Equitable Auctions

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November 12, 2024

First version: March 12, 2024
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Job Market Paper

Abstract

We initiate the study of how auction design affects the division of surplus among buyers. We propose a parsimonious measure for equity and apply it to the family of standard auctions for homogeneous goods. Our surplus-equitable mechanism is efficient, Bayesian-Nash incentive compatible, and achieves surplus parity among winners ex-post. The uniform-price auction is equity-optimal if and only if buyers have a pure common value. Against intuition, the pay-as-bid auction is not always preferred in terms of equity if buyers have pure private values. In auctions with price mixing between pay-as-bid and uniform prices, we provide prior-free bounds on the equity-preferred pricing rule under a common regularity condition on signals.

Key words: auctions, equity, mechanism design, pay-as-bid, uniform price, common value
JEL codes: D44, D47, D63, D82

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1   Introduction

Equity concerns in auctions have increasingly entered policy debates and are of critical importance to participation and stability in downstream markets. In practice, auctions are used, for example, to sell government debt, electricity, emission permits, oil, timber, coffee, art, and production input factors. Although auction design is a cornerstone of economics research, *division of surplus between buyers* has not been studied. Generally, auctions are held to elicit agents' values or costs for buying or selling goods. However, the auction mechanism, consisting of an allocation and pricing rule, can result in an asymmetric distribution of welfare, even among winning bidders: for example, in single-price auctions, high-value bidders obtain a larger surplus (i.e., value minus price) than low-value bidders. This points to possibly unintended implications for the welfare distribution in the auction. Nonetheless, we show that auctions can be made equitable by design.[1]

This article initiates the study of surplus distribution between buyers in multi-unit auctions. We focus on the class of standard auctions with independent signals, which are revenue-equivalent and, under mild assumptions, even revenue-maximizing.[2] Therefore, designing the equity objective is costless in terms of revenue and efficiency. We propose a family of equity metrics that are based on parsimonious, pairwise comparisons of realized surpluses (utilities). First, we characterize the direct surplus-equitable mechanism. This incentive-compatible mechanism uses windfall subsidies (when little redistribution is needed) to achieve ex-post identical surpluses among winning bidders of any type realization. Second, we turn to uniform and pay-as-bid auctions and combinations of these, that is, mixed-price auctions. In this class, we derive prior-free results on equity-preferred pricing, with strong policy implications for multi-unit auctions used in practice.

The distribution of surplus is important for policy makers, and incomplete market design may lead to unforeseen consequences. In 2022, after the Russian invasion of Ukraine, electricity prices in Europe reached unprecedented highs. Under the uniform pricing rule currently used in most electricity markets, infra-marginal generators (i.e., those not setting the market clearing price) are remunerated at the marginal price. This price was elevated by the bids of natural gas power plants (which had to recover their costs), leaving producers of lignite and renewable energy with extraordinary windfall profits. Most European countries imposed high taxes on these profits, mainly to mitigate the burden of increasing energy costs on the consumer side.[3] However, the discrepancy in profits between generators of different power sources also raises the question of how auction design impacts the distribution of surplus among participating bidders. The taxation of infra-marginal rents in electricity markets was debated, for example, in the context of recovering infrastructure investment costs (New Zealand Electricity Authority, 2014; Ruddell, Philpott and Downward, 2017).[4,5] The proposed tax on companies' observable surplus

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[1] Redistribution after the auction, withstanding legal feasibility, may distort bidding incentives and efficiency.

[2] See also the discussion in Section 6.5.

[3] By taxing 90% of the revenue exceeding a cap of €180/MWh on electricity prices, an estimated €106 billion were levied from power companies in 2022 (EU Council, 2022; Nicolay et al., 2023).

[4] Via comparison with hypothetical pre-investment market outcomes, the tax was to be applied to power generators who benefited from the new infrastructure.

[5] Electricity markets generally are subject to complex constraints and resulting incentives, and designing prices is inherently a multi-objective problem (Ahunbay, Bichler and Knörr, 2024).

is equivalent to a mixed auction design, in which prices are set by a combination of uniform and pay-as-bid pricing.[6] This class of mechanisms is one of the focal points of this article.

In practice, uniform pricing and pay-as-bid (discriminatory) pricing are the prevalent multi-unit auction formats.[7] In the uniform-price auction, all winners pay the first rejected bid,[8] and in the pay-as-bid auction, all winners pay the price they bid. The question of which pricing rule leads to more efficient power market outcomes, less collusion, and more revenue has been debated for decades (e.g., Kahn et al. 2001; Ausubel et al. 2014), but fairness and redistribution concerns have received little attention. Next to electricity markets, these concerns are central in many other auction environments. For example, the Small Business Act in the US requires the government to award 23% of procurement contracts each year to small businesses that are socially and economically disadvantaged or minority-owned.[9] In spectrum auctions, allocative fairness in the distribution of licenses is particularly important, as it affects competition in the downstream market (GSMA 2021; Kasberger 2023). Similarly, the distribution of surplus can influence market stability and competition in post-auction markets: bidders disadvantaged in the surplus distribution may face higher borrowing costs, especially in inefficient capital markets. This is aggravated in auctions that occur only once or a few times, as a low surplus can be detrimental to a company's survival in the post-auction market.[10] In addition to competition concerns, heterogeneity in marginal products of capital has been linked to lower total factor productivity (Hsieh and Klenow, 2009).

Although the focus of our article is on equity in the context of auctions, our insights apply more generally. First, the equity concerns we address in this work are fundamentally about equality or inequality in surplusoes, which could be seen as more neutral terminology. Second, our insights are not limited to auctions and apply to markets in general. In B2B interactions, for example, producers want to sell to a variety of customers and avoid being dependent on a single client. Accepting different prices from different clients, allowing them to derive more equal surpluses, is a way of helping those clients stay competitive in the B2C market. On the other hand, a uniform sales price might disadvantage smaller customers. Relying on multiple vendors is particularly important in government and defense procurement. For example, NASA is contracting with Blue Origin and SpaceX for lunar landing systems (NASA, 2023), and the Pentagon eventually split its $9 billion cloud computing contracts between Amazon, Google, Microsoft and Oracle (New York Times, 2022).

Our setup is as follows. We consider standard and winners-pay multi-unit auctions (so-called k-unit auctions) for the sale of indivisible, identical goods with a composition of private and common values. Each buyer has unit demand and receives a private and independent signal drawn from a publicly known distribution.[11] A buyer's value linearly interpolates between the

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[6] This design was also suggested for electricity markets by Holmberg and Tangerås (2023).

[7] In treasury auctions both designs are common OECD (2021), but most electricity markets feature the uniform pricing rule, with the exceptions of England&Wales, Mexico, Peru, and Panama. Further examples are auctions for emission certificates, e.g., the EU emission trading system or the California Cap and Trade market, or online advertisement.

[8] or last accepted bid

[9] See U.S. Small Business Administration (2024). Pai and Vohra (2012) give further examples across the world.

[10] Even in repeated auctions, e.g., for electricity, leaving high-value bidders systematically with higher rents implies long-term inequity.

[11] We note the assumption of unit demand is fairly strong and we provide a discussion in Section 6.4.

extremes of pure private and pure common value, that is, between their private signal and the average signal in the market.[12] We study direct incentive-compatible mechanisms and the class of mixed k-unit auctions. Mixed auctions combine uniform and pay-as-bid auctions and incorporate those as special cases. For a given δ, we call the convex combination of uniform and pay-as-bid pricing δ-*mixed pricing*, and the corresponding auction δ-*mixed auction*. The parameter δ describes the degree of price discrimination: $\delta = 0$ corresponds to uniform pricing and $\delta = 1$ to pay-as-bid pricing. In mixed auctions, we study the symmetric Bayesian equilibrium, which is found to be unique.[13] All considered auctions, under classical assumptions, achieve the same expected revenue (Milgrom and Segal, 2002) and allocate items to the highest-value buyers; thus, there are no trade-offs with revenue or efficiency and optimizing for surplus equity is costless.

1.1 Contributions

We consider our first conceptual contribution to be the introduction of a new, parsimonious measure of surplus equity, *dominance in pairwise differences*: Auction A dominates auction B in pairwise differences, if, in equilibrium, all absolute pairwise differences in ex-post utilities of winning bidders are weakly smaller in auction A than in auction B (Definition 2), with one pairwise comparison being strict. Equivalently, we say that A is *equity-preferred* to B. An auction is *equity-preferred* (in the class of mixed auctions) if and only if it dominates all other mixed auctions. Dominance in pairwise differences implies a partial order and is a strong requirement.

Our results hold for the family of equity metrics that are constructed by aggregation of pairwise differences with any increasing function. This family includes, e.g., the empirical variance, the Gini index or a comparison between the top and bottom deciles.[14] As an example of an aggregator, we study the *winners' empirical variance (WEV)* of surplus. In addition to anonymity, which is satisfied by the entire family of equity metrics, we show that WEV also satisfies the Pigou-Dalton principle (cf. Moulin, 2004), that is, monotonicity with respect to transfers from richer to poorer agents (Proposition 3). The empirical variance combines *within-bidder variation* and *across-bidder correlation* of surpluses. The analysis of within-agent variation addresses a bidder's individual risk-attitude and goes back to Vickrey (1961).[15] In contrast, an equity measure must take into account the correlation of surpluses between bidders.

In the following section, we summarize our most important contributions. We call the interpolation parameter c the *common value proportion*, or simply the *common value*, and its complement $1 - c$ the *private value proportion*, or the *private value*.

1. We characterize the direct and Bayes-Nash incentive-compatible mechanism that distrib-

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[12] This model can represent common resale opportunities. For example, an emission certificate is valuable for a company's production process (private value), but it can also be resold after the auction, with the resale value being common to the market. The resale model appears in previous work by, for example, Bikhchandani and Riley (1991), Klemperer (1998), Bulow and Klemperer (2002) and Goeree and Offerman (2003).

[13] Empirical evidence suggests that markets with symmetric bidders are relevant in practice. See, e.g., Armantier and Lafhel (2009) for auctions by the Bank of Canada, Hortaçsu, Kastl and Zhang (2018) for U.S. Treasury short-term securities auctions, or Hattori and Takahashi (2022) for Japanese treasury auctions.

[14] For prominent inequality measures in wealth or income, cf., e.g., Lorenz (1905), Gini (1912), Gini (1921), Pigou (1912), Dalton (1920), Atkinson (1970) and Sen and Foster (1973).

[15] In the appendix of his famous article, Vickrey showed that, in a *single-unit auction*, the ex ante variance of surplus is lower under the first-price than the second-price rule, given uniform distributions of private values. The result is generalized in Krishna (2009) who shows that the distribution of equilibrium prices in a second-price auction is a mean-preserving spread of that in a first-price auction, given any distribution of private values.

utes realized surpluses equitably among the winning bidders in the class of standard winners-pay mechanisms (Theorem 1). The surplus-equitable mechanism allocates the items to the highest bidders and charges them a payment consisting of three components: firstly, each bidder pays their private value, thus equalizing ex-post utilities; secondly, a uniform payment that cancels out the idiosyncratic payment in expectation; and finally, the expected value corresponding to the first rejected bid, akin to a "second-price" payment. The key and surprising insight is that there exists a uniform payment that cancels out the idiosyncratic payment part, *for any given signal*, in expectation.

2. We prove that the uniform-price auction is equity-preferred (dominant in pairwise differences) if and only if the bidders' values are pure common value (Theorem 2). In this case, the surplus-equitable mechanism is equivalent to the first-rejected-bid uniform-price auction, which, as any other uniform-price auction, equalizes bidders' realized surpluses. By contrast, for pure private values the pay-as-bid auction is not generally equity-preferred in the class of mixed auctions (Proposition 5); however, it is preferred if signals are drawn from a log-concave distribution (Corollary 1).

3. Given any proportion of private values $(1-c)$, the $(1-c)$-mixed auction is equity-preferred over any mixed auction with less than a $(1-c)$ share of price discrimination, assuming that the signal distributions are log-concave (Theorem 3). That is, the pricing for goods with a higher proportion of private value should contain more price discrimination in order to achieve an equitable distribution of realized surpluses.

4. Given any positive proportion of private values $(1-c)$, any level of price discrimination up to $2(1-c)$ is equity-preferred to uniform pricing if signals are drawn from a log-concave distribution (Theorem 4). In other words, in any scenario where the goods for sale have some private value, even a large extent of price discrimination is more equitable than uniform pricing. Theorems 3 and 4 are *prior-free* in the class of log-concave signal distributions.

Finally, we investigate equity in terms of winners' empirical variance (WEV) in numerical experiments. For a variety of signal distributions and common value proportions, we compute the landscape of WEV-minimal mixed pricing, which can be seen to be unique.

1.2 Related literature

This article relates and contributes to several strands of existing work, including a recent literature on redistributive market design, the study of fairness concerns and allocative equity in auctions, and of fair allocations more generally. Further, we contribute to the mechanism design literature on ex-post payment design, the analysis of uniform, pay-as-bid, and, in particular, mixed-price auctions, and the literature on redistribution in public finance and optimal taxation.

Broadly, our contribution fits into a recent strand of the economic literature on redistributive concerns in market design. In this literature, the focus is often on efficiency and equity trade-offs; e.g., in a large buyer-seller market for a single object, with agents differing in their marginal utilities of money (and values), Dworczak, Kominers and Akbarpour (2021) characterize the

optimal efficiency-equity trade-off, and Akbarpour, Dworczak and Kominers (2024), characterize when non-market mechanisms, as opposed to market-clearing prices, are optimal for a designer to allocate a fixed supply (also in a large market). Such non-market mechanisms forgo efficiency for the sake of improving equity. Our approach differs in that we focus on a class of efficient mechanisms in a small market with a finite number of buyers and demonstrate how to improve equity, up to achieving perfect surplus parity. Finally, Reuter and Groh (2020) determine the utilitarian optimal mechanism for the allocation of a finite number of objects to a finite number of heterogeneous agents. Crucially, they study when the objective is maximized in expectation, whereas we employ a stronger, ex-post notion of equity and redistribution.

Fairness concerns in auctions have been addressed through design instruments such as subsidies and set-asides. In a model with explicit target group favoritism, Pai and Vohra (2012) show that the optimal mechanism is a flat or a type-dependent subsidy, depending on the precise nature of the favoritism constraint. Athey, Coey and Levin (2013) come to similar conclusions in an empirical study of US Forest Service timer auctions, where set-asides for small bidders would reduce efficiency and revenue, while subsidies would increase revenue and profits of small bidders with little detriment to efficiency. In contrast with this literature, we consider only the pricing rules as a design instrument to achieve a more equitable distribution of surplus. However, a superficially "fair" pricing rule can lead to inequitable outcomes: e.g., Deb and Pai (2017) demonstrate that anonymous symmetric sealed-bid pricing leaves ample space for bidder discrimination through the seller. Crucially, we demonstrate that even with ex-ante symmetric bidders, the auction design can discriminate against bidders with different signals ex-post.

Allocative equity in auctions and the welfare generated in the post-auction market have been studied in Kasberger (2023), micro-founding the question when an equitable distribution of the auctioned objects themselves is beneficial for consumer welfare in downstream markets. Related results were developed in Janssen and Karamychev (2010), who show that the correlation between signals induced by the revealed auction outcome can affect post-auction market competition. In the literature on auctions for online advertisement, fairness for users has become an important concern, in that similar user should see similar ads (Celis, Mehrotra and Vishnoi, 2019; Chawla and Jagadeesan, 2022). In contrast with these works, we focus on equity in surpluses rather than in allocations.

The literature on fair allocation introduced many concepts of fairness, including, for example, *envy-freeness*, *equal division*, or *no domination* (for a survey, see Thomson 2011). Our notion of fairness is orthogonal to envy-freeness, which requires that no agent prefers another agent's allocation (object and price). In our market, in which a finite number of identical objects are sold, uniform pricing is the unique pricing scheme that results in envy-freeness among winners, and with pure private values, it results in envy-freeness among all participants.[16] However, envy-freeness does not take into account that bidders may have different signal (value) realizations. Subscribing to the notion of envy-freeness, we would accept that realized utilities may be very unequal, depending on the realization of the private value component. In contrast to this view, we consider the realization of utilities as the baseline for fairness considerations, relating to *equal treatment of equals* (cf., e.g., Thomson, 2011) in an ex-post view.

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[16] With a proportion of common value, depending on the realization of signals, winners may experience the winners' curse and prefer not to have won an item.

In practice, fairness is an important issue in spectrum auctions (GSMA, 2021; Myers, 2023). Formats such as the Simultaneous Multiple-Round Auction, the Combinatorial Multiple-Round Auction, or the Combinatorial Clock Auction are used. However, some of these formats have led bidders to pay different prices for identical licenses, which has been perceived as unfair (Myers, 2023).[17,18] Our work challenges this common notion of fairness, highlighting the importance of distinguishing between private and common values in the bidders' value and information structure. With pure private values, our view of equity requires strong buyers (with high-value realizations) to pay disproportionately more than weak buyers (with low-value realizations).

We also contribute to a strand of the mechanism design literature that developed important, nuanced ex-post implementations of truthful mechanisms. In their seminal article, d'Aspremont and Gérard-Varet (1979) show that ex-post budget balance can be achieved in a direct truthful mechanism. In a similar vein, Esö and Futó (1999) prove that for every incentive-compatible mechanism there exists a mechanism which provides deterministically the same revenue. We contribute to this literature by designing the payment rule that distributes bidder surplus equally between the winners in standard k-unit auctions.

Broadly, we also relate to the literature on public finance and optimal taxation which has long debated redistribution. Full redistribution resulting in equal surpluses, as our surplus-equitable mechanism does, is the solution to the classical utilitarian social welfare objective with concave homogeneous utility functions among individuals (Piketty and Saez, 2013; Edgeworth, 1897). An alternative and more general approach to social welfare functions, the so-called generalized social welfare weights, were suggested by Saez and Stantcheva (2016). Although an aggregation of our proposed pairwise differences with weighting factors akin to social welfare weights is possible, the economic interpretation remains different, as we aggregate differences in utilities. A crucial difference of our work and public finance is that our mechanism design approach takes inequality in the form of different signal realizations as given, whereas incomes are endogenous in most optimal taxation models. Remarkably, mixed auctions as we study in this article have a particular relationship with taxes. In uniform-price auctions, the seller observes a lower bound on the winning bidders' surpluses, the difference between observed winning bids and the clearing price, the so-called apparent surplus (Ruddell, Philpott and Downward, 2017). Applying a percentage tax on apparent surplus is equivalent to employing the mixed-price rule, charging each bidder a portion of the market clearing price *and* a portion of their submitted bid.

1.3 Outline

The remainder of the article is organized as follows. In Section 2, we introduce the model and derive equilibrium bidding strategies in mixed auctions. We introduce our notion of surplus equity in Section 3 and develop the surplus-equitable mechanism in Section 4. In Section 5, we describe our prior-free results for uniform, pay-as-bid, and mixed auctions and prove them in Appendix A. Section 6 provides a discussion and Section 7 concludes.

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[17] As a consequence, in versions of the Combinatorial Clock Auction, core prices are selected in a second stage based on selection criteria that guarantee fairness and stability (cf. e.g., Day and Milgrom, 2008; Erdil and Klemperer, 2010).

