EQUITABLE AUCTIONS

Simon Finster* Patrick Loiseau† Simon Mauras‡ Mathieu Molina§
Bary Pradelski¶

November 18, 2025

First version: March 12, 2024

Abstract

We initiate the study of how auction design affects the division of surplus among bidders. We propose a parsimonious measure for equity and apply it to standard auctions for homogeneous goods. The uniform price auction is equity-optimal if and only if bidders have a common value. With private values, the pay-as-bid auction is preferred with respect to equity under a common regularity condition on signals, but not in general. In auctions with price mixing between pay-as-bid and uniform prices, we provide bounds on equity-preferred pricing that are prior-free up to signal regularity. Moreover, we characterize direct mechanisms that achieve ex-post surplus parity among winning bidders. Our family of mechanisms can interpolate between efficient and fair allocations and is Bayesian-Nash incentive compatible and budget-balanced.

Key words: auctions, equity, mechanism design, pay-as-bid, uniform price, common value
JEL codes: D44, D47, D63, D82

*Johannes Kepler University Linz, simon.finster@jku.at
†Inria/FairPlay, patrick.loiseau@inria.fr
‡Inria/FairPlay, simon.mauras@inria.fr
§Tel Aviv University, mathieum@tauex.tau.ac.il
¶Maison Française d'Oxford,CNRS and Department of Economics, University of Oxford, bary.pradelski@cnrs.fr

Acknowledgments: We are grateful for feedback and comments from Pierre Boyer, Julien Combe, Péter Esö, Pär Holmberg, Atulya Jain, Simon Jantschgi, Bernhard Kasberger, Paul Klemperer, Yves Le Yaouanq, Laurent Linnemer, Bing Liu, Simon Loertscher, Matías Núñez, Sander Onderstal, Michael Ostrovsky, Ludvig Sinander, Alex Teytelboym, Kyle Woodward, and audiences at the Simons Laufer Mathematical Sciences Institute (Berkeley), CIRM (Marseille), CREST (Paris), NASMES 2024 (Nashville), CMID 2024 (Budapest), EARIE 2024 (Amsterdam), and Match-up 2024 (Oxford).

This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 and by the Alfred P. Sloan Foundation under grant G-2021-16778, while Simon Finster and Bary Pradelski were in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the Fall 2023 semester. This work was partially supported by the French National Research Agency (ANR) through grants ANR-20-CE23-0007 and ANR-23-CE23-0002 and through the PEPR IA FOUNDRY project (ANR-23-PEIA-0003). Simon Mauras received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 866132), as a postdoctoral fellow at Tel Aviv University.

1 Introduction

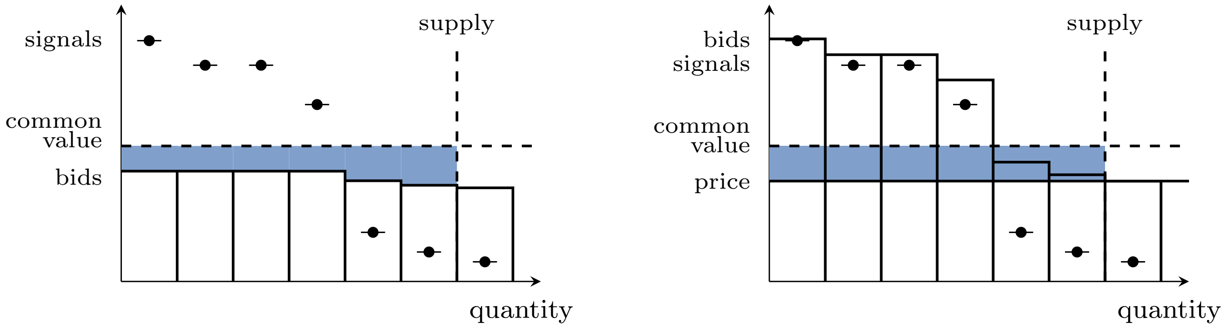
Markets allocate goods, but they also allocate rents. In auctions, the way surplus is divided can determine which firms thrive, which exit, and how downstream markets evolve. However, while efficiency and revenue have dominated in guiding auction design, the division of surplus between bidders has received almost no attention. Auctions are used to sell government debt, electricity, spectrum licenses, emission permits, oil, timber, coffee, art, and production inputs. The auction mechanism—its allocation and pricing rule—can produce large asymmetries in bidders' welfare, even among winners. For example, in single-price auctions, high-value bidders obtain larger surplus. These asymmetries have implications for participation, competition, and post-auction market stability. In this article, we show that auctions can be made more equitable by design. We propose a simple, intuitive measure of equity based on pairwise surplus comparisons. Based on this measure, we identify which auction formats used in practice distribute rents most evenly, and we characterize direct mechanisms that achieve perfect surplus parity.

A growing literature explores equity and redistributional concerns in market and non-market institutions (cf., e.g., Dworczak, Kominers and Akbarpour, 2021; Akbarpour, Dworczak and Kominers, 2024; Jeong and Pycia, 2023). These themes are increasingly salient in practice: the U.S. Small Business Act mandates that 23% of federal procurement contracts go to disadvantaged or minority-owned firms (Pai and Vohra, 2012; U.S. SBA, 2024), and fairness in spectrum license allocation shapes competition in the downstream telecommunications market (Kasberger, 2023; GSMA, 2021). Related concerns arise in electricity and environmental auctions, where unequal surplus shares may distort market entry and financing, as disadvantaged bidders face higher borrowing costs, especially in inefficient capital markets. Equity also matters in airport slot allocation because scarce landing/take-off rights shape market access and competition (Marra, 2024). Finally, normative concerns can also lead to equity considerations that call for redistribution. Despite these debates, theory lacks a tractable notion of surplus equity in multi-unit auctions. To this end, we introduce a new notion of surplus equity based on pairwise ex-post differences. We thus nest prominent inequality measures that aggregate pairwise differences, for example, the Gini index and the empirical variance.

We apply our concept to standard multi-unit auctions with risk-neutral unit-demand bidders and i.i.d. signals. Values may be private or common. To build intuition, consider two prominent auction formats used in practice—the uniform (first-rejected-bid) and the discriminatory (pay-as-bid) auction—and how each shapes the distribution of bidder surplus.[1] In the uniform auction, all winners pay the first rejected bid and truthful bidding is a weakly dominant strategy. In the pay-as-bid auction, each winning bidder pays their bid, which induces bid shading in the unique Bayes-Nash equilibrium. With a pure common value, uniform pricing is more surplus-equitable than pay-as-bid pricing because all winners obtain identical surpluses with the former, while, with the latter, high-signal bidders suffer from a winner's curse effect. Fig. 1 illustrates this: all winners' realized surpluses are equal in the uniform auction, contrasting pay-as-bid pricing. This holds for any realization of signals. Surpluses for small signals are similar between the two auction formats, but for larger signals pay-as-bid pricing penalizes realized surpluses.

[1] In treasury auctions both designs are common (OECD, 2021), while most electricity markets feature the uniform pricing rule. Further examples include auctions for emission certificates and online advertisement.

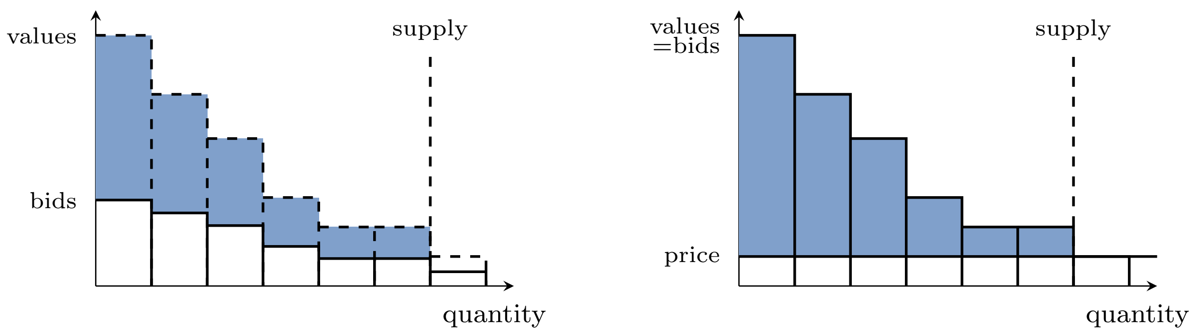
Note that revenue equivalence does not hold ex-post, but only ex-ante.



(a) Pay-as-bid auction (b) Uniform price auction

Figure 1: Illustrative example of ex-post realized surpluses in the pay-as-bid and uniform price auctions for **a pure common value** and uniformly distributed signals. The x-axis shows the quantity and the y-axis shows the signals (-●-), bids (bars), price, and common value. The shaded areas show the bidders' surplus. Notice that in the uniform price auction all winners' surpluses are equal, while in the pay-as-bid auction higher-signal bidders receive a lower payoff than lower-signal bidders—a "winner's curse" effect.

Next, consider pure private values. Intuitively, the logic runs in reverse, and the uniform auction is less equitable than the pay-as-bid auction. Fig. 2 illustrates that surplus in the uniform auction is more dispersed than in the pay-as-bid auction, and hence the realized utilities between different bidders are more unequal. We show that this intuition is not generally true. However, with assumptions on the signal distribution—which are satisfied, e.g., for uniformly distributed signals—we prove that pay-as-bid pricing is more equitable than uniform pricing, in the sense of pairwise comparisons of ex-post surpluses, for any signal realizations.



(a) Pay-as-bid auction (b) Uniform price auction

Figure 2: Illustrative example of ex-post realized surpluses in the pay-as-bid and uniform price auctions for **pure private values** and uniformly distributed signals. The x-axis shows the quantity and the y-axis shows the values (dashed bars), bids (solid bars), and price. The shaded areas show the bidders' surpluses. Notice that the uniform price auction is less equitable than the pay-as-bid auction.

Beyond uniform and pay-as-bid auctions, we study the mixed-price auction in which prices are set by a combination of uniform and pay-as-bid pricing, a class of mechanisms proposed for electricity markets (Holmberg and Tangerås, 2023) and further studied in Ruddell, Philpott and Downward (2017) and Woodward (2021). Turning to mechanism design, we identify surplus equity-optimal mechanisms in the class of efficient mechanisms and for randomized, allocation-fair assignments.

1.1 Our contribution

We consider standard and winners-pay multi-unit auctions for indivisible, identical goods with a combination of private and common values. Each bidder has unit demand[2] and receives a private, independent signal from a commonly known distribution. Values range from purely private to purely common.[3]

Our first conceptual contribution is the definition of a parsimonious equity comparison based on dominance in pairwise differences: mechanism A dominates B if all absolute pairwise differences in ex-post realized surpluses are weakly smaller in A than in B, and strictly smaller for at least one pair (Definition 2). This notion is consistent with any anonymous equity metric aggregating pairwise differences through an increasing function. Examples include the empirical variance, the Gini index, or top–bottom decile comparisons.[4]

Our main results address auction formats used in practice as well as surplus-equitable mechanism design. First, we consider uniform price and pay-as-bid auctions, and their convex combination, mixed auctions. We study the symmetric Bayesian equilibrium, which is found to be unique. Under classical assumptions, mixed auctions achieve the same expected revenue and allocate items to the highest-value bidders; thus, there are no trade-offs with revenue or efficiency (see Milgrom and Segal, 2002).

We show that uniform pricing is equity-optimal if and only if values are purely common (Theorem 1). In that case, the efficient, surplus-equitable mechanism coincides with the first-rejected-bid uniform price auction, and—as in all uniform auctions—surpluses of winners are equal ex-post. For pure private values, pay-as-bid pricing is not always equity-preferred within the class of mixed auctions (Proposition 5). However, with log-concave signal distributions, we obtain sharp prior-free rankings: for any common-value proportion c, the mixed auction with a $(1-c)$ share of price discrimination is equity-preferred to all mechanisms with less price discrimination (Theorem 2). Thus, as values are increasingly private, more price discrimination is required to achieve an equitable distribution of realized surpluses. In particular, for pure private values, pay-as-bid pricing becomes equity-preferred (Corollary 1). Any level of price discrimination up to $2(1-c)$ is equity-preferred to uniform pricing. These results hold for all equity metrics built from pairwise differences. As a concrete example of an aggregator, we study the winners' empirical variance (WEV), which satisfies monotonicity in transfers from richer to poorer agents, the Pigou–Dalton principle (Proposition 3).

Turning to equitable mechanism design, we contribute three mechanisms that achieve equal ex-post surpluses. Our first mechanism implements the efficient allocation, is Bayesian-Nash incentive-compatible and distributes realized surpluses equally among the winning bidders (Theorem 3). Losing bidders pay nothing, and winners obtain a transfer consisting of three compon-

[2] See Section 6.3 for a discussion of this assumption.

[3] For instance, private-common values can represent resale opportunities; see Bikhchandani and Riley (1991), Klemperer (1998), Bulow and Klemperer (2002) and Goeree and Offerman (2003). In markets for pollution permits, a certificate is valuable for production (private value) but also has a resale value common to the market.

[4] See Lorenz (1905), Gini (1912), Gini (1921), Pigou (1912), Dalton (1920), Atkinson (1970) and Sen and Foster (1973).

ents: firstly, each winner pays their private value, thus equalizing ex-post utilities; secondly, a uniform transfer that cancels out the idiosyncratic payment in expectation; and finally, a payment to ensure Bayesian-Nash incentive-compatibility. The three components of the transfer are weighted by the private and common value share. In our pure common value setting, the first and second component of the payment disappear and the equity-optimal mechanism boils down to the uniform auction. Our key and surprising insight is that the uniform payments cancel out the idiosyncratic part of the transfer that depends on a bidder's own signal—for any given signal—in expectation. For the seller, this mechanism is revenue-equivalent to a standard auction, e.g., the uniform price auction.

Our second and third mechanisms address surplus equity in the context of allocative equity. If the designer was ready to trade off efficiency with allocative equity, could surplus equity still hold? Our answer is affirmative. We demonstrate that surplus equity and allocative equity are two complementary notions that may be implemented simultaneously, albeit at the expense of efficiency and expected revenue. Our second mechanism is allocation-fair, i.e., it allocates items uniformly at random between all bidders. The mechanism is dominant-strategy incentive-compatible, individually rational, and ex-ante budget balanced, i.e., the seller makes zero revenue in expectation (Proposition 6). Losers pay nothing, and winners are charged a weighted aggregation of the average signal and their competitors' realized signals. Our third mechanism is also allocation-fair, and achieves ex-post equal surpluses between all participating bidders, including losers. It is Bayesian-Nash incentive compatible, individually rational, and ex-ante budget balanced (Proposition 7).

Finally, we demonstrate in Section 5.2 that any compromise between efficient and allocation-fair allocations can be achieved by using a convex combination of our payment rules from Theorem 3 and Proposition 6.

We also provide several extensions and a detailed discussion of the robustness of our results. We investigate surplus equity in terms of WEV in numerical experiments. For a variety of signal distributions and common value proportions, we compute the landscape of WEV-minimal mixed pricing, which can be seen to be unique (see Section 4.4). In Section 6.1, we discuss the distinction between equity and risk preferences. In Section 6.2, we illustrate why it would be difficult to generalize our equity rankings beyond log-concave signal distributions. We further discuss robustness to the unit-demand assumption in Section 6.3 and demonstrate that some of our results on pay-as-bid and uniform price auctions extend to settings with reserve prices.

1.2 Related Literature

We relate and contribute to several strands of existing work, including a recent literature on redistributive market design, fairness concerns and allocative equity in auctions, and the study of fair allocations more generally. Further, we contribute to the mechanism design literature on ex-post payment design and the analysis of uniform, pay-as-bid, and mixed-price auctions.

Broadly, our contribution fits into a recent strand of the economic literature on redistributive concerns in market design. In this literature, the focus is often on efficiency and equity trade-offs; e.g., in a large buyer-seller market for a single object, with agents differing in their marginal utilities of money (and values), Dworczak, Kominers and Akbarpour (2021) characterize the optimal efficiency-equity trade-off, and Akbarpour, Dworczak and Kominers (2024), characterize when non-market mechanisms, as opposed to market-clearing prices, are optimal for a designer to allocate a fixed supply (also in a large market). Such non-market mechanisms forgo efficiency for the sake of improving equity. Our approach differs in that we focus on a small market with a finite number of bidders and demonstrate how to improve equity, up to achieving perfect surplus parity. In a recent working paper, Jeong and Pycia (2023) show how first-price auctions can be augmented with appropriately chosen restrictions on the bid space to achieve different designer objectives, including notions of distributional fairness. Our contribution is different in two key respects. First, we study equity in ex-post realized surpluses—an intrinsically multi-unit and signal-dependent notion—not designer objectives defined ex-ante. Second, we preserve a standard, flexible bid space and show that pricing rules alone can deliver sharp improvements in surplus equity, up to fully equalizing realized surpluses. In this sense, our results complement theirs by demonstrating the redistributive reach of canonical multi-unit auctions without altering the strategic message space.

Fairness concerns in auctions have been addressed through design instruments such as subsidies and set-asides. In a model with explicit target group favoritism, Pai and Vohra (2012) show that the optimal mechanism is a flat or a type-dependent subsidy, depending on the precise nature of the favoritism constraint. Athey, Coey and Levin (2013) come to similar conclusions in an empirical study of US Forest Service timer auctions, where set-asides for small bidders would reduce efficiency and revenue, while subsidies would increase revenue and profits of small bidders with little detriment to efficiency. In contrast with this literature, we focus on equity among winners and show that a carefully designed pricing rule can achieve greater surplus equity.

The literature on fair allocation has introduced many concepts of fairness, including, for example, *envy-freeness*, *equal division*, or *no domination* (for a survey, see Thomson, 2011). Our notion of fairness is orthogonal to envy-freeness, which requires that no agent prefers another agent's allocation (object and price). In our market, uniform pricing is the unique pricing scheme that results in envy-freeness among winners, and with pure private values, it results in envy-freeness among all participants.[5] However, envy-freeness does not take into account that bidders may have different signal (value) realizations. Subscribing to the notion of envy-freeness, we would accept that realized utilities may be very unequal, depending on the realization of the private value component. In contrast to this view, we consider the realization of utilities as the baseline for fairness considerations, which relates to *equal treatment of equals* (cf., e.g., Thomson, 2011) in an ex-post view.

The analysis of surplus variance within a given bidder in a single-item auction goes back to Vickrey (1961), who showed that the first-price rule generates a smaller variance than the second-price rule for uniform signals. In recent work, McAfee et al. (2025) show that the first-price rule minimizes the variance of surplus (and any convex risk measure) within a given bidder. Our work differs in our focus on surplus equity *between* winning bidders. This question is inherently linked to a multi-unit auction with at least two items for sale and three bidders. However, empirical variance, which is part of the family of our considered equity measures, combines aspects of within-bidder variation and between-bidder correlation of surpluses (see Lemma 7).

[5] With a proportion of common value, depending on the realization of signals, winners may experience the winners' curse and prefer not to have won an item.

Our ex-post surplus-equalizing payment rules relate, broadly speaking, to the mechanism design literature on ex-post implementations of truthful mechanisms. For example, d'Aspremont and Gérard-Varet (1979) show that ex-post budget balance can be achieved in a direct truthful mechanism, and Esö and Futó (1999) prove that for every incentive-compatible mechanism there exists a mechanism which yields deterministically the same revenue.

Finally, we also relate to the literature on public finance and optimal taxation which has long debated redistribution. Full redistribution resulting in equal surpluses is the solution to the classical utilitarian social welfare objective with concave homogeneous utility functions among individuals (Piketty and Saez, 2013; Edgeworth, 1897). An alternative and more general approach to social welfare functions, the so-called generalized social welfare weights, were suggested by Saez and Stantcheva (2016). Although an aggregation of our proposed pairwise differences with weighting factors akin to social welfare weights is possible, the economic interpretation remains different, as we aggregate differences in utilities. A crucial difference of our work and public finance is that our mechanism design approach takes inequality in the form of different signal realizations as given, whereas incomes are endogenous in most optimal taxation models.

