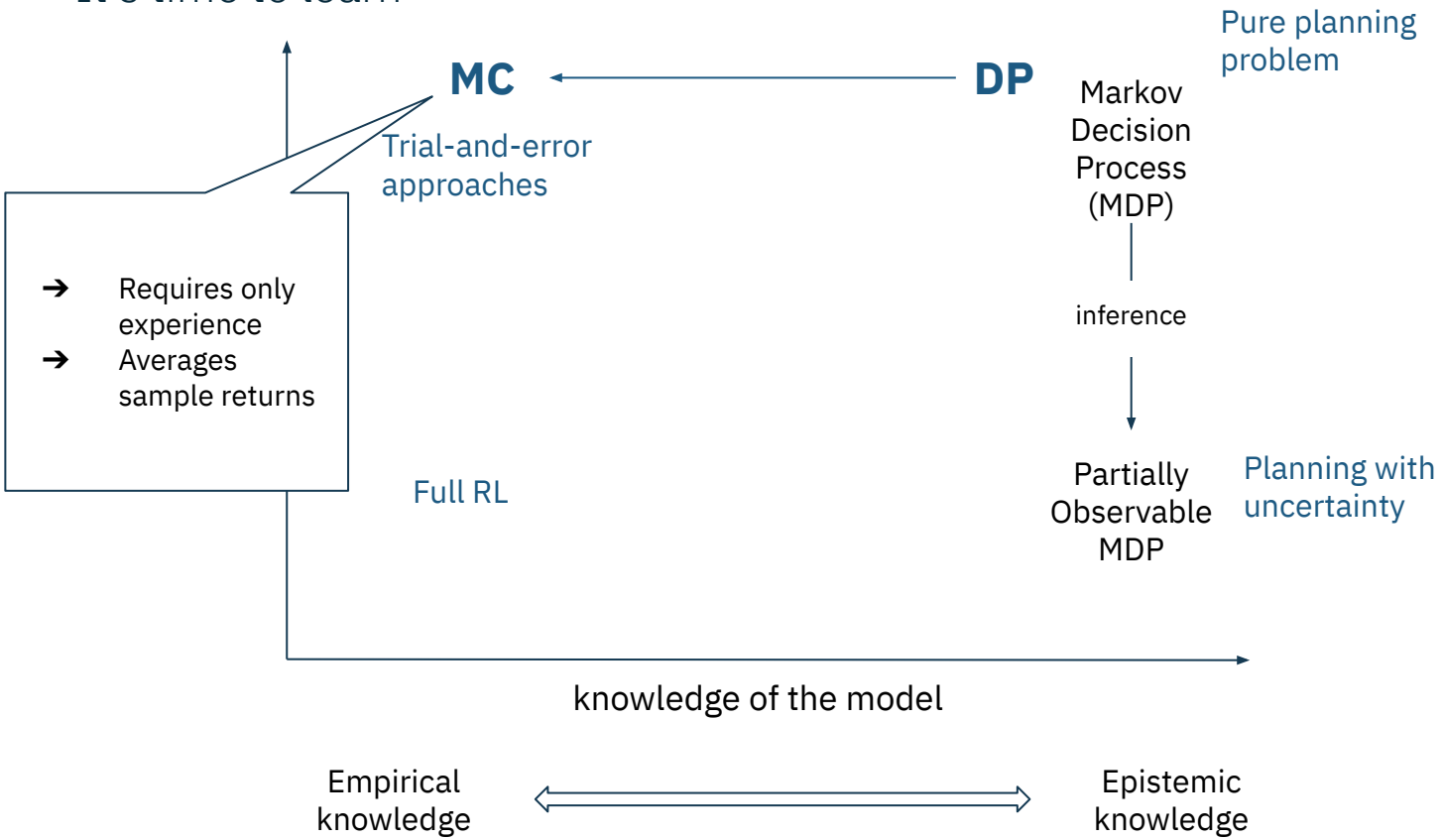
# Monte Carlo Methods

It’s time to learn

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# Monte Carlo Methods

## First-visit MC prediction idea

| | |
|---|---|
| Episode 0 | $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T_0-1}, A_{T_0-1}, R_{T_0}$ |
| Episode 1 | $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T_1-1}, A_{T_1-1}, R_{T_1}$ |
| Episode 2 | $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T_2-1}, A_{T_2-1}, R_{T_2}$ |
| Episode 3 | $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T_3-1}, A_{T_3-1}, R_{T_3}$ |