[18] In the Combinatorial Multi-Round Ascending Auction, there may also exist equilibria in which bidders pay different prices for an identical good, see, e.g., Kasberger and Teytelboym (2024).

2 Setup

2.1 Model

A finite number of bidders $[n] := \{1, \ldots, n\}$ compete for a fixed supply of items $[k] := \{1, \ldots, k\}$, where $2 \leq k < n$. Each bidder only demands one item. Bidder i receives a private signal s_i, which is drawn independently from a positive and bounded or unbounded support; denote its upper limit by $\bar{s}$. Signals are iid with an absolutely continuous probability distribution F with density f. We call $(0, \bar{s})$, i.e., all signals s so that $0 < F(s) < 1$, the open support of F, and assume that $f > 0$ over $(0, \bar{s})$. We also assume that the signals have a finite second moment $\mathbb{E}[s^2] < \infty$.

For $\boldsymbol{s} := \{s_i\}_{i \in [n]}$, a collection of iid signals, we denote by $Y_m(\boldsymbol{s})$ the m-th highest value of the collection $\boldsymbol{s}$ (with n entries). For example, $Y_1(\boldsymbol{s})$ is the maximum and $Y_n(\boldsymbol{s})$ is the minimum of the collection $\boldsymbol{s}$. Note that $Y_m(\boldsymbol{s})$ is a random variable, and we denote its probability distribution $G_m^n(y)$ with corresponding density $g_m^n(y)$. G_m^n is given by

$$G_m^n(y) = \sum_{j=0}^{m-1} \binom{n}{j} F(y)^{n-j} (1 - F(y))^j. \tag{1}$$

where each summand is the probability that j signals are above y. An expression for $g_m^n(y)$ is given in Appendix C.6. The value of bidder i for an item is given by the valuation function $v(s_i, \boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i} := (s_j)_{j \neq i}$, and $v(s_i, \boldsymbol{s}_{-i})$ is symmetric in other bidders' signals $\boldsymbol{s}_{-i}$.

Assumption 1. Values $v(s_i, \boldsymbol{s}_{-i})$ are given by

$$v(s_i, \boldsymbol{s}_{-i}) = (1 - c)s_i + \frac{c}{n} \sum_{j \in [n]} s_j, \tag{2}$$

where $c \in [0, 1]$ is the *proportion of the common value*.

Our model interpolates between a common value and private values, where the proportion of the common value c encodes to what extent the value of any given bidder is influenced by the signals of the other bidders. In particular, $c = 1$ defines a pure common value and $c = 0$ pure private values.[19] We note that the value function satisfies the *single-crossing* condition as for all $i, j \in [n]$, $i \neq j$, and for all $\boldsymbol{s}$, $\partial v(s_i, \boldsymbol{s}_{-i}) / \partial s_i \geq \partial v(s_j, \boldsymbol{s}_{-j}) / \partial s_i$.

Auction mechanisms. Auction mechanisms are represented by allocations and transfers $\{\pi_i(s_i, \boldsymbol{s}_{-i}), p_i(s_i, \boldsymbol{s}_{-i})\}_{i \in [n]}$, where $\pi_i(s_i, \boldsymbol{s}_{-i})$ is defined as the probability that an item is allocated to the bidder i when the reported signals are s_i and $\boldsymbol{s}_{-i}$, and $p_i(s_i, \boldsymbol{s}_{-i})$ is the corresponding price charged to the bidder, which is symmetric in its second argument.[20] We require that auction mechanisms be standard and winners-pay. An auction is *standard* if the k highest bids win

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[19] In an alternative model, the common value might be distributed according to some prior distribution, and the bidders' private signals are drawn conditional on the realization of this common value. The alternative model has identical qualitative characteristics (Goeree and Offerman, 2003): (i) the items are valued equally by all bidders in the common value component, and (ii) the winner's curse is present, i.e., winning an item is "bad news", in that the winner's expectation of the item's value was likely too optimistic.

[20] Symmetric means that $p_i(s_i, \boldsymbol{s}_{-i}) = p_i(s_i, s'_{-i})$ for all permutations s'_{-i} of s_{-i}.

the items,[21] and *winners-pay* if only the winners pay and no more than their bid. Any standard auction, in any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is *efficient* (Krishna, 2009), i.e., the bidders with the k highest values $v(s_i, \boldsymbol{s}_{-i})$ are assigned the items.

We consider two classes of mechanisms. First, we consider truthful, direct mechanisms in which bidders submit their signal. Second, we consider k-unit *mixed auctions* (defined below) in which each bidder submits a bid b_i, resulting in the vector of submitted bids $\mathbf{b}$. Restricting our attention to symmetric and monotonically increasing bidding strategies $b_i = b(s_i)$, we can write allocations and prices in both classes of mechanism as functions of signals only. The allocation of bidder i is given by $\pi_i(s_i, \boldsymbol{s}_{-i}) = \mathbb{1}\{s_i > Y_k(\boldsymbol{s}_{-i})\}$ when signals s_i and $\boldsymbol{s}_{-i}$ are reported. A bidder's utility (or surplus) when reporting signal $\hat{s}_i$, and the remaining $n-1$ bidders reporting signals $\boldsymbol{s}_{-i}$, is given by

$$u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i}) = \mathbb{1}\{\hat{s}_i > Y_k(\boldsymbol{s}_{-i})\} \cdot v(s_i, \boldsymbol{s}_{-i}) - p_i(\hat{s}_i, \boldsymbol{s}_{-i}). \tag{3}$$

Given a signal s_i, recall that we denote by $Y_k(\boldsymbol{s}_{-i})$ the k-th highest among the signals $\boldsymbol{s}_{-i}$. $Y_k(\boldsymbol{s}_{-i})$ has probability distribution G_k^{n-1} and density g_k^{n-1}.

Furthermore, we denote equilibrium bidding strategies by $(\beta_i)_{i \in [n]} = \boldsymbol{\beta}$.

Definition 1 (Mixed auctions). In the k-unit δ-*mixed auction*, parameterized by a given $\delta \in [0,1]$, each bidder i pays $p_i(\mathbf{b}) = (\delta b_i + (1-\delta)Y_{k+1}(\mathbf{b}))\mathbb{1}\{b_i > Y_{k+1}(\mathbf{b})\}$.

At one boundary, for $\delta = 0$, this resolves to *first-rejected-bid uniform pricing* or short *uniform pricing*, where each winning bidder i pays the $(k+1)$-th highest bid $Y_{k+1}(\mathbf{b})$. At the other boundary, for $\delta = 1$, this resolves to *pay-as-bid pricing*, where each winning bidder i pays their bid b_i. Finally, if $\delta \in (0,1)$, we say that the auction and the pricing are *strictly mixed*.[22]

2.2   Interim values, payments, and utilities

For all $x, y \in [0,1]$, we define the expected value given $s_i = x$ and $Y_k(\boldsymbol{s}_{-i}) = y$ as follows:

$$\widetilde{V}(x, y) := \mathbb{E}_{\boldsymbol{s}}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = x, Y_k(\boldsymbol{s}_{-i}) = y]. \tag{4}$$

The expected value is taken over $n-2$ signals not including the bidder's own signal and the k-th highest among their $n-1$ opponents. Observe that because $v(s_i, \boldsymbol{s}_{-i})$ is continuous and non-decreasing, $\widetilde{V}(x, y)$ is continuous and non-decreasing in x and y.[23] We define $V(y) := \widetilde{V}(y, y)$, the expectation of the value of an item conditional on the bidder winning against the relevant competing signal, the k-th highest among its competitors. Furthermore, we introduce *interim payments* $P_i(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s_i, \boldsymbol{s}_{-i})]$ and *interim utilities* $U_i(s_i, \hat{s}_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i})]$ with $U_i(x)$ being the shorthand of $U_i(x, x)$.

Interim incentive compatibility (IC) requires $U_i(s_i, s_i) \geq U_i(s_i, \hat{s}_i)$ for all $s_i, \hat{s}_i$, and interim individual rationality (IR) demands $U_i(s_i, s_i) \geq 0$ for all s_i. It is standard from an application

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[21] Cf. Krishna (2009).

[22] Mixed-price auctions, originating from Wang and Zender (2002) and Viswanathan and Wang (2002), have also appeared in a series of articles modeling a divisible and stochastic supply (Ruddell, Philpott and Downward, 2017; Marszalec, Teytelboym and Laksá, 2020; Woodward, 2021).

[23] In fact, it is strictly increasing in x.

of the envelope theorem (Milgrom and Segal, 2002) that the auctions we consider result in the same expected payment for each bidder.[24] The interim utility is given by

$$U_i(s_i, \hat{s}_i) = \mathbb{E}_{y=Y_k(\boldsymbol{s}_{-i})}[\mathbb{1}\{\hat{s}_i \geq y\} \widetilde{V}(s_i, y)] - P_i(\hat{s}_i). \tag{5}$$

Letting $G := G_k^{n-1}$ and $g := g_k^{n-1}$, we obtain $\partial_1 U_i(s_i, \hat{s}_i) = \int_0^{\hat{s}_i} \partial_1 \widetilde{V}(s_i, y) g(y) \, \mathrm{d}y$. Note that the expression is simple because, although values are not private, signals are independent. Let $U_i(s_i) = \max_{\hat{s}_i} U_i(s_i, \hat{s}_i)$ in the direct incentive-compatible mechanism, in which the maximum is obtained at $\hat{s}_i = s_i$ due to incentive compatibility. Then, by the envelope theorem, we must have $U_i'(s_i) = \partial_1 U_i(s_i, s_i)$. Thus, we have $U_i(s_i) = U_i(0) + \int_0^{s_i} U_i'(x) \, \mathrm{d}x$, and consequently $P_i(s_i) = \int_0^{s_i} \widetilde{V}(s_i, y) g(y) \, \mathrm{d}y - U_i(0) - \int_0^{s_i} U_i'(x) \, \mathrm{d}x$. From *winners-pay* and the continuity of the signals follows $U_i(0) = P_i(0) = 0$. The revenue equivalence extends to any standard auctions with independent signals in which winners pay, including mixed auctions, as the allocation rule is identical.

2.3 Equilibrium bidding

We derive the unique Bayes-Nash equilibrium in increasing and symmetric bidding strategies in δ-mixed auctions. This equilibrium is the center of our analysis of surplus equity in mixed auctions in Section 5.

Proposition 1 (Krishna 2009)**.** *The unique equilibrium bidding strategy in the uniform-price auction, i.e., the case $\delta = 0$, is given by $\beta^U(s) := \widetilde{V}(s, s) = \mathbb{E}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = s, Y_k(\boldsymbol{s}_{-i}) = s]$.*

Note that the equilibrium is unique in the class of increasing and symmetric strategies and weakly dominant with pure private values (Krishna, 2009).

Proposition 2. *The unique symmetric equilibrium bidding strategy in the δ-mixed auction, for $\delta \in (0, 1]$, is given by*

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G_k^{n-1}(y)^{\frac{1}{\delta}} \, \mathrm{d}y}{G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{6}$$

Proof: See Appendix C.2.

Note that β^δ converges to β^U as $\delta \to 0$. We illustrate it in the following example.

Example 1. We consider the simplest, non-degenerate setting with $n = 3$ bidders competing for $k = 2$ items. The bidders' signals are distributed uniformly on the support $[0, 1]$. First, we compute $V(s)$, the expected value of a bidder with signal s conditional on tying with the first rejected bid. As there are only three signals, the only remaining signal not fixed by the conditioning in V is the a signal known to be greater than s, i.e., with an expectation of $(s+1)/2$. Thus, we obtain $V(s) = (1-c)s + \frac{c}{6}(5s + 1)$. Further, we compute G and g, the cdf and pdf of the minimum of the three signals, and obtain $G(y) = y(2-y)$ and $g = 2(1-y)$. For the uniform-price auction, one can easily compute $\beta^0(s) = V(s)$ and $\beta^1(s) = \left(1 - \frac{c}{6}\right) \frac{s - \frac{2}{3}s^2}{2-s} + \frac{c}{6}$. Note that

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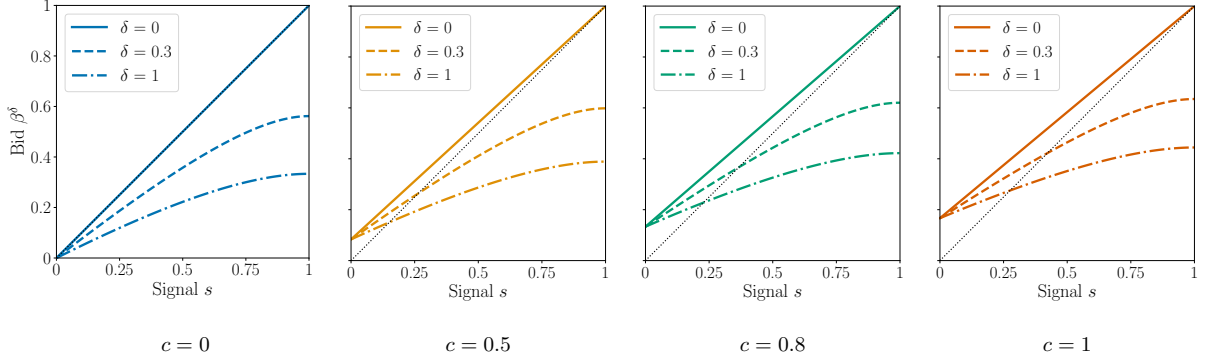β^0 is linear due to uniform signals. Fig. 1 illustrates the bid functions for four different values of c.

Figure 1: Equilibrium bid functions, β^δ, for uniform signal distributions as a function of the signal, s, for common value parameters $c \in \{0, 0.5, 0.8, 1\}$.

In the case of pure private values, bidding truthfully is a dominant strategy in the uniform-price auction. Increasing price discrimination, δ, decreases the bid corresponding to a given signal below the private value (this is also called "bid shading"). The extent of bid shading increases in δ. A higher common value component shifts equilibrium bids for low signal realizations above the diagonal, because a low-signal bidder's expectation of the average signal is higher than their private signal. For $s \to 1$, however, the expected value of an item conditional on tying with the price-setting competing signal converges to s, i.e., truthful bidding. Note that the latter need not be true for different n and k.

With a pure common value, the winner's curse becomes especially apparent. In equilibrium, bidders are attempting to salvage the winner's curse but cannot escape it. In fact, with a pure common value, winners' ex-post utilities are *decreasing in signals* as long as $\delta > 0$.

3 Surplus equity

We propose a notion of equity that we call *dominance in (absolute) pairwise differences*, or short *pairwise differences*. Our results hold for the family of equity measures that are defined by any increasing function of pairwise differences in ex-post surpluses (utilities). We call the collection $\{u_i\}_{i \in [n]}$ of ex-post utilities an *outcome*.

Definition 2 (Dominance in pairwise differences)**.** An outcome $\{u_i\}_{i \in [n]}$ dominates another outcome $\{u'_i\}_{i \in [n]}$ in pairwise differences iff, for all $i, j \in [n]$, it holds that $|u_i - u_j| \le |u'_i - u'_j|$ with one inequality strict.

We say that, for a family of parameterized outcomes $\{u_i^\delta\}_{i \in [n]}$, $\delta \in \Delta$, δ^* is dominant in pairwise differences if u^{δ^*} dominates all outcomes u^δ, $\delta \ne \delta^*$, $\delta \in \Delta$. Pairwise differences induces a partial dominance ranking over outcomes and therefore a dominant δ^* may not always exist.

In our auction setup, ex-post utilities depend on the collection of signals $\boldsymbol{s}$. Thus, we consider that dominance in pairwise differences holds as long as it holds almost surely. In addition, in

the class of standard auctions, items are assigned to the same buyers. Therefore, we focus on winning buyers in most of our analysis and adapt the definition of pairwise differences as follows.

Definition 3 (Dominance in pairwise differences among winners)**.** An outcome $\{u_i(\boldsymbol{s})\}_{i\in[n]}$ dominates another outcome $\{u_i'(\boldsymbol{s})\}_{i\in[n]}$ in pairwise differences iff, for all winning signals s_i, s_j with opponents' signals $\boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $i,j \in [n]$, it holds that $|u_i(s_i, \boldsymbol{s}_{-i}) - u_j(s_j, \boldsymbol{s}_{-j})| \leq |u_i'(s_i, \boldsymbol{s}_{-i}) - u_j'(s_j, \boldsymbol{s}_{-j})|$, almost surely and with one inequality strict.

Several prominent equity axioms, cf., e.g., (Patty and Penn, 2019), hold for pairwise differences. First, we note that anonymity is maintained. Any reordering of individuals in the population $[n]$ has no consequence, as pairwise comparisons must hold for any two bidders.[25] The Pigou-Dalton transfer principle asserts that any transfer from a wealthier agent to a poorer one must reduce inequality, provided the original welfare ranking between the two agents is maintained, that is, the wealthier agent does not become poorer than the previously poorer agent after the transfer (cf. Moulin, 2004). Since dominance in pairwise differences does not establish a complete order of outcomes, a Pigou-Dalton transfer may result in a decrease in some pairwise differences while others increase.

However, our results allow the classification of δ-mixed pricing rules based on pairwise differences and *any increasing function* of pairwise differences, where Pigou-Dalton transfer principle holds.[26] For example, the top decile of realized utilities can be compared to the lowest or the bottom decile of realized utilities, and classic inequity measures such as the Gini index can be constructed.[27] Larger differences can receive a higher weight than smaller ones, e.g., by squaring each pairwise difference.

To exemplify the aggregation of pairwise differences, we focus on the *expected empirical variance* of surplus between the winners, or *winners' empirical variance (WEV)* for short. This metric is defined in expectation, ensuring that it provides a ranking of pricing formats regardless of signal realizations.

Definition 4 (Winners' empirical variance)**.**

$$\text{WEV} \ = E_s\left[\frac{1}{k(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(u_i(\boldsymbol{s}) - u_j(\boldsymbol{s}))^2}{2}\ \middle|\ s_1,\ldots,s_k > Y_{k+1}(\boldsymbol{s})\right].^{28} \tag{7}$$

In addition to being a natural and well-known metric, this aggregation is attractive for two reasons: First, it ensures compliance with the Pigou-Dalton transfer principle, and second, the empirical variance is linked to surplus variance and correlation of surpluses among bidders.

Proposition 3. *The* winners' empirical variance *satisfies the* Pigou-Dalton principle.

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[25] We note that replication invariance and mean independence are not relevant in our setup, as we keep the population size (number of bidders) as well as the endowments (value distributions) fixed.

[26] A related aggregation is used by Feldman and Kirman (1974), who aggregate positive pairwise differences for a measure of envy per player. Contrasting our measure, Feldman and Kirman (1974) consider pairwise differences of hypothetical (if an agent had received another agent's bundle) and realized utilities.

[27] In our setting with uncertainty about signal realizations, one could define the expected Gini index among winners $G = \frac{1}{2n^2 E_s[u_1(\boldsymbol{s})|1\text{ wins}]}E_s[\sum_{i=1}^{k}\sum_{j=1}^{k}|u_i(\boldsymbol{s}) - u_j(\boldsymbol{s})| \mid s_1,\ldots,s_k > Y_{k+1}(\boldsymbol{s})]$. A small distinction is the normalization by the expected surplus.

