Equitable Pricing in Auctions[*]

Simon Finster[†] Patrick Loiseau[‡] Simon Mauras[§] Mathieu Molina[¶]
Bary Pradelski[‖]

July 10, 2024
(First version: March 12, 2024)

Abstract

How does auction design affect the division of surplus among buyers? We propose a parsimonious measure for equity and apply it to multi-unit auctions, in which unit demand buyers with private-common-values pay mixtures of uniform and pay-as-bid pricing. We show that uniform pricing is equity-optimal if and only if buyers have a pure common value. Surprisingly, however, with pure private values, pay-as-bid pricing may not be optimal, and uniform pricing can achieve higher surplus equity. We turn to the class of log-concave signal distributions and provide bounds on the equity-optimal pricing rule.

JEL codes: D44, D47, D63
Key words: Multi-unit auctions, equity, pay-as-bid, uniform pricing, common values

1 Introduction

Multi-unit auctions are prevalent in various sectors, with significant applications in the public domain, such as the sale of government debt, electricity, and carbon emission certificates, and in the private sector, including auctions for oil, timber, coffee, fractionalized art, and procurement. The study of pricing rules and their impact on efficiency and revenue has a long tradition in both the theoretical and applied economic literature (Wilson 1979, Back & Zender 1993, Armantier & Sbaï 2009, Ausubel et al. 2014, Hortaçsu et al. 2018). Much of this work has focused on revenue

[*]We are grateful for feedback and comments to Julien Combe, Simon Jantschgi, Paul Klemperer, and audiences at the Simons Laufer Mathematical Sciences Institute (Berkeley), CIRM (Marseille), CREST (Paris), and NASMES 2024 (Nashville).
This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 and by the Alfred P. Sloan Foundation under grant G-2021-16778, while Simon Finster and Bary Pradelski were in residence at the Simons Laufer Mathematical Sciences Institute (formerly MSRI) in Berkeley, California, during the Fall 2023 semester. Simon Mauras received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 866132), as a postdoctoral fellow at Tel Aviv University.

[†]CREST-ENSAE and Inria/FairPlay, simon.finster@ensae.fr
[‡]Inria/FairPlay, patrick.loiseau@inria.fr
[§]Inria/FairPlay and Tel-Aviv University, simon.mauras@inria.fr
[¶]Inria/FairPlay, mathieu.molina@inria.fr
[‖]Maison Française d'Oxford,CNRS and Department of Economics, University of Oxford, bary.pradelski@cnrs.fr

and welfare objectives, i.e., the allocation of surplus to the seller and the market as a whole. On the other hand, the distribution of surplus *among buyers*[1] has received little attention. In this article, we provide the first insights in equity-optimal pricing in multi-unit auctions.

In practice, auctions are primarily held when buyers' valuations for goods need to be discovered, and one goal often is, especially in the public sector, an efficient allocation. For example, licenses to build wind farms should go to enterprises that can build and operate those farms the most effectively. The elicitation of values is facilitated by prices, but this may introduce a redistribution of welfare: e.g., in single-price auctions, stronger, high-value bidders may obtain a larger surplus (that is, value minus price) than weaker, low-value bidders. Such redistribution cannot be easily reversed, as a transfer scheme after the auction may distort bidding incentives and prevent an efficient allocation. However, as we argue in this article, a comparatively more equitable distribution of surplus can be achieved in the auction by design.

An equitable allocation of surplus among buyers may be important to create long-term incentives for participation. Moreover, it may influence market stability and promote competition between auction participants, as those disadvantaged in the surplus redistribution may face higher borrowing costs in inefficient capital markets. For example, major financial institutions, acting as primary dealers in high-stake treasury auctions, play an intermediary role between governments and retail consumers.[2] The government may prefer an equitable distribution of surplus between the primary dealers for more effective competition in the retail market. Similarly, electricity wholesale companies engage in spot and forward markets for power generation. Emission certificates, vital inputs for production, are routinely distributed through auctions, impacting the participating companies' ability to produce and compete. Especially in public sector auctions, welfare considerations naturally enter the picture, but even private companies, for example those auctioning web space for online advertisement, may care about equity, contributing to the long-run satisfaction of their clients.

Our contribution

We study surplus considerations among buyers in multi-unit auctions and characterize how pricing affects the surplus distribution.

We consider standard multi-unit auctions for the sale of indivisible, identical goods with a composition of private and common values: part of the good's value is private to a buyer and information about it is only known to them, while another part of the good's value is common to all buyers. For example, an emission certificate may be necessary for a company to complete their production process, and the company derives a certain, privately known value from it. However, the emission certificate can also be resold after the auction, and its resale value is common to all potential buyers in the market. Buyers are assumed to have unit demand and their values linearly interpolate between the extremes of pure private values and pure common value. The latter is defined as the average of all private values.[3] We call the interpolation parameter the

[1] or sellers in a reverse auction

[2] The OECD (2021) finds in a survey that both pay-as-bid and uniform pricing are used: 25 out of 36 countries use pay-as-bid pricing and 21 out of 36 countries use uniform pricing (some use both). The Spanish government uses a hybrid format in which those bidding higher than the weighted average winning bid price (WAP) pay the WAP, while winners bidding below the WAP pay their bid (Álvarez & Mazón 2007).

[3] This model represents an approximate valuation of resale opportunities that is common to all bidders. It has

private-common-value proportion, or simply the *common value*. We study symmetric Bayesian equilibria as is common in the literature due to tractability and its relevance in practice.[4]

We consider a range of pricing formats, where each extremity represents a pricing rule commonly used in practice, namely uniform pricing, where all winners pay the first rejected bid, and pay-as-bid (or discriminatory) pricing, where winners pay the price they bid. For a given δ we call the convex combinations of uniform and pay-as-bid pricing *δ-mixed pricing*. The parameter δ describes the degree of price discrimination: $\delta = 0$ corresponds to uniform pricing and $\delta = 1$ to discriminatory pricing.[5] Note that Spanish treasury auctions are using a pricing format with partial price discrimination, in which only high-price bidders pay a uniform price (cf. Footnote 2).

All considered auctions, under classical assumptions, achieve the same expected revenue and allocate items to the highest value buyers; thus, there are no potential revenue and efficiency trade-offs. This allows us to focus on the study of equity.

To evaluate surplus equity, we introduce dominance in pairwise differences: Auction A is more equitable than auction B, i.e., A dominates B in pairwise differences, if, in equilibrium, all absolute pairwise differences in ex-post utilities of winning bidders are smaller in auction A than in auction B (Definition 1). Dominance in pairwise differences is a strong requirement and lends itself to aggregation by any increasing function, for example, the Gini index or a comparison between the top and bottom deciles.[6] As an aggregator, we choose the *winners' empirical variance (WEV)* of surplus and show that it satisfies the Pigou-Dalton principle (cf. Moulin 2004), that is, monotonicity with respect to transfers from richer to poorer agents. Moreover, WEV is a combined measure of *within-bidder variation* and *across-bidder correlation* of surpluses (for a detailed discussion on surplus equity, cf. Section 2.3).

Our results are as follows. First, we show, without further assumptions, that the uniform pricing rule is dominant in pairwise differences (and thus also minimizes WEV) if and only if the bidders' values are a pure common value (Theorem 1). Consequently, in most real world scenarios, that is, when the goods carry some private value, uniform pricing is not optimal in terms of equity. Next, we ask if mixed pricing can be justified. Indeed, there exists a range of private-common-value proportions such that mixed pricing minimizes WEV (Theorem 2).[7] Considering pure private values, one might be tempted to arrive at a quick conclusion in favor of pay-as-bid pricing. Surprisingly, however, this fails. We prove that, for certain signal distributions and pure private values, uniform pricing achieves lower WEV than pay-as-bid pricing (Proposition 4). By Theorem 1, uniform pricing is not WEV-minimal.

been used in previous work by, e.g., Bikhchandani & Riley (1991), Krishna & Morgan (1997), Klemperer (1998), Bulow & Klemperer (2002), Goeree & Offerman (2003).

[4]Empirical evidence suggests that the symmetric model is relevant in practice: for example, Armantier & Lafhel (2009) find that information between participants in auctions by the Bank of Canada is nearly symmetric, and Hortaçsu et al. (2018) find bidder symmetry in U.S. Treasury short-term securities auctions. Hattori & Takahashi (2022) find symmetry between bidders in Japanese treasury auctions. Bidder asymmetries have been documented in French (Armantier & Sbaï 2006) and Mexican (Cole et al. 2022) treasury auctions.

[5]Identical and closely related versions of mixed pricing have been considered, e.g., by Wang & Zender (2002), Viswanathan & Wang (2002), Armantier & Sbaï (2009), Ruddell et al. (2017), Woodward (2021).

[6]For prominent inequality measures in wealth or income, cf., e.g., Lorenz (1905), Gini (1912, 1921), Pigou (1912), Dalton (1920), Atkinson (1970), Sen & Foster (1973).

[7]We cannot guarantee the existence of mixed pricing dominant in pairwise differences as it only induces a partial ordering. See also further discussion in Section 2.3.

We thus impose a regularity condition on signal distributions, *log-concavity*, and prove an intuitive result: Goods with a higher proportion of private value should be sold with more pay-as-bid pricing with regard to equity concerns. In contrast, if goods have a higher common value, more uniform pricing distributes surplus more equitably. We show that equity-dominant (pairwise differences) pricing requires at least a $(1-c)$ share of price discrimination, where c represents the proportion of common value (Theorem 3). Further, uniform pricing is dominated in pairwise differences by many alternatives: Any level of price discrimination up to $2(1-c)$ is more equitable than uniform pricing (Theorem 4). In addition, we investigate equity in terms of WEV in numerical experiments. For a variety of signal distributions and proportions of common values, we compute the landscape of WEV-minimal mixed pricing, which can be seen to be unique.

Relationship to the literature

Surplus variation in auctions. The analysis of within-agent variation of surplus goes back to Vickrey (1961), who showed that the ex-ante variance of surplus is lower in a single-unit first-price than in a single-unit second-price auction for uniform distributions of private values. More generally, the distribution of equilibrium prices in a second-price auction is a mean-preserving spread of the distribution of equilibrium prices in a first-price auction (Krishna 2009), which implies that the variance of bidder surplus is greater in the second-price than in the first-price auction for any private-value distribution. While the within-agent variation relates to risk attitudes, the study of equity concerns must take into account across-bidder variation of surplus, the subject of our analysis. Crucially, we study how the proportion of common value affects surplus equity.

Common value auctions. The study of common values in auctions goes back to Wilson (1969, 1977). Wilson (1979) showed that share auctions may result in significantly lower revenue, regardless of uniform or discriminatory pricing, than an auction for an indivisible unit. Milgrom & Weber (1982) consider the statistical correlation between bidders' signals and provide revenue comparisons in a more general framework of private, common, and correlated values for single-unit auctions. Klemperer (1998) demonstrates that in ascending common-value auctions, multiple equilibria may exist, and, under revenue considerations, a first-price auction may be better for (almost) common-value goods. We add the new dimension of surplus equity as an objective, making a case for uniform pricing in auctions for predominantly common-value goods.

Multi-unit auctions. Our multi-unit auction model with unit demand bidders is based on the work of Ortega-Reichert (1968) and Krishna (2009). Generalizing their results, we derive explicit formulas for bid functions in the δ-mixed auction with private and common values. Our model of private and common values resembles the one used in Goeree & Offerman (2003) who study the uncertainty about the common value component. Moreover, our work shares the assumption of log-concave signal distributions for some of our results. A series of articles analyzes strategic bidding in multi-unit auctions with multi-unit demand, in which supply is assumed to be continuous and stochastic (cf., e.g., Ausubel et al. 2014, Pycia & Woodward

2018, Woodward 2021). A common finding in these general models, next to the difficulty of explicitly computing equilibrium bidding functions, is the multiplicity of equilibria and varying efficiency and revenue dependent on auction formats and value distributions. By focusing on unit demand, next to allowing the explicit derivation of equilibrium bidding functions, we have that revenue equivalence and efficiency hold across the considered pricing formats. This allows us to focus on equity concerns.

Fair allocation. A comprehensive survey on fair allocation can be found in Thomson (2011), covering the concepts of envy-freeness, equal division, no domination, and many others. Our notion of fairness is orthogonal to envy-freeness. Envy-freeness requires that no agent prefers another agent's allocation (object and price). As identical objects are sold, uniform pricing is the unique pricing scheme that results in envy-freeness among winners, and, with pure private values, it results in envy-freeness among all participants.[8] In contrast, we consider the realization of utilities. This relates to *equal treatment of equals* (cf., e.g., Thomson 2011) in an ex-post view. Although buyers are ex-ante symmetric, our criterion of pairwise differences implies that only buyers with ex-post identical signals should be treated as equal.

In practice, fairness is an important issue in spectrum auctions (cf. Myers 2023). Formats such as the Simultaneous Multiple-Round Auction, the Combinatorial Multiple-Round Auction, or the Combinatorial Clock Auction are used. However, some of these formats have led bidders to pay different prices for identical licenses, which has been perceived as unfair (Myers 2023).[9] Our work challenges this common notion of fairness, as we note the importance of distinguishing between private and common values. With pure private values, our view of equity requires strong buyers (with high values) to pay more than weak buyers (with low values).

Redistribution in markets. Allocative equity in auctions and the welfare generated in the post-auction market are studied in Kasberger (2023), micro-founding the question when an equitable distribution of the auctioned objects themselves is beneficial for consumer welfare in downstream markets. In contrast to our study, they focus on equity in terms of allocation rather than surplus. Redistributive concerns in markets are also discussed from a mechanism design perspective. Akbarpour et al. (2024) consider "non-market" mechanisms which allocate goods at below-market-clearing prices. They characterize when non-market mechanisms are optimal for a designer to allocate a fixed supply of goods of different qualities. Reuter & Groh (2020) study the allocation of a number of goods to unit demand agents. The agents are heterogeneous in their marginal utilities for money and, additionally, the designer has an ex-ante budget constraint. They contrast the utilitarian optimal allocation with the ex-post efficient allocation and consider "within-agent" and "between-agents" redistributive motives, reminiscent of the within-bidder variation and across-bidder variation of surplus in our work. In a similar vein, Dworczak et al. (2021) characterize a mechanism that yields the optimal trade-off between efficiency and redistribution in a buyer-seller market. Our work resembles this strand

[8] With a proportion of common value, depending on the realization of signals, winners may experience the winners' curse and prefer not to have won an item.

[9] As a consequence, in versions of the Combinatorial Clock Auction, core prices are selected in a second stage based on selection criteria that guarantee fairness and stability (cf. e.g., Day & Milgrom 2008, Erdil & Klemperer 2010).

of literature in spirit, but we take a market design, rather than a mechanism design perspective, initiating the study of surplus equity in auctions and providing guidance on how to improve common multi-unit auctions in practice.

Outline

The remainder of the article is organized as follows. In Section 2 we introduce the model, derive equilibrium bidding strategies, and introduce our notion of surplus equity. We state our main results in Section 3 and prove them in Section 4. Section 5 provides a discussion and Section 6 concludes.

2 Setup

2.1 Model

There are n bidders $i \in \{1, ..., n\} =: [n]$ competing for a fixed supply of k identical items $j \in \{1, ..., k\} =: [k]$, where $k < n$. Each bidder only demands one item. Bidder i receives a private signal s_i, which is drawn independently from a positive and bounded or unbounded support; denote its upper limit by $\bar{v}$. Signals are iid with an absolutely continuous probability distribution F with density f. We call $(0, \bar{v})$, i.e., all signals s so that $0 < F(s) < 1$, the open support of F, and we assume that $f > 0$ over $(0, \bar{v})$. We also assume that the signals have a finite second moment $\mathbb{E}[s^2] < \infty$.

For $\boldsymbol{s} := (s_i)_{i \in [n]}$, a collection of iid signals, we denote by $Y_m(\boldsymbol{s})$ the m-th highest value of the collection $\boldsymbol{s}$, e.g., $Y_1(\boldsymbol{s})$ is the maximum and $Y_n(\boldsymbol{s})$ is the minimum, and by $G^n_m(\boldsymbol{s})$ its distribution with corresponding density $g^n_m(\boldsymbol{s})$. G^n_m is given by

$$G^n_m(s) = \sum_{j=0}^{m-1} \binom{n}{j} F(s)^{n-j}(1 - F(s))^j.$$

where each summand is the probability that exactly j signals are above s. An expression for $g^n_m(\boldsymbol{s})$ is given in Appendix B.4. Bidder i's value for an item is given by the valuation function $v(s_i, \boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i} := \{s_j\}_{j \neq i}$. The value of bidder i is symmetric in other bidders' signals.

