Selling Multiple Complements with Packaging Costs*

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Abstract

We consider a package assignment problem with multiple units of indivisible items. The seller specifies preferences over partitions (between buyers) of their supply as packaging costs. To express these preferences, we propose incremental costs together with a graph that defines cost interdependence. This facilitates using linear programming to find anonymous and package-linear Walrasian equilibrium prices. We provide necessary and sufficient conditions for the existence of Walrasian equilibria, as well as additional sufficient conditions. Furthermore, our cost framework ensures fair and transparent dual pricing and admits preferences over the concentration of allocated bundles in the market.

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1 Introduction

In many multi-item markets, e.g., combinatorial auctions, buyers express preferences for bundles of indivisible items, and the seller may care about who wins which bundle. So far, despite attested preferences by real-world sellers, e.g., caps on the number of spectrum licenses in broadcasting frequency auctions (Cramton et al. 2011, Kasberger 2023, Myers 2023), the economic literature has failed to provide for general form preferences over the supply partitioning in indivisible goods markets.[1] A key challenge with partition preferences is their representation, as the domain, the universe of all possible supply partitions, is much larger than the universe of bundles. Furthermore, in markets for indivisible goods, even without partition preferences, the existence of a stable outcome in the form of a Walrasian equilibrium is guaranteed only under specific assumptions on preferences (e.g., Gul & Stacchetti (1999), Sun & Yang (2006), Milgrom & Strulovici (2009), Hatfield et al. (2013)).

This article proposes a general, yet parsimonious, framework to express partition preferences. In this framework, we characterize Walrasian equilibria and provide conditions for their existence. Our setting is a competitive market for multiple indivisible goods, including multiple buyers with values over bundles of items, and a seller with preferences over *anonymous* partitions (which will be defined shortly) of their supply between buyers. Even with anonymous partitions, the universe of such partitions remains large. To address this, we propose *incremental cost functions* which specify the cost (savings) of combining two or more items, together with a graph structure that defines the cost interdependencies of related bundles. Buyers in our model have a rich set of preferences over bundles that can incorporate complements and substitutes.

The combination of items in bundles may result in cost savings or additional expenses for the seller. These are independent of the buyers' identity, i.e., anonymous, but the seller does care about whether any two or more items are allocated to separate buyers or, packaged, to a single buyer. Thus, we name these costs *packaging costs*. Depending on the application, they may also be related to transaction costs, realized synergies, or indirect subsidies.

Consider, for example, the reallocation of land plots to farmers via an auction by the government. The government favors the allocation of two complementary land plots as a bundle, to support farmers and to encourage defragmentation (cf. Bryan et al. (2024)). However, it wishes to allocate the two land plots separately if two individual buyers have sufficiently high values for them. In our framework, it can do so by defining a negative packaging cost, offering the bundle more cheaply than the sum of costs of items contained in the bundle.[2] There may also be a small number of special, highly productive plots that, for fairness reasons, the government wishes to sell separately. Then, this preference can be expressed as a (strongly) positive packaging cost.

[1]The seller's values or costs are commonly set to zero or assumed to be symmetric to buyers.

[2]This can also be interpreted as an indirect subsidy. In auctions for biodiversity conservation contracts (e.g., Stoneham et al. (2003)), the government may wish to subsidize bundles if conservation measures implemented on the same piece of land are synergetic.

Further applications of our framework are found in procurement,[3,4] the insurance industry,[5] the transport sector (e.g., Cantillon & Pesendorfer (2005)), or broadcasting frequency auctions.

With the proposed representation of the seller's cost at hand, we search for a socially efficient allocation, and the immediate question is whether a Walrasian equilibrium exists. First, we generalize a classical result by Bikhchandani & Mamer (1997) and Bikhchandani & Ostroy (2002) on the existence of competitive equilibrium, which requires a novel proof strategy due to the generality of the framework. In contrast, the existing works address an exchange economy and a seller with zero costs, respectively.[6] We demonstrate that our preference structure can be embedded into the integer social welfare maximization problem and its linear programming relaxation.

However, this poses the additional challenge of characterizing the pricing function, which is non-trivial and involves a dual formulation of the preference structure. Our characterization reveals that the pricing function is anonymous and linear in packages,[7] ensuring fairness, and reflects the interdependencies of incremental costs, guaranteeing transparency in pricing. The characterization of the pricing function is crucial in arguing that those prices indeed support a Walrasian equilibrium allocation and in proving the equivalence between the integrality of solutions to the linear program and the existence of a Walrasian equilibrium.

Our next result is a sufficient condition for the existence of a Walrasian equilibrium. We assume a supply of one indivisible item per variety, given any number of varieties, and require that buyers have superadditive values (all items be complements) and the seller has subadditive costs (preferring coarser partitions of her supply). The seller's cost function domain again does not match the buyers' value domain. We show that, under these assumptions, a package-linear Walrasian equilibrium exists. This generalizes a result by Sun & Yang (2014), who establish equilibrium existence when buyers have superadditive values, and the seller, maximizing its revenue plus the value from unsold items, has superadditive values for those unsold items too.[8]

We explore the connection between their market and ours and prove a duality relation between their revenue-maximizing and our (cost-based) utility-maximizing seller. Moreover, we show that superadditive values of a revenue-maximizing seller are equivalent to *set-cover submodular*[9] costs of a utility-maximizing seller, noting that set-cover submodular cost functions

[3] Note that in procurement, in a market with multiple suppliers, the *buyer* has partition preferences. For example, a single supplier could provide maintenance and employee training for a machine more efficiently than two different suppliers.

[4] Some decision support systems used in practice allow for different types of discounts and sophisticated bids (Giunipero et al. 2009, Bichler et al. 2011). Bichler et al. (2011) propose a bidding language that accommodates various types of discounts a procurement manager may want to offer, although not in the context of Walrasian equilibrium.

[5] In the insurance industry, the risk between and across bundled products is interdependent. Representing such interdependencies while maintaining tractability is a key challenge in the partitioning problem that we address in our framework.

[6] The model by Bikhchandani & Mamer (1997) can be straightforwardly extended to costs that are additive between items, but these do not admit preferences over the partitioning of supply between buyers.

[7] Under package-linear pricing, the same price applies to identical packages, and the price of a collection of several packages equals the sum of prices of the packages contained in the collection. A package-linear pricing function is non-linear in items, i.e., the price of a bundle may be different to the sum of prices of the items contained.

[8] Sun & Yang (2014) also develop an incentive compatible mechanism for their setting. Results in a similar spirit are established in Fujishige & Yang (2020) for more general preferences.

[9] We define the notion of set-cover submodularity, which is weaker than submodularity, in Definition 12. Given a ground set N, the familiar submodularity condition is required for any two elements of 2^N whose union is N.

allow for subadditive as well as superadditive elements. However, the preferences of such a utility-maximizing seller do not take into account the partitions of allocated items between buyers.

On the buyer side, we allow substitute trade-offs between any bundles, thus encompassing substitute and complement preferences.[10] If a buyer is allocated a collection of bundles, their value corresponds to the value-maximal matching of those bundles to fictitious unit-demand agents, where each unit-demand agent is matched with most one bundle. With multiple fictitious agents corresponding to one buyer, the seller can ensure that bundled items are allocated to a single buyer, but separate items might be allocated to two fictitious agents corresponding to the same buyer. Thus, if the seller cares about the separation of items, only one fictitious agent per buyer is allowed (cf. Note 20). Our buyer preferences generalize assignment valuations (Shapley 1962, Shapley & Shubik 1971) and those admissible in the Product-Mix auction (Klemperer 2008, 2010, 2018),[11] in which buyers express substitute trade-offs between items.

Similarly to Product-Mix auctions, which implement a Walrasian equilibrium assuming competitive behavior of participants, our market can be implemented as a sealed-bid auction if agents act (approximately) as price takers and truthfully submit their preferences. The buyers' value functions and the seller's incremental cost functions and admissible graph structures, being parsimonious in the universe of anonymous partitions, are easy to communicate to the auctioneer. Fixing the number of distinct packages that can be demanded or supplied, our framework is also computationally tractable.[12] Moreover, in markets for complements in which the seller prefers coarser partitions, we provide an implementation as an ascending auction. Our extended ascending auction strictly generalizes the ascending auction by Sun & Yang (2014), accommodating a buyer-symmetric seller and an auctioneer with partition preferences.

Our framework of partition preferences implies new market design applications. The cost function graph we introduce facilitates modeling a wide range of cost interdependencies between related packages.[13] Furthermore, the incremental costs can encourage or discourage bundle allocation, but always provide flexibility to bundle in the opposite way, i.e., to sell items separately or together, if demand requires it. To illustrate this, consider the seller in a spectrum auction with concerns about an asymmetric distribution of licenses between buyers (Ofcom 2017, GSMA 2021). Such preferences would commonly be expressed through (hard) spectrum caps, i.e., each bidder can win at most a fixed number of licenses for each variety (frequency band) (Cramton 2013, Kasberger 2023). While caps are typically set to mitigate market power in the downstream market, "their drawback is that they may prohibit efficient aggregation of spectrum" (Cramton et al. 2011). The auctioneer can implement softer caps with our incremental cost structure, by penalizing bundle allocations with higher packaging costs. A buyer could then still obtain a

[10] Combinatorial preferences appear in many auctions in theory and practice, for example in the early package auction iBundle (Parkes 1999). A comprehensive account of (combinatorial) bidding languages is given, e.g., in Nisan (2000) and Lehmann et al. (2006).

[11] The Product-Mix auction has been in use by the Bank of England following the financial crisis in 2007 until today.

[12] Valuations over bundles are inherently exponential in the number of distinct items in the market, as is the pricing function. However, treating the number of packages as a constant, the linear program is a computationally benevolent formulation. Even for the social welfare integer program, duality theory provides avenues for practical computation (e.g., Vries & Vohra (2003)).

[13] For example, the bundling of ABC may be more expensive if several copies of AB are sold as well, but it may be independent of bundles of type AC.

bundled set of licenses, but the soft caps would require their bundle value to be high enough to outweigh the seller's preference for an equitable allocation. More generally, the auctioneer can steer market outcomes towards their preferred (less costly) allocation using our cost structure, where hard caps are a special case corresponding to prohibitively high costs for unequal allocations.

Walrasian equilibrium in markets in which agents view some or all items as complementary have been studied, e.g., by Sun & Yang (2006) and Teytelboym (2014), who establish equilibrium existence results, and by Sun & Yang (2009), who develop a Walrasian tâtonnement process. Baldwin & Klemperer (2019*b*) introduce the concept of demand types and establish with their unimodularity theorem the existence of a Walrasian equilibrium for many classes of complements and substitutes preferences. Furthermore, Candogan et al. (2015) show the existence of a linear pricing Walrasian equilibrium for the class of sign-consistent tree valuations[14] and Candogan et al. (2018) study pricing equilibria when buyers have graphical valuations. In all of those studies, partition preferences over the market supply are not considered. We also note that partition preferences with subadditive marginal costs, as well as (the buyers') superadditive values over bundles, are orthogonal to the well-known classes of gross substitutes (Kelso & Crawford 1982) and strong substitutes (Milgrom & Strulovici 2009) which admit a linear pricing Walrasian equilibrium.

The remainder of this article is structured as follows. In Section 2, we describe our partition preference framework and the general market setting. In Section 3, we characterize Walrasian equilibria, including the derivation of the pricing function, and develop our first main result on the existence of equilibrium. In Section 4, we prove sufficient conditions for the existence of a Walrasian equilibrium and explore the dual relationship between revenue-maximizing and utility-maximizing sellers. Section 5 provides a brief discussion and Section 6 concludes.

2 A model of a competitive market with packaging costs

2.1 Preliminaries

There are n indivisible, distinguishable varieties (or items) in the economy, identified with $j \in N := \{1, \dots, n\}$, with Ω_j units for each variety $j \in N$.

Definition 1 (Package)**.** A *package* is a subset of N, i.e., an element $S \in 2^N$.

Packages allow only a single unit of each variety to be bundled together.[15] In contrast, multisets generalize subsets to allow multiple copies of items by including the multiplicity of the elements of their ground set N (with multiplicity 0 for elements of N not present in the multiset). Occurrences of the same element are indistinguishable.[16] We denote by $\mathbb{Z}_+$ the set of positive integers including zero.

Definition 2 (Multiset)**.** Given a finite ground set $A = \{a_1, \dots, a_n\}$, a *multiset* of A is defined as a mapping $m : A \to \mathbb{Z}_+$, and represented by $\boldsymbol{m} := (m(a_1), \dots, m(a_n))$.

[14]In this class, each two goods must be either substitutes or complements for all buyers.

[15]This is without loss of generality, since with appropriate labeling packages containing identical varieties can be mapped to our model (we exemplify this in Appendix B.1).

[16]See, e.g., Blizard (1989) for a detailed treatment of multisets.

The image $m(a)$, or m_a, denotes the multiplicity of element a in $\boldsymbol{m}$. A *variety multiset* is of the ground set N. We also introduce *package multisets* of the power set of N, in which case the ground set is 2^N. Package multisets are denoted $\boldsymbol{k} = (k_{S_1}, \ldots, k_{S_{2^n}}) \in \mathbb{Z}_+^{2^n}$, where each k_S denotes the multiplicity of package S.

Definition 3 (Feasible multisets)**.** The universe of all feasible package multisets $\mathcal{K}$ is given by

$$\mathcal{K} := \left\{ \boldsymbol{k} \in \mathbb{Z}_+^{2^n} : \sum_{S \in 2^N} k_S \mathbb{1}_{j \in S} \leq \Omega_j \quad \forall j \in N \right\}.$$

A package multiset is equivalent to an "anonymous" partition in which the identity of the agents receiving an element of the partition does not matter. Henceforth, in the interest of conciseness, we refer to anonymous partitions simply as partitions.

If a package multiset $\boldsymbol{k}$ can be identified to a set S (i.e., $k_S = 1$ for exactly one $S \in 2^N$ and $k'_S = 0$ for all other S'), we will abuse the formal definition and write $\boldsymbol{k} = S$ for the sake of clarity. In that case, given a function $f : \mathbb{Z}_+^{2^n} \to \mathbb{R}$, $f(\boldsymbol{k})$ is written simply as $f(S)$. The set of package multisets over 2^N can be endowed with basic operations and functions.

Definition 4. Fix package multisets $\boldsymbol{k}$ and $\boldsymbol{k}'$.

(i) Sum. The sum $\boldsymbol{k}'' = \boldsymbol{k} + \boldsymbol{k}'$ is defined by $k''_S = k_S + k'_S$ for all $S \in 2^N$.

(ii) Multiplication with a scalar. $\alpha \boldsymbol{k} = (\alpha k_S)_{S \in 2^N}$ for any $\alpha \in \mathbb{R}$

(iii) Unpacking. $\boldsymbol{k}^\star = (m_j)_{j \in N}$ with $m_j := \sum_{S \in 2^N} k_S \mathbb{1}_{j \in S}$ for all $j \in N$

(iv) Cardinality. $|\boldsymbol{k}| = \sum_{S \in 2^N} k_S$

The unpacking operator * unpacks a package multiset $\boldsymbol{k}$ into the multiset of varieties contained in $\boldsymbol{k}$. For the seller, we call the unpacked multiset a supply vector.

To simplify notation, we often use implicit summation $f(X, Y) = \sum_{x \in X, y \in Y} f(x, y)$ for any finite sets X and Y. We let $[X] := \{1, \ldots, X\}$.

2.2 Agents and preferences

There is one seller ("she") in the economy, denoted 0, and a set of L buyers ("he") denoted $l \in [L] := \{1, \ldots, L\}$. The set of all agents is $[L]_0 := [L] \cup \{0\}$. The preferences of each buyer are specified by a value function $V^l : \mathbb{Z}_+^{2^n} \to \mathbb{Z}_+$ with $V^l(\emptyset) = 0$. If a buyer demands at most one package, the value function is given by $V^l : 2^N \to \mathbb{Z}_+$. Such an agent is called a "unit-demand" agent, where the "unit" refers to a package. We restrict V^l as follows: each buyer's value function is the aggregate of values (defined below) of a finite number $\overline{Q}^l$ of fictitious unit-demand agents, and $\overline{Q} := \max_{l \in [L]} \overline{Q}^l$. We sometimes denote by (q, l) the qth unit-demand agent of buyer l.