Proof: See Appendix C.1.

In expectation, equilibrium surplus varies due to a bidder's own and the competitors' signals, and surplus between winners may be correlated. As we consider efficient auctions, surplus only varies among the winning bidders. Among those, WEV measures variation and correlation of surplus, that is, it measures the dispersion of surplus *across* bidders.

Lemma 1. *An equivalent expression for the winners' empirical variance is given by* WEV $=$ Var$[u_1|1\ wins] -$ Cov$[u_1, u_2|1\ and\ 2\ win]$.

Proof: See Appendix C.1.

In contrast, the ex-ante variance, Var$_{\boldsymbol{s}}[u_i(\boldsymbol{s})]$, measures surplus variation *within* a given bidder, and is more adequate to measure risk, e.g., across a series of identical, repeated auctions, in which a given bidder redraws their signal in every auction. With pure private values and thus ex-post individual rationality, rankings of auction formats in terms of ex-ante variance or winners' ex-ante variance are identical. Rankings with respect to the empirical variance, however, may differ depending on if only winners are considered, or all bidders. A formal lemma and proof are given in Appendix C.1.

Example 1 (Continued). We revisit the example with uniformly distributed signals, $n = 3$ bidders, and $k = 2$ items. We compute WEV numerically and illustrate it for different values of the common-value proportion c in Fig. 2 below. For pure private and intermediate common
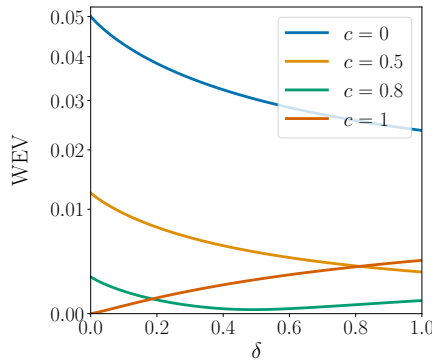
Figure 2: WEV as a function of δ for uniform signals and various common value proportions c

values, the pay-as-bid auction ($\delta = 1$) minimizes WEV. For $c = 0.8$, we observe an interior optimum, and for a pure common value, uniform pricing ($\delta = 0$) minimizes WEV. Note that WEV is not necessarily convex in δ for a fixed c.

As the considered mechanisms are revenue-equivalent and efficient (see also Appendix B), we can focus on the question of surplus distribution among buyers more succinctly, without considering potential trade-offs.

4   The surplus-equitable mechanism

In the class of efficient auctions, it is possible to distribute surplus among the winning bidders equitably. This means pairwise differences in ex-post utilities are zero, as is any aggregate metric such as the winners' empirical variance. We let $y := Y_k(\boldsymbol{s}_{-i})$, $G := G_k^{n-1}$, and $g := g_k^{n-1}$.

Theorem 1. In the class of standard k-unit auctions, there exists an incentive-compatible direct mechanism that distributes surpluses equitably among winners. Losers pay nothing, and the corresponding payments are given by

$$\widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = \left((1-c) \left(s_i - y - \frac{G(y)}{g(y)} \right) + V(y) \right) \mathbb{1}\{s_i > y\}. \tag{8}$$

The ex-post payment in Eq. (8) is constructed in a simple but powerful way. First, the term $(1-c)s_i$ removes the idiosyncratic part of each bidder's realized value due to their own signal. To align interim incentives, this term is adjusted by a uniform subsidy $(1-c)(y+G(y)/g(y))$, which cancels the idiosyncratic payment in expectation. Thus, the second-price payment $V(y)$, the expected value of the $(k+1)$th-highest signal conditional on tying with the kth-highest, induces truthful reporting for any given signal s_i, interim. If the first rejected signal is high, winning signals have to be paid subsidies. However, very high subsidy payments are low probability events, as the realizations of all winning signals and the first rejected signal must be high. Further intuition for the surplus-equitable payment is given in the continuation of Example 1 below. We also note that the surplus-equitable mechanism is ex-post individually rational in the pure private value case. Indeed, $\widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = (s_i - G(y)/g(y))\mathbb{1}\{s_i > y\} \leq s_i \mathbb{1}\{s_i > y\}$.

We now prove Theorem 1.

Proof. The payment rule $\widetilde{p}$ results in identical ex-post surpluses of winners. This follows directly from the definition of ex-post surplus under truthful reporting $u_i(s_i, \boldsymbol{s}_{-i}) = v(s_i, \boldsymbol{s}_{-i}) - \widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = \frac{c}{n}\sum_{j\in[n]} s_j + (1-c)(y - \frac{G(y)}{g(y)}) + V(y)$, for all $s_i > y$, which is independent of the bidder's identity as the first rejected signal is the same for any winner.

Furthermore, the payment $\widetilde{p}$ is interim incentive-compatible. First, we compute the expected payment

$$\widetilde{P}_i(s_i) = (1-c)\left(s_i G(s_i) - \int_0^{s_i} (yg(y) + G(y))\,\mathrm{d}y \right) + \int_0^{s_i} \widetilde{V}(y,y)g(y)\,\mathrm{d}y = \int_0^{s_i} \widetilde{V}(y,y)g(y)\,\mathrm{d}y.$$

The left-hand term is equal to 0 by integration by parts. Because losers pay nothing, the overall expected payment is equal to the expected payment of winners. The expected utility of a bidder who has signal s_i and reports $\hat{s}_i$ is given by

$$U_i(s_i, \hat{s}_i) = \int_0^{\hat{s}_i} \left(\widetilde{V}(s_i, y) - \widetilde{V}(y, y) \right) g(y)\,\mathrm{d}y, \tag{9}$$

As $\widetilde{V}(s_i, y)$ is increasing in s_i the integrand is positive for $\hat{s}_i < s_i$ and negative for $\hat{s}_i > s_i$ and $g > 0$ almost everywhere, the function $U_i(s_i, \hat{s}_i)$ is uniquely maximized at $\hat{s}_i = s_i$. $\qquad\square$

Distributing surplus equitably ex-post is the strongest of implementations, while an implementation of equal interim surpluses is infeasible in winners-pay auctions. If interim surpluses were equalized across different signals, incentive compatibility cannot hold.[29]

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[29] The winners-pay assumption is indispensable. Without it, interim surpluses among winners can indeed be equal. In the example with $n = 3$ and $k = 2$, uniform signals and pure private values ($c = 0$), it can be verified that an interim payment of $L(s_i) = s_i(5s_i - s_i^2 - 4)/3(1-s_i)^2$ charged to the losing bidders achieves equal expected surpluses of the winners of $\frac{2}{3}$. However, then the losing bidders would naturally enter equity considerations.

Example 1 (Continued). We continue the example with $n = 3$ bidders competing for $k = 2$ items and signals distributed uniformly on the support $[0, 1]$. As previously calculated, we have $V(s) = (1 - c)s + \frac{c}{6}(5s + 1)$, and $\frac{G}{g}(y) = \frac{y(2-y)}{2(1-y)}$. Together, we obtain

$$\widetilde{p}_i(s, y) = \left((1 - c) \left(s - \frac{y(2 - y)}{2(1 - y)} \right) + \frac{c}{6}(5y + 1) \right) \mathbb{1}\{s > y\}. \tag{10}$$

We illustrate the payment for signals $s = 0.2, 0.5, 0.8$ as a function of y in Fig. 3 below. Note that each payment corresponding to a signal s is only plotted for $y \leq s$, as for $y > s$ the signal s does not win and the payment is zero. The payment addresses two countervailing incentives, the first stemming from the private-value component, and the second stemming from the common-value component.

With pure private values, if the first rejected signal is high relative to s, the bidder winning with signal s is paid a subsidy. This subsidy compensates bidders in order to induce them to report truthfully even with high private signals. If a high-signal bidder wins *together* with a low-signal bidder, a large surplus (due to the private signal) is levied in order to equalize ex-post surpluses. Naturally, this would create an incentive to under-report your signal. However, in those cases where all winners' signals are high, no taxation is needed to equalize their surplus, and a subsidy is paid to winning bidders, in order to restore incentive compatibility ex-ante.

With a pure common value, the payment is increasing in the first rejected signal y. In this case, no taxation is needed, as the ex-post surpluses are equal under any uniform payment rule. However, truthful reporting must be incentivized, which a second-price rule (adjusted for the common value) achieves. Recall that $V(y)$ is the expected value of a bidder with signal y conditional on tying with the k-th highest among their $n - 1$ competitors.

With a stronger common value component, the taxation of the surplus due to the private signal is less important. Instead, as seen in Fig. 3 with $c = 0.8$, the payment is increasing in the first rejected signal. However, for high signals s, there exists an interior maximum of $\widetilde{p}$ in y, a turning point of the countervailing incentives. The taxation of the private signal still hurts the bidder with a higher signal, so a steep subsidy, on a small range of first rejected signals y, must be paid to level incentives in expectation.
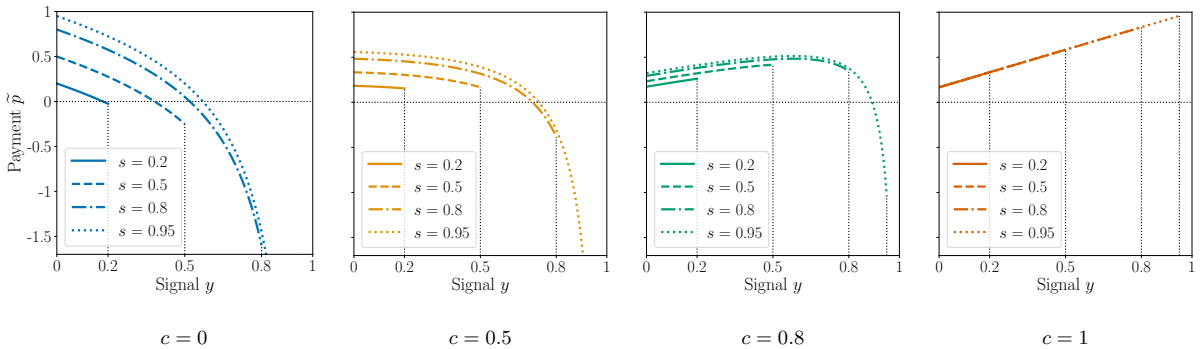

Figure 3: Surplus-equitable payments, $\widetilde{p}$, for uniform signal distributions as a function of the signal s and the first rejected signal, y, for common value parameters $c \in \{0, 0.5, 0.8, 1\}$.

The surplus-equitable mechanism relies on the signal prior, which, in practice, is often un-

known. However, as we show in the following section, even in the realm of the common uniform and pay-as-bid pricing, it is possible to improve surplus equity. Those results remain, in a large class of signal distributions, prior-free.

5 Uniform, pay-as-bid, and mixed auctions

The equity-preferred pricing rule in the mixed auction class crucially depends on the extent of the common value c. As seen for the winners' empirical variance in Example 1, with uniform signals, for some interior values of c, strictly mixed pricing is optimal. We formalize this fact in Section 5.1 for any signal distributions. Example 1 is in line with the general intuition that pay-as-bid pricing may be more equitable with higher private values, and uniform pricing with higher common values. However, as we show in Section 5.2, this is not true in general. Thus, additional distributional assumptions are necessary for a characterization of equity-preferred auctions. In Section 5.3, we consider log-concave signal distributions and provide simple and prior-free bounds on the auction that is dominant in pairwise differences.

5.1 Equity comparisons in uniform, pay-as-bid, and mixed auctions

We first consider the case of a pure common value ($c = 1$). As every bidder has the same ex-post realized value, ex-post utilities among winners are equalized if everyone pays the same price. This results in pairwise differences in utilities of zero. Once the private value component enters the value function with a non-zero weight, the picture is less clear: it may be pay-as-bid pricing that is dominant in pairwise differences, or it may be some degree of mixed pricing; however, it cannot be uniform pricing.

Theorem 2. The uniform-price auction is dominant in pairwise differences iff the common value proportion equals one (pure common value).

Proof: See Appendix C.3.

Furthermore, we show that, without any additional assumptions, strictly interior δ-mixed pricing minimizes WEV for a range of common values.

Proposition 4. *For any signal distribution, there exists $c^* < 1$, such that for common values in the interval $(c^*, 1)$, there exist δ-mixed auctions with lower WEV than pay-as-bid and uniform auctions.*

Proof: See Appendix C.3.

The intuitive notion that uniform pricing equitably distributes surplus under a pure common value may lead us to assume that pay-as-bid auctions are equity-preferred under private values. However, in the following section, we demonstrate a scenario where it fails and show that, with pure private values, uniform pricing can result in lower WEV than pay-as-bid pricing.

5.2 Challenging the intuition: private values and uniform pricing

To understand the reversal of the intuition, consider pairwise differences in utility, the building block for WEV. If ex-post absolute differences in utility are greater under uniform pricing than under pay-as-bid pricing for signal pairs with sufficient probability mass, then the reversal may also hold in expectation. To start with, consider any two winning signals $s_i > s_j$, $s_i, s_j \in [0, \bar{s})$ and private values only, i.e., $c = 0$. Let $u_i^0(s_i, \boldsymbol{s}_{-i})$ and $u_i^1(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price and pay-as-bid auction, respectively. Moreover, β^0 and β^1 denote the corresponding symmetric equilibrium bid functions and $Y_{k+1}(\boldsymbol{\beta})$ the first rejected bid. For $\delta \in [0, 1]$ and $c = 0$, we have $u_i^\delta(s_i, \boldsymbol{s}_{-i}) = s_i - \delta\beta^\delta(s_i) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$. Thus, we have $\Delta u^0 := |u_i^0 - u_j^0| = |s_i - s_j|$ and $\Delta u^1 := |u_i^1 - u_j^1| = |s_i - \beta^1(s_i) - (s_j - \beta^1(s_j))| = |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))|$. It holds that

$$\Delta U^0 < \Delta U^1 \tag{11}$$

$$\Leftrightarrow s_i - s_j < |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))| \tag{12}$$

$$\Rightarrow 2(s_i - s_j) < \beta^1(s_i) - \beta^1(s_j). \tag{13}$$

As bid functions are increasing, if $s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))$ was positive, Eq. (12) could never hold. Thus, Eq. (13) follows as a necessary condition for uniform pricing to have lower pairwise differences than pay-as-bid pricing. For the same statement to hold for WEV, it must be that the bid function has a slope of at least 2 for a sufficient mass of signals s_i and s_j. Bid function slopes greater than 2 imply that high-signal bidders shade their bids much less, proportionally to their value, than bidders with lower signals. Consequently, the differential in ex-post surplus with pay-as-bid pricing, comparing two sufficiently different signals, are higher than the differential in signals. The latter equals the surplus difference in the uniform-price auction.

With this intuition, we now prove that it is indeed possible to construct an *equilibrium* bid function with a slope greater than 2 for a sufficient mass of signals. For this, we require an extreme signal distribution where, broadly speaking, signals are equal to zero with probability ε and equal to one with probability $1 - \varepsilon$. However, to compute a Bayes-Nash equilibrium, we need a continuous signal distribution (with respect to the Lebesgue measure, without mass points) with connected support to solve the first-order condition. Thus, we add a small perturbation.

Example 2. Consider an auction with n bidders and $k = n - 1$ items. Each bidder i has a pure private value ($c = 0$) given by its signal s_i. The signal is equal to the sum of a Bernoulli random variable with parameter $\varepsilon > 0$ and a random perturbation drawn from $\text{Beta}(1, 1/\eta)$, with $\eta > 0$. The resulting signal distribution is continuous, with support $[0, 2]$. We formally state the signal cdf in Appendix C.4 and the quantile function F^{-1} below. As signals can be mapped one-to-one to quantiles, these can be used equivalently, simplifying the analysis in this example

$$\forall x \in [0, 1], \qquad F^{-1}(x) = \mathbb{1}\{x \geq \varepsilon\} + \gamma_\eta(x) \qquad \text{where} \qquad \gamma_\eta(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^\eta & \text{if } x < \varepsilon \\ 1 - \left(1 - \frac{x - \varepsilon}{1 - \varepsilon}\right)^\eta & \text{if } x \geq \varepsilon. \end{cases}$$

We fix $\varepsilon = 0.1/n$ and choose $\eta > 0$ to be an arbitrarily small constant. We consider the order

statistics with respect to the quantiles,[30] and we denote by $\widetilde{G}$ (resp. $\widetilde{g}$) the distribution function (resp. density) of the k-th highest quantile among $n-1$ buyers. We plot the signal cdf and the quantile function in Fig. 10 in Appendix C.4.

Applying the formula from Proposition 2, we can derive the equilibrium bid $b_\eta^\delta(x)$ of a bidder with quantile x (recall that equilibrium bids as functions of signals are denoted by β^δ). We state the bid function below and illustrate it in Fig. 4. For details, we refer to Appendix C.4. For $\delta = 0$ we have that $b_\eta^0(x) = \beta^0(F^{-1}(x)) = \mathbb{1}\{x \geq \varepsilon\} + \gamma_\eta(x)$; and for all $\delta > 0$ we have

$$\forall x \in [0,1], \qquad b_\eta^\delta(x) := \beta^\delta(F^{-1}(x)) = b_0^\delta(x) + \xi_\eta^\delta(x)$$

$$\text{where} \qquad b_0^\delta(x) := \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{G(\varepsilon)}{G(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases} \qquad \text{and} \qquad \xi_\eta^\delta(x) := \frac{\int_0^x \gamma_\eta(y) g(y) G(y)^{\frac{1}{\delta}-1} \, \mathrm{d}y}{\delta G(x)}.$$
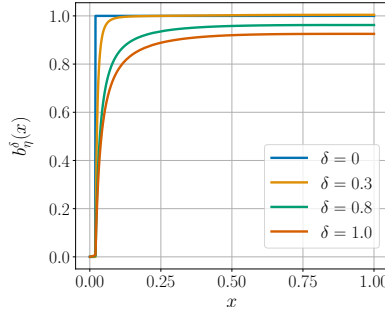

Figure 4: Equilibrium bid as a function of quantiles for $n = 5$ and $\eta = 0.01$

Next, we define the function $\phi_\eta^\delta(x) := F^{-1}(x) - \delta b_\eta^\delta(x)$ and WEV_η^δ, the winners' empirical variance in a δ-mixed auction with noise level η. For all $\delta \in [0,1]$ and for all $\eta > 0$, we have that $2 \cdot \mathrm{WEV}_\eta^\delta = \mathbb{E}_{\mathbf{x}}\left[(\phi_\eta^\delta(x_1) - \phi_\eta^\delta(x_2))^2 \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi_\eta^\delta(x)$ converge towards 0 when taking η arbitrarily small. Therefore, WEV_η^δ converges towards WEV_0^δ, formally defined in Appendix C.4.

We constructed the above example such that uniform pricing achieves lower WEV than pay-as-bid pricing, even with pure private values. The result is stated in the proposition below and holds in the limit as $\eta \to 0$, for any number of bidder $n \geq 5$. In the proof, we show that $\mathrm{WEV}_0^0 \leq \frac{0.005}{n}$ and $\mathrm{WEV}_0^1 \geq \frac{0.01}{n}$.