Assumption 1. Values $v(s_i, \boldsymbol{s}_{-i})$ are given by

$$v(s_i, \boldsymbol{s}_{-i}) = (1 - c)s_i + \frac{c}{n} \sum_{j \in [n]} s_j,$$

where $c \in [0, 1]$ is the proportion of the *common value*.

Our model interpolates between a common value and private values, where the proportion of common value c encodes to what extent the other bidders' signals influence the value of any bidder. In particular, $c = 1$ is the pure common value case and $c = 0$ is the pure private value case.[10]

[10]In an alternative model, the common value might be distributed according to some prior distribution, and the bidders' private signals are drawn conditional on the realization of this common value. The model where the

Auction mechanisms. We consider auctions that are *standard*, i.e., the k highest bids win the items,[11] and *winners pay*, i.e., only winners pay and no more than their bid. An auction is *efficient*, if, in equilibrium, the bidders with the k highest values $v(s_i, \boldsymbol{s}_{-i})$ are allocated the items. Submitting a bid b_i, the price for bidder i in an auction is given by $p_i(\mathbf{b})$, which is symmetric across buyers.[12]

We consider *δ-mixed pricing.* For a given $\delta \in [0, 1]$, each winning bidder i pays $p_i(\mathbf{b}) = \delta b_i + (1 - \delta)Y_{k+1}(\mathbf{b})$. At one boundary, for $\delta = 0$, this resolves to *first-rejected-bid uniform pricing* or short *uniform pricing*, where each winning bidder i pays the $(k + 1)$-th highest bid $Y_{k+1}(\mathbf{b})$. At the other boundary, for $\delta = 1$, this resolves to *pay-as-bid pricing*, where each winning bidder i pays their bid b_i. Finally, if $\delta \in (0, 1)$, we say that the auction and the pricing are *strictly mixed*.

Utilities and expected values. Bidders' utility is quasi-linear, i.e., when winning an item of value v_i and paying price p_i the utility (or surplus) is given by $u_i(\boldsymbol{s}, \mathbf{b}) = v(s_i, \boldsymbol{s}_{-i}) - p_i(\mathbf{b})$. We restrict our attention to symmetric and monotonically increasing bidding strategies $b_i = b(s_i)$. We denote equilibrium bidding strategies by $(\beta_i)_{i\in[n]} = \boldsymbol{\beta}$. As the k highest bids win, receiving an item only depends on bids, thus utilities are a function of signals:

$$u_i(\boldsymbol{s}) = \mathbb{1}[s_i > Y_k(\boldsymbol{s}_{-i})] \cdot (v(s_i, \boldsymbol{s}_{-i}) - p_i(\boldsymbol{s}))\,.$$

Given a buyer with signal s_i, recall that we denote by $Y_k(\boldsymbol{s}_{-i})$ the k-th highest signal among the $n - 1$ other signals. $Y_k(\boldsymbol{s}_{-i})$ has probability distribution G_k^{n-1} and density g_k^{n-1}. We define the expected value given $s_i = x$ and $Y_k(\boldsymbol{s}_{-i}) = y$ as follows:

$$\forall x, y \in [0, 1], \quad \tilde{v}(x, y) := \mathbb{E}_{\boldsymbol{s}}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = x, Y_k(\boldsymbol{s}_{-i}) = y]$$

The expected value is taken over $n - 2$ signals not including the bidder's own signal and the k-th highest among their $n - 1$ opponents. Observe that because $v(s_i, \boldsymbol{s}_{-i})$ is continuous and non-decreasing, $\tilde{v}(x, y)$ is continuous and non-decreasing in x and y.[13] We define $V(y) := \tilde{v}(y, y)$, the expectation of the value of an item conditional on the bidder winning against the relevant competing signal, the k-th highest among its competitors.

Revenue equivalence and efficiency. As shown in Krishna (2009), the mechanisms we consider are revenue equivalent, as we assume iid signals, standard auctions, and symmetric and increasing equilibrium strategies with an expected payment of zero at value zero.[14] We further note that our value function $v(\boldsymbol{s})$ satisfies the *single-crossing* condition as for all $i, j \in [n]$, $i \neq j$, and for all $\boldsymbol{s}$, $\frac{\partial v(s_i, \boldsymbol{s}_{-i})}{\partial s_i} \geq \frac{\partial v(s_j, \boldsymbol{s}_{-j})}{\partial s_i}$. As shown in Krishna (2009), any standard auction, under

common value component is the average of all bidders' signals and the alternative model have identical qualitative characteristics (Goeree & Offerman 2003): (i) the items are valued equally by all bidders in the common value component, and (ii) the winner's curse is present, i.e., winning an item is "bad news', in that the winner's expectation of the item's value was likely too optimistic.

[11] Cf. Krishna (2009).

[12] Symmetry means that $p_i(\mathbf{b}) = p(b_i, \{b_j\}_{j\neq i})$ for some pricing function p.

[13] In fact, it is strictly increasing in x.

[14] Note that in settings where signals are affiliated revenue equivalence fails (Krishna 2009, Chapter 6.5).

any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is efficient.[15]

Given revenue equivalence and efficiency, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs.

2.2 Equilibrium bidding

We now derive the unique Bayes-Nash equilibrium in symmetric and increasing bidding strategies, which forms the center of our analysis of surplus equity. All proofs are given in Appendix B.

Proposition 1 (Krishna 2009)**.** *The equilibrium bidding strategy in the uniform price auction, i.e., the case* $\delta = 0$*, is given by* $\beta^U(s) := \tilde{v}(s, s) = \mathbb{E}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = s, Y_k(\boldsymbol{s}_{-i}) = s]$.

Note that the equilibrium is unique in the class of increasing and symmetric strategies. With pure private values, the equilibrium strategy is weakly dominant (see also Krishna 2009).

Proposition 2. *The unique symmetric equilibrium bidding strategy in the* δ*-mixed-price auction, for* $\delta \in (0, 1]$*, is given by*

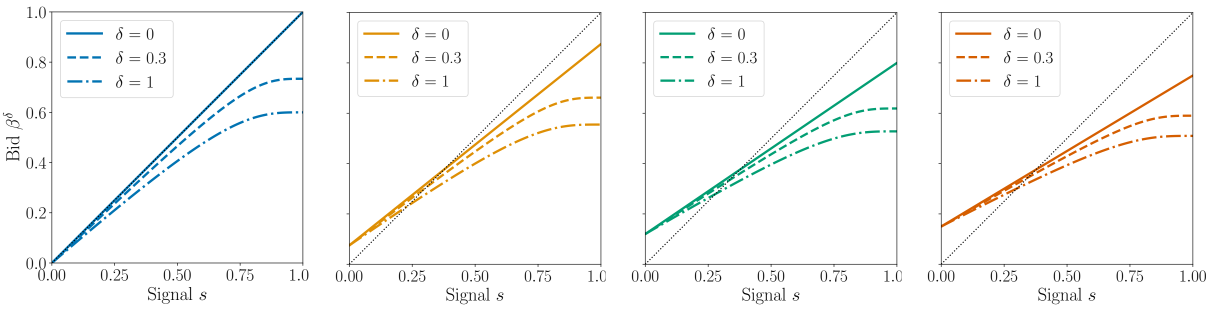
$$\beta^\delta(s) = \frac{\int_0^s V(y) g_k^{n-1}(y) G_k^{n-1}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{1}$$

An alternative representation is given by

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G_k^{n-1}(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{2}$$

We illustrate the bid function in the following example.

Example 1. There are $n = 10$ bidders competing for $k = 4$ items, with signals being sampled uniformly from $[0, 1]$. For the uniform price auction, one can easily compute $\beta^0(s) = V(s) = (1-c)s + c\left(\frac{3}{5}s + \frac{3}{20}\right)$, and accordingly β^δ. Note that β^0 is linear due to uniform signals. Fig. 1 illustrates the bid functions for four different values of c. In the case of pure private values,



$c = 0$ $c = 0.5$ $c = 0.8$ $c = 1$

Figure 1: Equilibrium bid functions, β^δ, for uniform signal distributions as a function of the signal, s, for common value parameters $c \in \{0, 0.5, 0.8, 1\}$.

[15] As detailed in Appendix A, revenue equivalence and efficiency even hold in a more general environment.

bidding truthfully is a dominant strategy in the uniform price auction. Increasing δ increases bid shading in equilibrium. Note that increasing the common value component shifts equilibrium bids for low signal realizations above $\beta^{\delta}(s) = s$. A bidder with a low signal has an expectation of the average signal that is higher than their bid. Equally, for high signals, the bid in the uniform price auction shifts below the 45-degree line. A bidder with a high signal knows that the average signal is lower than their own. With a pure common value, the winner's curse becomes especially apparent. In equilibrium, bidders are attempting to salvage the winner's curse but cannot escape it. In fact, with a pure common value, ex-post utilities are *decreasing in signals* as long as $\delta > 0$.

As the above example shows, the equilibrium bid functions exhibit several monotonicity properties which hold beyond uniform signal distributions. The following monotonicity properties are crucial in our analysis and their proof is given in Appendix B.2. By assumption, equilibrium bids are increasing in the bidder's own signal. For any given signal, they are also decreasing in the extent of price discrimination; naturally, if a higher proportion of one's own bid affects the price paid, the incentive to bid shade increases. The change in the payment corresponding to a bidder's own bid, due to a signal increase, is increasing in the weight of price discrimination, and vice versa.

Proposition 3. *The equilibrium bid functions satisfy the following monotonicity properties:*

1. *$\beta^{\delta}(s)$ is strictly increasing in s, for all fixed $\delta \in [0,1]$ (by assumption), and is strictly decreasing in δ, for all fixed $s \in (0, \bar{v})$.*
2. *$\frac{\partial(\delta\beta^{\delta}(s))}{\partial\delta}$ is strictly increasing in s, for all fixed $\delta \in [0,1]$, and $\frac{\partial(\delta\beta^{\delta}(s))}{\partial s}$ is strictly increasing in δ, for all fixed $s \in (0, \bar{v})$.*

2.3 Surplus equity

In the following we define our equity metric *dominance in pairwise differences.*

Definition 1 (Dominance in pairwise differences)**.** An outcome $\{u_i\}_{i\in[n]}$ dominates ($\succ$) another outcome $\{u'_i\}_{i\in[n]}$ in pairwise differences iff u results in smaller absolute pairwise differences of outcomes than u'. That is, $u \succ u'$ iff, for all $i, j \in [n], i \neq j$, it holds that $|u_i - u_j| \leq |u'_i - u'_j|$ with one inequality strict. We say that, for a family of parameterized outcomes $\{u^{\delta}_i\}_{i\in[n]}$, $\delta \in \Delta$, δ^* is dominant in pairwise differences if u^{δ^*} dominates all outcomes u^{δ}, $\delta \neq \delta^*$, $\delta \in \Delta$.

In the following, we will consider that pairwise dominance holds as long as it holds almost surely. Note that pairwise differences induces a partial dominance ranking over outcomes, and thus, a dominant δ^* may not always exist. It is straightforward to construct two outcomes such that neither dominates the other in pairwise differences.

In our auction setup, utilities are dependent on signals. As all auctions considered allocate the items to the same buyers, in most of our analysis, we focus on winning buyers. Thus, we adapt the definition of pairwise differences as follows: outcome $\{u_i(\boldsymbol{s})\}_{i\in[n]}$ dominates outcomes $\{u'_i(\boldsymbol{s})\}_{i\in[n]}$ in pairwise differences iff, for all winning signals s_i, s_j with opponents' signals $\boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $i \neq j \in [n]$, it holds that $|u_i(s_i, \boldsymbol{s}_{-i}) - u_j(s_j, \boldsymbol{s}_{-j})| \leq |u'_i(s_i, \boldsymbol{s}_{-i}) - u'_j(s_j, \boldsymbol{s}_{-j})|$ with one inequality strict.

We now discuss several prominent equity axioms (cf., e.g., Patty & Penn 2019) in relation to pairwise differences. First, we note that anonymity is maintained. Any reordering of individuals in the population $[n]$ has no consequence, as pairwise comparisons must hold for all individuals. Furthermore, we note that replication invariance and mean independence are not relevant in our setup, as we keep the population size (number of bidders) as well as the endowments (value distributions) fixed. The Pigou-Dalton transfer principle asserts that any transfer from a wealthier agent to a poorer one must reduce inequality, provided the original welfare ranking between the two agents is maintained, that is, the wealthier agent does not become poorer than the previously poorer agent after the transfer (cf. Moulin 2004). Since dominance in pairwise differences does not establish a complete ordering of outcomes, some Pigou-Dalton transfers may result in an increase in some pairwise differences while others decrease.

However, from pairwise differences, various equity metrics, which regulators might be interested in, can be constructed using an aggregator function. For example, the top decile of utilities can be compared to the lowest realized utility or to the bottom decile of utilities. A related aggregation is given by Feldman & Kirman (1974), who aggregate positive pairwise differences for a measure of envy per player.[16] Moreover, "classic" inequity measures such as the Gini coefficient can be constructed from pairwise differences.[17] Furthermore, one might wish to prioritize larger differences over smaller ones by, for instance, squaring each pairwise difference.

Adopting an ex-post perspective to compare utilities in pairs, our results allow the classification of δ-mixed pricing rules based on pairwise differences and *any increasing function* of pairwise differences. Furthermore, we propose a natural function to aggregate pairwise differences, the *expected empirical variance of surplus among the winners*, or *winners' empirical variance (WEV)* for short. This metric is defined in expectation, ensuring that it provides a ranking of pricing formats regardless of signal realizations. Furthermore, the aggregated sum of squared pairwise differences of utilities ensures compliance with the Pigou-Dalton transfer principle. Finally, the empirical variance is linked to surplus variance and correlation of surpluses among bidders.

Definition 2 (Winners' empirical variance)**.**

$$WEV \;= E_s \left[\frac{1}{k(k-1)} \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(u_i(\boldsymbol{s}) - u_j(\boldsymbol{s}))^2}{2} \middle| s_1, \dots, s_k > Y_{k+1}(\boldsymbol{s}) \right]$$

The empirical variance among all bidders (thus including losers) in the auction is given by $\mathrm{EV} = E_{\boldsymbol{s}}[\frac{1}{n(n-1)} \sum_{i=1}^{n} (u_i(\boldsymbol{s}) - u_j(\boldsymbol{s}))^2]$. The ex-ante variance of a winner is $\mathrm{Var}_{\boldsymbol{s}}[u_i(\boldsymbol{s})|i \text{ wins}]$. Using symmetry between bidders, an alternative expression for the ex-ante variance is given by $E_{\boldsymbol{s}}[u_1^2|i \text{ wins}] - E_{\boldsymbol{s}}[u_1|i \text{ wins}]^2$, and analogous definitions hold when not conditioning on winning an item.[18]

Observation 1. The *winners' empirical variance* satisfies the *Pigou-Dalton principle.*

[16] Contrasting our measure, Feldman & Kirman (1974) consider pairwise differences of hypothetical (if an agent had received another agent's bundle) and realized utilities.

[17] In our setting with uncertainty about signal realizations, one could define the expected Gini coefficient among winners $G = \frac{1}{2n^2 E_s[u_1(\boldsymbol{s})|1 \text{ wins}]} E_s[\sum_{i=1}^{k} \sum_{j=1}^{k} |u_i(\boldsymbol{s}) - u_j(\boldsymbol{s})| \mid s_1, \dots, s_k > Y_{k+1}(\boldsymbol{s})]$. A small distinction is the normalization by the expected surplus.

[18] Note that the notions that concern only winning bidders can only be sensibly defined for *winners pay auctions*.

Proof. Without loss of generality, consider an outcome profile u with three outcomes, u_i, u_j and U, where $u_i > u_j$, and U is arbitrary. Induce a Pigou-Dalton transfer $t > 0$ such that $u_i' = u_i - t > u_j$ and $u_j' = u_j + t < u_i$, and U remains the same. The outcome profile after the transfer is denoted u'. We show that the ranking between u and u' according to WEV coincides with what the Pigou-Dalton principle requires, namely it must be that $WEV(u') < WEV(u)$. Let $W := (u_i - U)^2 + (u_j - U)^2$. Then

$$
\begin{aligned}
& (u_i' - U)^2 + (u_j' - U)^2 \\
= & (u_i - t - U)^2 + (u_j + t - U)^2 \\
= & (u_i - U)^2 - 2t(u_i - U) + t^2 + (u_j - U)^2 + 2t(u_j - U) + t^2 \\
= & W + 2t\,(t - u_i + U + u_j - U) \\
= & W + 2t\,(u_j - (u_i - t)) \\
< & W
\end{aligned}
$$

The final inequality follows by the assumption that the transfer does not make i poorer than j was to start with. As U was arbitrarily chosen and, to compute WEV, expectations are taken around the sum of squared differences of the realized utilities, the result follows. □

In expectation, equilibrium surplus varies due to different factors: a bidder's own and their competitors' signals, and surplus between winners may be correlated. As we consider efficient auctions, surplus only varies among the winners in the auctions. Among those winners, WEV measures variation and correlation of surplus, i.e., it measures surplus dispersion *across* bidders. In contrast, the ex-ante variance measures only surplus variation *within* a given bidder, and is more adequate to measure risk across a series of identical, repeated auctions, in which a given bidder redraws their signal in every auction.

