Are Digital Monopolies Exploitative?

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Abstract

In markets with multiple divisible goods and budget-constrained buyers, competitive equilibrium may not be efficient. We study the notion of constrained efficiency that, in the presence of budgets, is implemented by prices. Firstly, we show that competitive equilibrium maximizes constrained social welfare and the constrained efficient allocation can be supported by competitive prices. Secondly, if buyers have linear substitutes valuations, the unique constrained efficient outcome also maximises the seller's revenue. Our proof develops a novel characterisation of the set of feasible prices at which demand does not exceed supply.

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JEL codes: D40, D44, D50

1 Introduction

Competition in digital markets is a pressing issue for regulators around the world. Platforms like Amazon, Google, Instagram, JD.com, and Tmall use online advertising as a source of revenue, based on the vast data they collect from customers. When these platforms sell advertising space to businesses, they can exert nearly monopolistic market power. A first-order regulatory concern is whether these ad allocations lead to inefficient and potentially exploitative outcomes, and to what extent the monopolists' objectives deviate from social welfare considerations. However, this question is complicated by the presence of budget constraints. In classic economic theory, a socially optimal outcome is not necessarily implemented by a competitive equilibrium (CE) when agents are budget-constrained.

The inefficiency of CE in budget-constrained markets is well documented and remedies have been developed: e.g., Che et al. (2013) characterize the socially optimal mechanism that achieves

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an efficient assignment of a homogeneous supply to a finite mass of agents relying on in-kind and cash subsidies. Although efficient, such mechanisms are too sophisticated to be practical in fast-paced and large-scale environments such as ad auctions where, adding further complexity, buyers have trade-offs between substitute products.[1] Moreover, in situations where the seller has no or limited control over resale, e.g., in markets for financial assets,[2] a mechanism involving random assignment and cash transfers may attract speculators in pursuit of resale profits. Instead, a price-based mechanism with a competitive assignment of goods is often preferable.

Restricting to price-only mechanisms in the presence of budget constraints, it is natural to demand a concept of efficiency. We consider the notion of *constrained efficiency*, which allows us to characterize a constrained socially optimal allocation and prices that achieve maximal social welfare subject to respecting the agents' budgets and demands. Firstly, we show that maximal constrained social welfare is attained by CE (which respects the agents' choice of good), and by CE only. This result holds for any preferences with quasi-linear utilities. Secondly, we show that, in the quasi-Fisher setting with linear valuations, the unique market outcome that is constrained efficient also maximises revenue. Our second result suggests that a digital monopolist can set prices which are both revenue maximizing and socially optimal within the feasible bounds of advertiser budgets, hence need not be exploitative. However, it does not imply that the monopolist always sets socially optimal prices, since not all revenue-maximizing prices are also socially optimal.

The notion of constrained efficiency provides a compelling and practical benchmark for efficiency in markets with budgets where CE allocations are not classically efficient. Constrained social welfare maximisation is especially important in settings where it would be hard to accept breaking or circumventing budget constraints for the sake of efficiency. This is the case in our two motivating examples, digital monopolistic ad auctions, and financial asset exchanges. In an ad auction, although a nameless small company may derive a high value from an ad placement, only their *ability to pay* can be relevant to a for-profit digital platform. Similarly, in the exchange of financial assets, the budget of a buyer represents the nominal amount limit of their asset that is to be exchanged, and thus a regulator would not want to grant an allocation of substitute assets exceeding this limit.[3]

Our market contains multiple buyers and one seller, who supplies multiple divisible goods in finite quantities. Agents have quasi-linear utilities. The seller's costs are normalized to zero. Each buyer has a budget constraint, i.e., an upper limit on the amount of money they can spend. We consider monotone and concave valuations, as well as linear valuations. The latter correspond to a fixed per-unit value for each good. The seller sets a uniform market price for each of the goods supplied, and each buyer demands one, several, or none of the goods at the given prices. The buyers' demands constrains the feasibility of outcomes; we say that an outcome consisting of prices and allocations (of goods to buyers) is feasible if supply is not exceeded and the allocated goods are demanded by the receiving buyers. In setting the market prices, the seller can have different objectives (possibly imposed by a regulator): We consider constrained

[1] See Conitzer et al. (2021); Dobzinski et al. (2012); Murray et al. (2020).