Definition 5 (Unit demand)**.** The *unit-demand valuations* of the fictitious agents associated with buyer l are defined as $v^l : 2^N \times \mathbb{Z}_+ \to \mathbb{Z}_+$, where $v^l(S, q)$ is the value of bundle S for the q^{th} unit-demand agent associated with buyer l.

A buyer's value for a multiset $\boldsymbol{k}$ is obtained by a value-maximizing matching of the contained bundles to his fictitious unit-demand agents,[17] where each fictitious unit-demand agent is assigned at most one package. Formally, we aggregate unit-demand values as follows. Let S_q be the bundle assigned to unit-demand agent q and $e^{S_q} \in \{0,1\}^{2^n}$ be the indicator vector with value 1 for bundle S_q and value 0 for all other bundles. We assume $v^l(S,q) = 0$ for all $q > \overline{Q}^l$.

Definition 6 (Unit-demand value aggregation)**.**

$$V^l(\mathbf{k}) := \max_{S_q \subseteq 2^N} \sum_{q \in [\overline{Q}^l]} v^l(S_q, q) \quad \text{s.t.} \sum_{q \in [\overline{Q}^l]} e^{S_q} \leq \mathbf{k}.$$

A *pricing function* is a function $p : \mathbb{Z}_+^{2^n} \to \mathbb{R}$ with $p(0) = 0$. It is *nonlinear* in varieties $j \in N$, i.e., for any package $S \in 2^N$, we may have $p(S) \neq \sum_{j \in S} p(j)$. A pricing function $p : \mathbb{Z}_+^{2^n} \to \mathbb{R}$ is *package-linear* if and only if, for all $\boldsymbol{k} \in \mathbb{Z}_+^{2^n}$, $p(\boldsymbol{k}) = \sum_{S \in 2^N} k_S p(S)$. Thus, a package-linear pricing function can be represented as a mapping $p : 2^N \to \mathbb{R}$.[18]

Each agent's utility is quasi-linear and given by $u^l(\boldsymbol{k}, p) = V^l(\boldsymbol{k}) - p(\boldsymbol{k})$ when they receive a package multiset $\boldsymbol{k}$ at package-linear prices p. Note that a buyer who receives a package $\boldsymbol{k}$ cannot unpack $\boldsymbol{k}$ (in the sense of Definition 4(iii)), i.e., the individual bundles of $\boldsymbol{k}$ are fixed. Example 1 below illustrates the aggregation of unit-demand values.

Example 1. Consider the sale of two units of good A and two units of good B, which may be sold separately or in packages. The set $\{AB\}$ is considered a package. Suppose that there is one buyer with two corresponding unit-demand buyers. The unit-demand value functions are given in Table 1 and aggregated according to Definition 6. For legibility, here we write multisets not as vectors $\boldsymbol{k}$ but as sets. We have, e.g., $v^1(\{A\}) = 3$, $v^1(\{B\}) = 5$, $v^1(\{A, B\}) = \max(3 + 2, 5 + 1) = 6$, $v^1(\{A, AB\}) = 3 + 9$, $v^1(\{B, AB\}) = 5 + 9$, and $v^1(\{A, B, AB\}) = \max(3 + 2, 5 + 1, 3 + 9, 5 + 9, 1 + 9, 2 + 9) = 14$. Note that with only two unit-demand buyers, the third and any further bundles contribute a value of zero.

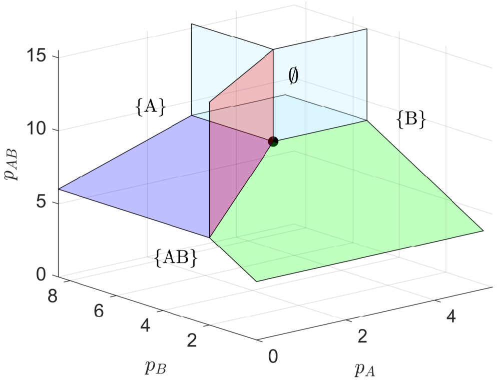

Figure 1: Unit demand in price space

q	$v^1(q, A)$	$v^1(q, B)$	$v^1(q, AB)$
1	3	5	9
2	1	2	9

Table 1: Unit-demand values

[17] Such preferences are also known as assignment valuations (Shapley 1962, Shapley & Shubik 1971), and the value-maximizing matching is sometimes called a maximum-weight matching. In the terminology of Lehmann et al. (2006), unit-demand valuations are combined by an "inclusive-or"-operation (see also Nisan (2000)). A generalized version of assignment valuations are the assignment messages in Milgrom (2009). Note that each of our unit-demand agents may be assigned a bundle of items.

[18] Sun & Yang (2014) call a pricing function $p : 2^N \to \mathbb{R}$ *non-linear*. Note that they do not consider partition pricing.

The preferences of unit-demand agent $q = 1$ are illustrated in price space in Fig. 1 and represented by the black dot at $(3, 5, 9)$. At prices beyond the light-blue facets, the agent demands nothing. The remaining area is divided into three. At prices above the blue and left of the red facet $\{A\}$ is demanded, above the green and right of the red facet $\{B\}$ is demanded, and at prices below the blue and the green facet $\{AB\}$ is demanded. On the facets or their intersections, the agent is indifferent between the bundles demanded in adjacent regions.[19]

The seller's preferences are over partitions of her supply, specified by a *cost function* C^0. She obtains the quasi-linear utility $u^0(\boldsymbol{k}, p) = p(\boldsymbol{k}) - C^0(\boldsymbol{k})$ when she sells the package multiset $\boldsymbol{k}$ at a prices p. Any package multiset induces a partition of the contained bundles between buyers, where the identity of the buyers who receive a bundle is irrelevant.[20]

The cost function C^0 consists of two elements: (i) *incremental costs* expressing additional costs or cost savings from bundling varieties in a package, and (ii) a graph with *cost connections* specifying cost interdependencies between related bundles. For single-item packages S ($|S|= 1$), incremental costs are simply costs.

Definition 7. A *cost function graph (CFG)* is a directed graph (G, A) with vertices labeled with elements of 2^N representing the distinct packages, and arcs A defining the cost connections between packages such that

(i) if $(T, S) \in A$, then $S \subset T$, and

(ii) every package S is connected to every single variety $\{j\}$, $j \in S$.

The CFG is a tree due to property (i). Formally, we say that S_1 and S_t are (cost-)connected and write $\exists(S_1...S_t)$ whenever there exists a sequence of vertices $(S_1, ..., S_t)$ such that $(S_1, S_2), ..., (S_{t-1}, S_t) \in A$. Given a path $H := (S_1, ..., S_t)$, $|H|= t - 1$ denotes the length of path H. Lastly, node S is connected to itself.

The *successors* $R^+_0(S) := \{S' \in 2^N : \exists(S...S')\}$ and the *strict successors* $R^+(S) := \{S' \in 2^N : \exists(S...S'), S' \neq S\}$ of package S are sets of packages reachable from S in the CFG. Likewise, the *predecessors* $R^-_0(S) := \{S' \in 2^N : \exists(S'...S)$ and the *strict predecessors* $R^-(S) := \{S' \in 2^N : \exists(S'...S), S' \neq S\}$ of S are sets of packages from which S can be reached. Furthermore, $N^+(S) := \{S' \in 2^N : (S, S') \in A\}$ and $N^-(S) := \{S' \in 2^N : (S', S) \in A\}$ are the neighbors, the *direct successors* and *direct predecessors* of S, and $N^+_0 := N^+ \cup \{S\}$ and $N^-_0 := N^- \cup \{S\}$.

When selling a supply partition, the seller incurs costs related to the individual elements in the partition as well as their interaction. Each node (package) S in the graph is associated with an incremental cost function. All predecessors $S' \in R^-_0(S)$ of some node $S \in 2^N$ are *cost-connected* to S. Economically, this means that an incremental cost corresponding to S also contributes to the cost of S'. Only bundles that are a subset or superset of each other can be cost-connected (Definition 7 (i)), and for each bundle the seller incurs at least the cost the individual varieties contained in the bundle (Definition 7 (ii)).

[19] The demand of multiple unit-demand agents can also be geometrically aggregated. Baldwin & Klemperer (2019*b*) illustrate this two-dimensional price space (not allowing a separate price for $\{AB\}$).

[20] We note that, with multi-unit-demand buyers (more than one corresponding unit-demand agent), there is an asymmetry in the interpretation of the seller's preferences: bundling items guarantees that a single buyer receives the bundle. However, separate items may still be received by unit-demand agents belonging to the same buyer. With only one unit-demand agent per buyer, separate items must go to different buyers.

Definition 8. Incremental costs are defined as $\Delta c : 2^N \times \mathbb{N} \to \mathbb{Z}$, where $\Delta c(S, r)$ is the cost increase that the seller incurs from selling a copy of some package T due to its cost connection to package S, when she sells $r - 1$ copies of other packages cost-connected to S.

Note that $\Delta c(\{j\}, r) := \infty$ for $r > \Omega_j$, for all $j \in N$ (similarly to a capacity constraint). Negative incremental costs, i.e., cost savings, are allowed. The total cost of a supply partition is obtained by adding all incremental costs associated with the bundles contained in the partition. For each bundle, the cost function graph defines the set of cost-connected bundles.

Definition 9. The cost of selling a partition of packages (package multiset) $\boldsymbol{k} \in \mathcal{K}$ is defined as $C^0 : \mathcal{K} \to \mathbb{Z}_+$. Given a cost function graph and associated incremental costs,

$$C^0(\boldsymbol{k}) = \sum_{S \in 2^N} \Delta c(S, [r_S]), \qquad \text{with } r_S := \sum_{S' \in R_0^-(S)} k_{S'}.$$

Note that r_S counts the copies of predecessors of S, i.e., cost-connected packages, that are sold in the partition $\boldsymbol{k}$. To further illustrate the cost aggregation, suppose a single copy of bundle S is sold and nothing else. Then the cost of S is obtained by adding all incremental costs $\Delta c(S', 1)$ of cost-connected bundles S', i.e., the cost of S is $\sum_{S' \in R_0^-(S)} \Delta c(S', 1)$. We often use the implicit summation $\Delta c(R_0^-(S), r)$. While negative incremental costs are allowed, we assume that costs are non-negative, i.e., $C^0(\boldsymbol{k}) \geq 0$ for all $\boldsymbol{k} \in \mathcal{K}$, and make the subsequent monotonicity assumption on incremental costs.

Assumption 1 (Increasing incremental cost)**.** For any package $S \in 2^N$ and for all $r \geq 1$, it holds that $\Delta c(S, r) \leq \Delta c(S, r + 1)$.

Intuitively, the more packages cost-connected to some package S are sold, the more costly it becomes to sell an additional package cost-connected to S.[21] The following observations further illustrate the properties of cost function graphs.

Observation 1.

(a) Node N is always a source; but a CFG may contain other sources (Definition 7 (i)).

(b) A CFG is acylic (Definition 7 (i)).

(c) Node $S \in 2^N$ is a sink iff $|S| = 1$ (Definition 7 (i) and (ii)).

(d) A CFG is weakly connected (Definition 7 (ii)).

(e) The cost of a partition is at least the sum of costs of the packages contained (Assumption 1).

In Example 1, we show the simplest cost function graph and associated incremental cost functions with two distinct items A and B. However, the expressive power of cost function graphs is much greater with at least three items A, B, and C, as the costs of packages with overlapping subsets of varieties may be correlated. For example, the cost savings of grouping ABC together may depend on the number of units of package AB that are sold simultaneously in the market. We illustrate this in Example 2.

[21] This assumption is reminiscent but not identical to one of increasing marginal costs. We note that, due to potential cost interdependencies, marginal costs can only be defined as a function of the entire sold partition.

Example 1 (Continued)**.** The seller's preferences can be summarized by the incremental cost function in Table 2 and the cost function graph in Fig. 2. Note that with only two goods, there exists only one valid cost function graph, and we illustrate more complex graphs in Example 2. For legibility, we write multisets not as vectors but in set notation.

The total cost is given, as defined in Definition 9, as $C^0(\{A\}) = 1$, $C^0(\{B\}) = 1$, $C^0(\{AB\}) = 1+1-1$, $C^0(\{A, A\}) = C^0(\{B, B\} = 1+2$, $C^0(\{A, B\}) = 1+1$, $C^0(\{A, AB\}) = C^0(\{B, AB\}) = 1 + 2 + 1 - 1$, $C^0(\{A, B, AB\} = 1 + 2 + 1 + 2 - 1$, and $C^0(\{AB, AB\}) = 1 + 2 + 1 + 2 - 1 + 0$.

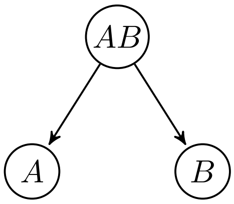

Figure 2: Cost function graph

r	$\Delta c(A, r)$	$\Delta c(B, r)$	$\Delta c(AB, r)$
1	1	1	-1
2	2	2	0

Table 2: Incremental cost

Example 2. There are three goods $N = \{A, B, C\}$ that can be bundled as any package $S \in 2^N$. We consider two different cost function graphs, shown in Fig. 3 and Fig. 5. In Fig. 3, each bundle S, $|S| \geq 2$, when allocated to a buyer, creates a cost corresponding to its own incremental cost function and costs related to its *subsets of a single variety.* With this type of packaging cost the overall allocation of each variety affects the cost of related (superset) bundles. However, packaging costs are independent between bundles that consist of more than one item. This is illustrated in Fig. 4 with the cost of selling the partition (or multiset) $\{A, B, C, AB, AC, BC, BC, ABC\}$.

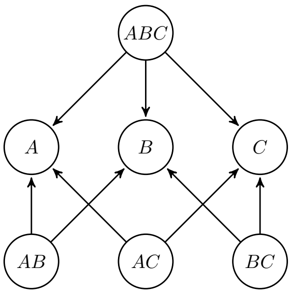

Figure 3: Cost function graph

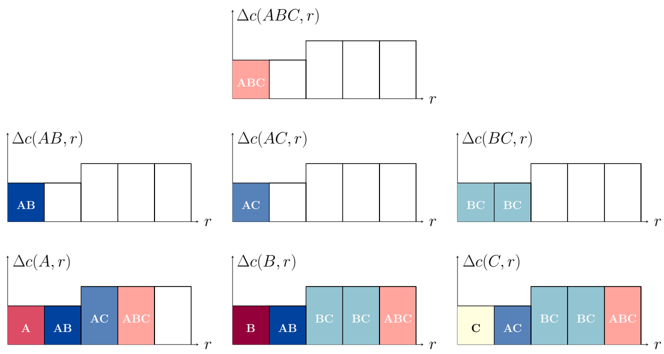

Figure 4: Cost of allocation $\{A, B, C, AB, AC, BC, BC, ABC\}$

In Fig. 5, each bundle S, $|S| \geq 2$, when allocated to a buyer, creates a cost corresponding to its own incremental cost function and costs related to *all of its subsets.* This means that the packaging cost of the sale of the package ABC also depends on how many units of AB (and AC and BC) are being sold. Incremental cost functions are weakly increasing, so the more of, e.g., AB is sold, the more expensive it becomes to sell the bundle ABC. We illustrate this further in Fig. 6.