Mixed auctions as they appear in our work also have an interpretation as ex-post taxation. In uniform auctions, the seller observes "apparent surplus", the difference between winning bids and the clearing price, which is a lower bound on a winning bidder's actual surplus (Ruddell, Philpott and Downward, 2017). Applying a percentage tax on apparent surplus is equivalent to charging a price according to the mixed-price rule. Such taxation of rents, which infra-marginal bidders earn in electricity auctions, was debated in the context of recovering infrastructure investment costs in power grids (New Zealand Electricity Authority, 2014; Ruddell, Philpott and Downward, 2017).

The remainder of the article is organized as follows. In Section 2, we introduce the model and derive equilibrium bidding strategies in mixed auctions. We introduce and illustrate our notion of surplus equity in Section 3. In Section 4, we develop results for uniform, pay-as-bid, and mixed auctions and prove them in Appendix A. In Section 5, we demonstrate three direct, incentive-compatible mechanisms that achieve ex-post surplus ex-post. Section 6 provides a discussion and Section 7 concludes.

2 Market Model and Equilibrium

A finite number of bidders $[n] := \{1, \dots, n\}$ compete for a fixed supply of items $[k] := \{1, ..., k\}$, where $2 \leq k < n$. Each bidder only demands one item. Bidder i receives a private signal s_i, which is drawn independently from a positive and bounded or unbounded support; denote its upper limit by $\bar{s}$. Signals are iid with an absolutely continuous probability distribution F with density f. We call $(0, \bar{s})$, i.e., all signals s so that $0 < F(s) < 1$, the open support of F, and assume that $f > 0$ over $(0, \bar{s})$. We also assume that the signals have a finite second moment $\mathbb{E}[s^2] < \infty$.

For $\boldsymbol{s} := \{s_i\}_{i \in [n]}$, a collection of iid signals, we denote by $Y_m(\boldsymbol{s})$ the m-th highest value of the collection $\boldsymbol{s}$ (with n entries). For example, $Y_1(\boldsymbol{s})$ is the maximum and $Y_n(\boldsymbol{s})$ is the minimum of the collection $\boldsymbol{s}$. Note that $Y_m(\boldsymbol{s})$ is a random variable, and we denote its probability distribution

$G^n_m(y)$ with corresponding density $g^n_m(y)$. G^n_m is given by

$$G^n_m(y) = \sum_{j=0}^{m-1} \binom{n}{j} F(y)^{n-j}(1 - F(y))^j. \tag{1}$$

where each summand is the probability that j signals are above y. An expression for $g^n_m(y)$ is given in Appendix B.8. The value of bidder i for an item is given by the valuation function $v(s_i, \boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i} := (s_j)_{j \neq i}$, and $v(s_i, \boldsymbol{s}_{-i})$ is symmetric in other bidders' signals $\boldsymbol{s}_{-i}$.

Assumption 1. Values $v(s_i, \boldsymbol{s}_{-i})$ are given by

$$v(s_i, \boldsymbol{s}_{-i}) = (1 - c)s_i + \frac{c}{n} \sum_{j \in [n]} s_j, \tag{2}$$

where $c \in [0, 1]$ is the *proportion of the common value.*

Our model interpolates between a common value and private values, where the proportion of the common value c encodes to what extent the value of any given bidder is influenced by the signals of the other bidders. In particular, $c = 1$ defines a pure common value and $c = 0$ pure private values.[6] We note that the value function satisfies the *single-crossing* condition as for all $i, j \in [n]$, $i \neq j$, and for all $\boldsymbol{s}$, $\partial v(s_i, \boldsymbol{s}_{-i})/\partial s_i \geq \partial v(s_j, \boldsymbol{s}_{-j})/\partial s_i$.

Auction mechanisms. Auction mechanisms are represented by allocations and transfers $\{\pi_i(s_i, \boldsymbol{s}_{-i}), p_i(s_i, \boldsymbol{s}_{-i})\}_{i \in [n]}$, where $\pi_i(s_i, \boldsymbol{s}_{-i})$ is defined as the probability that an item is allocated to the bidder i when the reported signals are s_i and $\boldsymbol{s}_{-i}$, and $p_i(s_i, \boldsymbol{s}_{-i})$ is the corresponding price charged to the bidder, which is symmetric in its second argument.[7] We require that auction mechanisms be standard and winners-pay. An auction is *standard* if the k highest bids win the items,[8] and *winners-pay* if only the winners pay and no more than their bid. Any standard auction, in any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is *efficient* (Krishna, 2009), i.e., the bidders with the k highest values $v(s_i, \boldsymbol{s}_{-i})$ are assigned the items.

We consider two classes of mechanisms. First, we consider truthful, direct mechanisms in which bidders submit their signal. Second, we consider k-unit *mixed auctions* (defined below) in which each bidder submits a bid b_i, resulting in the vector of submitted bids $\mathbf{b}$. Restricting our attention to symmetric and monotonically increasing bidding strategies $b_i = b(s_i)$, we can write allocations and prices in both classes of mechanism as functions of signals only. The allocation of bidder i is given by $\pi_i(s_i, \boldsymbol{s}_{-i}) = \mathbb{1}\{s_i > Y_k(\boldsymbol{s}_{-i})\}$ when signals s_i and $\boldsymbol{s}_{-i}$ are reported. A bidder's utility (or surplus) when reporting signal $\hat{s}_i$, and the remaining $n - 1$ bidders reporting

[6] In an alternative model, the common value might be distributed according to some prior distribution, and the bidders' private signals are drawn conditional on the realization of this common value. The alternative model has identical qualitative characteristics (Goeree and Offerman, 2003): (i) the items are valued equally by all bidders in the common value component, and (ii) the winner's curse is present, i.e., winning an item is "bad news", in that the winner's expectation of the item's value was likely too optimistic.

[7] Symmetric means that $p_i(s_i, s_{-i}) = p(s_i, s'_{-i})$ for all permutations s'_{-i} of s_{-i}.

[8] Cf. Krishna (2009).

signals $\boldsymbol{s}_{-i}$, is given by

$$u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i}) = \mathbb{1}\{\hat{s}_i > Y_k(\boldsymbol{s}_{-i})\} \cdot v(s_i, \boldsymbol{s}_{-i}) - p_i(\hat{s}_i, \boldsymbol{s}_{-i}). \tag{3}$$

Given a signal s_i, recall that we denote by $Y_k(\boldsymbol{s}_{-i})$ the k-th highest among the signals $\boldsymbol{s}_{-i}$. $Y_k(\boldsymbol{s}_{-i})$ has probability distribution G_k^{n-1} and density g_k^{n-1}.

Furthermore, we denote equilibrium bidding strategies by $(\beta_i)_{i\in[n]} = \boldsymbol{\beta}$.

Definition 1 (Mixed auctions)**.** In the k-unit δ-*mixed auction*, parameterized by a given $\delta \in [0,1]$, each bidder i pays $p_i(\mathbf{b}) = (\delta b_i + (1-\delta)Y_{k+1}(\mathbf{b}))\,\mathbb{1}\{b_i > Y_{k+1}(\mathbf{b})\}$.

At one boundary, for $\delta = 0$, this resolves to *first-rejected-bid uniform pricing* or short *uniform pricing*, where each winning bidder i pays the $(k+1)$-th highest bid $Y_{k+1}(\mathbf{b})$. At the other boundary, for $\delta = 1$, this resolves to *pay-as-bid pricing*, where each winning bidder i pays their bid b_i. Finally, if $\delta \in (0,1)$, we say that the auction and the pricing are *strictly mixed*.[9]

2.1 Interim Values, Payments, and Utilities

For all $x, y \in [0,1]$, we define the expected value given $s_i = x$ and $Y_k(\boldsymbol{s}_{-i}) = y$ as follows:

$$\widetilde{V}(x,y) := \mathbb{E}_{\boldsymbol{s}}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = x, Y_k(\boldsymbol{s}_{-i}) = y]. \tag{4}$$

The expected value is taken over $n-2$ signals not including the bidder's own signal and the k-th highest among their $n-1$ opponents. Observe that because $v(s_i, \boldsymbol{s}_{-i})$ is continuous and non-decreasing, $\widetilde{V}(x,y)$ is continuous and non-decreasing in x and y.[10] We define $V(y) := \widetilde{V}(y,y)$, the expectation of the value of an item conditional on the bidder winning against the relevant competing signal, the k-th highest among its competitors. Furthermore, we introduce *interim payments* $P_i(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s_i, \boldsymbol{s}_{-i})]$ and *interim utilities* $U_i(s_i, \hat{s}_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i})]$ with $U_i(x)$ being the shorthand of $U_i(x,x)$.

Interim incentive compatibility (IC) requires $U_i(s_i, s_i) \geq U_i(s_i, \hat{s}_i)$ for all $s_i, \hat{s}_i$, and interim individual rationality (IR) demands $U_i(s_i, s_i) \geq 0$ for all s_i. It is standard from an application of the envelope theorem (Milgrom and Segal, 2002) that the auctions we consider result in the same expected payment for each bidder.[11] The interim utility is given by

$$U_i(s_i, \hat{s}_i) = \mathbb{E}_{y=Y_k(\boldsymbol{s}_{-i})}[\mathbb{1}\{\hat{s}_i \geq y\}\,\widetilde{V}(s_i, y)] - P_i(\hat{s}_i). \tag{5}$$

Letting $G := G_k^{n-1}$ and $g := g_k^{n-1}$, we obtain $\partial_1 U_i(s_i, \hat{s}_i) = \int_0^{\hat{s}_i} \partial_1 \widetilde{V}(s_i, y) g(y)\,\mathrm{d}y$. Note that the expression is simple because, although values are not private, signals are independent. Let $U_i(s_i) = \max_{\hat{s}_i} U_i(s_i, \hat{s}_i)$ in the direct incentive-compatible mechanism, in which the maximum is obtained at $\hat{s}_i = s_i$ due to incentive compatibility. Then, by the envelope theorem, we must have $U_i'(s_i) = \partial_1 U_i(s_i, s_i)$. Thus, we have $U_i(s_i) = U_i(0) + \int_0^{s_i} U_i'(x)\,\mathrm{d}x$, and consequently

[9] Mixed-price auctions, originating from Wang and Zender (2002) and Viswanathan and Wang (2002), have also appeared in a series of articles modeling a divisible and stochastic supply (Ruddell, Philpott and Downward, 2017; Marszalec, Teytelboym and Laksá, 2020; Woodward, 2021).

[10] In fact, it is strictly increasing in x.

[11] This is also shown differently in Krishna (2009) for the single-unit auction. Note that in settings where signals are affiliated revenue equivalence fails (Krishna, 2009, Chapter 6.5).

$P_i(s_i) = \int_0^{s_i} \widetilde{V}(s_i, y)g(y)\,\mathrm{d}y - U_i(0) - \int_0^{s_i} U_i'(x)\,\mathrm{d}x$. From *winners-pay* and the continuity of the signals follows $U_i(0) = P_i(0) = 0$. The revenue equivalence extends to any standard auctions with independent signals in which winners pay, including mixed auctions, as the allocation rule is identical.

2.2 Equilibrium Bidding

We derive the unique Bayes-Nash equilibrium in increasing and symmetric bidding strategies in δ-mixed auctions. This equilibrium is the center of our analysis of surplus equity in mixed auctions in Section 4.

Proposition 1 (e.g., Krishna 2009)**.** *The unique equilibrium bidding strategy in the uniform price auction, i.e., the case $\delta = 0$, is given by $\beta^U(s) := \widetilde{V}(s,s) = \mathbb{E}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = s, Y_k(\boldsymbol{s}_{-i}) = s]$.*

Note that the equilibrium is unique in the class of increasing and symmetric strategies and weakly dominant with pure private values (Krishna, 2009).

Proposition 2. *The unique symmetric equilibrium bidding strategy in the δ-mixed auction, for $\delta \in (0,1]$, is given by*

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G_k^{n-1}(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{6}$$

The proof is given in Appendix B.3. Note that β^δ converges to β^U as $\delta \to 0$. We illustrate it in the following example. In the case of pure private values, bidding truthfully is a dominant strategy in the uniform price auction. Increasing price discrimination, δ, decreases the bid corresponding to a given signal below the private value—also called "bid shading"—and the extent of bid shading increases in δ. With a pure common value, the winner's curse becomes especially apparent as bidders cannot escape it. In particular, then winners' ex-post utilities are *decreasing in signals* as long as $\delta > 0$.

3 Surplus Equity

We propose an equity notion that we call *dominance in pairwise differences*, or short *pairwise differences*. Our results hold for the family of equity measures that are defined by any increasing function of pairwise differences in ex-post surpluses. We call the collection $\{u_i(\boldsymbol{s})\}_{i\in[n]}$ of ex-post utilities an *outcome*, where each utility depends on the collection of signals $\boldsymbol{s}$.

Definition 2 (Dominance in pairwise differences among winners)**.** An outcome $\{u_i(\boldsymbol{s})\}_{i\in[n]}$ dominates another outcome $\{u_i'(\boldsymbol{s})\}_{i\in[n]}$ in pairwise differences iff, for all winning signals s_i, s_j with opponents' signals $\boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $i, j \in [n]$, it holds that $|u_i(s_i, \boldsymbol{s}_{-i}) - u_j(s_j, \boldsymbol{s}_{-j})| \leq |u_i'(s_i, \boldsymbol{s}_{-i}) - u_j'(s_j, \boldsymbol{s}_{-j})|$, almost surely and with one inequality strict.

Naturally, dominance pairwise differences can be defined similarly based on pairwise comparisons among all bidders. To avoid issues with ties, we consider that dominance in pairwise differences holds as long as it holds almost surely. Furthermore, we say that, for a family of parameterized outcomes $\{u_i^\delta\}_{i\in[n]}$, $\delta \in \Delta$, δ^* is dominant in pairwise differences if u^{δ^*} dominates all outcomes u^δ, $\delta \neq \delta^*$, $\delta \in \Delta$. Pairwise differences induces a partial dominance ranking over outcomes and therefore a dominant δ^* may not always exist.

Several prominent equity axioms (cf., e.g., Patty and Penn, 2019) hold for pairwise differences. First, we note that anonymity is maintained. Any reordering of individuals in the population $[n]$ has no consequence, as pairwise comparisons must hold for any two bidders.[12] The Pigou-Dalton transfer principle requires that any transfer from a wealthier agent to a poorer one must reduce inequality, provided the original welfare ranking between the two agents is maintained, that is, the wealthier agent does not become poorer than the previously poorer agent after the transfer (cf., e.g., Moulin, 2004). Since dominance in pairwise differences does not establish a complete order of outcomes, a Pigou-Dalton transfer may result in a decrease in some pairwise differences while others increase.

However, our results allow the classification of δ-mixed pricing rules based on pairwise differences and *any increasing function* of pairwise differences.[13] For example, the top decile of realized utilities can be compared to the lowest or the bottom decile of realized utilities, and classic inequity measures such as the Gini index can be constructed. [14] Larger differences can receive a higher weight than smaller ones, e.g., by squaring each pairwise difference.

To exemplify the aggregation of pairwise differences, we focus on the *expected empirical variance* of surplus between the winners, or *winners' empirical variance (WEV)* for short.[15] This metric is defined in expectation, ensuring that it provides a ranking of auction designs for any signal realizations.

Definition 3 (Winners' empirical variance)**.**

$$\text{WEV } = E_s\left[\frac{1}{2k(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{k}(u_i(\boldsymbol{s})-u_j(\boldsymbol{s}))^2 \middle| s_1,\ldots,s_k > Y_{k+1}(\boldsymbol{s})\right]. \tag{7}$$

In addition to being a natural and well-known metric, this aggregation is attractive for two reasons: First, it ensures compliance with the Pigou-Dalton transfer principle, and second, the empirical variance is linked to surplus variance and correlation of surpluses among bidders.

Proposition 3. *The* winners' empirical variance *satisfies the* Pigou-Dalton principle*.*

The proof is given in Appendix B.2. In expectation, equilibrium surplus varies due to a bidder's own and the competitors' signals, and surplus between winners may be correlated. As we consider efficient auctions, surplus only varies among the winning bidders. Among those,

[12] We note that replication invariance and mean independence are not relevant in our setup, as we keep the population size (number of bidders) as well as the endowments (value distributions) fixed.

[13] A related aggregation is used by Feldman and Kirman (1974), who aggregate positive pairwise differences for a measure of envy per player. Contrasting our measure, Feldman and Kirman (1974) consider pairwise differences of hypothetical (if an agent had received another agent's bundle) and realized utilities.

[14] In our setting with uncertainty about signal realizations, one could define the expected Gini index among winners $G = \frac{1}{2n^2 E_s[u_1(\boldsymbol{s})|1 \text{ wins}]} E_s[\sum_{i=1}^{k}\sum_{j=1}^{k} |u_i(\boldsymbol{s}) - u_j(\boldsymbol{s})| \mid s_1,\ldots,s_k > Y_{k+1}(\boldsymbol{s})]$. A small distinction is the normalization by the expected surplus.

[15] The empirical variance among all bidders (thus including losers) in the auction is EV $= E_{\boldsymbol{s}}[\frac{1}{n(n-1)}\sum_{i=1}^{n}(u_i(\boldsymbol{s})-u_j(\boldsymbol{s}))^2]$.

WEV measures *within-bidder variation* and *across-bidder correlation* of surpluses (see Lemma 1 below). The analysis of within-agent variation addresses a bidder's individual risk-attitude and goes back to Vickrey (1961).[16] In contrast, an equity measure must take into account the correlation of surpluses between bidders.

Lemma 1. *An equivalent expression for the winners' empirical variance is given by* $\mathrm{WEV} = \mathrm{Var}[u_1|1 \textit{ wins}] - \mathrm{Cov}[u_1, u_2|1 \textit{ and } 2 \textit{ win}]$.

The proof is in Appendix B.2. In contrast, the ex-ante variance, $\mathrm{Var}_{\boldsymbol{s}}[u_i(\boldsymbol{s})]$, measures surplus variation *within* a given bidder, and is more adequate to measure risk, e.g., across a series of identical, repeated auctions, in which a given bidder redraws their signal in every auction. With pure private values and thus ex-post individual rationality, rankings of auction formats in terms of ex-ante variance or winners' ex-ante variance are identical. Rankings with respect to the empirical variance, however, may differ depending on if only winners are considered, or all bidders. A formal lemma and proof are given in Appendix B.2.

Example 1. We consider the simplest, non-degenerate setting with $n = 3$ bidders competing for $k = 2$ items. The bidders' signals are distributed uniformly on the support $[0, 1]$. We illustrate WEV for different values of price-mixing and the common-value proportion c (Fig. 3). For pure

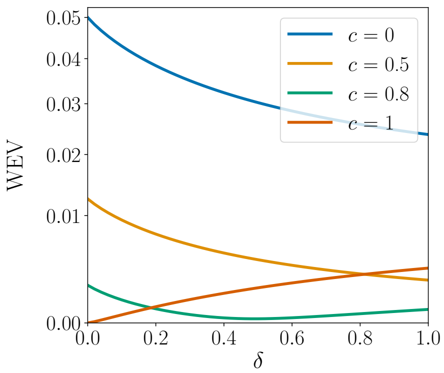


Figure 3: WEV as a function of δ for uniform signals and various common value proportions c

private and intermediate common values, the pay-as-bid auction ($\delta = 1$) minimizes WEV. For $c = 0.8$, we observe an interior optimum, and for a pure common value, uniform pricing ($\delta = 0$) minimizes WEV. Note that WEV is not necessarily convex in δ for a fixed c.

As the considered mechanisms are revenue equivalent and efficient (see also Appendix B.1), we can focus on the surplus distribution among bidders without considering potential trade-offs.

Remark. *In our market, any standard uniform price auction generates the same pairwise differences in winners' surpluses as all winners pay the same price. Consequently, any result that compares the FRB uniform price auction to another format (such as a discriminatory auction or the mixed-price auction defined earlier) extends directly to all uniform price auctions.*

[16] In the appendix of his famous article, Vickrey showed that, in a *single-unit auction*, the ex ante variance of surplus is lower under the first-price than the second-price rule, given uniform distributions of private values. See also Krishna (2009).