Proposition 5. *Let the values be distributed according to the the quantile function F^{-1} defined above. For $n \geq 5$, there exists η^*, such that for all $\eta \leq \eta^*$ it holds that the winners' empirical variance under uniform pricing is lower than under pay-as-bid pricing.*

Proof: See Appendix C.4.

Thus, in order to characterize equity-optimal pricing further, we need additional assumptions. In the next section, we show that, for a large class of signal distributions, simple bounds tell us which auction designs are candidates for being equity-preferred in the class of mixed auctions.

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[30] $\widetilde{G}$ and $\widetilde{g}$ correspond to the definitions of G_k^{n-1} and g_k^{n-1} with uniform signals, where $F = F^{-1}$.

5.3 Equity-optimal pricing for log-concave signal distributions

For our subsequent results, we assume a regularity condition on the bidders' signal distributions, *log-concavity*. The family of log-concave distributions contains many common distributions, for example uniform, normal, exponential, logistic or Laplace distributions (Bagnoli and Bergstrom, 2005).[31]

Definition 5. A real-valued function $h \in \mathbb{R}^{\mathbb{R}}$ is *log-concave* if $\log(h)$ is concave.

In this class of signal distributions, simple and prior-free bounds characterize the equity-preferred auction design in the class of mixed auctions. The proofs of Theorem 3 and Theorem 4 are developed in Appendix A.

Theorem 3. Let signals be drawn from a log-concave distribution. Then, for a given private value proportion $1 - c$, the mixed auction with price discrimination $\delta = 1 - c$ is equity-preferred among all mixed auctions with price discrimination of less than $1 - c$.

The equity-preferred pricing rule dominates in pairwise differences all pricing rules with less price discrimination. In other words, Theorem 3 provides a lower bound on the amount of price discrimination required to rule out dominated mixed auctions. We illustrate Theorem 3 in Fig. 5. All pricing rules in the shaded area in red are dominated by the diagonal $1 - c$, given any log-concave distribution of bidders' signals.

Furthermore, uniform pricing is dominated in pairwise differences by many alternative pricing rules, i.e., these pricing rules are preferred to uniform pricing in terms of equity.

Theorem 4. Let signals be drawn from a log-concave distribution. Then the uniform-price auction is dominated in pairwise differences by any strictly mixed pricing with price discrimination of up to $\min\{1, 2(1 - c)\}$.
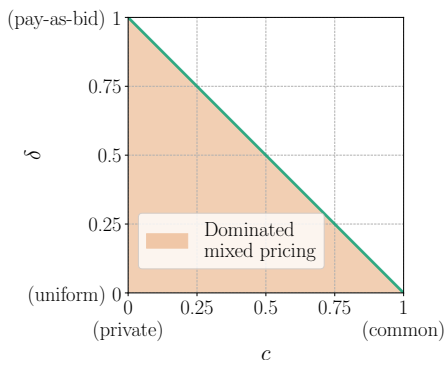
Figure 5: Bounds on equity-optimal combinations of c and δ
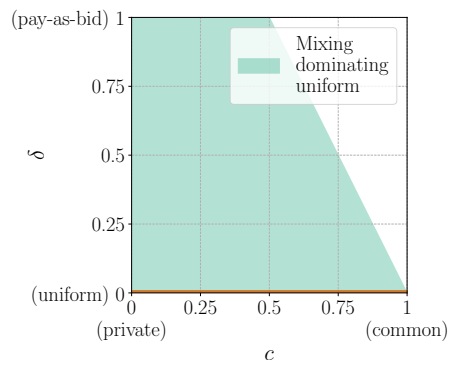
Figure 6: Range of price discrimination dominating uniform pricing

Theorem 4 is illustrated in Fig. 6, in which any pricing rule in the shaded area in green dominates uniform pricing for a given common value c.

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[31] Also χ distribution with degrees of freedom ≥ 1, gamma with shape parameter ≥ 1, χ^2 distribution with degree of freedom ≥ 2, beta with both shape parameters ≥ 1, Weibull with shape parameter ≥ 1, and others.

The intuition behind Theorem 3 is simple. As we show in Appendix A, pairwise differences are, for any given common value c, monotonically decreasing in the extent of price discrimination δ as long as δ is between zero and $1 - c$. Moreover, we show the equivalence of this result with ex-post utilities that increase in signals. As long as higher signals obtain a higher surplus, more equity can be achieved by taxing higher signals more than lower signal. Because the change in the δ-weighted bid in δ is increasing in a bidder's signal (as stated in Lemma 3), increasing the extent of price discrimination will have the desired effect.

A similar intuition explains Theorem 4, where the benefit of higher price discrimination compared to the absence of price discrimination can be realized up to a certain threshold. As long as utilities are increasing in signals, increasing price discrimination results in surplus taxation that benefits equity (cf. Theorem 3). We show in Appendix A that increasing ex-post utilities is equivalent to the slope of equilibrium bid functions being bounded $(1 - c)/\delta$. With steeper bid functions, the utilities might decrease in the signals. So, while increasing price discrimination might locally, in a neighborhood of δ, increase pairwise differences, price discrimination is still beneficial compared to uniform pricing. However, for $\delta \geq 2(1 - c)$, the bid functions are so steep that an increase in price discrimination results in an absolute utility gap between a high signal and a low signal bidder that is greater than under uniform pricing. With such price discrimination, the higher signal bidder is worse off than the low signal bidder.

With Theorems 3 and 4, we can now revisit the question: In terms of equity, should one use pay-as-bid pricing if bidders' values are pure private values? The answer is yes if the signal distributions are log-concave. Moreover, if the common value is small, pay-as-bid pricing is guaranteed to be more equitable than uniform pricing. We state this formally in the corollary below.

Corollary 1. Assume signals are drawn from a log-concave distribution. Then, for pure private values, pay-as-bid pricing is dominant in pairwise differences, and for a common value $c < \frac{1}{2}$, pay-as-bid pricing dominates uniform pricing in pairwise differences.

The first part of the corollary follows by setting $c = 0$ in Theorem 3. The second part follows by setting $\delta = 1$ in Theorem 4. Our numerical experiments in Section 6.1 show that, for $c < \frac{1}{2}$, pay-as-bid pricing in fact minimizes WEV for several common distributions. The intuition in the pure private value case carries through under the qualifying assumption of log-concave signals, and it may fail for very concentrated signal distributions. In the latter case, it is important that sufficient probability mass is gathered around higher signals, inducing a bidding equilibrium in which ex-post utilities are decreasing in signals for sufficiently many signal realizations.[32]

For specific signal distributions, we can extend the region where pairwise differences are monotonically decreasing slightly beyond the diagonal $1 - c$, as exemplified in the following proposition.

Proposition 6. *For uniformly distributed signals, any pricing dominant in pairwise differences contains a discriminatory proportion of at least $\frac{2n(1-c)}{2n-c(n-2)}$, and for exponentially distributed signals at least $\frac{2n(1-c)}{2n-c(n-(k+1))}$.*

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[32] For example, with a β-distribution as steep as illustrated in Fig. 8, Section 6, clearly violating log-concavity, pay-as-bid pricing is still optimal for a range of common values including pure private values.

Proof: See Appendix C.6.[33]

Note that both bounds converge to $\frac{1-c}{1-c/2}$ as the number of bidders goes to infinity (and the number of items k is kept constant). We illustrate the bound for signals uniformly distributed on $[0,1]$ and $n = 3$ bidders and $k = 2$ items in Fig. 7 below, together with the equity-preferred pricing in terms of WEV. The figure demonstrates that, for high values of c, this bound may be a good heuristic for the optimal pricing rule.
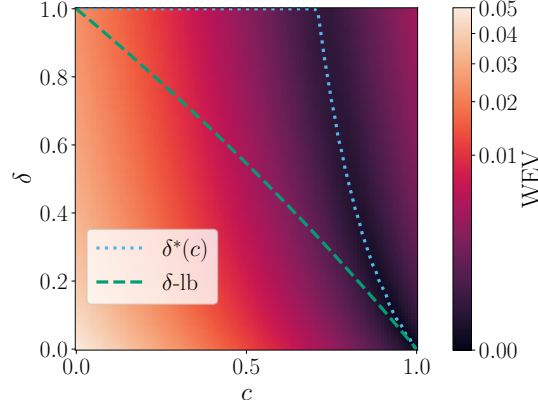

Figure 7: A lower bound δ-lb for pricing candidates dominant in pairwise differencesand the equity-optimal pricing δ^* in terms of WEV

6   Discussion

Our main results hold for all equity metrics that are based on pairwise differences, and, as discussed in the Introduction and in Section 3, the winners' empirical variance is particularly attractive as an aggregated one-dimensional metric. We illustrate WEV further in a series of numerical experiments and discuss how it relates to the within-bidder variation of surplus, as well as the empirical variance of surplus between all bidders. We also explain why the regularity assumption of log-concavity is necessary for our argument.

6.1   Numerical experiments

We further illustrate the effect of the common value on surplus equity by presenting several numerical examples. Similarly to Fig. 7, we compute the WEV-minimal pricing $\delta^*(c)$ for any given proportion of the private-common value c. We also illustrate bounds for WEV-minimal pricing and the condition of monotone ex-post utility (MEU). All of our experiments are based on equilibrium bid functions, whose calculation is computationally very expensive. Thus, we rely on theoretical simplifications, such as Lemma 4 and Lemma 11 (Appendix C.7). The simulations are performed through numerical integration of our analytical formulae.[34] Finally,

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[33] The proof should be read in conjunction with Theorem 3 as it follows a similar reasoning.

[34] The efficiency and accuracy of the code rely on various techniques. Most importantly, we rewrite all multidimensional expectations as nested one-dimensional integrals (with variable bounds), which we compute by integrating polynomial interpolations. Second, the code ensures that each quantity is computed at most once, using memorization. Integration is not computationally heavy at all and achieves high precision.

some quantities (such as bidding functions) have multiple analytical expressions, among which we choose the most appropriate for accuracy and speed, depending on the value of the signal (e.g., Eq. (6) can be integrated more efficiently than Eq. (18), but is less accurate for small signals). Our code is available on github.

We consider three signal distributions, a truncated exponential and a truncated normal distribution (both log-concave), as well as a Beta distribution with shape parameters $(0.5, 0.5)$, which is not log-concave. WEV-minimal pricing, a lower bound on the minimizer, and combinations of common value shares and mixed pricing for which MEU holds are shown in Fig. 8 for a market with $n = 10$ bidders and $k = 4$ items.

<div style="text-align:center">Truncated exponential   Truncated normal   Beta$(0.5, 0.5)$</div>
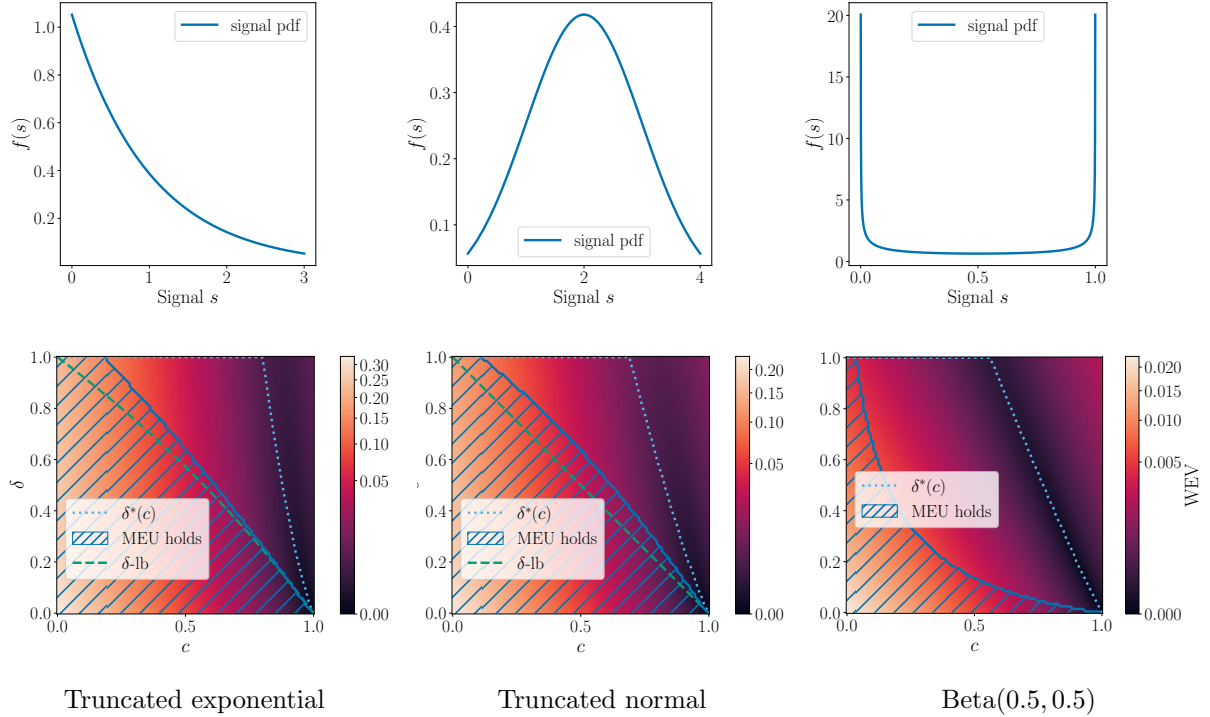
Figure 8: WEV-minimizing design $\delta^*(c)$, monotone ex-post utility (MEU), and lower bounds on $\delta^*(c)$ (δ-lb) for truncated exponential, truncated normal, and Beta$(0.5, 0.5)$ signal distributions

For the truncated exponential distribution, we show the lower bound of $\frac{2n(1-c)}{2n-c(n-(k+1))}$ (cf., Proposition 6) on $\delta^*(c)$, and for the normal distribution we show the general lower bound $1 - c$ (cf., Theorem 3). Each of these bounds dominates any extent of price discrimination below it. Note that for the Beta distribution, we cannot provide a theoretical lower bound on the WEV-minimal design δ^*, as the distribution is not log-concave. However, the region where MEU holds can be determined numerically, and its "frontier" provides a lower bound for the WEV-minimal design δ^*. Illustrating this for all three distributions, we observe that the area is much smaller for the Beta distribution. However, MEU is only a sufficient condition for the monotonicity of WEV (while it is necessary and sufficient for the monotonicity of pairwise differences). From the heat maps in Fig. 8, it is evident that WEV is monotone in δ for any given c up to δ^*.

Finally, we show the WEV-minimal pricing rule $\delta^*(c)$ for each signal distribution. The curve is qualitatively similar in each plot. In line with Theorem 3 — noting that the exponential and normal distribution are log-concave — the figure illustrates that with a high private value

component (low c), pay-as-bid pricing ($\delta = 1$) minimizes WEV; with higher common value components (high c), strictly mixed pricing for some $\delta \in (0, 1)$ minimizes WEV (cf., Proposition 4); and with a pure common value ($c = 1$), uniform pricing ($\delta = 0$) minimizes WEV (cf., Theorem 2). Analogous interpretations hold for the Beta distribution, although we cannot give theoretical guarantees.

For small common values, MEU holds for any δ and thus pay-as-bid pricing is dominant in pairwise differences (cf., Proposition 9). Even for larger common value parameters the WEV-minimal pricing is still pay-as-bid, but eventually strictly mixing ($\delta \in (0, 1)$) is required to minimize WEV. For a pure common value, uniform pricing is WEV-minimal regardless of the signal distribution. Notice also that WEV at the minimal δ^* decreases in c. Naturally, with a higher common value share, bidders' values given different signal realizations as well the corresponding bids move closer together, thus explaining smaller differences in utilities (ex-post and in expectation).

6.2 Variance and risk preferences

Surplus equity and distributional concerns are distinct from questions of within-agent variation and associated risk preferences. An appropriate measure to assess the latter is, e.g., the ex-ante variance of bidder surplus. While the two notions are distinct, the measures are linked through the covariance (see also Lemma 1). In addition, for the pure private value setting, we derive the following result:

Proposition 7. *With pure private values ($c = 0$), the pay-as-bid auction minimizes the ex-ante variance of surplus among all standard auctions with increasing equilibrium bid functions.*

Proof: See Appendix C.8.

Because of revenue equivalence, note that the previous proposition also implies that $\mathbb{E}[u_i^2]$ is minimal in the pay-as-bid auction among standard auctions. The second moment of surplus links the winners' empirical variance and the empirical variance among all bidders, as shown in Lemma 7 in the Appendix. As a consequence of Lemma 7, surplus equity rankings with respect to the winners' empirical variance and the empirical variance among all bidders may not be equivalent. However, applying Proposition 10 to the pure private value case, we have the following corollary:

Corollary 2. Assuming pure private values ($c = 0$), consider any δ-mixed auction, $\delta \in (0, 1]$, and suppose that the equilibrium bid β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2}{\delta}$ for all signals $s \in [0, \bar{s})$. Then, the empirical variance (among all bidders) is lower for δ-mixed pricing than for uniform pricing.

Although this result shows that Theorem 4 can be used to extend equity rankings under pure private values to the empirical variance *among all bidders*, this may not hold in the general case.

6.3 Beyond log-concave distributions

A crucial ingredient for Theorem 3 is that the derivative of the equilibrium bid function is bounded by 1, which holds for log-concave distributions by Proposition 11. In particular,

the density of the first rejected signal must be log-concave. In the following, we provide some insights as to why it is difficult to generalize this result beyond log-concave distributions.

For simplicity, consider the pay-as-bid and the uniform-price auction. Considering log-concave signal distributions, we note that log-concavity is equivalent to (A, G) concavity (a generalization of convexity, see Anderson, Vamanamurthy and Vuorinen (2007)), and $\frac{\partial \beta^\delta}{\partial s} \leq 1$ is thus equivalent to (A, G) concavity of $s \mapsto \int_0^s G_k^{n-1}$. One idea to extend our results could then be to consider other generalizations of convexity. Considering Proposition 10, one might attempt to bound the slope of the bid functions by 2. It holds that $\frac{\partial \beta^\delta}{\partial s} \leq 2$ is equivalent to (A, H) concavity of the same function where H is the harmonic mean. But contrary to (A, G) concave functions, there are no simple group closure properties that allow for the (A, H) concavity of f to always imply that of $\int_0^s G_k^{n-1}$. Thus, this route of inquiry does not carry fruits.

We also note that conditions similar to MEU such that uniform pricing yields lower pairwise differences (or WEV) than pay-as-bid pricing are much more difficult to attain. Why? If we follow the same main ideas as in the proof of Proposition 10, a similar condition using the mean value theorem would be that, for all $s \in (0, \bar{s})$, φ is an expansive mapping, translating into $|(1 - c) - \frac{\partial \beta^\delta}{\partial s}| \geq 1 - c$. As $\frac{\partial \beta^\delta}{\partial s}$ is strictly positive, we must have $\frac{\partial \beta^\delta}{\partial s} > 1 - c$. For signal distributions with bounded density, g_k^{n-1} is close to zero near $\bar{s}$ (this follows from the definition of order statistics), and therefore $\frac{\partial \beta^\delta}{\partial s}$ is close to zero for a nonzero interval of signals. Thus, $\frac{\partial \beta^\delta}{\partial s} > 1 - c$ cannot hold for all signals on the support, and we cannot rely on similar proof techniques to produce the desired conditions.