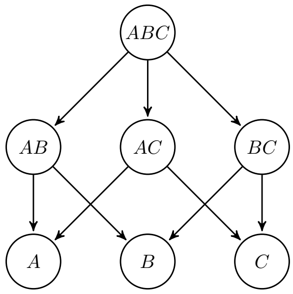

Figure 5: Complete cost function graph

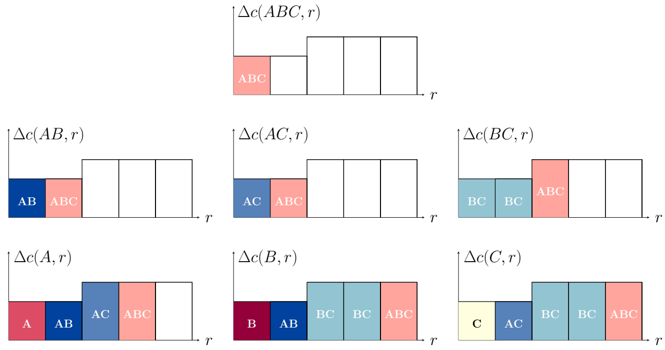

Figure 6: Cost of allocation $\{A, B, C, AB, AC, BC, BC, ABC\}$ (corresp. to complete graph)

The two cost function graphs in the example above illustrate the most extreme cases of no cost interdependence and cost interdependence between all available bundles. More generally, the cost functions satisfying Definition 7 allow for a wide range of packaging (or partitioning) costs, which may be important for the sale of complex bundles.

2.3 Demand, supply, and equilibrium

We now define demand, supply, pricing functions, and competitive equilibria. In our market, equilibrium prices are *package-linear*, i.e., the price of a partition is linear in packages. The seller chooses how to partition and allocate their supply of individual varieties to buyers, i.e., she chooses a feasible multiset (anonymous partition) $\boldsymbol{k} \in \mathcal{K}$ of packages to sell. An allocation of items in Ω is defined as an assignment $\pi = (\pi(l))_{l\in[L]_0}$ of these items between the buyers and the seller, such that $\sum_{l\in[L]} \pi(l) = \boldsymbol{k}$ and $\pi(0) = \Omega - \boldsymbol{k}^\star$. Recall that the unpacking operator $\star$ unpacks $\boldsymbol{k}$ into a vector of individual varieties. $\pi(l)$ is the package multiset assigned to agent l under the allocation π, where $\pi(l)$ may be the empty set, and $\pi(0) \neq \emptyset$ means that the items in $\pi(0)$ are not sold. For all agents $l \in [L]_0$, the demand correspondences (or supply correspondence for the seller), and indirect utilities, are defined as $D^l(p) := \arg\max_{\boldsymbol{k}\in\mathcal{K}} u^l(\boldsymbol{k}, p)$ and $\mathcal{V}^l(p) := \max_{\boldsymbol{k}\in\mathcal{K}} u^l(\boldsymbol{k}, p)$. An allocation π is *efficient* if, for every allocation π', it holds that $\sum_{l\in[L]} V^l(\pi(l)) - C^0(\boldsymbol{k}) \geq \sum_{l\in[L]} V^l(\pi'(l)) - C^0(\boldsymbol{k'})$, where $\boldsymbol{k} = \sum_{l\in[L]} \pi(l)$ and $\boldsymbol{k'} = \sum_{l\in[L]} \pi'(l)$. Given an efficient allocation π, the market value is defined as $V(\Omega) := \sum_{l\in[L]} V^l(\pi(l)) - C^0(\boldsymbol{k})$.

A *package-linear pricing Walrasian equilibrium* is a tuple (p^*, π^*), composed of a *package-linear* pricing function $p^* : 2^N \to \mathbb{R}$ and an allocation π^* such that $\sum_{l\in[L]} \pi^*(l) \in D^0(p^*)$ and $\pi^*(l) \in D^l(p^*)$ for every buyer $l \in [L]$. If the pricing function were linear in varieties, the package-linear Walrasian equilibrium would reduce to the well-known linear pricing Walrasian equilibrium.

3 Walrasian equilibrium

We formulate the social welfare maximization problem and use linear programming duality to characterize Walrasian equilibria. A crucial step towards this characterization is a closed form expression of equilibrium prices. Furthermore, we generalize the well-known result by Bikhchandani & Mamer (1997), providing necessary and sufficient conditions for the existence of a Walrasian equilibrium. Our first result asserts that the first and second welfare theorem hold in our market.[22]

Proposition 1. *If (p^*, π^*) is a package-linear Walrasian equilibrium, π^* is an efficient allocation. Furthermore, if π' is another efficient allocation, (p^*, π') is a package-linear Walrasian equilibrium as well.*

Proof. See Appendix A.1. □

3.1 Social welfare maximization

To formulate the social welfare maximization problem, we must aggregate the buyers' and the seller's preferences. As a first step, we reformulate the indirect utilities.

The buyers' value function can be rewritten using binary variables $x(S, q, l) \in \{0, 1\}$, indicating if fictitious unit-demand agent q is assigned bundle S or not. We rewrite the indicator vectors in Definition 6 as the constraint $\sum_{S \in 2^N} x(S, q, l) \leqslant 1$, $\forall q \in [\overline{Q}]$. Then, the buyers' indirect utility is given by

$$
\begin{aligned}
V^l(\boldsymbol{k}) = & \max_{\{x(S,q,l), S \in 2^N, q \in [\overline{Q}]\}} \sum_{q \in [\overline{Q}], S \in 2^N} v^l(S, q) x(S, q, l) - \sum_{S \in 2^N} k_S p(S) \\
& \text{s.t.} \quad x(S, [\overline{Q}], l) \leq k_S \ \forall S, \qquad x(2^N, q, l) \leq 1 \ \forall q, \qquad x(S, q, l) \in \{0, 1\} \ \forall S, q
\end{aligned} \tag{1}
$$

We denote by $\boldsymbol{x}(S, l)$ the vector $(x(S, l, q))_{q \in [\overline{Q}]}$, by $\boldsymbol{x}(q, l)$ the vector $(x(S, l, q))_{S \in 2^N}$, and by $\boldsymbol{x}(l)$ the vector $(x(l))_{S \in 2^N, q \in [\overline{Q}]}$.

For the seller, we also introduce a binary variable $y(S, r)$. This variable indicates if partition $\boldsymbol{k}$ invokes the assignment of step r of the incremental cost function associated with bundle S. The cost function graph is encoded in the constraint in the seller's cost function in Definition 9, given by $r_s = \sum_{S' \in R_0^-(S)} k_{S'}$. Formally, $y(S, r) = 1$ if $r \leq r_s$ and zero otherwise. We assume that there exists a finite number of steps such that $\Delta c(S, r) < \infty$. We denote this number by $\overline{R} := \max_{S,r}\{r : \Delta c(S, r) < \infty\}$ and $[\overline{R}] := \{1, \ldots, \overline{R}\}$.

Observation 2. If package S appears in the seller's partition, then one unit step on every incremental cost function corresponding to a successor of S, i.e., a package $S' \in R^+(S)$, must be assigned $y(S', \cdot) = 1$.

Proof. Observation 2 follows directly from the definition of r_S in Definition 9 and the definition of the $y(S, r)$ above. □

Observation 2 implies that the assignment on incremental cost function $\Delta c(S, \cdot)$ is limited by the minimum number of steps with finite height among all incremental cost functions $\Delta c(S', \cdot)$, $S' \in R_0^+(S)$. We write the seller's indirect utility as

$$
V^0(\boldsymbol{k}) = \max_{\boldsymbol{k} \in \mathcal{K}} \sum_{S \in 2^N} k_S p(S) - \sum_{S \in 2^N, r \in [r_s]} \Delta c(S, r) y(S, r) \quad \text{s.t.} \ r_s = \sum_{S' \in R_0^-(S)} k_{S'}, y(S, r) = 1 \ \forall r \leq r_s
$$

[22] The proof is standard but more general than the existing literature. E.g., Sun & Yang (2014) show the result for a market with one unit per per variety and no seller's costs.

Given a partition $\boldsymbol{k}$, $y(S,r)$ for $S \in 2^N$ and $r \in \mathbb{N}$ is uniquely defined. We also prove that the converse holds, i.e., each CFG allocation $y(S,r)$ for $S \in 2^N$ and $r \in \mathbb{N}$ can be mapped into a unique partition of packages $\boldsymbol{k}$. Crucially, this mapping is linear. This is formally stated in the following lemma.

Lemma 1. *Given a cost function graph G according to Definition 7, there exists a linear one-to-one mapping between an incremental cost function assignment $\{y(S,r)\}_{S\in 2^N, r\leq \bar{R}}$ and a corresponding package multiset $\boldsymbol{k}$. This mapping, the "characteristic function" $\phi^G : \mathbb{Z}_+^{2^n \times \bar{R}} \to \mathbb{Z}^{2^n}$, is inductively defined by Algorithm 1.*

Proof. See Appendix A.2. □

Intuitively, Algorithm 1 starts with a source $\mathcal{S}$ in the graph, for which $k_{\mathcal{S}} = y(S, [\bar{R}])$. Then, it selects some node S for which all predecessors have already been visited. Because the predecessors have been visited, we can compute $k_S = y(S, [\bar{R}]) - \sum_{S'\in R^-(S)} k_{S'}$. Then, it selects another node for which all predecessors have been visited, and so forth. For brevity, we write $\phi_S^G(\{S, [\bar{R}]\})$, $S \in 2^N$ simply as $\phi_S^G(\{S, [\bar{R}]\})$.

Algorithm 1: Construct partition from CFG assignment

Input: Cost function graph G with assignment $(y(S,r))_{S\in G, r\leq \bar{R}}$

Initialize list of successfully visited nodes $\mathscr{V} := \emptyset$.

while $V \neq \mathscr{V}$ **do**

  Select a node $S \in V \setminus \mathscr{V}$

  if $\exists S' \in V \setminus \mathscr{V} \cap R^-(S)$ **then** skip S;

  else

    set $k_S = y(S, [\bar{R}]) - \sum_{S' R^-(S)} k_{S'}$ and $\mathscr{V} = \mathscr{V} \cup S$

  end

end

return $(k_S)_{S\in 2^N} = (\phi_S^G(\{y(S, [\bar{R}])\}))_{S\in 2^N} = \phi^G(\{y(S, [\bar{R}])\})$

Using Lemma 1, we rewrite the seller's problem as follows.

$$\begin{aligned} V^0(\boldsymbol{k}) = &\max_{\{y(S,r), S\in 2^N, r\in[\bar{R}]\}} \sum_{S\in 2^N} k_S p(S) - \sum_{S\in 2^N, r\in[\bar{R}]} \Delta c(S,r) y(S,r) \\ &\text{s.t.} \quad k_S = \phi_S^G(\{y(S,r)\})\ \forall S, \qquad y(S,r) \in \{0,1\}\ \forall S, r \end{aligned} \tag{2}$$

With the reformulation of the buyers' and the seller's utility maximization problem given in Eqs. (1) and (2), we can now state the social welfare maximization problem. The objective is to find the *partition* $\boldsymbol{k} \in \mathcal{K}$ of supply between buyers that maximizes the sum of the buyers' and the seller's utilities, and thus the buyers' values minus the seller's costs. Because of Lemma 1, we can write the problem as an allocation problem with only binary decision variables and substitute k_S. The welfare maximization problem is named "SWP". For all constraints, we write short $\forall S$ for $\forall S \in 2^N$, $\forall q$ for $\forall q \in [\bar{Q}]$, $\forall r$ for $\forall r \in [\bar{R}]$, and $\forall l$ for $\forall l \in [L]$, and analogously for summation.

SWP

$$\max_{\{x(S,q,l),y(S,r),S\in 2^N,q\in\overline{Q},r\in\overline{R}\}} \left[\sum_{S,q,l} v^l(S,q)x(S,q,l) - \sum_{S,r} \Delta c(S,r)y(S,r)\right]$$

$$\begin{aligned} x(2^N,q,l) &\leq 1 && \forall q,l && (3)\\ x(S,[\overline{Q}],[L]) - \phi_S^G(\{y(S,r)\}) &\leq 0 && \forall S && (4)\\ x(S,q,l),y(S,r) &\in \{0,1\} && \forall S,q,r,l && (5) \end{aligned}$$

Note that to cancel the prices from the objective function we use the fact that, in equilibrium, equally many packages of type S must be sold as demanded. Relaxing the integrality constraints, we write the integer program as the linear program "SWLP" with corresponding dual variables listed next to the constraints. The feasible set of the SWLP is a non-empty, convex polytope and therefore an optimal solution always exists; by strong duality, an optimal solution for its dual "DSWLP" also exists.

SWLP

$$\max_{\{x(S,q,l),y(S,r),S\in 2^N,q\in\overline{Q},r\in\overline{R}\}} \left[\sum_{S,q,l} v^l(S,q)x(S,q,l) - \sum_{S,r} \Delta c(S,r)y(S,r)\right]$$

s.t.

$$\begin{aligned} x(2^N,q,l) &\leq 1 && \forall q,l && [b(q,l)] && (6)\\ x(S,[\overline{Q}],[L]) - \phi_S^G(\{y(S,r)\}) &\leq 0 && \forall S && [p(S)] && (7)\\ y(S,r) &\leq 1 && \forall S,r && [d(S,r)] && (8)\\ x(S,q,l),y(S,r) &\geq 0 && \forall S,q,r && && (9) \end{aligned}$$

Note that the constraint $x(S,q,l) \leq 1$ is implied by the first constraint and can thus be omitted. The corresponding dual problem "DSWLP" is constructed using the characteristic function dual $\psi^G(\{p(S)\})$ of $\phi^G(\{y(S,r)\})$. Formally, let $\phi^G(\{y(S,r)\}) = \Phi \boldsymbol{y}^\intercal$, where Φ is a $2^n \times 2^n$-matrix determined by Algorithm 1, and $\boldsymbol{y} = \left(y(S,[\overline{R}])\right)_{S\in 2^N}$, i.e., a row vector each entry of which contains the total quantity allocated on incremental cost function $\Delta c(S,\cdot)$. Thus, we have $\phi_S^G(\{y(S,r)\}) = \Phi_S \boldsymbol{y}^\intercal$, i.e., the row corresponding to package S of Φ multiplied by $\boldsymbol{y}^\intercal$. Finally, we can define the characteristic function dual $\psi^G(\{p(S)\}) = \Phi^\intercal \boldsymbol{p}^\intercal$, where $\boldsymbol{p} = (p(S))_{S\in 2^N}$.

DSWLP

$$\min_{\{b(q,l),p(S),d(S,r),S\in 2^N,q\in[\overline{Q}],r\in[\overline{R}],l\in[L]\}} \left[\sum_{q,l} b(q,l) + \sum_{S,r} d(S,r)\right]$$

s.t.

$$\begin{aligned} b(q,l) + p(S) &\geq v^l(S,q) && \forall S,q,l && [x(S,q,l)] && (10)\\ d(S,r) - \psi_S^G(\{p(S)\}) &\geq -\Delta c(S,r) && \forall S,r && [y(S,r)] && (11)\\ b(q,l),d(S,r),p(S) &\geq 0 && \forall S,q,r,l && && (12) \end{aligned}$$

The pricing function is given as part of the solution of the DSWLP: $p(S)$ is the value of the last "unit" of a given bundle S the seller allocates to a buyer. The dual variable $b(q,l)$

is fictitious agent q's (of buyer l) surplus, and $d(S,r)$ is the seller's "incremental surplus" from selling a bundle with cost connection to S assuming a corresponding incremental cost of $\Delta c(S,r)$.

Importantly, the pricing function is anonymous, facilitated by the preferences over *anonymous partitions*. As these prices are shown to be competitive equilibrium prices, they take into account interactions of package S with the aggregate package multiset, or partition, $\boldsymbol{k}$ the seller allocates. The price of the supplied package multiset, as well as the price for any buyer demanding a package multiset, is given by $p(\boldsymbol{k}) = \sum_{S\in 2^N} k_S p(S)$. We formally state the price structure in Proposition 2 below. We define $\widetilde{r}_S := \arg\max_r \{y(S,r) : y(S,r) > 0\}$, the last step on incremental cost function corresponding to bundle S, on which a positive quantity is allocated, and $\widetilde{r} := \min_{S\in 2^N} \widetilde{r}_S$.

