Equitable Pricing in Auctions*

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Abstract

How does auction design affect the division of surplus among buyers? We propose a parsimonious measure for equity and apply it to a market in which multiple identical items are sold to unit-demand buyers with private-common values. We characterize the efficient, incentive-compatible, and individually rational mechanism that achieves ex-post surplus parity. The uniform-price auction is equity-optimal if and only if buyers have a pure common value. In auctions with price mixing between pay-as-bid and uniform prices, we provide prior-free bounds on the equity-preferred pricing rule under a regularity condition on signals.

JEL codes: D44, D47, D63
Key words: Multi-unit auctions, equity, pay-as-bid, uniform pricing, common values

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1 Introduction

Multi-unit auctions are prevalent in various sectors, with significant applications in the public and private domain. The study of pricing rules and their impact on efficiency and revenue has a long tradition in both the theoretical and applied economic literature (e.g., Wilson 1979, Back & Zender 1993, Armantier & Sbaï 2009, Ausubel et al. 2014, Hortaçsu et al. 2018). Much of this work has focused on the distribution of surplus between the seller and the market as a whole, i.e., revenue and welfare. However, the distribution of surplus *among buyers*[1] has received little attention.

In practice, auctions are held primarily when agents' values or costs for buying or selling goods need to be discovered. Especially in the public sector, allocative efficiency is the goal, e.g., licenses to build wind farms should go to enterprises that can build and operate those farms the most effectively. The elicitation of values is facilitated by prices, but this can introduce an asymmetric distribution of welfare, even among winners in the auction: e.g., in single-price auctions, stronger, high-value bidders obtain a larger surplus (that is, value minus price) than weaker, low-value bidders. Such redistribution cannot be easily reversed, as a transfer scheme after the auction may distort bidding incentives and prevent an efficient allocation. However, as this article demonstrates, an equitable distribution of surplus can be achieved in the auction by design.

We initiate the study of surplus equity in auctions, focusing on the class of efficient auctions with independent but not exclusively private signals. In this class of mechanisms, the design of the equity objective is costless in terms of efficiency or revenue. We propose a family of equity metrics that are based on parsimonious, pairwise comparisons of realized ex-post utilities. First, we characterize the direct surplus-equitable mechanism. This mechanism achieves ex-post identical surplus among bidders of any type (signal) realization, while guaranteeing truthful reporting. Second, we turn to the prevalent uniform and pay-as-bid pricing rules as well as combinations of these, that is, mixed pricing. In this class, we derive prior-free results on equity-preferred pricing, with strong policy implications for multi-unit auctions used in practice.[2]

Fairness is a central objective in many auction markets. For example, the Small Business Act in the US requires the government to award 23% of procurement contracts each year to small businesses that are socially and economically disadvantaged, owned by veterans, women, or located in historically underutilized business zones.[3] In spectrum auctions, allocative fairness in the distribution of licenses is particularly important, as it affects competition in the downstream market (GSMA 2021, Kasberger 2023). Similarly, the distribution of surplus can influence market stability and competition in post-auction markets: bidders disadvantaged in the surplus distribution may face higher borrowing costs, especially in inefficient capital markets. As we demonstrate in this article, the pricing rule itself crucially affects the ex-post surplus distribution between bidders.

[1] or sellers in a reverse auction

[2] In the public sector, multi-unit auctions are used to sell government debt, electricity, and carbon emission certificates. In the private sector, auctions are used in markets for oil, timber, coffee, fractionalized art, and procurement.

[3] See U.S. Small Business Administration (2024). Pai & Vohra (2012) give further examples across the world.

In multi-unit auctions in practice, uniform pricing, where the first rejected bid[4] sets the price for all winning bids, and pay-as-bid (discriminatory) pricing, where every winner pays the price they bid, are the prevalent auction designs. While in treasury auctions, e.g., both design are common,[5] most electricity markets feature the uniform pricing rule.[6,7] The question of which pricing rule leads to more efficient outcomes, less collusion, and more revenue has been debated for decades (e.g., Kahn et al. 2001, Ausubel et al. 2014), but distributional concerns among generators have received little attention in this regard, with one exception. In 2014, the New Zealand Electricity Authority debated a proposal to recover infrastructure investment costs via a tax on "apparent surplus" (New Zealand Electricity Authority 2014, Ruddell et al. 2017).[8] This tax is equivalent to a mixed auction where the price is given by a convex combination of uniform and pay-as-bid pricing. This class of mechanisms forms one of the focal points of this article.

We consider standard and winners pay multi-unit auctions for the sale of indivisible, identical goods with a composition of private and common values. Each buyer has unit demand and receives a private signal, drawn from a publicly known distribution. A buyer's value linearly interpolates between the extremes of pure private and pure common value, that is, between their private signal and the average signal in the market.[9] We call the interpolation parameter c the *common value proportion*, or simply the *common value*, and its complement $1 - c$ the *private value proportion*, or the *private value*.

For our first result, we consider the class of direct incentive-compatible mechanisms. The subsequent results focus on the class of mixed auctions. Mixed auctions combine the pricing rules of uniform and pay-as-bid auctions, which are widely used in practice, and incorporate those as special cases. In the uniform-price auction, all winners pay the first rejected bid, and in the pay-as-bid (or discriminatory) auction, all winners pay the price they bid. For a given δ, we call the convex combination of uniform and pay-as-bid pricing δ*-mixed pricing*, and the mechanism using this pricing rule δ*-mixed auction*. The parameter δ describes the degree of price discrimination: $\delta = 0$ corresponds to uniform pricing and $\delta = 1$ to discriminatory pricing.[10] In the class of mixed auctions, we study the symmetric Bayesian equilibrium, which is found to be unique.[11] All considered auctions, under classical assumptions, achieve the same expected revenue (Milgrom & Segal 2002) and allocate items to the highest value buyers; thus, there are no potential revenue and efficiency trade-offs and optimizing for equity is costless. Implied by revenue (or payment)

[4] or the last accepted bid

[5] The OECD (2021) finds that in treasury auctions, 25 out of 36 countries use pay-as-bid pricing and 21 out of 36 countries use uniform pricing (some use both).

[6] Some exceptions are electricity markets in England&Wales, Mexico, Peru, and Panama.

[7] Further examples are auctions for emission certificates, e.g., the EU emission trading system or the California Cap and Trade market, or online advertisement.

[8] Via comparison with hypothetical pre-investment market outcomes, the tax would have been only applied to power generators who benefited from the infrastructure.

[9] This model can represent common resale opportunities. For example, an emission certificate is valuable for a company's production process (private value), but it can also be resold after the auction, with the resale value being common for the market. The resale model appears in previous work by, e.g., Bikhchandani & Riley (1991), Krishna & Morgan (1997), Klemperer (1998), Bulow & Klemperer (2002), Goeree & Offerman (2003).

[10] Identical and closely related versions of mixed pricing have been considered, e.g., by Wang & Zender (2002), Viswanathan & Wang (2002), Armantier & Sbaï (2009), Ruddell et al. (2017), Woodward (2021).

[11] Empirical evidence suggests that markets with symmetric bidders are relevant in practice. See, e.g., Armantier & Lafhel (2009) for auctions by the Bank of Canada, Hortaçsu et al. (2018) for U.S. Treasury short-term securities auctions, or Hattori & Takahashi (2022) for Japanese treasury auctions.

equivalence is also that a bidder with any given signal achieves identical ex-interim surplus across mechanism. However, ex-post surplus realizations differ across mechanisms for a given signal. Moreover, the differences in surplus between bidders with different signals will differ across mechanisms.

To evaluate surplus equity, we introduce the concept *dominance in pairwise differences*: Auction A dominates auction B in pairwise differences, if, in equilibrium, all absolute pairwise differences in ex-post utilities of winning bidders are weakly smaller in auction A than in auction B (Definition 2), with one pairwise comparison being strict.[12] Equivalently, we say that A is *equity-preferred* to B. An auction is *equity-preferred* (in the class of mixed auctions) if and only if it dominates all other mixed auctions. Dominance in pairwise differences implies a partial order and is a strong requirement.

Our results hold for the family of equity metrics that are constructed by aggregation of pairwise differences with any increasing function. This family includes, e.g., the empirical variance, the Gini index, or a comparison between the top and bottom deciles.[13] We apply our metric only to winning bidders, as in the class of efficient, standard, and winners pay auctions, surpluses only differ between auction formats only among winners. We also discuss the *winners' empirical variance (WEV)* of surplus as an exemplary aggregator. Next to anonymity, which is satisfied by the entire family of metrics, we show that WEV also satisfies the Pigou-Dalton principle (cf. Moulin 2004), that is, monotonicity with respect to transfers from richer to poorer agents. The empirical variance is also interesting as an equity metric because it combines *within-bidder variation* and *across-bidder correlation* of surpluses. While the analysis of within-agent variation of surplus goes back to Vickrey (1961),[14] relating to risk attitudes, an equity measure must take into account the correlation of surpluses between bidders. A detailed discussion of surplus equity follows in Section 3.

Contributions

1. We characterize the direct and Bayes-Nash incentive-compatible mechanism that distributes realized surpluses equitably among the winning bidders in the class of standard winners pay mechanisms (Theorem 1). The surplus-equitable mechanism allocates the items to the highest winning bidders and charges them a payment consisting of three components: firstly, each bidder pays their private value, thus equalizing ex-post utilities; secondly, a uniform payment that cancels out the idiosyncratic payment in expectation; and finally, the expected value corresponding to the first rejected bid, akin to a "second-price"payment. The key and surprising insight is that there exists a uniform payment that cancels out the idiosyncratic payment part, *for any given signal*, in expectation.

2. Turning to the class of mixed auctions, we prove that the uniform-price auction is dom-

[12] Put differently, the maximal absolute difference in realized utilities in auction A is smaller than the minimal difference in auction B.

[13] For prominent inequality measures in wealth or income, cf., e.g., Lorenz (1905), Gini (1912, 1921), Pigou (1912), Dalton (1920), Atkinson (1970), Sen & Foster (1973).

[14] In the appendix of his famous article, Vickrey showed that, in a *single-unit auction*, the ex-ante variance of surplus is lower under the first-price than the second-price rule, given uniform distributions of private values. The result is generalized in Krishna (2009) who show that the distribution of equilibrium prices in a second-price auction is a mean-preserving spread of that in a first-price auction, given any distribution of private values.

inant in pairwise differences if and only if the bidders' values are pure common value (Theorem 2). In this case, the surplus-equitable mechanism is equivalent to the first-rejected-bid uniform-price auction, which, as any other uniform-price auction, equalizes bidders' realized surpluses.

3. Given any proportion of private values $(1-c)$, the $(1-c)$-mixed auction is equity-preferred over any mixed auction with less than a $(1-c)$ share of price discrimination, assuming that the signal distributions are log-concave (Theorem 3). As a corollary, the pay-as-bid auction is equity-preferred in the class of mixed auctions in the pure private value case if the signal regularity holds. That is, the pricing for goods with a higher proportion of private value should contain more price discrimination in order to achieve an equitable distribution of realized surpluses.

4. Given any positive proportion of private values $(1-c)$, any level of price discrimination up to $2(1-c)$ is equity-preferred to uniform pricing if signals are drawn from a log-concave distribution (Theorem 4). In other words, in any scenario where the goods for sale have some private value, even a large extent of price discrimination is more equitable than uniform pricing. Importantly, Theorems 3 and 4 are *prior-free* in the class of log-concave signal distributions.

Considering pure private values, one might be tempted to arrive at a quick conclusion in favor of pay-as-bid pricing. Surprisingly, however, this fails; the regularity condition is important for sufficiency. If log-concavity is not satisfied, it is not automatic that uniform is equity-preferred to pay-as-bid pricing; our metric of dominance in pairwise differences is too strong for such conclusion. However, we demonstrate that it is possible to construct signal distributions such that uniform pricing achieves lower WEV than pay-as-bid pricing in the pure private value case (Proposition 5). Of course, by Theorem 2, uniform pricing is not WEV-minimal. Finally, we investigate equity in terms of WEV in numerical experiments. For a variety of signal distributions and proportions of common values, we compute the landscape of WEV-minimal mixed pricing, which can be seen to be unique.

Although the focus of our analysis is on auctions, the insights apply to more general B2B interactions. For example, producers of consumer goods aim to sell to a variety of customers and avoid being dependent on a two few clients. Enabling the client base to derive equitable surpluses, e.g., by accepting different prices from different clients (assuming private values), is one way to helping those clients to stay competitive in the B2C market, whereas a uniform sales price might disadvantage smaller customers.

Related literature

This article relates and contributes to several strands of existing works, including the recent literature on redistributed market design, the study of fairness concerns and allocative equity in auctions, the mechanism design literature on ex-post payment design, and more generally, the study of fair allocations.

Broadly, our contribution fits into a recent strand of the economic literature on redistributive concerns in market design. In this literature, the focus is often on efficiency and equity trade-offs; e.g., in a large buyer-seller market for a single object, with agents differing in their marginal utilities of money (and their values), Dworczak et al. (2021) characterize the optimal efficiency-equity trade-off, and Akbarpour et al. (2024), characterize when non-market mechanisms, as opposed to market-clearing prices, are optimal for a designer to allocate a fixed supply (also in a large market). Such non-market mechanisms forgo efficiency for the sake of improving equity. Our approach differs in that we focus on a class of efficient mechanisms in a small market with a finite number of buyers and demonstrate how to improve equity, up to achieving perfect surplus parity, within this class. Most related in this literature is Reuter & Groh (2020), who determine the utilitarian optimal mechanism for the allocation of a finite number of objects to a finite number of heterogeneous agents. Again, the agents differ in their marginal utilities for money and, additionally, the designer has an ex-ante budget constraint. Crucially, the utilitarian objective including redistribution concerns in Reuter & Groh (2020) is maximized in expectation, whereas we employ a stronger, ex-post notion of equity and redistribution. The auction implementations of their optimal mechanism considered by Reuter & Groh (2020) restrict to the allocation of a single object and use minimum bids and subsidies, further contrasting our simple yet powerful pricing rule.

Fairness concerns in auctions have been addressed through design instruments like subsidies and set-asides. In a model with explicit target group favoritism, Pai & Vohra (2012) show that the optimal mechanism is a flat or a type-dependent subsidy, depending on the precise nature of the constraint. Athey et al. (2013) come to similar conclusions in an empirical study of US Forest Service timer auctions, where set-asides for small bidders would reduce efficiency and revenue, while subsidies would increase revenue and profits of small bidders and little detriment to efficiency. Further empirical studies of affirmative action procurement auctions and bid credits in simultaneous ascending auctions have been conducted in Rosa (2020) and Rosa (2022). In contrast with this literature, we consider only the pricing rules as a design instrument to achieve a more equitable distribution of surplus. While our bidders are ex-ante symmetric, a low signal realization can reveal the corresponding bidder as weak in a given auction instance.

Allocative equity in auctions and the welfare generated in the post-auction market have been studied in Kasberger (2023), micro-founding the question when an equitable distribution of the auctioned objects themselves is beneficial for consumer welfare in downstream markets. In contrast to our study, they focus on equity in terms of allocation rather than surplus. Related results were developed in Janssen & Karamychev (2010), who showed that the correlation between signals induced by the revealed auction outcome can affect post-auction market competition.

In the mechanism design literature, some attention has been given to nuanced ex-post implementations of truthful mechanisms. In their seminal article, d'Aspremont & Gérard-Varet (1979) show that ex-post budget balance can be achieved in direct truthful mechanism. In a similar vein, Esö & Futó (1999) prove that for every incentive-compatible mechanism there exists a mechanism which provides deterministically the same revenue. We contribute to this literature by designing the mechanism, and in particular the payment rule that distributes bidder surplus between the winners in the auction in a perfectly equitable way.

Our result are derived in a generalized, canonical auction model (Ortega-Reichert 1968, Krishna 2009) for bidders with unit demand. We enrich the model with a private-common-value structure and the considered class of mechanisms extends to all mixed-price auctions. A series of articles analyzes strategic bidding in mixed-price auctions with multi-unit demand, in which supply is assumed to be continuous and stochastic (cf., e.g., Ruddell et al. 2017, Marszalec et al. 2020, Woodward 2021). A common finding in these general models, next to the difficulty of explicitly computing equilibrium bidding functions, is the multiplicity of equilibria and varying efficiency and revenue dependent on auction formats and value distributions. By focusing on unit demand, next to allowing the explicit derivation of equilibrium bidding functions, revenue equivalence and efficiency hold across standard winners pay auctions with independent signals. Thus, optimizing in terms of equity is costless in terms of revenue and efficiency.

A comprehensive survey on fair allocation can be found in Thomson (2011), covering the concepts of envy-freeness, equal division, no domination, and many others. Our notion of fairness is orthogonal to envy-freeness. Envy-freeness requires that no agent prefers another agent's allocation (object and price). As identical objects are sold, uniform pricing is the unique pricing scheme that results in envy-freeness among winners, and, with pure private values, it results in envy-freeness among all participants.[15] In contrast, we consider the realization of utilities. This relates to *equal treatment of equals* (cf., e.g., Thomson 2011) in an ex-post view. Although buyers are ex-ante symmetric, our criterion of pairwise differences implies that only buyers with ex-post identical signals should be treated as equal.