6.4 Multi-unit demand

A natural question is how robust the equity dominance results for pay-as-bid, uniform, and mixed auctions are with respect to the assumption of unit demand. A simple generalization are flat d_i-unit demands for some $d_> 1$, $i \in [n]$, meaning that bidders have the same constant marginal value for the first d_i items obtained and 0 after. There are two main challenges to extending our results to this setting, both leading to potential equity-efficiency trade-offs. Firstly, each bidder may win a different number of items, and thus equity could be studied with respect to a bidder's total surplus or average per-unit surplus. The latter allows for some efficiency concerns. The second challenge is behavioral concerning the multiplicity of equilibria in uniform-price auctions: Noussair (1995) show that even for two bidders and two-unit demand, there is no unique symmetric equilibrium as there exists an inefficient, zero-revenue symmetric equilibrium, as well as an efficient, zero-utility symmetric equilibrium.

Ausubel et al. (2014) show in their Theorem 1 that a necessary condition for the existence of an ex-post efficient equilibrium in multi-unit auctions is that the total supply of items is an integer multiple of a homogeneous demand, i.e., k/d is an integer ("demand divides supply"). Under this strong assumption, our results extend immediately without ambiguity if we consider a mixed pricing between pay-as-bid and Vickrey pricing. Under Vickrey pricing, a bidder who wins m items must pay the sum of the m highest losing bids, not including the bidder's own losing bids. This payment rule is exactly that of the VCG mechanism; hence, the Vickrey auction is incentive-compatible and efficient. When demand divides supply supply, all winners of the Vickrey auction will receive exactly d items, and each winner pays the same price, since

they face the same losing bids. This renders the Vickrey payment into a uniform-pricing rule. Under pay-as-bid pricing, the equilibrium bids for the first d-items are the same as for the 1-unit demand case, and thus each bidder also receives exactly d items (see also Ausubel et al. 2014). The consequence of demand dividing supply in both the pay-as-bid and Vickrey auction formats is that items are sold in bundles of size d, with payments and values scaled by d. Therefore, all results of the unit-demand case hold with flat d-unit demand when d divides k.

When demand does not divide supply, our results do not extend to the multi-unit case. In the example in Ausubel et al. (2014) with two bidders with flat two-unit demand and pure private values, there exists an inefficient equilibrium in the uniform-price auction. In this equilibrium, each bidder is allocated one unit, which is always more surplus-equitable than the efficient equilibrium of the pay-as-bid auction which allocates both items to the higher-value bidder.

6.5   Revenue maximization and reserve prices

Standard auctions lead to an efficient allocation of items in our model. Moreover, in a large class of mechanisms, these auctions are also revenue-maximizing. This class of mechanisms is described in Bulow and Klemperer (1996), which, in addition to our assumption of independent signals, requires that the bidders' marginal revenues are increasing in signals and weakly positive.[35] Marginal revenues are positive in many applications where bidders' values are high, i.e., the support of signals is strictly positive.[36] In those cases, it is optimal for the seller to sell their entire supply. In practice, this is observed in my many high-stake auctions, e.g., for spectrum licenses. Our class of auctions is also revenue-maximizing if the seller is legally required to sell their entire supply. More generally, maximizing seller revenue at the expense of efficiency is discussed in Riley and Samuelson (1981) and Myerson (1981). A common tool to raise revenue is setting a reserve price $r > 0$ such that only bids exceeding r can win and pay the auction price, or at least r, whichever is higher. Myerson (1981) and Riley and Samuelson (1981) show that, with pure private values, an optimal reserve price maximizes the seller's expected revenue.

Some of our results on equity-preferred pricing extend to pay-as-bid and uniform-price auctions with an additional reserve price. With a price, the number of winners, although identical in different standard auctions with identical reserve prices, is not necessarily equal to the number of items k and becomes a random variable. In the uniform-price auction, the standard derivation leads to an equilibrium bid of $V(s)$ for $s > s_r$, where s_r is some threshold, and 0 otherwise. Using revenue equivalence, we obtain the equilibrium bid in the pay-as-bid auction $\beta_r^{\delta=1}(s) = (r - V(s_r))G(s_r)/G(s) + V(s) - \int_{s_r}^s V'(y)G(y)dy/G(s)$ for $s > s_r$ and 0 otherwise. For the equity comparison of pay-as-bid and uniform pricing, we establish the following property.

Proposition 8. *If monotone ex-post utility (MEU, Definition 6) holds in the pay-as-bid auction without a reserve price, then it also holds with a strictly positive reserve price.*

Proof: See Appendix C.8.

Then, the following corollary is immediate from Proposition 8 (a slightly weaker statement than Proposition 9).

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[35] With independent signals, the marginal revenue of bidder i given a realization of signals $\boldsymbol{s}$ is given by $MR_i(s_i) = \frac{-1}{f(s_i)} \frac{\partial}{\partial s_i} (v(s_i)(1 - F(s_i)))$.

[36] In our model, without loss of generality, the signal support includes the lower bound zero.

Corollary 3. Suppose MEU holds in the equilibrium of the pay-as-bid auction without reserve price. Then the pay-as-bid auction with a given reserve price is equity-preferred to (dominates in pairwise differences) any uniform-price auction with the same reserve price.

7 Conclusion

This article studies the division of surplus between buyers in auction design. We introduce a family of equity measures that are based on absolute pairwise differences in realized utilities. Our equity family includes well-known metrics, such as the empirical variance and the expected Gini index. Considering standard and winners-pay auctions in an independent signal setting with single-crossing values, focusing on equity as a single design objective is costless in terms of potential trade-offs with efficiency and revenue.

First, we design the surplus-equitable mechanism, a direct and truthful mechanism that efficiently allocates the items for sale and charges each winner a personalized price. This price equalizes the winners' realized utilities, while losers pay nothing. Turning to the class of uniform, pay-as-bid, and mixed auctions, we show that, in most cases, some degree of price discrimination is beneficial in terms of equity. We show that equity-preferred pricing crucially depends on the common value proportion in the buyers' value structure. Moreover, our results have significant implications for the design of multi-unit auctions in practice. By carefully selecting a pricing mixture based on (an estimate of) the common value, auctioneers can achieve a more equitable division of surplus among buyers.

Future research could explore the trade-offs between efficiency, revenue, and equity, or extend our analysis to other types of auctions and value distributions. For example, in multi-unit demand settings in which items may be allocated inefficiently (Section 6.4), trade-offs become relevant. In practice, other designs such as dynamic auctions or the Spanish auction[37] are used, and understanding the impact of such designs on surplus equity remains an open question.

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[37] Spanish treasury auctions are a hybrid design with partial price discrimination, in which only high-price bidders pay a uniform price. Those bidding higher than the weighted average winning bid price (WAP) pay the WAP, while winners bidding below the WAP pay their bid (Álvarez and Mazón, 2007).

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Appendices

A Structural insights and proofs of Theorems 3 and 4

In this section, we provide an overview of the proofs of Theorems 3 and 4. Each of these theorems is proved by combining a proposition on monotonicity of pairwise differences and dominance of pairwise differences, respectively, with a third proposition that bounds the slope of bid functions. In particular, we identify the property of *monotone ex-post utility* as a fundamental and sufficient condition for our dominance results.

Definition 6 (Monotone ex-post utility). The ex-post utility $u(\boldsymbol{s})$ satisfies *monotone ex-post utility (MEU)* iff, for any two signals $s_i, s_j \in [0, \bar{s})$ and $\forall \, \boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $s_i \leq s_j \Leftrightarrow u_i(s_i, \boldsymbol{s}_{-i}) \leq u_j(s_j, \boldsymbol{s}_{-j})$.

Monotone ex-post utility (MEU) relates to the slope of equilibrium bids as follows.

Lemma 2. *An equilibrium satisfies monotone ex-post utility iff equilibrium bid functions β^δ satisfy $\frac{\partial \beta^\delta}{\partial s} \leq \frac{1-c}{\delta}$ for all signals $s \in [0, \bar{s})$.*

Proof: See Appendix C.5.

The ex-post difference in utilities depends only on the private value proportion $(1-c)s$ and the discriminatory part of the payment $\delta\beta^\delta$. Thus, as long as the discriminatory payment does not grow faster in the signal than the private-value share, ex-post utilities are monotone.

As seen in Fig. 1, the equilibrium exhibit several monotonicity properties, and these hold beyond uniform signals. By assumption, equilibrium bids are increasing in the bidder's own signal. Equilibrium bids are also decreasing in the extent of price discrimination: if a higher proportion of one's own bid affects the price, the incentive to bid-shade increases. Finally, the change in the payment corresponding to a bidder's own bid, due to a signal increase, is increasing in the weight of price discrimination, and vice versa. The latter monotonicity is crucial for Proposition 9.

Lemma 3. *The equilibrium bid functions satisfy the following monotonicity properties:*

1. *$\beta^\delta(s)$ is strictly increasing in s, for all fixed $\delta \in [0,1]$ (consistent with the assumption), and is strictly decreasing in δ, for all fixed $s \in (0, \bar{s})$.*

2. *$\frac{\partial(\delta\beta^\delta(s))}{\partial\delta}$ is strictly increasing in s, for all fixed $\delta \in [0,1]$, and $\frac{\partial(\delta\beta^\delta(s))}{\partial s}$ is strictly increasing in δ, for all fixed $s \in (0, \bar{s})$.*

Proof: See Appendix C.2.

We now characterize the fundamental role of monotone ex-post utility: it is equivalent to the monotonicity property of pairwise differences.

Proposition 9. *For a given common value c and for some $\bar{\delta} \in [0,1]$, pairwise differences are monotonically decreasing over $[0, \bar{\delta}]$ if and only if the equilibrium (which depends on c and $\bar{\delta}$) satisfies MEU.*

Proof: See Appendix C.5.

The equivalence between decreasing pairwise differences and MEU being satisfied in equilibrium is crucial in our proof of Theorem 3. When MEU holds, the slope of the equilibrium bid function is sufficiently flat and more price discrimination impacts higher signal bidders more than lower signal bidders. In contrast, including more uniform pricing in the price mix will, proportionally to the change in δ, offer higher signal bidders a higher discount than lower signal bidders and thus does not improve surplus equity. A similar intuition holds for Proposition 10 below (for details on the intuition, see Section 5.3).

Proposition 10. *For a given common value c, consider any δ-mixed auction, $\delta \in (0,1]$, and suppose the equilibrium bidding function β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2(1-c)}{\delta}$ for all signals $s \in [0, \bar{s})$. Then, δ-mixed pricing dominates uniform pricing in pairwise differences.*

Proof: See Appendix C.5.

Example 1 (Continued)**.** Whether MEU is satisfied can be verified numerically, either by computing differences in realized utilities for every pair of signals or simply by checking the derivative of the bid function. We illustrate this for the example of uniform signal distributions and $n = 3$ and $k = 2$ in Fig. 9 below. For example, with $c = 0.8$ and $\delta = 0.3$, close to the MEU boundary in Fig. 9, the derivative of the bid function cannot be larger than $0.667 = \frac{1-0.8}{0.3}$. From Fig. 1, the slope of the bid function with $c = 0.8$ and $\delta = 0.3$ is close to 0.68 for low signals. Thus, for this combination of c and δ, MEU is not satisfied.
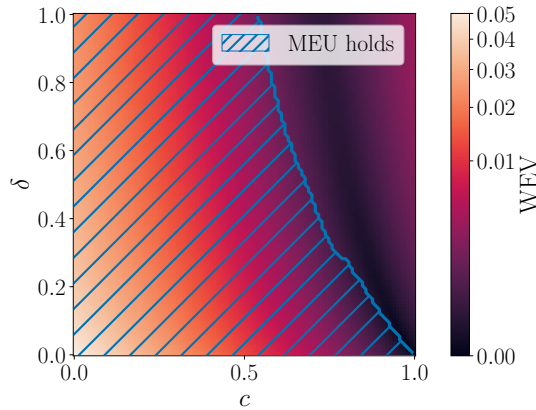
Figure 9: Monotone ex-post utility for common value c and price discrimination δ

The final crucial proposition bounds the slope of the equilibrium bid functions by 1 for the family of log-concave signal distribution.

Proposition 11. *If the signal density f is log-concave, then $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ for all signals $s \in [0, \bar{s})$.*

The proof is given in Appendix A.1 below. With pure private values, this bound implies that ex-post utility is non-decreasing in signals for log-concave signal distributions. Indeed, for $u(s) = s - \delta\beta^\delta(s) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have that $\frac{\partial u}{\partial s} = 1 - \delta\frac{\partial \beta^\delta(s)}{\partial s} \geq 0$. A similar reasoning

leads to Theorems 3 and 4.

Proof of Theorems 3 and 4. Because of Proposition 11, we have that under log-concave signal distributions MEU holds if $\delta \leq (1-c)$, as $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{(1-c)}{\delta}$ (see Lemma 2). Thus, applying Proposition 9, pairwise differences are monotonically decreasing for $\delta \in [0, 1-c]$ and Theorem 3 follows.

Similarly, because of Proposition 11, it holds that with log-concave signals, $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{2(1-c)}{\delta}$ if $\delta \leq 2(1-c)$. Applying Proposition 10, it follows that any mixed pricing with $\delta \in (0, 2(1-c)]$ dominates uniform pricing in pairwise differences. □

A.1 Proving the bound on bid function slopes

Bounding the bid function slope for log-concave distributions requires three main observations, which we detail in the lemmas below and then use to prove Proposition 11.

The first lemma establishes a simplified expression of $V(s)$ which allows to bound $V'(s)$ by 1.

Lemma 4. *Assuming common values ($c = 1$), $V(s)$ is differentiable on $(0, \bar{s})$, and can be expressed as*

$$V(s) = \frac{2}{n}s + \frac{n-k-1}{n} \frac{\int_0^s t f(t)\, \mathrm{d}t}{F(s)} + \frac{k-1}{n} \frac{\int_s^{\bar{s}} t f(t)\, \mathrm{d}t}{1 - F(s)}$$

Moreover, if the signal density f is log-concave, then $V'(s) \leq 1$ for all signals $s \in [0, \bar{s})$.

Proof: See Appendix C.6.

The proof proceeds by noticing that order statistics conditioned on other order statistics behave just like order statistics of a truncation of the original distribution. Thus, a more tractable expression of the expected valuation V can be derived for the pure common value case ($c = 1$). Together with results on log-concavity by Bagnoli and Bergstrom (2005), we use this expression to show that $V' \leq 1$ for all signals $s \in (0, \bar{s})$.

The next lemma establishes a sufficient condition for the equilibrium bid function slope to be bounded by 1 in the pure private value case. Differentiating twice $\int_0^s G^{1/\delta}$, we establish that its log-concavity is equivalent to $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$.

Lemma 5. *Assuming private values ($c = 0$), for any $\delta \in (0, 1]$, $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ iff $\int_0^s G^{\frac{1}{\delta}}(y)\, \mathrm{d}y$ is log-concave.*

Proof: See Appendix C.6.

Finally, we establish that a log-concave signal density is sufficient for the integral of their order statistics to be log-concave, using closure properties of product and integration of log-concave distributions, and results by Bagnoli and Bergstrom (2005).

Lemma 6. *If the density of signals f is log-concave, then so is $\int_0^s G^{\frac{1}{\delta}}(y)\, \mathrm{d}y$.*

Proof: See Appendix C.6.

With the three lemmas above, we can prove Proposition 11.

Proof of Proposition 11. First, we recall the expression of the derivative of the bid function for any $s \in (0, \bar{s})$:

$$\frac{\partial \beta^{\delta}(s)}{\partial s} = \frac{g(s)}{G(s)} \frac{\int_0^s V'(y) G^{\frac{1}{\delta}}(y) \, \mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \tag{14}$$

Note that for any $c \in [0, 1]$, $V(s)$ is a linear combination of s and V. In the case of a pure common value, the derivative of the latter is bounded by 1 by Lemma 4. Hence for any c, $V'(s) \leq 1$. Moreover, because of Lemma 6, we know that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave, and we can therefore apply Lemma 5. Hence using the above results,

$$\frac{\partial \beta^{\delta}(s)}{\partial s} \leq \frac{g(s)}{G(s)} \frac{\int_0^s \max_t V'(t) G^{\frac{1}{\delta}}(y) \, \mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq \frac{g(s)}{G(s)} \frac{\int_0^s 1 \cdot G^{\frac{1}{\delta}}(y) \, \mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq 1. \tag{15}$$

$\square$

B Revenue equivalence and efficiency

We recall results from Krishna (2009) that show that the auctions we consider exhibit revenue equivalence and (allocative) efficiency.

Proposition (Revenue equivalence, Krishna 2009)**.** *Assuming iid signals, any standard auction, under any symmetric and increasing equilibrium with an expected payment of zero at value zero, yields the same expected revenue to the seller.*

We note that the crucial assumption for revenue equivalence is the independence of signals. In settings where signals are correlated, revenue equivalence fails Krishna, 2009, Chapter 6.5. It can be further shown that a bidder with signal s_i has an expected surplus

$$\tilde{u}(s_i) := \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] = \int_0^{s_i} (\widetilde{V}(s_i, y) - \widetilde{V}(y, y)) g_k^{n-1}(y) \, \mathrm{d}y$$

A value function $v(\boldsymbol{s})$ satisfies the *single crossing* condition if for all $i, j \neq i \in [n]$ and for all $\boldsymbol{s}$, $\frac{\partial v(s_i, \boldsymbol{s}_{-i})}{\partial s_i} \geq \frac{\partial v(s_j, \boldsymbol{s}_{-j})}{\partial s_i}$, and the value function v as given in Assumption 1 satisfies this condition.

Proposition (Efficiency, Krishna 2009)**.** *Any standard auction, under any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is efficient.*

Given the prior propositions, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs.

C Proofs

C.1 Surplus equity

Proof of Lemma 1. The empirical variance of surplus can be transformed as follows.

$$\mathbb{E}_s \left[\frac{1}{n-1} \sum_i^n \left(u_i - \frac{1}{n} \sum_j^n u_j \right)^2 \right] = \mathbb{E}_s \left[\frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (u_i - u_j)^2 \right]$$
$$= \frac{\mathbb{E}_s \left[(u_1 - u_2)^2 \right]}{2}$$
$$= \mathbb{E}_s[u_1^2] - \mathbb{E}_s[u_1 u_2]$$
$$= \mathrm{Var}(u_1) - \mathrm{Cov}(u_1, u_2)$$

Similarly, the empirical variance conditioned on winning can be written as

$$\mathbb{E}_s \left[\frac{1}{k-1} \sum_{i=1}^k \left(u_i - \frac{1}{k} \sum_{j=1}^k u_j \right)^2 \middle| 1, \ldots, k \text{ win} \right] = \frac{\mathbb{E}_s \left[(u_1 - u_2)^2 \mid 1 \text{ and } 2 \text{ win} \right]}{2}$$
$$= \mathbb{E}_s \left[u_1^2 \mid 1 \text{ wins} \right] - \mathbb{E}_s \left[u_1 u_2 \mid 1 \text{ and } 2 \text{ win} \right]$$
$$= \mathrm{Var}(u_1 \mid 1 \text{ wins}) - \mathrm{Cov}(u_1, u_2 \mid 1 \text{ and } 2 \text{ win}).$$

$\square$

With pure private values, ex-post individual rationality holds. The lemma below shows that, in this case, any ranking of auction formats in terms of ex-ante variance (Var) or winners' ex-ante variance (WV) is identical. In contrast, a ranking with respect to empirical variance (EV) can differ depending on whether only winners are considered or all bidders.