[2] See Klemperer (2018)

[3] In general, the analysis of social welfare that transcends agents' budget constraints may be more relevant when these budget constraints are linked to individuals with unequal endowments or socially indispensable businesses with unequal access to capital markets.

social welfare maximization, where the buyers' aggregate values are maximized among feasible outcomes. In particular, respecting the buyers' budgets and demands constrains the social welfare objective. We also consider competitive equilibrium, where the seller clears the market subject to maintaining feasibility. Finally, the seller can maximize her revenue, retaining some of her supply to raise prices.

Contributions.

Our contribution is twofold. Firstly, we show that analogues of the first and second welfare theorems hold for constrained social welfare and competitive equilibrium, i.e., any competitive equilibrium is constrained-efficient, and any feasible, constrained-efficient outcome is a competitive equilibrium (Theorem 1).[4] Secondly, we show that the two objectives of maximising revenue and finding a competitive equilibrium coincide. The unique market-clearing prices are buyer-optimal among all revenue-maximising prices in the sense that they maximise the quantities allocated to each buyer, and thus maximise buyer utilities. To arrive at this result, we introduce a novel geometric object, the *feasible region*, which is defined as the set of prices at which, for every good, either the market clears (through buyers' demand, respecting budgets) or there is excess supply. First we show that this non-convex region has elementwise-smallest prices. We then complete the argument by showing that these prices are constrained efficient and maximise the seller's revenue.

Applications.

The quasi-Fisher market is a simple model of many ad auctions as they occur in practice (Conitzer et al., 2022). When businesses compete for advertising space online, the decision of which publisher (i.e., product) to bid for is non-trivial. It is intuitive to choose an advertising budget and state demand in terms of "limit market prices" for multiple, distinct products. The seller or platform assigns to each buyer those products that yield the highest value for money to them. Budget-constrained buyers also appear in exchanges for financial assets. For example, Klemperer (2018) introduces the "arctic auction", originally developed for the government of Iceland, who planned to use this auction to exchange blocked accounts for other financial assets such as cash or bonds. Buyers could submit a budget and their trade-offs between different assets, and the auction was solved to maximise the seller's revenue. Quasi-Fisher markets can be interpreted as a special case of this auction, as we discuss in Section 6. Further applications include debt restructuring and the (re-)division of firms between shareholders (see also Klemperer 2018; Baldwin et al. 2024).

1.1 Related literature

The practical relevance of our market is highlighted in Conitzer et al. (2022), whose setting is particularly inspired by online ad auctions. The basic properties of the market and budget-constrained buyers are identical to ours, although they consider only linear valuations. Contrasting our setting, each divisible good is sold in an independent, single-unit first-price

[4] We note that any feasible allocation is inherently tied to prices in our model, as those determine feasibility. Thus, we state that the constrained-efficient outcome *is* indeed a CE.

auction, in which only the highest bidders can win a positive quantity. Conitzer et al. (2022) introduce the solution concept of *first price pacing equilibria* (FPPE), in which the submitted bids correspond to the buyers' values scaled (uniformly for each buyer) by a pacing multiplier.[5] Interestingly, this at first sight unrelated auction procedure can also be solved using the modified Eisenberg-Gale convex programme of Chen et al. (2007). Moreover, Conitzer et al. (2022) show that the unique FPPE corresponds to a competitive equilibrium in the sense of our setting; that is, in the overarching market for all goods with budget-constrained, quasi-linear buyers. While they show that the FPPE is revenue-maximal among all budget-feasible pacing multipliers and corresponding allocations, our work implies that the FPPE is indeed revenue-maximising in the entire market.

Several papers have studied optimal mechanism and auction design in the presence of budget constraints. A crucial distinction between those mechanisms and ours is the focus on incentives that arise from the presence and extent of private information. For example, Laffont and Robert (1996) characterize the optimal auction, and Maskin (2000) designed the constrained-efficient mechanism when budgets are known. The optimal and constrained-efficient mechanism in different settings with private budgets and values, including a single buyer, multiple buyers, or a population of buyers, are developed, for example, in Che and Gale (2000); Che et al. (2013); Pai and Vohra (2014); Richter (2019). In contrast to this literature, our approach is more practical, focusing on a market with a supply of differentiated goods in which the seller is restricted to a price-only mechanism (see also Section 1), and buyers behave non-strategically.