Observation 3. For all $S \in 2^N$ and for all $S' \in R^+(S)$, it holds that $\widetilde{r}_S \leq \widetilde{r}_{S'}$.

Proof. As noted in Observation 2, the assignment $y(S,r) = 1$ of a step on some incremental cost function $\Delta c(S,\cdot)$ requires the assignment of some step r' such that $y(S',r') = 1$ on every incremental cost function $\Delta c(S',\cdot)$, if S has a cost connection to S', i.e., $S' \in R^+(S)$. Therefore, for any given partition, on any incremental cost function corresponding to S', there must be at least as many steps with $y(S',\cdot) = 1$ as on any incremental cost function corresponding to S. □

3.2 Competitive equilibrium and its pricing function

As is standard in linear programming, we work with complementary slackness conditions to establish a competitive equilibrium. Because the seller's language allows for preferences of partitions, this requires additional techniques, compared to, e.g., Bikhchandani & Mamer (1997), Bikhchandani & Ostroy (2002). Most importantly, we characterize in a first step the pricing function that supports, as we later demonstrate, a competitive equilibrium allocation.

Proposition 2. *Fixing any $r \leq \overline{R}$, it holds that $p(S) \leq \Delta c(R_0^+(S), r) + d(R_0^+(S), r)$ for all $S \in 2^N$. Moreover, for all $S \in 2^N$ and $r \leq \widetilde{r}_S$, it holds that $p(S) = \Delta c(R_0^+(S), r) + d(R_0^+(S), r)$.*

Proof. See Appendix A.2. □

From Proposition 2 it follows that the price for a package S, of which a positive amount is allocated in the market, is composed of the prices of packages which S has a cost connection to. Note that while $\Delta c(S,r)$ is the incremental cost corresponding to package S, the dual variable $d(S,r)$ is an additional variable allowing price flexibility, e.g., to set agents not receiving bundle S indifferent.

Corollary 1. The pricing function $p(\boldsymbol{k})$ is uniform, anonymous, and package-linear.

Corollary 1 is immediate from Proposition 2. The pricing function is a generalization of the uniform pricing rule for a market with multiple packages of different varieties. In Example 1, we further illustrate the equilibrium pricing function given by Proposition 2.

Note that competitive equilibrium prices are not necessarily unique. This creates flexibility for the seller to choose from a set of equilibrium prices, and she can specify additional rules to do so.[23] A set of lowest equilibrium prices on *all* packages may not always exist (cf. Example 1 below), but the seller may, for example, choose equilibrium prices that are lowest given a lexicographic ordering of bundles.

The term $\psi_S^G(\{p(S)\})$ in Eq. (11) defines the price gain from combining the individual items in S together in package S (as opposed to selling them separately), subject to also satisfying all other cost connections. Exploiting the price structure given by Proposition 2 and Lemma 5 (in the appendix[24]) we can further characterize the dual variable ψ_S^G.

Lemma 2. *For all $r \leq \widetilde{r}_S$, it holds that*

$$\psi_S^G(\{p(S)\}) = p(S) - \big(d(R^+(S), r) + \Delta c(R^+(S), r)\big) \, .$$

Moreover, $d(S, \widetilde{r}_S) + \Delta c(S, \widetilde{r}_S) \leq \Delta c(S, \widetilde{r}_S + 1)$.

Proof. See Appendix A.2. □

If the price is strictly less than $\Delta c(S, r)$, the seller prefers not to sell the package. If the price gain is strictly greater than $\Delta c(S, r+1)$, the seller prefers to sell an additional package S. We use Proposition 2, Lemma 2, and Lemma 5 to show the characterization of a competitive equilibrium by primal-dual solutions in Proposition 3 below.

Proposition 3. *Prices $\{p(S)\}$, which are part of an optimal solution of DSWLP, support the allocation $\{x(S, q, l), y(S, r)\}$, given by an optimal solution of SWLP, as a package-linear pricing Walrasian equilibrium.*

Proof. See Appendix A.2. □

Example 1 (Continued)**.** To illustrate the computation and existence of a Walrasian equilibrium, suppose that there are two additional unit-demand buyers labeled 3 and 4. All unit-demand values are given in Table 3.

The seller has cost savings if she sells one unit of A and one unit of B as a package; hence, agent 4 obtains $\{AB\}$ (breaking the tie between an alternative allocation of A to agent 4 and B to agent 1). For the second unit of A and B, the seller is indifferent between selling items A and B separately and selling them as a bundle, so agent 1 and agent 3 win. The assignment of sold bundles on incremental cost functions is shown in Fig. 7. The cost of supplying bundle AB consist of the cost of supplying A, B, and the packaging cost (savings) $\Delta c(AB, 1)$.

Prices must satisfy the set of equations that correspond to the constraints of DSWLP. We can verify that the set of equilibrium prices is given by $(p(A), p(B), p(AB) \in \{(4, 5, 9), (5, 4, 9), (5, 5, 9), (5, 5, 10)\}$, which all support the unique equilibrium allocation. From Proposition 2 we have that, e.g., $p(AB) = p(A) + p(B) + \Delta c(AB, 1) + d(AB, 1)$. Lemma 2 implies that $\Delta c(AB, 1) + d(AB, 1) \leq \Delta c(AB, 2) = 0$. Together with $d(AB, 1) \geq 0$ it follows that equilibrium prices must satisfy either $p(A) + p(B) = p(AB)$ or $p(A) + p(B) = p(AB) - 1$. The surplus constraints of DSWLP can be used to precisely pin down the set of equilibrium prices.

[23] Note that modern LP-solvers can return the set of all integer solutions.

[24] Lemma 5 establishes several useful facts that must hold in a competitive equilibrium. Analogous facts in a simpler setup are established as "characteristic conditions" in Baldwin & Klemperer (2019*a*).

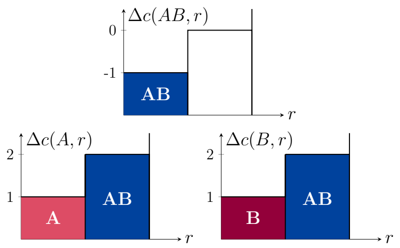

Figure 7: Incremental cost functions

q	$v^1(q,A)$	$v^1(q,B)$	$v^1(q,AB)$
1	3	$\underline{5}$	9
2	1	2	9
3	$\underline{5}$	3	8
4	6	2	$\underline{11}$

Table 3: Unit-demand values

From Proposition 3, we know that the defined auction prices support a package-linear Walrasian equilibrium. The following theorem asserts that a package-linear Walrasian equilibrium exists *if and only if* it can be characterized by an optimal solution of SWLP.

Theorem 1. A package-linear pricing Walrasian equilibrium exists if and only if any optimal solution to SWP is also an optimal solution to SWLP, i.e., the optimum values of SWP and SWLP coincide.

Proof. See Appendix A.2. □

Theorem 1 generalizes the well-known results by Bikhchandani & Mamer (1997) and Bikhchandani & Ostroy (2002): seller preferences in our model concern partitions. This introduces a richer and more complex price structure, the identification of which, in Proposition 2 and Lemma 2, is crucial for our results, including Lemma 5 and Proposition 3. Our Theorem 1 is first to accommodate preferences over partitions of supply in the primal-dual characterization and existence equivalence of Walrasian equilibrium.

The Walrasian equilibrium we describe has package prices which are identical for each given bundle. In this market, supply is given in terms of individual items, and the seller determines an efficient partition; thus, not only units of the same variety, but also packages composed of different varieties compete against each other for an efficient allocation. Our market can also be seen as 2^n simultaneous markets with competition between markets (or packages). Thus, the results from Swinkels (2001) on asymptotic efficiency apply. Under asymptotic environmental similarity[25] and other standard assumptions, they show that any equilibrium must be asymptotically ex-ante efficient. If a large number of buyers *for each package* participate in our market, with the assumptions of Swinkels (2001) being satisfied for each package separately, our market is also asymptotically ex-ante efficient.[26]

3.3 Closed form solutions for complete graphs

In general, cost function graphs are characterized by their characteristic function, which can be computed using Algorithm 1. For complete graphs, we derive a closed form solution of the characteristic function. To simplify notation, we first define "levels" within a graph.

[25] Asymptotic environmental similarity requires the probability of some buyer i winning the hth unit with bid b to converge uniformly to the probability of any buyer j winning the h'th unit with bid b, for all i, j, h, h', b. Bids in our market setup correspond to the preference of a unit-demand agent.

[26] Related results have been shown by Cripps & Swinkels (2006) and Fan et al. (2003).

Definition 10. Let $x, y \in 2^N$. Then $x \subset_t y := \{x \mid x \subseteq y, |y| - |x| = t\}$. y is said to be r levels above x, and x is r levels below y.

Similarly, we define $x \subset_{\geq t} y$ and $x \subset_{\leq t} y$, whereby $|y| - |x| = t$ is replaced with $|y| - |x| \geq t$ and $|y| - |x| \leq t$, respectively. $x \supset_0 y$ implies $x = y$. In a complete cost function graph, each bundle has a cost connection to all of its subsets. This can be achieved with a minimal number of edges, i.e., there are directed edges from each package S to its subsets of size $|S| - 1$ only. An example of a complete cost function graph was shown in Fig. 5.

Proposition 4. *Let G be a complete cost function graph. Then its characteristic function is given by* $\phi_S^G(\{y(S, r)\}) = \sum_{t=0}^{n-|S|} \sum_{r, S' \supset_t S} (-1)^t y(S', r)$.

Proof. See Appendix A.2. □

We also derive a closed form solution for the characteristic function dual.

Corollary 2. Let G be a complete cost function graph. Then its characteristic function dual is given by $\psi_S^G(\{p(S)\}) = \sum_{t=0}^{|S|-1} \sum_{S' \subset_t S} (-1)^t p(S')$.

Complete cost function graphs may be especially relevant in the procurement of factor inputs: if services A and B are delivered by the same provider, the manufacturer incurs cost savings, and similarly for services A and C. However, if services B and C are delivered by the same provider, additional costs arise, e.g., because of substitutability between B and C paired with a high contingency risk of the provider. If all three services A, B, and C were delivered by the same provider, cost savings from A and B, A and C, and additional costs from pairing B and C enter the cost function, plus an additional cost savings term to account for the interaction of A, B, and C together.

4 Equilibrium existence

In this section, we study a special case of the market introduced earlier. In this market, the seller has only one item per variety to sell and prefers coarser partitions of her supply between buyers to less coarse partitions. In addition, buyers regard all available items as complements. We prove that under these conditions, a package-linear Walrasian equilibrium exists.

The supply is given by $\Omega_j = 1$ for all items $j \in N$. The seller chooses a supply partition, or package multiset, which is now given by $\boldsymbol{k} \in \{0, 1\}^{2^n}$. Each buyer has only one corresponding unit-demand agent with valuation $v^l : 2^N \to \mathbb{Z}_+$. The utility of a buyer is given by $u^l(S, p) = v^l(S) - p(S)$. The seller's cost can also be simplified. Instead of incremental costs, we define the seller's marginal cost of selling a package $S \in 2^N$ as $c^0 : 2^N \to \mathbb{Z}_+$. The aggregate cost of selling partition $\boldsymbol{k}$ is given by $C^0(\boldsymbol{k}) = \sum_{S \in 2^N} c^0(S) k_S$. To further simplify notation in this section, we interpret $\boldsymbol{k}$ directly as a partition of N into packages, i.e., $\boldsymbol{k} = \{S_1, \ldots, S_k\}$ where $S_t \in 2^N, t = 1, \ldots, k$, $S_{t_1} \cap S_{t_2} = \emptyset$ for all $S_{t_1} \neq S_{t_2}$, and $\bigcup_{t=1}^k S_t \in 2^N$. An allocation $\pi = (\pi(0), \pi(1), \ldots, \pi(L))$ can also be interpreted as a partition $\boldsymbol{k}$ of N, where $\pi(l)$ is removed if it is the empty set. As before, $\mathcal{K}$ denotes the universe of all partitions of N. Thus, the seller's utility is given by $u^0(\boldsymbol{k}, p) = \sum_{S \in \boldsymbol{k}} \big(p(S) - c^0(S)\big)$ when she sells the set of packages $\boldsymbol{k}$ at prices p. The demand and supply correspondences and indirect utilities remain unchanged.

Assumption 2. The buyers' valuations are superadditive, i.e., for all disjoint $S_1, S_2 \in 2^N$ and for all buyers $l \in [L]$, $v^l(S_1) + v^l(S_2) \leq v^l(S_1 \cup S_2)$.

Superadditivity is the most general concept of complementarity, which contains supermodularity and gross complements (Samuelson 1974, Sun & Yang 2014).

Assumption 3. The seller's marginal costs are subadditive, i.e., for all disjoint $S_1, S_2 \in 2^N$, $c^0(S_1) + c^0(S_2) \geq c^0(S_1 \cup S_2)$.

In this market, we are able to show that a package-linear Walrasian equilibrium exists.

Theorem 2. If the buyers' value functions satisfy superadditivity and the seller's marginal cost function satisfies subadditivity, there exists a package-linear Walrasian equilibrium.

The proof of Theorem 2 proceeds by constructing an equilibrium with a modified version of the ascending auction by Sun & Yang (2014) (henceforth SY).[27] We also extend this modified ascending auction to a version that strictly generalizes the auction by SY. We first state the procedure and then investigate the relationship of our market to the model and auction by SY.

4.1 Constructing an equilibrium

In each round of the ascending auction, the seller states her supply, and buyers state their demand at current prices. If a package is overdemanded, the price of this package increases by one in the next round. The procedure stops as soon as no package is overdemanded. We let $p(t, S)$ denote the price of bundle $S \in 2^N$ at time t. The seller and buyers behave straightforwardly. Buyer l bids straightforwardly with respect to his values v^l if, at every time $t \in \mathbb{Z}_+$ and for any prices $p(t)$, he demands $S^l(t) \in D^l(p(t)) = \arg\max_{S \in 2^N} \left\{v^l(S) - p(t, S)\right\}$, where $S^l(t) = \emptyset$ when $\emptyset \in D^l(p(t))$. That is, in each round, he demands a bundle that maximizes his utility given the current auction prices. We denote aggregate demand by the partition $\boldsymbol{k}^D(t) := \{S^1(t), \ldots, S^L(t)\}$. Aggregate demand can also be interpreted as a vector $\boldsymbol{k}^D \in \mathbb{Z}_+^{2^n}$, where each element counts how many buyers demand a package $S \in 2^N$. The seller behaves straightforwardly with respect to her cost C^0 if, at every time $t \in \mathbb{Z}_+$ and for any prices $p(t)$, she chooses a supply partition $\boldsymbol{k}(t) \in D^0(p(t)) = \arg\max_{\boldsymbol{k} \in \mathcal{K}} \sum_{S \in \boldsymbol{k}} \big(p(t, S) - c^0(S)\big)$. We say that a package is overdemanded at time t iff $k_S^D(t) > k_S(t)$, where $\boldsymbol{k}^D(t)$ and $\boldsymbol{k}(t)$ are the aggregate demand and the supply reported at time t, respectively. Formally, the procedure is described in Algorithm 2.

[27] In the market with only one unit per variety, the notion of package-linear pricing coincides with the nonlinear pricing in Sun & Yang (2014).

Algorithm 2: Ascending auction

Seller states initial reserve prices c^0 setting $t = 0$ and initial prices $p(0, S) = c^0(S)$ $\forall S \in 2^N$.

All buyers $l \in [L]$ report a demanded bundle $S^l(0)$ and the seller chooses a supply partition $\boldsymbol{k}(0)$.

while *some package* S *is overdemanded at time* t **do**

  Set $p(t + 1, S) = p(t, S) + 1$ for all overdemanded bundles S, i.e., those with $k_S^D(t) > k_S(t)$.

  Set $p(t + 1, S') = p(t, S')$ for all bundles S' that are not overdemanded at t.