In practice, fairness is an important issue in spectrum auctions (cf. Myers 2023). Formats such as the Simultaneous Multiple-Round Auction, the Combinatorial Multiple-Round Auction, or the Combinatorial Clock Auction are used. However, some of these formats have led bidders to pay different prices for identical licenses, which has been perceived as unfair (Myers 2023).[16] Our work challenges this common notion of fairness, as we note the importance of distinguishing between private and common values. With pure private values, our view of equity requires strong buyers (with high values) to pay disproportionately more than weak buyers (with low values) for equity in ex-post surpluses to improve.

Outline

The remainder of the article is organized as follows. In Section 2 we introduce the model, derive equilibrium bidding strategies, and introduce our notion of surplus equity. We state the surplus-equitable mechanism in Section 3. In Section 4, we describe our prior-free results on uniform, pay-as-bid, and mixed auctions and prove them in Section 5. Section 6 provides a discussion and Section 7 concludes.

[15] With a proportion of common value, depending on the realization of signals, winners may experience the winners' curse and prefer not to have won an item.

[16] As a consequence, in versions of the Combinatorial Clock Auction, core prices are selected in a second stage based on selection criteria that guarantee fairness and stability (cf. e.g., Day & Milgrom 2008, Erdil & Klemperer 2010).

2 Setup

2.1 Model

A finite number of bidders bidders $[n] := \{1, \ldots, n\}$ competes for a fixed supply $[k] := \{1, ..., k\}$, where $k < n$. Each bidder only demands one item. Bidder i receives a private signal s_i, which is drawn independently from a positive and bounded or unbounded support; denote its upper limit by $\bar{v}$. Signals are iid with an absolutely continuous probability distribution F with density f. We call $(0, \bar{v})$, i.e., all signals s so that $0 < F(s) < 1$, the open support of F, and we assume that $f > 0$ over $(0, \bar{v})$. We also assume that the signals have a finite second moment $\mathbb{E}[s^2] < \infty$.

For $\boldsymbol{s} := (s_i)_{i \in [n]}$, a collection of iid signals, we denote by $Y_m(\boldsymbol{s})$ the m-th highest value of the collection $\boldsymbol{s}$, e.g., $Y_1(\boldsymbol{s})$ is the maximum and $Y_n(\boldsymbol{s})$ is the minimum, and by $G_m^n(\boldsymbol{s})$ its distribution with corresponding density $g_m^n(\boldsymbol{s})$. G_m^n is given by

$$G_m^n(s) = \sum_{j=0}^{m-1} \binom{n}{j} F(s)^{n-j} (1 - F(s))^j.$$

where each summand is the probability that j signals are above s. An expression for $g_m^n(\boldsymbol{s})$ is given in Appendix B.4. The value of bidder i for an item is given by the valuation function $v(s_i, \boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i} := (s_j)_{j \neq i}$. The value of bidder i is symmetric in other bidders' signals.

Assumption 1. Values $v(s_i, \boldsymbol{s}_{-i})$ are given by

$$v(s_i, \boldsymbol{s}_{-i}) = (1 - c)s_i + \frac{c}{n} \sum_{j \in [n]} s_j,$$

where $c \in [0, 1]$ is the *proportion of the common value.*

Our model interpolates between a common value and private values, where the proportion of the common value c encodes to what extent the value of any given bidder is influenced by the signals of the other bidders. In particular, $c = 1$ defines a pure common value and $c = 0$ pure private values.[17] We note that the value function satisfies the *single-crossing* condition as for all $i, j \in [n]$, $i \neq j$, and for all $\boldsymbol{s}$, $\partial v(s_i, \boldsymbol{s}_{-i})/\partial s_i \geq \partial v(s_j, \boldsymbol{s}_{-j})/\partial s_i$.

Auction mechanisms. Auction mechanisms are represented by allocations and transfers $\{\pi_i(s_i, \boldsymbol{s}_{-i}), p_i(s_i, \boldsymbol{s}_{-i})\}_{i \in [n]}$, where $\pi_i(s_i, \boldsymbol{s}_{-i})$ is defined as the probability that an item is allocated to the bidder i when the reported signals are s_i and $\boldsymbol{s}_{-i}$, and $p_i(s_i, \boldsymbol{s}_{-i})$ is the corresponding price charged to the bidder, which is symmetric in its second argument.[18] We require that auction mechanisms be standard and winners pay. An auction is *standard* if the k highest bids win the items,[19] and *winners pay* if only the winners pay and no more than their bid. Any standard auction, in any symmetric and increasing equilibrium and values satisfying the single-crossing

[17] In an alternative model, the common value might be distributed according to some prior distribution, and the bidders' private signals are drawn conditional on the realization of this common value. The alternative model has identical qualitative characteristics (Goeree & Offerman 2003): (i) the items are valued equally by all bidders in the common value component, and (ii) the winner's curse is present, i.e., winning an item is "bad news", in that the winner's expectation of the item's value was likely too optimistic.

[18] Symmetric means that $p_i(s_i, s_{-i}) = p(s_i, s'_{-i})$ for all permutations s'_{-i} of s_{-i}.

[19] Cf. Krishna (2009).

condition, is *efficient* (Krishna 2009), i.e., the bidders with the k highest values $v(s_i, \boldsymbol{s}_{-i})$ are assigned the items.

We consider two classes of mechanisms. First, we consider truthful, direct mechanisms in which bidders submit their signal. Second, we consider *mixed auctions* (defined below) in which each bidder submits a bid b_i, resulting in the vector of submitted bids $\mathbf{b}$. Restricting our attention to symmetric and monotonically increasing bidding strategies $b_i = b(s_i)$, we can write allocations and prices in both classes of mechanism as functions of signals only. The allocation of bidder i is given by $\pi_i(s_i, \boldsymbol{s}_{-i}) = \mathbb{1}[s_i > Y_k(\boldsymbol{s}_{-i})]$ when signals s_i and $\boldsymbol{s}_{-i}$ are reported. A bidder's utility (or surplus) when reporting signal $\hat{s}_i$, and the remaining $n-1$ bidders reporting signals $\boldsymbol{s}_{-i}$, is given by

$$u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i}) = \mathbb{1}[\hat{s}_i > Y_k(\boldsymbol{s}_{-i})] \cdot (v(s_i, \boldsymbol{s}_{-i}) - p_i(\hat{s}_i, \boldsymbol{s}_{-i}))\,.$$

Given a signal s_i, recall that we denote by $Y_k(\boldsymbol{s}_{-i})$ the k-th highest among the signals $\boldsymbol{s}_{-i}$. $Y_k(\boldsymbol{s}_{-i})$ has probability distribution G_k^{n-1} and density g_k^{n-1}.

Furthermore, we denote equilibrium bidding strategies by $(\beta_i)_{i\in[n]} = \boldsymbol{\beta}$.

Definition 1 (Mixed auctions)**.** In the δ-*mixed auction*, parameterized by a given $\delta \in [0,1]$, each winning bidder i pays $p_i(\mathbf{b}) = \delta b_i + (1-\delta)Y_{k+1}(\mathbf{b})$.

At one boundary, for $\delta = 0$, this resolves to *first-rejected-bid uniform pricing* or short *uniform pricing*, where each winning bidder i pays the $(k+1)$-th highest bid $Y_{k+1}(\mathbf{b})$. At the other boundary, for $\delta = 1$, this resolves to *pay-as-bid pricing*, where each winning bidder i pays their bid b_i. Finally, if $\delta \in (0,1)$, we say that the auction and the pricing are *strictly mixed.*

Ex-interim values, payments, and utilities. For all $x, y \in [0,1]$, we define the expected value given $s_i = x$ and $Y_k(\boldsymbol{s}_{-i}) = y$ as follows:

$$\widetilde{V}(x,y) := \mathbb{E}_{\boldsymbol{s}}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = x, Y_k(\boldsymbol{s}_{-i}) = y]$$

The expected value is taken over $n-2$ signals not including the bidder's own signal and the k-th highest among their $n-1$ opponents. Observe that because $v(s_i, \boldsymbol{s}_{-i})$ is continuous and non-decreasing, $\widetilde{V}(x,y)$ is continuous and non-decreasing in x and y.[20] We define $V(y) := \widetilde{V}(y,y)$, the expectation of the value of an item conditional on the bidder winning against the relevant competing signal, the k-th highest among its competitors. Furthermore, we introduce *interim payments* $P_i(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s_i, \boldsymbol{s}_{-i})]$ and *interim utilities* $U_i(s_i, \hat{s}_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \hat{s}_i, \boldsymbol{s}_{-i})]$ with $U_i(x)$ being the shorthand of $U_i(x,x)$.

Revenue equivalence. Ex-interim incentive compatibility (IC) requires $U_i(s_i, s_i) \geq U_i(s_i, \hat{s}_i)$ for all $s_i, \hat{s}_i$, and ex-interim individual rationality (IR) demands $U_i(s_i, s_i) \geq 0$ for all s_i. It is standard from an application of the envelope theorem (Milgrom & Segal 2002) that the auctions we consider result in the same expected payment for each bidder.[21] The interim utility is given

[20] In fact, it is strictly increasing in x.

[21] This is also shown differently in Krishna (2009) for the single-unit auction. Note that in settings where signals are affiliated revenue equivalence fails (Krishna 2009, Chapter 6.5).

by

$$U_i(s_i, \hat{s}_i) = \mathbb{E}_{y=Y_k(\boldsymbol{s}_{-i})}[\mathbb{1}\{\hat{s}_i \geq y\}\widetilde{V}(s_i, y)] - P_i(\hat{s}_i).$$

Letting $G := G_k^{n-1}$ and $g := g_k^{n-1}$, we obtain $\partial_1 U_i(s_i, \hat{s}_i) = \int_0^{\hat{s}_i} \widetilde{V}'(s_i, y)g(y)\,\mathrm{d}y$. Note that the expression is simple because while values are not private, signals are independent. By the envelope theorem, any incentive-compatible, direct mechanism must satisfy $U_i(s_i) = U_i(0) + \int_0^{s_i} U_i'(x)\,\mathrm{d}x$ and thus $P_i(s_i) = \int_0^{s_i} \widetilde{V}(s_i, y)g(y)\,\mathrm{d}y - U_i(0) - \int_0^{s_i} U_i'(x)\,\mathrm{d}x$. From *winners pay* and the continuity of signals follows $U_i(0) = P_i(0) = 0$. The revenue equivalence extends to any standard auctions with independent signals in which winners pay, including mixed auctions, as the allocation rule is identical.

2.2 Equilibrium bidding

We derive the unique Bayes-Nash equilibrium in increasing and symmetric bidding strategies in mixed auctions with price discrimination $\delta \in [0, 1]$. The equilibrium bid function forms the center of our analysis of surplus equity in Sections 3 and 4.

Proposition 1 (Krishna 2009)**.** *The unique equilibrium bidding strategy in the uniform price auction, i.e., the case $\delta = 0$, is given by $\beta^U(s) := \widetilde{V}(s, s) = \mathbb{E}[v(s_i, \boldsymbol{s}_{-i}) \mid s_i = s, Y_k(\boldsymbol{s}_{-i}) = s]$.*

Note that the equilibrium is unique in the class of increasing and symmetric strategies and weakly dominant with pure private values (Krishna 2009).

Proposition 2. *The unique symmetric equilibrium bidding strategy in the δ-mixed auction, for $\delta \in (0, 1]$, is given by*

$$\beta^\delta(s) = \frac{\int_0^s V(y)g_k^{n-1}(y)G_k^{n-1}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{1}$$

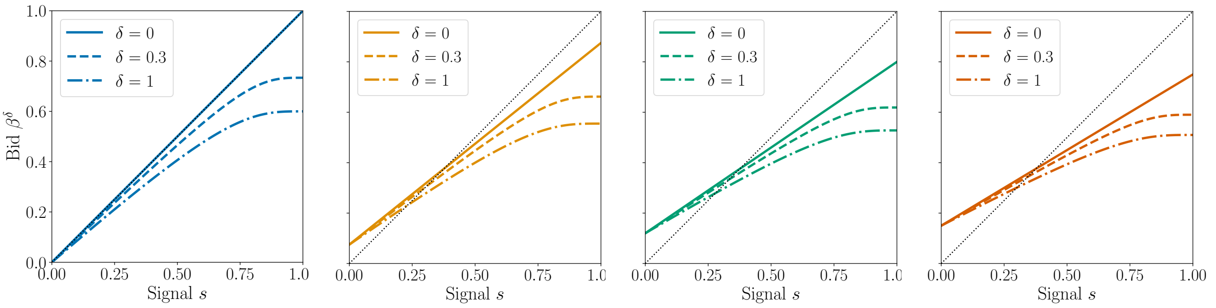
An alternative representation is given by

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y)G_k^{n-1}(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{G_k^{n-1}(s)^{\frac{1}{\delta}}}. \tag{2}$$

Proof. See Appendix B.2. □

Note that β^δ converges to β^U as $\delta \to 0$. We illustrate the bid function in the following example.

Example 1. There are $n = 10$ bidders competing for $k = 4$ items, with signals being sampled uniformly from $[0, 1]$. For the uniform price auction, one can easily compute $\beta^0(s) = V(s) = (1-c)s + c\left(\frac{3}{5}s + \frac{3}{20}\right)$, and accordingly β^δ. Note that β^0 is linear due to uniform signals. Fig. 1 illustrates the bid functions for four different values of c. In the case of pure private values, bidding truthfully is a dominant strategy in the uniform price auction. Increasing δ increases bid shading in equilibrium. Note that increasing the common value component shifts equilibrium bids for low signal realizations above $\beta^\delta(s) = s$. A bidder with a low signal has an expectation



$c = 0$ $\quad$ $c = 0.5$ $\quad$ $c = 0.8$ $\quad$ $c = 1$

Figure 1: Equilibrium bid functions, β^δ, for uniform signal distributions as a function of the signal, s, for common value parameters $c \in \{0, 0.5, 0.8, 1\}$.

of the average signal that is higher than their bid. Equally, for high signals, the bid in the uniform price auction shifts below the 45-degree line. A bidder with a high signal knows that the average signal is lower than their own. With a pure common value, the winner's curse becomes especially apparent. In equilibrium, bidders are attempting to salvage the winner's curse but cannot escape it. In fact, with a pure common value, ex-post utilities are *decreasing in signals* as long as $\delta > 0$.

As the above example shows, the equilibrium bid functions exhibit several monotonicity properties which hold beyond uniform signal distributions. By assumption, equilibrium bids are increasing in the bidder's own signal. For any given signal, they are decreasing in the extent of price discrimination; naturally, if a higher proportion of one's own bid affects the price paid, the incentive to bid shade increases. The change in the payment corresponding to a bidder's own bid, due to a signal increase, is increasing in the weight of price discrimination, and vice versa. The latter monotonicity is crucial for our result on prior-free dominance in mixed auctions.

Proposition 3. *The equilibrium bid functions satisfy the following monotonicity properties:*

1. $\beta^\delta(s)$ *is strictly increasing in* s*, for all fixed* $\delta \in [0, 1]$ *(consistent with the assumption), and is strictly decreasing in* δ*, for all fixed* $s \in (0, \bar{v})$.
2. $\frac{\partial(\delta\beta^\delta(s))}{\partial\delta}$ *is strictly increasing in* s*, for all fixed* $\delta \in [0, 1]$*, and* $\frac{\partial(\delta\beta^\delta(s))}{\partial s}$ *is strictly increasing in* δ*, for all fixed* $s \in (0, \bar{v})$.

Proof. See Appendix B.2. □

2.3 Surplus equity

Given revenue equivalence and efficiency, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs. In the following we define our equity metric *dominance in pairwise differences.*

Definition 2 (Dominance in pairwise differences)**.** An outcome $\{u_i\}_{i\in[n]}$ dominates another outcome $\{u'_i\}_{i\in[n]}$ in pairwise differences iff, for all $i, j \in [n]$, it holds that $|u_i - u_j| \leq |u'_i - u'_j|$ with one inequality strict.

We say that, for a family of parameterized outcomes $\{u_i^\delta\}_{i\in[n]}$, $\delta \in \Delta$, δ^* is dominant in pairwise differences if u^{δ^*} dominates all outcomes u^δ, $\delta \neq \delta^*$, $\delta \in \Delta$. In the following, we will consider that pairwise dominance holds as long as it holds almost surely. Note that pairwise differences induces a partial dominance ranking over results and therefore a dominant δ^* may not always exist. It is straightforward to construct two outcomes so that none dominates the other in pairwise differences.

In our auction setup, utilities are dependent on signals. As all auctions considered allocate the items to the same buyers, in most of our analysis, we focus on winning buyers. Thus, we adapt the definition of pairwise differences as follows:

Definition 3. An outcome $\{u_i(\boldsymbol{s})\}_{i\in[n]}$ dominates another outcome $\{u'_i(\boldsymbol{s})\}_{i\in[n]}$ in pairwise differences iff, for all winning signals s_i, s_j with opponents' signals $\boldsymbol{s}_{-i}, \boldsymbol{s}_{-j}$, $i, j \in [n]$, it holds that $|u_i(s_i, \boldsymbol{s}_{-i}) - u_j(s_j, \boldsymbol{s}_{-j})| \leq |u'_i(s_i, \boldsymbol{s}_{-i}) - u'_j(s_j, \boldsymbol{s}_{-j})|$, almost surely and with one inequality strict.