The market with budget-constrained buyers with quasi-linear utilities and linear valuations has also appeared as a quasi-Fisher market (Murray et al., 2020), as it can be considered a generalisation of standard Fisher markets (Brainard and Scarf, 2005). Quasi-Fisher markets have appeared in various guises, mainly in a computational context. The first results on quasi-linear Fisher markets were developed by Chen et al. (2007), who showed that competitive equilibria can be computed in polynomial time, with several others to follow.[6] In standard, linear Fisher markets, buyers spend their entire budget at any market prices, and so revenue is constant at all prices. In contrast, buyers with quasi-linear utility and budget constraints spend nothing when prices are unacceptably high. Hence, the notion of maximising revenue becomes a viable objective for the seller to pursue. Contrasting the literature on quasi-Fisher markets, our paper considers not only competitive equilibrium, but also maximising revenue. Our unifying result on constrained efficiency, competitive equilibrium, and revenue demonstrates the importance of the quasi-linear setting from a theoretical perspective as well as in applications.

Organisation.

The remainder of the paper is structured as follows. In Section 2, we motivate the research with several examples where buyers' values are not linear, and the coincidence of revenue-optimality and constrained efficiency fails. Section 3 describes the model, and Section 4 defines

[5] An FPPE is defined as a set of pacing multipliers (one for each buyer) and allocations that satisfy the allocation and pricing rule of standard first-price auctions, as well as budget feasibility, supply feasibility, market clearing for demanded goods, and 'no unnecessary pacing', i.e. a buyer's multiplier equals one if she has unspent budget.

[6] A more comprehensive overview of computational contributions on linear and quasi-linear Fisher markets is given in the version of this article that appeared at WINE'23.

market outcomes and their properties. In Section 4.1, we state the analogue of the welfare theorems, and in Section 5, we prove the coincidence of constrained efficiency and revenue-optimality in the linear case. Section 6 establishes the connection between quasi-Fisher markets and the Arctic auction, and Section 7 concludes.

Notation. For any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we write $\boldsymbol{v} \cdot \boldsymbol{w}$ for their dot product, and $\boldsymbol{v} \leq \boldsymbol{w}$ when the inequality holds element-wise. For any $j \in \{1, \dots, n\}$, $\boldsymbol{e}^j$ denotes the n-dimensional indicator vector with $e^j_j = 1$ and e^j_k for all $k \neq i$. We also define $\boldsymbol{e}^0 := \boldsymbol{0}$.

2 Examples: Efficiency, constrained efficiency, and revenue

We illustrate in different examples the objectives of maximizing efficiency, constrained efficiency, and the seller's revenue. Example 1 shows that competitive equilibrium is not necessarily fully efficient with budget constraints, but constrained efficient. In Example 2, we show that with diminishing marginal values, the seller's revenue is not maximised in a constrained-efficient allocation, and Example 3 illustrates, in a setting with two goods and constant marginal values, the coincidence of competitive equilibrium and revenue-optimality.

Example 1. Consider a market with two buyers and a seller with one good in unit supply and zero costs. The buyers' per-unit values are $v^1 > v^2$ and their budgets are $\beta^1 > \beta^2$, and we assume $\beta^1 + \beta^2 \leq v^2$. The fully efficient allocation gives the entire unit to the first buyer. However, due to their budgets, for any price $p \leq v^2$, each agent i demands quantity b^i/p. To clear the market, the price must satisfy $\frac{b^1}{p} + \frac{b^2}{p} = 1$, i.e., $p = b^1 + b^2$. The allocation to agent 1 is therefore $\frac{b^1}{b^1+b^2} < 1$.

The CE in Example 1 is not efficient. However, we define notion of constrained efficiency, which maximises social welfare respecting the agents' budgets. The budget constraints intrinsically link constrained efficiency to prices, as only in the context of prices budgets are meaningful. We say that an outcome, consisting of an allocation and prices, is feasible if it respects the agents' demands, budgets, and the supply constraint. The constrained-efficient outcome is defined as maximizing social welfare among all feasible outcomes. In the example, social welfare is maximised by assigning as much as possible to buyer 1. However, if we cannot violate agents' budgets, increasing the quantity beyond $\frac{b^1}{b^1+b^2}$ requires decreasing the price. This would again increase the demand of buyer 2 and violate the supply constraint. Thus, the competitive equilibrium is constrained efficient. The seller's revenue in the CE is $b^1 + b^2$. This is clearly optimal, as the seller cannot hope to obtain more money than is available in the market.