  Set $t = t + 1$.

end

The auction terminates at $t = t^*$. Every bundle $S \in S^l(t^*)$ is allocated to buyer l at price $p(t^*, S)$.

for *any* $S \in \boldsymbol{k}(t^*)$ *not demanded by any buyer at* $p(t^*)$ **do**

  if $p(t^*, S) = c^0(S)$ **then** S remains with the seller;

  else

    S is allocated at price $p(t^*, S)$ to a buyer who demanded and was among the last to forfeit S in a previous round.

  end

end

return Walrasian equilibrium prices $p(t^*, S)$, $S \in 2^N$ and allocation π

Proposition 5. *If all buyers bid straightforwardly, the ascending auction given by Algorithm 2 terminates in a package-linear pricing Walrasian equilibrium after a finite number of rounds.*

Proof. See Appendix A.3. □

Theorem 2 follows directly from Proposition 5. The ascending auction specified by Algorithm 2 is modified from the ascending auction in SY by the participation of a seller with partition preferences. We also extend the procedure to fully generalize the ascending auction of SY in Section 4.3. To do so, we first demonstrate the key difference in the seller's preferences. In the next section, we investigate how these preferences and their properties relate to ours.

4.2 Revenue vs. utility maximization

The market studied by SY includes a revenue-maximizing seller whose preferences are symmetric to buyers. If this seller does not sell some bundle S, she derives a utility $v^0(S)$ from it. The function $v^0 : 2^N \to \mathbb{Z}_+$ is called a "reserve price" and is assumed to be superadditive. The seller obtains the market price of all sold bundles, and the sum of these prices plus the value of the unsold bundle is defined as the seller's revenue.[28] In contrast, our seller has *preferences over partitions* expressed through a cost function. In our model, the seller's cost of supplying several bundles is the sum of their marginal costs. The total cost depends on the partitioning of the supply, and the seller's utility is defined as the sum of obtained prices minus her total cost.

In the following, we provide an equivalence characterization between revenue maximization as in SY and utility maximization based on cost functions. This equivalence implies fundamentally different cost functions in the model by SY and ours, and it reveals a new class of preferences and may thus be of independent interest.

[28]With bundling, true reserve prices are not straightforward. Consider the supply of two goods A and B, which can be bundled as AB. Suppose that the package AB is worth $v(AB)$ to the seller and that A and B are worth $v(A)$ and $v(B)$, respectively. At prices $p(A), p(B), p(AB)$, the seller chooses to sell A alone only if $p(A) + v(B) \geq v(AB)$ and $p(A) + v(B) \geq p(AB)$. Thus, her true reserve price for A is $v(AB) - v(B)$. At this price, the seller is indifferent between selling A and keeping the bundle AB.

First, we introduce the set function dual.[29] Let S^c denote the complement for any set $S \in 2^N$, i.e., $S^c = N \setminus S$.

Definition 11. For any bundle $S \in 2^N$, given a set function $f : 2^N \to \mathbb{R}$ with $f(\emptyset) = 0$, define the transformation $g(f, S) = f(N) - f(S^c)$. $g(f, \cdot)$ is called the *set function dual* of f.

Note that $g(f, N) = f(N)$. With the set function dual, we are equipped to formalize the relationship between revenue maximization and utility maximization.

Proposition 6. *Given a value function $v^0 : 2^N \to \mathbb{Z}_+$ with $v^0(\emptyset) = 0$, the objective of maximizing revenue is equivalent to maximizing utility, where the marginal costs $c^0 : 2^N \to \mathbb{Z}_+$ are given by the dual of v^0, and the total cost of any given partition $\boldsymbol{k}$ are $C^0(\boldsymbol{k}) = c^0\left(\bigcup_{S \in \boldsymbol{k}} S\right)$.*

Proof. See Appendix A.3. □

We introduce a new notion of submodularity to further characterize the revenue-maximizing seller's cost function.

Definition 12 (Set-cover submodularity)**.** Given a finite set N, a function $f : 2^N \to \mathbb{R}$ is *set-cover submodular* if $\forall\ S_1, S_2 \in 2^N$ with $S_1 \cup S_2 = N$,

$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2).$$

If the inequality sign is reversed, f is *set-cover supermodular*, and if replaced with an equality sign, f is *set-cover modular*. The following lemma makes the crucial connection between superadditive and set-cover submodular functions.[30]

Lemma 3. *Given a superadditive (subadditive) function $f : 2^N \to \mathbb{R}$, its dual $g(f, \cdot)$ is set-cover submodular (set-cover supermodular).*

Proof. See Appendix A.3. □

From Lemma 3 and Proposition 6 the subsequent corollary is immediate.

Corollary 3. The revenue-maximizing seller's marginal costs are set-cover submodular.

Set-cover submodularity is weaker than submodularity, because it is only required for every two subsets of N, the union of which fully covers N. In particular, set-cover submodularity does not imply subadditivity (unlike proper submodularity, which does imply subadditivity). A set-cover submodular cost function can have strictly subadditive and strictly superadditive elements, as shown in Example 3 below. A revenue maximizer only takes into account the *set* of sold items, whereas a utility maximizer with a cost function as we define it cares about the *partition* of sold items between buyers.

Example 3. The function v in Table 4 is superadditive and its dual c is set-cover submodular. The dual c is strictly subadditive with respect to A and BC $(2 + 3 > 4)$, but superadditive with respect to items B and C $(2 + 0 < 3)$. Selling any partition of $\{A, B, C\}$ at 4 is cheaper than selling $\{B, C\}$ (3) and $\{A\}$ (2).

[29] The set function dual is common in matroid theory, see e.g., Gul & Stacchetti (2000) or Fujishige (2005).

[30] Set-cover submodular and set-cover supermodular functions can both be subadditive or superadditive.

S	A	B	C	AB	AC	BC	ABC
v	1	2	0	4	2	2	4
c	2	2	0	4	2	3	4

Table 4: Superadditive function v and set-cover submodular dual c

Our seller's preferences are substantially different from those in SY, as shown in Proposition 6. Moreover, as the revenue-maximizing seller is symmetric to buyers, we can generalize the ascending auction to incorporate agents with both types of cost functions.

4.3 The extended ascending auction

In the extended auction, the rules for buyers are identical to the ascending auction in SY, but differ for the sellers. A revenue-maximizing seller as well as an additional seller with its own supply, or an additional agent with partition preferences over the original seller's supply may participate in the auction.

The revenue-maximizing seller surrenders her supply to the auctioneer and disguises herself as a buyer in the auction, attempting to buy back her supply. Her bids serve as reserve prices. In addition to potentially winning a bundle herself, she receives the price of every bundle (of her original supply) sold to other agents. We formally state this in Proposition 7. Example 5 further illustrates that the extended ascending auction may lead to different outcomes depending on the total cost function used, even when marginal costs and values are dual to each other.

In the following, we consider the extended ascending auction with a revenue-maximizing seller and an auctioneer with partition preferences over the seller's supply. The variation of the market in which the auctioneer sells an additional set of items that is disjoint from the other seller's supply is analogous, as long as items from the two sellers are not bundled together.[31] The following lemma and proposition show that our procedure strictly generalizes the ascending auction by SY.

Lemma 4. *If the auctioneer's marginal costs are zero, the extended ascending auction and the ascending auction by SY terminate in the same allocation (up to ties).*

Proof. See Appendix A.3. □

Proposition 7. *Suppose the auctioneer's marginal costs are zero and the extended ascending auction starts at prices $p(-1, S) = v^0(S) - 1\ \forall S \in 2^N$. Then, there exists a price path that is identical in the extended ascending auction and in the ascending auction by SY, resulting in final prices that support the same allocation in both auctions.*

Proof. See Appendix A.3. □

With non-zero marginal costs, the auctioneer can express additional preferences. The existence of a package-linear Walrasian equilibrium is also guaranteed in the extended ascending auction. In the literature, the existence of a Walrasian equilibrium has been demonstrated for

[31] If multiple sellers were participating in the auction, a rule for surplus sharing would have to be defined. SY propose Shapley's allocation rule for sharing surplus in an extension to multiple sellers.

markets with only substitutes (linear pricing) or only complements (non-linear pricing, cf. SY), and some markets with a mixture of substitutes and complements (see also the Introduction). To our knowledge, we are the first to prove the existence of package-linear Walrasian equilibrium with preferences over partitions. Subadditive cost functions are especially relevant in markets with a high degree of fragmentation in which the auctioneer wishes to encourage bundle allocations. We note the generality of (weak) superadditivity of values, as they allow buyers to be interested in a single item only or to have merely additive values.

5 Discussion

5.1 Implementation as a sealed-bid auction

Under the assumption of approximately competitive behavior, the Product-Mix Auction Klemperer (2008, 2010, 2018) implements a competitive equilibrium with uniform, competitive prices in markets for substitutes. The Bank of England has been using this sealed-bid procedure to allocate loans to commercial banks against different types of collateral since the financial crisis in 2007. Similarly, our market may be implemented as a sealed-bid auction, assuming a large market with many buyers, in which all agents, including the seller, behave non-strategically. Our buyer preferences are related to the preferences buyers can submit in the Product-Mix auction, where each buyer submits a finite list of "paired bids" bids to the auctioneer. Such a list of bids can also be seen as the aggregation of multiple unit-demand valuations, and our buyer preferences can be interpreted as a generalization of (some) Product-Mix preferences to incorporate packaged items.

Our seller's preferences also lend themselves for implementation as a bidding language. The seller can simply submit their cost function graph and incremental cost functions to a clearing house or auctioneer. In general, in combinatorial allocation problems, there is no hope to achieve better than exponential communication, even if agents truthfully report their preferences (Nisan & Segal 2006). The aggregation of incremental costs significantly reduces the communication in the context bundles, as only the incremental cost functions of single varieties are mandatory for the allocation to be solvable. Cost savings or increments for the combination of single varieties can be specified only for relevant bundles and omitted for those the seller is indifferent about.

5.2 The seller's demand type

If all agents have strong substitutes valuations and supply is fixed, it is well known that a linear pricing Walrasian equilibrium exists (Milgrom & Strulovici 2009).[32] The notion of strong substitutes can be readily extended to packages using the concept of demand types (Baldwin & Klemperer 2019*b*). An agent's demand type defines how their demand may change due to small price changes, i.e., which trade-offs between items (in our setup, bundles) are allowed. A valuation is strong substitutes if and only if it is concave and these trade-offs are one-to-one.[33]

[32] See also (Danilov et al. 2001) and Baldwin & Klemperer (2019*b*).

[33] Baldwin & Klemperer (2019*b*) show that a valuation is strong substitutes iff it is concave and corresponds to a strong substitutes demand type. A demand type is specified for quasi-linear utilities by a list of vectors describing the directions in which demand could change due to an infinitesimal generic price change. A strong substitutes demand type is defined by vectors with at most one +1 entry, at most one -1 entry, and no other

Our buyers' preferences correspond to aggregations of unit-demand agents with one-to-one trade-offs and can thus be said to be "strong substitutes between packages". However, for this analogy between items and bundles to be complete, one would also require a fixed supply of bundles. In our market, supply is given in terms of items, and the seller partitions these items into packages. Thus, in the context of package-linear pricing, her preferences are not strong substitutes between packages, and a package-linear pricing Walrasian equilibrium may not exist.

We illustrate this with Example 2 from Sun & Yang (2014) (also Beviá et al. (1999)). Three buyers 1, 2, and 3 are interested in purchasing three items A, B, and C with values given in Table 5. The seller's costs are zero. Walrasian equilibrium does not exist, either with linear pricing (Beviá et al. 1999) or with non-linear pricing (Sun & Yang 2014). All buyers demand only one package, hence their valuations are strong substitutes between packages. However, induced by an infinitesimal price change, the seller may wish to, e.g., sell partition $\{AB\}$ instead of $\{A, B\}$.[34] Therefore, the seller's demand type is not strong substitutes between packages.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Buyer 1	0	10	8	2	13	11	9	14
Buyer 2	0	8	5	10	13	14	13	15
Buyer 3	0	1	1	8	2	9	9	10

Table 5: Buyers' valuations of bundles

6 Conclusion

In many markets for bundled items, it is natural for the seller to have preferences over the partitioning of items between buyers. We analyze a competitive market in which such preferences are modeled as incremental cost functions together with a graph that defines cost interdependencies. To our knowledge, ours is the first study of Walrasian equilibrium and its existence in the presence of partition preferences. Our characterization and existence results are facilitated by our proposed cost framework, including incremental costs and a cost function graph, which possesses a characteristic function and its corresponding dual. An additional existence result relies on an ascending auction. Moreover, we uncover a dual relationship between valuations and costs defined on sets of indivisible goods.

Although our results are theoretical, this article also aims to inspire market design in practice, especially in (near-)competitive environments. Our setting allows the seller to express vastly richer preferences than described in the previous literature and used in present-day auction design. The package-linear Walrasian equilibrium prices we describe have a modular structure: the price of a given package is tied to the marginal costs of varieties, or subsets of varieties, contained in that package (based on the cost interdependencies specified by the seller). This ensures transparency in pricing between related packages. Moreover, in combinatorial auctions in practice, this guarantees that even if not all bundles are bid for, plausible and transparent

non-zero entries. For details, we refer to Baldwin & Klemperer (2019*b*).

[34] In the example, the seller's "demand" type characterization would contain the vectors $\pm(1, 1, 0, -1, 0, 0, 0)$. The entries of the vector correspond to the change in her supply of packages $(A, B, C, AB, AC, BC, ABC)$ induced by an arbitrary, small price change.

equilibrium bundle prices can be easily constructed. Finally, our framework of preferences may be of independent interest in other allocation problems, where the distribution of goods, services, or matchings in the market matters.

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Appendix

A Proofs

A.1 Proofs for Section 2

Proof of Proposition 1. Let $\mathcal{A}$ denote the universe of all feasible allocations, i.e.,

$$\mathcal{A} := \left\{ \pi \in \mathbb{Z}_+^{2^n \times |[L]_0|} : \sum_{l \in [L]} \pi(l) = \boldsymbol{k} \text{ for some } \boldsymbol{k} \in \mathcal{K}, \text{ and } \pi(0) = \Omega - \boldsymbol{k}^* \right\}$$

(p^*, π^*) is a package-linear pricing Walrasian equilibrium, so for any buyer $l \in [L]$ and any allocation $\pi' \in \mathcal{A}$, we have

$$V^l(\pi^*(l)) - \sum_{S \in \pi^*(l)} p^*(S) \geq V^l(\pi'(l)) - \sum_{S \in \pi'(l)} p^*(S)$$

Let $\sum_{l \in [L]} \pi^*(l) = \boldsymbol{k}$ and $\sum_{l \in [L]} \pi'(l) = \boldsymbol{k}'$. We sum over $l \in [L]$ and add and subtract the seller's cost

$$\begin{aligned} &\sum_{l \in [L]} V^l(\pi^*(l)) - C^0(\boldsymbol{k}) - \left(\sum_{l \in [L]} V^l(\pi'(l)) - C^0(\boldsymbol{k}') \right) \\ &\geq \sum_{l \in [L]} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\boldsymbol{k}) - \left(\sum_{l \in [L]} \sum_{S \in \pi'(l)} p^*(S) - C^0(\boldsymbol{k}') \right) \end{aligned} \tag{13}$$

Because $\pi^* \in D^0(p^*)$, we have, for all $\pi' \in \mathcal{A}$,

$$\sum_{l \in [L]} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\boldsymbol{k}) \geq \sum_{l \in [L]} \sum_{S \in \pi'(l)} p^*(S) - C^0(\boldsymbol{k}')$$

From Eq. (13), it follows that, for all $\pi' \in \mathcal{A}$,

$$\sum_{l \in [L]} V^l(\pi^*(l)) - C^0(\boldsymbol{k}) - \left(\sum_{l \in [L]} V^l(\pi'(l)) - C^0(\boldsymbol{k}') \right) \geq 0$$

and so π^* is efficient.