Lemma 1. *The empirical variance can be written as* $EV = Var[u_1] - \mathrm{Cov}[u_1, u_2]$*, and the winners' empirical variance as* $WEV = Var[u_1|1 \text{ wins}] - \mathrm{Cov}[u_1, u_2|1 \text{ and } 2 \text{ win}]$.

Lemma 1 establishes that the empirical variance measures variation within bidder and the correlation between surpluses. A related expression involves the expected second moment of surplus and is given in the proof in Appendix B.1. Finally, we note that, with pure private values and thus ex-post individual rationality, a ranking of auction formats in terms of ex-ante variance or winners' ex-ante variance is identical. A ranking with respect to the empirical variance, however, may differ depending on if only winners are considered, or all bidders. A formal lemma and proof are given in Appendix B.1.

Example 1 (Continued)**.** In the example with uniformly distributed signals, $n = 10$ bidders, and $k = 4$ items, we can compute WEV numerically. We illustrate this for four values of the common-value proportion c in Fig. 2 below. Note that for pure private and intermediate common values, the pay-as-bid auction minimizes WEV. For $c = 0.8$, we observe an interior optimum, and for a pure common value, uniform pricing minimizes WEV.

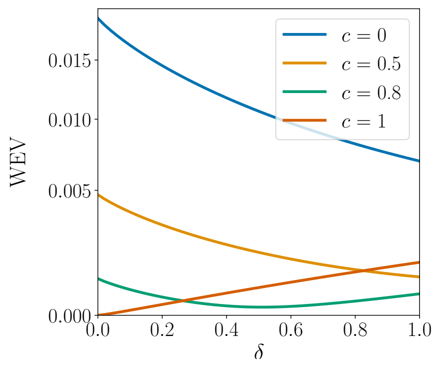


Figure 2: WEV as a function of δ for uniform signals and various common value proportions c

3 Equity optimal pricing

The equity-optimal pricing rule in terms of dominance in pairwise differences crucially depends on the extent of the common value, c. As seen for WEV in Example 1 (Fig. 2), with uniform signals, for some values of c, strictly mixed pricing is optimal. We formalize this fact in Section 3.1 for any signal distributions. Example 1 is in line with the general intuition that pay-as-bid pricing may be more equitable with higher private values, and uniform pricing with higher common values. However, as we show in Example 2, this is not true in general for any signal distribution. Thus, additional distributional assumptions are necessary to arrive at some characterization. In Section 3.3, we consider log-concave signal distributions and provide very simple bounds on the pricing rule that is dominant in pairwise differences.

3.1 Equity comparisons in general mixed auctions

We first consider the case of a pure common value ($c = 1$). As every bidder has the same ex-post realized value, ex-post utilities among winners are equalized if everyone pays the same price. This results in pairwise differences in utilities of zero. Once the private value component enters the value function with a non-zero weight, the picture is less clear: it may be pay-as-bid pricing that is dominant in pairwise differences, or it may be some degree of mixed pricing; however, it cannot be uniform pricing.

Theorem 1. The uniform price auction is dominant in pairwise differences iff the common value proportion equals one (pure common value).

Proof. To prove the "if" direction, note that for $c = 1$, the realized value is identical for all bidders $i \in [n]$ as $v(\boldsymbol{s}) = \frac{1}{n}\sum_{j\in[n]} s_j$. Thus, with a uniform price that is identical between bidders, they all have identical surplus. For any $\delta > 0$, the payment differs between the winners at least for some signal realizations.

To prove the "only if" let $\varphi^\delta(s) = (1-c)\cdot s - \delta\beta^\delta(s)$. We then have $(u(s_i) - u(s_j))^2 = (\varphi^\delta(s_i) - \varphi^\delta(s_j))^2$ for two winning bids s_i, s_j (see the proof of Proposition 7 for details). Thus, it holds that

$$\frac{\partial}{\partial\delta}(\varphi^\delta(s_i) - \varphi^\delta(s_j))^2 = -2(\varphi^\delta(s_i) - \varphi^\delta(s_j))\left(\beta^\delta(s_i) - \beta^\delta(s_j) + \delta\frac{\partial\beta^\delta(s_i)}{\partial\delta} - \delta\frac{\partial\beta^\delta(s_j)}{\partial\delta}\right).$$

Using Lemma 8, we take the limit of β^δ, $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$, and $\varphi^\delta(s)$, as δ goes to 0. We have that $(\varphi^\delta(s_i)-\varphi^\delta(s_j)) \to (1-c)(s_i-s_j)$ and $(\beta^\delta(s_i)-\beta^\delta(s_j)+\delta\frac{\partial\beta^\delta(s_i)}{\partial\delta}-\delta\frac{\partial\beta^\delta(s_j)}{\partial\delta}) \to (V(s_i)-V(s_j))$. As V is increasing, the product $(V(s_j)-V(s_i))(s_i-s_j)$ is strictly negative almost surely, which concludes the proof. □

Furthermore, we show that, without any further assumptions, strictly interior δ-mixed pricing minimizes WEV for a range of private-common-value proportions.

Theorem 2. For any signals distribution, there exists $c^* < 1$, such that for private-common-value proportions in the interval $(c^*, 1)$, there exist δ-mixed auctions with lower WEV than pay-as-bid and uniform auctions.

Proof. We show that for any $c \in (c^*, 1)$, pay-as-bid pricing does not minimize WEV. From this and the "only if" statement in the proof of Theorem 1, the result follows. Note that WEV is continuous in c and at $c = 1$ it is strictly lower for uniform pricing ($\delta = 0$) than for pay-as-bid pricing ($\delta = 1$) by Theorem 1. Thus, by the mean value theorem, there exists an open interval $C = (c^*, 1)$, $c^* < 1$, such that, for any $c \in C$, WEV remains strictly lower under uniform pricing than pay-as-bid pricing. □

The intuitive notion that uniform pricing equitably distributes surplus under a pure common value may lead us to assume that pay-as-bid auctions achieve maximal equity under private values. However, in the following section, we demonstrate a scenario where it fails. We present a detailed example to prove that uniform pricing can result in lower WEV than pay-as-bid pricing with pure private values.

3.2 Challenging the intuition: pay-as-bid pricing with pure private values

To understand the reversal of the intuition, consider pairwise differences in utility, the building block for WEV. If ex-post absolute differences in utility are greater under uniform pricing than under pay-as-bid pricing for signal pairs with sufficient probability mass, then the reversal may also hold in expectation. To start with, consider any two winning signals $s_i > s_j$, $s_i, s_j \in [0, \bar{v})$ and private values only, i.e., $c = 0$. Let $u_i^0(s_i, \boldsymbol{s}_{-i})$ and $u_i^1(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price and pay-as-bid auction, respectively. Moreover, β^0 and β^1 denote the corresponding symmetric equilibrium bid functions and $Y_{k+1}(\boldsymbol{\beta})$ the first rejected bid. For $\delta \in [0,1]$ and $c = 0$, we have $u_i^\delta(s_i, \boldsymbol{s}_{-i}) = s_i - \delta\beta^\delta(s_i) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$. Thus, we have $\Delta u^0 := |u_i^0 - u_j^0| = |s_i - s_j|$ and $\Delta u^1 := |u_i^1 - u_j^1| = |s_i - \beta^1(s_i) - (s_j - \beta^1(s_j))| = |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))|$. It holds that

$$\Delta U^0 < \Delta U^1 \tag{3}$$

$$\Leftrightarrow s_i - s_j < |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))| \tag{4}$$

$$\Rightarrow 2(s_i - s_j) < \beta^1(s_i) - \beta^1(s_j) \tag{5}$$

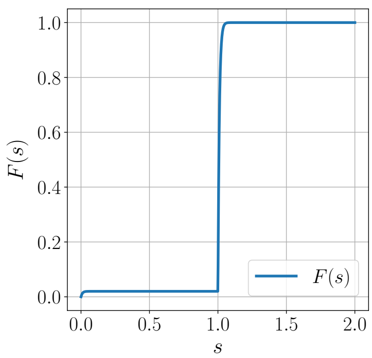
As bid functions are increasing, if $s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))$ was positive, Eq. (4) could never hold. Thus, Eq. (5) follows as a necessary condition for uniform pricing to have lower pairwise differences than pay-as-bid pricing. For the same statement to hold for WEV, it must be that the bid function has a slope of at least 2 for a sufficient mass of signals s_i and s_j. Bid function slopes greater than 2 imply that high-signal bidders shade their bids much less, proportionally to their value, than lower-signal bidders. Consequently, the differential in ex-post surplus with pay-as-bid pricing, comparing two sufficiently different signals, are higher than the differential in signals. The latter equals the surplus difference in the uniform price auction.

With this intuition, we now prove that it is indeed possible to construct an *equilibrium* bid function with a slope greater than 2 for a sufficient mass of signals. For this, we require an extreme signal distribution where, broadly speaking, signals are equal to zero with probability ε and equal to one with probability $1-\varepsilon$. However, to compute a Bayes-Nash equilibrium, we need a continuous signal distribution (with respect to the Lebesgue measure, without mass points) with connected support (to solve the first-order condition). Thus, we add a small perturbation between 0 and 1.

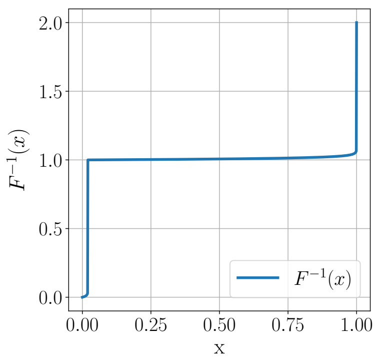
Example 2. Consider an auction with n bidders and $k = n-1$ items. Each bidder i has a pure private value ($c = 0$) given by its signal s_i. The signal is equal to the sum of a Bernoulli random variable with parameter $\varepsilon > 0$ and a random perturbation drawn from $\mathrm{Beta}(1, 1/\eta)$, with $\eta > 0$. The resulting signal distribution is continuous, with support $[0, 2]$. We formally state the signal cdf F in Appendix B.3 and give the quantile function below. The example builds on the quantile function throughout, which simplifies the analysis. This is without loss of generality, since signals can be mapped one-to-one to quantiles.

$$\forall x \in [0,1], \qquad F^{-1}(x) = \mathbb{1}[x \geq \varepsilon] + \gamma_\eta(x) \qquad \text{where} \qquad \gamma_\eta(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^\eta & \text{if } x < \varepsilon \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^\eta & \text{if } x \geq \varepsilon \end{cases}$$

We fix $\varepsilon = 0.1/n$ and choose $\eta > 0$ to be an arbitrarily small constant. We consider the order



Signal cdf



Quantile function

Figure 3: Bidder signals and quantiles for $n = 5$ and $\eta = 0.01$

statistics with respect to the quantiles,[19] and we denote by $\widetilde{G}$ (resp. $\widetilde{g}$) the distribution function (resp. density) of the k-th highest quantile among $n-1$ buyers. We plot the signal cdf and the quantile function in Fig. 3.

Applying the formula from Proposition 2, we can derive the equilibrium bid $b^\delta_\eta(x)$ of a bidder with quantile x (recall that equilibrium bids as functions of signals are denoted by β^δ). We state

[19] $\widetilde{G}$ and $\widetilde{g}$ correspond to the definitions of G^{n-1}_k and g^{n-1}_k with uniform signals, where $F = F^{-1}$.

the bid function below and illustrate it in Fig. 4. For details, we refer to Appendix B.3. For $\delta = 0$ we have that $b^0_\eta(x) = \beta^0(F^{-1}(x)) = \mathbb{1}[x \geq \varepsilon] + \gamma_\eta(x)$; and for all $\delta > 0$ we have

$$\forall x \in [0,1], \qquad b^\delta_\eta(x) := \beta^\delta(F^{-1}(x)) = b^\delta_0(x) + \xi^\delta_\eta(x)$$

$$\text{where} \qquad b^\delta_0(x) := \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{G(\varepsilon)}{G(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases} \qquad \text{and} \qquad \xi^\delta_\eta(x) \; := \frac{\int_0^x \gamma_\eta(y) g(y) G(y)^{\frac{1}{\delta}-1} \, \mathrm{d}y}{\delta G(x)}.$$

Next, we define the function $\phi^\delta_\eta(x) := F^{-1}(x) - \delta b^\delta_\eta(x)$, denoting the utility of a bidder

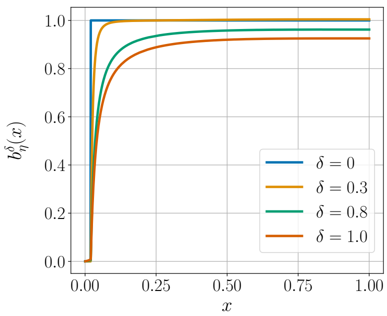


Figure 4: Equilibrium bid as a function of quantiles for $n = 5$ and $\eta = 0.01$

as a function of their quantile, without the uniform payment which cancels out in the computation of WEV. Finally, we define WEV^δ_η, the winners' empirical variance in a δ-mixed auction with noise level η. For all $\delta \in [0,1]$ and for all $\eta > 0$, we have that $2 \cdot \mathrm{WEV}^\delta_\eta = \mathbb{E}_{\mathbf{x}}\left[(\phi^\delta_\eta(x_1) - \phi^\delta_\eta(x_2))^2 \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi^\delta_\eta(x)$ converge towards 0 when taking η arbitrarily small. Therefore, WEV^δ_η converges towards WEV^δ_0, formally defined in Appendix B.3.

We constructed the above example such that uniform pricing achieves lower WEV than pay-as-bid pricing, even with pure private values. The result is stated in the proposition below and holds in the limit as $\eta \to 0$, for any number of bidder $n \geq 5$.

Proposition 4. *For $n \geq 5$, there exists η^*, such that for all $\eta \leq \eta^*$ it holds that the* winners' empirical variance *under uniform pricing is lower than under pay-as-bid pricing.*

The proof is given in Appendix B.3, and shows that $\mathrm{WEV}^0_0 \leq \frac{0.005}{n}$ and $\mathrm{WEV}^1_0 \geq \frac{0.01}{n}$.

Thus, in order to characterize equity-optimal pricing further, we need additional assumptions. In the next section, we show that, for a large class of signal distributions, simple bounds tell us which pricing rules are candidates for being dominant in pairwise differences, and which pricings are dominated.

3.3 Equity-optimal pricing for log-concave signal distributions

For the class of *log-concave* signal distributions, simple bounds characterize a pricing rule that is dominant in pairwise differences.

Definition 3. A real-valued function $h \in \mathbb{R}^\mathbb{R}$ is *log-concave* if $\log(h)$ is concave.

The family of log-concave distributions contains many common distributions, for example uniform, normal, exponential, logistic or Laplace distributions (Bagnoli & Bergstrom 2005).[20]

The proofs of the following two theorems are given in Section 4.

Theorem 3. Assume that signals are drawn from a log-concave distribution. Then, for a given common value component c, there exists a pricing rule which dominates in pairwise differences any pricing rule with a discriminatory proportion of less than $1 - c$.

Theorem 3 provides a lower bound on the amount of price discrimination required to rule out dominated pricing formats. The following theorem states that uniform pricing is dominated in pairwise differences by many alternative pricing rules.

Theorem 4. Assume signals are drawn from a log-concave distribution. Then the uniform price auction is dominated in pairwise differences by any mixed pricing with price discrimination of up to $2(1 - c)$.