We now discuss several prominent equity axioms (cf., e.g., Patty & Penn 2019) in relation to pairwise differences. First, we note that anonymity is maintained. Any reordering of individuals in the population $[n]$ has no consequence, as pairwise comparisons must hold for all individuals. Furthermore, we note that replication invariance and mean independence are not relevant in our setup, as we keep the population size (number of bidders) as well as the endowments (value distributions) fixed. The Pigou-Dalton transfer principle asserts that any transfer from a wealthier agent to a poorer one must reduce inequality, provided the original welfare ranking between the two agents is maintained, that is, the wealthier agent does not become poorer than the previously poorer agent after the transfer (cf. Moulin 2004). Since dominance in pairwise differences does not establish a complete ordering of outcomes, some Pigou-Dalton transfers may result in an increase in some pairwise differences while others decrease.

However, from pairwise differences, various equity metrics, which regulators might be interested in, can be constructed using an aggregator function. For example, the top decile of utilities can be compared to the lowest realized utility or to the bottom decile of utilities. A related aggregation is given by Feldman & Kirman (1974), who aggregate positive pairwise differences for a measure of envy per player.[22] Moreover, "classic" inequity measures such as the Gini coefficient can be constructed from pairwise differences.[23] Furthermore, one might wish to prioritize larger differences over smaller ones by, for instance, squaring each pairwise difference.

Adopting an ex-post perspective to compare utilities in pairs, our results allow the classification of δ-mixed pricing rules based on pairwise differences and *any increasing function* of pairwise differences. Furthermore, we propose a natural function to aggregate pairwise differences, the *expected empirical variance of surplus among the winners*, or *winners' empirical variance (WEV)* for short. This metric is defined in expectation, ensuring that it provides a ranking of pricing formats regardless of signal realizations. Furthermore, the aggregated sum of squared pairwise

[22] Contrasting our measure, Feldman & Kirman (1974) consider pairwise differences of hypothetical (if an agent had received another agent's bundle) and realized utilities.

[23] In our setting with uncertainty about signal realizations, one could define the expected Gini coefficient among winners $G = \frac{1}{2n^2 E_s[u_1(\boldsymbol{s})|1 \text{ wins}]} E_s[\sum_{i=1}^k \sum_{j=1}^k |u_i(\boldsymbol{s}) - u_j(\boldsymbol{s})| \mid s_1, \ldots, s_k > Y_{k+1}(\boldsymbol{s})]$. A small distinction is the normalization by the expected surplus.

differences of utilities ensures compliance with the Pigou-Dalton transfer principle. Finally, the empirical variance is linked to surplus variance and correlation of surpluses among bidders.

Definition 4 (Winners' empirical variance)**.**

$$WEV \; = E_s\left[\frac{1}{k(k-1)}\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{(u_i(\boldsymbol{s})-u_j(\boldsymbol{s}))^2}{2}\middle| s_1,\ldots,s_k > Y_{k+1}(\boldsymbol{s})\right]$$

The empirical variance among all bidders (thus including losers) in the auction is given by $\text{EV} = E_{\boldsymbol{s}}[\frac{1}{n(n-1)}\sum_{i=1}^{n}(u_i(\boldsymbol{s})-u_j(\boldsymbol{s}))^2]$. The ex-ante variance of a winner is $\text{Var}_{\boldsymbol{s}}[u_i(\boldsymbol{s})|i \text{ wins}]$. Using symmetry between bidders, an alternative expression for the ex-ante variance is given by $E_{\boldsymbol{s}}[u_1^2|i \text{ wins}] - E_{\boldsymbol{s}}[u_1|i \text{ wins}]^2$, and analogous definitions hold when not conditioning on winning an item.[24]

Observation 1. The *winners' empirical variance* satisfies the *Pigou-Dalton principle.*

Proof. See Appendix B.1. □

In expectation, equilibrium surplus varies due to different factors: a bidder's own and their competitors' signals, and surplus between winners may be correlated. As we consider efficient auctions, surplus only varies among the winners in the auctions. Among those winners, WEV measures variation and correlation of surplus, i.e., it measures surplus dispersion *across* bidders. In contrast, the ex-ante variance measures only surplus variation *within* a given bidder, and is more adequate to measure risk across a series of identical, repeated auctions, in which a given bidder redraws their signal in every auction.

Lemma 1. *The empirical variance can be written as* $EV = Var[u_1] - \text{Cov}[u_1, u_2]$*, and the winners' empirical variance as* $WEV \; = Var[u_1|1 \; wins] - \text{Cov}[u_1, u_2|1 \text{ and } 2 \; win]$.

Proof. See Appendix B.1. □

Lemma 1 establishes that the empirical variance measures variation within a bidder and the correlation between surpluses. With pure private values and thus ex-post individual rationality, rankings of auction formats in terms of ex-ante variance or winners' ex-ante variance are identical. Rankings with respect to the empirical variance, however, may differ depending on if only winners are considered, or all bidders. A formal lemma and proof are given in Appendix B.1.

Example 1 (Continued)**.** In the example with uniformly distributed signals, $n = 10$ bidders, and $k = 4$ items, we can compute WEV numerically. We illustrate this for four values of the common-value proportion c in Fig. 2 below. Note that for pure private and intermediate common values, the pay-as-bid auction minimizes WEV. For $c = 0.8$, we observe an interior optimum, and for a pure common value, uniform pricing minimizes WEV.

[24] Note that the notions that concern only winning bidders can only be sensibly defined for *winners pay auctions*.

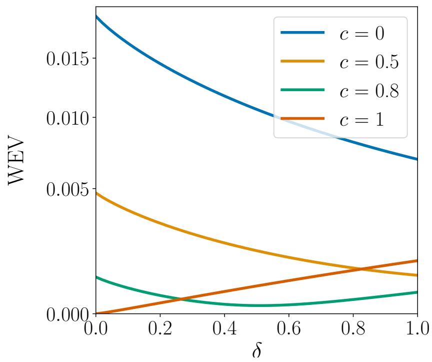


Figure 2: WEV as a function of δ for uniform signals and various common value proportions c

3 The surplus equitable mechanism

In the class of efficient auctions, it is possible to distribute surplus among the winning bidders equitably. This means pairwise differences in ex-post utilities are zero, as is any aggregate metric such as the winners' empirical variance.

Theorem 1. In the class of efficient k-unit auctions, there exists an incentive-compatible direct mechanism that results in identical surpluses of winners. Letting $y := Y_k(\boldsymbol{s}_{-i})$, the payment rule is given by

$$\widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = (1-c)(s_i - y - \frac{G(y)}{g(y)}) + V(y) \tag{3}$$

The ex-post payment in Eq. (3) is constructed in a simple but powerful way. First, the term $(1-c)s_i$ removes the idiosyncratic part of each bidder's realized value due to their own signal. To help align ex-interim incentives this term is adjusted by a uniform bonus $(1-c)(y+G(y)/g(y)$, which cancels the idiosyncratic payment in expectation. Thus, the second-price payment $V(y)$, the expected value of the $(k+1)$th-highest signal conditional on tying with the kth-highest, induces truthful reporting for any given signal s_i, ex-interim. Further details are provided in the formal proof below.

Proof. The payment rule $\widetilde{p}$ results in identical ex-post surpluses of winners. This follows directly from the definition of ex-post surplus under truthful reporting $u_i(s_i, \boldsymbol{s}_{-i}) = v(s_i, \boldsymbol{s}_{-i}) - \widetilde{p}_i(s_i, \boldsymbol{s}_{-i}) = \frac{c}{n}\sum_{j\in[n]} s_j + (1-c)(y - \frac{G(y)}{g(y)}) + V(y)$, which is independent of s_i.

Furthermore, the payment $\widetilde{p}$ is ex-interim incentive-compatible. First, we compute the expected payment (conditional on winning)

$$\widetilde{P}_i(s_i) = (1-c)\left(s_iG(s_i) - \int_0^{s_i} yg(y) + G(y))\,\mathrm{d}y\right) + \int_0^{s_i} \widetilde{V}(y,y)g(y)\,\mathrm{d}y = \int_0^{s_i} \widetilde{V}(y,y)g(y)\,\mathrm{d}y$$

Because losers pay nothing the expected payment conditional on winning equals the unconditional expected payment. The expected utility of a bidder with signal s_i and report $\hat{s}_i$ is given by

$$U_i(s_i, \hat{s}_i) = \int_0^{\hat{s}_i} \left(\widetilde{V}(s_i, y) - \widetilde{V}(y, y)\right) g(y)\,\mathrm{d}y,$$

which simplifies to

$$U_i(s_i, \hat{s}_i) = \mathbb{E}_{\boldsymbol{s}_{-i,-k}} \int_0^{\hat{s}_i} \left(1 - c + \frac{c}{n}\right)(s_i - y) g(y)\,\mathrm{d}y,$$

As the integrand is positive for $\hat{s}_i < s_i$ and negative for $\hat{s}_i > s_i$ and $g > 0$, the function $U_i(s_i, \hat{s}_i)$ is uniquely maximized at $\hat{s}_i = s_i$. □

The surplus-equitable mechanism relies on the signal priors, which, in practice, is often unknown. In the following sections, we state and prove prior-free results on equity-preferred auction design.

4 Mixed auctions

The equity-preferred pricing rule in terms of dominance in pairwise differences crucially depends on the extent of the common value, c. As seen for WEV in Example 1 (Fig. 2), with uniform signals, for some values of c, strictly mixed pricing is optimal. We formalize this fact in Section 4.1 for any signal distributions. Example 1 is in line with the general intuition that pay-as-bid pricing may be more equitable with higher private values, and uniform pricing with higher common values. However, as we show in Example 2, this is not true in general for any signal distribution. Thus, additional distributional assumptions are necessary to arrive at some characterization. In Section 4.3, we consider log-concave signal distributions and provide very simple bounds on the pricing rule that is dominant in pairwise differences.

4.1 Equity comparisons in uniform, pay-as-bid, and mixed auctions

We first consider the case of a pure common value ($c = 1$). As every bidder has the same ex-post realized value, ex-post utilities among winners are equalized if everyone pays the same price. This results in pairwise differences in utilities of zero. Once the private value component enters the value function with a non-zero weight, the picture is less clear: it may be pay-as-bid pricing that is dominant in pairwise differences, or it may be some degree of mixed pricing; however, it cannot be uniform pricing.

Theorem 2. The uniform price auction is dominant in pairwise differences iff the common value proportion equals one (pure common value).

Proof. To prove the "if" direction, note that for $c = 1$, the realized value is identical for all bidders $i \in [n]$ as $v(\boldsymbol{s}) = \frac{1}{n}\sum_{j\in[n]} s_j$. Thus, with a uniform price that is identical between bidders, they all have identical surplus. For any $\delta > 0$, the payment differs between the winners at least for some signal realizations.

To prove the "only if" let $\varphi^\delta(s) = (1 - c) \cdot s - \delta\beta^\delta(s)$. We then have $(u(s_i) - u(s_j))^2 = (\varphi^\delta(s_i) - \varphi^\delta(s_j))^2$ for two winning bids s_i, s_j (see the proof of Proposition 8 for details). Thus, it holds that

$$\frac{\partial}{\partial\delta}(\varphi^\delta(s_i)-\varphi^\delta(s_j))^2 = -2(\varphi^\delta(s_i)-\varphi^\delta(s_j))\left(\beta^\delta(s_i)-\beta^\delta(s_j)+\delta\frac{\partial\beta^\delta(s_i)}{\partial\delta}-\delta\frac{\partial\beta^\delta(s_j)}{\partial\delta}\right).$$

Using Lemma 8, we take the limit of β^δ, $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$, and $\varphi^\delta(s)$, as δ goes to 0. We have that $(\varphi^\delta(s_i)-\varphi^\delta(s_j)) \to (1-c)(s_i-s_j)$ and $(\beta^\delta(s_i)-\beta^\delta(s_j)+\delta\frac{\partial\beta^\delta(s_i)}{\partial\delta}-\delta\frac{\partial\beta^\delta(s_j)}{\partial\delta}) \to (V(s_i)-V(s_j))$. As V is increasing, the product $(V(s_j)-V(s_i))(s_i-s_j)$ is strictly negative almost surely, which concludes the proof. □

Furthermore, we show that, without any further assumptions, strictly interior δ-mixed pricing minimizes WEV for a range of common values.

Proposition 4. *For any signals distribution, there exists $c^* < 1$, such that for common values in the interval $(c^*, 1)$, there exist δ-mixed auctions with lower WEV than pay-as-bid and uniform auctions.*

Proof. We show that for any $c \in (c^*, 1)$, pay-as-bid pricing does not minimize WEV. From this and the "only if" statement in the proof of Theorem 2, the result follows. Note that WEV is continuous in c and at $c = 1$ it is strictly lower for uniform pricing ($\delta = 0$) than for pay-as-bid pricing ($\delta = 1$) by Theorem 2. Thus, by the mean value theorem, there exists an open interval $C = (c^*, 1)$, $c^* < 1$, such that, for any $c \in C$, WEV remains strictly lower under uniform pricing than pay-as-bid pricing. □

The intuitive notion that uniform pricing equitably distributes surplus under a pure common value may lead us to assume that pay-as-bid auctions achieve maximal equity under private values. However, in the following section, we demonstrate a scenario where it fails. We present a detailed example to prove that uniform pricing can result in lower WEV than pay-as-bid pricing with pure private values.

4.2 Challenging the intuition: pay-as-bid pricing with pure private values

To understand the reversal of the intuition, consider pairwise differences in utility, the building block for WEV. If ex-post absolute differences in utility are greater under uniform pricing than under pay-as-bid pricing for signal pairs with sufficient probability mass, then the reversal may also hold in expectation. To start with, consider any two winning signals $s_i > s_j$, $s_i, s_j \in [0, \bar{v})$ and private values only, i.e., $c = 0$. Let $u_i^0(s_i, \boldsymbol{s}_{-i})$ and $u_i^1(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price and pay-as-bid auction, respectively. Moreover, β^0 and β^1 denote the corresponding symmetric equilibrium bid functions and $Y_{k+1}(\boldsymbol{\beta})$ the first rejected bid. For $\delta \in [0, 1]$ and $c = 0$, we have $u_i^\delta(s_i, \boldsymbol{s}_{-i}) = s_i - \delta\beta^\delta(s_i) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$. Thus, we have $\Delta u^0 := |u_i^0 - u_j^0| = |s_i - s_j|$ and $\Delta u^1 := |u_i^1 - u_j^1| = |s_i - \beta^1(s_i) - (s_j - \beta^1(s_j))| = |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))|$. It holds that

$$\Delta U^0 < \Delta U^1 \tag{4}$$
$$\Leftrightarrow s_i - s_j < |s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))| \tag{5}$$
$$\Rightarrow 2(s_i - s_j) < \beta^1(s_i) - \beta^1(s_j) \tag{6}$$

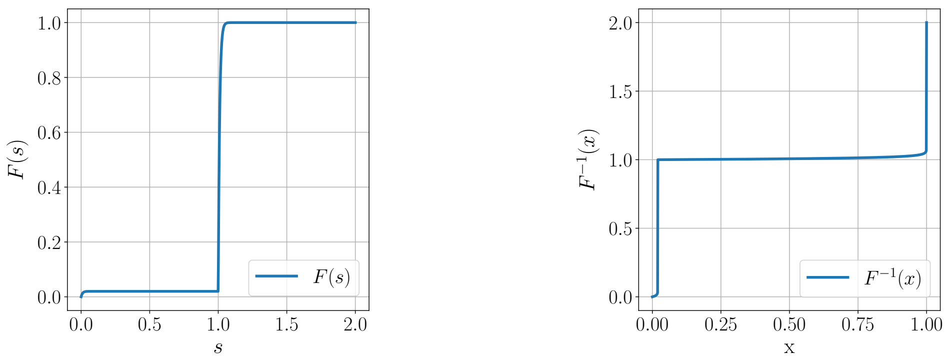
As bid functions are increasing, if $s_i - s_j - (\beta^1(s_i) - \beta^1(s_j))$ was positive, Eq. (5) could never hold. Thus, Eq. (6) follows as a necessary condition for uniform pricing to have lower pairwise differences than pay-as-bid pricing. For the same statement to hold for WEV, it must be that the bid function has a slope of at least 2 for a sufficient mass of signals s_i and s_j. Bid function slopes greater than 2 imply that high-signal bidders shade their bids much less, proportionally to their value, than lower-signal bidders. Consequently, the differential in ex-post surplus with pay-as-bid pricing, comparing two sufficiently different signals, are higher than the differential in signals. The latter equals the surplus difference in the uniform price auction.

With this intuition, we now prove that it is indeed possible to construct an *equilibrium* bid function with a slope greater than 2 for a sufficient mass of signals. For this, we require an extreme signal distribution where, broadly speaking, signals are equal to zero with probability ε and equal to one with probability $1-\varepsilon$. However, to compute a Bayes-Nash equilibrium, we need a continuous signal distribution (with respect to the Lebesgue measure, without mass points) with connected support (to solve the first-order condition). Thus, we add a small perturbation between 0 and 1.