Lemma 7. *Assuming that the auction is a* winners-pay auction, *the empirical variance and the ex-ante variance can be decomposed, respectively, as* $\mathrm{EV} = \frac{k(k-1)}{n(n-1)} \cdot \mathrm{WEV} + \left(1 - \frac{k-1}{n-1} \right) E_{\boldsymbol{s}}[u_1^2]$ *and* $\mathrm{Var} = \frac{k}{n} \cdot \mathrm{WV} + \left(\frac{n}{k} - 1 \right) \cdot E_{\boldsymbol{s}}[u_1]^2$.

Recall that $E_{\boldsymbol{s}}[u_1]$ does not depend on the auction format (by revenue equivalence), while $E_{\boldsymbol{s}}[u_1^2]$ does.

Proof of Lemma 7. We first note that

$$\mathrm{WV} = E_{\boldsymbol{s}}[u_1^2 \mid 1 \text{ wins}] - E_{\boldsymbol{s}}[u_1 \mid 1 \text{ wins}]^2 = \frac{n}{k} E_{\boldsymbol{s}}[u_1^2] - \left(\frac{n}{k} \right)^2 E_{\boldsymbol{s}}[u_1]^2$$

For the ex-ante variance, we write:

$$
\begin{aligned}
\mathrm{Var} = E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1]^2 &= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \mid 1\ \mathrm{wins}] - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \mid 1\ \mathrm{wins}] - P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1 \mid 1\ \mathrm{wins}]^2 + P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1 \mid 1\ \mathrm{wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot \mathrm{WV} + P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1 \mid 1\ \mathrm{wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot \mathrm{WV} + \frac{P_{\boldsymbol{s}}[1\ \mathrm{wins}]^2}{P_{\boldsymbol{s}}[1\ \mathrm{wins}]} \cdot E_{\boldsymbol{s}}[u_1 \mid 1\ \mathrm{wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot \mathrm{WV} + \left(\frac{n}{k} - 1\right) \cdot E_{\boldsymbol{s}}[u_1]^2
\end{aligned}
$$

For the empirical variance, we write:

$$
\begin{aligned}
\mathrm{WEV} = E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1 u_2] &= P_{\boldsymbol{s}}[1\ \mathrm{wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \mid 1\ \mathrm{wins}] - P_{\boldsymbol{s}}[1\ \mathrm{and}\ 2\ \mathrm{win}] \cdot E_{\boldsymbol{s}}[u_1 u_2 \mid 1\ \mathrm{and}\ 2\ \mathrm{win}] \\
&= P_{\boldsymbol{s}}[1\ \mathrm{and}\ 2\ \mathrm{win}] \cdot \mathrm{WEV} + \left(1 - \frac{P_{\boldsymbol{s}}[1\ \mathrm{and}\ 2\ \mathrm{win}]}{P_{\boldsymbol{s}}[1\ \mathrm{wins}]}\right) \cdot E_{\boldsymbol{s}}[u_1^2]
\end{aligned}
$$

$\square$

Proof of Proposition 3. Without loss of generality, consider an outcome profile u with three outcomes, u_i, u_j and U, where $u_i > u_j$, and U is arbitrary. Induce a Pigou-Dalton transfer $t > 0$ such that $u_i' = u_i - t > u_j$ and $u_j' = u_j + t < u_i$, and U remains the same. The outcome profile after the transfer is denoted u'. We show that the ranking between u and u' according to WEV coincides with what the Pigou-Dalton principle requires, namely it must be that $WEV(u') < \mathrm{WEV}(u)$. Let $W := (u_i - U)^2 + (u_j - U)^2$. Then

$$
\begin{aligned}
& (u_i' - U)^2 + (u_j' - U)^2 \\
={}& (u_i - t - U)^2 + (u_j + t - U)^2 \\
={}& (u_i - U)^2 - 2t(u_i - U) + t^2 + (u_j - U)^2 + 2t(u_j - U) + t^2 \\
={}& W + 2t\left(t - u_i + U + u_j - U\right) \\
={}& W + 2t\left(u_j - (u_i - t)\right) \\
<{}& W
\end{aligned}
$$

The final inequality follows by the assumption that the transfer does not make i poorer than j was to start with. As U was arbitrarily chosen and, to compute WEV, expectations are taken around the sum of squared differences of the realized utilities, the result follows. $\square$

C.2 Equilibrium bidding

Proof of Proposition 1. Consider bidder i and let all bidders $j \neq i$ follow the bidding strategy $\beta^U(s_j) = \widetilde{V}(s_j, s_j)$. First, observe that β^U is continuous and increasing. Then bidder i's expected payoff when their signal is s_i and bidding $\beta^U(z)$ is given by

$$
U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \widetilde{V}(y, y)\right) g_k^{n-1}(y)\,\mathrm{d}y
$$

Because $\widetilde{V}(s_i, y)$ is increasing in s_i, it holds for all $y < s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) > 0$, and for

all $y > s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) < 0$. Therefore, choosing $z = s_i$ maximizes bidder i' expected payoff $U(s_i, z)$. $\qquad\square$

Proof of Proposition 2. First, observe that β^δ is continuous. We verify that it is also monotone: writing $G_k^{n-1} =: G$, $g_k^{n-1} =: g$, and $\widetilde{V}(s, s) =: V(s)$, an alternative expression for β^δ is given by

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G(y)^{\frac{1}{\delta}} \, \mathrm{d}y}{G(s)^{\frac{1}{\delta}}}. \tag{16}$$

In particular, it is differentiable almost everywhere and we can compute its derivative.

$$\beta^{\delta\prime}(s) = \frac{g(s) \int_0^s V'(y) G(y)^{\frac{1}{\delta}} \, \mathrm{d}y}{\delta G(s)^{1+\frac{1}{\delta}}} \tag{17}$$

which it positive almost everywhere. Next, assume that all bidders $j \neq i$ follow the bidding strategy β^δ, and let $\beta^\delta(z)$ be bidder i's bid, whose expected utility is given by

$$U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \delta\beta^\delta(z) - (1 - \delta)\beta^\delta(y) \right) g(y) \, \mathrm{d}y$$

The derivative of $U(s_i, z)$ is

$$\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, z) = \widetilde{V}(s_i, z)g(z) - \delta\beta^{\delta\prime}(z)G(z) - \delta\beta^\delta(z)g(z) - (1 - \delta)\beta^\delta(z)g(z)$$
$$= (\widetilde{V}(s_i, z) - \beta^\delta(z))g(z) - \delta\beta^{\delta\prime}(z)G(z).$$

In equilibrium, the first order condition requires $\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, s_i) = 0$. We solve this differential equation using $G^{\frac{1}{\delta}-1}$ as the integrating factor. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[G(z)^{\frac{1}{\delta}} \beta^\delta(z) \right] = \left(\frac{1}{\delta} G(z)^{\frac{1}{\delta}-1} \right) \cdot (\beta^\delta(z)g(z) + \delta\beta^{\delta\prime}G(z)) = \left(\frac{1}{\delta} G(z)^{\frac{1}{\delta}-1} \right) \cdot \widetilde{V}(s_i, z)g(z).$$

Solving for β^δ, we obtain

$$\beta^\delta(s) = \frac{\int_0^s V(y) g_k^{n-1}(y) G_k^{n-1}(y)^{\frac{1}{\delta}-1} \, \mathrm{d}y}{\delta G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{18}$$

Using equations (16) and (17), and the fact that $\widetilde{V}(s_i, z)$ is increasing in s_i, we have that $\frac{\mathrm{d}U}{\mathrm{d}z}$ is positive when $z \leq s_i$ and negative when $z \geq s_i$. Therefore, choosing $z = s_i$ maximizes i's expected payoff $U(s_i, z)$.

Finally, we derive the expression for β^δ stated in the the proposition from Eq. (18). Writing $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$, observe that the derivative of $\delta G^{\frac{1}{\delta}}$ is $gG^{\frac{1}{\delta}-1}$. Using integration by parts and a change of variable, we obtain

$$\int_0^s V(y) g(y) G(y)^{\frac{1}{\delta}-1} \, \mathrm{d}y = \left[\delta V(y) G(y)^{\frac{1}{\delta}} \right]_0^s - \delta \int_0^s V'(y) G(y)^{\frac{1}{\delta}} \, \mathrm{d}y$$
$$= \delta V(s) G(s)^{\frac{1}{\delta}} - \delta \int_{V(0)}^{V(s)} G(V^{-1}(y))^{\frac{1}{\delta}} \, \mathrm{d}y.$$

Dividing by $\delta G^{\frac{1}{\delta}}$ gives the result. $\qquad\qquad\square$

Lemma 8. *For any continuous function* $\varphi : [0, \bar{s}) \to \mathbb{R}$, *and for all* $s \in (0, \bar{s})$, *we have*

$$\lim_{\delta \to 0} \int_0^s \frac{\varphi(t)}{\delta} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \varphi(s) \cdot \frac{G(s)}{g(s)}$$

$$\lim_{\delta \to 0} \int_0^s \ln\left(\frac{G(s)}{G(t)}\right) \frac{\varphi(t)}{\delta^2} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \varphi(s) \cdot \frac{G(s)}{g(s)}$$

where $G_k^{n-1} =: G$ *and* $g_k^{n-1} =: g$.

Proof. Fix $\delta > 0$, and let $\psi : (0, 1] \to \mathbb{R}$ be a continuous function, such that $\psi(u) = O(1/u)$ when $u \to 0$. Using the change of variable $u = v^\delta$, we have that

$$\int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \mathrm{d}u = \int_0^1 \psi(v^\delta) v^\delta dv$$

$$\int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \mathrm{d}u = \int_0^1 \ln(1/v) \psi(v^\delta) v^\delta dv.$$

Observe that for all fixed $v \in (0, 1]$, and taking $\delta \to 0$, the first (resp. second) integrand converges towards $\psi(1)$ (resp., $\psi(1) \ln(1/v)$). We define the constant $M = \sup_{u \in (0,1]} u\psi(u)$, we bound the first integrand by M (resp.the second integrand by $M \ln(1/v)$), and we use the theorem of dominated convergence, which gives

$$\lim_{\delta \to 0} \int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \mathrm{d}u = \int_0^1 \psi(1) \, \mathrm{d}v = \psi(1)$$

$$\lim_{\delta \to 0} \int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \mathrm{d}u = \int_0^1 \psi(1) \ln(1/v) \, \mathrm{d}v = \psi(1)$$

To prove the lemma, observe that with the change of variable $u = \frac{G(t)}{G(s)}$, we have

$$\int_0^s \frac{\varphi(t)}{\delta} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \int_0^1 \frac{\psi(u)}{\delta} u^{\frac{1}{\delta}} \mathrm{d}u$$

$$\int_0^s \ln\left(\frac{G(s)}{G(t)}\right) \frac{\varphi(t)}{\delta^2} \left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}t = \int_0^1 \ln(1/u) \frac{\psi(u)}{\delta^2} u^{\frac{1}{\delta}} \mathrm{d}u$$

where we define

$$\psi(u) := G(s) \cdot \frac{\varphi(G^{-1}(uG(s))}{g(G^{-1}(uG(s))}.$$

Finally, it remains to prove that $\psi(u) = O(1/u)$ when $u \to 0$. First, observe that φ is bounded on $[0, s]$. Second, observe that we have

$$\frac{u}{g(G^{-1}(uG(s)))} = \frac{1}{G(s)} \frac{G(x)}{g(x)},$$

where $x = G^{-1}(uG(s)) \to 0$. Because g is positive and integrable in 0, we have that G/g is bounded. Therefore, the overall limit when $\delta \to 0$ is equal to $\psi(1)$, which concludes the proof. $\qquad\square$

Lemma 9. *The following formulas can be derived:*

$$\beta^\delta(s) = \begin{cases} V(s) & \text{if } \delta = 0 \\ V(s) - \int_0^s V'(y) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} dy & \text{if } \delta > 0 \end{cases}$$

$$\frac{\partial(\beta^\delta(s))}{\partial s} = \begin{cases} V'(s) & \text{if } \delta = 0, s > 0 \\ \frac{g(s)}{G(s)} \int_0^s \frac{V'(y)}{\delta} \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} dy & \text{if } \delta > 0 \end{cases}$$

$$\frac{\delta\partial(\beta^\delta(s))}{\partial\delta} = \begin{cases} 0 & \text{if } \delta = 0 \text{ or } s = 0 \\ \int_0^s V'(y) \ln\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} dy & \text{for } \delta, s > 0 \end{cases}$$

$$\frac{\partial^2(\delta\beta^\delta(s))}{\partial s \partial\delta} = \frac{-g(s)}{\delta G(s)} \int_0^s V'(y) \log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} dy \text{ for } \delta, s > 0$$

Proof. In order to derive the value of these functions at points where they are not directly, defined, we will use the dominated convergence theorem.

(1) Let $s \in (0, \bar{s})$. We first look at $\beta^\delta(s) = V(s) - \int_0^s V'(y) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} dy$. Let $h(\delta, y)$ be the function under the integral. Because G is increasing, for $y < s$ we have that $G(y)/G(s) < 1$. Hence h is dominated by V', and $\lim_{\delta \to 0} h(\delta, y) = 0$, hence by dominated convergence $\beta^\delta(s) = V(s)$ when $\delta = 0$, and the function is separately continuous over $[0, 1] \times [0, \bar{s})$.

(2) We now consider the derivative of β^δ with respect to s. Let $s \in (0, \bar{s})$. There exists $M > m > 0$ such that $s \in [m, M]$. We focus on the derivative of the integral part:

$$-\frac{\partial}{\partial s} V'(y) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} = V'(y) \frac{g(s) G^{1/\delta}(y)}{G^{1/\delta+1}(s)} \leq V'(y) \frac{g(s)}{G(s)} \leq V'(y) \frac{\sup_{t \in [m,M]} g(t)}{G(m)},$$

where the $\sup_{t \in [m,M]} g(t)$ is finite as g is continuous. Because V' is integrable, we can use dominated convergence. Using the Leibniz integral rule yields the result. The limit as δ goes to 0 can be computed by applying Lemma 8.

(3) Let us now compute the derivative of β^δ with respect to δ. Again, we use a dominated convergence property to show that the integral and derivative can be inverted. It is easier to show that this can be done for the function $\delta\beta^\delta(s)$, and we have

$$\frac{\partial\delta\beta^\delta(s)}{\partial\delta} = \beta^\delta + \delta\frac{\partial\beta^\delta(s)}{\partial\delta}.$$

Computing the derivative of $\delta\beta^\delta(s)$, we can recover that of $\beta^\delta(s)$.

Let $h(\delta, y, s) = \delta V'(y)(G(y)/G(s))^{1/\delta}$ be the function under the integral part of $\delta\beta^\delta$. We

have

$$\frac{\partial h(\delta, y, s)}{\partial \delta} = V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y)\frac{\delta}{\delta^2}\log\left(\frac{G(y)}{G(s)}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}$$
$$= V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}.$$

The first part is again dominated by V', which is integrable. Focusing on the second part, we define for $0 < u < w < 1$ the function $\psi(u, w) = (u/w)\log(w/u)$. Note that $0 < s < y < \bar{s}$ implies that for $u = G^{1/\delta}(y)$ and $w = G^{1/\delta}(s)$, we have $0 < u < w < 1$ as G is increasing and takes values in $(0, 1)$ over $(0, \bar{s})$ by definition. Fix w, and take the derivative with respect to u: we obtain that $\psi'(u, w) = (\log(w/u) - 1)/w$ which is positive as long as $u \le w/e$ and negative otherwise. The maximum of ψ for $u < w$ is at $u = w/e$ and $\psi(w/e, w) = 1/e$. This shows the right-hand side of h is smaller than $V'(y)/e$, which is also integrable. Overall by dominated convergence we can invert derivative and integral: $\frac{\partial}{\partial\delta}\int h = \int \frac{\partial}{\partial\delta}h$. Thus

$$\frac{\partial\delta\beta^\delta}{\partial\delta} = V(s) - \int_0^s V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}dy + \int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta},$$

and we recover

$$\delta\frac{\partial\beta^\delta(s)}{\partial\delta} = \int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}.$$

Using the same upper bound on ψ, we can show that the integrand of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ is smaller than $V'(y)/e$ which allows for domination both in small δ and small s. By dominated convergence, we obtain that the limit of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ as either δ or s go to 0 is 0.

(4) Finally, let us compute the cross derivative. The integrand of $\frac{\partial\beta^\delta(s)}{\partial s}$ is $h(\delta, y, s) = V'(y)(G(y)/G(s))^{1/\delta}$, and its derivative with respect to delta is $-\frac{1}{\delta}V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)$. Because this function is continuous on the open set $(0, 1) \times [0, \bar{s})$, we can apply dominated convergence to show that the order of derivative and integral can be reversed. Therefore

$$\frac{\partial^2\delta\beta^\delta}{\partial\delta\partial s} = \frac{g(s)}{G(s)}\frac{-G^{1/\delta}(s)\frac{1}{\delta}\int_0^s V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)\,dy + \frac{1}{\delta}\log(G^{1/\delta}(s))G^{1/\delta}(s)\int_0^s V'(y)G^{1/\delta}(y)\,dy}{G^{2/\delta}}$$
$$= \frac{-g(s)}{\delta G(s)}\int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\,dy.$$

$\square$

Lemma 10. *Consider a function $\varphi : [0, 1] \times (0, \bar{s}) \to \mathbb{R}_+$, such that*

- *$\varphi_\delta : s \mapsto \varphi(\delta, s)$ is continuous over $(0, \bar{s})$ for all fixed $\delta \in [0, 1]$,*

- *$\varphi_s : \delta \mapsto \varphi(\delta, s)$ is continuous over $[0, 1]$ for all fixed $s \in (0, \bar{s})$,*

- *either all φ_s's are monotone or all φ_δ's are monotone,*

then φ is jointly continuous in δ and s.

Proof. The proof on the open set $(0,1) \times (0, \bar{s})$ is written in Kruse and Deely (1969), and directly generalizes to $\delta = 0$ and $\delta = 1$ given that φ is separately continuous in those points. $\qquad \square$

Proof of Lemma 3. Monotonicity follows from the derivatives computed in Lemma 9. $\qquad \square$

C.3 Equity comparisons in uniform, pay-as-bid and mixed auctions

Proof of Theorem 2. To prove the "if" direction, note that for $c = 1$, the realized value is identical for all bidders $i \in [n]$ as $v(\boldsymbol{s}) = \frac{1}{n} \sum_{j \in [n]} s_j$. Thus, with a uniform price that is identical between bidders, they all have identical surplus. For any $\delta > 0$, the payment differs between the winners at least for some signal realizations.