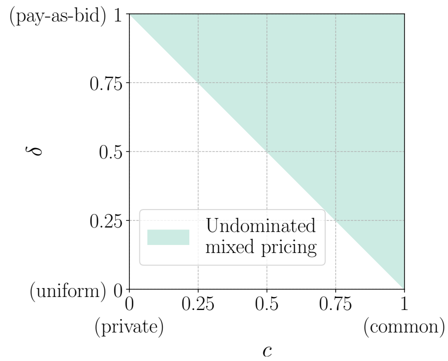


Figure 5: Bounds on equity-optimal combinations of c and δ

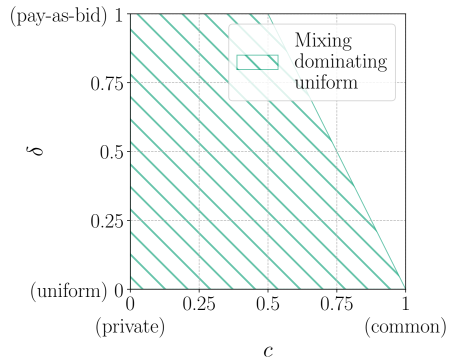


Figure 6: Range of price discrimination dominating uniform pricing

We illustrate Theorem 3 in Fig. 5. The pricing rule that is dominant in pairwise differences must lie within the shaded green area, given any log-concave distribution of bidders' signals. In particular, $\delta = 1 - c$ dominates all pricing rules with $\delta < 1 - c$. Similarly, Fig. 6 illustrates Theorem 4. Any pricing rule within the hatched green area dominates uniform pricing.

The intuition behind Theorem 3 is simple. As we show in Section 4 below, pairwise differences are, for any given common value c, monotonically decreasing in the extent of price discrimination δ as long as δ is between zero and $1 - c$. Moreover, we show the equivalence of this result with ex-post utilities that increase in signals. As long as higher signals obtain a higher surplus, more equity can be achieved by taxing higher signals more than lower signal. Because the change in the δ-weighted bid in δ is increasing in a bidder's signal (as stated in Proposition 3), increasing the extent of price discrimination will have the desired effect.

A similar intuition explains Theorem 4, where the benefit of higher price discrimination compared to the absence of price discrimination can be realized up to a certain threshold. As long as utilities are increasing in signals, increasing price discrimination results in surplus taxation that benefits equity (cf. Theorem 3). We show in Section 4 that increasing ex-post utilities is

[20] Also χ distribution with degrees of freedom ≥ 1, gamma with shape parameter ≥ 1, χ^2 distribution with degree of freedom ≥ 2, beta with both shape parameters ≥ 1, Weibull with shape parameter ≥ 1, and others.

equivalent to the slope of equilibrium bid functions being bounded $(1-c)/\delta$. With steeper bid functions, the utilities might decrease in the signals. So, while increasing price discrimination might locally, in a neighborhood of δ, increase pairwise differences, price discrimination is still beneficial compared to uniform pricing. However, for $\delta \geq 2(1-c)$, the bid functions are so steep that an increase in price discrimination results in an absolute utility gap between a high signal and a low signal bidder that is greater than under uniform pricing. With such price discrimination, the higher signal bidder is worse off than the low signal bidder.

With Theorems 3 and 4, we can now revisit the question: In terms of equity, should one use pay-as-bid pricing if bidders' values are pure private values? The answer is yes, if the signal distributions are log-concave. Moreoever, if the common value is small, pay-as-bid pricing is guaranteed to be more equitable than uniform pricing. We state this formally in the corollary below.

Corollary 1. Assume signals are drawn from a log-concave distribution. Then, for pure private values, pay-as-bid pricing is dominant in pairwise differences, and for a private-common-value proportion $c < \frac{1}{2}$, pay-as-bid pricing dominates uniform pricing in pairwise differences.

The first part of the corollary follows by setting $c = 0$ in Theorem 3. The second part follows by setting $\delta = 1$ in Theorem 4. Our numerical experiments in Section 5.1 show that, for $c < \frac{1}{2}$, pay-as-bid pricing in fact minimizes WEV for several common distributions. The intuition in the pure private value case carries through under the qualifying assumption of log-concave signals, and it may fail for very concentrated signal distributions. In the latter case, it is important that sufficient probability mass is gathered around higher signals, inducing a bidding equilibrium in which ex-post utilities are decreasing in signals for sufficiently many signal realizations.[21]

For specific signal distributions, we can extend the region where pairwise differences are monotonically decreasing slightly beyond the diagonal $1-c$, as exemplified in the following proposition. The proof is given in Appendix B.5.[22]

Proposition 5. *For uniformly distributed signals, any pricing dominant in pairwise differences contains a discriminatory proportion of at least* $\frac{2n(1-c)}{2n-c(n-2)}$*, and for exponentially distributed signals at least* $\frac{2n(1-c)}{2n-c(n-(k+1))}$.

Note that both bounds converge to $\frac{1-c}{1-c/2}$ as the number of bidders goes to infinity (and the number of items k is kept constant). We illustrate the bound for the uniform distribution in Fig. 7 below, together with the equity-optimal pricing in terms of WEV. The figure demonstrates that, for high values of c, this bound may be a good heuristic for the optimal pricing rule.

4 Structural insights and proofs of Theorems 3 and 4

In this section, we provide an overview of the proofs of Theorems 3 and 4. Each of these theorems is proved by combining a proposition on monotonicity of pairwise differences and dominance of pairwise differences, respectively, with a third proposition that bounds the slope of bid functions.

[21]For example, with a β-distribution as steep as illustrated in Fig. 9, Section 5, clearly violating log-concavity, pay-as-bid pricing is still optimal for a range of private-common-value proportions including pure private values.
[22]The proof should be read in conjunction with Theorem 3 as it follows a similar reasoning.

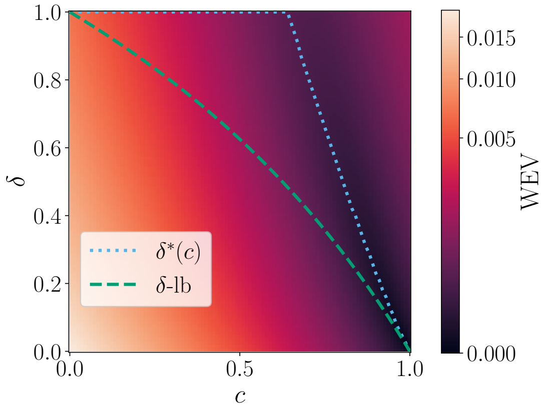


Figure 7: A lower bound for pricing candidates dominant in pairwise differencesand the equity-optimal pricing in terms of WEV

In particular, we identify the property of *monotone ex-post utility* as a fundamental and sufficient condition for our dominance results.

Definition 4 (Monotone ex-post utility)**.** The ex-post utility $u(\boldsymbol{s})$ satisfies *monotone ex-post utility (MEU)* iff, for any two signals $s_i, s_j \in [0, \bar{v})$ and $\forall\ \boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $s_i \leq s_j \Leftrightarrow u_i(s_i, \boldsymbol{s}_{-i}) \leq u_j(s_j, \boldsymbol{s}_{-j})$.

Monotone ex-post utility (MEU) relates to the slope of equilibrium bid functions by the following lemma. The proof is given in Appendix B.4.

Lemma 2. *An equilibrium satisfies monotone ex-post utility iff equilibrium bid functions β^δ satisfy $\frac{\partial \beta^\delta}{\partial s} \leq \frac{1-c}{\delta}$ for all signals $s \in [0, \bar{v})$.*

The ex-post difference in utilities depends only on the private value proportion $(1-c)s$ and the discriminatory part of the payment $\delta\beta^\delta$. Thus, as long as the discriminatory payment does not grow faster in the signal than the private-value share, ex-post utilities are monotone.

We now characterize the fundamental role of monotone ex-post utility: it is equivalent to the monotonicity property of pairwise differences.

Proposition 6. *For a given common value c and for some $\bar{\delta} \in [0, 1]$, pairwise differences are monotonically decreasing over $[0, \bar{\delta}]$ if and only if the equilibrium (which depends on c and $\bar{\delta}$ satisfies MEU.*

The proof is given in Appendix B.4.

The equivalence between decreasing pairwise differences and MEU being satisfied in equilibrium is crucial in our proof of Theorem 3. When MEU holds, the slope of the equilibrium bid function is sufficiently flat and more price discrimination impacts higher signal bidders more than lower signal bidders. In contrast, including more uniform pricing in the price mix will, proportionally to the change in δ, offer higher signal bidders a higher discount than lower signal bidders and thus does not improve surplus equity. A similar intuition holds for Proposition 7 below (for details on the intuition, see Section 3.3). The proof is given in Appendix B.4.

Proposition 7. *For a given common value c, consider any δ-mixed auction, $\delta \in (0,1]$, and suppose the equilibrium bidding function β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2(1-c)}{\delta}$ for all signals $s \in [0,\bar{v})$. Then, δ-mixed pricing dominates uniform pricing in pairwise differences.*

Example 1 (Continued)**.** Whether MEU is satisfied can be verified numerically, either by computing differences in realized utilities for every pair of signals or simply by checking the derivative of the bid function. We illustrate this for the example of uniform signal distributions in Fig. 8 below. For example, with $c = 0.8$ and $\delta = 0.3$, close to the MEU boundary in Fig. 8, the derivative of the function cannot be larger than $0.667 = \frac{1-0.8}{0.3}$. From Fig. 1, the slope of the bid function with $c = 0.8$ and $\delta = 0.3$ is close to 0.68 for low signals. Thus, for this combination of c and δ MEU is not satisfied.

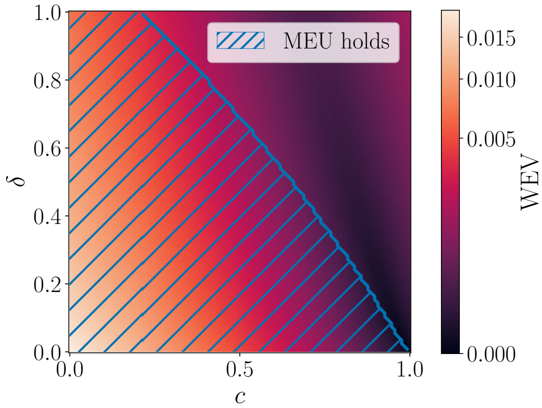


Figure 8: Monotone ex-post utility for combinations of common value c and price discrimination δ

The final crucial proposition bounds the slope of the equilibrium bid functions by 1 for the family of log-concave signal distribution. The proof is given in Section 4.1 below.

Proposition 8. *If the signal density f is log-concave, then $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ for all signals $s \in [0,\bar{v})$.*

With pure private values, this bound implies that ex-post utility is non-decreasing in signals for log-concave signal distributions. Indeed, for $u(s) = s - \delta\beta^\delta(s) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have that $\frac{\partial u}{\partial s} = 1 - \delta\frac{\partial \beta^\delta(s)}{\partial s} \geq 0$. A similar reasoning leads to Theorems 3 and 4.

Proof of Theorems 3 and 4. Because of Proposition 8, we have that under log-concave signal distributions MEU holds if $\delta \leq (1-c)$, as $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{(1-c)}{\delta}$ (see Lemma 2). Thus, applying Proposition 6, pairwise differences are monotonically decreasing for $\delta \in [0, 1-c]$ and Theorem 3 follows.

Similarly, because of Proposition 8, it holds that with log-concave signals, $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{2(1-c)}{\delta}$ if $\delta \leq 2(1-c)$. Applying Proposition 7, it follows that any mixed pricing with $\delta \in (0, 2(1-c)]$ dominates uniform pricing in pairwise differences. $\square$

4.1 Proving the bound on bid function slopes

Bounding the bid function slope for log-concave distributions requires three main observations, which we detail in the lemmas below and then use to prove Proposition 8. All proofs are relegated to Appendix B.5.

The first lemma establishes a simplified expression of $V(s)$ which allows to readily bound $V'(s)$ by 1. The proof can be found here.

Lemma 3. *Assuming common values ($c = 1$), $V(s)$ is differentiable on $(0, \bar{v})$, and can be expressed as*

$$V(s) = \frac{2}{n} s + \frac{n-k-1}{n} \frac{\int_0^s t f(t)\,\mathrm{d}t}{F(s)} + \frac{k-1}{n} \frac{\int_s^{\bar{v}} t f(t)\,\mathrm{d}t}{1 - F(s)}$$

Moreover, if the signal density f is log-concave, then $V'(s) \leq 1$ for all signals $s \in [0, \bar{v})$.

The proof proceeds by noticing that order statistics conditioned on other order statistics behave just like order statistics of a truncation of the original distribution. Thus, a more tractable expression of the expected valuation V can be derived for the pure common value case ($c = 1$). Together with results on log-concavity by Bagnoli & Bergstrom (2005), we use this expression to show that $V' \leq 1$ for all signals $s \in (0, \bar{v})$.

The next lemma establishes a sufficient condition for the equilibrium bid function slope to be bounded by 1 in the pure private value case. Differentiating twice $\int_0^s G^{1/\delta}$, we establish that its log-concavity is equivalent to $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$. The proof is given here.

Lemma 4. *Assuming private values ($c = 0$), for any $\delta \in (0, 1]$, $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ iff $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is log-concave.*

Finally, we establish that a log-concave signal density is sufficient for the integral of their order statistics to be log-concave, using closure properties of product and integration of log-concave distributions, and results by Bagnoli & Bergstrom (2005). The proof is also given in the appendix.

Lemma 5. *If the density of signals f is log-concave, then so is $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$.*

With the three lemmas above, we can prove Proposition 8.

Proof of Proposition 8. First, we recall the expression of the derivative of the bid function for any $s \in (0, \bar{v})$:

$$\frac{\partial \beta^\delta(s)}{\partial s} = \frac{g(s)}{G(s)} \frac{\int_0^s V'(y) G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \tag{6}$$

Note that for any $c \in [0, 1]$, $V(s)$ is a linear combination of s and V. In the case of a pure common value, the derivative of the latter is bounded by 1 by Lemma 3. Hence for any c, $V'(s) \leq 1$. Moreover, because of Lemma 5, we know that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave, and we can therefore apply Lemma 4. Hence using the above results,

$$\frac{\partial \beta^\delta(s)}{\partial s} \leq \frac{g(s)}{G(s)} \frac{\int_0^s \max_t V'(t) G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq \frac{g(s)}{G(s)} \frac{\int_0^s 1 \cdot G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq 1. \tag{7}$$

□

5 Discussion

Our main results hold for all equity metrics that are based on pairwise differences, and, as discussed in the Introduction and in Section 2.3, the winners' empirical variance is particularly attractive as an aggregated one-dimensional metric. We illustrate WEV further in a series of numerical experiments and discuss how it relates to the within-bidder variation of surplus, as well as the empirical variance of surplus between all bidders. We also explain why the regularity assumption of log-concavity is necessary for our argument.

5.1 Numerical experiments

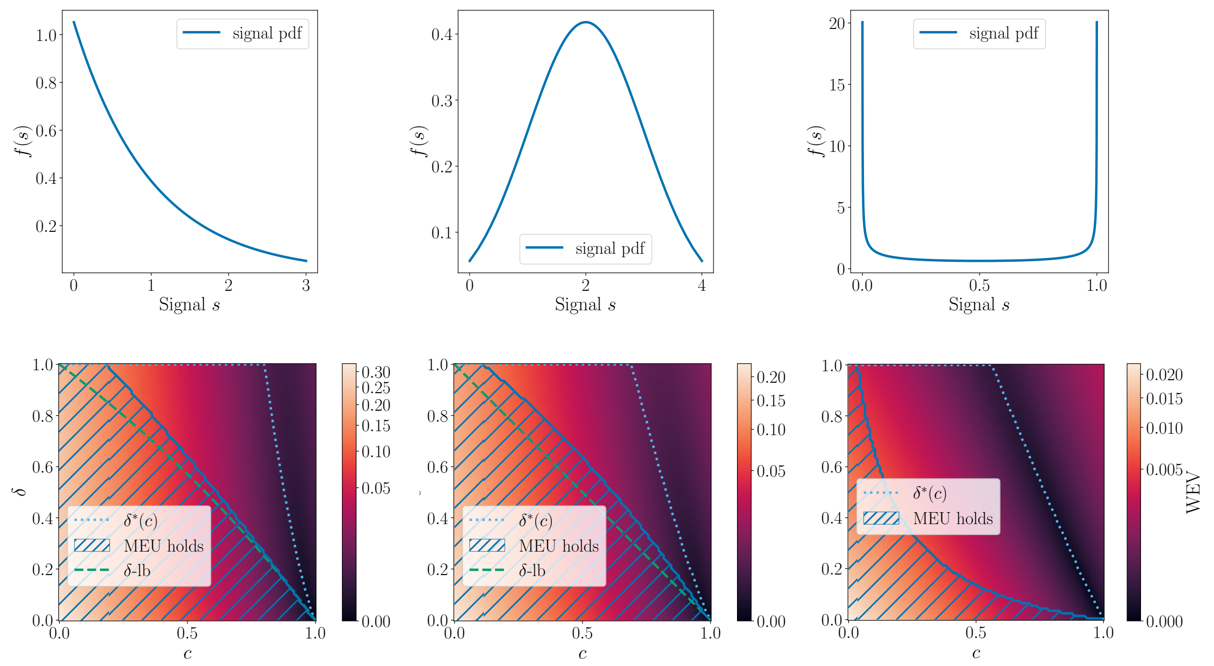
We further illustrate the effect of the common value on surplus equity by presenting several numerical examples. Similarly to Fig. 7, we compute the WEV-minimal pricing $\delta^*(c)$ for any given proportion of the private-common value c. We also illustrate bounds for WEV-minimal pricing and the condition of monotone ex-post utility (MEU). All of our experiments are based on equilibrium bid functions, whose calculation is computationally very expensive. Thus, we rely on theoretical simplifications, such as Lemma 3 and Lemma 10 (Appendix B.6). The simulations are performed through numerical integration of our analytical formulae.[23] Finally, some quantities (such as bidding functions) have multiple analytical expressions, among which we choose the most appropriate for accuracy and speed, depending on the value of the signal (e.g., Eq. (2) can be integrated more efficiently than Eq. (1), but is less accurate for small signals). Our code is available on github.