Example 2. Consider an auction with n bidders and $k = n-1$ items. Each bidder i has a pure private value ($c = 0$) given by its signal s_i. The signal is equal to the sum of a Bernoulli random variable with parameter $\varepsilon > 0$ and a random perturbation drawn from Beta$(1, 1/\eta)$, with $\eta > 0$. The resulting signal distribution is continuous, with support $[0, 2]$. We formally state the signal cdf F in Appendix B.3 and give the quantile function below. The example builds on the quantile function throughout, which simplifies the analysis. This is without loss of generality, since signals can be mapped one-to-one to quantiles.

$$\forall x \in [0,1], \qquad F^{-1}(x) = \mathbb{1}[x \geq \varepsilon] + \gamma_\eta(x) \qquad \text{where} \qquad \gamma_\eta(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^\eta & \text{if } x < \varepsilon \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^\eta & \text{if } x \geq \varepsilon \end{cases}$$

We fix $\varepsilon = 0.1/n$ and choose $\eta > 0$ to be an arbitrarily small constant. We consider the order



Signal cdf　　　　Quantile function

Figure 3: Bidder signals and quantiles for $n = 5$ and $\eta = 0.01$

statistics with respect to the quantiles,[25] and we denote by $\widetilde{G}$ (resp. $\widetilde{g}$) the distribution function (resp. density) of the k-th highest quantile among $n-1$ buyers. We plot the signal cdf and the

[25] $\widetilde{G}$ and $\widetilde{g}$ correspond to the definitions of G_k^{n-1} and g_k^{n-1} with uniform signals, where $F = F^{-1}$.

quantile function in Fig. 3.

Applying the formula from Proposition 2, we can derive the equilibrium bid $b^\delta_\eta(x)$ of a bidder with quantile x (recall that equilibrium bids as functions of signals are denoted by β^δ). We state the bid function below and illustrate it in Fig. 4. For details, we refer to Appendix B.3. For $\delta = 0$ we have that $b^0_\eta(x) = \beta^0(F^{-1}(x)) = \mathbb{1}[x \geq \varepsilon] + \gamma_\eta(x)$; and for all $\delta > 0$ we have

$$\forall x \in [0,1], \qquad b^\delta_\eta(x) := \beta^\delta(F^{-1}(x)) = b^\delta_0(x) + \xi^\delta_\eta(x)$$

$$\text{where} \qquad b^\delta_0(x) := \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{G(\varepsilon)}{G(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases} \qquad \text{and} \qquad \xi^\delta_\eta(x) \; := \frac{\int_0^x \gamma_\eta(y) g(y) G(y)^{\frac{1}{\delta}-1}\, \mathrm{d}y}{\delta G(x)}.$$

Next, we define the function $\phi^\delta_\eta(x) := F^{-1}(x) - \delta b^\delta_\eta(x)$, denoting the utility of a bidder

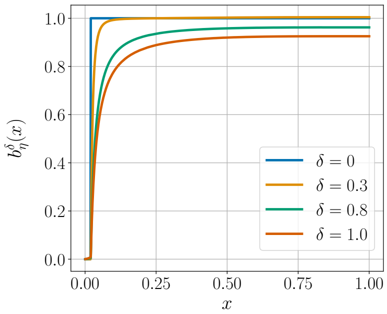


Figure 4: Equilibrium bid as a function of quantiles for $n = 5$ and $\eta = 0.01$

as a function of their quantile, without the uniform payment which cancels out in the computation of WEV. Finally, we define WEV^δ_η, the winners' empirical variance in a δ-mixed auction with noise level η. For all $\delta \in [0,1]$ and for all $\eta > 0$, we have that $2 \cdot \mathrm{WEV}^\delta_\eta = \mathbb{E}_{\mathbf{x}}\left[(\phi^\delta_\eta(x_1) - \phi^\delta_\eta(x_2))^2 \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi^\delta_\eta(x)$ converge towards 0 when taking η arbitrarily small. Therefore, WEV^δ_η converges towards WEV^δ_0, formally defined in Appendix B.3.

We constructed the above example such that uniform pricing achieves lower WEV than pay-as-bid pricing, even with pure private values. The result is stated in the proposition below and holds in the limit as $\eta \to 0$, for any number of bidder $n \geq 5$.

Proposition 5. *For $n \geq 5$, there exists η^*, such that for all $\eta \leq \eta^*$ it holds that the* winners' empirical variance *under uniform pricing is lower than under pay-as-bid pricing.*

The proof is given in Appendix B.3, and shows that $\mathrm{WEV}^0_0 \leq \frac{0.005}{n}$ and $\mathrm{WEV}^1_0 \geq \frac{0.01}{n}$.

Thus, in order to characterize equity-optimal pricing further, we need additional assumptions. In the next section, we show that, for a large class of signal distributions, simple bounds tell us which pricing rules are candidates for being dominant in pairwise differences, and which pricings are dominated.

4.3 Equity-optimal pricing for log-concave signal distributions

For the class of *log-concave* signal distributions, simple bounds characterize a pricing rule that is dominant in pairwise differences.

Definition 5. A real-valued function $h \in \mathbb{R}^{\mathbb{R}}$ is *log-concave* if $\log(h)$ is concave.

The family of log-concave distributions contains many common distributions, for example uniform, normal, exponential, logistic or Laplace distributions (Bagnoli & Bergstrom 2005).[26]

The proofs of the following two theorems are given in Section 5.

Theorem 3. Assume that signals are drawn from a log-concave distribution. Then, for a given common value component c, there exists a pricing rule which dominates in pairwise differences any pricing rule with a discriminatory proportion of less than $1 - c$.

Theorem 3 provides a lower bound on the amount of price discrimination required to rule out dominated pricing formats. The following theorem states that uniform pricing is dominated in pairwise differences by many alternative pricing rules.

Theorem 4. Assume signals are drawn from a log-concave distribution. Then the uniform price auction is dominated in pairwise differences by any strictly mixed pricing with price discrimination of up to $\min\{1, 2(1 - c)\}$.

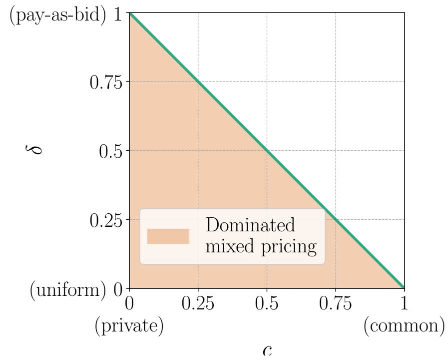


Figure 5: Bounds on equity-optimal combinations of c and δ

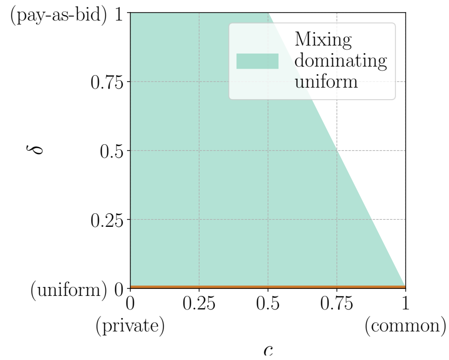


Figure 6: Range of price discrimination dominating uniform pricing

We illustrate Theorem 3 in Fig. 5. All pricing rules in the shaded area in red are dominated by the diagonal $1 - c$, given any log-concave distribution of bidders' signals. Similarly, Fig. 6 illustrates Theorem 4. Any pricing rule in the shaded area in green dominates uniform pricing for a given common value c.

The intuition behind Theorem 3 is simple. As we show in Section 5 below, pairwise differences are, for any given common value c, monotonically decreasing in the extent of price discrimination δ as long as δ is between zero and $1 - c$. Moreover, we show the equivalence of this result with ex-post utilities that increase in signals. As long as higher signals obtain a higher surplus, more equity can be achieved by taxing higher signals more than lower signal.

[26] Also χ distribution with degrees of freedom ≥ 1, gamma with shape parameter ≥ 1, χ^2 distribution with degree of freedom ≥ 2, beta with both shape parameters ≥ 1, Weibull with shape parameter ≥ 1, and others.

Because the change in the δ-weighted bid in δ is increasing in a bidder's signal (as stated in Proposition 3), increasing the extent of price discrimination will have the desired effect.

A similar intuition explains Theorem 4, where the benefit of higher price discrimination compared to the absence of price discrimination can be realized up to a certain threshold. As long as utilities are increasing in signals, increasing price discrimination results in surplus taxation that benefits equity (cf. Theorem 3). We show in Section 5 that increasing ex-post utilities is equivalent to the slope of equilibrium bid functions being bounded $(1-c)/\delta$. With steeper bid functions, the utilities might decrease in the signals. So, while increasing price discrimination might locally, in a neighborhood of δ, increase pairwise differences, price discrimination is still beneficial compared to uniform pricing. However, for $\delta \geq 2(1-c)$, the bid functions are so steep that an increase in price discrimination results in an absolute utility gap between a high signal and a low signal bidder that is greater than under uniform pricing. With such price discrimination, the higher signal bidder is worse off than the low signal bidder.

With Theorems 3 and 4, we can now revisit the question: In terms of equity, should one use pay-as-bid pricing if bidders' values are pure private values? The answer is yes, if the signal distributions are log-concave. Moreoever, if the common value is small, pay-as-bid pricing is guaranteed to be more equitable than uniform pricing. We state this formally in the corollary below.

Corollary 1. Assume signals are drawn from a log-concave distribution. Then, for pure private values, pay-as-bid pricing is dominant in pairwise differences, and for a common value $c < \frac{1}{2}$, pay-as-bid pricing dominates uniform pricing in pairwise differences.

The first part of the corollary follows by setting $c = 0$ in Theorem 3. The second part follows by setting $\delta = 1$ in Theorem 4. Our numerical experiments in Section 6.1 show that, for $c < \frac{1}{2}$, pay-as-bid pricing in fact minimizes WEV for several common distributions. The intuition in the pure private value case carries through under the qualifying assumption of log-concave signals, and it may fail for very concentrated signal distributions. In the latter case, it is important that sufficient probability mass is gathered around higher signals, inducing a bidding equilibrium in which ex-post utilities are decreasing in signals for sufficiently many signal realizations.[27]

For specific signal distributions, we can extend the region where pairwise differences are monotonically decreasing slightly beyond the diagonal $1-c$, as exemplified in the following proposition. The proof is given in Appendix B.5.[28]

Proposition 6. *For uniformly distributed signals, any pricing dominant in pairwise differences contains a discriminatory proportion of at least* $\frac{2n(1-c)}{2n-c(n-2)}$*, and for exponentially distributed signals at least* $\frac{2n(1-c)}{2n-c(n-(k+1))}$.

Note that both bounds converge to $\frac{1-c}{1-c/2}$ as the number of bidders goes to infinity (and the number of items k is kept constant). We illustrate the bound for the uniform distribution in Fig. 7 below, together with the equity-optimal pricing in terms of WEV. The figure demonstrates that, for high values of c, this bound may be a good heuristic for the optimal pricing rule.

[27] For example, with a β-distribution as steep as illustrated in Fig. 9, Section 6, clearly violating log-concavity, pay-as-bid pricing is still optimal for a range of common values including pure private values.

[28] The proof should be read in conjunction with Theorem 3 as it follows a similar reasoning.

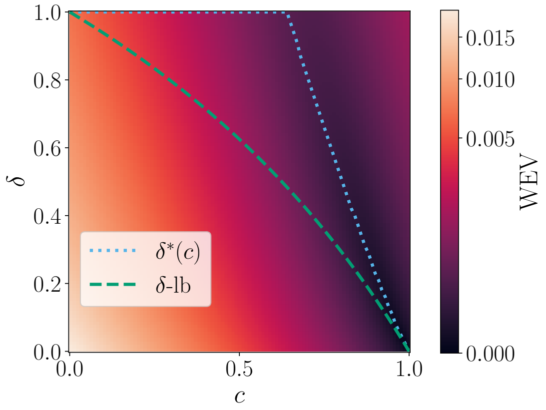


Figure 7: A lower bound for pricing candidates dominant in pairwise differencesand the equity-optimal pricing in terms of WEV

5 Structural insights and proofs of Theorems 3 and 4

In this section, we provide an overview of the proofs of Theorems 3 and 4. Each of these theorems is proved by combining a proposition on monotonicity of pairwise differences and dominance of pairwise differences, respectively, with a third proposition that bounds the slope of bid functions. In particular, we identify the property of *monotone ex-post utility* as a fundamental and sufficient condition for our dominance results.

Definition 6 (Monotone ex-post utility)**.** The ex-post utility $u(\boldsymbol{s})$ satisfies *monotone ex-post utility (MEU)* iff, for any two signals $s_i, s_j \in [0, \bar{v})$ and $\forall\ \boldsymbol{s}_{-i}, \boldsymbol{s}_{-j},\ s_i \leq s_j \Leftrightarrow u_i(s_i, \boldsymbol{s}_{-i}) \leq u_j(s_j, \boldsymbol{s}_{-j})$.

Monotone ex-post utility (MEU) relates to the slope of equilibrium bid functions by the following lemma. The proof is given in Appendix B.4.

Lemma 2. *An equilibrium satisfies monotone ex-post utility iff equilibrium bid functions β^δ satisfy $\frac{\partial \beta^\delta}{\partial s} \leq \frac{1-c}{\delta}$ for all signals $s \in [0, \bar{v})$.*

The ex-post difference in utilities depends only on the private value proportion $(1-c)s$ and the discriminatory part of the payment $\delta\beta^\delta$. Thus, as long as the discriminatory payment does not grow faster in the signal than the private-value share, ex-post utilities are monotone.

We now characterize the fundamental role of monotone ex-post utility: it is equivalent to the monotonicity property of pairwise differences.

Proposition 7. *For a given common value c and for some $\bar{\delta} \in [0, 1]$, pairwise differences are monotonically decreasing over $[0, \bar{\delta}]$ if and only if the equilibrium (which depends on c and $\bar{\delta}$ satisfies MEU.*

The proof is given in Appendix B.4.

The equivalence between decreasing pairwise differences and MEU being satisfied in equilibrium is crucial in our proof of Theorem 3. When MEU holds, the slope of the equilibrium bid function is sufficiently flat and more price discrimination impacts higher signal bidders more than lower signal bidders. In contrast, including more uniform pricing in the price mix will, proportionally to the change in δ, offer higher signal bidders a higher discount than lower signal bidders and thus does not improve surplus equity. A similar intuition holds for Proposition 8 below (for details on the intuition, see Section 4.3). The proof is given in Appendix B.4.

Proposition 8. *For a given common value c, consider any δ-mixed auction, $\delta \in (0,1]$, and suppose the equilibrium bidding function β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2(1-c)}{\delta}$ for all signals $s \in [0, \bar{v})$. Then, δ-mixed pricing dominates uniform pricing in pairwise differences.*

Example 1 (Continued)**.** Whether MEU is satisfied can be verified numerically, either by computing differences in realized utilities for every pair of signals or simply by checking the derivative of the bid function. We illustrate this for the example of uniform signal distributions in Fig. 8 below. For example, with $c = 0.8$ and $\delta = 0.3$, close to the MEU boundary in Fig. 8, the derivative of the function cannot be larger than $0.667 = \frac{1-0.8}{0.3}$. From Fig. 1, the slope of the bid function with $c = 0.8$ and $\delta = 0.3$ is close to 0.68 for low signals. Thus, for this combination of c and δ MEU is not satisfied.

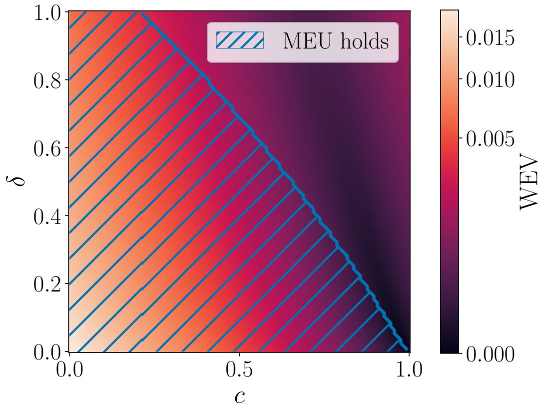


Figure 8: Monotone ex-post utility for combinations of common value c and price discrimination δ

The final crucial proposition bounds the slope of the equilibrium bid functions by 1 for the family of log-concave signal distribution. The proof is given in Section 5.1 below.

Proposition 9. *If the signal density f is log-concave, then $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ for all signals $s \in [0, \bar{v})$.*

With pure private values, this bound implies that ex-post utility is non-decreasing in signals for log-concave signal distributions. Indeed, for $u(s) = s - \delta\beta^\delta(s) - (1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have that $\frac{\partial u}{\partial s} = 1 - \delta\frac{\partial \beta^\delta(s)}{\partial s} \geq 0$. A similar reasoning leads to Theorems 3 and 4.

Proof of Theorems 3 and 4. Because of Proposition 9, we have that under log-concave signal distributions MEU holds if $\delta \leq (1-c)$, as $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{(1-c)}{\delta}$ (see Lemma 2). Thus, applying Proposition 7, pairwise differences are monotonically decreasing for $\delta \in [0, 1-c]$ and Theorem 3 follows.

Similarly, because of Proposition 9, it holds that with log-concave signals, $\frac{\partial \beta^\delta}{\partial s} \leq 1 \leq \frac{2(1-c)}{\delta}$ if $\delta \leq 2(1-c)$. Applying Proposition 8, it follows that any mixed pricing with $\delta \in (0, 2(1-c)]$ dominates uniform pricing in pairwise differences. $\square$

5.1 Proving the bound on bid function slopes

Bounding the bid function slope for log-concave distributions requires three main observations, which we detail in the lemmas below and then use to prove Proposition 9. All proofs are relegated to Appendix B.5.

The first lemma establishes a simplified expression of $V(s)$ which allows to readily bound $V'(s)$ by 1. The proof can be found here.