To prove the "only if" let $\varphi^\delta(s) = (1 - c) \cdot s - \delta \beta^\delta(s)$. We then have $(u(s_i) - u(s_j))^2 = (\varphi^\delta(s_i) - \varphi^\delta(s_j))^2$ for two winning bids s_i, s_j (see the proof of Proposition 10 for details). Thus, it holds that

$$\frac{\partial}{\partial \delta}(\varphi^\delta(s_i) - \varphi^\delta(s_j))^2 = -2(\varphi^\delta(s_i) - \varphi^\delta(s_j))\left(\beta^\delta(s_i) - \beta^\delta(s_j) + \delta \frac{\partial \beta^\delta(s_i)}{\partial \delta} - \delta \frac{\partial \beta^\delta(s_j)}{\partial \delta}\right).$$

Using Lemma 9, we take the limit of β^δ, $\delta \frac{\partial \beta^\delta(s)}{\partial \delta}$, and $\varphi^\delta(s)$, as δ goes to 0. We have that $(\varphi^\delta(s_i) - \varphi^\delta(s_j)) \to (1-c)(s_i - s_j)$ and $(\beta^\delta(s_i) - \beta^\delta(s_j) + \delta \frac{\partial \beta^\delta(s_i)}{\partial \delta} - \delta \frac{\partial \beta^\delta(s_j)}{\partial \delta}) \to (V(s_i) - V(s_j))$. As V is increasing, the product $(V(s_j) - V(s_i))(s_i - s_j)$ is strictly negative almost surely, which concludes the proof. $\qquad \square$

Proof of Proposition 4. We show that for any $c \in (c^*, 1)$, pay-as-bid pricing does not minimize WEV. From this and the "only if" statement in the proof of Theorem 2, the result follows. Note that WEV is continuous in c and at $c = 1$ it is strictly lower for uniform pricing ($\delta = 0$) than for pay-as-bid pricing ($\delta = 1$) by Theorem 2. Thus, by the mean value theorem, there exists an open interval $C = (c^*, 1)$, $c^* < 1$, such that, for any $c \in C$, WEV remains strictly lower under uniform pricing than pay-as-bid pricing. $\qquad \square$

C.4 Challenging the intuition: private values and uniform pricing

For Example 2, we consider the order statistics of quantiles $F^{-1}(x)$ and not of signals s. For convenience, we define the following distribution functions and densities.

$$\widetilde{G}(x) := G_k^{n-1}(F^{-1}(x)) = 1 - (1-x)^{n-1} \qquad \widetilde{g}(x) := (n-1)(1-x)^{n-2}$$
$$\widetilde{H}(x) := G_{k-1}^{n-2}(F^{-1}(x)) = 1 - (1-x)^{n-2} \qquad \widetilde{h}(x) := (n-2)(1-x)^{n-3}$$

We choose a continuous distribution of signals, with support $[0, 2]$, where each signal is given by the sum of a Bernoulli(ε) random variable and a random perturbation drawn from Beta($1, 1/\eta$), with $\varepsilon = 0.1/n$ and η a small constant. First, we compute the distribution function F and

quantile function F^{-1} of the signal distribution. Using the law of total probabilities, we have

$$\forall s \in [0, 2], \qquad F(s) = P[\mathrm{Bernoulli}(\varepsilon) + \mathrm{Beta}(1, 1/\eta) \leq s]$$
$$= P[\mathrm{Bernoulli}(\varepsilon) = 0] \cdot P[\mathrm{Beta}(1, 1/\eta) \leq s]$$
$$+ P[\mathrm{Bernoulli}(\varepsilon) = 1] \cdot P[\mathrm{Beta}(1, 1/\eta) \leq s - 1].$$

Simplifying this expression depending on the value of s, we get

$$\forall s \in [0, 2], \qquad F(s) = \begin{cases} \varepsilon \cdot (1 - (1 - s)^{1/\eta}) & \text{if } s \leq 1, \\ \varepsilon + (1 - \varepsilon) \cdot (1 - (2 - s)^{1/\eta}) & \text{if } s \geq 1. \end{cases}$$

Then, computing piece-by-piece the inverse of F, we obtain

$$\forall x \in [0, 1], \qquad F^{-1}(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x \leq \varepsilon, \\ 2 - \left(1 - \frac{x - \varepsilon}{1 - \varepsilon}\right)^{\eta} & \text{if } x \geq \varepsilon. \end{cases}$$

A bidder with quantile $x \in [0, 1]$ bids (truthfully) their signal $F^{-1}(x)$ in the uniform-price auction ($\delta = 0$), which we write as $b_\eta^0(x) := F^{-1}(x) = \mathbb{1}\{x \geq \varepsilon\} + \gamma_\eta(x)$, where

$$\forall x \in [0, 1], \qquad \gamma_\eta(x) := \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x < \varepsilon, \\ 1 - \left(1 - \frac{x - \varepsilon}{1 - \varepsilon}\right)^{\eta} & \text{if } x \geq \varepsilon. \end{cases}$$

For mixed auctions with $\delta > 0$, the equilibrium bid function is given by Proposition 2. Letting
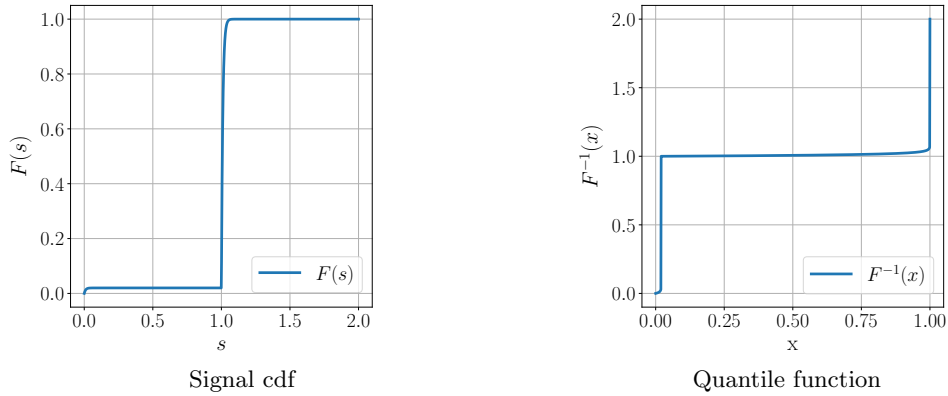
Figure 10: Bidder signals and quantiles for $n = 5$ and $\eta = 0.01$

$b_\eta^\delta(x) := \beta^\delta(F^{-1}(x))$ denote the equilibrium bid of a bidder with quantile $x \in [0, 1]$, we have

$$b_\eta^\delta(x) = \frac{\int_0^{F^{-1}(x)} V(s) g_k^{n-1}(s) G_k^{n-1}(s)^{\frac{1}{\delta} - 1} \, \mathrm{d}s}{\delta G_k^{n-1}(F^{-1}(x))} = \frac{\int_0^x F^{-1}(y) \widetilde{g}(y) \widetilde{G}(y)^{\frac{1}{\delta} - 1} \, \mathrm{d}y}{\delta \widetilde{G}(x)},$$

where we used the change of variable $y = F(s)$. Finally, using the additive form of F^{-1} we write the equilibrium bid function as $b_\eta^\delta(x) = b_0^\delta(x) + \xi_\eta^\delta(x)$, where

$$\forall x \in [0,1], \qquad b_0^\delta(x) := \frac{\int_\varepsilon^x \widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta\widetilde{G}(F^{-1}(x))} = \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases}$$

$$\xi_\eta^\delta(x) := \frac{\int_0^x \gamma_\eta(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta\widetilde{G}(x)}$$

Next, we define the function $\phi_\eta^\delta(x) := F^{-1}(x) - \delta b_\eta^\delta(x)$, the utility of a winning bidder as a function of their quantile. Denoting WEV_η^δ the *winners' empirical variance* in a δ-mixed auction with noise level η, we write

$$\forall \delta \in [0,1],\ \forall \eta > 0, \qquad \mathrm{WEV}_\eta^\delta = \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi_\eta^\delta(x_1) - \phi_\eta^\delta(x_2))^2}{2}\,\Big|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right].$$

where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi_\eta^\delta(x)$ converge towards 0 when taking η arbitrarily small, and thus $\phi_\eta^\delta(x)$ converges towards $\phi_0^\delta(x) := \mathbb{1}\{x \geq \varepsilon\} - \delta b_0^\delta(x)$. Therefore, WEV_η^δ converges towards WEV_0^δ, defined by

$$\forall \delta \in [0,1], \quad \mathrm{WEV}_0^\delta := \mathbb{E}_{\mathbf{x}}\left[\frac{((\mathbb{1}\{x_1 \geq \varepsilon\} - \delta b_0^\delta(x_1)) - (\mathbb{1}\{x_2 \geq \varepsilon\} - \delta b_0^\delta(x_2)))^2}{2}\,\Big|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$$

$$= \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi_0^\delta(x_1) - \phi_0^\delta(x_2))^2}{2}\,\Big|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \lim_{\eta \to 0} \mathrm{WEV}_\eta^\delta.$$

Proof of Proposition 5. We are now equipped to prove the proposition. We write $\mathrm{WEV}_0^\delta = \mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)^2\,|\,x_1 > Y_{k+1}(\mathbf{x})\right] - \mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)\phi_0^\delta(x_2)\,|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, with

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)^2\,|\,x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_0^1 \phi_0^\delta(x)^2 G(x)\,\mathrm{d}x$$

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^\delta(x_1)\phi_0^\delta(x_2)\,|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-2}\int_0^1 \left(\int_t^1 \phi_0^\delta(x)\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t$$

We next compute these quantities for uniform and discriminatory pricing. For uniform pricing ($\delta = 0$) we have that $\phi_0^0(x) = \mathbb{1}\{x \geq \varepsilon\}$. We derive

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^0(x_1)^2\,|\,x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \widetilde{G}(x)\,\mathrm{d}x = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1}$$

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^0(x_1)\phi_0^0(x_2)\,|\,x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-2}\int_0^\varepsilon (1-\varepsilon)^2\widetilde{h}(t)\,\mathrm{d}t + \frac{n}{n-2}\int_\varepsilon^1 (1-t)^2\widetilde{h}(t)\,\mathrm{d}t$$

$$= \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2}$$

and finally

$$\text{WEV}_0^0 = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1} - \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2}$$

$$= \frac{n\left[(1-\varepsilon)^n + (1-\varepsilon)(\varepsilon(n-1)-1)\right]}{(n-1)(n-2)}$$

$$\leq \frac{(\varepsilon n)^2/2}{n} = \frac{0.005}{n}$$

For discriminatory pricing ($\delta = 1$) we have that $\phi_0^1(x) = \mathbb{1}\{x \geq \varepsilon\} - b_0^1(x) = \mathbb{1}\{x \geq \varepsilon\}\frac{G(\varepsilon)}{G(x)}$. We will use the following bounds:

$$\int_\varepsilon^1 \frac{1}{\widetilde{G}(x)}\,dx = \int_\varepsilon^1 \frac{1}{1-(1-x)^{n-1}}\,dx = \int_\varepsilon^1 \sum_{i=0}^\infty (1-x)^{(n-1)i}\,dx$$

$$= \sum_{i=0}^\infty \frac{(1-\varepsilon)^{(n-1)i+1}}{(n-1)i+1} \geq (1-\varepsilon) + \sum_{i=1}^\infty \frac{(1-\varepsilon)^{ni}}{ni}$$

$$\geq (1-\varepsilon) + \frac{1}{n}\sum_{i=1}^\infty \frac{0.9^i}{i} = 1 - \frac{0.1}{n} - \frac{\ln(0.1)}{n} \geq 1 + \frac{2.2}{n}$$

$$\int_0^1 \frac{x}{\widetilde{G}(x)}\,dx = \int_0^1 \frac{x}{1-(1-x)^{n-1}}\,dx = \int_0^1 \sum_{i=0}^\infty x(1-x)^{(n-1)i}\,dx$$

$$= \frac{1}{2} + \sum_{i=1}^\infty \frac{1}{((n-1)i+1)((n-1)i+2)} \leq \frac{1}{2} + \frac{1.65}{n^2} \qquad \text{(when } n \geq 5\text{)}$$

Next, we write

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)^2 \mid x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)^2}{\widetilde{G}(x)}\,dx \geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1 + \frac{2.2}{n}\right)$$

and

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2) \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-2}\int_0^\varepsilon \left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx\right)^2 h(t)\,dt$$

$$+ \frac{n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx\right)^2 h(t)\,dt$$

$$= \underbrace{\frac{n\widetilde{H}(\varepsilon)}{n-2}\left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx\right)^2 + \frac{n}{n-2}\left[\left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx\right)^2 H(t)\right]_\varepsilon^1}_{=0}$$

$$+ \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,dt$$

$$= \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_\varepsilon^x \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,dt\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,dx$$

Next, we will use the upper bound $\widetilde{H}(t)/\widetilde{G}(t) \leq 1$, which is nearly tight as $\widetilde{H}(t)/\widetilde{G}(t)$ is increas-

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ing, and has the limit $(n-2)/(n-1)$ when $t \to 0$.

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2) \mid x_1, x_2 > Y_{k+1}(\mathbf{x})\right] \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\int_\varepsilon^1 \frac{x}{\widetilde{G}(x)}\,\mathrm{d}x \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2} + \frac{1.65}{n^2}\right)$$

Finally, we obtain

$$
\begin{aligned}
\mathrm{WEV}_0^1 &\geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1 + \frac{2.2}{n}\right) - \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2} + \frac{1.65}{n^2}\right) && \text{(when } n \geq 5) \\
&= n\widetilde{G}(\varepsilon)^2\left(\frac{2.2}{n(n-1)} - \frac{1}{(n-1)(n-2)} + \frac{3.3}{n^2(n-2)}\right) \\
&\geq \frac{0.01}{n} && \text{(when } n \geq 4)
\end{aligned}
$$

$\square$

C.5 Proving the main theorems

Proof of Lemma 2. Let $s_i \leq s_j = s_i + \epsilon$ for some $\epsilon > 0$. Then

$$
\begin{aligned}
\Leftrightarrow && u_i(s_i, \boldsymbol{s}_{-i}) &\leq u_j(s_j, \boldsymbol{s}_j) \\
\Leftrightarrow \quad (1-c)s_i - \delta\beta^\delta(s_i) &\leq (1-c)s_j - \delta\beta^\delta(s_j) \\
\Leftrightarrow \quad (1-c)(s_j - s_j) &\geq \delta(\beta^\delta(s_j) - \beta^\delta(s_i))
\end{aligned}
$$

Dividing by $s_j - s_i$ and taking $\epsilon \to 0$ concludes the proof. $\square$

Proof of Proposition 9. First, we prove that pairwise differences is locally decreasing in δ. Let s_i, s_j with $s_i \geq s_j$ denote the signals of two winning bidders and $\varphi^\delta(s) := (1-c)s - \delta\beta^\delta(s)$. Note that because of Lemma 3 (2), monotone ex-post utility holds for all $\delta \leq \bar{\delta}$. For all δ_1, δ_2, $0 \leq \delta_1 \leq \delta_2 \leq \bar{\delta}$, we have

$$|u^{\delta_1}(s_i) - u^{\delta_1}(s_j)| \geq |u^{\delta_2}(s_i) - u^{\delta_2}(s_j)| \tag{19}$$

$$\Leftrightarrow \quad |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \geq |\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_i)| \tag{20}$$

$$\Leftrightarrow \quad -\delta_1\left(\beta^{\delta_1}(s_i) - \beta^{\delta_1}(s_j)\right) \geq -\delta_2\left(\beta^{\delta_2}(s_i) - \beta^{\delta_2}(s_j)\right) \tag{21}$$

For the final equivalence, observe that monotone ex-post utility together with Lemma 3 (1) implies that $\frac{\delta}{1-c}\beta^\delta$ is non-expansive, allowing to remove the absolute value in Eq. (20). Lemma 3 (2) guarantees that Eq. (21) holds. As the ex-post difference in utilities (Eq. (19)) is decreasing in δ, so is its expectation. To establish global monotonicty on $[0, \bar{\delta}]$, note that if $\bar{\delta}\frac{\partial\beta^{\bar{\delta}}}{\partial s} \leq 1-c$ then it also holds for any $\delta < \bar{\delta}$ by Lemma 3 (2), concluding the proof. $\square$

Proof of Proposition 10. Let $u_i^\delta(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the δ-mixed auction, and let $u_i^U(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform-price auction. Now let $i, j \in [n]$ be two winning bidders. As above, β^δ (resp. β^U) denotes the symmetric equilibrium bid function in the δ-mixed (resp. uniform price) auction. Let $Y_{k+1}(\boldsymbol{\beta})$ denote the first rejected bid. Then,

canceling out $(1 - \delta)Y_{k+1}(\boldsymbol{\beta})$, we have

$$
\begin{aligned}
|u_i^\delta - u_j^\delta| &= |(v_i(s_i, \boldsymbol{s}_{-i}) - \delta\beta^\delta(s_i)) - (v_j(s_j, \boldsymbol{s}_{-j}) - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_i)) - ((1-c)s_j + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i - \delta\beta^\delta(s_i)) - ((1-c)s_j - \delta\beta^\delta(s_j))| \\
&= |\varphi^\delta(s_i) - \varphi^\delta(s_j)|,
\end{aligned}
$$

where $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. It also holds that

$$
|u_i^U - u_j^U| = |(v_i(s_i, \boldsymbol{s}_{-i}) - Y_{k+1}(\boldsymbol{\beta})) - (v_j(s_j, \boldsymbol{s}_{-j}) - Y_{k+1}(\boldsymbol{\beta}))| = |(1-c)(s_i - s_j)|.
$$

We will now show that $\frac{\varphi^\delta}{1-c}$ is a non-expansive mapping. Note that φ^δ can be increasing or decreasing, so we need to show that $|\frac{\partial\varphi^\delta}{\partial s}| \le 1 - c$. We have $\frac{\partial\varphi^\delta}{\partial s} = 1 - c - \delta\frac{\partial\beta^\delta}{\partial s}$. As β^δ is increasing in s, $|\frac{\partial\varphi^\delta}{\partial s}| \le 1 - c$ holds whenever $\delta\frac{\partial\beta^\delta}{\partial s} \le 2(1-c)$. Therefore

$$
|u_i^\delta - u_j^\delta| = |\varphi^\delta(v_i) - \varphi^\delta(v_j)| \le |(1-c)(s_i - s_j)| = |u_i^U - u_j^U| \tag{22}
$$

Taking the square of Eq. (22) we obtain the result point-wise, for each pair of winning signals s_i and s_j and, taking the expectation, the theorem follows. $\qquad\square$

Theorem 5. For a given common value component c, consider two δ-mixed auctions for $\delta_1 \le \delta_2$ and suppose the equilibrium bidding functions β^δ satisfies $\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c)$ for all signals $s \in [0, \bar{s})$. Then, WEV is lower for the δ_2-mixed auction than for the δ_1 one.