Now let π' be an efficient allocation. Then $V(\Omega) = \sum_{l \in [L]} V^l(\pi'(l)) - C^0(\boldsymbol{k}')$. It also holds that $V(\Omega) = \sum_{l \in [L]} V^l(\pi^*(l)) - C^0(\boldsymbol{k})$ because π^* is efficient as part of the equilibrium. The equilibrium is also buyer-optimal and seller-optimal, given prices. Thus, we obtain the following two inequalities:

$$\begin{aligned} &\mathcal{V}^l(p^*) \geq V^l(\pi'(l)) - \sum_{S \in \pi'(l)} p^*(S), \qquad \text{for all } l \in [L] \text{ and} \\ &\sum_{l \in [L]} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\boldsymbol{k}) = \mathcal{V}^0(p^*) \geq \sum_{l \in [L]} \sum_{S \in \pi'(l)} p^*(S) - C^0(\boldsymbol{k}') \end{aligned}$$

Suppose that one of these two inequalities were strict, then we would obtain

$$\begin{aligned}
V(\Omega) &= \sum_{l\in[L]} V^l(\pi^*(l)) - C^0(\boldsymbol{k}) \\
&= \sum_{l\in[L]} \left[V^l(\pi^*(l)) - \sum_{S\in\pi^*(l)} p^*(S) + \sum_{S\in\pi^*(l)} p^*(S) \right] - C^0(\boldsymbol{k}) \\
&= \sum_{l\in[L]} \mathcal{V}^l(p^*) + \mathcal{V}^0(p^*) \\
&> V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S) + \sum_{l\in[L]} \sum_{S\in\pi'(l)} p^*(S) - C^0(\boldsymbol{k}') \\
&= \sum_{l\in[L]} V^l(\pi'(l)) - C^0(\boldsymbol{k}') \\
&= V(\Omega),
\end{aligned}$$

This is a contradiction, and consequently it holds that

$$\begin{aligned}
\mathcal{V}^l(p^*) &= V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S), \qquad \text{for all } l \in [L] \text{ and },\\
\mathcal{V}^0(p^*) &= \sum_{l\in[L]} \sum_{S\in\pi'(l)} p^*(S) - C^0(\boldsymbol{k}'), \qquad \text{i.e., } \pi' \in D^0(p^*).
\end{aligned}$$

It follows that (p^*, π') is also a package-linear pricing Walrasian equilibrium. □

A.2 Proofs for Section 3

Proof of Lemma 1. First, we show that given a cost function graph G, any incremental cost function assignment $(y(S,r))_{S\in 2^N, r\leq \bar{R}}$ maps into a unique vector $(k_S)_{S\in 2^N}$. To proof that Algorithm 1 constructs a unique image from any permissible input, we demonstrate that (a) Algorithm 1 terminates after a finite number of steps, (b) each node S is successfully visited at some point, and (c) the time at which S is successfully visited is irrelevant. The mapping is linear by the definition of k_S in the algorithm.

(a) and (b) follow because Definition 7(i) implies that there are no cycles in G. Thus, as long as $V \neq \mathscr{V}$, there exists some $S \in V \setminus \mathscr{V}$ for which the if-condition is false. Consequently, all nodes are added to $\mathscr{V}$ at some point. It is without loss of generality to require that the algorithm does not get stuck in a trivial loop, i.e., it does not select a sequence of nodes that allow no successful visit and are skipped and revisited indefinitely. (c) follows because once the if-condition is false for a given node S, the set of nodes $A \in \mathscr{V} : \exists(A...S)$ remains unaltered. Once S *could* be successfully visited, is does not matter when it is actually selected for the successful visit, i.e., other nodes may be selected first.

The reverse mapping is straightforward. Given a cost function graph G, a partition $\boldsymbol{k} = (k_S)_{S\in 2^N}$ maps in the following incremental cost function assignment: for all $S \in 2^N$, $y(S, r_S) = 1 \; \forall r_S \leq \sum_{A:\exists(A...S)} k_A$ and $y(S, r_S) = 0 \; \forall r > \sum_{A:\exists(A...S)} k_A$. □

Proof of Proposition 2. The difficulty in proving this result is that, with the introduced formulation of the social welfare maximization problem, prices are nested in the dual characteristic function. To disentangle them, we use a different but equivalent formulation of the SWP. Recall that whenever a step on incremental cost function $\Delta c(S, \cdot)$ is allocated, a step on every incremental cost function $\Delta c(S', \cdot)$ for all $S' \in R^+(S)$ must be allocated also (see also Observation 3). Thus, on each incremental cost function associated with package S, some steps may be allocated that are directly linked to the allocation S and, in addition, some steps may be allocated that are related to the allocation of some package $S' \in R^-(S)$. To make this distinction, we denote a cost function step as $r(S', S)$ if it is allocated on cost function $\Delta c(S', \cdot)$ due to the allocation of package S (to a buyer). With this definition, $\sum_{r(S,S)} y(S, r(S,S)) = k_S$.

In the original SWLP, the cost function graph was encoded in the characteristic function ϕ^G. In the subsequent alternative formulation SWLP$'$, the cost function graph is encoded in the equality constraints which require that, for each incremental cost step corresponding to the allocation of package S, cost steps on all incremental cost functions corresponding to successors $R^+(S)$ are also allocated. Recall that $S \in R_0^+(S)$. In the notation of dual variables we write, e.g., $d(S', S, r(S', S))$ as $d(S', S, r)$ for brevity.

SWLP$'$

$$\max_{\substack{\{x(S,q,l), y(S', r(S',S)), r(S',S) \in [\bar{R}] \\ \forall S \in 2^N, S' \in R_0^+(S)), q \in [\bar{Q}]\}}} \left[\sum_{S,q,l} v^l(S,q) x(S,q,l) - \sum_{S, S' \in R_0^+(S), r(S',S)} y(S', r(S',S) \Delta c(S', r(S',S)) \right]$$

s.t.

$$\begin{array}{rcll} x(2^N, q, l) & \leq & 1 \;\; \forall q, l & b(q,l) \quad (14) \\ x(S, [\bar{Q}], [L]) - \sum_{r(S,S)} y(S, r(S,S)) & \leq & 0 \;\; \forall S & [p(S)] \quad (15) \\ y(S', r(S',S)) & \leq & 1 \;\; \forall S, S' \in R_0^+(S), r(S',S) & [d(S',S,r)] \quad (16) \\ y(S, r(S,S)) - y(S', r(S',S)) & = & 0 \;\; \forall S, S' \in R^+(S), r(S',S) & [u(S',S,r)] \quad (17) \\ x(S,q,l), y(S, r(S',S)) & \geq & 0 \;\; \forall S, S' \in R_0^+(S), q, r(S',S) & \quad (18) \end{array}$$

We formulate the dual of SWLP' as follows.

DSWLP$'$

$$\min_{\{b(q,l), p(S), d(S',S,r), u(S',S,r\}} \left[\sum_{q,l} b(q,l) + \sum_{S, S' \in R_0^+(S), r(S',S)} d(S',S,r) \right]$$

s.t.

$$\begin{array}{rlll} b(q,l) + p(S) \geq v^l(S,q) & \forall S, q, l & [x(S,q,l)] & (19) \\ -p(S) + d(S,S,r) + u(R^+(S), S, r) \geq -\Delta c(S, r(S,S)) & \forall S, r(S,S) & [y(S, r(S,S))] & (20) \\ d(S',S,r) - u(S',S,r) \geq -\Delta c(S', r(S',S)) & \forall S, S' \in R^+(S), r(S',S) & [y(S', r(S',S))] & (21) \\ b(q,l), d(S',S,r), p(S) \geq 0, u(S',S,r) \in \mathbb{R} & \forall S, S' \in R^+(S), q, r(S',S) & & (22) \end{array}$$

Substituting Eq. (21) into Eq. (20) and rearranging, we obtain $p(S) \leq \sum_{S' \in R_0^+(S)} d(S',S,r) + \Delta c(S', r(S',S))$. On any cost function step that is allocated a positive quantity $y(S', r(S',S)) > 0$ complementary slackness implies that constraints (20) and (21) hold with equality. While $r(S',S)$ designates which package is allocated on step r of incremental cost function $\Delta c(S', \cdot)$, the order in which packages are allocated on incremental cost functions does not matter; it is only the sum of all allocated incremental cost steps that determines the seller's costs. Because $r(S', S)$ could indeed be any of the steps on $\Delta c(S', \cdot)$, on which a positive amount is allocated, we can omit the specification of S and S'. The first statement of Proposition 2 holds then indeed for any step r, and the second statement for all steps r on which a positive quantity is allocated, i.e., all $r \leq \widetilde{r}_S$. □

Observation 4. Complementary slackness from LP duality implies the following observations, corresponding to the constraints in Eqs. (6) to (8), (10) and (11).

(i) If $x(2^N, q, l) < 1$, then $b(q, l) = 0 \; \forall q \in [\overline{Q}], l \in [L]$. (S1)

(ii) If $x(S, [\overline{Q}], [L]) - \phi_S^G(\{y(S, r)\}) < 0$, then $p(S) = 0 \; \forall S \in 2^N$. (S2)

(iii) If $y(S, r) < 1$, then $d(S, r) = 0 \; \forall S \in 2^N, r \leq \overline{R}$. (S3)

(iv) If $x(S, q, l) \neq 0$, then $b(q, l) = v^l(S, q) - p(S) \; \forall S \in 2^N, q \in [\overline{Q}], l \in [L]$. (D1)

(iv) If $y(S, r) \neq 0$, then $d(S, r) + \Delta c(S, r) = \psi_S^G(\{p(S)\}) \; \forall S \in 2^N, r \leq \overline{R}$. (D2)

Lemma 5. *If $\{x(S, q, l), y(S, r)\}$ and $\{b(q, l), d(S, r), p(S)\}$, $S \in 2^N, q \in [\overline{Q}], l \in [L], r \in [\overline{R}]$, are solutions to SWLP and DSWLP, respectively, the following facts hold.*

(i) $\sum_{S' \in R_0^-(S)} x(S', [\overline{Q}], [L]) \leq y(S, [\overline{R}])$, *for all* $S \in 2^N$. *(F1)*

(ii) *If* $p(S) > 0$, *then* $\sum_{S' \in R_0^-(S)} x(S', [\overline{Q}], [L]) = y(S, [\overline{R}])$, *for all* $S \in 2^N$. *(F2)*

(iii) *If* $v^l(S, q) - p(S) < \max_{S' \neq S}\{v^l(S', q) - p(S'), 0\}$, *then* $x(S, q, l) = 0$, *for all* S, q, l. *(F3)*

(iv) *If* $\max_{S \in 2^N}\{v^l(S, q) - p(S)\} > 0$, *then* $x(2^N, q, l) = 1$, *for all* $q \in [\overline{Q}], l \in [L]$. *(F4)*

(v) *If* $\Delta c(S, r) < \psi_S^G(\{p(S)\})$, *then* $y(S, r) = 1$, *for all* $S \in 2^N$, $r \leq \overline{R}$. *(F5)*

(vi) *If* $\Delta c(S, r) > \psi_S^G(\{p(S)\})$, *then* $y(S, r) = 0$, *for all* $S \in 2^N$, $r \leq \overline{R}$. *(F6)*

Proof of Lemma 5. (F1) is obtained by summing up the bundle S supply constraints from SWLP. In particular, we sum over all $S' \in R_0^-(S)$. By definition of the incremental cost functions, $\sum_{S' \in R_0^-(S)} Y_{S'} = \sum_{r \in [\overline{R}]} y(S, r)$ for any package $S \in 2^N$.

To show (F2), first note the contrapositive of (S2): if $p(S) > 0$, $x(S, [\overline{Q}], [L]) - \phi^G(\{y(S, r)\}) = 0$ (> 0 is ruled out by constraint Eq. (7) of the SWLP). We wish to sum this constraint across all packages $S' \in R_0^-(S)$ of which some positive quantity is allocated, i.e., $y(S', r) > 0$ for some step r. For $y(S', r) > 0$, we can apply Proposition 2, and with $C^0(\boldsymbol{k}) \geq 0 \; \forall \boldsymbol{k} \in \mathcal{K}$ and $d(S, r) \geq 0 \; \forall S, r$ it follows that $p(S') > 0$ for all $S' \in R_0^-(S)$. Therefore, we can take the sum of tight supply constraints corresponding to packages $S' \in R_0^-(S)$, noting that including those packages S' of which nothing is allocated, i.e., $y(S', r) = 0$ for all r (and thus also $x(S', q, l) = 0$ for all q, l), does not change the sum. Thus, (F2) follows.

(F3) is derived from (D1): Assume $x(S, q, l) > 0$ and $x(S'', q, l) > 0$, $S \neq S''$. Then, by (D1), $v^l(S, q) - p(S) = v^l(S'', q) - p(S'') = b(q, l)$. So if $v^l(S, q) - p(S) < v^l(S', q) - p(S')$ for some S', then $x(S, q, l) = 0$. Furthermore, $b(q, l) \geq 0$ and $b(q, l) \geq v^l(S, q) - p(S)$ by Eq. (12). Thus, if $v^l(S, q) - p(S) < 0$, then $v^l(S, q) - p(S) < b(q, l)$, and thus $x(S, q, l) = 0$. Together, we obtain (F3).

To show (F4), note that $b(q, l) \geq v^l(S, q) - p(S)$ implies that, if $\max_{S \in 2^N}\{v^l(S, q) - p(S)\} > 0$, then $b(q, l) > 0$. The contrapositive of (S1) then implies $x(2^N, q, l) = 1$, and thus (F4). (F5) follows by the contrapositive of (S3), as from the constraint $d(S, r) \geq \psi^G(\{p(S)\}) - \Delta c(S, r)$ it follows that if $\Delta c(S, r) < \psi^G(\{p(S)\})$, then $d(S, r) > 0$ for any $S \in 2^N$ and $r \in [\overline{R}]$. Finally, if $\Delta c(S, r) > \psi^G(\{p(S)\})$, then it must be that $d(S, r) > \psi^G(\{p(S)\}) - \Delta c(S, r)$, because $d(S, r)$

is non-negative. Hence, the contrapositive of (D2) implies $y(S,r)=0$, and therefore (F6). $\square$

Proof of Lemma 2. From Proposition 2, we have $p(S)=\Delta c(R_0^+(S),r)+d(R_0^+(S),r)$. Substituting (D2) into Proposition 2, the first statement follows. The contrapositive of (F5) gives $y(S,\widetilde{r}_S+1)=0 \Rightarrow \Delta c(S,\widetilde{r}_S+1)\geq \psi_S^G(\{p(S)\})$. As $y(S,\widetilde{r}_S)=1$, we can substitute (D2) and the second statement follows. $\square$

Proof of Proposition 3. Assume the allocation $\{x(S,q,l),y(S,r)\}$ and prices $\{p(S)\}$ are solutions of SWLP and DSWLP as defined above. By Lemma 5, conditions (F1) - (F6) hold. In the following, we show that, together with (S1) - (D2) and the constraints of SWLP and DSWLP, Lemma 5 implies that the prices $\{p(S)\}$ support $\{x(S,q,l),y(S,r)\}$ as a package-linear pricing Walrasian equilibrium.

We prove that (a) there is no surplus improvement possible for any fictitious agents q and any buyer l, (b) for a buyer who received a package multiset, no surplus improvement is possible from reassigning elements of the multiset to different unit-demand agents, (c) that, given an allocated supply partition and prices, the seller cannot improve her utility by allocating more or less of a given package, and (d) that, given an allocated supply partition and prices, the seller cannot improve her utility by choosing a different partition.

(a): If the surplus of unit-demand agent (q,l) is negative on all packages in its valuation $v^l(S,q)$, (F3) ensures that this unit-demand agent is not assigned anything. (F4) implies that, if a strictly positive surplus can be made on any bundle of some unit-demand agent, the maximum quantity of one is allocated to that agent, and from (F3) it follows that only bundles that maximize the surplus of agent (q,l) are allocated with a positive quantity. Eq. (6) ensures that not more than the maximum quantity of one is allocated.