Lemma 3. *Assuming common values ($c = 1$), $V(s)$ is differentiable on $(0, \bar{v})$, and can be expressed as*

$$V(s) = \frac{2}{n}s + \frac{n-k-1}{n}\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} + \frac{k-1}{n}\frac{\int_s^{\bar{v}} tf(t)\,\mathrm{d}t}{1-F(s)}$$

Moreover, if the signal density f is log-concave, then $V'(s) \leq 1$ for all signals $s \in [0, \bar{v})$.

The proof proceeds by noticing that order statistics conditioned on other order statistics behave just like order statistics of a truncation of the original distribution. Thus, a more tractable expression of the expected valuation V can be derived for the pure common value case ($c = 1$). Together with results on log-concavity by Bagnoli & Bergstrom (2005), we use this expression to show that $V' \leq 1$ for all signals $s \in (0, \bar{v})$.

The next lemma establishes a sufficient condition for the equilibrium bid function slope to be bounded by 1 in the pure private value case. Differentiating twice $\int_0^s G^{1/\delta}$, we establish that its log-concavity is equivalent to $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$. The proof is given here.

Lemma 4. *Assuming private values ($c = 0$), for any $\delta \in (0, 1]$, $\frac{\partial \beta^\delta(s)}{\partial s} \leq 1$ iff $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is log-concave.*

Finally, we establish that a log-concave signal density is sufficient for the integral of their order statistics to be log-concave, using closure properties of product and integration of log-concave distributions, and results by Bagnoli & Bergstrom (2005). The proof is also given in the appendix.

Lemma 5. *If the density of signals f is log-concave, then so is $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$.*

With the three lemmas above, we can prove Proposition 9.

Proof of Proposition 9. First, we recall the expression of the derivative of the bid function for any $s \in (0, \bar{v})$:

$$\frac{\partial \beta^\delta(s)}{\partial s} = \frac{g(s)}{G(s)}\frac{\int_0^s V'(y)G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \tag{7}$$

Note that for any $c \in [0, 1]$, $V(s)$ is a linear combination of s and V. In the case of a pure common value, the derivative of the latter is bounded by 1 by Lemma 3. Hence for any c, $V'(s) \leq 1$. Moreover, because of Lemma 5, we know that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave, and we can therefore apply Lemma 4. Hence using the above results,

$$\frac{\partial \beta^\delta(s)}{\partial s} \leq \frac{g(s)}{G(s)}\frac{\int_0^s \max_t V'(t)G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq \frac{g(s)}{G(s)}\frac{\int_0^s 1 \cdot G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}}(s)} \leq 1. \tag{8}$$

□

6 Discussion

Our main results hold for all equity metrics that are based on pairwise differences, and, as discussed in the Introduction and in Section 3, the winners' empirical variance is particularly attractive as an aggregated one-dimensional metric. We illustrate WEV further in a series of numerical experiments and discuss how it relates to the within-bidder variation of surplus, as well as the empirical variance of surplus between all bidders. We also explain why the regularity assumption of log-concavity is necessary for our argument.

6.1 Numerical experiments

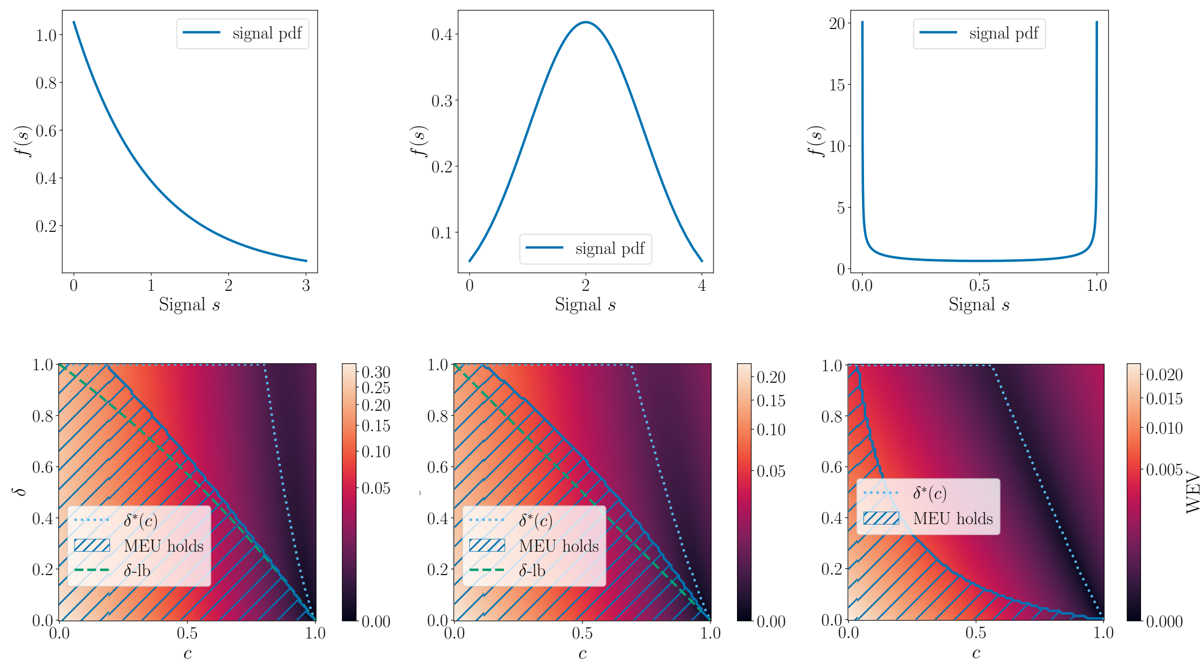
We further illustrate the effect of the common value on surplus equity by presenting several numerical examples. Similarly to Fig. 7, we compute the WEV-minimal pricing $\delta^*(c)$ for any given proportion of the private-common value c. We also illustrate bounds for WEV-minimal pricing and the condition of monotone ex-post utility (MEU). All of our experiments are based on equilibrium bid functions, whose calculation is computationally very expensive. Thus, we rely on theoretical simplifications, such as Lemma 3 and Lemma 10 (Appendix B.6). The simulations are performed through numerical integration of our analytical formulae.[29] Finally, some quantities (such as bidding functions) have multiple analytical expressions, among which we choose the most appropriate for accuracy and speed, depending on the value of the signal (e.g., Eq. (2) can be integrated more efficiently than Eq. (1), but is less accurate for small signals). Our code is available on github.

We consider three signal distributions, a truncated exponential and a truncated normal distribution (both log-concave), as well as a Beta distribution with shape parameters $(0.5, 0.5)$, which is not log-concave. WEV-minimal pricing, a lower bound on the minimizer, and combinations of common value shares and mixed pricing for which MEU holds are shown in Fig. 9.

For the truncated exponential distribution, we show the lower bound of $\frac{2n(1-c)}{2n-c(n-(k+1))}$ (cf., Proposition 6) on $\delta^*(c)$, and for the normal distribution we show the general lower bound $1-c$ (cf., Theorem 3). Each of these bounds dominates any extent of price discrimination below it. Note that for the Beta distribution, we cannot provide a theoretical lower bound on the WEV-minimal design δ^*, as the distribution is not log-concave. However, the region where MEU holds can be determined numerically, and its "frontier" provides a lower bound for the WEV-minimal design δ^*. Illustrating this for all three distributions, we observe that the area is much smaller for the Beta distribution. However, MEU is only a sufficient condition for the monotonicity of WEV (while it is necessary and sufficient for the monotonicity of pairwise differences). From the heat maps in Fig. 9, it is evident that WEV is monotone in δ for any given c up to δ^*.

Finally, we show the WEV-minimal pricing rule $\delta^*(c)$ for each signal distribution. The curve is qualitatively similar in each plot. In line with Theorem 3 — noting that the exponential and normal distribution are log-concave — the figure illustrates that with a high private value component (low c), pay-as-bid pricing ($\delta = 1$) minimizes WEV; with higher common value

[29] The efficiency and accuracy of the code rely on various techniques. Most importantly, we rewrite all multidimensional expectations as nested one-dimensional integrals (with variable bounds), which we compute by integrating polynomial interpolations. Second, the code ensures that each quantity is computed at most once, using memorization. Integration is not computationally heavy at all and achieves high precision.



Truncated exponential

Truncated normal

Beta$(0.5, 0.5)$

Figure 9: WEV-minimizing design $\delta^*(c)$, monotone ex-post utility (MEU), and lower bounds on $\delta^*(c)$ (δ-lb) for truncated exponential, truncated normal, and Beta$(0.5, 0.5)$ signal distributions

components (high c), strictly mixed pricing for some $\delta \in (0, 1)$ minimizes WEV (cf., Proposition 4); and with a pure common value ($c = 1$), uniform pricing ($\delta = 0$) minimizes WEV (cf., Theorem 2). Analogous interpretations hold for the Beta distribution, although we cannot give theoretical guarantees.

For small common values, MEU holds for any δ and thus pay-as-bid pricing is dominant in pairwise differences (cf., Proposition 7). Even for larger common value parameters the WEV-minimal pricing is still pay-as-bid, but eventually strictly mixing ($\delta \in (0, 1)$) is required to minimize WEV. For a pure common value, uniform pricing is WEV-minimal regardless of the signal distribution. Notice also that WEV at the minimal δ^* decreases in c. Naturally, with a higher common value share, bidders' values given different signal realizations as well the corresponding bids move closer together, thus explaining smaller differences in utilities (ex-post and in expectation).

6.2 Variance and risk preferences

Surplus equity and distributional concerns are distinct from questions of within-agent variation and associated risk preferences. An appropriate measure to assess the latter is, e.g., the ex-ante variance of bidder surplus. While the two notions are distinct, the measures are linked through the covariance (see also Lemma 1). In addition, for the pure private value setting, we derive the following result:

Proposition 10. *With pure private values ($c = 0$), the pay-as-bid auction minimizes the ex-ante variance of surplus among all standard auctions with increasing equilibrium bid functions.*

The proof is given in Appendix B.7. Because of revenue equivalence, note that the previous proposition also implies that $\mathbb{E}[u_i^2]$ is minimal in the pay-as-bid auction among standard auctions. The second moment of surplus links the winners' empirical variance and the empirical variance among all bidders, as shown in Lemma 6 in the Appendix. As a consequence of Lemma 6, surplus equity rankings with respect to the winners' empirical variance and the empirical variance among all bidders may not be equivalent. However, applying Proposition 8 to the pure private value case, we have the following corollary:

Corollary 2. Assuming pure private values ($c = 0$), consider any δ-mixed auction, $\delta \in (0, 1]$, and suppose that the equilibrium bid β^δ satisfies $\frac{\partial \beta^\delta}{\partial s} \leq \frac{2}{\delta}$ for all signals $s \in [0, \bar{v})$. Then, the empirical variance (among all bidders) is lower for δ-mixed pricing than for uniform pricing.

Although this result shows that Theorem 4 can be used to extend equity rankings under pure private values to the empirical variance *among all bidders*, this may not hold in the general case.

6.3 Beyond log-concave distributions

A crucial ingredient for Theorem 3 is that the derivative of the equilibrium bid function is bounded by 1, which holds for log-concave distributions by Proposition 9. In particular, the density of the first rejected signal must be log-concave. In the following, we provide some insights as to why it is difficult to generalize this result beyond log-concave distributions.

For simplicity, consider the pay-as-bid and the uniform price auction. Considering log-concave signal distributions, we note that log-concavity is equivalent to (A, G) concavity (a generalization of convexity, see Anderson et al. 2007), and $\frac{\partial \beta^\delta}{\partial s} \leq 1$ is thus equivalent to (A, G) concavity of $s \mapsto \int_0^s G_k^{n-1}$. One idea to extend our results could then be to consider other generalizations of convexity. Considering Proposition 8, one might attempt to bound the slope of the bid functions by 2. It holds that $\frac{\partial \beta^\delta}{\partial s} \leq 2$ is equivalent to (A, H) concavity of the same function where H is the harmonic mean. But contrary to (A, G) concave functions, there are no simple group closure properties that allow for the (A, H) concavity of f to always imply that of $\int_0^s G_k^{n-1}$. Thus, this route of inquiry does not carry fruits.

We also note that conditions similar to MEU such that uniform pricing yields lower pairwise differences (or WEV) than pay-as-bid pricing are much more difficult to attain. Why? If we follow the same main ideas as in the proof of Proposition 8, a similar condition using the mean value theorem would be that, for all $s \in (0, \bar{v})$, φ is an expansive mapping, translating into $|(1 - c) - \frac{\partial \beta^\delta}{\partial s}| \geq 1 - c$. As $\frac{\partial \beta^\delta}{\partial s}$ is strictly positive, we must have $\frac{\partial \beta^\delta}{\partial s} > 1 - c$. For signal distributions with bounded density, g_k^{n-1} is close to zero near $\bar{v}$ (this follows from the definition of order statistics), and therefore $\frac{\partial \beta^\delta}{\partial s}$ is close to zero for a nonzero interval of signals. Thus, $\frac{\partial \beta^\delta}{\partial s} > 1 - c$ cannot hold for all signals on the support, and we cannot rely on similar proof techniques to produce the desired conditions.

7 Conclusion

This article studies the division of surplus between buyers in auction design. We introduce a family of equity measures that are based on dominance in pairwise differences, including the empirical variance and other well-known metrics. Considering efficient and revenue equivalent auctions, we focus on a single design objective, equity.

First, we design the surplus-equitable mechanism, a direct and truthful mechanism that efficiently allocates the items for sale and charges each winner a personalized price. This price equalizes the winners' realized utilities, while losers pay nothing. We show that equity-optimal pricing crucially depends on the common value proportion in the buyers' value structure. Turning to the class of uniform, pay-as-bid, and mixed auctions, we demonstrate that in most cases, some degree of price discrimination is beneficial in terms of equity. Moreover, our results have significant implications for the design of multi-unit auctions in practice. By carefully selecting a pricing mixture based on (an estimate of) the common value, auctioneers can achieve a more equitable division of surplus among buyers.

Future research could explore the trade-offs between efficiency, revenue, and equity, or extend our analysis to other types of auctions and value distributions. For example, in multi-unit demand settings, where items may be allocated inefficiently (cf. Ausubel et al. 2014), trade-offs become relevant. Finally, our work generates several testable predictions which empirical studies could attempt to validate or disprove in real-world auctions.

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A Revenue equivalence and efficiency

We recall results from Krishna (2009) that show that the auctions we consider exhibit revenue equivalence and (allocative) efficiency.

Proposition (Revenue equivalence, Krishna 2009)**.** *Assuming iid signals, any standard auction, under any symmetric and increasing equilibrium with an expected payment of zero at value zero, yields the same expected revenue to the seller.*

We note that the crucial assumption for revenue equivalence is the independence of signals. In settings where signals are correlated, revenue equivalence fails (Krishna 2009, Chapter 6.5). It can be further shown that a bidder with signal s_i has an expected surplus

$$\tilde{u}(s_i) := \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] = \int_0^{s_i} (\widetilde{V}(s_i, y) - \widetilde{V}(y, y)) g_k^{n-1}(y) \, \mathrm{d}y$$

A value function $v(\boldsymbol{s})$ satisfies the *single crossing* condition if for all $i, j \neq i \in [n]$ and for all $\boldsymbol{s}$, $\frac{\partial v(s_i, \boldsymbol{s}_{-i})}{\partial s_i} \geq \frac{\partial v(s_j, \boldsymbol{s}_{-j})}{\partial s_i}$. Naturally, the value function v as given in Assumption 1 is single-crossing.

Proposition (Efficiency, Krishna 2009)**.** *Any standard auction, under any symmetric and increasing equilibrium and values satisfying the single-crossing condition, is efficient.*

Given the prior propositions, we can focus on the question of surplus distribution among buyers more succinctly without considering potential trade-offs.

B Proofs

B.1 Surplus equity

Proof of Lemma 1. The empirical variance of surplus can be transformed as follows.

$$\begin{aligned}
\mathbb{E}_s\left[\frac{1}{n-1}\sum_i^n \left(u_i - \frac{1}{n}\sum_j^n u_j\right)^2\right] &= \mathbb{E}_s\left[\frac{1}{2n(n-1)}\sum_{i=1}^n\sum_{j=1}^n (u_i - u_j)^2\right] \\
&= \frac{\mathbb{E}_s\left[(u_1 - u_2)^2\right]}{2} \\
&= \mathbb{E}_s[u_1^2] - \mathbb{E}_s[u_1 u_2] \\
&= \mathrm{Var}(u_1) - \mathrm{Cov}(u_1, u_2)
\end{aligned}$$

Similarly, the empirical variance conditioned on winning can be written as

$$\begin{aligned}
\mathbb{E}_s\left[\frac{1}{k-1}\sum_{i=1}^k \left(u_i - \frac{1}{k}\sum_{j=1}^k u_j\right)^2 \middle| 1, \ldots, k \text{ win}\right] &= \frac{\mathbb{E}_s\left[(u_1 - u_2)^2 \mid 1 \text{ and } 2 \text{ win}\right]}{2} \\
&= \mathbb{E}_s\left[u_1^2 \mid 1 \text{ wins}\right] - \mathbb{E}_s\left[u_1 u_2 \mid 1 \text{ and } 2 \text{ win}\right] \\
&= \mathrm{Var}(u_1 \mid 1 \text{ wins}) - \mathrm{Cov}(u_1, u_2 \mid 1 \text{ and } 2 \text{ win}).
\end{aligned}$$

□

With pure private values, ex-post individual rationality holds. The lemma below shows that, in this case, any ranking of auction formats in terms of ex-ante variance (Var) or winners' ex-ante variance (WV) is identical. In contrast, a ranking with respect to the empirical variance (EV) may differ depending on if only winners are considered or all bidders.