4 Uniform, Pay-as-bid, and Mixed Auctions

The equity-preferred pricing rule in the mixed auction class crucially depends on the extent of the common value c. As seen for the winners' empirical variance in Example 1, with uniform signals, for some interior values of c, strictly mixed pricing is optimal. We formalize this fact in Section 4.1 for any signal distributions. Example 1 is in line with the general intuition that pay-as-bid pricing may be more equitable with higher private values, and uniform pricing with higher common values. However, as we show in Section 4.2, this is not true in general. Thus, additional distributional assumptions are necessary for a characterization of equity-preferred auctions. In Section 4.3, we consider log-concave signal distributions and provide simple and prior-free bounds on the auction that is dominant in pairwise differences.

4.1 Equity Rankings

We first consider the case of a pure common value ($c = 1$). As every bidder has the same ex-post realized value, ex-post utilities among winners are equalized if everyone pays the same price. This results in pairwise differences in utilities of zero. Once the private value component enters the value function with a non-zero weight, the picture is less clear: it may be pay-as-bid pricing that is dominant in pairwise differences, or it may be some degree of mixed pricing; however, it cannot be uniform pricing. Proofs of this section are relegated to Appendix B.5.

Theorem 1. The uniform price auction is dominant in pairwise differences iff the common value proportion equals one (pure common value).

Furthermore, we show that, without any additional assumptions, strictly interior δ-mixed pricing minimizes WEV for a range of common values.

Proposition 4. *For any signal distribution, there exists $c^* < 1$, such that for common values in the interval $(c^*, 1)$, there exist δ-mixed auctions with lower* WEV *than pay-as-bid and uniform auctions.*

The intuitive notion that uniform pricing equitably distributes surplus under a pure common value may lead us to assume that pay-as-bid auctions are equity-preferred under private values. However, in the following section, we demonstrate a scenario where it fails and show that, with pure private values, uniform pricing can result in lower WEV than pay-as-bid pricing.

4.2 Challenging the Intuition: Private Values and Uniform Pricing

To understand the reversal of the intuition, consider pairwise differences in utility, the building block for WEV. If ex-post absolute differences in utility are greater under uniform pricing than under pay-as-bid pricing for signal pairs with sufficient probability mass, then the reversal may also hold in expectation. To start with, consider any two winning signals $s_i > s_j$, $s_i, s_j \in [0, \bar{s})$ and private values only, i.e., $c = 0$. Let $u_i^0(s_i, \boldsymbol{s}_{-i})$ and $u_i^1(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price and pay-as-bid auction, respectively. Moreover, β^0 and β^1 denote the corresponding symmetric equilibrium bid functions and $Y_{k+1}(\boldsymbol{\beta})$ the first rejected bid. For $\delta \in [0, 1]$ and $c = 0$, we have $u_i^\delta(s_i, \boldsymbol{s}_{-i}) = s_i - \delta\beta^\delta(s_i) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$. Thus, we have $\Delta u^0 :=$

$|u_i^0 - u_j^0| = |s_i - s_j|$ and $\Delta u^1 := |u_i^1 - u_j^1| = |s_i - \beta^1(s_i) - (s_j - \beta^1(s_j))| = |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))|$. It holds that

$$\Delta U^0 < \Delta U^1 \tag{8}$$
$$\Leftrightarrow s_i - s_j < |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))| \tag{9}$$
$$\Rightarrow 2(s_i - s_j) < \beta^1(s_i) - \beta^1(s_j). \tag{10}$$

As bid functions are increasing, if $s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))$ was positive, Eq. (9) could never hold. Thus, Eq. (10) follows as a necessary condition for uniform pricing to have lower pairwise differences than pay-as-bid pricing. For the same statement to hold for WEV, it must be that the bid function has a slope of at least 2 for a sufficient mass of signals s_i and s_j. Bid function slopes greater than 2 imply that high-signal bidders shade their bids much less, proportionally to their value, than bidders with lower signals. Consequently, the differential in ex-post surplus with pay-as-bid pricing, comparing two sufficiently different signals, are higher than the differential in signals. The latter equals the surplus difference in the uniform price auction.

The challenge in designing a counter example where WEV is lower in the uniform price auction than in the pay-as-bid auction is that the slope of β^δ at 0 must be smaller than 1, and thus cannot be greater than 2 for all signals. Thus, the probability mass on regions of the support with a slope of β^δ greater than 2 must be higher, but this also changes β^δ. Nonetheless, the following counter-example satisfies our requirements.

Example 2. Consider an auction with n bidders and $k = n - 1$ items. Each bidder i has a pure private value ($c = 0$) given by its signal s_i. The signal is equal to the sum of a Bernoulli random variable with parameter $\varepsilon > 0$ and a random perturbation drawn from $\text{Beta}(1, 1/\eta)$, with $\eta > 0$. The resulting signal distribution is continuous, with support $[0, 2]$. This yields the following quantile function:

$$\forall x \in [0,1], \qquad F^{-1}(x) = \mathbb{1}\{x \geq \varepsilon\} + \gamma_\eta(x) \qquad \text{where} \qquad \gamma_\eta(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^\eta & \text{if } x < \varepsilon \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^\eta & \text{if } x \geq \varepsilon. \end{cases}$$

Further derivations and the proof of the below proposition are given in Appendix B.6.

Proposition 5. *Let the values be distributed according to the quantile function F^{-1} defined above. For $n \geq 5$, there exists η^*, such that for all $\eta \leq \eta^*$ it holds that the* winners' empirical variance *under uniform pricing is lower than under pay-as-bid pricing.*

Thus, in order to characterize equity-preferred pricing further, we need additional assumptions. In the next section, we show that, for a large class of signal distributions, simple bounds tell us which auction designs are candidates for being equity-preferred in the class of mixed auctions.

4.3 Equity-Preferred Pricing for Log-Concave Signal Distributions

For our subsequent results, we assume a regularity condition on the bidders' signal distributions, *log-concavity*. The family of log-concave distributions contains many common distributions, for example uniform, normal, exponential, logistic or Laplace distributions (Bagnoli and Bergstrom, 2005).[17]

Definition 4. A real-valued function $h \in \mathbb{R}^{\mathbb{R}}$ is *log-concave* if $\log(h)$ is concave.

In this class of signal distributions, simple and prior-free bounds characterize the equity-preferred auction design in the class of mixed auctions. The proof of Theorem 2 is developed in Appendix A.

Theorem 2. Let signals be drawn from a log-concave distribution. Then, for a given private value proportion $1-c$, the mixed auction with price discrimination $\delta = 1-c$ is equity-preferred among all mixed auctions with price discrimination of less than $1-c$. Moreover, uniform pricing is dominated in pairwise differences by any strictly mixed pricing with price discrimination of up to $\min\{1, 2(1-c)\}$.

The equity-preferred pricing rule dominates in pairwise differences all pricing rules with less price discrimination. In other words, Theorem 2 provides a lower bound on the amount of price discrimination required to rule out dominated mixed auctions. We illustrate Theorem 2 in Fig. 4. All pricing rules in the shaded area in red are dominated by the diagonal $1-c$, given any log-concave distribution of bidders' signals.

Moreover, uniform pricing is dominated in pairwise differences by many alternative pricing rules, i.e., these pricing rules are preferred to uniform pricing in terms of equity. This is illustrated in Fig. 5, in which any pricing rule in the shaded area in green dominates uniform pricing for a given common value c.

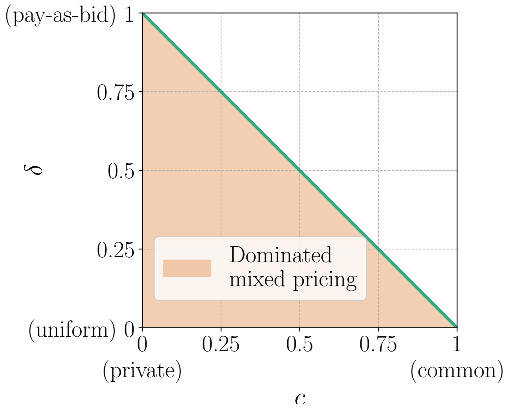


Figure 4: Dominated combinations of c and δ

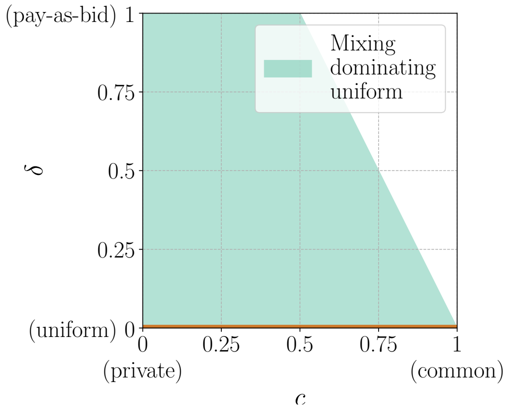


Figure 5: Mixed pricing dominating uniform pricing

The intuition behind Theorem 2 is simple. As we show in Appendix A, pairwise differences are, for any given common value c, monotonically decreasing in the extent of price discrimination δ as long as δ is between zero and $1-c$. Moreover, we show the equivalence of this result with ex-post utilities that increase in signals. As long as higher signals obtain a higher surplus, more equity can be achieved by taxing higher signals more than lower signal. Because

[17] Also χ distribution with degrees of freedom ≥ 1, gamma with shape parameter ≥ 1, χ^2 distribution with degree of freedom ≥ 2, beta with both shape parameters ≥ 1, Weibull with shape parameter ≥ 1, and others.

the change in the δ-weighted bid in δ is increasing in a bidder's signal (as stated in Lemma 3), increasing the extent of price discrimination will have the desired effect.

A similar intuition explains the dominance of mixed pricing over uniform pricing, where the benefit of higher price discrimination compared to the absence of price discrimination can be realized up to a certain threshold. As long as utilities are increasing in signals, increasing price discrimination results in surplus taxation that benefits equity. We show in Appendix A that increasing ex-post utilities is equivalent to the slope of equilibrium bid functions being bounded $(1-c)/\delta$. With steeper bid functions, the utilities might decrease in the signals. So, while increasing price discrimination might locally, in a neighborhood of δ, increase pairwise differences, price discrimination is still beneficial compared to uniform pricing. However, for $\delta \geq 2(1-c)$, the bid functions are so steep that an increase in price discrimination results in an absolute utility gap between a high signal and a low signal bidder that is greater than under uniform pricing. With such price discrimination, the higher signal bidder is worse off than the low signal bidder.

With Theorem 2, we can now revisit the question: In terms of equity, should one use pay-as-bid pricing if bidders' values are pure private values? The answer is yes if the signal distributions are log-concave. Moreover, if the common value is small, pay-as-bid pricing is guaranteed to be more equitable than uniform pricing. We state this formally in the corollary below.

Corollary 1. Assume signals are drawn from a log-concave distribution. Then, for pure private values, pay-as-bid pricing is dominant in pairwise differences, and for a common value $c < \frac{1}{2}$, pay-as-bid pricing dominates uniform pricing in pairwise differences.

The first part of the corollary follows by setting $c = 0$ and the second part follows by setting $\delta = 1$ in Theorem 2. Our numerical experiments in Section 4.4 show that, for $c < \frac{1}{2}$, pay-as-bid pricing in fact minimizes WEV for several common distributions. The intuition in the pure private value case carries through under the qualifying assumption of log-concave signals, and it may fail for very concentrated signal distributions. In the latter case, it is important that sufficient probability mass is gathered around higher signals, inducing a bidding equilibrium in which ex-post utilities are decreasing in signals for sufficiently many signal realizations.[18]

For specific signal distributions, we can extend the region where pairwise differences are monotonically decreasing slightly beyond the diagonal $1-c$. In particular we show that for uniformly distributed signals any pricing dominant in pairwise differences contains a discriminatory proportion of at least $\frac{2n(1-c)}{2n-c(n-2)}$; and for exponentially distributed signals at least $\frac{2n(1-c)}{2n-c(n-(k+1))}$ (see Corollary 5).[19] Note that both bounds converge to $\frac{1-c}{1-c/2}$ as the number of bidders goes to infinity (and the number of items k is kept constant). We illustrate the bounds for 10 bidders and 4 items in Fig. 6 in the next section, together with the equity-preferred pricing in terms of WEV.

[18] For example, with a β-distribution as steep as illustrated in Fig. 6, Section 6, clearly violating log-concavity, pay-as-bid pricing is still optimal for a range of common values including pure private values.

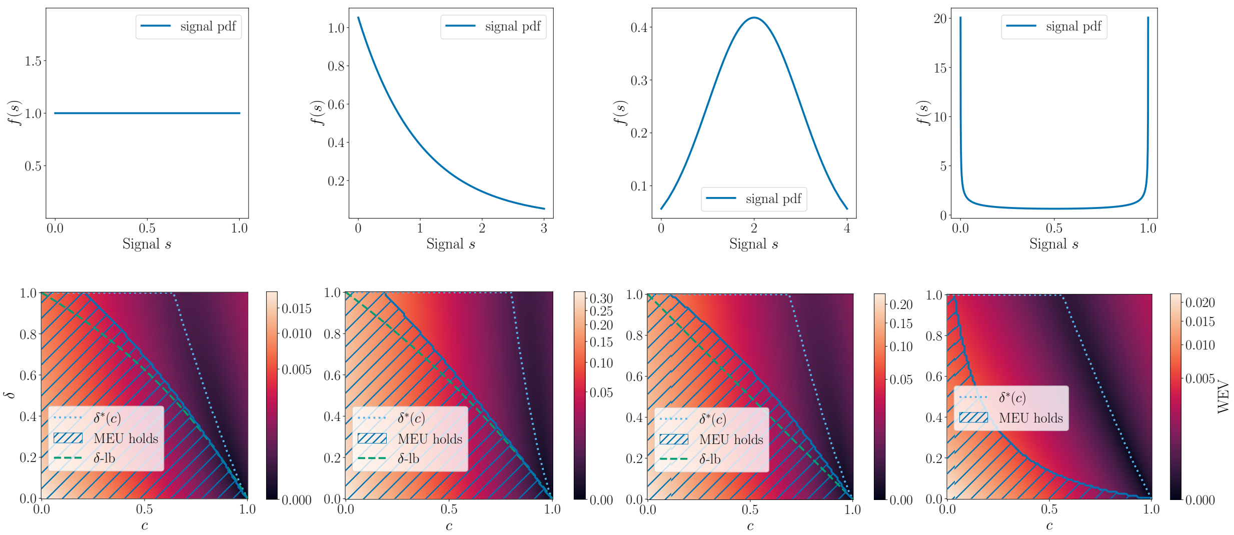
[19] The proof should be read in conjunction with Theorem 2 as it follows a similar reasoning.

4.4 Simulation Results

In this section, we present several numerical examples. We compute the WEV-minimal pricing $\delta^*(c)$ for any given proportion of the private-common value c and illustrate the aforementioned bounds for WEV-minimal pricing and the condition of monotone ex-post utility (MEU).

All of our experiments are based on equilibrium bid functions, whose calculation is computationally very expensive. Thus, we rely on theoretical simplifications, such as Lemma 4 and Lemma 11 (Appendix C). The simulations are performed through numerical integration of our analytical formulae.[20] Finally, some quantities (such as bidding functions) have multiple analytical expressions, among which we choose the most appropriate for accuracy and speed, depending on the value of the signal (e.g., Eq. (6) can be integrated more efficiently than Eq. (19), but is less accurate for small signals). Our code is available on github.

We consider four signal distributions, a uniform, a truncated exponential and a truncated normal distribution (both log-concave), as well as a Beta distribution with shape parameters $(0.5, 0.5)$, which is not log-concave. WEV-minimal pricing, a lower bound on the minimizer, and combinations of common value shares and mixed pricing for which MEU holds are shown in Fig. 6 for a market with $n = 10$ bidders and $k = 4$ items.



Uniform[0,1] Truncated exponential Truncated normal Beta$(0.5, 0.5)$

Figure 6: WEV-minimizing design $\delta^*(c)$, monotone ex-post utility (MEU), and lower bounds on $\delta^*(c)$ (δ-lb) for uniform, truncated exponential, truncated normal, and Beta$(0.5, 0.5)$ signal distributions

For the uniform and the truncated exponential distribution, we show the lower bounds on $\delta^*(c)$ given in Corollary 5, while for the normal distribution we show the general lower bound $1 - c$ (cf., Theorem 2). Each of these bounds dominates any extent of price discrimination below it. For the Beta distribution, we cannot provide a theoretical lower bound on the WEV-

[20] The efficiency and accuracy of the code rely on various techniques. Most importantly, we rewrite all multidimensional expectations as nested one-dimensional integrals (with variable bounds), which we compute by integrating polynomial interpolations. Second, the code ensures that each quantity is computed at most once, using memorization. Integration is not computationally heavy at all and achieves high precision.

minimal design δ^*, as the distribution is not log-concave. The region where MEU holds can be determined numerically, and its "frontier" also provides a lower bound for the WEV-minimal design δ^*. Illustrating this for all four distributions, we observe that the area is much smaller for the Beta distribution. However, note that MEU is only a sufficient condition for the monotonicity of WEV (while it is necessary and sufficient for the monotonicity of pairwise differences). From the heat maps in Fig. 6, it is evident that WEV is monotone in δ for any given c up to δ^*.

Finally, we show the WEV-minimal pricing rule $\delta^*(c)$ for each signal distribution. The curve is qualitatively similar in each plot. In line with Theorem 2 — noting that the exponential and normal distribution are log-concave — the figure illustrates that with a high private value component (low c), pay-as-bid pricing ($\delta = 1$) minimizes WEV; with higher common value components (high c), strictly mixed pricing for some $\delta \in (0, 1)$ minimizes WEV (cf., Proposition 4); and with a pure common value ($c = 1$), uniform pricing ($\delta = 0$) minimizes WEV (cf., Theorem 1). Analogous interpretations hold for the Beta distribution, although we cannot give theoretical guarantees.

For small common values, MEU holds for any δ and thus pay-as-bid pricing is dominant in pairwise differences (cf., Proposition 10). Even for larger common values the WEV-minimal pricing is pay-as-bid, but eventually strictly mixing ($\delta \in (0, 1)$) minimizes WEV. For a pure common value, uniform pricing always is WEV-minimal. Notice also that WEV at the minimal δ^* decreases in c. Naturally, with a higher common value share, bidders' values given different signal realizations as well the corresponding bids move closer together, thus explaining smaller differences in utilities (ex-post and in expectation).

While for high values of c the bound and the optimal pricing rule are close, for most common value parameters, a high degree of price discrimination is needed to achieve the equity-preferred pricing among mixed pricing rules. In fact, "super-mixed" pricing can in some instances improve equity further. Super-mixing requires bidders to pay their own bid plus δ times the difference between their own and the last rejected bid, where $\delta \geq 0$. We illustrate this in Figs. 11 and 12 in Appendix C.

5 Surplus-Equitable Mechanisms

In the class of efficient auctions, it is possible to distribute surplus among the winning bidders equitably. This means pairwise differences in ex-post utilities are zero, as is any aggregate metric such as the winners' empirical variance. We let $y := Y_k(\boldsymbol{s}_{-i})$, $G := G_k^{n-1}$, and $g := g_k^{n-1}$.

Theorem 3. In the class of standard k-unit auctions, there exists an incentive-compatible direct mechanism that distributes surpluses equitably among winners. Losers pay nothing, and the corresponding payments are given by

$$\widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = \left((1-c)\left(s_i - y - \frac{G(y)}{g(y)}\right) + V(y)\right) \mathbb{1}\{s_i > y\}. \tag{11}$$

The proof of Theorem 3 is given in Appendix B.4 and relies on constructing ex-post payments, which are simple and powerful. First, the term $(1-c)s_i$ removes the idiosyncratic part of each bidder's realized value due to their own signal. To align interim incentives, this term is adjusted by a uniform subsidy $(1-c)(y+G(y)/g(y))$, which cancels the idiosyncratic payment in expectation. Thus, the second-price payment $V(y)$, the expected value of the $(k+1)$th-highest signal conditional on tying with the kth-highest, induces truthful reporting for any given signal s_i, interim. If the first rejected signal is high, winning signals have to be paid subsidies. However, very high subsidy payments are low probability events, as the realizations of all winning signals and the first rejected signal must be high. Further intuition for the surplus-equitable payment is given in the continuation of Example 1 below.