We consider three signal distributions, a truncated exponential and a truncated normal distribution (both log-concave), as well as a Beta distribution with shape parameters $(0.5, 0.5)$, which is not log-concave. WEV-minimal pricing, a lower bound on the minimizer, and combinations of common value shares and mixed pricing for which MEU holds are shown in Fig. 9.

For the truncated exponential distribution, we show the lower bound of $\frac{2n(1-c)}{2n-c(n-(k+1))}$ (cf., Proposition 5) on $\delta^*(c)$, and for the normal distribution we show the general lower bound $1-c$ (cf., Theorem 3). Each of these bounds dominates any extent of price discrimination below it. Note that for the Beta distribution, we cannot provide a theoretical lower bound on the WEV-minimal design δ^*, as the distribution is not log-concave. However, the region where MEU holds can be determined numerically, and its "frontier" provides a lower bound for the WEV-minimal design δ^*. Illustrating this for all three distributions, we observe that the area is much smaller for the Beta distribution. However, MEU is only a sufficient condition for the monotonicity of WEV (while it is necessary and sufficient for the monotonicity of pairwise differences). From the heat maps in Fig. 9, it is evident that WEV is monotone in δ for any given c up to δ^*.

Finally, we show the WEV-minimal pricing rule $\delta^*(c)$ for each signal distribution. The curve is qualitatively similar in each plot. In line with Theorem 3 — noting that the exponential and normal distribution are log-concave — the figure illustrates that with a high private value component (low c), pay-as-bid pricing ($\delta = 1$) minimizes WEV; with higher common value

[23] The efficiency and accuracy of the code rely on various techniques. Most importantly, we rewrite all multidimensional expectations as nested one-dimensional integrals (with variable bounds), which we compute by integrating polynomial interpolations. Second, the code ensures that each quantity is computed at most once, using memorization. Integration is not computationally heavy at all and achieves high precision.



Truncated exponential | Truncated normal | Beta$(0.5, 0.5)$

Figure 9: WEV-minimizing design $\delta^*(c)$, monotone ex-post utility (MEU), and lower bounds on $\delta^*(c)$ (δ-lb) for truncated exponential, truncated normal, and Beta$(0.5, 0.5)$ signal distributions

components (high c), strictly mixed pricing for some $\delta \in (0, 1)$ minimizes WEV (cf., Theorem 2); and with a pure common value ($c = 1$), uniform pricing ($\delta = 0$) minimizes WEV (cf., Theorem 1). Analogous interpretations hold for the Beta distribution, although we cannot give theoretical guarantees.

For small common values, MEU holds for any δ and thus pay-as-bid pricing is dominant in pairwise differences (cf., Proposition 6). Even for larger common value parameters the WEV-minimal pricing is still pay-as-bid, but eventually strictly mixing ($\delta \in (0, 1)$) is required to minimize WEV. For a pure common value, uniform pricing is WEV-minimal regardless of the signal distribution. Notice also that WEV at the minimal δ^* decreases in c. Naturally, with a higher common value share, bidders' values given different signal realizations as well the corresponding bids move closer together, thus explaining smaller differences in utilities (ex-post and in expectation).

5.2 Variance and risk preferences

Surplus equity and distributional concerns are distinct from questions of within-agent variation and associated risk preferences. An appropriate measure to assess the latter is, e.g., the ex-ante variance of bidder surplus. While the two notions are distinct, the measures are linked through the covariance (see also Lemma 1). In addition, for the pure private value setting, we derive the following result:

Proposition 9. *With pure private values ($c = 0$), the pay-as-bid auction minimizes the ex-ante variance of surplus among all standard auctions with increasing equilibrium bid functions.*

The proof is given in Appendix B.7. Because of revenue equivalence, note that the previous proposition also implies that $\mathbb{E}[u_i^2]$ is minimal in the pay-as-bid auction among standard auctions. The second moment of surplus links the winners' empirical variance and the empirical variance among all bidders, as shown in Lemma 6 in the Appendix. As a consequence of Lemma 6, surplus equity rankings with respect to the winners' empirical variance and the empirical variance among all bidders may not be equivalent. However, applying Proposition 7 to the pure private value case, we have the following corollary:

Corollary 2. Assuming pure private values ($c = 0$), consider any δ-mixed auction, $\delta \in (0, 1]$, and suppose that the equilibrium bid β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2}{\delta}$ for all signals $s \in [0, \bar{v})$. Then, the empirical variance (among all bidders) is lower for δ-mixed pricing than for uniform pricing.

Although this result shows that Theorem 4 can be used to extend equity rankings under pure private values to the empirical variance *among all bidders*, this may not hold in the general case.

5.3 Beyond log-concave distributions

A crucial ingredient for Theorem 3 is that the derivative of the equilibrium bid function is bounded by 1, which holds for log-concave distributions by Proposition 8. In particular, the density of the first rejected signal must be log-concave. In the following, we provide some insights as to why it is difficult to generalize this result beyond log-concave distributions.

For simplicity, consider the pay-as-bid and the uniform price auction. Considering log-concave signal distributions, we note that log-concavity is equivalent to (A, G) concavity (a generalization of convexity, see Anderson et al. 2007), and $\frac{\partial \beta^\delta}{\partial s} \leq 1$ is thus equivalent to (A, G) concavity of $s \mapsto \int_0^s G_k^{n-1}$. One idea to extend our results could then be to consider other generalizations of convexity. Considering Proposition 7, one might attempt to bound the slope of the bid functions by 2. It holds that $\frac{\partial \beta^\delta}{\partial s} \leq 2$ is equivalent to (A, H) concavity of the same function where H is the harmonic mean. But contrary to (A, G) concave functions, there are no simple group closure properties that allow for the (A, H) concavity of f to always imply that of $\int_0^s G_k^{n-1}$. Thus, this route of inquiry does not carry fruits.

We also note that conditions similar to MEU such that uniform pricing yields lower pairwise differences (or WEV) than pay-as-bid pricing are much more difficult to attain. Why? If we follow the same main ideas as in the proof of Proposition 7, a similar condition using the mean value theorem would be that, for all $s \in (0, \bar{v})$, φ is an expansive mapping, translating into $|(1 - c) - \frac{\partial \beta^\delta}{\partial s}| \geq 1 - c$. As $\frac{\partial \beta^\delta}{\partial s}$ is strictly positive, we must have $\frac{\partial \beta^\delta}{\partial s} > 1 - c$. For signal distributions with bounded density, g_k^{n-1} is close to zero near $\bar{v}$ (this follows from the definition of order statistics), and therefore $\frac{\partial \beta^\delta}{\partial s}$ is close to zero for a nonzero interval of signals. Thus, $\frac{\partial \beta^\delta}{\partial s} > 1 - c$ cannot hold for all signals on the support, and we cannot rely on similar proof techniques to produce the desired conditions.

6 Conclusion

This article proposes the study of a novel objective in auction design: the division of surplus among buyers. We introduce dominance in pairwise differences as a fundamental measure of surplus equity. Pairwise differences, in turn, can be the basis for the construction of many equity metrics, exemplified by the winners' empirical variance (WEV) of surplus.

Considering efficient and revenue equivalent auctions, we focus on a single design objective, equity. Our findings suggest that equity-optimal pricing crucially depends on the common value proportion and is not necessarily located at the extremes of pay-as-bid or uniform pricing. Moreover, our results have significant implications for the design of multi-unit auctions in practice. By carefully selecting a pricing mixture based on (an estimate of) the private-common-value proportion, auctioneers can achieve a more equitable division of surplus among buyers.

Future research could explore the trade-offs between efficiency, revenue, and equity, or extend our analysis to other types of auctions and value distributions. For example, in multi-unit demand settings, where items may be allocated inefficiently (cf. Ausubel et al. 2014), trade-offs become relevant. Finally, our work generates several testable predictions which empirical studies could attempt to validate or disprove in real-world auctions.

References

Akbarpour, M., Dworczak, P. & Kominers, S. D. (2024), 'Redistributive allocation mechanisms', *Journal of Political Economy* . Forthcoming.

Anderson, G. D., Vamanamurthy, M. & Vuorinen, M. (2007), 'Generalized convexity and inequalities', *Journal of Mathematical Analysis and Applications* **335**, 1294–1308.

Armantier, O. & Lafhel, N. (2009), Comparison of auction formats in Canadian government auctions, Technical report, Bank of Canada Working Paper, https://www.bankofcanada.ca/2009/01/working-paper-2009-5/.

Armantier, O. & Sbaï, E. (2006), 'Estimation and comparison of treasury auction formats when bidders are asymmetric', *Journal of Applied Econometrics* **21**(6), 745–779.

Armantier, O. & Sbaï, E. (2009), 'Comparison of alternative payment mechanisms for french treasury auctions', *Annals of Economics and Statistics/Annales d'Économie et de Statistique* pp. 135–160.

Arnold, B. C., Balakrishnan, N. & Nagaraja, H. N. (2008), A first course in order statistics (classics in applied mathematics).

Atkinson, A. B. (1970), 'On the measurement of inequality', *Journal of Economic Theory* **2**(3), 244–263.

Ausubel, L. M., Cramton, P., Pycia, M., Rostek, M. & Weretka, M. (2014), 'Demand reduction and inefficiency in multi-unit auctions', *Review of Economic Studies* **81**, 1366–1400.

Back, K. & Zender, J. F. (1993), 'Auctions of divisible goods: On the rationale for the treasury experiment', *The Review of Financial Studies* **6**(4), 733–764.

Bagnoli, M. & Bergstrom, T. (2005), 'Log-concave probability and its applications', *Economic Theory* **26**(2), 445–469.

Bikhchandani, S. & Riley, J. G. (1991), 'Equilibria in open common value auctions', *Journal of Economic Theory* **53**(1), 101–130.

Bulow, J. & Klemperer, P. (2002), 'Prices and the winner's curse', *The RAND Journal of Economics* **33**(1), 1–21.

Cole, H., Neuhann, D. & Ordoñez, G. (2022), 'Asymmetric information and sovereign debt: Theory meets mexican data', *Journal of Political Economy* **130**(8), 2055–2109.

Dalton, H. (1920), 'The measurement of the inequality of incomes', *The Economic Journal* **30**(119), 348–361.

Day, R. & Milgrom, P. (2008), 'Core-selecting package auctions', *International Journal of Game Theory* **36**(3-4), 393–407.

Dworczak, P., Kominers, S. D. & Akbarpour, M. (2021), 'Redistribution through markets', *Econometrica* **89**(4), 1665–1698.

Erdil, A. & Klemperer, P. (2010), 'A new payment rule for core-selecting package auctions', *Journal of the European Economic Association* **8**(2-3), 537–547.

Feldman, A. & Kirman, A. (1974), 'Fairness and envy', *The American Economic Review* **64**(6), 995–1005.

Fisz, M. (1965), *Probability Theory and Mathematical Statistics*, Wiley, New York, 3rd edition.

Gini, C. (1912), *Variabilità e mutabilità: contributo allo studio delle distribuzioni e delle relazioni statistiche. [Fasc. I.]*, Studi economico-giuridici pubblicati per cura della facoltà di Giurisprudenza della R. Università di Cagliari, Tipogr. di P. Cuppini.

Gini, C. (1921), 'Measurement of inequality of incomes', *The Economic Journal* **31**(121), 124–126.

Goeree, J. K. & Offerman, T. (2003), 'Competitive bidding in auctions with private and common values', *The Economic Journal* **113**(489), 598–613.

Hattori, T. & Takahashi, S. (2022), Discriminatory versus uniform auctions under non-competitive auction: Evidence from japan, Technical report, Available at SSRN.

Hortaçsu, A., Kastl, J. & Zhang, A. (2018), 'Bid shading and bidder surplus in the US treasury auction system', *The American Economic Review* **108**(1), 147–169.

Kasberger, B. (2023), 'When can auctions maximize post-auction welfare?', *International Journal of Industrial Organization* **89**, 102972.

Klemperer, P. (1998), 'Auctions with almost common values: The 'wallet game' and its applications', *European Economic Review* **42**(3), 757–769.

Krishna, V. (2009), *Auction theory*, Academic press.

Krishna, V. & Morgan, J. (1997), *(Anti-) Competitive Effects of Joint Bidding and Bidder Restrictions*, Discussion paper in economics. Woodrow Wilson school of public and international affairs, Woodrow Wilson School, Princeton University.

Kruse, R. L. & Deely, J. J. (1969), 'Joint continuity of monotonic functions', *The American Mathematical Monthly* **76**(1), 74–76.

Lorenz, M. O. (1905), 'Methods of measuring the concentration of wealth', *Publications of the American Statistical Association* **9**(70), 209–219.

Milgrom, P. R. & Weber, R. J. (1982), 'A theory of auctions and competitive bidding', *Econometrica: Journal of the Econometric Society* pp. 1089–1122.

Moulin, H. (2004), *Fair division and collective welfare*, MIT press.

Myers, G. (2023), *Spectrum Auctions*, LSE Press, London.

OECD (2021), *OECD Sovereign Borrowing Outlook 2021*.

Ortega-Reichert, A. (1968), Models for competitive bidding under uncertainty, PhD thesis.

Patty, J. W. & Penn, E. M. (2019), 'Measuring fairness, inequality, and big data: Social choice since arrow', *Annual Review of Political Science* **22**(1), 435–460.

Pigou, A. C. (1912), *Wealth and welfare*, London: Macmillan and Company, limited.

Pycia, M. & Woodward, K. (2018), A case for pay-as-bid auctions, Working paper, https://sites.econ.uzh.ch/pycia/pycia-woodward-paba.pdf.

Reuter, M. & Groh, C.-C. (2020), Mechanism design for unequal societies, Working paper, https://ssrn.com/abstract=3688376.

Ruddell, K., Philpott, A. & Downward, A. (2017), 'Supply function equilibrium with taxed benefits', *Operations Research* **65**(1), 1–18.

Sen, A. & Foster, J. (1973), *On Economic Inequality*, Oxford University Press.

Thomson, W. (2011), Chapter twenty-one - fair allocation rules, *in* K. J. Arrow, A. Sen & K. Suzumura, eds, 'Handbook of Social Choice and Welfare', Vol. 2 of *Handbook of Social Choice and Welfare*, Elsevier, pp. 393–506.

Vickrey, W. (1961), 'Counterspeculation, auctions, and competitive sealed tenders', *The Journal of Finance* **16**(1), 8–37.

Viswanathan, S. & Wang, J. J. D. (2002), 'Market architecture: limit-order books versus dealership markets', *Journal of Financial Markets* **5**(2), 127–167.

Wang, J. J. D. & Zender, J. F. (2002), 'Auctioning divisible goods', *Economic Theory* **19**(4), 673–705.

Wilson, R. (1969), 'Competitive bidding with disparate information', *Management Science* **15**(7), 446–448.

Wilson, R. (1977), 'A bidding model of perfect competition', *The Review of Economic Studies* **44**(3), 511–518.

Wilson, R. (1979), 'Auctions of shares', *The Quarterly Journal of Economics* **93**(4), 675–689.

Woodward, K. (2021), Mixed-price auctions for divisible goods, Working paper, https://kylewoodward.com/ research/auto/woodward-2021a.pdf.