(b): If a buyer receives a multiset of items $\boldsymbol{k}$, it is value-maximally assigned to his corresponding unit-demand agents by Definition 6. Recall that the unpacking of multisets is not allowed by our model assumptions. To see that *at the given auction prices*, a buyer has no incentive to reassign elements of the allocated multiset between his unit-demand agents, let $\boldsymbol{k}$ denote the multiset of items received by buyer l and let $\mathcal{Q}^l$ denote the set of the corresponding winning unit-demand agents, i.e., $\boldsymbol{k}=\left(\sum_{q\in\mathcal{Q}^l}x(S,q,l)\right)_{S\in 2^N}$. The buyer's utility is given by $u^l(\boldsymbol{k},p)=\sum_{q\in\mathcal{Q}^l,S}(v^l(S,q)-p(S))x(S,q,l)$. Suppose that in an alternative assignment $\widetilde{\boldsymbol{x}}(l)\neq\boldsymbol{x}(l)$ such that $\left(\sum_{q\in\widetilde{\mathcal{Q}}^l}\widetilde{x}(S,q,l)\right)_{S\in 2^N}=\boldsymbol{k}$, which gives strictly higher utility, i.e., $\widetilde{u}(\boldsymbol{k},p)=\sum_{q\in\mathcal{Q}^l,S}(v^l(S,q)-p(S))\widetilde{x}(S,q,l)>u(\boldsymbol{k},p)$. Then, there exist at least two unit-demand agents $i,j\in\widetilde{\mathcal{Q}}$, for which $1=\widetilde{x}(\widetilde{S},q,l)\neq x(\widetilde{S},q,l)=0$, $(q,\widetilde{S})=(i,\widetilde{S}_i),(j,\widetilde{S}_j)$ and $0=\widetilde{x}(S,q,l)\neq x(S,q,l)=1$, $(q,S)=(i,S_i),(j,S_j)$. Moreover, $v^l(\widetilde{S}_i,i)-p(\widetilde{S}_i)+v^l(\widetilde{S}_j,j)-p(\widetilde{S}_j)>v^l(S_i,i)-p(S_i)+v^l(S_j,j)-p(S_j)=b(i,l)+b(j,l)$. However, by Eq. (10), we must have $v^l(S,q)-p(S)\leq b(q,l)$ for all $S\in 2^N$, a contradiction. Thus, no additional surplus can be generated by shifting allocations between winning unit-demand bidders. Moreover, no additional surplus can be generated by allocating to non-winning unit-demand agents, as, by the contraposition of (F4) we must have $p(S)\geq v^l(S,q)$ for all $S\in 2^N$, for all $q\in[\overline{Q}]\setminus\widetilde{\mathcal{Q}}$. Thus, $\widetilde{u}(\boldsymbol{k},p)\leq u(\boldsymbol{k},p)$.

(c): If a step in the incremental cost function $\Delta c(S,\cdot)$ is allocated, i.e., $y(S,r)>0$, then the contrapositive of (F6) implies $\psi^G(\{p(S)\}) \geq \Delta c(S,r)$. Together with Lemma 2 we have $p(S) \geq (d(R^+(S),r) + \Delta c(R^+(S),r)) + \Delta c(S,r)$, i.e., the seller always sells package S at a weakly positive surplus. Furthermore, if $p(S) > 0$, it follows by (F2) that $\sum_{S' \in R_0^-(S)} x(S', [\overline{Q}], [L]) = y(S, [\overline{R}])$, for all $S \in 2^N$, i.e., the amount of all packages with cost connection to S sold equals $y(S, [\overline{R}])$ and no package assigned on some incremental cost function goes to waste.

Now we show that, if a positive surplus can be made on some incremental cost function step r corresponding to package S, then it is assigned $y(S,r) = 1$. Suppose $p(S) > \Delta c(S,r) + d(R^+(S),r) + \Delta c(R^+(S),r)$, i.e., a strictly positive surplus is made on package S (recall $d(S,r) \geq 0$ for all S, r). Then, we have

$$\begin{aligned}
d(S,r) &= d(R_0^+(S),r) + \Delta c(R_0^+(S),r) - \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) - \Delta c(S,r) \\
&\geq p(S) - \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) - \Delta c(S,r) \\
&> 0
\end{aligned}$$

By the contrapositive of (S3), $d(S,r) > 0$ implies that $y(S,r) = 1$.

We also show that, if a loss would be made by assigning step r corresponding to package S, then $y(S,r) = 0$. Note that the loss is compared to not assigning the items in S at all, or compared to assigning the items contained in S as a different partition (with elements that are strict subsets of S). Thus, the shadow prices $d(S,r)$, which capture potential gains in these subsets, appear in the following equation. Let $p(S) < d(R^+(S),r) + \Delta c(R^+(S),r) + \Delta c(S,r)$, i.e., assigning package S is not profitable at the given prices. For contradiction, suppose $y(S,r) > 0$. Then, Lemma 2 applies, and

$$\begin{aligned}
\psi^G(\{p(S)\}) &= p(S) - \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) \\
&< d(R^+(S),r) + \Delta c(R^+(S),r) + \Delta c(S,r) - \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) \\
&= \Delta c(S,r)
\end{aligned}$$

Then, by (F6) we must have $y(S,r) = 0$.

(d) We claim that, given a partition of supply that is a solution to SWLP, the seller cannot improve her utility by choosing a different partition of supply. To see this, recall Lemma 1, which states that the mapping from the assignment on incremental cost functions $(y(S,r))_{S \in 2^N, r \leq \overline{R}}$ to a package multiset $(k_S)_{S \in 2^N}$ is one-to-one. In (c), we have shown that the seller cannot improve her surplus given the assignment on incremental cost functions, given the dual prices. By Lemma 1, the resulting partition of supply is unique and also optimal for the seller. □

Proof of Theorem 1. Let O_{SWP} (O_{SWLP}, O_{DSWLP}) denote the value of an optimal solution to SWP (SWLP, DSWLP), respectively. First, suppose $O_{\text{SWP}} = O_{\text{SWLP}}$. Then, there exists $\{x(S,q,l), y(S,r)\}$ as an optimal solution to SWP and SWLP, and $\{x(S,q,l), y(S,r)\}$ is efficient. By Proposition 3, the dual variables $\{p(S)\}$ of DSWLP support this allocation as a package-linear pricing Walrasian equilibrium.

Now suppose that there exists an equilibrium, i.e., prices $\{p(S)\} \geq 0$ that support $\{x(S,q,l), y(S,r)\}$ as an equilibrium allocation. By Proposition 1, the allocation is efficient. Let $b(q,l) :=$

$v^l(S,q) - p(S)$ for all $q \in [\bar{Q}], l \in [L] : x(S,q,l) > 0$, and let $d(S,r) := \psi_S^G(\{p(S)\}) - \Delta c(S,r)$ for all $S \in 2^N, r \leq \widetilde{r}_S : y(S,r) > 0$. The dual variables $d(S,r)$, $S \in 2^N, r \leq \widetilde{r}_S$, are defined recursively by Lemma 2: for $j \in N$, we have $\psi_S^G(\{p(j)\}) = p(j)$, so, by Proposition 2, $d(j,r) := p(j) - \Delta c(j,r)$ for all $j \in N, r \leq \bar{R}$. Given $d(j,r)$, one can go on to define $d(S,r)$ for all $S \in N^-(j), j \in N$, etc. The $d(S,r)$ are the seller's surplus on each individual supply step, and the $b(q,l)$ are the surplus of each unit-demand agent. Because $\{p(S)\}$ are Walrasian equilibrium prices, it must be that the surpluses are positive, i.e., $d(S,r), b(q,l) \geq 0$. Together with their definitions, this implies that $b(q,l), d(S,r)$, and $p(S)$ are feasible in DSWLP.

We now show that the seller's revenue $\sum_S p(S)Y_S$ is equivalent to the sum of the seller's revenue corresponding to each incremental cost function, $\sum_{S,r:y(S,r)=1} \psi_S^G(\{p(S)\})$. Using Lemma 6, we have

$$\sum_{S,r:y(S,r)=1} \psi_S^G(\{p(S)\}) = \sum_{S,r:y(S,r)=1} p(S) - \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big)$$

Moreover, recall that the quantity assigned on an incremental cost function $\Delta c(S,\cdot)$ equals the sum of all packages sold that have a cost connection to S, i.e., $\sum_r y(S,r) = \sum_{S' \in R_0^-(S)} Y_{S'} = Y_S + \sum_{S' \in R^-(S)} Y_{S'}$. Thus, we further write

$$\sum_{S,r:y(S,r)=1} \psi_S^G(\{p(S)\}) = \sum_S p(S) Y_S + F, \quad \text{where}$$

$$F := \sum_S p(S) \left(\sum_{S' \in R^-(S)} Y_{S'} \right) - \sum_S \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) \left(\sum_{S' \in R_0^-(S)} Y_{S'} \right)$$

Using Proposition 2, we substitute for $p(S)$) and obtain

$$\begin{aligned} F &= \sum_S \big(d(R_0^+(S),r) + \Delta c(R_0^+(S),r)\big) \left(\sum_{S' \in R^-(S)} Y_{S'} \right) \\ &\quad - \sum_S \big(d(R^+(S),r) + \Delta c(R^+(S),r)\big) \left(\sum_{S' \in R_0^-(S)} Y_{S'} \right) \\ &= \sum_S (d(S,r) + \Delta c(S,r)) \left(\sum_{S' \in R^-(S)} Y_{S'} \right) \\ &\quad - \sum_S \left(\sum_{S' \in R^+(S)} d(S',r) + \Delta c(S',r) \right) Y_S \end{aligned}$$

The term F is equal to zero because of symmetry: the sum of the products of each node with the sum of its predecessors is equal to the sum of the products of each node with its successors. Because we are taking the sum across all nodes, it does not matter in which direction we traverse the directed graph. Thus, we have shown that $\sum_S p(S)Y_S = \sum_{S,r:y(S,r)=1} \psi_S^G(\{p(S)\})$.

Finally, efficiency of the package-linear Walrasian equilibrium (cf. Proposition 1) implies that the allocation $\{x(S,q,l), y(S,r)\}$ is optimal in SWLP. By strong duality, $O_{\text{SWLP}} = O_{\text{DSWLP}}$

holds. Thus, we have

$$
\begin{aligned}
O_{\text{SWLP}} &= O_{\text{DSWLP}} \\
&\overset{(1)}{\leq} \sum_{q,l} b(q,l) + \sum_{S,r} d(S,r) \\
&\overset{(2)}{=} \sum_{S,q,l:x(S,q,l)=1} (v^l(S,q) - p(S)) + \sum_{S,r:y(S,r)=1} (\psi_S^G(\{p(S)\}) - \Delta c(S,r)) \\
&\overset{(3)}{=} \sum_{q,l,S} v^l(S,q)x(S,q,l) - \sum_{S,r} \Delta c(S,r)y(S,r) \\
&\overset{(4)}{=} O_{SWP}
\end{aligned}
$$

(1) follows from the objective function of DSWLP. (2) follows by definition of $b(q,l)$ and $d(S,r)$ above. (3) follows because $\sum_{S,q,l:x(S,q,l)} p(S) = \sum_S Y_S p(S)$. Finally, (4) follows from the definition of SWP. Overall, $O_{\text{SWLP}} \leq O_{SWP}$. It also holds that $O_{\text{SWLP}} \geq O_{SWP}$ because any solution of SWP is feasible in SWLP, and the claim follows. $\square$

Lemma 6. *Given are sets $x \subseteq z \in 2^N$, and a number q with $|x| \leq q \leq |z|$. Let $R := \{y \in 2^N | x \subseteq y \subseteq z, |y| = q\}$. Then, $|R| = \binom{|z|-|x|}{q-|x|}$.*

Proof. This is a standard combinatorics problem. First, note that $q - |x|$ items can be added to x such that y contains q items. These items also need to be different from those contained in x, and they need to be contained in z. Hence, there are $|z| - |x|$ different items, of which $q - |x|$ many can be added to x. This is possible in $\binom{|z|-|x|}{q-|x|}$ different ways. $\square$

Proof of Proposition 4. Let $W \subseteq S \subseteq S' \in 2^N$, and let $y(S, S', t, W)$ denote the amount allocated on incremental cost function $\Delta c(S, \cdot)$ that is due to the cost connection of allocated bundle $S' \supset_t W$, t levels above W. Let $dist(x, y) := ||x| - |y||$ for any $x, y \in 2^N$. We first establish a series of facts.

Fact (1). Given any reference supply set $W \subseteq S$, we can write the amount allocated on incremental cost function $\Delta c(S, \cdot)$ as

$$\sum_q y(S, r) = \sum_{t=dist(S,W)}^{n-|W|} \sum_{S' \supset_t W} y(S, S', t, W).$$

Fact (2). Given S, S', W, let $r := dist(S, W)$ and $t := dist(S', W)$. By Lemma 6 above, there exist $\binom{t}{r}$ incremental cost functions $\Delta c(S, \cdot)$ relative to $\Delta c(W, \cdot)$, on which the amount $y(S, S', t, W)$ is allocated.

Fact (3). $y(S, S', t, W)$ does not depend on S. If a step on incremental cost function $\Delta c(S', \cdot)$ is allocated, then a step on each incremental cost function $S \subseteq S'$ is allocated. Thus, the allocation of bundles on incremental cost functions in the graph “between” W and S' *due to the allocation of bundle S'* has to be the same amount.

Fact (4). For any $t \geq 1$, we have, by the binomial theorem, $\sum_{z=0}^{t} \binom{t}{z} (-1)^z = 0$.

Using the facts above (references in the equation below), we have

$$
\begin{aligned}
& \sum_{t=0}^{n-|S|} \sum_{r} \sum_{S' \supset_t S} (-1)^t y(S', r) \\
= \quad & \sum_{t=0}^{n-|S|} \sum_{S' \supset_t S} (-1)^t \sum_{r} y(S', r) \\
\overset{(1)}{=} \quad & \sum_{t=0}^{n-|S|} \sum_{S' \supset_t S} (-1)^t \sum_{t=dist(S',S)}^{n-|S|} \sum_{S'' \supset_t S} y(S', S'', t, S) \\
= \quad & \sum_{t=0}^{n-|S|} \sum_{S' \supset_t S} (-1)^t \sum_{z=t}^{n-|S|} \sum_{S'' \supset_z S} y(S', S'', z, S) \\
\overset{(2),(3)}{=} \quad & \sum_{t=0}^{n-|S|} (-1)^t \sum_{z=t}^{n-|S|} \binom{z}{t} \sum_{S'' \supset_z S} y(\cdot, S'', z, S) \\
\overset{(4)}{=} \quad & \sum_{t=0}^{n-|S|} \sum_{S'' \supset_t S} y(\cdot, S'', t, S) \sum_{z=0}^{t} (-1)^z \binom{t}{z} \\
= \quad & \sum_{S'' \supset_0 S} y(\cdot, S'', 0, S) \sum_{t=0}^{0} (-1)^t \binom{0}{t} \\
= \quad & y(\cdot, S, 0, S) \\
= \quad & \phi_S^G(\{y(S, r)\})
\end{aligned}
$$

□

A.3 Proofs for Section 4

Proof of Proposition 5. The proof proceeds by analogy with Sun & Yang (2014), but the detailed arguments differ. The auction terminates at some time t^*, because buyers' values are finite, i.e., demand ceases at some point. The empty package is always priced at zero.