While competitive equilibrium (if it exists) is always constrained efficient, with any number of goods, agents, and irrespective of the valuation, the coincidence of revenue-optimality and constrained efficiency is more fragile. The following example shows that, with diminishing marginal values, it may fail.

Example 2. The seller provides a single good with supply $s \in \mathbb{R}_+$ at zero costs. There is one buyer with a continuous valuation $v : \mathbb{R} \to \mathbb{R}$ for the good and quasi-linear utility $u(p, x) = v(x) - px$, where p denotes the unit price of the good, and a budget $\beta \in \mathbb{R}_+ \cup \{\infty\}$. We aim to find outcomes (p, x) that maximise constrained efficiency and the seller's revenue, respectively. The buyer demands quantity x at price p if $x \in \arg\max_x v(x) - px$ such that $px \leq \beta$.

Maximising revenue with a price-only mechanism, the seller sets a price in anticipation of the buyer's demand. Thus, the seller's revenue is maximised at $(p, x) \in \arg\max_{p,x} px$ such that $x \in \arg\max_x v(x) - px$, $px \leq \beta$, and $x \leq s$. Social welfare, on the other hand, is maximised at $(p, x) \in \arg\max_{p,x} v(x)$ such that $x \in \arg\max_x v(x) - px$, $px \leq \beta$, and $x \leq s$.

To derive allocations and prices, we first have to make some assumptions on the buyer's valuation. A typical assumption are diminishing marginal values. In that case, the buyer always demands a quantity such that their marginal utility at this quantity is zero. Assuming $v'(s) \cdot s \leq \beta$, social welfare is maximised if $v'(x) = p$ and $x = s$, and markets clear. However, if the seller were allowed to adjust the price, anticipating the buyer's demand, she might be able to extract more revenue (and not sell the entire supply). The following proposition shows that this is indeed the case for all strongly concave value functions if the buyer's budget is large. The proof is given in Appendix 7, together with Example 4, which further illustrates the case with large budgets.

Proposition 1. *Let v be differentiable and strongly concave with parameter m for some $m > 0$, and let $\beta = \infty$. Then there exists some supply $s \in \mathbb{R}$ so that revenue is not maximised at the market-clearing price.*

When the buyer has a finite budget, the situation is more intricate. Indeed, supply can only lie in the finite interval $X := [0, \max\{x \mid xv'(x) \leq \beta\}]$. We define $\tilde{x}$ implicitly by $\tilde{x}v'(\tilde{x}) = \beta$.

Proposition 2. *Suppose the buyer has a strongly concave valuation v with parameter m and a finite budget β. If supply s is contained in X with $v'(s) < ms$, or if $m > \frac{v'(\tilde{x})}{\tilde{x}}$, then revenue is not maximised at the market-clearing price.*

We prove Proposition 2 in Section 7. In other words, for any strongly concave valuation, we can find a combination of budget and supply such that the maximisers of the constrained social welfare and the revenue maximisation problem do not coincide. For example, we may require the valuation to be sufficiently concave relative to the budget. Example 4 shows a specific valuation for a single good for which revenue and welfare do not coincide. Note that the budget constraint never binds, so this example also applies to general quasi-linear utilities without budget constraints.

In light of the above propositions, we consider the class of constant marginal values, i.e. $v(x) = vx$ for some scalar v.

Proposition 3. *Suppose the buyer has valuation $v(x) = vx$ and budget β. Then the seller's revenue is maximised at market-clearing prices.*

The above proposition is straightforward. Constrained social welfare is maximised at $p = \min\{v, \frac{\beta}{s}\}$ and $x = s$. Because the buyer demands a bundle that exhausts their entire budget, the seller cannot extract more revenue, and the constrained-efficient allocation and price are also revenue-optimal.

An immediate question to ask is whether this reasoning extends to more general environments and preferences. The answer we present in this paper is affirmative: if any number of buyers have quasi-linear, budget-constrained utility and linear values for any number of goods, then seller-optimaal revenue and constrained efficiency are attained at the unique set of elementwise-minimal prices. This result, however, is not immediate. In the following, we illustrate the difficulty in another simple example with two goods.