Álvarez, F. & Mazón, C. (2007), 'Comparing the spanish and the discriminatory auction formats: A discrete model with private information', *European Journal of Operational Research* **179**(1), 253–266.

A Revenue equivalence and efficiency

We recall results from Krishna (2009) that show that the auctions we consider exhibit revenue equivalence and (allocative) efficiency.

Proposition (Revenue equivalence, Krishna 2009)**.** *Assuming iid signals, any standard auction, under any symmetric and increasing equilibrium with an expected payment of zero at value zero, yields the same expected revenue to the seller.*

We note that the crucial assumption for revenue equivalence is the independence of signals. In settings where signals are correlated, revenue equivalence fails (Krishna 2009, Chapter 6.5). It can be further shown that a bidder with signal s_i has an expected surplus

$$\tilde{u}(s_i) := \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] = \int_0^{s_i} (\tilde{v}(s_i, y) - \tilde{v}(y, y)) g_k^{n-1}(y)\, \mathrm{d}y$$

A value function $v(\boldsymbol{s})$ satisfies the *single crossing* condition if for all $i, j \neq i \in [n]$ and for all $\boldsymbol{s}$, $\frac{\partial v(s_i, \boldsymbol{s}_{-i})}{\partial s_i} \geq \frac{\partial v(s_j, \boldsymbol{s}_{-j})}{\partial s_i}$. Naturally, the value function v as given in Assumption 1 is single-crossing.

Proposition (Efficiency, Krishna 2009)**.** *Any standard auction, under any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is efficient.*

Given the prior propositions, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs.

B Proofs

B.1 Surplus equity

Proof of Lemma 1. The empirical variance of surplus can be transformed as follows.

$$\begin{aligned}
\mathbb{E}_s\left[\frac{1}{n-1}\sum_i^n\left(u_i - \frac{1}{n}\sum_j^n u_j\right)^2\right] &= \mathbb{E}_s\left[\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n (u_i-u_j)^2\right] \\
&= \frac{\mathbb{E}_s\left[(u_1-u_2)^2\right]}{2} \\
&= \mathbb{E}_s[u_1^2] - \mathbb{E}_s[u_1u_2] \\
&= \mathrm{Var}(u_1) - \mathrm{Cov}(u_1,u_2)
\end{aligned}$$

Similarly, the empirical variance conditioned on winning can be written as

$$\begin{aligned}
\mathbb{E}_s\left[\frac{1}{k-1}\sum_{i=1}^k\left(u_i - \frac{1}{k}\sum_{j=1}^k u_j\right)^2 \middle| 1,\dots,k \text{ win}\right] &= \frac{\mathbb{E}_s\left[(u_1-u_2)^2 \mid 1 \text{ and } 2 \text{ win}\right]}{2} \\
&= \mathbb{E}_s\left[u_1^2 \mid 1 \text{ wins}\right] - \mathbb{E}_s\left[u_1u_2 \mid 1 \text{ and } 2 \text{ win}\right] \\
&= \mathrm{Var}(u_1 \mid 1 \text{ wins}) - \mathrm{Cov}(u_1,u_2 \mid 1 \text{ and } 2 \text{ win}).
\end{aligned}$$

□

With pure private values, ex-post individual rationality holds. The lemma below shows that, in this case, any ranking of auction formats in terms of ex-ante variance (Var) or winners' ex-ante variance (WV) is identical. In contrast, a ranking with respect to the empirical variance (EV) may differ depending on if only winners are considered or all bidders.

Lemma 6. *Assuming that the auction is a* winners pay auction*, the empirical variance and the ex-ante variance can be decomposed, respectively, as* $EV = \frac{k(k-1)}{n(n-1)} \cdot WEV \ + \left(1 - \frac{k-1}{n-1}\right) E_{\boldsymbol{s}}[u_1^2]$ *and* $Var = \frac{k}{n} \cdot WV + \left(\frac{n}{k} - 1\right) \cdot E_{\boldsymbol{s}}[u_1]^2$.

Recall that $E_{\boldsymbol{s}}[u_1]$ does not depend on the auction format (by revenue equivalence), while $E_{\boldsymbol{s}}[u_1^2]$ does.

Proof of Lemma 6. We first note that

$$\mathrm{WV} = E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 = \frac{n}{k} E_{\boldsymbol{s}}[u_1^2] - \left(\frac{n}{k}\right)^2 E_{\boldsymbol{s}}[u_1]^2$$

For the ex-ante variance, we write:

$$
\begin{aligned}
\mathrm{Var} &= E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1]^2 = P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 + P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \mathrm{WV} + P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \mathrm{WV} + \frac{P_{\boldsymbol{s}}[1 \text{ wins}]^2}{P_{\boldsymbol{s}}[1 \text{ wins}]} \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \mathrm{WV} + \left(\frac{n}{k} - 1\right) \cdot E_{\boldsymbol{s}}[u_1]^2
\end{aligned}
$$

For the empirical variance, we write:

$$
\begin{aligned}
WEV &= E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1 u_2] = P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot E_{\boldsymbol{s}}[u_1 u_2 \,|\, 1 \text{ and } 2 \text{ win}] \\
&= P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot \mathrm{WEV} + \left(1 - \frac{P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}]}{P_{\boldsymbol{s}}[1 \text{ wins}]}\right) \cdot E_{\boldsymbol{s}}[u_1^2]
\end{aligned}
$$

□

B.2 Equilibrium bidding

Proof of Proposition 1. Consider bidder i and let all bidders $j \neq i$ follow the bidding strategy $\beta^U(s_j) = \tilde{v}(s_j, s_j)$. First, observe that β^U is continuous and increasing. Then bidder i's expected payoff when their signal is s_i and bidding $\beta^U(z)$ is given by

$$
U(s_i, z) := \int_0^z \left(\tilde{v}(s_i, y) - \tilde{v}(y, y)\right) g_k^{n-1}(y)\, \mathrm{d}y
$$

Because $\tilde{v}(s_i, y)$ is increasing in s_i, it holds for all $y < s_i$ that $\tilde{v}(s_i, y) - \tilde{v}(y, y) > 0$, and for all $y > s_i$ that $\tilde{v}(s_i, y) - \tilde{v}(y, y) < 0$. Therefore, choosing $z = s_i$ maximizes bidder i' expected payoff $U(s_i, z)$. □

Proof of Proposition 2. First, observe that β^δ is continuous. We verify that it is also monotone: writing $G_k^{n-1} =: G$, $g_k^{n-1} =: g$, and $\tilde{v}(s, s) =: V(s)$, an alternative expression for β^δ is given by

$$
\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G(y)^{\frac{1}{\delta}}\, \mathrm{d}y}{G(s)^{\frac{1}{\delta}}}. \tag{8}
$$

In particular, it is differentiable almost everywhere and we can compute its derivative.

$$
\beta^{\delta\prime}(s) = \frac{g(s) \int_0^s V'(y) G(y)^{\frac{1}{\delta}}\, \mathrm{d}y}{\delta G(s)^{1+\frac{1}{\delta}}} \tag{9}
$$

which it positive almost everywhere. Next, assume that all bidders $j \neq i$ follow the bidding strategy β^δ, and let $\beta^\delta(z)$ be bidder i's bid, whose expected utility is given by

$$
U(s_i, z) := \int_0^z \left(\tilde{v}(s_i, y) - \delta \beta^\delta(z) - (1-\delta)\beta^\delta(y)\right) g(y)\, \mathrm{d}y
$$

The derivative of $U(s_i, z)$ is

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, z) &= \tilde{v}(s_i, z)g(z) - \delta\beta^{\delta\prime}(z)G(z) - \delta\beta^{\delta}(z)g(z) - (1-\delta)\beta^{\delta}(z)g(z) \\ &= (\tilde{v}(s_i, z) - \beta^{\delta}(z))g(z) - \delta\beta^{\delta\prime}(z)G(z).\end{aligned}$$

In equilibrium, the first order condition requires $\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, s_i) = 0$. Solving this differential equation yields the stated form for β^{δ}. Using $G^{\frac{1}{\delta}-1}$ as the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[G(z)^{\frac{1}{\delta}}\beta^{\delta}(z)\right] = \left(\frac{1}{\delta}G(z)^{\frac{1}{\delta}-1}\right)\cdot(\beta^{\delta}(z)g(z) + \delta\beta^{\delta\prime}G(z)) = \left(\frac{1}{\delta}G(z)^{\frac{1}{\delta}-1}\right)\cdot\tilde{v}(s_i, z)g(z).$$

Using equations (8) and (9), and the fact that $\tilde{v}(s_i, z)$ is increasing in s_i, we obtain that $\frac{\mathrm{d}U}{\mathrm{d}z}$ is positive when $z \le s_i$ and negative when $z \ge s_i$. Therefore, choosing $z = s_i$ maximizes i's expected payoff $U(s_i, z)$.

Writing $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$, observe that the derivative of $\delta G^{\frac{1}{\delta}}$ is $gG^{\frac{1}{\delta}-1}$. Using integration by parts and a change of variable, we obtain

$$\begin{aligned}\int_0^s V(y)g(y)G(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y &= \left[\delta V(y)G(y)^{\frac{1}{\delta}}\right]_0^s - \delta\int_0^s V'(y)G(y)^{\frac{1}{\delta}}\,\mathrm{d}y \\ &= \delta V(s)G(s)^{\frac{1}{\delta}} - \delta\int_{V(0)}^{V(s)} G(V^{-1}(y))^{\frac{1}{\delta}}\,\mathrm{d}y.\end{aligned}$$

Dividing by $\delta G^{\frac{1}{\delta}}$ gives the result. $\square$

Lemma 7. *For any continuous function $\varphi : [0, \bar{v}) \to \mathbb{R}$, and for all $s \in (0, \bar{v})$, we have*

$$\lim_{\delta\to 0}\int_0^s \frac{\varphi(t)}{\delta}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$

$$\lim_{\delta\to 0}\int_0^s \ln\left(\frac{G(s)}{G(t)}\right)\frac{\varphi(t)}{\delta^2}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$

where $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$.

Proof. Fix $\delta > 0$, and let $\psi : (0, 1] \to \mathbb{R}$ be a continuous function, such that $\psi(u) = O(1/u)$ when $u \to 0$. Using the change of variable $u = v^{\delta}$, we have that

$$\int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(v^{\delta})v^{\delta}dv$$

$$\int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \ln(1/v)\psi(v^{\delta})v^{\delta}dv.$$

Observe that for all fixed $v \in (0, 1]$, and taking $\delta \to 0$, the first (resp. second) integrand converges towards $\psi(1)$ (resp., $\psi(1)\ln(1/v)$). We define the constant $M = \sup_{u\in(0,1]} u\psi(u)$, we bound the first integrand by M (resp.the second integrand by $M\ln(1/v)$), and we use the theorem of dominated convergence, which gives

$$\lim_{\delta\to 0}\int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(1)\,\mathrm{d}v = \psi(1)$$
$$\lim_{\delta\to 0}\int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(1)\ln(1/v)\,\mathrm{d}v = \psi(1)$$

To prove the lemma, observe that with the change of variable $u = \frac{G(t)}{G(s)}$, we have

$$\int_0^s \frac{\varphi(t)}{\delta}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u$$
$$\int_0^s \ln\left(\frac{G(s)}{G(t)}\right)\frac{\varphi(t)}{\delta^2}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u$$

where we define

$$\psi(u) := G(s)\cdot\frac{\varphi(G^{-1}(uG(s))}{g(G^{-1}(uG(s))}.$$

Finally, it remains to prove that $\psi(u) = O(1/u)$ when $u \to 0$. First, observe that φ is bounded on $[0, s]$. Second, observe that we have

$$\frac{u}{g(G^{-1}(uG(s)))} = \frac{1}{G(s)}\frac{G(x)}{g(x)},$$

where $x = G^{-1}(uG(s)) \to 0$. Because g is positive and integrable in 0, we have that G/g is bounded. Therefore, the overall limit when $\delta \to 0$ is equal to $\psi(1)$, which concludes the proof. $\square$

Lemma 8. *The following derivate formulas can be derived:*

$$\beta^\delta(s) = \begin{cases} V(s) & \text{if } \delta = 0 \\ V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\partial(\beta^\delta(s))}{\partial s} = \begin{cases} V'(s) & \text{if } \delta = 0, s > 0 \\ \frac{g(s)}{G(s)}\int_0^s \frac{V'(y)}{\delta}\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\delta\partial(\beta^\delta(s))}{\partial\delta} = \begin{cases} 0 & \text{if } \delta = 0 \text{ or } s = 0 \\ \int_0^s V'(y)\ln\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}y & \text{for } \delta, s > 0 \end{cases}$$

$$\frac{\delta\partial^2(\beta^\delta(s))}{\partial s\partial\delta} = \frac{\delta\partial^2(\beta^\delta(s))}{\partial\delta\partial s} = \frac{-g(s)}{\delta G(s)}\int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\mathrm{d}y \text{ for } \delta, s > 0$$

Proof. In order to derive the value of these functions at points where they are not directly, defined, we will use the dominated convergence theorem multiple times.

(1) Let $s \in (0, \bar{v})$. We first look at $\beta^\delta(s) = V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}y$. Let $h(\delta, y)$ be the function under the integral. Clearly because G is increasing, for $y < s$ we have that

$G(y)/G(s) < 1$. Hence h is dominated by V', and $\lim_{\delta\to 0} h(\delta, y) = 0$, hence by dominated convergence $\beta^\delta(s) = V(s)$ when $\delta = 0$, and the function is separately continuous over $[0,1]\times[0,\bar{v})$.

(2) We now consider the derivative of β^δ with respect to s. Let $s \in (0, \bar{v})$. There exists $m > 0$ and $M > 0$ such that $s \in [m, M]$. We focus on the derivative of the integral part:

$$-\frac{\partial}{\partial s} V'(y) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} = V'(y) \frac{g(s) G^{1/\delta+1}(y)}{G^{1/\delta}(s)} \le V'(y) \frac{g(s)}{G(s)} \le V'(y) \frac{\sup_{t\in[m,M]} g(t)}{G(m)},$$

where the $\sup_{t\in[m,M]} g(t)$ is finite as g is continuous. Because V' is integrable, we can use dominated convergence. Using Leibniz integral rule yields the formula. The formula as δ goes to 0 can be computed with Lemma 7.

(3) Le us now compute the derivative of $\delta\beta^\delta$ with respect to δ. Note that we do a careful derivation as we are also interested in the value of this derivative at $\delta = 0$. Let $h(\delta, y, s) = \delta V'(y)(G(y)/G(s))^{1/\delta}$. We have

$$\begin{aligned}
\frac{\partial h(\delta, y, s)}{\partial \delta} &= V'(y) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y) \frac{\delta}{\delta^2} \log\left(\frac{G(y)}{G(s)}\right) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} \\
&= V'(y) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} + V'(y) \log\left(\left(\frac{G(s)}{G(y)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{1/\delta}.
\end{aligned}$$

The first part is again dominated by V' which is integrable, we focus on the second part: define for $0 < u < w < 1$ the function $\psi(u, w) = (u/w)\log(w/u)$. Note that $0 < s < y < \bar{v}$ implies that for $u = G^{1/\delta}(y)$ and $w = G^{1/\delta}(s)$, we have $0 < u < w < 1$ as G is increasing and takes values in $(0, 1)$ over $(0, \bar{v})$ by definition. Fix w, and take the derivative with respect to u: we obtain that $\psi'(u, w) = (\log(w/u) - 1)/w$ which is positive as long as $u \le w/e$ and negative otherwise. The maximum of ψ for $u < w$ is at $u = w/e$ and $\psi(w/e, w) = 1/e$. This shows that the second part is smaller that $V'(y)/e$ which is also integrable. Overall by dominated convergence we can invert derivative and integral: $\frac{\partial}{\partial\delta}\int h = \int \frac{\partial}{\partial\delta} h$. Using that $\delta\beta^\delta(s) = \delta V'(s) - \int_0^s h(\delta, y, s)\,\mathrm{d}y$ we can conclude by $\frac{\partial}{\partial\delta}\delta = \beta^\delta + \delta\frac{\partial}{\partial\delta}\beta^\delta$ that

$$\frac{\partial\beta^\delta(s)}{\partial\delta} = -\int_0^s \frac{V'(y)}{\delta} \ln\left(\left(\frac{G(s)}{G(y)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y.$$

Using the same upper bound on ψ, we can show that the integrand of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ is smaller than $V'(s)/e$ which allows for domination both in small δ and small s. By dominated convergence once more, we obtain that the limit of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ as either δ or s go to 0 is 0.