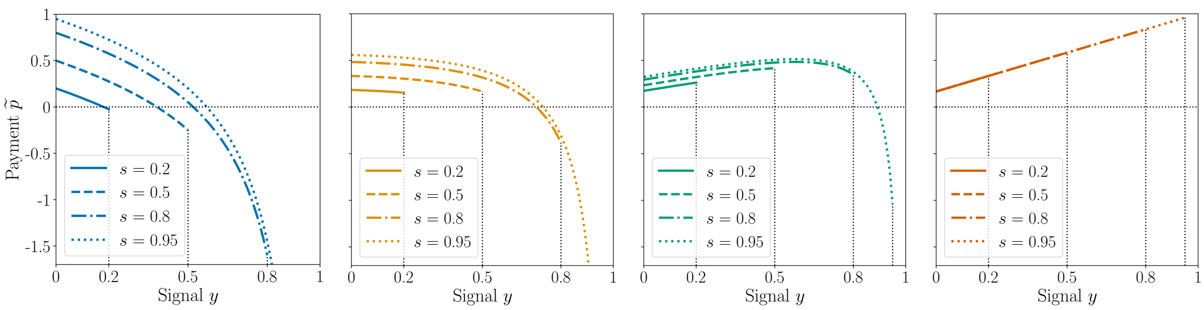
We also note that the surplus-equitable mechanism is ex-post individually rational for bidders in the pure private value case. Indeed, $\widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = (s_i - G(y)/g(y))\mathbb{1}\{s_i > y\} \leq s_i \mathbb{1}\{s_i > y\}$. With a common value component we only have interim individual rationality. The seller can afford the subsidies in expectation as the mechanism is revenue-equivalent to a standard auction; however, the subsidies may be expensive ex-post for certain signal realizations.

Distributing surplus equitably ex-post is the strongest of implementations, while an implementation of equal interim surpluses is infeasible in winners-pay auctions. If interim surpluses were equalized across different signals, incentive compatibility cannot hold.[21]

Example 1 (Continued)**.** We continue the example with $n = 3$ bidders competing for $k = 2$ items and signals distributed uniformly on the support $[0, 1]$. As previously calculated, we have $V(s) = (1-c)s + \frac{c}{6}(5s+1)$, and $\frac{G}{g}(y) = \frac{y(2-y)}{2(1-y)}$. Together, we obtain

$$\widetilde{p}_i(s, y) = \left((1-c)\left(s - \frac{y(2-y)}{2(1-y)}\right) + \frac{c}{6}(5y+1)\right)\mathbb{1}\{s > y\}. \tag{12}$$

We illustrate the payment for signals $s = 0.2, 0.5, 0.8$ as a function of y in Fig. 7 below. Note that each payment corresponding to a signal s is only plotted for $y \leq s$, as for $y > s$ the signal s does not win and the payment is zero.



$c = 0$ $c = 0.5$ $c = 0.8$ $c = 1$

Figure 7: Surplus-equitable payments, $\widetilde{p}$, for uniform signal distributions as a function of the signal s and the first rejected signal, y, for common value parameters $c \in \{0, 0.5, 0.8, 1\}$.

The payment addresses two countervailing incentives, the first stemming from the private-value component, and the second stemming from the common-value component.

[21] The winners-pay assumption is indispensable. Without it, interim surpluses among winners can indeed be equal. In the example with $n = 3$ and $k = 2$, uniform signals and pure private values ($c = 0$), it can be verified that an interim payment of $L(s_i) = s_i(5s_i - s_i^2 - 4)/3(1-s_i)^2$ charged to the losing bidders achieves equal expected surpluses of the winners of $\frac{2}{3}$. However, then the losing bidders would naturally enter equity considerations.

With pure private values, if the first rejected signal is high relative to s, the bidder winning with signal s is paid a subsidy. This subsidy compensates bidders in order to induce them to report truthfully even with high private signals. If a high-signal bidder wins *together* with a low-signal bidder, a large surplus (due to the private signal) is levied in order to equalize ex-post surpluses. Naturally, this would create an incentive to under-report your signal. However, in those cases where all winners' signals are high, no taxation is needed to equalize their surplus, and a subsidy is paid to winning bidders, in order to restore incentive compatibility ex-ante.

With a pure common value, the payment is increasing in the first rejected signal y. In this case, no taxation is needed, as the ex-post surpluses are equal under any uniform payment rule. However, truthful reporting must be incentivized, which a second-price rule (adjusted for the common value) achieves. Recall that $V(y)$ is the expected value of a bidder with signal y conditional on tying with the k-th highest among their $n-1$ competitors.

With a stronger common value component, the taxation of the surplus due to the private signal is less important. Instead, as seen in Fig. 7 with $c = 0.8$, the payment is increasing in the first rejected signal. However, for high signals s, there exists an interior maximum of $\widetilde{p}$ in y, a turning point of the countervailing incentives. The taxation of the private signal still hurts the bidder with a higher signal, so a steep subsidy, on a small range of first rejected signals y, must be paid to level incentives in expectation.

5.1 Allocative Equity

So far, we focused on efficient mechanisms only and on equity in bidders' surpluses. A different, but natural notion of equity concerns the allocation of items itself – ensuring each bidder has the same chance of receiving an item, regardless of their private value. While the efficient allocation maximizes overall social welfare, it generally conflicts with allocative equity. However, we demonstrate that, relaxing efficiency, allocative equity can be complementary to surplus equity. Indeed, our following result show that they can be achieved simultaneously.

The uniform lottery is a natural, equitable allocation rule distributing the k items uniformly at random among the n bidders. Under this allocation rule, each bidder receives an item with probability k/n, independently of their signal. The key question is whether such equity is compatible with surplus equity. Proposition 6 shows that there exists a winners-pay mechanism that delivers both ex-ante allocative equity and ex-post surplus equity.

Proposition 6. *Consider the mechanism that allocates items by a uniform lottery, so that each agent receives an item with probability k/n, together with the following payment rule:*

$$p_i(s_i, s_{-i}) = (1-c)\left((n-1)\mathbb{E}[s_1] - \sum_{j\neq i} s_j\right) \mathbb{1}[i \textit{ wins an item}]. \tag{13}$$

Then the surplus of winners is equalized ex-post, and the mechanism is weakly DSIC, individually rational, and ex-ante budget balanced.

The proof is given in Appendix C. Proposition 6 establishes that there is no inherent trade-off between allocative equity among all bidders and surplus equity among winners. However, allocative equity drives expected revenue to zero, as we show in the proof. This is in stark contrast to the surplus-equitable, efficient mechanism, which is revenue-equivalent to a uniform price auction. We note that ex-post individual rationality could also be guaranteed, but only at the cost of a budget deficit.

Next, it is natural to ask whether in a uniformly random allocation one can equalize surplus across all bidders, winners and losers alike. The following proposition shows this is also achievable in a Bayes-Nash incentive-compatible mechanism with a suitable modification of payments.

Proposition 7. *Consider the mechanism that allocates items by a uniform lottery, so that each agent receives an item with probability k/n, and the following payment rule:*

$$p_i(s_i, s_{-i}) = \left((1-c)\left(s_i + \frac{k}{n}(n-1)\mathbb{E}[s_1] - \frac{k}{n}\sum_{j=1}^{n} s_j\right) + \frac{c}{n}\left(1 - \frac{k}{n}\right)\sum_{j=1}^{n} s_j\right)\mathbb{1}[i \textit{ wins an item}] + \left((1-c)\frac{k}{n}\left((n-1)\mathbb{E}[s_1] - \sum_{j=1}^{n} s_j\right) - \frac{ck}{n^2}\sum_{j=1}^{n} s_j\right)\mathbb{1}[i \textit{ does not win an item}] \tag{14}$$

Then the surplus of all *bidders is equalized ex-post. The mechanism is Bayes–Nash IC, individually rational, and ex-ante budget balanced.*

The proof is given in Appendix C. Notice that, in contrast to Proposition 6, with a pure common value ($c = 1$) the winners' payments are no longer zero, reflecting the stronger requirement of equalizing surplus across both winners and losers.

In both cases of Proposition 6 and Proposition 7, the seller extracts no revenue in expectation and instead functions purely as an intermediary that balances allocative and surplus equity. These results together with our surplus-equitable mechanism from Section 5 demonstrate a nuanced set of trade-offs: surplus equity alone (among winners) can be implemented without sacrificing efficiency or revenue. By contrast, allocative equity necessitates abandoning efficiency, and the seller must forgo expected revenue. Taken together, regardless of whether one prioritizes efficiency or allocative equity in the first instance, surplus equity can always be achieved at no additional cost beyond the one associated to the allocation rule.

5.2 Between Efficiency and Allocative Equity

Suppose we seek a mechanism that interpolates between allocative equity and efficiency. In this case, a simple convex combination of the two corresponding mechanisms suffices.

A mechanism is characterized by a randomized allocation rule $\mathbf{s} \mapsto \mathbf{x}(\mathbf{s}) \in \operatorname{conv}\{\mathbf{1}_S : S \subseteq [n],\, |S| \leq k\}$, which assigns, for each signal profile $\mathbf{s}$, a distribution over feasible allocations, and a randomized payment rule $\mathbf{s} \mapsto \mathbf{p}(\mathbf{s}) \in \mathbb{R}^n$. Here $\mathbf{p}(\mathbf{s})$ is a random variable, and can be correlated to $\mathbf{x}$. The expected utility of agent i given the signal profile $\mathbf{s}$ is then $u_i(s_i, s_{-i}) = x_i(s_i, s_{-i}) \cdot v_i(s_i, s_{-i}) - p_i(s_i, s_{-i})$.

Given two mechanisms $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$, and a mixing parameter $\lambda \in [0, 1]$, define their convex combination as

$$(\mathbf{x}^\lambda, \mathbf{p}^\lambda) = \big(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}',\, \lambda\mathbf{p} + (1-\lambda)\mathbf{p}'\big),$$

where $\mathbf{x}^\lambda$ corresponds to the mixture between the two allocation rules: with probability λ the allocation $\mathbf{x}$ is implemented, and with probability $1-\lambda$ the allocation $\mathbf{x}'$ is implemented. Agents are aware of λ but not of the realized mechanism prior to observing the outcome. This lottery over mechanisms is one possible implementation of the mixed distribution x^λ, but it need not be unique.

Corollary 2. The convex combination of the efficient, surplus-equitable mechanism and the allocatively equitable, surplus-equitable mechanism is itself Bayes–Nash incentive compatible, interim individually rational, and surplus equitable.

Proof. Surplus equitability holds trivially under mixtures, as the property is preserved under convex combinations of outcomes. For incentive compatibility, note that if both $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$ satisfy Bayes–Nash incentive compatibility, then for all agents $i \in [n]$ and for all $s_i, z_i \in [0, \bar{s})$,

$$\mathbb{E}[x_i(s_i, s_{-i})v_i(s_i, s_{-i}) - p_i(s_i, s_{-i}) \mid s_i] \geq \mathbb{E}[x_i(z_i, s_{-i})v_i(s_i, s_{-i}) - p_i(z_i, s_{-i}) \mid s_i],$$

and the same holds for $(\mathbf{x}', \mathbf{p}')$. Taking a convex combination of these two inequalities (with weights λ and $1-\lambda$) immediately implies Bayes–Nash incentive compatibility of $(\mathbf{x}^\lambda, \mathbf{p}^\lambda)$. The same argument can be applied to prove interim individual rationality. □

We note that by the same argument, this new mechanism preserves ex-post individual rationality for pure private values.

6 Discussion

In this section, we discuss a few extensions of our results. First, we explain how surplus equity and in particular WEV relates to the within-bidder variation of surplus, as well as the empirical variance of surplus between all bidders. We explain why the regularity assumption of log-concavity is necessary for our argument. We discuss the what happens if unit demand is generalized to multi-unit demand. Finally, we consider how our results extend to auctions with reserve prices.

6.1 Variance and Risk Preferences

Surplus equity and distributional concerns are distinct from questions of within-agent variation and associated risk preferences. An appropriate measure to assess the latter is, e.g., the ex-ante variance of bidder surplus. While the two notions are distinct, the measures are linked through the covariance (see also Lemma 1). In addition, for the pure private value setting, we derive the following result. The proof is deferred to Appendix C.

Proposition 8. *With pure private values ($c = 0$), the pay-as-bid auction minimizes the ex-ante variance of surplus among all standard auctions with increasing equilibrium bid functions.*

Because of revenue equivalence, note that the previous proposition also implies that $\mathbb{E}[u_i^2]$ is minimal in the pay-as-bid auction among standard auctions. The second moment of surplus links the winners' empirical variance and the empirical variance among all bidders, as shown in Lemma 7 in the Appendix. As a consequence of Lemma 7, surplus equity rankings with respect to the winners' empirical variance and the empirical variance among all bidders may not be equivalent. However, applying Proposition 11 to the pure private value case, we have the following corollary:

Corollary 3. Assuming pure private values ($c = 0$), consider any δ-mixed auction, $\delta \in (0, 1]$, and suppose that the equilibrium bid β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2}{\delta}$ for all signals $s \in [0, \bar{s})$. Then, the empirical variance (among all bidders) is lower for δ-mixed pricing than for uniform pricing.

Although this result shows that we can extend equity rankings under pure private values to the empirical variance *among all bidders*, this may not hold in the general case.

6.2 Beyond Log-Concave Distributions

A crucial ingredient for Theorem 2 is that the derivative of the equilibrium bid function is bounded by 1, which holds for log-concave distributions by Proposition 12. In particular, the density of the first rejected signal must be log-concave. In the following, we provide some insights as to why it is difficult to generalize this result beyond log-concave distributions.

For simplicity, consider the pay-as-bid and the uniform price auction. Considering log-concave signal distributions, we note that log-concavity is equivalent to (A, G) concavity (a generalization of convexity, see Anderson, Vamanamurthy and Vuorinen (2007)), and $\frac{\partial \beta^\delta}{\partial s} \leq 1$ is thus equivalent to (A, G) concavity of $s \mapsto \int_0^s G_k^{n-1}$. One idea to extend our results could then be to consider other generalizations of convexity. Considering Proposition 11, one might attempt to bound the slope of the bid functions by 2. It holds that $\frac{\partial \beta^\delta}{\partial s} \leq 2$ is equivalent to (A, H) concavity of the same function where H is the harmonic mean. But contrary to (A, G) concave functions, there are no simple group closure properties that allow for the (A, H) concavity of f to always imply that of $\int_0^s G_k^{n-1}$. Thus, this route of inquiry is not fruitful.

We also note that conditions similar to MEU such that uniform pricing yields lower pairwise differences (or WEV) than pay-as-bid pricing are much more difficult to attain. Why? If we follow the same main ideas as in the proof of Proposition 11, a similar condition using the mean value theorem would be that, for all $s \in (0, \bar{s})$, φ is an expansive mapping, translating into $|(1 - c) - \frac{\partial \beta^\delta}{\partial s}| \geq 1 - c$. As $\frac{\partial \beta^\delta}{\partial s}$ is strictly positive, we must have $\frac{\partial \beta^\delta}{\partial s} > 1 - c$. For signal distributions with bounded density, g_k^{n-1} is close to zero near $\bar{s}$ (this follows from the definition of order statistics), and therefore $\frac{\partial \beta^\delta}{\partial s}$ is close to zero for a nonzero interval of signals. Thus, $\frac{\partial \beta^\delta}{\partial s} > 1 - c$ cannot hold for all signals on the support, and we cannot rely on similar proof techniques to produce the desired conditions.

6.3 Multi-Unit Demand

A natural question is how robust the equity dominance results for pay-as-bid, uniform, and mixed auctions are with respect to the assumption of unit demand. In this section, we discuss a simple generalization of our model to flat d-unit demands for some integer d, where all bidders have the same constant marginal value for the first d items obtained and 0 after.

There are two main considerations to extending our results to this setting, both leading to potential equity-efficiency trade-offs. Firstly, each bidder may win a different number of items, and thus equity could be studied with respect to a bidder's total surplus or average per-unit surplus. The former equity metric interacts with efficiency. Secondly, we have a multiplicity of equilibria in uniform price auctions: Noussair (1995) show that even for two bidders and two-unit demand, there is no unique symmetric equilibrium; instead, there exists an inefficient, zero-revenue symmetric equilibrium, as well as an efficient, zero-utility symmetric equilibrium.

As noted earlier in our analysis, any comparison with respect to the first-rejected-bid uniform price auction carries over to any other uniform price auction. Ausubel et al. (2014) show in their Theorem 1 that a necessary condition for the existence of an ex-post efficient equilibrium in multi-unit auctions is given by the condition "demand divides supply", which requires the total supply of items to be an integer multiple of a homogeneous demand d, i.e., k/d to be integer. Under this strong assumption, considering mixed pricing between pay-as-bid and Vickrey pricing (which, as we argue below, coincides with uniform pricing under the aforementioned condition), our results extend immediately. Without this condition, for $d > 1$, Vickrey pricing does not coincide with the FRB uniform price auction.

Under Vickrey pricing, a bidder who wins m items must pay the sum of the m highest losing bids, not including their own losing bids. This payment rule is exactly that of the VCG mechanism; hence, the auction is incentive-compatible and efficient. When demand divides supply, all winners of the VCG mechanism will receive exactly d items, and each winner pays the same price, since they face the same losing bids. This renders the Vickrey payment into a uniform-pricing rule. Under pay-as-bid pricing with constant marginal values for d items and zero after, the equilibrium bids for these first d-items are the same as for the 1-unit demand case, and thus each bidder also receives exactly d items (see also Ausubel et al. 2014). Overall, as a consequence of "demand divides supply", in both the pay-as-bid and the Vickrey auction, items are sold in bundles of size d with payments and values scaled by d. Therefore, all results of the unit-demand case hold with flat d-unit demand when d divides k in the comparison between pay-as-bid and Vickrey pricing.

When demand does not divide supply, our results do not extend to the multi-unit case. In the example in Ausubel et al. (2014) with two bidders with flat two-unit demand and pure private values, there exists an inefficient equilibrium in the uniform price auction. In this equilibrium, each bidder is allocated one unit, which is always more surplus-equitable (with respect to a bidder's overall surplus) than the efficient equilibrium of the pay-as-bid auction which allocates both items to the higher-value bidder.

6.4 Revenue Maximization and Reserve Prices

Standard auctions lead to an efficient allocation of items in our model. Moreover, for a large class of probability distributions (and common values), these auctions are also revenue maximizing (see Bulow and Klemperer, 1996). They show that if, in addition to independent signals, the bidders' marginal revenues are increasing in signals and weakly positive, standard auctions are revenue maximizing.[22] Marginal revenues are positive in many applications where bidders' values are high, i.e., the support of signals is sufficiently positive.[23] In those cases, it is optimal for the seller to sell their entire supply. Our class of auctions is also revenue maximizing if the seller is legally required to sell their entire supply. In practice, either (or both) of the two conditions are observed in many high-stake auctions, e.g., for spectrum licenses.

More generally, seller revenue can be maximized at the expense of efficiency (Riley and Samuelson, 1981; Myerson, 1981). A common tool to raise revenue is to set a reserve price $r > 0$ such that only bids exceeding r can win and pay the auction price, or at least r, whichever is higher. Myerson (1981) and Riley and Samuelson (1981) show that, with pure private values, an optimal reserve price maximizes the seller's expected revenue.

Some of our results on equity-preferred pricing extend to pay-as-bid and uniform price auctions with an additional reserve price. With a reserve price, the number of winners, although identical in different standard auctions with identical reserve prices, is not necessarily equal to the number of items k and becomes a random variable. In the uniform price auction, the standard derivation leads to an equilibrium bid of $V(s)$ for $s > s_r$, where s_r is some threshold, and 0 otherwise. Using revenue equivalence, we obtain the equilibrium bid in the pay-as-bid auction $\beta_r^{\delta=1}(s) = (r - V(s_r))G(s_r)/G(s) + V(s) - \int_{s_r}^{s} V'(y)G(y)dy/G(s)$ for $s > s_r$ and 0 otherwise. For the equity comparison of pay-as-bid and uniform pricing, we establish the following property.