First-visit MC prediction algorithm

Identify the first time a state is visited and average the following returns

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# Monte Carlo Methods

## First-visit MC prediction idea

v(s) = mean of

Episode

s, a, r, s, a, r, s, a, r ,s, a, r, s ,a, r, s (terminal state)

v(s) = mean of

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# Monte Carlo Methods

## First-visit MC prediction algorithm

Episode

s, a, r, s, a, r, s, a, r ,s, a, r, s ,a, r, s (terminal state)

**First-visit MC prediction, for estimating $V \approx v_\pi$**

Input: a policy $\pi$ to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following $\pi$: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless $S_t$ appears in $S_0, S_1, \ldots, S_{t-1}$:

Append $G$ to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

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# Monte Carlo Methods

## How to identify the optimal policy?

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Greedy policy approach

$$
\begin{aligned}
\pi'(s) &\doteq \underset{a}{\arg\max}\, q_\pi(s,a) \\
&= \underset{a}{\arg\max}\, \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \underset{a}{\arg\max} \sum_{s',r} p(s',r \mid s,a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

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# Monte Carlo Methods

## How to identify the optimal policy?

We do not have a mode!

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Greedy policy approach

$$
\begin{aligned}
\pi'(s) &\doteq \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

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# Monte Carlo Methods

## How to identify the optimal policy?

We need to use the Q-value!

| | | | |
|---|---|---|---|
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Greedy policy approach

$$
\begin{aligned}
\pi'(s) &\doteq \boxed{\arg\max_a q_\pi(s,a)} \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}
$$

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# Monte Carlo Methods

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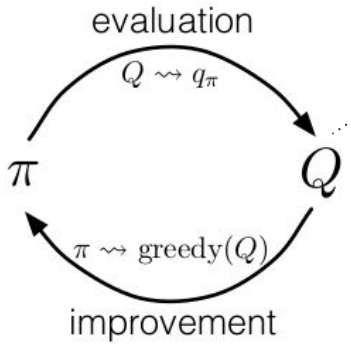
| 0.0 | -14. | -20. | -22. |
| -14. | -18. | -20. | -20. |
| -20. | -20. | -18. | -14. |
| -22. | -20. | -14. | 0.0 |

Greedy policy approach

$$\begin{aligned}\pi'(s) &\doteq \boxed{\arg\max_a q_\pi(s,a)} \\ &= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma v_\pi(s')\big],\end{aligned}$$

We need to estimate the action-value function!




The same idea of policy iteration of DP but with Q + estimation

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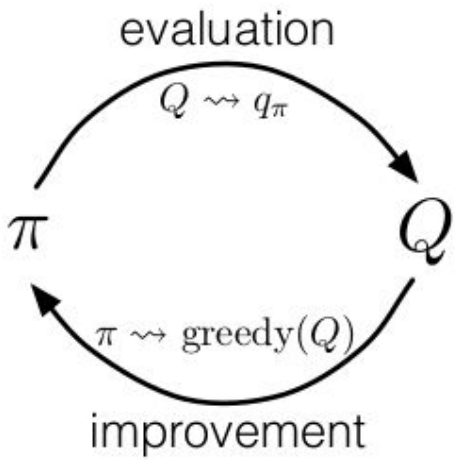
# Monte Carlo Methods

## How to identify the optimal policy?




Episode s, a, r, **s**, a, r, **s**, a, r ,**s**, a, r, **s** ,a, r, s (terminal state)

v(s) = mean of

Episode s, a, r, **s**, **a**, r, **s**, **a**, r ,**s**, **a**, r, **s** ,**a**, r, s (terminal state)

q(s,a) = mean of

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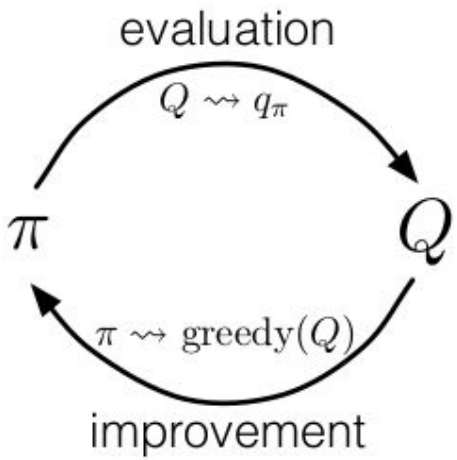
# Monte Carlo Methods

## How to identify the optimal policy?




Episode s, a, r, s, a, r, s, a, r ,s, a, r, s ,a, r, s (terminal state)

v(s) = mean of

Episode s, a, r, s, a, r, s, a, r ,s, a, r, s ,a, r, s (terminal state)

q(s,a) = mean of

A policy might not generate all the pairs!
How can we guarantee exploration?