(4) Finally, let us compute the cross derivative. The integrand of $\frac{\partial\beta^\delta(s)}{\partial s}$ is $h(\delta, y, s) = V'(y)(G(y)/G(s))^{1/\delta}$, which derivative with respect to delta is $-\frac{1}{\delta}V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)$. Because this function is continuous on the open set $(0, 1) \times [0, \bar{v})$, we can as done previously

apply dominated convergence to show that derivation and integral can be inverted. Therefore

$$\begin{aligned}\frac{\partial^2 \delta\beta^\delta}{\partial\delta\partial s} &= \frac{g(s)}{G(s)} \frac{-G^{1/\delta}(s)\frac{1}{\delta}\int_0^s V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)\,\mathrm{d}y + \frac{1}{\delta}\log(G^{1/\delta}(s))G^{1/\delta}(s)\int_0^s V'(y)G^{1/\delta}(y)\,\mathrm{d}y}{G^{2/\delta}} \\ &= \frac{-g(s)}{\delta G(s)}\int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\mathrm{d}y.\end{aligned}$$

□

Lemma 9. *Consider a function $\varphi : [0,1]\times(0,\bar{v}) \to \mathbb{R}_+$, such that*

- *$\varphi_\delta : s \mapsto \varphi(\delta, s)$ is continuous over $(0,\bar{v})$ for all fixed $\delta \in [0,1]$,*
- *$\varphi_s : \delta \mapsto \varphi(\delta, s)$ is continuous over $[0,1]$ for all fixed $s \in (0,\bar{v})$,*
- *either all φ_s's are monotone or all φ_δ's are monotone,*

then φ is jointly continuous in δ and s.

Proof. The proof on the open set $(0,1)\times(0,\bar{v})$ is written in Kruse & Deely (1969), and directly generalizes to $\delta = 0$ and $\delta = 1$ given that φ is separately continuous in those points. □

Proof of Proposition 3. Monotonicity follow from the derivatives computed in Lemma 8. □

B.3 Challenging the intuition: private values and uniform pricing

For Example 2, we consider the order statistics of quantiles $F^{-1}(x)$ and not of signals s. For convenience, we define the following distribution functions and densities.

$$\begin{aligned}\widetilde{G}(x) &:= G_k^{n-1}(F^{-1}(x)) = 1-(1-x)^{n-1} & \widetilde{g}(x) &:= (n-1)(1-x)^{n-2} \\ \widetilde{H}(x) &:= G_{k-1}^{n-2}(F^{-1}(x)) = 1-(1-x)^{n-2} & \widetilde{h}(x) &:= (n-2)(1-x)^{n-3}\end{aligned}$$

We choose a continuous distribution of signals, with support $[0,2]$, where each signal is given by the sum of a Bernoulli(ε) random variable and a random perturbation drawn from Beta$(1,1/\eta)$, with $\varepsilon = 0.1/n$ and η a small constant. First, we compute the distribution function F and quantile function F^{-1} of the signal distribution. Using the law of total probabilities, we have

$$\begin{aligned}\forall s\in[0,2],\qquad F(s) &= P[\text{Bernoulli}(\varepsilon) + \text{Beta}(1,1/\eta) \le s] \\ &= P[\text{Bernoulli}(\varepsilon) = 0]\cdot P[\text{Beta}(1,1/\eta)\le s] \\ &\quad + P[\text{Bernoulli}(\varepsilon) = 1]\cdot P[\text{Beta}(1,1/\eta)\le s-1].\end{aligned}$$

Simplifying this expression depending on the value of s, we get

$$\forall s\in[0,2],\qquad F(s) = \begin{cases}\varepsilon\cdot(1-(1-s)^{1/\eta}) & \text{if } s\le 1,\\ \varepsilon + (1-\varepsilon)\cdot(1-(2-s)^{1/\eta}) & \text{if } s\ge 1.\end{cases}$$

Then, computing piece-by-piece the inverse of F, we obtain

$$\forall x \in [0,1], \qquad F^{-1}(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x \leq \varepsilon, \\ 2 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^{\eta} & \text{if } x \geq \varepsilon. \end{cases}$$

A bidder with quantile $x \in [0,1]$ bids (truthfully) their signal $F^{-1}(x)$ in the uniform price auction ($\delta = 0$), which we write as $b^0_\eta(x) := F^{-1}(x) = \mathbb{1}[x \geq \varepsilon] + \gamma_\eta(x)$, where

$$\forall x \in [0,1], \qquad \gamma_\eta(x) := \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x < \varepsilon, \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^{\eta} & \text{if } x \geq \varepsilon. \end{cases}$$

For mixed auctions with $\delta > 0$, the equilibrium bid function is given by Proposition 2. Letting $b^\delta_\eta(x) := \beta^\delta(F^{-1}(x))$ denote the equilibrium bid of a bidder with quantile $x \in [0,1]$, we have

$$b^\delta_\eta(x) = \frac{\int_0^{F^{-1}(x)} V(s) g_k^{n-1}(s) G_k^{n-1}(s)^{\frac{1}{\delta}-1}\,\mathrm{d}s}{\delta G_k^{n-1}(F^{-1}(x))} = \frac{\int_0^x F^{-1}(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(x)},$$

where we used the change of variable $y = F(s)$. Finally, using the additive form of F^{-1} we write the equilibrium bid function as $b^\delta_\eta(x) = b^\delta_0(x) + \xi^\delta_\eta(x)$, where

$$\forall x \in [0,1], \qquad b^\delta_0(x) := \frac{\int_\varepsilon^x \widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(F^{-1}(x))} = \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases}$$

$$\xi^\delta_\eta(x) := \frac{\int_0^x \gamma_\eta(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(x)}$$

Next, we define the function $\phi^\delta_\eta(x) := F^{-1}(x) - \delta b^\delta_\eta(x)$, the utility of a winning bidder as a function of their quantile. Denoting WEV^δ_η the *winners' empirical variance* in a δ-mixed auction with noise level η, we write

$$\forall \delta \in [0,1],\ \forall \eta > 0, \qquad \mathrm{WEV}^\delta_\eta = \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi^\delta_\eta(x_1) - \phi^\delta_\eta(x_2))^2}{2} \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right].$$

where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi^\delta_\eta(x)$ converge towards 0 when taking η arbitrarily small, and thus $\phi^\delta_\eta(x)$ converges towards $\phi^\delta_0(x) := \mathbb{1}[x \geq \varepsilon] - \delta b^\delta_0(x)$. Therefore, WEV^δ_η converges towards WEV^δ_0, defined by

$$\begin{aligned}\forall \delta \in [0,1], \quad \mathrm{WEV}^\delta_0 :=& \mathbb{E}_{\mathbf{x}}\left[\frac{((\mathbb{1}[x_1 \geq \varepsilon] - \delta b^\delta_0(x_1)) - (\mathbb{1}[x_2 \geq \varepsilon] - \delta b^\delta_0(x_2)))^2}{2} \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] \\ =& \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi^\delta_0(x_1) - \phi^\delta_0(x_2))^2}{2} \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \lim_{\eta \to 0} \mathrm{WEV}^\delta_\eta.\end{aligned}$$

Proof of Proposition 4. We are now equipped to prove the proposition. We write $WEV^\delta_0 =$

$\mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)^2 \mid x_1 > Y_{k+1}(\mathbf{x})\right] - \mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)\phi_0^\delta(x_2) \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, with

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)^2 \mid x_1 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-1}\int_0^1 \phi_0^\delta(x)^2 G(x)\,\mathrm{d}x \\
\mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)\phi_0^\delta(x_2) \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^1 \left(\int_t^1 \phi_0^\delta(x)\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t
\end{aligned}$$

We next compute these quantities for uniform and discriminatory pricing. For uniform pricing ($\delta = 0$) we have that $\phi_0^0(x) = \mathbb{1}[x \geq \varepsilon]$. We derive

$$\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[\phi_0^0(x_1)^2 \mid x_1 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-1}\int_\varepsilon^1 \widetilde{G}(x)\,\mathrm{d}x = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1} \\
\mathbb{E}_{\mathbf{x}}\left[\phi_0^0(x_1)\phi_0^0(x_2) \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^\varepsilon (1-\varepsilon)^2\widetilde{h}(t)\,\mathrm{d}t + \frac{n}{n-2}\int_\varepsilon^1 (1-t)^2\widetilde{h}(t)\,\mathrm{d}t \\
&= \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2}
\end{aligned}$$

and finally

$$\begin{aligned}
\mathrm{WEV}_0^0 &= \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1} - \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2} \\
&= \frac{n\left[(1-\varepsilon)^n + (1-\varepsilon)(\varepsilon(n-1) - 1)\right]}{(n-1)(n-2)} \\
&\leq \frac{(\varepsilon n)^2/2}{n} = \frac{0.005}{n}
\end{aligned}$$

For discriminatory pricing ($\delta = 1$) we have that $\phi_0^1(x) = \mathbb{1}[x \geq \varepsilon] - b_0^1(x) = \mathbb{1}[x \geq \varepsilon]\frac{G(\varepsilon)}{G(x)}$. We will use the following bounds:

$$\begin{aligned}
\int_\varepsilon^1 \frac{1}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_\varepsilon^1 \frac{1}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_\varepsilon^1 \sum_{i=0}^\infty (1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \sum_{i=0}^\infty \frac{(1-\varepsilon)^{(n-1)i+1}}{(n-1)i+1} \geq (1-\varepsilon) + \sum_{i=1}^\infty \frac{(1-\varepsilon)^{ni}}{ni} \\
&\geq (1-\varepsilon) + \frac{1}{n}\sum_{i=1}^\infty \frac{0.9^i}{i} = 1 - \frac{0.1}{n} - \frac{\ln(0.1)}{n} \geq 1 + \frac{2.2}{n} \\
\int_0^1 \frac{x}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_0^1 \frac{x}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_0^1 \sum_{i=0}^\infty x(1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \frac{1}{2} + \sum_{i=1}^\infty \frac{1}{((n-1)i+1)((n-1)i+2)} \leq \frac{1}{2} + \frac{1.65}{n^2} \qquad \text{(when } n \geq 5\text{)}
\end{aligned}$$

Next, we write

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)^2 \mid x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)^2}{\widetilde{G}(x)}\,\mathrm{d}x \geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1 + \frac{2.2}{n}\right)$$

and

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2)\,|\,x_1,x_2>Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^{\varepsilon}\left(\int_{\varepsilon}^{1}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&+ \frac{n}{n-2}\int_{\varepsilon}^{1}\left(\int_{t}^{1}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&= \underbrace{\frac{n\widetilde{H}(\varepsilon)}{n-2}\left(\int_{\varepsilon}^{1}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 + \frac{n}{n-2}\left[\left(\int_{t}^{1}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 H(t)\right]_{\varepsilon}^{1}}_{=0} \\
&+ \frac{2n}{n-2}\int_{\varepsilon}^{1}\left(\int_{t}^{1}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t \\
&= \frac{2n}{n-2}\int_{\varepsilon}^{1}\left(\int_{\varepsilon}^{x}\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x
\end{aligned}
$$

Next, we will use the upper bound $\widetilde{H}(t)/\widetilde{G}(t) \le 1$, which is nearly tight as $\widetilde{H}(t)/\widetilde{G}(t)$ is increasing, and has the limit $(n-2)/(n-1)$ when $t \to 0$.

$$
\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2)\,|\,x_1,x_2>Y_{k+1}(\mathbf{x})\right] \le \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\int_{\varepsilon}^{1}\frac{x}{\widetilde{G}(x)}\,\mathrm{d}x \le \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2}+\frac{1.65}{n^2}\right)
$$

Finally, we obtain

$$
\begin{aligned}
\mathrm{WEV}_0^1 &\ge \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1+\frac{2.2}{n}\right) - \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2}+\frac{1.65}{n^2}\right) && \text{(when } n \ge 5) \\
&= n\widetilde{G}(\varepsilon)^2\left(\frac{2.2}{n(n-1)} - \frac{1}{(n-1)(n-2)} + \frac{3.3}{n^2(n-2)}\right) \\
&\ge \frac{0.01}{n} && \text{(when } n \ge 4)
\end{aligned}
$$

□

B.4 Proving the main theorems

Proof of Lemma 2. Let $s_i \le s_j = s_i + \epsilon$ for some $\epsilon > 0$. Then

$$
\begin{array}{lrcl}
\Leftrightarrow & u_i(s_i, \boldsymbol{s}_{-i}) & \le & u_j(s_j, \boldsymbol{s}_j) \\
\Leftrightarrow & (1-c)s_i - \delta\beta^{\delta}(s_i) & \le & (1-c)s_j - \delta\beta^{\delta}(s_j) \\
\Leftrightarrow & (1-c)(s_j - s_j) & \ge & \delta(\beta^{\delta}(s_j) - \beta^{\delta}(s_i))
\end{array}
$$

Dividing by $s_j - s_i$ and taking letting $\epsilon \to 0$ concludes the proof. □

Proof of Proposition 6. First, we prove that pairwise differences is locally decreasing in δ. Let s_i, s_j with $s_i \ge s_j$ denote the signals of two winning bidders and $\varphi^{\delta}(s) := (1-c)s - \delta\beta^{\delta}(s)$. Note that because of Proposition 3 (2), monotone ex-post utility holds for all $\delta \le \bar{\delta}$. For all

δ_1, δ_2, $0 \le \delta_1 \le \delta_2 \le \bar{\delta}$, we have

$$
\begin{aligned}
& |u^{\delta_1}(s_i) - u^{\delta_1}(s_j)| \ge |u^{\delta_2}(s_i) - u^{\delta_2}(s_j)| && (10)\\
\Leftrightarrow \quad & |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \ge |\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_i)| && (11)\\
\Leftrightarrow \quad & -\delta_1 \left(\beta^{\delta_1}(s_i) - \beta^{\delta_1}(s_j)\right) \ge -\delta_2 \left(\beta^{\delta_2}(s_i) - \beta^{\delta_2}(s_j)\right) && (12)
\end{aligned}
$$

For the final equivalence, observe that monotone ex-post utility together with Proposition 3 (1) implies that $\frac{\delta}{1-c}\beta^{\delta}$ is non-expansive, allowing to remove the absolute value in Eq. (11). Proposition 3 (2) guarantees that Eq. (12) holds. As the ex-post difference in utilities (Eq. (10)) is decreasing in δ, so is its expectation. To establish global monotonicty on $[0, \bar{\delta}]$, note that if $\bar{\delta}\frac{\partial \beta^{\bar{\delta}}}{\partial s} \le 1 - c$ then it also holds for any $\delta < \bar{\delta}$ by Proposition 3 (2), concluding the proof. $\square$

Proof of Proposition 7. Let $u_i^{\delta}(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the δ-mixed auction, and let $u_i^U(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price auction. Now let $i, j \in [n]$ be two winning bidders. As above, β^{δ} (resp. β^U) denotes the symmetric equilibrium bid function in the δ-mixed (resp. uniform price) auction. Let $Y_{k+1}(\boldsymbol{\beta})$ denote the first rejected bid. Then, canceling out $(1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have

$$
\begin{aligned}
|u_i^{\delta} - u_j^{\delta}| &= |(v_i(s_i, \boldsymbol{s}_{-i}) - \delta\beta^{\delta}(s_i)) - (v_j(s_j, \boldsymbol{s}_{-j}) - \delta\beta^{\delta}(s_j))| \\
&= |((1-c)s_i + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^{\delta}(s_i)) - ((1-c)s_j + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^{\delta}(s_j))| \\
&= |((1-c)s_i - \delta\beta^{\delta}(s_i)) - ((1-c)s_j - \delta\beta^{\delta}(s_j))| \\
&= |\varphi^{\delta}(s_i) - \varphi^{\delta}(s_j)|,
\end{aligned}
$$

where $\varphi^{\delta}(s) = (1-c)s - \delta\beta^{\delta}(s)$. It also holds that

$$
|u_i^U - u_j^U| = |(v_i(s_i, \boldsymbol{s}_{-i}) - Y_{k+1}(\boldsymbol{\beta})) - (v_j(s_j, \boldsymbol{s}_{-j}) - Y_{k+1}(\boldsymbol{\beta}))| = |(1-c)(s_i - s_j)|.
$$

We will now show that $\frac{\varphi^{\delta}}{1-c}$ is a non-expansive mapping. Note that φ^{δ} can be increasing or decreasing, so we need to show that $|\frac{\partial\varphi^{\delta}}{\partial s}| \le 1-c$. We have $\frac{\partial\varphi^{\delta}}{\partial s} = 1 - c - \delta\frac{\partial\beta^{\delta}}{\partial s}$. As β^{δ} is increasing in s, $|\frac{\partial\varphi^{\delta}}{\partial s}| \le 1-c$ holds whenever $\delta\frac{\partial\beta^{\delta}}{\partial s} \le 2(1-c)$. Therefore

$$
|u_i^{\delta} - u_j^{\delta}| = |\varphi^{\delta}(v_i) - \varphi^{\delta}(v_j)| \le |(1-c)(s_i - s_j)| = |u_i^U - u_j^U| \tag{13}
$$

Taking the square of Eq. (13) we obtain the result point-wise, for each pair of winning signals s_i and s_j and, taking the expectation, the theorem follows. $\square$

Theorem 5. For a given common value component c, consider two δ-mixed auctions for $\delta_1 \le \delta_2$ and suppose the equilibrium bidding functions β^{δ} satisfies $\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c)$ for all signals $s \in [0, \bar{v})$. Then, WEV is lower for the δ_2-mixed auction than for the δ_1 one.