Proposition 9. *If monotone ex-post utility (MEU, Definition 5) holds in the pay-as-bid auction without a reserve price, then it also holds with a strictly positive reserve price.*

The proof is given in Appendix C. Then, the following corollary is immediate from Proposition 9 (a slightly weaker statement than Proposition 10).

Corollary 4. Suppose MEU holds in the equilibrium of the pay-as-bid auction without reserve price. Then the pay-as-bid auction with a given reserve price is equity-preferred to (dominates in pairwise differences) any uniform price auction with the same reserve price.

7 Conclusion and Future Research

We provide the first results surplus equity between bidders in auction design. We introduce a family of equity measures, based on absolute pairwise differences in realized utilities, that includes popular metrics such as the empirical variance and the expected Gini index. Considering standard and winners-pay auctions in an independent signal setting with single-crossing values, the equity design objective is costless in terms of potential trade-offs with efficiency and revenue.

In the class of uniform, pay-as-bid, and mixed auctions, we show that, in most cases, some degree of price discrimination is beneficial in terms of equity. The equity-preferred auction design crucially depends on the common value proportion in the bidders' value structure, and this result is prior-free up to log-concavity of signal distributions. Our results have substantial implications for auction design in practice. By carefully selecting a pricing mixture based on

[22] With independent signals, the marginal revenue of bidder i given a realization of signals $\boldsymbol{s}$ is given by $MR_i(s_i) = \frac{-1}{f(s_i)} \frac{\partial}{\partial s_i} (v(s_i)(1 - F(s_i)))$.

[23] In our model, without loss of generality, the signal support includes the lower bound zero.

(an estimate of) the common value, auctioneers can achieve a more equitable division of surplus among winning bidders.

Turning to more general mechanisms, we design an efficient, surplus-equitable mechanism, a direct and truthful mechanism that efficiently allocates the items for sale and equalizes the winners' realized utilities. Each winner pays a personalized price, while losers pay nothing. We further demonstrate that allocation-fair mechanisms with uniformly random allocation rules can be made surplus-equitable.

Future research could further explore the trade-offs between efficiency and equity (see also Section 5.1). Extensions could also cover other types of auctions and value distributions. For example, in multi-unit demand settings in which items may be allocated inefficiently (Section 6.3), trade-offs become relevant. In practice, other designs such as dynamic auctions or the Spanish auction (Álvarez and Mazón, 2007) are used, and understanding the impact of such designs on surplus equity remains an open question.

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Appendices

A Structural Insights and Proof of Theorem 2

In this section, we provide an overview of the proof of Theorem 2. We combine two propositions on monotonicity of pairwise differences and dominance of pairwise differences, respectively, with a third proposition that bounds the slope of bid functions. In particular, we identify the property of *monotone ex-post utility* as a fundamental and sufficient condition for our dominance results.

Definition 5 (Monotone ex-post utility)**.** The ex-post utility $u(\boldsymbol{s})$ satisfies *monotone ex-post utility (MEU)* iff, for any two signals $s_i, s_j \in [0, \bar{s})$ and $\forall\ \boldsymbol{s}_{-i}, \boldsymbol{s}_{-j},\ s_i \leq s_j \Leftrightarrow u_i(s_i, \boldsymbol{s}_{-i}) \leq u_j(s_j, \boldsymbol{s}_{-j})$.

Monotone ex-post utility (MEU) relates to the slope of equilibrium bids as follows.

Lemma 2. *An equilibrium satisfies monotone ex-post utility iff equilibrium bid functions β^δ satisfy $\frac{\partial \beta^\delta}{\partial s} \leq \frac{1-c}{\delta}$ for all signals $s \in [0, \bar{s})$.*

Proof: See Appendix B.7.

The ex-post difference in utilities depends only on the private value proportion $(1-c)s$ and the discriminatory part of the payment $\delta\beta^\delta$. Thus, as long as the discriminatory payment does not grow faster in the signal than the private-value share, ex-post utilities are monotone.

As seen in **??**, the equilibrium exhibit several monotonicity properties, and these hold beyond uniform signals. By assumption, equilibrium bids are increasing in the bidder's own signal. Equilibrium bids are also decreasing in the extent of price discrimination: if a higher proportion of one's own bid affects the price, the incentive to bid-shade increases. Finally, the change in the payment corresponding to a bidder's own bid, due to a signal increase, is increasing in the weight of price discrimination, and vice versa. The latter monotonicity is crucial for Proposition 10.

Lemma 3. *The equilibrium bid functions satisfy the following monotonicity properties:*

1. *$\beta^\delta(s)$ is strictly increasing in s, for all fixed $\delta \in [0,1]$ (consistent with the assumption), and is strictly decreasing in δ, for all fixed $s \in (0, \bar{s})$.*
2. *$\frac{\partial(\delta\beta^\delta(s))}{\partial \delta}$ is strictly increasing in s, for all fixed $\delta \in [0,1]$, and $\frac{\partial(\delta\beta^\delta(s))}{\partial s}$ is strictly increasing in δ, for all fixed $s \in (0, \bar{s})$.*

Proof: See Appendix B.3.

We now characterize the fundamental role of monotone ex-post utility: it is equivalent to the monotonicity property of pairwise differences.

Proposition 10. *For a given common value c and for some $\bar{\delta} \in [0,1]$, pairwise differences are monotonically decreasing over $[0, \bar{\delta}]$ if and only if the equilibrium (which depends on c and $\bar{\delta}$) satisfies MEU.*

Proof: See Appendix B.7.

The equivalence between decreasing pairwise differences and MEU being satisfied in equilibrium is crucial in our proof of Theorem 2. When MEU holds, the slope of the equilibrium bid function is sufficiently flat and more price discrimination impacts higher signal bidders more than lower signal bidders. In contrast, including more uniform pricing in the price mix will, proportionally to the change in δ, offer higher signal bidders a higher discount than lower signal bidders and thus does not improve surplus equity. A similar intuition holds for Proposition 11 below (for details on the intuition, see Section 4.3).

Proposition 11. *For a given common value c, consider any δ-mixed auction, $\delta \in (0, 1]$, and suppose the equilibrium bidding function β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2(1-c)}{\delta}$ for all signals $s \in [0, \bar{s})$. Then, δ-mixed pricing dominates uniform pricing in pairwise differences.*

Proof: See Appendix B.7.

Example 1 (Continued)**.** Whether MEU is satisfied can be verified numerically, either by computing differences in realized utilities for every pair of signals or simply by checking the derivative of the bid function. We illustrate this for the example of uniform signal distributions and $n = 3$ and $k = 2$ in Fig. 8 below. For example, with $c = 0.8$ and $\delta = 0.3$, close to the MEU boundary in Fig. 8, the derivative of the bid function cannot be larger than $0.667 = \frac{1-0.8}{0.3}$. From **??**, the slope of the bid function with $c = 0.8$ and $\delta = 0.3$ is close to 0.68 for low signals. Thus, for this combination of c and δ, MEU is not satisfied.

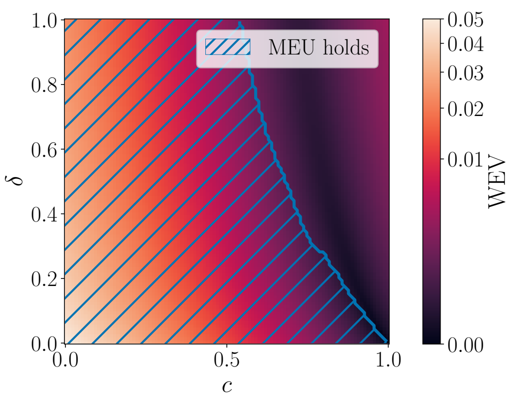

Figure 8: Monotone ex-post utility for common value c and price discrimination δ

The final crucial proposition bounds the slope of the equilibrium bid functions by 1 for the family of log-concave signal distribution.

Proposition 12. *If the signal density f is log-concave, then $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ for all signals $s \in [0, \bar{s})$.*

The proof is given in Appendix A.1 below. With pure private values, this bound implies that ex-post utility is non-decreasing in signals for log-concave signal distributions. Indeed, for $u(s) = s - \delta\beta^\delta(s) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have that $\frac{\partial u}{\partial s} = 1 - \delta\frac{\partial \beta^\delta(s)}{\partial s} \geq 0$. A similar reasoning leads to Theorem 2.

Proof of Theorem 2. Because of Proposition 12, we have that under log-concave signal distributions MEU holds if $\delta \leq (1-c)$, as $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{(1-c)}{\delta}$ (see Lemma 2). Thus, applying Proposition 10, pairwise differences are monotonically decreasing for $\delta \in [0, 1-c]$ and Theorem 2 follows.

Similarly, because of Proposition 12, it holds that with log-concave signals, $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{2(1-c)}{\delta}$ if $\delta \leq 2(1-c)$. Applying Proposition 11, it follows that any mixed pricing with $\delta \in (0, 2(1-c)]$ dominates uniform pricing in pairwise differences. □

A.1 Proving the Bound on Bid Function Slopes

Bounding the bid function slope for log-concave distributions requires three main observations, which we detail in the lemmas below and then use to prove Proposition 12.

The first lemma establishes a simplified expression of $V(s)$ which allows to bound $V'(s)$ by 1.

Lemma 4. *Assuming common values ($c = 1$), $V(s)$ is differentiable on $(0, \bar{s})$, and can be expressed as*

$$V(s) = \frac{2}{n}s + \frac{n-k-1}{n}\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} + \frac{k-1}{n}\frac{\int_s^{\bar{s}} tf(t)\,\mathrm{d}t}{1-F(s)}$$

Moreover, if the signal density f is log-concave, then $V'(s) \leq 1$ for all signals $s \in [0, \bar{s})$.

Proof: See Appendix B.8.

The proof proceeds by noticing that order statistics conditioned on other order statistics behave just like order statistics of a truncation of the original distribution. Thus, a more tractable expression of the expected valuation V can be derived for the pure common value case ($c = 1$). Together with results on log-concavity by Bagnoli and Bergstrom (2005), we use this expression to show that $V' \leq 1$ for all signals $s \in (0, \bar{s})$.

The next lemma establishes a sufficient condition for the equilibrium bid function slope to be bounded by 1 in the pure private value case. Differentiating twice $\int_0^s G^{1/\delta}$, we establish that its log-concavity is equivalent to $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$.

Lemma 5. *Assuming private values ($c = 0$), for any $\delta \in (0, 1]$, $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ iff $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is log-concave.*

Proof: See Appendix B.8.

Finally, we establish that a log-concave signal density is sufficient for the integral of their order statistics to be log-concave, using closure properties of product and integration of log-concave distributions, and results by Bagnoli and Bergstrom (2005).

Lemma 6. *If the density of signals f is log-concave, then so is $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$.*

Proof: See Appendix B.8.

With the three lemmas above, we can prove Proposition 12.

Proof of Proposition 12. First, we recall the expression of the derivative of the bid function for any $s \in (0, \bar{s})$:

$$\frac{\partial \beta^\delta(s)}{\partial s} = \frac{g(s)}{G(s)} \frac{\int_0^s V'(y) G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \tag{15}$$

Note that for any $c \in [0,1]$, $V(s)$ is a linear combination of s and V. In the case of a pure common value, the derivative of the latter is bounded by 1 by Lemma 4. Hence for any c, $V'(s) \leq 1$. Moreover, because of Lemma 6, we know that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave, and we can therefore apply Lemma 5. Hence using the above results,

$$\frac{\partial \beta^\delta(s)}{\partial s} \leq \frac{g(s)}{G(s)} \frac{\int_0^s \max_t V'(t) G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq \frac{g(s)}{G(s)} \frac{\int_0^s 1 \cdot G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq 1. \tag{16}$$

□

B Proofs

B.1 Revenue Equivalence and Efficiency

We recall results from Krishna (2009) that show that the auctions we consider exhibit revenue equivalence and (allocative) efficiency.

Proposition (Revenue equivalence, Krishna 2009)**.** *Assuming iid signals, any standard auction, under any symmetric and increasing equilibrium with an expected payment of zero at value zero, yields the same expected revenue to the seller.*

We note that the crucial assumption for revenue equivalence is the independence of signals. In settings where signals are correlated, revenue equivalence fails Krishna, 2009, Chapter 6.5. It can be further shown that a bidder with signal s_i has an expected surplus

$$\tilde{u}(s_i) := \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] = \int_0^{s_i} (\widetilde{V}(s_i, y) - \widetilde{V}(y, y)) g_k^{n-1}(y)\,\mathrm{d}y$$

A value function $v(\boldsymbol{s})$ satisfies the *single crossing* condition if for all $i, j \neq i \in [n]$ and for all $\boldsymbol{s}$, $\frac{\partial v(s_i, \boldsymbol{s}_{-i})}{\partial s_i} \geq \frac{\partial v(s_j, \boldsymbol{s}_{-j})}{\partial s_i}$, and the value function v as given in Assumption 1 satisfies this condition.

Proposition (Efficiency, Krishna 2009)**.** *Any standard auction, under any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is efficient.*

Given the prior propositions, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs.

B.2 Surplus Equity

Proof of Lemma 1. The empirical variance of surplus can be transformed as follows.

$$\begin{aligned}\mathbb{E}_s\left[\frac{1}{n-1}\sum_i^n\left(u_i-\frac{1}{n}\sum_j^n u_j\right)^2\right] &= \mathbb{E}_s\left[\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n(u_i-u_j)^2\right]\\ &= \frac{\mathbb{E}_s\left[(u_1-u_2)^2\right]}{2}\\ &= \mathbb{E}_s[u_1^2]-\mathbb{E}_s[u_1u_2]\\ &= \mathrm{Var}(u_1)-\mathrm{Cov}(u_1,u_2)\end{aligned}$$

Similarly, the empirical variance conditioned on winning can be written as

$$\begin{aligned}\mathbb{E}_s\left[\frac{1}{k-1}\sum_{i=1}^k\left(u_i-\frac{1}{k}\sum_{j=1}^k u_j\right)^2\middle|\, 1,\ldots,k\text{ win}\right] &= \frac{\mathbb{E}_s\left[(u_1-u_2)^2\mid 1\text{ and }2\text{ win}\right]}{2}\\ &= \mathbb{E}_s\left[u_1^2\mid 1\text{ wins}\right]-\mathbb{E}_s\left[u_1u_2\mid 1\text{ and }2\text{ win}\right]\\ &= \mathrm{Var}(u_1\mid 1\text{ wins})-\mathrm{Cov}(u_1,u_2\mid 1\text{ and }2\text{ win}).\end{aligned}$$

□

With pure private values, ex-post individual rationality holds. The lemma below shows that, in this case, any ranking of auction formats in terms of ex-ante variance (Var) or winners' ex-ante variance (WV) is identical. In contrast, a ranking with respect to empirical variance (EV) can differ depending on whether only winners are considered or all bidders.

Lemma 7. *Assuming that the auction is a* winners-pay auction, *the empirical variance and the ex-ante variance can be decomposed, respectively, as* $\mathrm{EV}=\frac{k(k-1)}{n(n-1)}\cdot\mathrm{WEV}+\left(1-\frac{k-1}{n-1}\right)E_{\boldsymbol{s}}[u_1^2]$ *and* $\mathrm{Var}=\frac{k}{n}\cdot\mathrm{WV}+\left(\frac{n}{k}-1\right)\cdot E_{\boldsymbol{s}}[u_1]^2$.

Recall that $E_{\boldsymbol{s}}[u_1]$ does not depend on the auction format (by revenue equivalence), while $E_{\boldsymbol{s}}[u_1^2]$ does.

Proof of Lemma 7. We first note that

$$\mathrm{WV}=E_{\boldsymbol{s}}[u_1^2\,|\,1\text{ wins}]-E_{\boldsymbol{s}}[u_1\,|\,1\text{ wins}]^2=\frac{n}{k}E_{\boldsymbol{s}}[u_1^2]-\left(\frac{n}{k}\right)^2E_{\boldsymbol{s}}[u_1]^2$$

For the ex-ante variance, we write:

$$\begin{aligned}\mathrm{Var}=E_{\boldsymbol{s}}[u_1^2]-E_{\boldsymbol{s}}[u_1]^2&=P_{\boldsymbol{s}}[1\text{ wins}]\cdot E_{\boldsymbol{s}}[u_1^2\,|\,1\text{ wins}]-E_{\boldsymbol{s}}[u_1]^2\\ &=P_{\boldsymbol{s}}[1\text{ wins}]\cdot E_{\boldsymbol{s}}[u_1^2\,|\,1\text{ wins}]-P_{\boldsymbol{s}}[1\text{ wins}]\cdot E_{\boldsymbol{s}}[u_1\,|\,1\text{ wins}]^2+P_{\boldsymbol{s}}[1\text{ wins}]\cdot E_{\boldsymbol{s}}[u_1\,|\,1\text{ wins}]^2-E_{\boldsymbol{s}}[u_1]^2\\ &=P_{\boldsymbol{s}}[1\text{ wins}]\cdot\mathrm{WV}+P_{\boldsymbol{s}}[1\text{ wins}]\cdot E_{\boldsymbol{s}}[u_1\,|\,1\text{ wins}]^2-E_{\boldsymbol{s}}[u_1]^2\\ &=P_{\boldsymbol{s}}[1\text{ wins}]\cdot\mathrm{WV}+\frac{P_{\boldsymbol{s}}[1\text{ wins}]^2}{P_{\boldsymbol{s}}[1\text{ wins}]}\cdot E_{\boldsymbol{s}}[u_1\,|\,1\text{ wins}]^2-E_{\boldsymbol{s}}[u_1]^2\\ &=P_{\boldsymbol{s}}[1\text{ wins}]\cdot\mathrm{WV}+\left(\frac{n}{k}-1\right)\cdot E_{\boldsymbol{s}}[u_1]^2\end{aligned}$$

For the empirical variance, we write:

$$\begin{aligned}\text{WEV} &= E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1 u_2] = P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot E_{\boldsymbol{s}}[u_1 u_2 \,|\, 1 \text{ and } 2 \text{ win}] \\ &= P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot \text{WEV} + \left(1 - \frac{P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}]}{P_{\boldsymbol{s}}[1 \text{ wins}]}\right) \cdot E_{\boldsymbol{s}}[u_1^2]\end{aligned}$$

$\square$

Proof of Proposition 3. Without loss of generality, consider an outcome profile u with three outcomes, u_i, u_j and U, where $u_i > u_j$, and U is arbitrary. Induce a Pigou-Dalton transfer $t > 0$ such that $u_i' = u_i - t > u_j$ and $u_j' = u_j + t < u_i$, and U remains the same. The outcome profile after the transfer is denoted u'. We show that the ranking between u and u' according to WEV coincides with what the Pigou-Dalton principle requires, namely it must be that $WEV(u') < \text{WEV}(u)$. Let $W := (u_i - U)^2 + (u_j - U)^2$. Then

$$\begin{aligned}&(u_i' - U)^2 + (u_j' - U)^2 \\ =\ & (u_i - t - U)^2 + (u_j + t - U)^2 \\ =\ & (u_i - U)^2 - 2t(u_i - U) + t^2 + (u_j - U)^2 + 2t(u_j - U) + t^2 \\ =\ & W + 2t\,(t - u_i + U + u_j - U) \\ =\ & W + 2t\,(u_j - (u_i - t)) \\ <\ & W\end{aligned}$$

The final inequality follows by the assumption that the transfer does not make i poorer than j was to start with. As U was arbitrarily chosen and, to compute WEV, expectations are taken around the sum of squared differences of the realized utilities, the result follows. $\square$

B.3 Equilibrium Bidding

Proof of Proposition 1. Consider bidder i and let all bidders $j \neq i$ follow the bidding strategy $\beta^U(s_j) = \widetilde{V}(s_j, s_j)$. First, observe that β^U is continuous and increasing. Then bidder i's expected payoff when their signal is s_i and bidding $\beta^U(z)$ is given by

$$U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \widetilde{V}(y, y)\right) g_k^{n-1}(y)\,\mathrm{d}y$$

Because $\widetilde{V}(s_i, y)$ is increasing in s_i, it holds for all $y < s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) > 0$, and for all $y > s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) < 0$. Therefore, choosing $z = s_i$ maximizes bidder i' expected payoff $U(s_i, z)$. $\square$

Proof of Proposition 2. First, observe that β^δ is continuous. We verify that it is also monotone: writing $G_k^{n-1} =: G$, $g_k^{n-1} =: g$, and $\widetilde{V}(s, s) =: V(s)$, an alternative expression for β^δ is given by

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{G(s)^{\frac{1}{\delta}}}. \tag{17}$$

In particular, it is differentiable almost everywhere and we can compute its derivative.

$$\beta^{\delta\prime}(s) = \frac{g(s)\int_0^s V'(y)G(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{\delta G(s)^{1+\frac{1}{\delta}}} \tag{18}$$

which it positive almost everywhere. Next, assume that all bidders $j \neq i$ follow the bidding strategy β^δ, and let $\beta^\delta(z)$ be bidder i's bid, whose expected utility is given by

$$U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \delta\beta^\delta(z) - (1-\delta)\beta^\delta(y)\right) g(y)\,\mathrm{d}y$$

The derivative of $U(s_i, z)$ is

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, z) &= \widetilde{V}(s_i, z)g(z) - \delta\beta^{\delta\prime}(z)G(z) - \delta\beta^\delta(z)g(z) - (1-\delta)\beta^\delta(z)g(z)\\ &= (\widetilde{V}(s_i, z) - \beta^\delta(z))g(z) - \delta\beta^{\delta\prime}(z)G(z).\end{aligned}$$

In equilibrium, the first order condition requires $\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, s_i) = 0$. We solve this differential equation using $G^{\frac{1}{\delta}-1}$ as the integrating factor. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[G(z)^{\frac{1}{\delta}}\beta^\delta(z)\right] = \left(\frac{1}{\delta}G(z)^{\frac{1}{\delta}-1}\right)\cdot(\beta^\delta(z)g(z) + \delta\beta^{\delta\prime}G(z)) = \left(\frac{1}{\delta}G(z)^{\frac{1}{\delta}-1}\right)\cdot\widetilde{V}(s_i, z)g(z).$$

Solving for β^δ, we obtain

$$\beta^\delta(s) = \frac{\int_0^s V(y)g_k^{n-1}(y)G_k^{n-1}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{19}$$

Using equations (17) and (18), and the fact that $\widetilde{V}(s_i, z)$ is increasing in s_i, we have that $\frac{\mathrm{d}U}{\mathrm{d}z}$ is positive when $z \leq s_i$ and negative when $z \geq s_i$. Therefore, choosing $z = s_i$ maximizes i's expected payoff $U(s_i, z)$.