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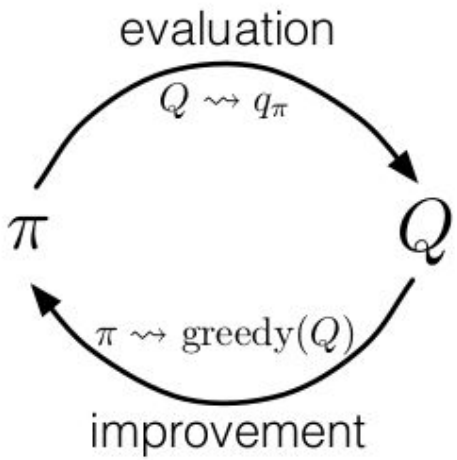
# Monte Carlo Methods

## How to identify the optimal policy?




Episode s, a, r, **s**, a, r, **s**, a, r ,**s**, a, r, **s** ,a, r, s (terminal state)

v(s) = mean of

Episode s, a, r, **s**, **a**, r, **s**, **a**, r ,**s**, **a**, r, **s** ,**a**, r, s (terminal state)

q(s,a) = mean of

A policy might not generate all the pairs!
How can we guarantee exploration?

Exploring start approach or ε-soft policy

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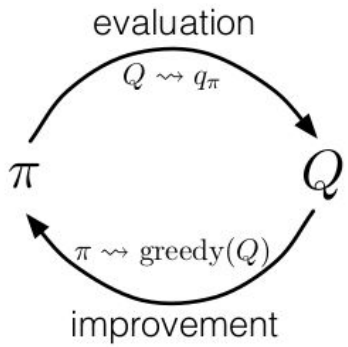
# Monte Carlo Methods

## Control: how to find the optimal policy?




**Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$**

Initialize:
- $\pi(s) \in \mathcal{A}(s)$ (arbitrarily), for all $s \in \mathcal{S}$
- $Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$
- $Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Loop forever (for each episode):
- Choose $S_0 \in \mathcal{S}$, $A_0 \in \mathcal{A}(S_0)$ randomly such that all pairs have probability $> 0$
- Generate an episode from $S_0, A_0$, following $\pi$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
- $G \leftarrow 0$
- Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
  - $G \leftarrow \gamma G + R_{t+1}$
  - Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:
    - Append $G$ to $Returns(S_t, A_t)$
    - $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$
    - $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$

Exploring start approach

Evaluation

Improvement

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# Monte Carlo Methods

## Control: how to find the optimal policy?

**On-policy first-visit MC control (for $\varepsilon$-soft policies), estimates $\pi \approx \pi_*$**

Algorithm parameter: small $\varepsilon > 0$

Initialize:

- $\pi \leftarrow$ an arbitrary $\varepsilon$-soft policy
- $Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$
- $Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

- Generate an episode following $\pi$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
- $G \leftarrow 0$
- Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
  - $G \leftarrow \gamma G + R_{t+1}$
  - Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:
    - Append $G$ to $Returns(S_t, A_t)$
    - $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$
    - $A^* \leftarrow \text{argmax}_a Q(S_t, a)$ (with ties broken arbitrarily)
    - For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

suboptimal

ε-soft policy

Evaluation

Improvement

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# Monte Carlo Methods

## On-policy vs off-policy algorithms

**<u>Learning control methods dilemma</u>**
learning action-value of an optimal policy means also exploring ...

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# Monte Carlo Methods

## On-policy vs off-policy algorithms

**Learning control methods dilemma**
learning action-value of an optimal policy means also exploring ...

**On-policy algorithm**
learning action-value for suboptimal policy

SARSA

**Off-policy algorithm**
Use two policies. One for exploring and one for searching the optimal policy

Q-learning

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# Temporal-Difference Learning

Combining Monte Carlo and Dynamic Programming

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# Temporal Difference Learning

## Combining two ideas for prediction

Wait until the end of the episode

Monte Carlo

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big]$$

Wait until the next step

Temporal Difference Learning

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\Big]$$

One step sampling

Estimating (as DP)

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# Temporal Difference Learning