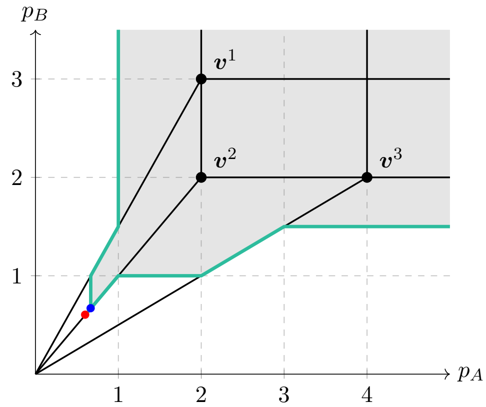

Figure 1: The feasible region in price space corresponding to Example 3.

Example 3. Two goods, A and B, are for sale with a supply of $s_A = 3$ and $s_B = 2$. There are three buyers $1, 2$, and 3 with the following marginal values: $\boldsymbol{v}^1 = (v_A^1, v_B^1) = (2, 3)$, $\boldsymbol{v}^2 = (v_A^2, v_B^2) = (2, 2)$, and $\boldsymbol{v}^3 = (v_A^3, v_B^3) = (4, 2)$. Utilities are quasi-linear, i.e. $u^i(x, p) = \sum_j (v_j^i - p_j)x_j$ for buyer i. Each buyer has a budget of $\beta^1 = \beta^2 = \beta^3 = 1$. We can allocate the 3 units of A and 2 units of B among the three buyers to maximise either revenue or social welfare, but need to respect individual demand. It is not hard to check that at given prices (p_A, p_B) each buyer i will demand a good $j \in \arg\max_{j=1,2} \frac{v_j^i}{p_j}$ if $v_j^i \geq p_j$. Any good k with $v_k^i < p_k$ will never be demanded by buyer i. This kind of individual demand can be easily represented in price space (more detail in Section 6). At some prices, aggregate demand is too large to be satisfied by supply. Prices at which aggregate demand does *not* exceed supply are called *feasible*. The set of feasible prices makes up the *feasible region*. The bids (black dots) and the feasible region (in grey) are illustrated in Fig. 1. Note that the feasible region also includes a short line segment between $\boldsymbol{p}^* := (\frac{3}{5}, \frac{3}{5})$ (red dot) and $\boldsymbol{p}' := (\frac{2}{3}, \frac{2}{3})$ (blue dot). The feasible region has the key property that for any pair of feasible price vectors, the elementwise minimum of them also belongs to the region.

At prices $(\frac{3}{5}, \frac{3}{5})$, buyer 1 demands $\frac{5}{3}$ of B, buyer 3 demands $\frac{5}{3}$ of A, and buyer 2 demands $x_A^2 \in [0, \frac{5}{3}]$ copies of A and $\frac{5}{3} - x_A^2$ of B. With supply $(s_A, s_B) = (3, 2)$, set $x_A^2 = \frac{4}{3}$ to clear the market. It is easy to check that indeed any prices on $[p^*, p']$ induce a feasible allocation. All prices $[p^*, p']$ are revenue-maximising. However, only $\boldsymbol{p}^*$ clears the market.

3 The Market

We have m buyers $[m] := \{1, \ldots, m\}$, n divisible goods $[n] := \{1, \ldots, n\}$, and a (divisible) numeraire, denoted 0, that we call 'money'. Let $[n]_0 = \{0, \ldots, n\}$. A *bundle*, typically denoted $\boldsymbol{x}$ or $\boldsymbol{y}$, is an n-dimensional vector of non-negative reals whose entry $x_j \geq 0$ for each $j \in [n]$ denotes the *quantity* of good j. The seller has a *supply bundle* $\boldsymbol{s} \in \mathbb{R}_+^n$ that she wishes to sell, partially or completely, by setting uniform, non-negative market prices $\boldsymbol{p} \in \mathbb{R}^n_+$ for the goods. The price of money is fixed at $p_0 = 1$.