Finally, we derive the expression for β^δ stated in the the proposition from Eq. (19). Writing $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$, observe that the derivative of $\delta G^{\frac{1}{\delta}}$ is $gG^{\frac{1}{\delta}-1}$. Using integration by parts and a change of variable, we obtain

$$\begin{aligned}\int_0^s V(y)g(y)G(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y &= \left[\delta V(y)G(y)^{\frac{1}{\delta}}\right]_0^s - \delta\int_0^s V'(y)G(y)^{\frac{1}{\delta}}\,\mathrm{d}y\\ &= \delta V(s)G(s)^{\frac{1}{\delta}} - \delta\int_{V(0)}^{V(s)} G(V^{-1}(y))^{\frac{1}{\delta}}\,\mathrm{d}y.\end{aligned}$$

Dividing by $\delta G^{\frac{1}{\delta}}$ gives the result. □

Lemma 8. *For any continuous function $\varphi : [0, \bar{s}) \to \mathbb{R}$, and for all $s \in (0, \bar{s})$, we have*

$$\lim_{\delta\to 0}\int_0^s \frac{\varphi(t)}{\delta}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$

$$\lim_{\delta\to 0}\int_0^s \ln\left(\frac{G(s)}{G(t)}\right)\frac{\varphi(t)}{\delta^2}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$

where $G_k^{n-1} =: G$ *and* $g_k^{n-1} =: g$.

Proof. Fix $\delta > 0$, and let $\psi : (0,1] \to \mathbb{R}$ be a continuous function, such that $\psi(u) = O(1/u)$ when $u \to 0$. Using the change of variable $u = v^\delta$, we have that

$$\int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \,\mathrm{d}u = \int_0^1 \psi(v^\delta) v^\delta dv$$
$$\int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \,\mathrm{d}u = \int_0^1 \ln(1/v) \psi(v^\delta) v^\delta dv.$$

Observe that for all fixed $v \in (0,1]$, and taking $\delta \to 0$, the first (resp. second) integrand converges towards $\psi(1)$ (resp., $\psi(1)\ln(1/v)$). We define the constant $M = \sup_{u\in(0,1]} u\psi(u)$, we bound the first integrand by M (resp.the second integrand by $M\ln(1/v)$), and we use the theorem of dominated convergence, which gives

$$\lim_{\delta\to 0} \int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \,\mathrm{d}u = \int_0^1 \psi(1) \,\mathrm{d}v = \psi(1)$$
$$\lim_{\delta\to 0} \int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \,\mathrm{d}u = \int_0^1 \psi(1) \ln(1/v) \,\mathrm{d}v = \psi(1)$$

To prove the lemma, observe that with the change of variable $u = \frac{G(t)}{G(s)}$, we have

$$\int_0^s \frac{\varphi(t)}{\delta} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \,\mathrm{d}u$$
$$\int_0^s \ln\left(\frac{G(s)}{G(t)}\right) \frac{\varphi(t)}{\delta^2} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \,\mathrm{d}u$$

where we define

$$\psi(u) := G(s) \cdot \frac{\varphi(G^{-1}(uG(s))}{g(G^{-1}(uG(s))}.$$

Finally, it remains to prove that $\psi(u) = O(1/u)$ when $u \to 0$. First, observe that φ is bounded on $[0, s]$. Second, observe that we have

$$\frac{u}{g(G^{-1}(uG(s)))} = \frac{1}{G(s)} \frac{G(x)}{g(x)},$$

where $x = G^{-1}(uG(s)) \to 0$. Because g is positive and integrable in 0, we have that G/g is bounded. Therefore, the overall limit when $\delta \to 0$ is equal to $\psi(1)$, which concludes the proof. $\square$

Lemma 9. *The following formulas can be derived:*

$$\beta^\delta(s) = \begin{cases} V(s) & \text{if } \delta = 0 \\ V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\partial(\beta^\delta(s))}{\partial s} = \begin{cases} V'(s) & \text{if } \delta = 0, s > 0 \\ \frac{g(s)}{G(s)} \int_0^s \frac{V'(y)}{\delta}\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\delta\partial(\beta^\delta(s))}{\partial \delta} = \begin{cases} 0 & \text{if } \delta = 0 \text{ or } s = 0 \\ \int_0^s V'(y) \ln\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{for } \delta, s > 0 \end{cases}$$

$$\frac{\partial^2(\delta\beta^\delta(s))}{\partial s \partial \delta} = \frac{-g(s)}{\delta G(s)} \int_0^s V'(y) \log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} \mathrm{d}y \text{ for } \delta, s > 0$$

Proof. In order to derive the value of these functions at points where they are not directly, defined, we will use the dominated convergence theorem.

(1) Let $s \in (0, \bar{s})$. We first look at $\beta^\delta(s) = V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y$. Let $h(\delta, y)$ be the function under the integral. Because G is increasing, for $y < s$ we have that $G(y)/G(s) < 1$. Hence h is dominated by V', and $\lim_{\delta \to 0} h(\delta, y) = 0$, hence by dominated convergence $\beta^\delta(s) = V(s)$ when $\delta = 0$, and the function is separately continuous over $[0, 1] \times [0, \bar{s})$.

(2) We now consider the derivative of β^δ with respect to s. Let $s \in (0, \bar{s})$. There exists $M > m > 0$ such that $s \in [m, M]$. We focus on the derivative of the integral part:

$$-\frac{\partial}{\partial s} V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} = V'(y)\frac{g(s)G^{1/\delta}(y)}{G^{1/\delta+1}(s)} \leq V'(y)\frac{g(s)}{G(s)} \leq V'(y)\frac{\sup_{t \in [m,M]} g(t)}{G(m)},$$

where the $\sup_{t \in [m,M]} g(t)$ is finite as g is continuous. Because V' is integrable, we can use dominated convergence. Using the Leibniz integral rule yields the result. The limit as δ goes to 0 can be computed by applying Lemma 8.

(3) Let us now compute the derivative of β^δ with respect to δ. Again, we use a dominated convergence property to show that the integral and derivative can be inverted. It is easier to show that this can be done for the function $\delta\beta^\delta(s)$, and we have

$$\frac{\partial \delta\beta^\delta(s)}{\partial \delta} = \beta^\delta + \delta\frac{\partial \beta^\delta(s)}{\partial \delta}.$$

Computing the derivative of $\delta\beta^\delta(s)$, we can recover that of $\beta^\delta(s)$.

Let $h(\delta, y, s) = \delta V'(y)(G(y)/G(s))^{1/\delta}$ be the function under the integral part of $\delta\beta^\delta$. We have

$$\frac{\partial h(\delta,y,s)}{\partial\delta} = V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y)\frac{\delta}{\delta^2}\log\left(\frac{G(y)}{G(s)}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}$$
$$= V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}.$$

The first part is again dominated by V', which is integrable. Focusing on the second part, we define for $0 < u < w < 1$ the function $\psi(u,w) = (u/w)\log(w/u)$. Note that $0 < s < y < \bar{s}$ implies that for $u = G^{1/\delta}(y)$ and $w = G^{1/\delta}(s)$, we have $0 < u < w < 1$ as G is increasing and takes values in $(0,1)$ over $(0,\bar{s})$ by definition. Fix w, and take the derivative with respect to u: we obtain that $\psi'(u,w) = (\log(w/u) - 1)/w$ which is positive as long as $u \le w/e$ and negative otherwise. The maximum of ψ for $u < w$ is at $u = w/e$ and $\psi(w/e,w) = 1/e$. This shows the right-hand side of h is smaller than $V'(y)/e$, which is also integrable. Overall by dominated convergence we can invert derivative and integral: $\frac{\partial}{\partial\delta}\int h = \int \frac{\partial}{\partial\delta}h$. Thus

$$\frac{\partial\delta\beta^\delta}{\partial\delta} = V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} dy + \int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta},$$

and we recover

$$\delta\frac{\partial\beta^\delta(s)}{\partial\delta} = \int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}.$$

Using the same upper bound on ψ, we can show that the integrand of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ is smaller than $V'(y)/e$ which allows for domination both in small δ and small s. By dominated convergence, we obtain that the limit of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ as either δ or s go to 0 is 0.

(4) Finally, let us compute the cross derivative. The integrand of $\frac{\partial\beta^\delta(s)}{\partial s}$ is $h(\delta,y,s) = V'(y)(G(y)/G(s))^{1/\delta}$, and its derivative with respect to delta is $-\frac{1}{\delta}V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)$. Because this function is continuous on the open set $(0,1)\times[0,\bar{s})$, we can apply dominated convergence to show that the order of derivative and integral can be reversed. Therefore

$$\frac{\partial^2\delta\beta^\delta}{\partial\delta\partial s} = \frac{g(s)}{G(s)}\frac{-G^{1/\delta}(s)\frac{1}{\delta}\int_0^s V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)\,\mathrm{d}y + \frac{1}{\delta}\log(G^{1/\delta}(s))G^{1/\delta}(s)\int_0^s V'(y)G^{1/\delta}(y)\,\mathrm{d}y}{G^{2/\delta}}$$
$$= \frac{-g(s)}{\delta G(s)}\int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\mathrm{d}y.$$

□

Lemma 10. *Consider a function $\varphi : [0,1]\times(0,\bar{s}) \to \mathbb{R}_+$, such that*

- *$\varphi_\delta : s \mapsto \varphi(\delta,s)$ is continuous over $(0,\bar{s})$ for all fixed $\delta \in [0,1]$,*
- *$\varphi_s : \delta \mapsto \varphi(\delta,s)$ is continuous over $[0,1]$ for all fixed $s \in (0,\bar{s})$,*
- *either all φ_s's are monotone or all φ_δ's are monotone,*

then φ is jointly continuous in δ and s.

Proof. The proof on the open set $(0,1)\times(0,\bar{s})$ is written in Kruse and Deely (1969), and directly generalizes to $\delta=0$ and $\delta=1$ given that φ is separately continuous in those points. $\square$

Proof of Lemma 3. Monotonicity follows from the derivatives computed in Lemma 9. $\square$

B.4 The Surplus-Equitable Payment Rule

Proof of Theorem 3. The payment rule $\widetilde{p}$ results in identical ex-post surpluses of winners. This follows directly from the definition of ex-post surplus under truthful reporting $u_i(s_i,\boldsymbol{s}_{-i}) = v(s_i,\boldsymbol{s}_{-i})-\widetilde{p}_i(s_i,\boldsymbol{s}_{-i}) = \frac{c}{n}\sum_{j\in[n]} s_j+(1-c)(y-\frac{G(y)}{g(y)})+V(y)$, for all $s_i>y$, which is independent of the bidder's identity as the first rejected signal is the same for any winner.

Furthermore, the payment $\widetilde{p}$ is interim incentive-compatible. First, we compute the expected payment

$$\widetilde{P}_i(s_i) = (1-c)\left(s_iG(s_i) - \int_0^{s_i}(yg(y)+G(y))\,\mathrm{d}y\right) + \int_0^{s_i}\widetilde{V}(y,y)g(y)\,\mathrm{d}y = \int_0^{s_i}\widetilde{V}(y,y)g(y)\,\mathrm{d}y.$$

The left-hand term is equal to 0 by integration by parts. Because losers pay nothing, the overall expected payment is equal to the expected payment of winners. The expected utility of a bidder who has signal s_i and reports $\hat{s}_i$ is given by

$$U_i(s_i,\hat{s}_i) = \int_0^{\hat{s}_i}\left(\widetilde{V}(s_i,y)-\widetilde{V}(y,y)\right)g(y)\,\mathrm{d}y, \tag{20}$$

As $\widetilde{V}(s_i,y)$ is increasing in s_i the integrand is positive for $\hat{s}_i<s_i$ and negative for $\hat{s}_i>s_i$ and $g>0$ almost everywhere, the function $U_i(s_i,\hat{s}_i)$ is uniquely maximized at $\hat{s}_i=s_i$. $\square$

B.5 Equity Comparisons in Uniform, Pay-as-bid and Mixed Auctions

Proof of Theorem 1. To prove the "if" direction, note that for $c=1$, the realized value is identical for all bidders $i\in[n]$ as $v(\boldsymbol{s})=\frac{1}{n}\sum_{j\in[n]}s_j$. Thus, with a uniform price that is identical between bidders, they all have identical surplus. For any $\delta>0$, the payment differs between the winners at least for some signal realizations.

To prove the "only if" let $\varphi^\delta(s)=(1-c)\cdot s-\delta\beta^\delta(s)$. We then have $(u(s_i)-u(s_j))^2 = (\varphi^\delta(s_i)-\varphi^\delta(s_j))^2$ for two winning bids s_i,s_j (see the proof of Proposition 11 for details). Thus, it holds that

$$\frac{\partial}{\partial\delta}(\varphi^\delta(s_i)-\varphi^\delta(s_j))^2 = -2(\varphi^\delta(s_i)-\varphi^\delta(s_j))\left(\beta^\delta(s_i)-\beta^\delta(s_j)+\delta\frac{\partial\beta^\delta(s_i)}{\partial\delta}-\delta\frac{\partial\beta^\delta(s_j)}{\partial\delta}\right).$$

Using Lemma 9, we take the limit of β^δ, $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$, and $\varphi^\delta(s)$, as δ goes to 0. We have that $(\varphi^\delta(s_i)-\varphi^\delta(s_j))\to(1-c)(s_i-s_j)$ and $(\beta^\delta(s_i)-\beta^\delta(s_j)+\delta\frac{\partial\beta^\delta(s_i)}{\partial\delta}-\delta\frac{\partial\beta^\delta(s_j)}{\partial\delta})\to(V(s_i)-V(s_j))$. As V is increasing, the product $(V(s_j)-V(s_i))(s_i-s_j)$ is strictly negative almost surely, which concludes the proof. $\square$

Proof of Proposition 4. We show that for any $c\in(c^*,1)$, pay-as-bid pricing does not minimize WEV. From this and the "only if" statement in the proof of Theorem 1, the result follows. Note that WEV is continuous in c and at $c = 1$ it is strictly lower for uniform pricing ($\delta = 0$) than for pay-as-bid pricing ($\delta = 1$) by Theorem 1. Thus, by the mean value theorem, there exists an open interval $C = (c^*, 1)$, $c^* < 1$, such that, for any $c \in C$, WEV remains strictly lower under uniform pricing than pay-as-bid pricing. □

B.6 Challenging the Intuition: Private Values and Uniform Pricing

We have to show that it is indeed possible to construct an *equilibrium* bid function with a slope greater than 2 for a sufficient mass of signals. For this, we require an extreme signal distribution where, broadly speaking, signals are equal to zero with probability ε and equal to one with probability $1 - \varepsilon$. However, to compute a Bayes-Nash equilibrium, we need a continuous signal distribution (with respect to the Lebesgue measure, without mass points) with connected support to solve the first-order condition. Thus, we add a small perturbation.

For Example 2, we consider the order statistics of quantiles $F^{-1}(x)$ and not of signals s. For convenience, we define the following distribution functions and densities.

$$\widetilde{G}(x) := G_k^{n-1}(F^{-1}(x)) = 1 - (1-x)^{n-1} \qquad \widetilde{g}(x) := (n-1)(1-x)^{n-2}$$
$$\widetilde{H}(x) := G_{k-1}^{n-2}(F^{-1}(x)) = 1 - (1-x)^{n-2} \qquad \widetilde{h}(x) := (n-2)(1-x)^{n-3}$$

We choose a continuous distribution of signals, with support $[0, 2]$, where each signal is given by the sum of a Bernoulli(ε) random variable and a random perturbation drawn from Beta$(1, 1/\eta)$, with $\varepsilon = 0.1/n$ and η a small constant. First, we compute the distribution function F and quantile function F^{-1} of the signal distribution. Using the law of total probabilities, we have

$$\begin{aligned} \forall s \in [0,2], \qquad F(s) &= P[\text{Bernoulli}(\varepsilon) + \text{Beta}(1, 1/\eta) \leq s] \\ &= P[\text{Bernoulli}(\varepsilon) = 0] \cdot P[\text{Beta}(1, 1/\eta) \leq s] \\ &+ P[\text{Bernoulli}(\varepsilon) = 1] \cdot P[\text{Beta}(1, 1/\eta) \leq s - 1]. \end{aligned}$$

Simplifying this expression depending on the value of s, we get

$$\forall s \in [0,2], \qquad F(s) = \begin{cases} \varepsilon \cdot (1 - (1-s)^{1/\eta}) & \text{if } s \leq 1, \\ \varepsilon + (1-\varepsilon) \cdot (1 - (2-s)^{1/\eta}) & \text{if } s \geq 1. \end{cases}$$

Then, computing piece-by-piece the inverse of F, we obtain

$$\forall x \in [0,1], \qquad F^{-1}(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x \leq \varepsilon, \\ 2 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^{\eta} & \text{if } x \geq \varepsilon. \end{cases}$$

See Figure 9 for the CDF and quantile function of the signals.