Let $p^* = p(t^*)$ and let $S^{l*} = S^l(t^*)$. Furthermore, let $\boldsymbol{k}^* = \boldsymbol{k}(t^*) \in D^0(p^*)$ denote the supply set in $D^0(p^*)$ that is chosen at time t^* by the seller. First, we establish an allocation π^* such that (p^*, π^*) constitutes a package-linear Walrasian equilibrium. Because at p^* no package is overdemanded, for any buyer $l \in [L]$ with $S^{l*} \neq \emptyset$, his demand S^{l*} must be in $\boldsymbol{k}^*$. If $\bigcup_{l \in [L]} S^{l*} = N$ holds, let $\pi^*(l) = S^{l*}$ for all $l \in [L]$ and $\pi^*(0) = \emptyset$. Then (p^*, π^*) is a package-linear Walrasian equilibrium. If $\bigcup_{l \in [L]} S^{l*} \subset N$, there is at least one package B in the chosen supply set $\boldsymbol{k}^*$ which is not demanded by any buyer at time t^*. By analogy with SY, B is called a *squeezed-out* package. We distinguish multiple cases:

Case 1: $p^*(B) = c^0(B)$. The final price of bundle B is still fixed at the starting price, so B was never overdemanded. If a buyer demanded it in some earlier round, this buyer demands now a different, more profitable package. Let $\boldsymbol{k}_0^* = \{B \in \boldsymbol{k}^* \mid p^*(B) = c^0(B) \text{ and } B \neq S^{l*} \text{ for all } l \in [L]\}$ be the set of all squeezed-out packages. Let $\pi^*(0) = \bigcup_{B \in \boldsymbol{k}_0^*} B$ and allocate $\pi^*(0)$ to the seller at zero cost. Let $\mathcal{K}_0^*$ denote the universe of all partitions of the items contained in $\boldsymbol{k}_0^*$.

Because $\boldsymbol{k}^* \in D^0(p^*)$, we have

$$\sum_{B \in \gamma} \left[p^*(B) - c^0(B)\right] \leq \sum_{B \in \boldsymbol{k}_0^*} \left[p^*(B) - c^0(B)\right] = 0$$

for all $\gamma \in \mathcal{K}_0^*$. Hence, the seller is indifferent between selling $\pi^*(0)$ or not.

Case 2: $p^*(B) > c^0(B)$. Package B was demanded by some buyer in an earlier round. Denoting by t the last round in which B was demanded by some buyer l, B may be allocated to buyer l at the current price $p^*(B)$ by the auction rule. Thus, we need to demonstrate that it is still utility-maximizing for buyer l to receive package B at the current price. By the auction rule, we must have $\mathcal{V}^l(p(t)) = u^l(B, p(t)) = v^l(B) - p(t, B) \geq 1$ and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. Thus, we have for buyer l, who is allocated the squeezed-out package B,

$$u^l(B, p^*) = v^l(B) - p^*(B) \geq 0 \tag{23}$$

Now two sub-cases need to be distinguished:

Case 2A: If $S^{l*} = \emptyset$, assign buyer l the squeezed-out bundle, i.e., $\pi^*(l) = B$. $S^{l*} \in D^l(p^*)$ and $S^{l*} = \emptyset$ imply that $\mathcal{V}^l(p^*) = 0$. By definition of $\mathcal{V}^l$ we have $\mathcal{V}^l(p^*) \geq u^l(B, p^*)$. Together with Eq. (23) this implies $u^l(B, p^*) = 0$, and hence $\pi^*(l) \in D^l(p^*)$.

Case 2B: If $S^{l*} \neq \emptyset$, assign buyer l what he demanded at time t^* and the squeezed-out bundle, i.e., $\pi^*(l) = S^{l*} \cup B$. Because the seller chose a supply set $\boldsymbol{k}^* \ni \{S^{l*}, B\}$, we have

$$p^*(S^{l*}) - c^0(S^{l*}) + p^*(B) - c^0(B) \geq p^*(\pi^*(l)) - c^0(\pi^*(l)). \tag{24}$$

Superadditivity of l's utility, subadditivity of the seller's cost, and $S^{l*} \in D^l(p^*)$ imply

$$v^l(\pi^*(l)) \geq v^l(S^{l*}) + v^l(B) \tag{25}$$

$$c^0(\pi^*(l)) \leq c^0(S^{l*}) + c^0(B) \tag{26}$$

$$v^l(S^{l*}) - p^*(S^{l*}) \geq v^l(\pi^*(l)) - p^*(\pi^*(l)). \tag{27}$$

From Eqs. (24) and (26) follows

$$p^*(S^{l*}) + p^*(B) \geq p^*(\pi^*(l)). \tag{28}$$

Then, using equation Eq. (28), Eq. (25), and Eq. (23) (in this order), we obtain

$$\begin{aligned} v^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq v^l(\pi^*(l)) - \left[p^*(S^{l*}) + p^*(B)\right] \\ &\geq \left[v^l(S^{l*}) - p^*(S^{l*})\right] + \left[v^l(B) - p^*(B)\right] \\ &\geq v^l(S^{l*}) - p^*(S^{l*}). \end{aligned}$$

By Eq. (27), $v^l(\pi^*(l)) - p^*(\pi^*(l)) = v^l(\pi^*(l)) - \left[p^*(S^{l*}) + p^*(B)\right] = v^l(S^{l*}) - p^*(S^{l*})$, and thus

$$p^*(\pi^*(l)) = p^*(S^{l*}) + p^*(B). \tag{29}$$

Buyer l is therefore happy to receive bundle B in addition to his demanded bundle A_k^*, and pay the price that is set for the bundle $\pi^*(l)$. This process can be repeated for every squeezed-out bundle B with $p^*(B) > c^0(B)$. Every buyer l who is not allocated any squeezed-out bundle receives his demanded package, i.e., $\pi^*(l) = S^{l*}$. $\boldsymbol{k}^*$ is a partition of N chosen by the seller, and thus $(\pi^*(0), \ldots, \pi^*(L))$ is an allocation of N. By Eq. (29), the seller's utility is

$$\sum_{l\in[L]} \left[p^*(\pi^*(l)) - c^0(\pi^*(l))\right] = \sum_{A\in \boldsymbol{k}^*} \left[p^*(A) - c^0(A)\right] = \mathcal{V}^0(p^*)$$

It follows that (p^*, π^*) is a package-linear pricing Walrasian equilibrium. $\square$

Proof of Proposition 6. In SY, the seller's supply correspondence is defined as

$$S(p) = \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}} p(A) \right\}$$

In our ascending auction the seller's supply correspondence is defined as

$$D^0(p) = \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}} \left(p(A) - c^0(A)\right) \right\}$$

In SY's ascending auction, in holds that $p(B) = v^0(B)$ for any bundle B that is assigned to the seller during the procedure and $p(\pi(0)) = v^0(\pi(0))$. Hence, we have

$$\begin{aligned} S(p) &= \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}\setminus B} p(A) + v^0(B) \right\} \\ &= \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}\setminus B} p(A) + v^0(B) - v^0(N) \right\} \\ &= \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}\setminus B} p(A) - c^*(N\setminus B) \right\} \\ &= \arg\max_{\boldsymbol{k}\in\mathcal{K}} \left\{ \sum_{A\in\boldsymbol{k}\setminus B} p(A) - c^*\left(\bigcup_{A\in\boldsymbol{k}\setminus B} A\right) \right\} \end{aligned}$$

c^* is by definition the dual of v^0, and $c^*\left(\bigcup_{A\in\boldsymbol{k}\setminus B} A\right)$ may be interpreted as the seller's cost function. Thus, part (i) and (ii) of the proposition follow. $\square$

Proof of Lemma 3. To simplify notation, we write $c^*(v^0, S)$ as $c^*(S)$ for any $S \in 2^N$. Let $S_1^c, S_2^c \in 2^N$ and $S_1^c \cap S_2^c = \emptyset$. Note that $c^*(N) = v^0(N)$ and $S_1^c \cap S_2^c = \emptyset \Leftrightarrow S_1 \cup S_2 = N$.

Because v^0 is superadditive we have

$$\begin{aligned} & v^0(S_1^c \cup S_2^c) \geq v^0(S_1^c) + v^0(S_2^c) \\ \Leftrightarrow \quad & v^0(N) - c^*((S_1^c \cup S_2^c)^c) \geq 2v^0(N) - c^*(S_1) - c^*(S_2) \\ \Leftrightarrow \quad & c^*(S_1) + c^*(S_2) \geq c^*(S_1 \cup S_2) + c^*(S_1 \cap S_2) \end{aligned}$$

The proof for subadditive v^0 is analogous. □

Proof of Lemma 4. With a revenue-maximizing seller, an allocation π is efficient if it holds for every allocation π' that

$$\sum_{l \in [L]_0} \left[v^l(\pi(l)) \right] \geq \sum_{l \in [L]_0} \left[v^l(\pi'(l)) \right] \tag{30}$$

With a utility-maximizing seller, an allocation π is efficient if for every allocation π' it holds that

$$\sum_{l \in [L]} \left[v^l(\pi(l)) - c^0(\pi(l)) \right] \geq \sum_{l \in [L]} \left[v^l(\pi'(l)) - c^0(\pi'(l)) \right] \tag{31}$$

SY's ascending auction terminates in an efficient allocation π in the sense of Eq. (30). The extended ascending auction terminates in an efficient allocation π in the sense of Eq. (31). If the auctioneer's marginal costs are zero, the efficient allocation in the extended ascending auction is equivalent to the efficient allocation in the sense of Eq. (30): running the auction with the set of buyers $[L]' = [L]+\{0\} = [L]_0$, Eq. (30) and Eq. (31) are equivalent and the claim follows. □

Proof of Proposition 7. Note that every conventional buyer bids identically in the ascending auction and the extended ascending auction, up to ties. We split the revenue-maximizing seller into 2^n dummy buyers, denoted $l_S, S \in 2^N$. Define dummy l_S's utility function as follows:

$$v^{l_S}(B) := \begin{cases} v^0(S) & \text{if } B \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

Each dummy l_S has the highest bid on bundle S among all dummies because v^0 is superadditive. Let dummy l_S demand bundle S whenever he weakly prefers S to any other bundle except the empty set and let him demand the empty set when she weakly prefers to do so.

Let the extended ascending auction start at $t = -1$ with starting prices $p(-1, S) = v^0(S) - 1 \; \forall S \in 2^N$. Let two instances of each dummy l_S participate. Dummies $l_S, S \in 2^N$ each demand bundle S. The auctioneer offers some supply set. Regardless of the non-dummy buyers' demand, each bundle $S \in 2^N$ is overdemanded in $t = -1$, so prices in $t = 0$ are increased by one. The dummies all demand the empty set for all $t = 0, 1, \ldots$, so if at some round $t \geq 0$ the auction ends with squeezed-out bundles, they can be allocated to the dummies if they were the last to demand them. It is without loss of generality to stipulate that the squeezed-out bundles *are* allocated to dummies in this case, and not to regular buyers who might have demanded them at $t = -1$ as well. Then, in all rounds $t = 0, 1, \ldots$, the supply correspondence and the demand correspondences are chosen to maximize identical utility functions in both auctions. Hence, the supply correspondence and demand correspondences are identical in every round of both auctions, and it follows that an identical price path resulting in the same allocation exists. □

B Additional examples

B.1 Packaging costs between identical items

If the seller has partition preferences over identical items (or if there are complementarities between identical items on the buyers' side), one can appropriately relabel items and adjust valuations and costs.[35] We illustrate this with an example.

Example 4. There are two items A and B supplied with $\Omega_A = \Omega_B = 2$. We wish to allow for the package $\{AA\}$ to have its own price $p(AA)$ not necessarily equal to $2p(A)$. The values are given by $v(A, q, l), v(B, q, l), v(AA, q, l), v(AB, q, l), v(AAB, q, l)$, $l = 1, 2$, $q = 1, 2$. We give each unit of A its own index, i.e., $N := \{A_1, A_2, B\}$, and obtain values $v(A_1, q, l)$,$v(A_2, q, l)$, $v(B, q, l), v(A_1, A_2, q, l)$, $v(A_1B, q, l)$, $v(A_2B, q, l)$, and $v(A_1A_2B, q, l)$, where $v(A_1, q, l) = v(A_2, q, l)$, $v(A_1A_2, q, l) = v(AA, q, l)$, $v(A_1B, q, l) = v(A_2B, q, l) = v(AB, q, l)$ and $v(A_1A_2B, q, l) = v(AAB, q, l)$.

The seller submits incremental cost functions $\Delta c(A, \cdot)$, $\Delta c(B,$ $\Delta c(AA, \cdot)$, $\Delta c(AB, \cdot)$, and $\Delta c(AAB, \cdot)$, and a cost function graph defining the cost connections between those packages as shown in Fig. B1. The transformed incremental cost functions $\Delta c(A_1, \cdot)$, $\Delta c(A_2, \cdot)$, $\Delta c(A_1A_2, \cdot)$, $\Delta c(A_1B, \cdot)$, $\Delta c(A_2B, \cdot)$, $\Delta c(A_1A_2B, \cdot)$ are such that $\Delta c(A_1, 1) = \Delta c(A, 1)$, $\Delta c(A_2, 1) = \Delta c(A, 2)$, $\Delta c(A_1B, 1) = \Delta c(AB, 1)$, $\Delta c(A_2B, 1) = \Delta c(AB, 2)$, $\Delta c(A_1A_2, 1) = \Delta c(AA, 1)$, and $\Delta c(A_1A_2B, 1) = \Delta c(AAB, 1)$. $\Delta c(B, \cdot)$ remains unchanged and all other $\Delta c(S, r)$ are set to ∞. The cost function graph is adjusted as shown in Fig. B2, where the outgoing edges are all implied by the original graph in Fig. B1

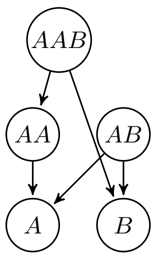

Figure B1: CFG with package AA

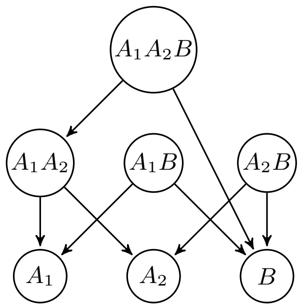

Figure B2: Augmented CFG

Note that the auction prices must satisfy $p(A) = \min\{p(A_1), p(A_2)\}$, $p(AA) = p(A_1A_2)$, and $p(AB) = \min\{p(A_1B), p(A_2B)\}$. If A_1 and A_2 are both allocated to buyers, it must hold that $p(A_1) = p(A_2)$, and similarly for $p(AB)$, because of $v(A_1, q, l) = v(A_2, q, l) = v(A, q, l)$ and the constraints of DSWLP.

[35] From a market design perspective, it is most efficient in terms of computational complexity to only relabel those items on which partition preferences or complementarities are expected.

B.2 Ascending auctions

Example 5. Buyers are labeled L1 to L6. Their values and the seller's values v^0 and dual marginal costs c^0 are given in Table B1. Table B2 details the ascending auction where the seller maximizes revenue or, equivalently, utility based on the cost function $C^0(\boldsymbol{k}) = c^0\left(\bigcup_{S\in\boldsymbol{k}} S\right)$, in each round. Table B3 details the extended ascending auction where the seller maximizes utility based on the cost function $\widetilde{C}^0(\boldsymbol{k}) = \sum_{S\in\boldsymbol{k}} c^0(S)$ in each round. If the seller maximizes revenue, the two individual items A and B are allocated, e.g., to L1 and L3. If the seller maximizes utility based on $\widetilde{C}^0$, bundle AB is allocated, e.g., to L5.

	A	B	AB
v^1	5	0	5
v^2	5	0	5
v^3	0	7	7
v^4	0	7	7
v^5	0	0	11
v^6	0	0	11
v^0	2	4	8
c^0	4	6	8

Table B1: Values and costs

current price	supply set	demand L1	L2	L3	L4	L5	L6	squeezed-out
$p(0) = (2, 4, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (3, 5, 9)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(2) = (4, 6, 10)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(3) = (5, 7, 11)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	A,B

Table B2: Ascending auction

current price	supply set	demand L1	L2	L3	L4	L5	L6	squeezed-out
$p(0) = (4, 6, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (5, 7, 9)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(2) = (5, 7, 10)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(3) = (5, 7, 11)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB

Table B3: Modified ascending auction