Each buyer $i \in [m]$ has a *budget* β^i and a valuation v^i mapping every bundle $\boldsymbol{x}$ to a value $v^i(\boldsymbol{x}) \in \mathbb{R}$. We assume that v^i is concave and monotone increasing. Buyers have *quasi-linear utility* $u^i(\boldsymbol{x}, \boldsymbol{p}) := v^i(\boldsymbol{x}) + \beta^i - \boldsymbol{p} \cdot \boldsymbol{x}$ from receiving bundle $\boldsymbol{x}$ at prices $\boldsymbol{p}$. A buyer's *demand* at prices $\boldsymbol{p}$ consists of the bundles $\boldsymbol{x}$ that maximise her utility $u^i(\cdot, \boldsymbol{p})$, subject to not exceeding her budget (expressed by the budget constraint $\boldsymbol{p} \cdot \boldsymbol{x} \leq \beta^i$). This leads to the budget-constrained *demand correspondence* $D^i(\boldsymbol{p}) := \arg\max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{p} \cdot \boldsymbol{x} \leq \beta^i} (v^i(\boldsymbol{x}) - \boldsymbol{p} \cdot \boldsymbol{x})$.

In Sections 5 and 6, we will consider the special case that agents have linear valuations $v^i(\boldsymbol{x}) = \boldsymbol{v}^i \cdot \boldsymbol{x}$ for some vector $\boldsymbol{v}^i$.

4 Market Outcomes

The seller solves the auction by determining a market *outcome* $(\boldsymbol{p}, (\boldsymbol{x}^i)_{i \in [m]})$, which consists of *market prices* $\boldsymbol{p}$ and an *allocation* $\boldsymbol{x}^i$ to each buyer $i \in [m]$. So x^i_j denotes the quantity of good j allocated to bidder i. For brevity, we drop the subscript $i \in [m]$ when denoting an allocation $(\boldsymbol{x}^i)$ to buyers, as the number of buyers is fixed throughout. Market prices $\boldsymbol{p}$ and an allocation $(\boldsymbol{x}^i)$ together specify the amount $\boldsymbol{p} \cdot \boldsymbol{x}^i$ that each buyer i spends. When determining a market outcome, the seller wishes not to exceed supply and to allocate each buyer a bundle they demand at the chosen market prices. We call such outcomes *feasible*.

Definition 1. An outcome $(\boldsymbol{p}, (\boldsymbol{x}^i)_{i \in [m]})$ is *feasible* if:

(i) the aggregate allocation of each good j does not exceed its supply, so $\sum_{i \in [m]} \boldsymbol{x}^i \leq \boldsymbol{s}$;

(ii) each buyer demands his allocation at $\boldsymbol{p}$, so $\boldsymbol{x}^i \in D^i(\boldsymbol{p})$ for all $i \in [m]$.

Prices are *feasible* if they can be extended to a feasible outcome with some allocation. In Section 5, we will study the geometry of the feasible region, which consists of the set of all feasible prices. Conversely, we say that an allocation $(\boldsymbol{x}^i)$ is *supported* by some prices $\boldsymbol{p}$ if $(\boldsymbol{p}, (\boldsymbol{x}^i))$ form a feasible outcome. This means that the allocation can be realised by some prices $\boldsymbol{p}$ at which each agent i demands bundle $\boldsymbol{x}^i$.

Definition 2. We say that prices $\boldsymbol{p}$ are *feasible* if there exists an allocation $(\boldsymbol{x}^i)_{i \in [m]}$ so that $(\boldsymbol{p}, (\boldsymbol{x}^i)_{i \in [m]})$ is a feasible outcome. The *feasible region* consists of all feasible prices.

The seller in our market typically wishes to maximise her revenue or to clear the market. This contrasts with the regulator, who may wish to impose outcomes that prioritise efficiency.

The *revenue* of an outcome $(\boldsymbol{p}, (\boldsymbol{x}^i))$ is given by $\sum_{i \in [m]} \boldsymbol{p} \cdot \boldsymbol{x}^i$. Under the *revenue objective*, the seller wishes to find a feasible outcome that maximises revenue. In order to achieve this, the seller may prefer to set prices at which her supply is not cleared.

In contrast, a *competitive equilibrium* consist of a feasible outcome that clears the market. The existence of competitive equilibrium is guaranteed in our market, as it can be seen as a special case of the Arrow-Debreu model (Arrow and Debreu, 1954; Chen et al., 2007).