A bidder with quantile $x \in [0, 1]$ bids (truthfully) their signal $F^{-1}(x)$ in the uniform price

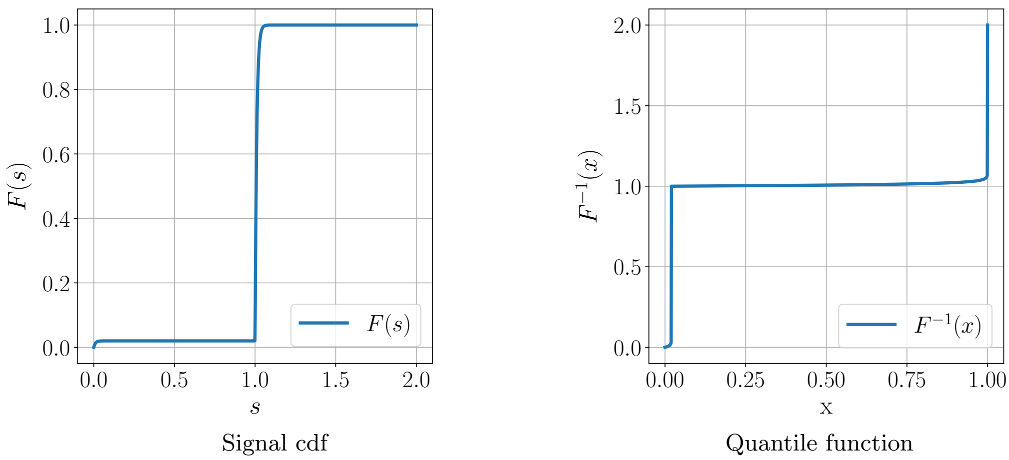

Figure 9: Bidder signals and quantiles for $n = 5$ and $\eta = 0.01$

auction ($\delta = 0$), which we write as $b^0_\eta(x) := F^{-1}(x) = \mathbb{1}\{x \geq \varepsilon\} + \gamma_\eta(x)$, where

$$\forall x \in [0,1], \qquad \gamma_\eta(x) := \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^\eta & \text{if } x < \varepsilon, \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^\eta & \text{if } x \geq \varepsilon. \end{cases}$$

For mixed auctions with $\delta > 0$, the equilibrium bid function is given by Proposition 2. Letting $b^\delta_\eta(x) := \beta^\delta(F^{-1}(x))$ denote the equilibrium bid of a bidder with quantile $x \in [0,1]$, we have

$$b^\delta_\eta(x) = \frac{\int_0^{F^{-1}(x)} V(s) g_k^{n-1}(s) G_k^{n-1}(s)^{\frac{1}{\delta}-1}\,\mathrm{d}s}{\delta G_k^{n-1}(F^{-1}(x))} = \frac{\int_0^x F^{-1}(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(x)},$$

where we used the change of variable $y = F(s)$. Finally, using the additive form of F^{-1} we write the equilibrium bid function as $b^\delta_\eta(x) = b^\delta_0(x) + \xi^\delta_\eta(x)$, where

$$\forall x \in [0,1], \qquad b^\delta_0(x) := \frac{\int_\varepsilon^x \widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(F^{-1}(x))} = \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases}$$

$$\xi^\delta_\eta(x) := \frac{\int_0^x \gamma_\eta(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(x)}$$

See Figure 10 for the bid functions.

Next, we define the function $\phi^\delta_\eta(x) := F^{-1}(x) - \delta b^\delta_\eta(x)$, the utility of a winning bidder as a function of their quantile. Denoting WEV^δ_η the *winners' empirical variance* in a δ-mixed auction with noise level η, we write

$$\forall \delta \in [0,1],\ \forall \eta > 0, \qquad \mathrm{WEV}^\delta_\eta = \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi^\delta_\eta(x_1) - \phi^\delta_\eta(x_2))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right].$$

where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi^\delta_\eta(x)$ converge towards 0 when taking η arbitrarily small, and thus $\phi^\delta_\eta(x)$ converges towards $\phi^\delta_0(x) := \mathbb{1}\{x \geq \varepsilon\} - \delta b^\delta_0(x)$. Therefore,

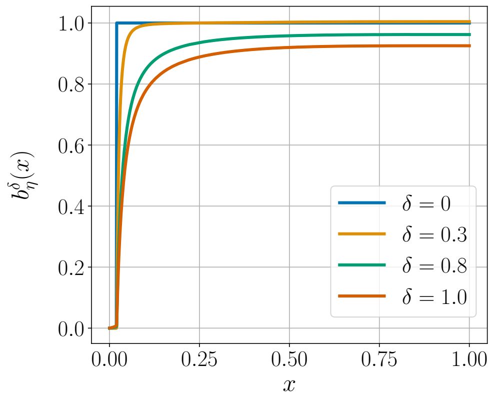

Figure 10: Equilibrium bid as a function of quantiles for $n = 5$ and $\eta = 0.01$

WEV_η^δ converges towards WEV_0^δ, defined by

$$\forall \delta \in [0,1], \quad \mathrm{WEV}_0^\delta := \mathbb{E}_{\mathbf{x}} \left[\frac{((\mathbb{1}\{x_1 \geq \varepsilon\} - \delta b_0^\delta(x_1)) - (\mathbb{1}\{x_2 \geq \varepsilon\} - \delta b_0^\delta(x_2)))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x}) \right]$$
$$= \mathbb{E}_{\mathbf{x}} \left[\frac{(\phi_0^\delta(x_1) - \phi_0^\delta(x_2))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x}) \right] = \lim_{\eta \to 0} \mathrm{WEV}_\eta^\delta.$$

Proof of Proposition 5. We are now equipped to prove the proposition. We write $\mathrm{WEV}_0^\delta = \mathbb{E}_{\mathbf{x}} \left[\phi_0^\delta(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x}) \right] - \mathbb{E}_{\mathbf{x}} \left[\phi_0^\delta(x_1)\phi_0^\delta(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x}) \right]$, with

$$\mathbb{E}_{\mathbf{x}} \left[\phi_0^\delta(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x}) \right] = \frac{n}{n-1} \int_0^1 \phi_0^\delta(x)^2 G(x) \,\mathrm{d}x$$
$$\mathbb{E}_{\mathbf{x}} \left[\phi_0^\delta(x_1)\phi_0^\delta(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x}) \right] = \frac{n}{n-2} \int_0^1 \left(\int_t^1 \phi_0^\delta(x) \,\mathrm{d}x \right)^2 h(t) \,\mathrm{d}t$$

We next compute these quantities for uniform and discriminatory pricing. For uniform pricing ($\delta = 0$) we have that $\phi_0^0(x) = \mathbb{1}\{x \geq \varepsilon\}$. We derive

$$\mathbb{E}_{\mathbf{x}} \left[\phi_0^0(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x}) \right] = \frac{n}{n-1} \int_\varepsilon^1 \widetilde{G}(x) \,\mathrm{d}x = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1}$$
$$\mathbb{E}_{\mathbf{x}} \left[\phi_0^0(x_1)\phi_0^0(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x}) \right] = \frac{n}{n-2} \int_0^\varepsilon (1-\varepsilon)^2 \widetilde{h}(t) \,\mathrm{d}t + \frac{n}{n-2} \int_\varepsilon^1 (1-t)^2 \widetilde{h}(t) \,\mathrm{d}t$$
$$= \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2}$$

and finally

$$\mathrm{WEV}_0^0 = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1} - \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2}$$
$$= \frac{n\left[(1-\varepsilon)^n + (1-\varepsilon)(\varepsilon(n-1) - 1)\right]}{(n-1)(n-2)}$$
$$\leq \frac{(\varepsilon n)^2/2}{n} = \frac{0.005}{n}$$

For discriminatory pricing ($\delta = 1$) we have that $\phi_0^1(x) = \mathbb{1}\{x \geq \varepsilon\} - b_0^1(x) = \mathbb{1}\{x \geq \varepsilon\} \frac{G(\varepsilon)}{G(x)}$. We will use the following bounds:

$$\begin{aligned}
\int_\varepsilon^1 \frac{1}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_\varepsilon^1 \frac{1}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_\varepsilon^1 \sum_{i=0}^{\infty}(1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \sum_{i=0}^{\infty} \frac{(1-\varepsilon)^{(n-1)i+1}}{(n-1)i+1} \geq (1-\varepsilon) + \sum_{i=1}^{\infty} \frac{(1-\varepsilon)^{ni}}{ni} \\
&\geq (1-\varepsilon) + \frac{1}{n}\sum_{i=1}^{\infty} \frac{0.9^i}{i} = 1 - \frac{0.1}{n} - \frac{\ln(0.1)}{n} \geq 1 + \frac{2.2}{n} \\
\int_0^1 \frac{x}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_0^1 \frac{x}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_0^1 \sum_{i=0}^{\infty} x(1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{((n-1)i+1)((n-1)i+2)} \leq \frac{1}{2} + \frac{1.65}{n^2} \qquad \text{(when } n \geq 5\text{)}
\end{aligned}$$

Next, we write

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)^2}{\widetilde{G}(x)}\,\mathrm{d}x \geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1 + \frac{2.2}{n}\right)$$

and

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^\varepsilon \left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&+ \frac{n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&= \underbrace{\frac{n\widetilde{H}(\varepsilon)}{n-2}\left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 + \frac{n}{n-2}\left[\left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 H(t)\right]_\varepsilon^1}_{=0} \\
&+ \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t \\
&= \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_\varepsilon^x \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x
\end{aligned}$$

Next, we will use the upper bound $\widetilde{H}(t)/\widetilde{G}(t) \leq 1$, which is nearly tight as $\widetilde{H}(t)/\widetilde{G}(t)$ is increasing, and has the limit $(n-2)/(n-1)$ when $t \to 0$.

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\int_\varepsilon^1 \frac{x}{\widetilde{G}(x)}\,\mathrm{d}x \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2} + \frac{1.65}{n^2}\right)$$

Finally, we obtain

$$
\begin{aligned}
\mathrm{WEV}_0^1 &\geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1+\frac{2.2}{n}\right)-\frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2}+\frac{1.65}{n^2}\right) && \text{(when } n \geq 5\text{)}\\
&= n\widetilde{G}(\varepsilon)^2\left(\frac{2.2}{n(n-1)}-\frac{1}{(n-1)(n-2)}+\frac{3.3}{n^2(n-2)}\right)\\
&\geq \frac{0.01}{n} && \text{(when } n \geq 4\text{)}
\end{aligned}
$$

□

B.7 Proving the Main Theorems

Proof of Lemma 2. Let $s_i \leq s_j = s_i + \epsilon$ for some $\epsilon > 0$. Then

$$
\begin{array}{rrcl}
\Leftrightarrow & u_i(s_i, \boldsymbol{s}_{-i}) & \leq & u_j(s_j, \boldsymbol{s}_j)\\
\Leftrightarrow & (1-c)s_i - \delta\beta^\delta(s_i) & \leq & (1-c)s_j - \delta\beta^\delta(s_j)\\
\Leftrightarrow & (1-c)(s_j - s_j) & \geq & \delta(\beta^\delta(s_j) - \beta^\delta(s_i))
\end{array}
$$

Dividing by $s_j - s_i$ and taking $\epsilon \to 0$ concludes the proof. □

Proof of Proposition 10. First, we prove that pairwise differences is locally decreasing in δ. Let s_i, s_j with $s_i \geq s_j$ denote the signals of two winning bidders and $\varphi^\delta(s) := (1-c)s - \delta\beta^\delta(s)$. Note that because of Lemma 3 (2), monotone ex-post utility holds for all $\delta \leq \bar{\delta}$. For all δ_1, δ_2, $0 \leq \delta_1 \leq \delta_2 \leq \bar{\delta}$, we have

$$
\begin{aligned}
& |u^{\delta_1}(s_i) - u^{\delta_1}(s_j)| \geq |u^{\delta_2}(s_i) - u^{\delta_2}(s_j)| && (21)\\
\Leftrightarrow\quad & |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \geq |\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_i)| && (22)\\
\Leftrightarrow\quad & -\delta_1\left(\beta^{\delta_1}(s_i) - \beta^{\delta_1}(s_j)\right) \geq -\delta_2\left(\beta^{\delta_2}(s_i) - \beta^{\delta_2}(s_j)\right) && (23)
\end{aligned}
$$

For the final equivalence, observe that monotone ex-post utility together with Lemma 3 (1) implies that $\frac{\delta}{1-c}\beta^\delta$ is non-expansive, allowing to remove the absolute value in Eq. (22). Lemma 3 (2) guarantees that Eq. (23) holds. As the ex-post difference in utilities (Eq. (21)) is decreasing in δ, so is its expectation. To establish global monotonicty on $[0, \bar{\delta}]$, note that if $\bar{\delta}\frac{\partial\beta^{\bar{\delta}}}{\partial s} \leq 1-c$ then it also holds for any $\delta < \bar{\delta}$ by Lemma 3 (2), concluding the proof. □

Proof of Proposition 11. Let $u_i^\delta(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the δ-mixed auction, and let $u_i^U(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price auction. Now let $i, j \in [n]$ be two winning bidders. As above, β^δ (resp. β^U) denotes the symmetric equilibrium bid function in the δ-mixed (resp. uniform price) auction. Let $Y_{k+1}(\boldsymbol{\beta})$ denote the first rejected bid. Then, canceling out $(1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have

$$\begin{aligned}
|u_i^\delta - u_j^\delta| &= |(v_i(s_i,\boldsymbol{s}_{-i}) - \delta\beta^\delta(s_i)) - (v_j(s_j,\boldsymbol{s}_{-j}) - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_i)) - ((1-c)s_j + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i - \delta\beta^\delta(s_i)) - ((1-c)s_j - \delta\beta^\delta(s_j))| \\
&= |\varphi^\delta(s_i) - \varphi^\delta(s_j)|,
\end{aligned}$$

where $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. It also holds that

$$|u_i^U - u_j^U| = |(v_i(s_i,\boldsymbol{s}_{-i}) - Y_{k+1}(\boldsymbol{\beta})) - (v_j(s_j,\boldsymbol{s}_{-j}) - Y_{k+1}(\boldsymbol{\beta}))| = |(1-c)(s_i - s_j)|.$$

We will now show that $\frac{\varphi^\delta}{1-c}$ is a non-expansive mapping. Note that φ^δ can be increasing or decreasing, so we need to show that $|\frac{\partial\varphi^\delta}{\partial s}| \le 1-c$. We have $\frac{\partial\varphi^\delta}{\partial s} = 1-c-\delta\frac{\partial\beta^\delta}{\partial s}$. As β^δ is increasing in s, $|\frac{\partial\varphi^\delta}{\partial s}| \le 1-c$ holds whenever $\delta\frac{\partial\beta^\delta}{\partial s} \le 2(1-c)$. Therefore

$$|u_i^\delta - u_j^\delta| = |\varphi^\delta(v_i) - \varphi^\delta(v_j)| \le |(1-c)(s_i - s_j)| = |u_i^U - u_j^U| \tag{24}$$

Taking the square of Eq. (24) we obtain the result point-wise, for each pair of winning signals s_i and s_j and, taking the expectation, the theorem follows. □

Theorem 4. For a given common value component c, consider two δ-mixed auctions for $\delta_1 \le \delta_2$ and suppose the equilibrium bidding functions β^δ satisfies $\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c)$ for all signals $s \in [0,\bar{s})$. Then, WEV is lower for the δ_2-mixed auction than for the δ_1 one.

Proof. Let $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. We have $u_i^\delta(\mathbf{s}) - u_j^\delta(\mathbf{s}) = \varphi^\delta(s_i) - \varphi^\delta(s_j)$. Let $\delta_1 \le \delta_2$. By the generalized Cauchy mean value Theorem, we have that there exists $\xi \in [s_i, s_j]$ such that

$$|\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_j)|\left|\frac{\partial\varphi^{\delta_1}(\xi)}{\partial s}\right| = |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)|\left|\frac{\partial\varphi^{\delta_2}(\xi)}{\partial s}\right|.$$

Hence if $|\frac{\partial\varphi^{\delta_2}}{\partial s}|/|\frac{\partial\varphi^{\delta_1}}{\partial s}| \le 1$ then we have lower WEV for the δ_2 mixed auction. We have the following chain of equivalences:

$$\begin{aligned}
&\left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right| \le \left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right|, \forall s \in (0,\bar{s}) \\
\iff\quad &\left|(1-c) - \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s}\right| \le \left|(1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}\right|, \forall s \in (0,\bar{s}) \\
\iff\quad &\delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} - (1-c) \le (1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}, \forall s \in (0,\bar{s}) \\
\iff\quad &\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c), \forall s \in (0,\bar{s}),
\end{aligned}$$

where the third equations comes from the monotonicity of $\delta\frac{\partial\beta^\delta}{\partial s}$ in δ from Lemma 3. □

B.8 Proving the Bound on Bid Function Slopes

Proof of Lemma 4. We first rewrite $\tilde{v}(x,y)$ for $c=1$ in terms of all the order-statistics of s_{-i}.

$$\begin{aligned}
\tilde{v}(x,y) &= \mathbb{E}[v(s_i,s_{-i}) \mid s_i = x, Y_k = y] \\
&= \mathbb{E}[\frac{1}{n}\sum_{j\in[n]} s_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n}\sum_{\substack{j\in[n],\\ j\neq i}} s_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n}\sum_{j\in[n-1]} Y_j \mid s_i = x, Y_k = y] && \text{(Ordering the signals)} \\
&= \frac{x}{n} + \frac{y}{n} + \mathbb{E}[\frac{1}{n}\sum_{\substack{j\in[n-1],\\ j\neq k}} Y_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \frac{y}{n} + \frac{1}{n}\sum_{j=1}^{k-1}\mathbb{E}[Y_j \mid s_i = x, Y_k = y] + \frac{1}{n}\sum_{j=k+1}^{n-1}\mathbb{E}[Y_j \mid s_i = x, Y_k = y]
\end{aligned}$$

Note that the previous decomposition is similar to the equilibrium bid in an English auction given that k bidders have dropped out in Goeree and Offerman (2003). However, we offer a careful derivation in the multi-unit setting of our model. We now use Theorem 2.4.1 and Theorem 2.4.2 from Arnold, Balakrishnan and Nagaraja (2008) on the conditional distribution of order statistics. They state that, for $j<k$, the distribution of Y_j given $Y_k = y$ is the same as the distribution of the j-th order statistic of $k-1$ independent samples of the original distribution left-truncated at y, and we denote Z_j^l a random variable drawn according to this distribution. Hence, for $j<k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^l]$. Similarly for $j>k$ we have that the distribution of Y_j given $Y_k = y$ is the same as the distribution of the $j-k$-th order statistic of $n-k-1$ independent samples of the original distribution right-truncated at y, and we denote by Z_j^r a random variable drawn according to this distribution. Hence, for $j>k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^r]$. Notice that summing all order statistics drawn from some samples recovers exactly the sum of original samples. Thus we obtain

$$\sum_{j=1}^{k-1}\mathbb{E}[Y_j \mid s_i = x, Y_k = y] = \sum_{j=1}^{k-1}\mathbb{E}[Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} s_j \mid \forall j\in[k-1], s_j \geq y] = \sum_{j=1}^{k-1}\mathbb{E}[s_j \mid s_j \geq y].$$

The same can be done for the Z_j^r. Finally, the s_j are iid and thus have identical conditional expectations. We obtain

$$\begin{aligned}
\tilde{v}(x,y) &= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\mathbb{E}[s_j \mid s_j \leq y] + \frac{k-1}{n}\mathbb{E}[s_j \mid s_j \geq y] && (25)\\
&= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\frac{\int_0^y t f(t)\,\mathrm{d}t}{F(y)} + \frac{k-1}{n}\frac{\int_y^{\bar{s}} t f(t)\,\mathrm{d}t}{1-F(y)}, && (26)
\end{aligned}$$

which readily yields a formula for $V(s) = \tilde{v}(s,s)$. Clearly, the above function is well defined and differentiable on the open support of F.