## TD(0) for prediction

**Tabular TD(0) for estimating $v_\pi$**

Input: the policy $\pi$ to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop for each episode:
  Initialize $S$
  Loop for each step of episode:
    $A \leftarrow$ action given by $\pi$ for $S$
    Take action $A$, observe $R$, $S'$
    $V(S) \leftarrow V(S) + \alpha\big[R + \gamma V(S') - V(S)\big]$
    $S \leftarrow S'$
  until $S$ is terminal

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# Temporal Difference Learning

## SARSA: on-policy TD control

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\Big]$$

**Sarsa (on-policy TD control) for estimating $Q \approx q_*$**

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize $S$
  Choose $A$ from $S$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
  Loop for each step of episode:
    Take action $A$, observe $R$, $S'$
    Choose $A'$ from $S'$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma Q(S', A') - Q(S, A)\big]$
    $S \leftarrow S'$; $A \leftarrow A'$;
  until $S$ is terminal

Q is updated using the action A' derived from the actual policy

On-policy update

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# Temporal Difference Learning

## Q-learning: off-policy TD control

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\Big[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\Big]$$

**Q-learning (off-policy TD control) for estimating $\pi \approx \pi_*$**

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
  Initialize $S$
  Loop for each step of episode:
    Choose $A$ from $S$ using policy derived from $Q$ (e.g., $\varepsilon$-greedy)
    Take action $A$, observe $R$, $S'$
    $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma \max_a Q(S', a) - Q(S, A)\big]$
    $S \leftarrow S'$
  until $S$ is terminal

Off-policy update

Q is updated using the greedy action a

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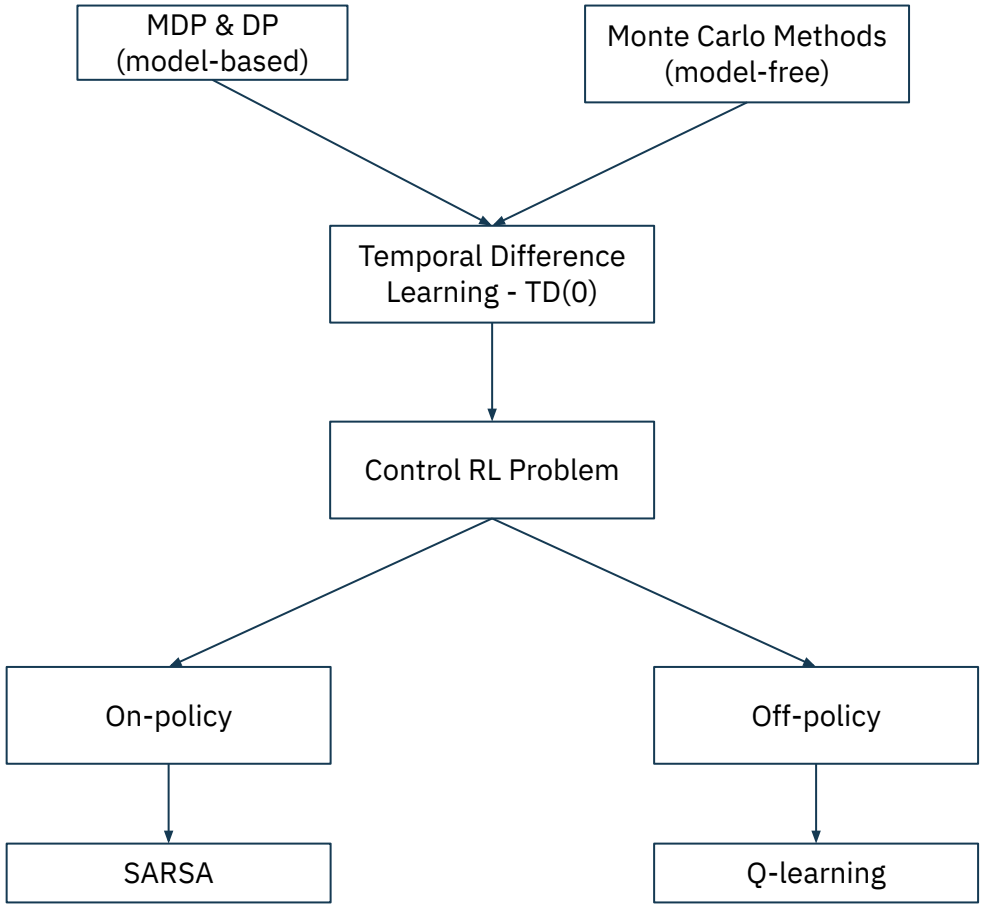
# Recap

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# Thank you

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