Lemma 6. *Assuming that the auction is a* winners pay auction*, the empirical variance and the ex-ante variance can be decomposed, respectively, as* $EV = \frac{k(k-1)}{n(n-1)} \cdot WEV \ + \left(1 - \frac{k-1}{n-1}\right) E_{\boldsymbol{s}}[u_1^2]$ *and* $Var = \frac{k}{n} \cdot WV + \left(\frac{n}{k} - 1\right) \cdot E_{\boldsymbol{s}}[u_1]^2$.

Recall that $E_{\boldsymbol{s}}[u_1]$ does not depend on the auction format (by revenue equivalence), while $E_{\boldsymbol{s}}[u_1^2]$ does.

Proof of Lemma 6. We first note that

$$\text{WV} = E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 = \frac{n}{k} E_{\boldsymbol{s}}[u_1^2] - \left(\frac{n}{k}\right)^2 E_{\boldsymbol{s}}[u_1]^2$$

For the ex-ante variance, we write:

$$\begin{aligned}
\text{Var} &= E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1]^2 = P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 + P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \text{WV} + P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \text{WV} + \frac{P_{\boldsymbol{s}}[1 \text{ wins}]^2}{P_{\boldsymbol{s}}[1 \text{ wins}]} \cdot E_{\boldsymbol{s}}[u_1 \,|\, 1 \text{ wins}]^2 - E_{\boldsymbol{s}}[u_1]^2 \\
&= P_{\boldsymbol{s}}[1 \text{ wins}] \cdot \text{WV} + \left(\frac{n}{k} - 1\right) \cdot E_{\boldsymbol{s}}[u_1]^2
\end{aligned}$$

For the empirical variance, we write:

$$\begin{aligned}
WEV \ &= E_{\boldsymbol{s}}[u_1^2] - E_{\boldsymbol{s}}[u_1 u_2] = P_{\boldsymbol{s}}[1 \text{ wins}] \cdot E_{\boldsymbol{s}}[u_1^2 \,|\, 1 \text{ wins}] - P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot E_{\boldsymbol{s}}[u_1 u_2 \,|\, 1 \text{ and } 2 \text{ win}] \\
&= P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}] \cdot \text{WEV} + \left(1 - \frac{P_{\boldsymbol{s}}[1 \text{ and } 2 \text{ win}]}{P_{\boldsymbol{s}}[1 \text{ wins}]}\right) \cdot E_{\boldsymbol{s}}[u_1^2]
\end{aligned}$$

□

Proof of Observation 1. Without loss of generality, consider an outcome profile u with three outcomes, u_i, u_j and U, where $u_i > u_j$, and U is arbitrary. Induce a Pigou-Dalton transfer $t > 0$ such that $u_i' = u_i - t > u_j$ and $u_j' = u_j + t < u_i$, and U remains the same. The outcome profile after the transfer is denoted u'. We show that the ranking between u and u' according to WEV coincides with what the Pigou-Dalton principle requires, namely it must be that $WEV(u') < WEV(u)$. Let $W := (u_i - U)^2 + (u_j - U)^2$. Then

$$\begin{aligned}
&(u_i' - U)^2 + (u_j' - U)^2 \\
&= (u_i - t - U)^2 + (u_j + t - U)^2 \\
&= (u_i - U)^2 - 2t(u_i - U) + t^2 + (u_j - U)^2 + 2t(u_j - U) + t^2 \\
&= W + 2t\,(t - u_i + U + u_j - U) \\
&= W + 2t\,(u_j - (u_i - t)) \\
&< W
\end{aligned}$$

The final inequality follows by the assumption that the transfer does not make i poorer than j was to start with. As U was arbitrarily chosen and, to compute WEV, expectations are taken around the sum of squared differences of the realized utilities, the result follows. □

B.2 Equilibrium bidding

Proof of Proposition 1. Consider bidder i and let all bidders $j \neq i$ follow the bidding strategy $\beta^U(s_j) = \widetilde{V}(s_j, s_j)$. First, observe that β^U is continuous and increasing. Then bidder i's expected payoff when their signal is s_i and bidding $\beta^U(z)$ is given by

$$U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \widetilde{V}(y, y)\right) g_k^{n-1}(y)\,\mathrm{d}y$$

Because $\widetilde{V}(s_i, y)$ is increasing in s_i, it holds for all $y < s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) > 0$, and for all $y > s_i$ that $\widetilde{V}(s_i, y) - \widetilde{V}(y, y) < 0$. Therefore, choosing $z = s_i$ maximizes bidder i' expected payoff $U(s_i, z)$. $\square$

Proof of Proposition 2. First, observe that β^δ is continuous. We verify that it is also monotone: writing $G_k^{n-1} =: G$, $g_k^{n-1} =: g$, and $\widetilde{V}(s, s) =: V(s)$, an alternative expression for β^δ is given by

$$\beta^\delta(s) = V(s) - \frac{\int_0^s V'(y) G(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{G(s)^{\frac{1}{\delta}}}. \tag{9}$$

In particular, it is differentiable almost everywhere and we can compute its derivative.

$$\beta^{\delta\prime}(s) = \frac{g(s) \int_0^s V'(y) G(y)^{\frac{1}{\delta}}\,\mathrm{d}y}{\delta G(s)^{1+\frac{1}{\delta}}} \tag{10}$$

which it positive almost everywhere. Next, assume that all bidders $j \neq i$ follow the bidding strategy β^δ, and let $\beta^\delta(z)$ be bidder i's bid, whose expected utility is given by

$$U(s_i, z) := \int_0^z \left(\widetilde{V}(s_i, y) - \delta\beta^\delta(z) - (1-\delta)\beta^\delta(y)\right) g(y)\,\mathrm{d}y$$

The derivative of $U(s_i, z)$ is

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, z) &= \widetilde{V}(s_i, z) g(z) - \delta\beta^{\delta\prime}(z) G(z) - \delta\beta^\delta(z) g(z) - (1-\delta)\beta^\delta(z) g(z) \\
&= (\widetilde{V}(s_i, z) - \beta^\delta(z)) g(z) - \delta\beta^{\delta\prime}(z) G(z).
\end{aligned}$$

In equilibrium, the first order condition requires $\frac{\mathrm{d}U}{\mathrm{d}z}(s_i, s_i) = 0$. Solving this differential equation yields the stated form for β^δ. Using $G^{\frac{1}{\delta}-1}$ as the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[G(z)^{\frac{1}{\delta}}\beta^\delta(z)\right] = \left(\frac{1}{\delta} G(z)^{\frac{1}{\delta}-1}\right) \cdot (\beta^\delta(z) g(z) + \delta\beta^{\delta\prime} G(z)) = \left(\frac{1}{\delta} G(z)^{\frac{1}{\delta}-1}\right) \cdot \widetilde{V}(s_i, z) g(z).$$

Using equations (9) and (10), and the fact that $\widetilde{V}(s_i, z)$ is increasing in s_i, we obtain that $\frac{\mathrm{d}U}{\mathrm{d}z}$ is positive when $z \leq s_i$ and negative when $z \geq s_i$. Therefore, choosing $z = s_i$ maximizes i's expected payoff $U(s_i, z)$.

Writing $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$, observe that the derivative of $\delta G^{\frac{1}{\delta}}$ is $g G^{\frac{1}{\delta}-1}$. Using integration by parts and a change of variable, we obtain

$$\begin{aligned}\int_0^s V(y)g(y)G(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y &= \left[\delta V(y)G(y)^{\frac{1}{\delta}}\right]_0^s - \delta\int_0^s V'(y)G(y)^{\frac{1}{\delta}}\,\mathrm{d}y\\ &= \delta V(s)G(s)^{\frac{1}{\delta}} - \delta\int_{V(0)}^{V(s)} G(V^{-1}(y))^{\frac{1}{\delta}}\,\mathrm{d}y.\end{aligned}$$

Dividing by $\delta G^{\frac{1}{\delta}}$ gives the result. ☐

Lemma 7. *For any continuous function $\varphi : [0,\bar{v}) \to \mathbb{R}$, and for all $s \in (0,\bar{v})$, we have*

$$\lim_{\delta\to 0}\int_0^s \frac{\varphi(t)}{\delta}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$
$$\lim_{\delta\to 0}\int_0^s \ln\left(\frac{G(s)}{G(t)}\right)\frac{\varphi(t)}{\delta^2}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \varphi(s)\cdot\frac{G(s)}{g(s)}$$

where $G_k^{n-1} =: G$ and $g_k^{n-1} =: g$.

Proof. Fix $\delta > 0$, and let $\psi : (0,1] \to \mathbb{R}$ be a continuous function, such that $\psi(u) = O(1/u)$ when $u \to 0$. Using the change of variable $u = v^\delta$, we have that

$$\int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(v^\delta)v^\delta dv$$
$$\int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \ln(1/v)\psi(v^\delta)v^\delta dv.$$

Observe that for all fixed $v \in (0,1]$, and taking $\delta \to 0$, the first (resp. second) integrand converges towards $\psi(1)$ (resp., $\psi(1)\ln(1/v)$). We define the constant $M = \sup_{u\in(0,1]} u\psi(u)$, we bound the first integrand by M (resp.the second integrand by $M\ln(1/v)$), and we use the theorem of dominated convergence, which gives

$$\lim_{\delta\to 0}\int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(1)\,\mathrm{d}v = \psi(1)$$
$$\lim_{\delta\to 0}\int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u = \int_0^1 \psi(1)\ln(1/v)\,\mathrm{d}v = \psi(1)$$

To prove the lemma, observe that with the change of variable $u = \frac{G(t)}{G(s)}$, we have

$$\int_0^s \frac{\varphi(t)}{\delta}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \int_0^1 \frac{\psi(u)}{\delta}u^{\frac{1}{\delta}}\,\mathrm{d}u$$
$$\int_0^s \ln\left(\frac{G(s)}{G(t)}\right)\frac{\varphi(t)}{\delta^2}\left(\frac{G(t)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}t = \int_0^1 \ln(1/u)\frac{\psi(u)}{\delta^2}u^{\frac{1}{\delta}}\,\mathrm{d}u$$

where we define

$$\psi(u) := G(s)\cdot\frac{\varphi(G^{-1}(uG(s))}{g(G^{-1}(uG(s))}.$$

Finally, it remains to prove that $\psi(u) = O(1/u)$ when $u \to 0$. First, observe that φ is bounded on $[0, s]$. Second, observe that we have

$$\frac{u}{g(G^{-1}(uG(s)))} = \frac{1}{G(s)} \frac{G(x)}{g(x)},$$

where $x = G^{-1}(uG(s)) \to 0$. Because g is positive and integrable in 0, we have that G/g is bounded. Therefore, the overall limit when $\delta \to 0$ is equal to $\psi(1)$, which concludes the proof. $\square$

Lemma 8. *The following derivate formulas can be derived:*

$$\beta^\delta(s) = \begin{cases} V(s) & \text{if } \delta = 0 \\ V(s) - \int_0^s V'(y) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\partial(\beta^\delta(s))}{\partial s} = \begin{cases} V'(s) & \text{if } \delta = 0, s > 0 \\ \frac{g(s)}{G(s)} \int_0^s \frac{V'(y)}{\delta} \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{if } \delta > 0 \end{cases}$$

$$\frac{\delta\partial(\beta^\delta(s))}{\partial \delta} = \begin{cases} 0 & \text{if } \delta = 0 \text{ or } s = 0 \\ \int_0^s V'(y) \ln\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y & \text{for } \delta, s > 0 \end{cases}$$

$$\frac{\delta\partial^2(\beta^\delta(s))}{\partial s \partial \delta} = \frac{\delta\partial^2(\beta^\delta(s))}{\partial \delta \partial s} = \frac{-g(s)}{\delta G(s)} \int_0^s V'(y) \log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} \mathrm{d}y \text{ for } \delta, s > 0$$

Proof. In order to derive the value of these functions at points where they are not directly, defined, we will use the dominated convergence theorem multiple times.

(1) Let $s \in (0, \bar{v})$. We first look at $\beta^\delta(s) = V(s) - \int_0^s V'(y) \left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}} \mathrm{d}y$. Let $h(\delta, y)$ be the function under the integral. Clearly because G is increasing, for $y < s$ we have that $G(y)/G(s) < 1$. Hence h is dominated by V', and $\lim_{\delta \to 0} h(\delta, y) = 0$, hence by dominated convergence $\beta^\delta(s) = V(s)$ when $\delta = 0$, and the function is separately continuous over $[0, 1] \times [0, \bar{v})$.

(2) We now consider the derivative of β^δ with respect to s. Let $s \in (0, \bar{v})$. There exists $m > 0$ and $M > 0$ such that $s \in [m, M]$. We focus on the derivative of the integral part:

$$-\frac{\partial}{\partial s} V'(y) \left(\frac{G(y)}{G(s)}\right)^{1/\delta} = V'(y) \frac{g(s) G^{1/\delta+1}(y)}{G^{1/\delta}(s)} \leq V'(y) \frac{g(s)}{G(s)} \leq V'(y) \frac{\sup_{t \in [m, M]} g(t)}{G(m)},$$

where the $\sup_{t \in [m, M]} g(t)$ is finite as g is continuous. Because V' is integrable, we can use dominated convergence. Using Leibniz integral rule yields the formula. The formula as δ goes to 0 can be computed with Lemma 7.

(3) Le us now compute the derivative of $\delta\beta^\delta$ with respect to δ. Note that we do a careful derivation as we are also interested in the value of this derivative at $\delta = 0$. Let $h(\delta, y, s) =$

$\delta V'(y)(G(y)/G(s))^{1/\delta}$. We have

$$\frac{\partial h(\delta,y,s)}{\partial \delta} = V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} - V'(y)\frac{\delta}{\delta^2}\log\left(\frac{G(y)}{G(s)}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}$$
$$= V'(y)\left(\frac{G(y)}{G(s)}\right)^{1/\delta} + V'(y)\log\left(\left(\frac{G(s)}{G(y)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}.$$

The first part is again dominated by V' which is integrable, we focus on the second part: define for $0<u<w<1$ the function $\psi(u,w)=(u/w)\log(w/u)$. Note that $0<s<y<\bar{v}$ implies that for $u=G^{1/\delta}(y)$ and $w=G^{1/\delta}(s)$, we have $0<u<w<1$ as G is increasing and takes values in $(0,1)$ over $(0,\bar{v})$ by definition. Fix w, and take the derivative with respect to u: we obtain that $\psi'(u,w)=(\log(w/u)-1)/w$ which is positive as long as $u\le w/e$ and negative otherwise. The maximum of ψ for $u<w$ is at $u=w/e$ and $\psi(w/e,w)=1/e$. This shows that the second part is smaller that $V'(y)/e$ which is also integrable. Overall by dominated convergence we can invert derivative and integral: $\frac{\partial}{\partial\delta}\int h=\int\frac{\partial}{\partial\delta}h$. Using that $\delta\beta^\delta(s)=\delta V'(s)-\int_0^s h(\delta,y,s)\,\mathrm{d}y$ we can conclude by $\frac{\partial}{\partial\delta}\delta=\beta^\delta+\delta\frac{\partial}{\partial\delta}\beta^\delta$ that

$$\frac{\partial\beta^\delta(s)}{\partial\delta} = -\int_0^s\frac{V'(y)}{\delta}\ln\left(\left(\frac{G(s)}{G(y)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{\frac{1}{\delta}}\mathrm{d}y.$$

Using the same upper bound on ψ, we can show that the integrand of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ is smaller than $V'(s)/e$ which allows for domination both in small δ and small s. By dominated convergence once more, we obtain that the limit of $\delta\frac{\partial\beta^\delta(s)}{\partial\delta}$ as either δ or s go to 0 is 0.

(4) Finally, let us compute the cross derivative. The integrand of $\frac{\partial\beta^\delta(s)}{\partial s}$ is $h(\delta,y,s)=V'(y)(G(y)/G(s))^{1/\delta}$, which derivative with respect to delta is $-\frac{1}{\delta}V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)$. Because this function is continuous on the open set $(0,1)\times[0,\bar{v})$, we can as done previously apply dominated convergence to show that derivation and integral can be inverted. Therefore

$$\frac{\partial^2\delta\beta^\delta}{\partial\delta\partial s} = \frac{g(s)}{G(s)}\frac{-G^{1/\delta}(s)\frac{1}{\delta}\int_0^s V'(y)\log(G^{1/\delta}(y))G^{1/\delta}(y)\,\mathrm{d}y+\frac{1}{\delta}\log(G^{1/\delta}(s))G^{1/\delta}(s)\int_0^s V'(y)G^{1/\delta}(y)\,\mathrm{d}y}{G^{2/\delta}}$$
$$= \frac{-g(s)}{\delta G(s)}\int_0^s V'(y)\log\left(\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\right)\left(\frac{G(y)}{G(s)}\right)^{1/\delta}\mathrm{d}y.$$

□

Lemma 9. *Consider a function $\varphi:[0,1]\times(0,\bar{v})\to\mathbb{R}_+$, such that*

- *$\varphi_\delta: s\mapsto\varphi(\delta,s)$ is continuous over $(0,\bar{v})$ for all fixed $\delta\in[0,1]$,*
- *$\varphi_s:\delta\mapsto\varphi(\delta,s)$ is continuous over $[0,1]$ for all fixed $s\in(0,\bar{v})$,*
- *either all φ_s's are monotone or all φ_δ's are monotone,*

then φ is jointly continuous in δ and s.