Proof. Let $\varphi^{\delta}(s) = (1-c)s - \delta\beta^{\delta}(s)$. We have $u_i^{\delta}(\mathbf{s}) - u_j^{\delta}(\mathbf{s}) = \varphi^{\delta}(s_i) - \varphi^{\delta}(s_j)$. Let $\delta_1 \le \delta_2$. By the generalized Cauchy mean value Theorem, we have that there exists $\xi \in [s_i, s_j]$ such that

$$
|\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_j)| \left|\frac{\partial\varphi^{\delta_1}(\xi)}{\partial s}\right| = |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \left|\frac{\partial\varphi^{\delta_2}(\xi)}{\partial s}\right|.
$$

Hence if $|\frac{\partial \varphi^{\delta_2}}{\partial s}|/|\frac{\partial \varphi^{\delta_1}}{\partial s}| \leq 1$ then we have lower WEV for the δ_2 mixed auction. We have the following chain of equivalences:

$$\begin{aligned} & \left|\frac{\partial \varphi^{\delta_2}(s)}{\partial s}\right| \leq \left|\frac{\partial \varphi^{\delta_2}(s)}{\partial s}\right|, \forall s \in (0, \bar{v}) \\ \iff \quad & \left|(1-c) - \delta_2 \frac{\partial \beta^{\delta_2}(s)}{\partial s}\right| \leq \left|(1-c) - \delta_1 \frac{\partial \beta^{\delta_1}(s)}{\partial s}\right|, \forall s \in (0, \bar{v}) \\ \iff \quad & \delta_2 \frac{\partial \beta^{\delta_2}(s)}{\partial s} - (1-c) \leq (1-c) - \delta_1 \frac{\partial \beta^{\delta_1}(s)}{\partial s}, \forall s \in (0, \bar{v}) \\ \iff \quad & \delta_1 \frac{\partial \beta^{\delta_1}(s)}{\partial s} + \delta_2 \frac{\partial \beta^{\delta_2}(s)}{\partial s} \leq 2(1-c), \forall s \in (0, \bar{v}), \end{aligned}$$

where the third equations comes from the monotonicity of $\delta \frac{\partial \beta^{\delta}}{\partial s}$ in δ from Proposition 3. □

Proof of Lemma 5. To prove Lemma 5, we will use properties of log-concave distributions from Bagnoli & Bergstrom (2005). Namely their Theorems 1 and 3 state together that log-concavity of a density f implies log-concavity of the corresponding cdf F and of the complementary cdf $1-F$, and that log-concavity of F or $1-F$ imply log-concavity of respectively $\int_0^s F$ or $\int_s^{\bar{v}} F$, where $\bar{v}$ is the upper limit of the support of f (either a constant or $+\infty$). Additionally, we also have that the product of two log-concave functions is log-concave also. Using the above properties, we have that F and $1-F$ are log-concave.

Moreover, alternative expression for the order statistics are given, e.g., in Fisz (1965).

$$G_m^n(s) = \frac{n!}{(n-m)!(m-1)!} \int_0^{F(s)} t^{n-m}(1-t)^{m-1} dt$$

and

$$g_m^n(s) = \frac{n!}{(n-m)!(m-1)!} F(s)^{n-m}(1-F(s))^{m-1} f(s). \tag{14}$$

Thus, the order statistics density g, given by Eq. (14), is a product of F, $1-F$, and f, and g as well as the corresponding cdf G are also log-concave. Furthermore, $G^{\frac{1}{\delta}}$ is log-concave because $\log(G^{\frac{1}{\delta}}) = \delta \log(G)$. Finally, we remark that $G^{\frac{1}{\delta}}$ is right-continuous non-decreasing by composition with $x \mapsto x^{\frac{1}{\delta}}$, which is continuous non-decreasing, and $G^{\frac{1}{\delta}}(0) = 0$, as well as $G^{\frac{1}{\delta}}(\bar{v}) = 1$ (if $\bar{v} = \infty$, the equality is understood as a limit). Therefore $G^{\frac{1}{\delta}}$ is a cdf, and applying one last time Bagnoli & Bergstrom (2005), we obtain that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave. □

B.5 Proving the bound on bid function slopes

Proof of Lemma 3. We first rewrite $\tilde{v}(x,y)$ for $c=1$ in terms of all the order-statistics of s_{-i}.

$$
\begin{aligned}
\tilde{v}(x,y) &= \mathbb{E}[v(s_i,s_{-i}) \mid s_i=x, Y_k=y] \\
&= \mathbb{E}[\frac{1}{n}\sum_{j\in[n]} s_j \mid s_i=x, Y_k=y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n}\sum_{\substack{j\in[n],\\ j\neq i}} s_j \mid s_i=x, Y_k=y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n}\sum_{j\in[n-1]} Y_j \mid s_i=x, Y_k=y] && \text{(Ordering the signals)} \\
&= \frac{x}{n} + \frac{y}{n} + \mathbb{E}[\frac{1}{n}\sum_{\substack{j\in[n-1],\\ j\neq k}} Y_j \mid s_i=x, Y_k=y] \\
&= \frac{x}{n} + \frac{y}{n} + \frac{1}{n}\sum_{j=1}^{k-1}\mathbb{E}[Y_j \mid s_i=x, Y_k=y] + \frac{1}{n}\sum_{j=k+1}^{n-1}\mathbb{E}[Y_j \mid s_i=x, Y_k=y]
\end{aligned}
$$

Note that the previous decomposition is similar the equilibrium bid in an English auction given that k bidders have dropped out in Goeree & Offerman (2003). However, we offer a careful derivation in the multi-unit setting of our model. We now use Theorem 2.4.1 and Theorem 2.4.2 from Arnold et al. (2008) on the conditional distribution of order statistics. They state that, for $j<k$, the distribution of Y_j given $Y_k=y$ is the same as the distribution of the j-th order statistic of $k-1$ independent samples of the original distribution left-truncated at y, and we denote Z_j^l a random variable drawn according to this distribution. Hence, for $j<k$, $\mathbb{E}[Y_j \mid Y_k=y]=\mathbb{E}[Z_j^l]$. Similarly for $j>k$ we have that the distribution of Y_j given $Y_k=y$ is the same as the distribution of the $j-k$-th order statistic of $n-k-1$ independent samples of the original distribution right-truncated at y, and we denote by Z_j^r a random variable drawn according to this distribution. Hence, for $j>k$, $\mathbb{E}[Y_j \mid Y_k=y]=\mathbb{E}[Z_j^r]$. Notice that summing all order statistics drawn from some samples recovers exactly the sum of original samples. Thus we obtain

$$
\sum_{j=1}^{k-1}\mathbb{E}[Y_j \mid s_i=x, Y_k=y] = \sum_{j=1}^{k-1}\mathbb{E}[Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} s_j \mid \forall j\in[k-1], s_j\geq y] = \sum_{j=1}^{k-1}\mathbb{E}[s_j \mid s_j \geq y].
$$

The same can be done for the Z_j^r. Finally, the s_j are iid and thus have identical conditional expectations. We obtain

$$
\begin{aligned}
\tilde{v}(x,y) &= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\mathbb{E}[s_j \mid s_j \leq y] + \frac{k-1}{n}\mathbb{E}[s_j \mid s_j \geq y] && (15)\\
&= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\frac{\int_0^y tf(t)\,\mathrm{d}t}{F(y)} + \frac{k-1}{n}\frac{\int_y^{\bar{v}} tf(t)\,\mathrm{d}t}{1-F(y)}, && (16)
\end{aligned}
$$

which readily yields a formula for $V(s)=\tilde{v}(s,s)$. Clearly, the above function is well defined and differentiable on the open support of F.

We now examine the derivative of $V(s)$ and prove that $V'(s) \leq 1$. First, we consider the derivatives of the two ratios with an integral in the numerator in Eq. (16). First, by integration by parts, we have

$$\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = \frac{[tF(t)]_0^s - \int_0^s F(t)\,\mathrm{d}t}{F(s)} = s - \frac{\int_0^s F(t)\,\mathrm{d}t}{F(s)},$$

and using that for positive random variables $\int_0^{\bar{v}} tf(t)\,\mathrm{d}t = \int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t = \mathbb{E}[s_i] < \infty$, which guarantees convergence of the integral, we have that

$$\begin{aligned}\frac{\int_s^{\bar{v}} tf(t)\,\mathrm{d}t}{1-F(s)} = \frac{\mathbb{E}[s_i] - \int_0^s tf(t)\,\mathrm{d}t}{1-F(s)} &= \frac{\int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t - sF(s) + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)} \\ &= \frac{\int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t + s(1-F(s)) - s + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)} \\ &= s + \frac{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\end{aligned}$$

Now, taking derivatives, we have

$$\frac{\partial}{\partial s}\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = 1 - \frac{F(s)^2 - f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2} = \frac{f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2}.$$

By a similar argument as in the proof of Lemma 4, using log-concavity of f, the above derivative is bounded by 1. Taking the derivative of the second ratio, we have

$$\frac{\partial}{\partial s}\left(s + \frac{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\right) = 1 + \frac{-(1-F(s))^2 + f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2} = \frac{f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2}. \tag{17}$$

To derivative of $\log(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t)$:

$$\frac{\partial^2}{(\partial s)^2}\log(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t) = \frac{\partial}{\partial s}\frac{-(1-F(s))}{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t} = \frac{f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t - (1-F(s))^2}{\left(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t\right)^2}. \tag{18}$$

Eq. (18) is negative iff $f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2 \leq 1$. This means that the log-concavity of $\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t$ is equivalent to Eq. (17) being smaller than 1. As the log-concavity of $\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t$ follows from the log-concavity of f and $(1-F)$ (Bagnoli & Bergstrom 2005, Theorem 3), $f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2 \leq 1$ is implied. Finally, using the above derivatives it is clear that $V'(s) > 0$, and

$$V'(s) \leq \frac{2}{n} + \frac{n-k-1}{n}\cdot 1 + \frac{k-1}{n}\cdot 1 = 1$$

□

Proof of Lemma 4. Let us compute the second derivative of the logarithm of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$:

$$\begin{aligned}
\frac{\partial^2 \log\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)}{(\partial s)^2} &= \frac{\partial}{\partial s}\left(\frac{G^{\frac{1}{\delta}}(s)}{\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}\right) \\
&= \frac{\frac{1}{\delta}g(s)G^{\frac{1}{\delta}-1}(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{2}{\delta}}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2} \\
&= \frac{G^{\frac{1}{\delta}-1}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\left(\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)\right).
\end{aligned}$$

Notice that the left-hand fraction is always positive. Hence log-concavity of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is equivalent to $\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)$ being negative. The latter is equivalent to

$$1 \geq \frac{g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}+1}(s)} = \frac{\partial \beta^\delta(s)}{\partial s}.$$

□

Proof of Proposition 5. While $\sup_{[0,\bar{v})}\frac{\partial\beta^\delta}{\partial s}$ can be difficult to compute analytically even for simple distributions, it is sometimes possible to compute $\sup_{[0,\bar{v})} V'(s)$. For the uniform distribution, we have $\sup_{[0,\bar{v})} V'(s) = 1 - c\frac{n-2}{2n}$. Thus, using the same argument as in the proof of Theorem 3, it follows that $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-2)} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. For the exponential distribution, we have $\sup_{[0,\bar{v})} V'(s) = 1 - c(\frac{1}{2} - \frac{k+1}{2n})$, and thus $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-(k+1))} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. □

B.6 Numerical experiments

Lemma 10. *Suppose an auction is a* winners pay auction. *Then we can write* $E_{\boldsymbol{s}}[u_1 \mid 1 \text{ wins}] = \frac{n}{k}E_{\boldsymbol{s}}[u_1]$, $E_{\boldsymbol{s}}[u_1^2 | 1 \text{ wins}] = \frac{n}{k}E_{\boldsymbol{s}}[u_1^2]$, *and* $E_{\boldsymbol{s}}[u_1u_2 | 1 \text{ and } 2 \text{ win}] = \frac{n(n-1)}{k(k-1)}E_{\boldsymbol{s}}[u_1u_2]$.

Proof. Observe that we have

$$\mathbb{E}_s[u_1^2 \mid 1 \text{ and } 2 \text{ win}] = \mathbb{E}_s[u_1^2 \mid 1 \text{ wins}] = \frac{\mathbb{E}[u_1^2]}{\mathbb{P}[1 \text{ wins}]} = \frac{n}{k}\cdot\mathbb{E}[u_1^2] \tag{19}$$

$$\mathbb{E}_s[u_1u_2 \mid 1 \text{ and } 2 \text{ win}] = \frac{\mathbb{E}[u_1u_2]}{\mathbb{P}[1 \text{ and } 2 \text{ win}]} = \frac{n(n-1)}{k(k-1)}\cdot\mathbb{E}[u_1u_2] \tag{20}$$

□

B.7 Discussion

Proof of Proposition 9. We define the probability that i wins $q_i(s_i) := \mathbb{P}_{\boldsymbol{s}_{-i}}[i \text{ wins}]$. Recall that $b^D(s_i)$ denotes the equilibrium bid in the pay-as-bid auction. Consider any standard auction, characterised by a payment rule $(p_1(s),\ldots,p_n(s))$. Revenue equivalence implies that

$$q_i(s_i)\cdot b^D(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[b^D(s_i)\cdot\mathbb{1}[i \text{ wins}]] = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s)]. \tag{21}$$

In particular, note that if p_i is chosen to be the uniform pricing rule, this formula can be used to compute $b^D(s_i)$. Now define the ex-post surplus $u_i(s_i, \boldsymbol{s}_{-i}) := v(s_i) \cdot \mathbb{1}[i \text{ wins}] - p_i(s)$. We write

$$u_i(s_i, \boldsymbol{s}_{-i}) = \underbrace{\mathbb{1}[i \text{ wins}] \cdot (v(s_i) - b^D(s_i))}_{u_i^D(s)} + \underbrace{\mathbb{1}[i \text{ wins}] \cdot b^D(s_i) - p_i(s)}_{\delta(s)}. \tag{22}$$

Now, observe that by revenue equivalence we have $\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})] = 0$ for all s_i. We write

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 \tag{23}$$

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] + 2\underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})]}_{\geq 0} + \underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})^2]}_{\geq 0} \tag{24}$$

To show that the extra terms are non-negative, notice that $\delta(s_i, \boldsymbol{s}_{-i})^2 \geq 0$, and that

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})] = \underbrace{(v(s_i) - b^D(s_i))}_{\geq 0} \cdot \underbrace{(q_i(s_i) \cdot b^D(s_i) - \mathbb{E}_{\boldsymbol{s}_{-i}}[\mathbb{1}[i \text{ wins}] \cdot p_i(s)])}_{\geq \mathbb{E}[\delta(s)] = 0} \tag{25}$$

Therefore, putting everything together, we obtain

$$\begin{aligned} \mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] &= \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 && (26) \\ &\geq \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 && (27) \\ &= \mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})] && (28) \end{aligned}$$

Finally, observe that an auction which minimize the interim variance also minimize the ex-ante variance. Denoting by u_i the utility of a bidder in the pay-as-bid auction, the law of total variance states

$$\mathrm{Var}_s[u_i] = \mathbb{E}_{s_i}[\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]] + \mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]. \tag{29}$$

By the revenue equivalence theorem, we know that $\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]$ is the same for all standard auctions, hence $\mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]$ is also the same for all standard auctions (it only depends on the distribution of the signals).

The interim variance $\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]$ is minimal point-wise (in s_i) for all standard auctions, hence is also minimal in expectation. Therefore, the ex-ante variance is minimal in the pay-as-bid auction among standard auctions. $\square$