Proof. Let $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. We have $u_i^\delta(\mathbf{s}) - u_j^\delta(\mathbf{s}) = \varphi^\delta(s_i) - \varphi^\delta(s_j)$. Let $\delta_1 \le \delta_2$. By the generalized Cauchy mean value Theorem, we have that there exists $\xi \in [s_i, s_j]$ such that

$$
|\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_j)| \left|\frac{\partial\varphi^{\delta_1}(\xi)}{\partial s}\right| = |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \left|\frac{\partial\varphi^{\delta_2}(\xi)}{\partial s}\right|.
$$

Hence if $|\frac{\partial\varphi^{\delta_2}}{\partial s}|/|\frac{\partial\varphi^{\delta_1}}{\partial s}| \le 1$ then we have lower WEV for the δ_2 mixed auction. We have the following chain of equivalences:

$$
\begin{aligned}
&\left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right| \le \left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right|, \forall s \in (0, \bar{s}) \\
\iff\ &\left|(1-c) - \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s}\right| \le \left|(1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}\right|, \forall s \in (0, \bar{s}) \\
\iff\ &\delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} - (1-c) \le (1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}, \forall s \in (0, \bar{s}) \\
\iff\ &\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c), \forall s \in (0, \bar{s}),
\end{aligned}
$$

where the third equations comes from the monotonicity of $\delta\frac{\partial\beta^\delta}{\partial s}$ in δ from Lemma 3. $\qquad\square$

C.6 Proving the bound on bid function slopes

Proof of Lemma 4. We first rewrite $\tilde{v}(x, y)$ for $c = 1$ in terms of all the order-statistics of s_{-i}.

$$\tilde{v}(x, y) = \mathbb{E}[v(s_i, s_{-i}) \mid s_i = x, Y_k = y]$$

$$= \mathbb{E}[\frac{1}{n} \sum_{j \in [n]} s_j \mid s_i = x, Y_k = y]$$

$$= \frac{x}{n} + \mathbb{E}[\frac{1}{n} \sum_{\substack{j \in [n], \\ j \neq i}} s_j \mid s_i = x, Y_k = y]$$

$$= \frac{x}{n} + \mathbb{E}[\frac{1}{n} \sum_{j \in [n-1]} Y_j \mid s_i = x, Y_k = y] \qquad \text{(Ordering the signals)}$$

$$= \frac{x}{n} + \frac{y}{n} + \mathbb{E}[\frac{1}{n} \sum_{\substack{j \in [n-1], \\ j \neq k}} Y_j \mid s_i = x, Y_k = y]$$

$$= \frac{x}{n} + \frac{y}{n} + \frac{1}{n} \sum_{j=1}^{k-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y] + \frac{1}{n} \sum_{j=k+1}^{n-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y]$$

Note that the previous decomposition is similar to the equilibrium bid in an English auction given that k bidders have dropped out in Goeree and Offerman (2003). However, we offer a careful derivation in the multi-unit setting of our model. We now use Theorem 2.4.1 and Theorem 2.4.2 from Arnold, Balakrishnan and Nagaraja (2008) on the conditional distribution of order statistics. They state that, for $j < k$, the distribution of Y_j given $Y_k = y$ is the same as the distribution of the j-th order statistic of $k-1$ independent samples of the original distribution left-truncated at y, and we denote Z_j^l a random variable drawn according to this distribution. Hence, for $j < k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^l]$. Similarly for $j > k$ we have that the distribution of Y_j given $Y_k = y$ is the same as the distribution of the $j - k$-th order statistic of $n - k - 1$ independent samples of the original distribution right-truncated at y, and we denote by Z_j^r a random variable drawn according to this distribution. Hence, for $j > k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^r]$. Notice that summing all order statistics drawn from some samples recovers exactly the sum of original samples. Thus we obtain

$$\sum_{j=1}^{k-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y] = \sum_{j=1}^{k-1} \mathbb{E}[Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} s_j \mid \forall j \in [k-1], s_j \geq y] = \sum_{j=1}^{k-1} \mathbb{E}[s_j \mid s_j \geq y].$$

The same can be done for the Z_j^r. Finally, the s_j are iid and thus have identical conditional expectations. We obtain

$$\tilde{v}(x, y) = \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n} \mathbb{E}[s_j \mid s_j \leq y] + \frac{k-1}{n} \mathbb{E}[s_j \mid s_j \geq y] \qquad (23)$$

$$= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n} \frac{\int_0^y t f(t) \, dt}{F(y)} + \frac{k-1}{n} \frac{\int_y^{\bar{s}} t f(t) \, dt}{1 - F(y)}, \qquad (24)$$

which readily yields a formula for $V(s) = \tilde{v}(s, s)$. Clearly, the above function is well defined and

differentiable on the open support of F.

We now examine the derivative of $V(s)$ and prove that $V'(s) \leq 1$. First, we consider the derivatives of the two ratios with an integral in the numerator in Eq. (24). First, by integration by parts, we have

$$\frac{\int_0^s t f(t) \, \mathrm{d}t}{F(s)} = \frac{[tF(t)]_0^s - \int_0^s F(t) \, \mathrm{d}t}{F(s)} = s - \frac{\int_0^s F(t) \, \mathrm{d}t}{F(s)},$$

and using that for positive random variables $\int_0^{\bar{s}} t f(t) \, \mathrm{d}t = \int_0^{\bar{s}} (1 - F(t)) \, \mathrm{d}t = \mathbb{E}[s_i] < \infty$, which guarantees convergence of the integral, we have that

$$\frac{\int_s^{\bar{s}} t f(t) \, \mathrm{d}t}{1 - F(s)} = \frac{\mathbb{E}[s_i] - \int_0^s t f(t) \, \mathrm{d}t}{1 - F(s)} = \frac{\int_0^{\bar{s}} (1 - F(t)) \, \mathrm{d}t - sF(s) + \int_0^s F(t) \, \mathrm{d}t}{1 - F(s)}$$

$$= \frac{\int_0^{\bar{s}} (1 - F(t)) \, \mathrm{d}t + s(1 - F(s)) - s + \int_0^s F(t) \, \mathrm{d}t}{1 - F(s)}$$

$$= s + \frac{\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t}{1 - F(s)}$$

Now, taking derivatives, we have

$$\frac{\partial}{\partial s} \frac{\int_0^s t f(t) \, \mathrm{d}t}{F(s)} = 1 - \frac{F(s)^2 - f(s) \int_0^s F(t) \, \mathrm{d}t}{F(s)^2} = \frac{f(s) \int_0^s F(t) \, \mathrm{d}t}{F(s)^2}.$$

By a similar argument as in the proof of Lemma 5, using log-concavity of f, the above derivative is bounded by 1. Taking the derivative of the second ratio, we have

$$\frac{\partial}{\partial s} \left(s + \frac{\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t}{1 - F(s)} \right) = 1 + \frac{-(1 - F(s))^2 + f(s) \int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t}{(1 - F(s))^2} = \frac{f(s) \int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t}{(1 - F(s))^2}.$$

(25)

To derivative of $\log(\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t)$:

$$\frac{\partial^2}{(\partial s)^2} \log\left(\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t \right) = \frac{\partial}{\partial s} \frac{-(1 - F(s))}{\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t} = \frac{f(s) \int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t - (1 - F(s))^2}{\left(\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t \right)^2}.$$

(26)

Eq. (26) is negative iff $f(s) \int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t / (1 - F(s))^2 \leq 1$. This means that the log-concavity of $\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t$ is equivalent to Eq. (25) being smaller than 1. As the log-concavity of $\int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t$ follows from the log-concavity of f and $(1 - F)$ (Bagnoli and Bergstrom, 2005, Theorem 3), $f(s) \int_s^{\bar{s}} (1 - F(t)) \, \mathrm{d}t / (1 - F(s))^2 \leq 1$ is implied. Finally, using the above derivatives it is clear that $V'(s) > 0$, and

$$V'(s) \leq \frac{2}{n} + \frac{n - k - 1}{n} \cdot 1 + \frac{k - 1}{n} \cdot 1 = 1.$$

$\square$

Proof of Lemma 5. Let us compute the second derivative of the logarithm of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$:

$$\frac{\partial^2 \log\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)}{(\partial s)^2} = \frac{\partial}{\partial s}\left(\frac{G^{\frac{1}{\delta}}(s)}{\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}\right)$$

$$= \frac{\frac{1}{\delta}g(s)G^{\frac{1}{\delta}-1}(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{2}{\delta}}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}$$

$$= \frac{G^{\frac{1}{\delta}-1}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\left(\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)\right).$$

Notice that the left-hand fraction is always positive. Hence log-concavity of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is equivalent to $\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y - G^{\frac{1}{\delta}+1}(s)$ being negative. The latter is equivalent to

$$1 \geq \frac{g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}+1}(s)} = \frac{\partial \beta^\delta(s)}{\partial s}.$$

$\square$

Proof of Proposition 6. While $\sup_{[0,\bar s)}\frac{\partial\beta^\delta}{\partial s}$ can be difficult to compute analytically even for simple distributions, it is sometimes possible to compute $\sup_{[0,\bar s)} V'(s)$. For the uniform distribution, we have $\sup_{[0,\bar s)} V'(s) = 1 - c\frac{n-2}{2n}$. Thus, using the same argument as in the proof of Theorem 3, it follows that $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-2)} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. For the exponential distribution, we have $\sup_{[0,\bar s)} V'(s) = 1 - c(\frac{1}{2} - \frac{k+1}{2n})$, and thus $\delta^*(c) \geq \frac{2n(1-c)}{2n-c(n-(k+1))} \to_{n\to\infty} \frac{(1-c)}{1-c/2}$. $\square$

Proof of Lemma 6. To prove Lemma 6, we will use properties of log-concave distributions from Bagnoli and Bergstrom (2005). Namely their Theorems 1 and 3 state together that log-concavity of a density f implies log-concavity of the corresponding cdf F and of the complementary cdf $1-F$, and that log-concavity of F or $1-F$ imply log-concavity of respectively $\int_0^s F$ or $\int_s^{\bar s} F$, where $\bar s$ is the upper limit of the support of f (either a constant or $+\infty$). Additionally, we also have that the product of two log-concave functions is log-concave also. Using the above properties, we have that F and $1-F$ are log-concave.

Moreover, alternative expression for the order statistics are given, e.g., in Fisz (1965).

$$G_m^n(s) = \frac{n!}{(n-m)!(m-1)!}\int_0^{F(s)} t^{n-m}(1-t)^{m-1}dt$$

and

$$g_m^n(s) = \frac{n!}{(n-m)!(m-1)!}F(s)^{n-m}(1-F(s))^{m-1}f(s). \tag{27}$$

Thus, the order statistics density g, given by Eq. (27), is a product of F, $1-F$, and f, and g as well as the corresponding cdf G are also log-concave. Furthermore, $G^{\frac{1}{\delta}}$ is log-concave because $\log(G^{\frac{1}{\delta}}) = \delta\log(G)$. Finally, we remark that $G^{\frac{1}{\delta}}$ is right-continuous non-decreasing by composition with $x \mapsto x^{\frac{1}{\delta}}$, which is continuous non-decreasing, and $G^{\frac{1}{\delta}}(0) = 0$, as well as $G^{\frac{1}{\delta}}(\bar s) = 1$ (if $\bar s = \infty$, the equality is understood as a limit). Therefore $G^{\frac{1}{\delta}}$ is a cdf, and applying

one last time Bagnoli and Bergstrom (2005), we obtain that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave. $\qquad\square$

C.7 Numerical experiments

Lemma 11. *Suppose an auction is a* winners-pay auction. *Then we can write* $E_{\boldsymbol{s}}[u_1 \mid 1 \text{ wins}] = \frac{n}{k}E_{\boldsymbol{s}}[u_1]$, $E_{\boldsymbol{s}}[u_1^2 \mid 1 \text{ wins}] = \frac{n}{k}E_{\boldsymbol{s}}[u_1^2]$, *and* $E_{\boldsymbol{s}}[u_1 u_2 \mid 1 \text{ and } 2 \text{ win}] = \frac{n(n-1)}{k(k-1)}E_{\boldsymbol{s}}[u_1 u_2]$.

Proof. Observe that we have

$$\mathbb{E}_s[u_1^2 \mid 1 \text{ and } 2 \text{ win}] = \mathbb{E}_s[u_1^2 \mid 1 \text{ wins}] = \frac{\mathbb{E}[u_1^2]}{\mathbb{P}[1 \text{ wins}]} = \frac{n}{k} \cdot \mathbb{E}[u_1^2] \tag{28}$$

$$\mathbb{E}_s[u_1 u_2 \mid 1 \text{ and } 2 \text{ win}] = \frac{\mathbb{E}[u_1 u_2]}{\mathbb{P}[1 \text{ and } 2 \text{ win}]} = \frac{n(n-1)}{k(k-1)} \cdot \mathbb{E}[u_1 u_2] \tag{29}$$

$\qquad\square$

C.8 Discussion

Proof of Proposition 7. We define the probability that i wins $q_i(s_i) := \mathbb{P}_{\boldsymbol{s}_{-i}}[i \text{ wins}]$. Recall that $b^D(s_i)$ denotes the equilibrium bid in the pay-as-bid auction. Consider any standard auction, characterised by a payment rule $(p_1(s), \ldots, p_n(s))$. Revenue equivalence implies that

$$q_i(s_i) \cdot b^D(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[b^D(s_i) \cdot \mathbb{1}\{i \text{ wins}\}] = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s)]. \tag{30}$$

In particular, note that if p_i is chosen to be the uniform pricing rule, this formula can be used to compute $b^D(s_i)$. Now define the ex-post surplus $u_i(s_i, \boldsymbol{s}_{-i}) := v(s_i) \cdot \mathbb{1}\{i \text{ wins}\} - p_i(s)$. We write

$$u_i(s_i, \boldsymbol{s}_{-i}) = \underbrace{\mathbb{1}\{i \text{ wins}\} \cdot (v(s_i) - b^D(s_i))}_{u_i^D(s)} + \underbrace{\mathbb{1}\{i \text{ wins}\} \cdot b^D(s_i) - p_i(s)}_{\delta(s)}. \tag{31}$$

Now, observe that by revenue equivalence we have $\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})] = 0$ for all s_i. We write

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 \tag{32}$$

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] + 2\underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})]}_{\geq 0} + \underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})^2]}_{\geq 0} \tag{33}$$

To show that the extra terms are non-negative, notice that $\delta(s_i, \boldsymbol{s}_{-i})^2 \geq 0$, and that

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})] = \underbrace{(v(s_i) - b^D(s_i))}_{\geq 0} \cdot \underbrace{(q_i(s_i) \cdot b^D(s_i) - \mathbb{E}_{\boldsymbol{s}_{-i}}[\mathbb{1}\{i \text{ wins}\} \cdot p_i(s)])}_{\geq \mathbb{E}[\delta(s)] = 0} \tag{34}$$

Therefore, putting everything together, we obtain

$$\text{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 \tag{35}$$

$$\geq \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 \tag{36}$$

$$= \text{Var}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})] \tag{37}$$

Finally, observe that an auction which minimize the interim variance also minimize the ex-ante variance. Denoting by u_i the utility of a bidder in the pay-as-bid auction, the law of total variance states

$$\text{Var}_s[u_i] = \mathbb{E}_{s_i}[\text{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]] + \text{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]. \tag{38}$$

By the revenue equivalence theorem, we know that $\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]$ is the same for all standard auctions, hence $\text{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]$ is also the same for all standard auctions (it only depends on the distribution of the signals).

The interim variance $\text{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]$ is minimal point-wise (in s_i) for all standard auctions, hence is also minimal in expectation. Therefore, the ex-ante variance is minimal in the pay-as-bid auction among standard auctions. □

Proof of Proposition 8. We first derive the equilibrium bid for the first-rejected-bid uniform auction with common values and reserve price $r > 0$. Fix a signal s. Let β be an increasing symmetric equilibrium, and let $s_r = \inf\{s \geq 0 \mid \beta(s_r) \geq r\}$ be the threshold signal for the bid to exceed a given reserve price r. For $z \geq s_r$, we consider $U(s_i, z)$, the expected payoff of bidding $\beta(z)$ with signal s_i:

$$\begin{aligned} U(s_i, z) &= \int_0^z \widetilde{V}(s_i, y)g(y)\,\mathrm{d}y - \int_0^{s_r} rg(y)\,\mathrm{d}y - \int_{s_r}^z \beta(y)g(y)\,\mathrm{d}y \\ &= \int_0^z \widetilde{V}(s_i, y)g(y)\,\mathrm{d}y - rG(s_r) - \int_{s_r}^z \beta(y)g(y)\,\mathrm{d}y. \end{aligned}$$

If $z < s_r$ then the bid is below the reserve price, no item is won, and $U(s_i, z) = 0$. If the payoff is maximized for $z \geq s_r$, then, by solving the first order condition, a bid of $\widetilde{V}(s_i, s_i) = V(s_i)$ is optimal. Hence, bidding $V(s_i)$ is preferred to bidding zero if the expected payoff is greater than zero. Because $V(s_i)$ is increasing and continuous, these two payoffs are equal for $s_i = s_r$ by definition: s_r corresponds to the threshold signal beyond which a positive bid of $V(s_i)$ is preferred to a zero profit. The equation

$$U(s_r, s_r) = \int_0^{s_r} \widetilde{V}(s_r, y)g(y) - rG(s_r) = 0, \tag{39}$$

implicitly characterizes s_r. The equilibrium bid is $\beta_r^{\delta=0} = V(s_i)$ for $s_i \geq s_r$ and $\beta_r^{\delta=0} = 0$ otherwise.

Using revenue equivalence, we derive the equilibrium bid in the pay-as-bid auction. We have that, for $s_i \geq s_r$,

$$\beta_r^{\delta=1}(s_i) = \int_0^{s_r} \frac{rg(y)}{G(s_i)}\,\mathrm{d}y + \int_{s_r}^{s_i} \frac{V(y)g(y)}{G(s_i)}\,\mathrm{d}y = V(s_i) + (r - V(s_r))\frac{G(s_r)}{G(s_i)} - \int_{s_r}^{s_i} \frac{V'(y)G(y)}{G(s_i)}\,\mathrm{d}y. \tag{40}$$

Taking the derivative yields

$$
\begin{aligned}
\frac{\partial \beta_r^{\delta=1}(s_i)}{\partial s_i} &= \frac{g(s_i)}{G^2(s_i)} \left((V(s_r) - r)G(s_r) + \int_{s_r}^{s_i} V'(y)G(y)\,\mathrm{d}y \right) \\
&= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} V(y)g(y)\,\mathrm{d}y - rG(S_r) + \int_0^{s_i} V'(y)G(y)\,\mathrm{d}y \right) \\
&= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} V(y)g(y)\,\mathrm{d}y - rG(S_r) + \int_0^{s_i} V'(y)G(y)\,\mathrm{d}y \right) \\
&= \frac{g(s_i)}{G^2(s_i)} \left(\int_0^{s_r} (\widetilde{V}(y,y) - \widetilde{V}(s_r,y))g(y) + \int_0^{s_i} V'(y)G(y)\,\mathrm{d}y \right) \\
&\leq \frac{g(s_i)}{G^2(s_i)} \int_0^{s_i} V'(y)G(y)\,\mathrm{d}y \\
&= \frac{\partial \beta_{r=0}^{\delta=1}(s_i)}{\partial s}.
\end{aligned}
$$

We use Eq. (39) for the second-to-last equality, and the fact that $\widetilde{V}(y,y) \leq \widetilde{V}(s_r,y)$ for $y \leq s_r$, by monotonicity of $\widetilde{V}$, for the inequality. $\qquad\square$