Proof. The proof on the open set $(0,1)\times(0,\bar{v})$ is written in Kruse & Deely (1969), and directly generalizes to $\delta=0$ and $\delta=1$ given that φ is separately continuous in those points. □

Proof of Proposition 3. Monotonicity follows from the derivatives computed in Lemma 8. □

B.3 Challenging the intuition: private values and uniform pricing

For Example 2, we consider the order statistics of quantiles $F^{-1}(x)$ and not of signals s. For convenience, we define the following distribution functions and densities.

$$\widetilde{G}(x) := G_k^{n-1}(F^{-1}(x)) = 1 - (1-x)^{n-1} \qquad \widetilde{g}(x) := (n-1)(1-x)^{n-2}$$
$$\widetilde{H}(x) := G_{k-1}^{n-2}(F^{-1}(x)) = 1 - (1-x)^{n-2} \qquad \widetilde{h}(x) := (n-2)(1-x)^{n-3}$$

We choose a continuous distribution of signals, with support $[0,2]$, where each signal is given by the sum of a Bernoulli(ε) random variable and a random perturbation drawn from Beta$(1, 1/\eta)$, with $\varepsilon = 0.1/n$ and η a small constant. First, we compute the distribution function F and quantile function F^{-1} of the signal distribution. Using the law of total probabilities, we have

$$\begin{aligned}\forall s \in [0,2], \qquad F(s) &= P[\text{Bernoulli}(\varepsilon) + \text{Beta}(1,1/\eta) \le s] \\ &= P[\text{Bernoulli}(\varepsilon) = 0] \cdot P[\text{Beta}(1,1/\eta) \le s] \\ &\quad + P[\text{Bernoulli}(\varepsilon) = 1] \cdot P[\text{Beta}(1,1/\eta) \le s-1].\end{aligned}$$

Simplifying this expression depending on the value of s, we get

$$\forall s \in [0,2], \qquad F(s) = \begin{cases} \varepsilon \cdot (1 - (1-s)^{1/\eta}) & \text{if } s \le 1, \\ \varepsilon + (1-\varepsilon) \cdot (1 - (2-s)^{1/\eta}) & \text{if } s \ge 1. \end{cases}$$

Then, computing piece-by-piece the inverse of F, we obtain

$$\forall x \in [0,1], \qquad F^{-1}(x) = \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x \le \varepsilon, \\ 2 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^{\eta} & \text{if } x \ge \varepsilon. \end{cases}$$

A bidder with quantile $x \in [0,1]$ bids (truthfully) their signal $F^{-1}(x)$ in the uniform price auction ($\delta = 0$), which we write as $b_\eta^0(x) := F^{-1}(x) = \mathbb{1}[x \ge \varepsilon] + \gamma_\eta(x)$, where

$$\forall x \in [0,1], \qquad \gamma_\eta(x) := \begin{cases} 1 - \left(1 - \frac{x}{\varepsilon}\right)^{\eta} & \text{if } x < \varepsilon, \\ 1 - \left(1 - \frac{x-\varepsilon}{1-\varepsilon}\right)^{\eta} & \text{if } x \ge \varepsilon. \end{cases}$$

For mixed auctions with $\delta > 0$, the equilibrium bid function is given by Proposition 2. Letting $b_\eta^\delta(x) := \beta^\delta(F^{-1}(x))$ denote the equilibrium bid of a bidder with quantile $x \in [0,1]$, we have

$$b_\eta^\delta(x) = \frac{\int_0^{F^{-1}(x)} V(s) g_k^{n-1}(s) G_k^{n-1}(s)^{\frac{1}{\delta}-1} \, \mathrm{d}s}{\delta G_k^{n-1}(F^{-1}(x))} = \frac{\int_0^x F^{-1}(y) \widetilde{g}(y) \widetilde{G}(y)^{\frac{1}{\delta}-1} \, \mathrm{d}y}{\delta \widetilde{G}(x)},$$

where we used the change of variable $y = F(s)$. Finally, using the additive form of F^{-1} we write the equilibrium bid function as $b^\delta_\eta(x) = b^\delta_0(x) + \xi^\delta_\eta(x)$, where

$$\forall x \in [0,1], \qquad b^\delta_0(x) := \frac{\int_\varepsilon^x \widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(F^{-1}(x))} = \begin{cases} 0 & \text{if } x < \varepsilon \\ 1 - \left(\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\right)^{\frac{1}{\delta}} & \text{if } x \geq \varepsilon \end{cases}$$

$$\xi^\delta_\eta(x) := \frac{\int_0^x \gamma_\eta(y)\widetilde{g}(y)\widetilde{G}(y)^{\frac{1}{\delta}-1}\,\mathrm{d}y}{\delta \widetilde{G}(x)}$$

Next, we define the function $\phi^\delta_\eta(x) := F^{-1}(x) - \delta b^\delta_\eta(x)$, the utility of a winning bidder as a function of their quantile. Denoting WEV^δ_η the *winners' empirical variance* in a δ-mixed auction with noise level η, we write

$$\forall \delta \in [0,1],\ \forall \eta > 0, \qquad \mathrm{WEV}^\delta_\eta = \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi^\delta_\eta(x_1) - \phi^\delta_\eta(x_2))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right].$$

where $\mathbf{x}$ is a random vector of quantiles, with n independent coordinates distributed uniformly on $[0,1]$. For every $x \in [0,1)$, observe that $\gamma_\eta(x)$ and $\xi^\delta_\eta(x)$ converge towards 0 when taking η arbitrarily small, and thus $\phi^\delta_\eta(x)$ converges towards $\phi^\delta_0(x) := \mathbb{1}[x \geq \varepsilon] - \delta b^\delta_0(x)$. Therefore, WEV^δ_η converges towards WEV^δ_0, defined by

$$\begin{aligned} \forall \delta \in [0,1], \quad \mathrm{WEV}^\delta_0 :=& \mathbb{E}_{\mathbf{x}}\left[\frac{((\mathbb{1}[x_1 \geq \varepsilon] - \delta b^\delta_0(x_1)) - (\mathbb{1}[x_2 \geq \varepsilon] - \delta b^\delta_0(x_2)))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] \\ =& \mathbb{E}_{\mathbf{x}}\left[\frac{(\phi^\delta_0(x_1) - \phi^\delta_0(x_2))^2}{2} \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \lim_{\eta \to 0} \mathrm{WEV}^\delta_\eta. \end{aligned}$$

Proof of Proposition 5. We are now equipped to prove the proposition. We write $WEV^\delta_0 = \mathbb{E}_{\mathbf{x}}\left[\phi^\delta_0(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x})\right] - \mathbb{E}_{\mathbf{x}}\left[\phi^\delta_0(x_1)\phi^\delta_0(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right]$, with

$$\mathbb{E}_{\mathbf{x}}\left[\phi^\delta_0(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_0^1 \phi^\delta_0(x)^2 G(x)\,\mathrm{d}x$$

$$\mathbb{E}_{\mathbf{x}}\left[\phi^\delta_0(x_1)\phi^\delta_0(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-2}\int_0^1 \left(\int_t^1 \phi^\delta_0(x)\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t$$

We next compute these quantities for uniform and discriminatory pricing. For uniform pricing ($\delta = 0$) we have that $\phi^0_0(x) = \mathbb{1}[x \geq \varepsilon]$. We derive

$$\mathbb{E}_{\mathbf{x}}\left[\phi^0_0(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \widetilde{G}(x)\,\mathrm{d}x = \frac{n(1-\varepsilon) - (1-\varepsilon)^n}{n-1}$$

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}\left[\phi^0_0(x_1)\phi^0_0(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^\varepsilon (1-\varepsilon)^2 \widetilde{h}(t)\,\mathrm{d}t + \frac{n}{n-2}\int_\varepsilon^1 (1-t)^2 \widetilde{h}(t)\,\mathrm{d}t \\ &= \frac{n(1-\varepsilon)^2 - 2(1-\varepsilon)^n}{n-2} \end{aligned}$$

and finally

$$
\begin{aligned}
\mathrm{WEV}_0^0 &= \frac{n(1-\varepsilon)-(1-\varepsilon)^n}{n-1} - \frac{n(1-\varepsilon)^2-2(1-\varepsilon)^n}{n-2} \\
&= \frac{n\left[(1-\varepsilon)^n+(1-\varepsilon)(\varepsilon(n-1)-1)\right]}{(n-1)(n-2)} \\
&\le \frac{(\varepsilon n)^2/2}{n} = \frac{0.005}{n}
\end{aligned}
$$

For discriminatory pricing ($\delta = 1$) we have that $\phi_0^1(x) = \mathbb{1}[x \ge \varepsilon] - b_0^1(x) = \mathbb{1}[x \ge \varepsilon]\frac{G(\varepsilon)}{G(x)}$. We will use the following bounds:

$$
\begin{aligned}
\int_\varepsilon^1 \frac{1}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_\varepsilon^1 \frac{1}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_\varepsilon^1 \sum_{i=0}^\infty (1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \sum_{i=0}^\infty \frac{(1-\varepsilon)^{(n-1)i+1}}{(n-1)i+1} \ge (1-\varepsilon) + \sum_{i=1}^\infty \frac{(1-\varepsilon)^{ni}}{ni} \\
&\ge (1-\varepsilon) + \frac{1}{n}\sum_{i=1}^\infty \frac{0.9^i}{i} = 1 - \frac{0.1}{n} - \frac{\ln(0.1)}{n} \ge 1 + \frac{2.2}{n} \\
\int_0^1 \frac{x}{\widetilde{G}(x)}\,\mathrm{d}x &= \int_0^1 \frac{x}{1-(1-x)^{n-1}}\,\mathrm{d}x = \int_0^1 \sum_{i=0}^\infty x(1-x)^{(n-1)i}\,\mathrm{d}x \\
&= \frac{1}{2} + \sum_{i=1}^\infty \frac{1}{((n-1)i+1)((n-1)i+2)} \le \frac{1}{2} + \frac{1.65}{n^2} \qquad\qquad \text{(when } n \ge 5\text{)}
\end{aligned}
$$

Next, we write

$$
\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)^2 \,|\, x_1 > Y_{k+1}(\mathbf{x})\right] = \frac{n}{n-1}\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)^2}{\widetilde{G}(x)}\,\mathrm{d}x \ge \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1+\frac{2.2}{n}\right)
$$

and

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2) \,|\, x_1, x_2 > Y_{k+1}(\mathbf{x})\right] &= \frac{n}{n-2}\int_0^\varepsilon \left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&\quad + \frac{n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 h(t)\,\mathrm{d}t \\
&= \underbrace{\frac{n\widetilde{H}(\varepsilon)}{n-2}\left(\int_\varepsilon^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 + \frac{n}{n-2}\left[\left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)^2 H(t)\right]_\varepsilon^1}_{=0} \\
&\quad + \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_t^1 \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t \\
&= \frac{2n}{n-2}\int_\varepsilon^1 \left(\int_\varepsilon^x \frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(t)}\widetilde{H}(t)\,\mathrm{d}t\right)\frac{\widetilde{G}(\varepsilon)}{\widetilde{G}(x)}\,\mathrm{d}x
\end{aligned}
$$

Next, we will use the upper bound $\widetilde{H}(t)/\widetilde{G}(t) \le 1$, which is nearly tight as $\widetilde{H}(t)/\widetilde{G}(t)$ is increasing, and has the limit $(n-2)/(n-1)$ when $t \to 0$.

$$\mathbb{E}_{\mathbf{x}}\left[\phi_0^1(x_1)\phi_0^1(x_2)\,|\,x_1,x_2 > Y_{k+1}(\mathbf{x})\right] \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\int_{\varepsilon}^{1}\frac{x}{\widetilde{G}(x)}\,\mathrm{d}x \leq \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2}+\frac{1.65}{n^2}\right)$$

Finally, we obtain

$$\begin{aligned}
\mathrm{WEV}_0^1 &\geq \frac{n\widetilde{G}(\varepsilon)^2}{n-1}\left(1+\frac{2.2}{n}\right) - \frac{2n\widetilde{G}(\varepsilon)^2}{n-2}\left(\frac{1}{2}+\frac{1.65}{n^2}\right) && \text{(when } n \geq 5\text{)}\\
&= n\widetilde{G}(\varepsilon)^2\left(\frac{2.2}{n(n-1)} - \frac{1}{(n-1)(n-2)} + \frac{3.3}{n^2(n-2)}\right)\\
&\geq \frac{0.01}{n} && \text{(when } n \geq 4\text{)}
\end{aligned}$$

□

B.4 Proving the main theorems

Proof of Lemma 2. Let $s_i \leq s_j = s_i + \epsilon$ for some $\epsilon > 0$. Then

$$\begin{aligned}
\Leftrightarrow \quad u_i(s_i, \boldsymbol{s}_{-i}) &\leq u_j(s_j, \boldsymbol{s}_j)\\
\Leftrightarrow \quad (1-c)s_i - \delta\beta^\delta(s_i) &\leq (1-c)s_j - \delta\beta^\delta(s_j)\\
\Leftrightarrow \quad (1-c)(s_j - s_j) &\geq \delta(\beta^\delta(s_j) - \beta^\delta(s_i))
\end{aligned}$$

Dividing by $s_j - s_i$ and taking letting $\epsilon \to 0$ concludes the proof. □

Proof of Proposition 7. First, we prove that pairwise differences is locally decreasing in δ. Let s_i, s_j with $s_i \geq s_j$ denote the signals of two winning bidders and $\varphi^\delta(s) := (1-c)s - \delta\beta^\delta(s)$. Note that because of Proposition 3 (2), monotone ex-post utility holds for all $\delta \leq \bar{\delta}$. For all δ_1, δ_2, $0 \leq \delta_1 \leq \delta_2 \leq \bar{\delta}$, we have

$$\begin{aligned}
&|u^{\delta_1}(s_i) - u^{\delta_1}(s_j)| \geq |u^{\delta_2}(s_i) - u^{\delta_2}(s_j)| && (11)\\
\Leftrightarrow\quad &|\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \geq |\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_i)| && (12)\\
\Leftrightarrow\quad &-\delta_1\left(\beta^{\delta_1}(s_i) - \beta^{\delta_1}(s_j)\right) \geq -\delta_2\left(\beta^{\delta_2}(s_i) - \beta^{\delta_2}(s_j)\right) && (13)
\end{aligned}$$

For the final equivalence, observe that monotone ex-post utility together with Proposition 3 (1) implies that $\frac{\delta}{1-c}\beta^\delta$ is non-expansive, allowing to remove the absolute value in Eq. (12). Proposition 3 (2) guarantees that Eq. (13) holds. As the ex-post difference in utilities (Eq. (11)) is decreasing in δ, so is its expectation. To establish global monotonicty on $[0, \bar{\delta}]$, note that if $\bar{\delta}\frac{\partial\beta^{\bar{\delta}}}{\partial s} \leq 1-c$ then it also holds for any $\delta < \bar{\delta}$ by Proposition 3 (2), concluding the proof. □

Proof of Proposition 8. Let $u_i^\delta(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the δ-mixed auction, and let $u_i^U(s_i, \boldsymbol{s}_{-i})$ denote bidder i's utility in the uniform price auction. Now let $i, j \in [n]$ be two winning bidders. As above, β^δ (resp. β^U) denotes the symmetric equilibrium bid function in the δ-mixed (resp. uniform price) auction. Let $Y_{k+1}(\boldsymbol{\beta})$ denote the first rejected bid. Then, canceling out $(1-\delta)Y_{k+1}(\boldsymbol{\beta})$, we have

$$\begin{aligned}
|u_i^\delta - u_j^\delta| &= |(v_i(s_i, \boldsymbol{s}_{-i}) - \delta\beta^\delta(s_i)) - (v_j(s_j, \boldsymbol{s}_{-j}) - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_i)) - ((1-c)s_j + \frac{c}{n}\sum_{k\in[n]} s_k - \delta\beta^\delta(s_j))| \\
&= |((1-c)s_i - \delta\beta^\delta(s_i)) - ((1-c)s_j - \delta\beta^\delta(s_j))| \\
&= |\varphi^\delta(s_i) - \varphi^\delta(s_j)|,
\end{aligned}$$

where $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. It also holds that

$$|u_i^U - u_j^U| = |(v_i(s_i, \boldsymbol{s}_{-i}) - Y_{k+1}(\boldsymbol{\beta})) - (v_j(s_j, \boldsymbol{s}_{-j}) - Y_{k+1}(\boldsymbol{\beta}))| = |(1-c)(s_i - s_j)|.$$

We will now show that $\frac{\varphi^\delta}{1-c}$ is a non-expansive mapping. Note that φ^δ can be increasing or decreasing, so we need to show that $|\frac{\partial\varphi^\delta}{\partial s}| \le 1-c$. We have $\frac{\partial\varphi^\delta}{\partial s} = 1 - c - \delta\frac{\partial\beta^\delta}{\partial s}$. As β^δ is increasing in s, $|\frac{\partial\varphi^\delta}{\partial s}| \le 1-c$ holds whenever $\delta\frac{\partial\beta^\delta}{\partial s} \le 2(1-c)$. Therefore

$$|u_i^\delta - u_j^\delta| = |\varphi^\delta(v_i) - \varphi^\delta(v_j)| \le |(1-c)(s_i - s_j)| = |u_i^U - u_j^U| \tag{14}$$

Taking the square of Eq. (14) we obtain the result point-wise, for each pair of winning signals s_i and s_j and, taking the expectation, the theorem follows. ☐

Theorem 5. For a given common value component c, consider two δ-mixed auctions for $\delta_1 \le \delta_2$ and suppose the equilibrium bidding functions β^δ satisfies $\delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c)$ for all signals $s \in [0, \bar{v})$. Then, WEV is lower for the δ_2-mixed auction than for the δ_1 one.