We now examine the derivative of $V(s)$ and prove that $V'(s) \leq 1$. First, we consider the derivatives of the two ratios with an integral in the numerator in Eq. (26). First, by integration by parts, we have

$$\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = \frac{[tF(t)]_0^s - \int_0^s F(t)\,\mathrm{d}t}{F(s)} = s - \frac{\int_0^s F(t)\,\mathrm{d}t}{F(s)},$$

and using that for positive random variables $\int_0^{\bar{s}} tf(t)\,\mathrm{d}t = \int_0^{\bar{s}}(1-F(t))\,\mathrm{d}t = \mathbb{E}[s_i] < \infty$, which guarantees convergence of the integral, we have that

$$\begin{aligned}\frac{\int_s^{\bar{s}} tf(t)\,\mathrm{d}t}{1-F(s)} = \frac{\mathbb{E}[s_i] - \int_0^s tf(t)\,\mathrm{d}t}{1-F(s)} &= \frac{\int_0^{\bar{s}}(1-F(t))\,\mathrm{d}t - sF(s) + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)} \\ &= \frac{\int_0^{\bar{s}}(1-F(t))\,\mathrm{d}t + s(1-F(s)) - s + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)} \\ &= s + \frac{\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\end{aligned}$$

Now, taking derivatives, we have

$$\frac{\partial}{\partial s}\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = 1 - \frac{F(s)^2 - f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2} = \frac{f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2}.$$

By a similar argument as in the proof of Lemma 5, using log-concavity of f, the above derivative is bounded by 1. Taking the derivative of the second ratio, we have

$$\frac{\partial}{\partial s}\left(s + \frac{\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\right) = 1 + \frac{-(1-F(s))^2 + f(s)\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2} = \frac{f(s)\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2}. \tag{27}$$

To derivative of $\log(\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t)$:

$$\frac{\partial^2}{(\partial s)^2}\log\Big(\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t\Big) = \frac{\partial}{\partial s}\frac{-(1-F(s))}{\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t} = \frac{f(s)\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t - (1-F(s))^2}{\left(\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t\right)^2}. \tag{28}$$

Eq. (28) is negative iff $f(s)\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2 \leq 1$. This means that the log-concavity of $\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t$ is equivalent to Eq. (27) being smaller than 1. As the log-concavity of $\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t$ follows from the log-concavity of f and $(1-F)$ (Bagnoli and Bergstrom, 2005, Theorem 3), $f(s)\int_s^{\bar{s}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2 \leq 1$ is implied. Finally, using the above derivatives it is clear that $V'(s) > 0$, and

$$V'(s) \leq \frac{2}{n} + \frac{n-k-1}{n}\cdot 1 + \frac{k-1}{n}\cdot 1 = 1.$$

□

Proof of Lemma 5. Let us compute the second derivative of the logarithm of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$:

$$\begin{aligned}
\frac{\partial^2 \log\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)}{(\partial s)^2} &= \frac{\partial}{\partial s}\left(\frac{G^{\frac{1}{\delta}}(s)}{\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}\right)\\
&= \frac{\frac{1}{\delta}g(s)G^{\frac{1}{\delta}-1}(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{2}{\delta}}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\\
&= \frac{G^{\frac{1}{\delta}-1}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\left(\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)\right).
\end{aligned}$$

Notice that the left-hand fraction is always positive. Hence log-concavity of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is equivalent to $\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)$ being negative. The latter is equivalent to

$$1 \geq \frac{g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}+1}(s)} = \frac{\partial \beta^\delta(s)}{\partial s}.$$

□

Corollary 5. For uniformly distributed signals, any pricing dominant in pairwise differences contains a discriminatory proportion of at least $\frac{2n(1-c)}{2n-c(n-2)}$, and for exponentially distributed signals at least $\frac{2n(1-c)}{2n-c(n-(k+1))}$.

Proof of Corollary 5. While $\sup_{[0,\bar{s})}\frac{\partial\beta^\delta}{\partial s}$ can be difficult to compute analytically even for simple distributions, it is sometimes possible to compute $\sup_{[0,\bar{s})}V'(s)$. For the uniform distribution, we have $\sup_{[0,\bar{s})}V'(s) = 1 - c\frac{n-2}{2n}$. Thus, using the same argument as in the proof of Theorem 2, it follows that $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-2)} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. For the exponential distribution, we have $\sup_{[0,\bar{s})}V'(s) = 1 - c(\frac{1}{2} - \frac{k+1}{2n})$, and thus $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-(k+1))} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. □

Proof of Lemma 6. To prove Lemma 6, we will use properties of log-concave distributions from Bagnoli and Bergstrom (2005). Namely their Theorems 1 and 3 state together that log-concavity of a density f implies log-concavity of the corresponding cdf F and of the complementary cdf $1-F$, and that log-concavity of F or $1-F$ imply log-concavity of respectively $\int_0^s F$ or $\int_s^{\bar{s}} F$, where $\bar{s}$ is the upper limit of the support of f (either a constant or $+\infty$). Additionally, we also have that the product of two log-concave functions is log-concave also. Using the above properties, we have that F and $1-F$ are log-concave. Moreover, alternative expression for the order statistics are given, e.g., in Fisz (1965).

$$G_m^n(s) = \frac{n!}{(n-m)!(m-1)!}\int_0^{F(s)} t^{n-m}(1-t)^{m-1}dt$$

and

$$g_m^n(s) = \frac{n!}{(n-m)!(m-1)!}F(s)^{n-m}(1-F(s))^{m-1}f(s). \tag{29}$$

Thus, the order statistics density g, given by Eq. (29), is a product of F, $1-F$, and f, and g as well as the corresponding cdf G are also log-concave. Furthermore, $G^{\frac{1}{\delta}}$ is log-concave because $\log(G^{\frac{1}{\delta}}) = \delta \log(G)$. Finally, we remark that $G^{\frac{1}{\delta}}$ is right-continuous non-decreasing by composition with $x \mapsto x^{\frac{1}{\delta}}$, which is continuous non-decreasing, and $G^{\frac{1}{\delta}}(0) = 0$, as well as $G^{\frac{1}{\delta}}(\bar{s}) = 1$ (if $\bar{s} = \infty$, the equality is understood as a limit). Therefore $G^{\frac{1}{\delta}}$ is a cdf, and applying one last time Bagnoli and Bergstrom (2005), we obtain that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave. □

C Additional Figures

The following two figures illustrate instances in which "super-mixed" pricing (where we allow $\delta > 1$) may improve equity compared to the standard mixed pricing rule. In Fig. 11, we plot the equity-preferred pricing rule δ as a function of the common value c. For pure private values, charging a bidder their own bid plus roughly two times the difference between their bid and the first rejected bid minimizes WEV among all super-mixed pricing rules. In contrast, Fig. 12 shows that in some instances, the equity-preferred pricing rule for pure private values requires an infinite amount of price discrimination (leading to bid functions converging to zero).

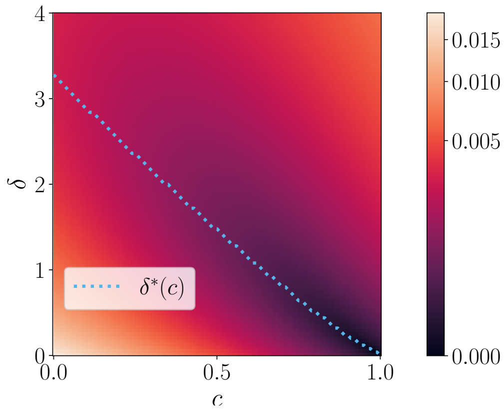

Figure 11: Super-mixed pricing instance with uniform signals and $n = 10$ bidders and $k = 4$ items

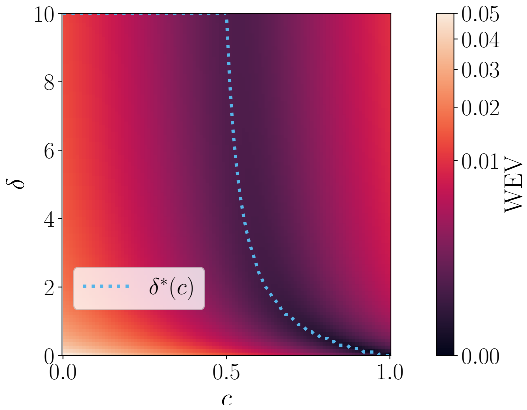

Figure 12: Super-mixed pricing instance with uniform signals and $n = 3$ bidders and $k = 2$ items

In our numerical simulations, we make use of the following lemma.

Lemma 11. *Suppose an auction is a* winners-pay auction. *Then we can write* $E_{\boldsymbol{s}}[u_1 \mid 1 \text{ wins}] = \frac{n}{k} E_{\boldsymbol{s}}[u_1]$, $E_{\boldsymbol{s}}[u_1^2 | 1 \text{ wins}] = \frac{n}{k} E_{\boldsymbol{s}}[u_1^2]$, *and* $E_{\boldsymbol{s}}[u_1 u_2 | 1 \text{ and } 2 \text{ win}] = \frac{n(n-1)}{k(k-1)} E_{\boldsymbol{s}}[u_1 u_2]$.

Proof.

$$\mathbb{E}_s[u_1^2 \mid 1 \text{ and } 2 \text{ win}] = \mathbb{E}_s[u_1^2 \mid 1 \text{ wins}] = \frac{\mathbb{E}[u_1^2]}{\mathbb{P}[1 \text{ wins}]} = \frac{n}{k} \cdot \mathbb{E}[u_1^2] \tag{30}$$

$$\mathbb{E}_s[u_1 u_2 \mid 1 \text{ and } 2 \text{ win}] = \frac{\mathbb{E}[u_1 u_2]}{\mathbb{P}[1 \text{ and } 2 \text{ win}]} = \frac{n(n-1)}{k(k-1)} \cdot \mathbb{E}[u_1 u_2] \tag{31}$$

□

Proof of Proposition 8. We define the probability that i wins $q_i(s_i) := \mathbb{P}_{\boldsymbol{s}_{-i}}[i \text{ wins}]$. Recall that $b^D(s_i)$ denotes the equilibrium bid in the pay-as-bid auction. Consider any standard auction, characterised by a payment rule $(p_1(s), \dots, p_n(s))$. Revenue equivalence implies that

$$q_i(s_i) \cdot b^D(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[b^D(s_i) \cdot \mathbb{1}\{i \text{ wins}\}] = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s)]. \tag{32}$$

In particular, note that if p_i is chosen to be the uniform pricing rule, this formula can be used to compute $b^D(s_i)$. Now define the ex-post surplus $u_i(s_i, \boldsymbol{s}_{-i}) := v(s_i) \cdot \mathbb{1}\{i \text{ wins}\} - p_i(s)$. We write

$$u_i(s_i, \boldsymbol{s}_{-i}) = \underbrace{\mathbb{1}\{i \text{ wins}\} \cdot (v(s_i) - b^D(s_i))}_{u_i^D(s)} + \underbrace{\mathbb{1}\{i \text{ wins}\} \cdot b^D(s_i) - p_i(s)}_{\delta(s)}. \tag{33}$$

Now, observe that by revenue equivalence we have $\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})] = 0$ for all s_i. We write

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 \tag{34}$$

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] + 2\underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})]}_{\geq 0} + \underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})^2]}_{\geq 0} \tag{35}$$

To show that the extra terms are non-negative, notice that $\delta(s_i, \boldsymbol{s}_{-i})^2 \geq 0$, and that

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})] = (\underbrace{v(s_i) - b^D(s_i)}_{\geq 0}) \cdot (\underbrace{q_i(s_i) \cdot b^D(s_i) - \mathbb{E}_{\boldsymbol{s}_{-i}}[\mathbb{1}\{i \text{ wins}\} \cdot p_i(s)]}_{\geq \mathbb{E}[\delta(s)] = 0}) \tag{36}$$

Therefore, putting everything together, we obtain

$$\begin{aligned}
\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] &= \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 && (37)\\
&\geq \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 && (38)\\
&= \mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})] && (39)
\end{aligned}$$

Finally, observe that an auction which minimize the interim variance also minimize the ex-ante variance. Denoting by u_i the utility of a bidder in the pay-as-bid auction, the law of total variance states

$$\mathrm{Var}_s[u_i] = \mathbb{E}_{s_i}[\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]] + \mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]. \tag{40}$$

By the revenue equivalence theorem, we know that $\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]$ is the same for all standard auctions, hence $\mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]$ is also the same for all standard auctions (it only depends on the distribution of the signals).

The interim variance $\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]$ is minimal point-wise (in s_i) for all standard auctions, hence is also minimal in expectation. Therefore, the ex-ante variance is minimal in the pay-as-bid auction among standard auctions. □

Proof of Proposition 9. We derive the equilibrium for the FRB uniform price auction with a common value and given reserve price $r > 0$. Fix signal s, let β be an increasing symmetric equilibrium, and let $s_r = \inf\{s \geq 0 \mid \beta(s_r) \geq r\}$ be the threshold at which β exceeds r. For

$z \geq s_r$, we consider $U(s_i, z)$, the expected payoff of bidding $\beta(z)$ with signal s_i:

$$\begin{aligned} U(s_i, z) &= \int_0^z \widetilde{V}(s_i, y) g(y)\, \mathrm{d}y - \int_0^{s_r} r g(y)\, \mathrm{d}y - \int_{s_r}^z \beta(y) g(y)\, \mathrm{d}y \\ &= \int_0^z \widetilde{V}(s_i, y) g(y)\, \mathrm{d}y - rG(s_r) - \int_{s_r}^z \beta(y) g(y)\, \mathrm{d}y. \end{aligned}$$

If $z < s_r$ then the bid is below the reserve price, no item is won, and $U(s_i, z) = 0$. If the payoff is maximized for $z \geq s_r$, then, by solving the first order condition, a bid of $\widetilde{V}(s_i, s_i) = V(s_i)$ is optimal. Hence, bidding $V(s_i)$ is preferred to bidding zero if the expected payoff is greater than zero. Because $V(s_i)$ is increasing and continuous, these two payoffs are equal for $s_i = s_r$ by definition: s_r corresponds to the threshold signal beyond which a positive bid of $V(s_i)$ is preferred to a zero profit. The equation

$$U(s_r, s_r) = \int_0^{s_r} \widetilde{V}(s_r, y) g(y) - rG(s_r) = 0, \tag{41}$$

implicitly characterizes s_r. The equilibrium bid is $\beta_r^{\delta=0} = V(s_i)$ for $s_i \geq s_r$ and $\beta_r^{\delta=0} = 0$ otherwise. For the pay-as-bid auction, by revenue equivalence, we have that, for $s_i \geq s_r$,

$$\beta_r^{\delta=1}(s_i) = \int_0^{s_r} \frac{rg(y)}{G(s_i)}\, \mathrm{d}y + \int_{s_r}^{s_i} \frac{V(y)g(y)}{G(s_i)}\, \mathrm{d}y = V(s_i) + (r - V(s_r)) \frac{G(s_r)}{G(s_i)} - \int_{s_r}^{s_i} \frac{V'(y)G(y)}{G(s_i)}\, \mathrm{d}y. \tag{42}$$

Taking the derivative yields

$$\begin{aligned} \frac{\partial \beta_r^{\delta=1}(s_i)}{\partial s_i} &= \frac{g(s_i)}{G^2(s_i)} \left((V(s_r) - r)G(s_r) + \int_{s_r}^{s_i} V'(y)G(y)\, \mathrm{d}y \right) \\ &= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} V(y)g(y)\, \mathrm{d}y - rG(S_r) + \int_0^{s_i} V'(y)G(y)\, \mathrm{d}y \right) \\ &= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} V(y)g(y)\, \mathrm{d}y - rG(S_r) + \int_0^{s_i} V'(y)G(y)\, \mathrm{d}y \right) \\ &= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} (\widetilde{V}(y, y) - \widetilde{V}(s_r, y))g(y) + \int_0^{s_i} V'(y)G(y)\, \mathrm{d}y \right) \\ &\leq \frac{g(s_i)}{G^2(s_i)} \int_0^{s_i} V'(y)G(y)\, \mathrm{d}y \\ &= \frac{\partial \beta_{r=0}^{\delta=1}(s_i)}{\partial s}. \end{aligned}$$

We use Eq. (41) for the second-to-last equality, and the fact that $\widetilde{V}(y, y) \leq \widetilde{V}(s_r, y)$ for $y \leq s_r$, by monotonicity of $\widetilde{V}$, for the inequality. □

Proof of Proposition 6. The mechanism is weakly DSIC as the payment for a winner doesn't depend on the winner's report. We compute the realized (ex-post) utility of a winner:

$$u_i(s_i, s_{-i}) = (1-c)s_i + \frac{c}{n}\sum_{j=1}^{n} s_j - (1-c)\left(s_i + (n-1)\mathbb{E}[s_1] - \sum_{j=1}^{n} s_j\right)$$
$$= \frac{c + n(1-c)}{n}\sum_{j=1}^{n} s_j - (1-c)(n-1)\mathbb{E}[s_1]$$

As the latter term only depends on the sum of signals, all winners' ex-post utilities are equal.

We now compute the expected payment of bidder i, using the fact that signals are i.i.d..

$$\mathbb{E}\left[p_i(s_i, s_{-i})\right] = \frac{k}{n}(1-c)\left((n-1)\mathbb{E}[s_1] - \mathbb{E}\left[\sum_{j\neq i} s_j\right]\right)$$
$$= \frac{k}{n}(1-c)\left((n-1)\mathbb{E}[s_1] - (n-1)\mathbb{E}[s_1]\right) = 0$$

As the total expected payment is 0, the seller's expected revenue is 0 (budget balance). Given positive values and zero expected payments, the mechanism is interim individually rational. □

Proof of Proposition 7. First, we compute ex-post utilities. If i wins an item, we have

$$u_i(s_i, s_{-i}) = (1-c)s_i + \frac{c}{n}\sum_{j=1}^{n} s_j - \left((1-c)\left(s_i + \frac{k}{n}(n-1)\mathbb{E}[s_1] - \frac{k}{n}\sum_{j=1}^{n} s_j\right) + \frac{c}{n}\left(1 - \frac{k}{n}\right)\sum_{j=1}^{n} s_j\right)$$
$$= \frac{ck}{n^2}\sum_{j=1}^{n} s_j - (1-c)\frac{k}{n}\left((n-1)\mathbb{E}[s_1] - \sum_{j=1}^{n} s_j\right),$$

This is identical to the utility of a losing bidder and only depends of the sum of signal, equalizing ex-post utilities. Finally, the expected payment given signal s_i and reporting z, when bidders $j \neq i$ truthfully report s_j, is given by

$$\mathbb{E}\left[p(z, s_{-i}) \mid s_i\right]$$
$$= \frac{k}{n}\left((1-c)\left(z + \frac{k}{n}(n-1)\mathbb{E}[s_1] - \frac{k}{n}(n-1)\mathbb{E}[s_1] - \frac{k}{n}z\right) + \frac{c}{n}\left(1 - \frac{k}{n}\right)(n-1)\mathbb{E}[s_1] + \frac{c}{n}\left(1 - \frac{k}{n}\right)z\right)$$
$$+ \left(1 - \frac{k}{n}\right)\left((1-c)\frac{k}{n}\left((n-1)\mathbb{E}[s_1] - (n-1)\mathbb{E}[s_1] - z\right) - \frac{ck}{n^2}(n-1)\mathbb{E}[s_1] - \frac{ck}{n^2}z\right)$$
$$= \frac{k}{n}\left((1-c)\left(1 - \frac{k}{n}\right)z + \frac{c}{n}\left(1 - \frac{k}{n}\right)z + \frac{c}{n}\left(1 - \frac{k}{n}\right)(n-1)\mathbb{E}[s_1]\right)$$
$$+ \left(1 - \frac{k}{n}\right)\left(-\frac{k}{n}(1-c)z - \frac{ck}{n^2}z - \frac{ck}{n^2}(n-1)\mathbb{E}[s_1]\right)$$
$$= \frac{k}{n}z\left((1-c)\left(1 - \frac{k}{n}\right) + \frac{c}{n}\left(1 - \frac{k}{n}\right) - \left(1 - \frac{k}{n}\right)(1-c) - \frac{c}{n}\left(1 - \frac{k}{n}\right)\right) = 0$$

As neither the expected payment or the expected value depend on a bidder's report z, reporting truthfully $z = s_i$ is a Bayes-Nash equilibrium. With an expected payment of zero individual rationality as well as budget balance follow. □