Proof. Let $\varphi^\delta(s) = (1-c)s - \delta\beta^\delta(s)$. We have $u_i^\delta(\mathbf{s}) - u_j^\delta(\mathbf{s}) = \varphi^\delta(s_i) - \varphi^\delta(s_j)$. Let $\delta_1 \le \delta_2$. By the generalized Cauchy mean value Theorem, we have that there exists $\xi \in [s_i, s_j]$ such that

$$|\varphi^{\delta_2}(s_i) - \varphi^{\delta_2}(s_j)| \left|\frac{\partial\varphi^{\delta_1}(\xi)}{\partial s}\right| = |\varphi^{\delta_1}(s_i) - \varphi^{\delta_1}(s_j)| \left|\frac{\partial\varphi^{\delta_2}(\xi)}{\partial s}\right|.$$

Hence if $|\frac{\partial\varphi^{\delta_2}}{\partial s}|/|\frac{\partial\varphi^{\delta_1}}{\partial s}| \le 1$ then we have lower WEV for the δ_2 mixed auction. We have the following chain of equivalences:

$$\begin{aligned}
& \left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right| \le \left|\frac{\partial\varphi^{\delta_2}(s)}{\partial s}\right|, \forall s \in (0, \bar{v}) \\
\iff\quad & \left|(1-c) - \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s}\right| \le \left|(1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}\right|, \forall s \in (0, \bar{v}) \\
\iff\quad & \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} - (1-c) \le (1-c) - \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s}, \forall s \in (0, \bar{v}) \\
\iff\quad & \delta_1\frac{\partial\beta^{\delta_1}(s)}{\partial s} + \delta_2\frac{\partial\beta^{\delta_2}(s)}{\partial s} \le 2(1-c), \forall s \in (0, \bar{v}),
\end{aligned}$$

where the third equations comes from the monotonicity of $\delta\frac{\partial\beta^\delta}{\partial s}$ in δ from Proposition 3. ☐

Proof of Lemma 5. To prove Lemma 5, we will use properties of log-concave distributions from Bagnoli & Bergstrom (2005). Namely their Theorems 1 and 3 state together that log-concavity of a density f implies log-concavity of the corresponding cdf F and of the complementary cdf $1-F$, and that log-concavity of F or $1-F$ imply log-concavity of respectively $\int_0^s F$ or $\int_s^{\bar{v}} F$, where $\bar{v}$ is the upper limit of the support of f (either a constant or $+\infty$). Additionally, we also have that the product of two log-concave functions is log-concave also. Using the above properties, we have that F and $1-F$ are log-concave.

Moreover, alternative expression for the order statistics are given, e.g., in Fisz (1965).

$$G_m^n(s) = \frac{n!}{(n-m)!(m-1)!} \int_0^{F(s)} t^{n-m}(1-t)^{m-1} dt$$

and

$$g_m^n(s) = \frac{n!}{(n-m)!(m-1)!} F(s)^{n-m}(1-F(s))^{m-1} f(s). \tag{15}$$

Thus, the order statistics density g, given by Eq. (15), is a product of F, $1-F$, and f, and g as well as the corresponding cdf G are also log-concave. Furthermore, $G^{\frac{1}{\delta}}$ is log-concave because $\log(G^{\frac{1}{\delta}}) = \delta \log(G)$. Finally, we remark that $G^{\frac{1}{\delta}}$ is right-continuous non-decreasing by composition with $x \mapsto x^{\frac{1}{\delta}}$, which is continuous non-decreasing, and $G^{\frac{1}{\delta}}(0) = 0$, as well as $G^{\frac{1}{\delta}}(\bar{v}) = 1$ (if $\bar{v} = \infty$, the equality is understood as a limit). Therefore $G^{\frac{1}{\delta}}$ is a cdf, and applying one last time Bagnoli & Bergstrom (2005), we obtain that $\int_0^s G^{\frac{1}{\delta}}$ is log-concave. ☐

B.5 Proving the bound on bid function slopes

Proof of Lemma 3. We first rewrite $\tilde{v}(x, y)$ for $c = 1$ in terms of all the order-statistics of s_{-i}.

$$\begin{aligned}
\tilde{v}(x,y) &= \mathbb{E}[v(s_i, s_{-i}) \mid s_i = x, Y_k = y] \\
&= \mathbb{E}[\frac{1}{n} \sum_{j \in [n]} s_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n} \sum_{\substack{j \in [n], \\ j \neq i}} s_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \mathbb{E}[\frac{1}{n} \sum_{j \in [n-1]} Y_j \mid s_i = x, Y_k = y] && \text{(Ordering the signals)} \\
&= \frac{x}{n} + \frac{y}{n} + \mathbb{E}[\frac{1}{n} \sum_{\substack{j \in [n-1], \\ j \neq k}} Y_j \mid s_i = x, Y_k = y] \\
&= \frac{x}{n} + \frac{y}{n} + \frac{1}{n} \sum_{j=1}^{k-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y] + \frac{1}{n} \sum_{j=k+1}^{n-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y]
\end{aligned}$$

Note that the previous decomposition is similar the equilibrium bid in an English auction given that k bidders have dropped out in Goeree & Offerman (2003). However, we offer a careful derivation in the multi-unit setting of our model. We now use Theorem 2.4.1 and Theorem 2.4.2 from Arnold et al. (2008) on the conditional distribution of order statistics. They state that, for $j < k$, the distribution of Y_j given $Y_k = y$ is the same as the distribution of the j-th order statistic of $k-1$ independent samples of the original distribution left-truncated at y, and we denote Z_j^l a random variable drawn according to this distribution. Hence, for $j<k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^l]$. Similarly for $j>k$ we have that the distribution of Y_j given $Y_k = y$ is the same as the distribution of the $j-k$-th order statistic of $n-k-1$ independent samples of the original distribution right-truncated at y, and we denote by Z_j^r a random variable drawn according to this distribution. Hence, for $j>k$, $\mathbb{E}[Y_j \mid Y_k = y] = \mathbb{E}[Z_j^r]$. Notice that summing all order statistics drawn from some samples recovers exactly the sum of original samples. Thus we obtain

$$\sum_{j=1}^{k-1} \mathbb{E}[Y_j \mid s_i = x, Y_k = y] = \sum_{j=1}^{k-1} \mathbb{E}[Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} Z_j^l] = \mathbb{E}[\sum_{j=1}^{k-1} s_j \mid \forall j \in [k-1], s_j \geq y] = \sum_{j=1}^{k-1} \mathbb{E}[s_j \mid s_j \geq y].$$

The same can be done for the Z_j^r. Finally, the s_j are iid and thus have identical conditional expectations. We obtain

$$\begin{aligned} \tilde{v}(x,y) &= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\mathbb{E}[s_j \mid s_j \leq y] + \frac{k-1}{n}\mathbb{E}[s_j \mid s_j \geq y] && (16)\\ &= \frac{x}{n} + \frac{y}{n} + \frac{n-k-1}{n}\frac{\int_0^y tf(t)\,\mathrm{d}t}{F(y)} + \frac{k-1}{n}\frac{\int_y^{\bar{v}} tf(t)\,\mathrm{d}t}{1-F(y)}, && (17)\end{aligned}$$

which readily yields a formula for $V(s) = \tilde{v}(s,s)$. Clearly, the above function is well defined and differentiable on the open support of F.

We now examine the derivative of $V(s)$ and prove that $V'(s) \leq 1$. First, we consider the derivatives of the two ratios with an integral in the numerator in Eq. (17). First, by integration by parts, we have

$$\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = \frac{[tF(t)]_0^s - \int_0^s F(t)\,\mathrm{d}t}{F(s)} = s - \frac{\int_0^s F(t)\,\mathrm{d}t}{F(s)},$$

and using that for positive random variables $\int_0^{\bar{v}} tf(t)\,\mathrm{d}t = \int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t = \mathbb{E}[s_i] < \infty$, which guarantees convergence of the integral, we have that

$$\begin{aligned}\frac{\int_s^{\bar{v}} tf(t)\,\mathrm{d}t}{1-F(s)} = \frac{\mathbb{E}[s_i] - \int_0^s tf(t)\,\mathrm{d}t}{1-F(s)} &= \frac{\int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t - sF(s) + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)}\\ &= \frac{\int_0^{\bar{v}}(1-F(t))\,\mathrm{d}t + s(1-F(s)) - s + \int_0^s F(t)\,\mathrm{d}t}{1-F(s)}\\ &= s + \frac{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\end{aligned}$$

Now, taking derivatives, we have

$$\frac{\partial}{\partial s}\frac{\int_0^s tf(t)\,\mathrm{d}t}{F(s)} = 1 - \frac{F(s)^2 - f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2} = \frac{f(s)\int_0^s F(t)\,\mathrm{d}t}{F(s)^2}.$$

By a similar argument as in the proof of Lemma 4, using log-concavity of f, the above derivative is bounded by 1. Taking the derivative of the second ratio, we have

$$\frac{\partial}{\partial s}\left(s+\frac{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{1-F(s)}\right)=1+\frac{-(1-F(s))^2+f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2}=\frac{f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}{(1-F(s))^2}. \tag{18}$$

To derivative of $\log(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t)$:

$$\frac{\partial^2}{(\partial s)^2}\log(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t)=\frac{\partial}{\partial s}\frac{-(1-F(s))}{\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t}=\frac{f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t-(1-F(s))^2}{\left(\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t\right)^2}. \tag{19}$$

Eq. (19) is negative iff $f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2\leq 1$. This means that the log-concavity of $\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t$ is equivalent to Eq. (18) being smaller than 1. As the log-concavity of $\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t$ follows from the log-concavity of f and $(1-F)$ (Bagnoli & Bergstrom 2005, Theorem 3), $f(s)\int_s^{\bar{v}}(1-F(t))\,\mathrm{d}t/(1-F(s))^2\leq 1$ is implied. Finally, using the above derivatives it is clear that $V'(s)>0$, and

$$V'(s)\leq\frac{2}{n}+\frac{n-k-1}{n}\cdot 1+\frac{k-1}{n}\cdot 1=1$$

□

Proof of Lemma 4. Let us compute the second derivative of the logarithm of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$:

$$\begin{aligned}\frac{\partial^2\log\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)}{(\partial s)^2}&=\frac{\partial}{\partial s}\left(\frac{G^{\frac{1}{\delta}}(s)}{\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}\right)\\&=\frac{\frac{1}{\delta}g(s)G^{\frac{1}{\delta}-1}(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y-G^{\frac{2}{\delta}}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\\&=\frac{G^{\frac{1}{\delta}-1}(s)}{\left(\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y\right)^2}\left(\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y-G^{\frac{1}{\delta}+1}(s)\right).\end{aligned}$$

Notice that the left-hand fraction is always positive. Hence log-concavity of $\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y$ is equivalent to $\frac{1}{\delta}g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y-G^{\frac{1}{\delta}+1}(s)$ being negative. The latter is equivalent to

$$1\geq\frac{g(s)\int_0^s G^{\frac{1}{\delta}}(y)\,\mathrm{d}y}{\delta G^{\frac{1}{\delta}+1}(s)}=\frac{\partial\beta^\delta(s)}{\partial s}.$$

□

Proof of Proposition 6. While $\sup_{[0,\bar{v})}\frac{\partial\beta^\delta}{\partial s}$ can be difficult to compute analytically even for simple distributions, it is sometimes possible to compute $\sup_{[0,\bar{v})}V'(s)$. For the uniform distribution, we have $\sup_{[0,\bar{v})}V'(s)=1-c\frac{n-2}{2n}$. Thus, using the same argument as in the proof of Theorem 3, it follows that $\delta^*(c)\geq\frac{2n(1-c)}{2n-c(n-2)}\rightarrow_{n\to\infty}\frac{(1-c)}{1-c/2}$. For the exponential distribution, we have $\sup_{[0,\bar{v})}V'(s)=1-c(\frac{1}{2}-\frac{k+1}{2n})$, and thus $\delta^*(c)\geq\frac{2n(1-c)}{2n-c(n-(k+1))}\rightarrow_{n\to\infty}\frac{(1-c)}{1-c/2}$. □

B.6 Numerical experiments

Lemma 10. *Suppose an auction is a* winners pay auction. *Then we can write* $E_{\boldsymbol{s}}[u_1 \, |1 \text{ wins}] = \frac{n}{k} E_{\boldsymbol{s}}[u_1]$, $E_{\boldsymbol{s}}[u_1^2 | 1 \text{ wins}] = \frac{n}{k} E_{\boldsymbol{s}}[u_1^2]$, *and* $E_{\boldsymbol{s}}[u_1 u_2 | 1 \text{ and } 2 \text{ win}] = \frac{n(n-1)}{k(k-1)} E_{\boldsymbol{s}}[u_1 u_2]$.

Proof. Observe that we have

$$\mathbb{E}_s[u_1^2 \mid 1 \text{ and } 2 \text{ win}] = \mathbb{E}_s[u_1^2 \mid 1 \text{ wins}] = \frac{\mathbb{E}[u_1^2]}{\mathbb{P}[1 \text{ wins}]} = \frac{n}{k} \cdot \mathbb{E}[u_1^2] \tag{20}$$

$$\mathbb{E}_s[u_1 u_2 \mid 1 \text{ and } 2 \text{ win}] = \frac{\mathbb{E}[u_1 u_2]}{\mathbb{P}[1 \text{ and } 2 \text{ win}]} = \frac{n(n-1)}{k(k-1)} \cdot \mathbb{E}[u_1 u_2] \tag{21}$$

□

B.7 Discussion

Proof of Proposition 10. We define the probability that i wins $q_i(s_i) := \mathbb{P}_{\boldsymbol{s}_{-i}}[i \text{ wins}]$. Recall that $b^D(s_i)$ denotes the equilibrium bid in the pay-as-bid auction. Consider any standard auction, characterised by a payment rule $(p_1(s), \ldots, p_n(s))$. Revenue equivalence implies that

$$q_i(s_i) \cdot b^D(s_i) = \mathbb{E}_{\boldsymbol{s}_{-i}}[b^D(s_i) \cdot \mathbb{1}[i \text{ wins}]] = \mathbb{E}_{\boldsymbol{s}_{-i}}[p_i(s)]. \tag{22}$$

In particular, note that if p_i is chosen to be the uniform pricing rule, this formula can be used to compute $b^D(s_i)$. Now define the ex-post surplus $u_i(s_i, \boldsymbol{s}_{-i}) := v(s_i) \cdot \mathbb{1}[i \text{ wins}] - p_i(s)$. We write

$$u_i(s_i, \boldsymbol{s}_{-i}) = \underbrace{\mathbb{1}[i \text{ wins}] \cdot (v(s_i) - b^D(s_i))}_{u_i^D(s)} + \underbrace{\mathbb{1}[i \text{ wins}] \cdot b^D(s_i) - p_i(s)}_{\delta(s)}. \tag{23}$$

Now, observe that by revenue equivalence we have $\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})] = 0$ for all s_i. We write

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 \tag{24}$$

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] = \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] + 2 \underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})]}_{\geq 0} + \underbrace{\mathbb{E}_{\boldsymbol{s}_{-i}}[\delta(s_i, \boldsymbol{s}_{-i})^2]}_{\geq 0} \tag{25}$$

To show that the extra terms are non-negative, notice that $\delta(s_i, \boldsymbol{s}_{-i})^2 \geq 0$, and that

$$\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i}) \cdot \delta(s_i, \boldsymbol{s}_{-i})] = \underbrace{(v(s_i) - b^D(s_i))}_{\geq 0} \cdot \underbrace{(q_i(s_i) \cdot b^D(s_i) - \mathbb{E}_{\boldsymbol{s}_{-i}}[\mathbb{1}[i \text{ wins}] \cdot p_i(s)])}_{\geq \mathbb{E}[\delta(s)] = 0} \tag{26}$$

Therefore, putting everything together, we obtain

$$\begin{aligned} \mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})] &= \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]^2 && (27) \\ &\geq \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})^2] - \mathbb{E}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})]^2 && (28) \\ &= \mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i^D(s_i, \boldsymbol{s}_{-i})] && (29) \end{aligned}$$

Finally, observe that an auction which minimize the interim variance also minimize the ex-ante variance. Denoting by u_i the utility of a bidder in the pay-as-bid auction, the law of total variance states

$$\mathrm{Var}_s[u_i] = \mathbb{E}_{s_i}[\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]] + \mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]. \tag{30}$$

By the revenue equivalence theorem, we know that $\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]$ is the same for all standard auctions, hence $\mathrm{Var}_{s_i}[\mathbb{E}_{\boldsymbol{s}_{-i}}[u_i]]$ is also the same for all standard auctions (it only depends on the distribution of the signals).

The interim variance $\mathrm{Var}_{\boldsymbol{s}_{-i}}[u_i(s_i, \boldsymbol{s}_{-i})]$ is minimal point-wise (in s_i) for all standard auctions, hence is also minimal in expectation. Therefore, the ex-ante variance is minimal in the pay-as-bid auction among standard auctions. □