Definition 3. An outcome $(\boldsymbol{p}, (\boldsymbol{x}^i))$ is a *competitive equilibrium* if it is feasible and the market clears, so $\sum_{i \in [m]} \boldsymbol{x}^i = \boldsymbol{s}$.

4.1 Constrained Efficiency

Regulators are typically interested in ensuring the efficiency of markets. The *social welfare* of an allocation $(\boldsymbol{x}^i)$ is $\sum_{i\in[m]} v^i(\boldsymbol{x}^i)$, and the social welfare of an outcome is the welfare of its allocation. In markets with unlimited spending power, the fundamental theorems of welfare economics state that competitive equilibrium allocations are efficient, that is, they maximise social welfare. When we introduce budgets, however, competitive equilibrium allocations may not maximise social welfare, as welfare-maximising allocations may not be supported by any market prices. Example 1 in Section 2 illustrates this in our model.

Instead, we consider the maximum social welfare achievable by an allocation $(\boldsymbol{x}^i)$ that is supported by some market prices; i.e., by a market outcome. We call a feasible outcome *constrained efficient* if it maximises social welfare among all feasible outcomes.[7] Our notion of constrained efficiency can thus be considered a sensible variation of efficiency in the presence of budgets and the requirement in price-only mechanisms that allocations are supported by prices.

Definition 4. An outcome $(\boldsymbol{p},(\boldsymbol{x}^i))$ is *constrained efficient* if it is feasible and maximises social welfare, $\sum_{i\in[m]} v^i(\boldsymbol{x}^i)$, among all feasible outcomes.

Our first main result establishes that analogues to the first and welfare theorems with respect to competitive equilibrium and constrained efficiency hold in our market: competitive equilibrium maximises constrained efficiency, and any feasible outcome maximising constrained efficiency constitutes a competitive equilibrium.

Theorem 1. *An outcome $(\boldsymbol{q},(\boldsymbol{y}^i))$ is a competitive equilibrium if and only if it is constrained efficient.*

Proof. Fix a competitive equilibrium $(\boldsymbol{p},(\boldsymbol{x}^i))$ and let $(\boldsymbol{q},(\boldsymbol{y}^i))$ be any feasible outcome. For convenience, define the aggregately allocated bundles $\boldsymbol{x} = \sum_{i\in[m]} \boldsymbol{x}^i$ and $\boldsymbol{y} = \sum_{i\in[m]} \boldsymbol{y}^i$. As $(\boldsymbol{p},(\boldsymbol{x}^i))$ is feasible, quasi-linearity of demand implies $v^i(\boldsymbol{x}^i) - \boldsymbol{p}\cdot\boldsymbol{x}^i \geq v^i(\boldsymbol{y}^i) - \boldsymbol{p}\cdot\boldsymbol{y}^i$ for every agent i, so

$$\sum_{i\in[m]} v^i(\boldsymbol{x}^i) - \sum_{i\in[m]} v^i(\boldsymbol{y}^i) \geq \sum_{i\in[m]} \boldsymbol{p}\cdot(\boldsymbol{x}^i - \boldsymbol{y}^i) = \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}). \tag{1}$$

As $(\boldsymbol{p},(\boldsymbol{x}^i))$ is a competitive equilibrium, it clears the market, so $\boldsymbol{x} = \boldsymbol{s}$. The feasibility of $(\boldsymbol{q},(\boldsymbol{y}^i))$ implies $\boldsymbol{y} \leq \boldsymbol{s} = \boldsymbol{x}$, so we see that $\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) \geq 0$. As $(\boldsymbol{q},(\boldsymbol{y}^i))$ was an arbitrary feasible outcome, (1) thus implies that $(\boldsymbol{p},(\boldsymbol{x}^i))$ maximizes social welfare among all feasible outcomes.

We now show that a constrained efficient outcome must also be a competitive equilibrium. Suppose that $(\boldsymbol{q},(\boldsymbol{y}^i))$ maximizes social welfare among all feasible outcomes, so $\sum_{i\in[m]} v^i(\boldsymbol{y}^i) = \sum_{i\in[m]} v^i(\boldsymbol{x}^i)$. By (1), we have

$$0 = \sum_{i\in[m]} v^i(\boldsymbol{x}^i) - \sum_{i\in[m]} v^i(\boldsymbol{y}^i) \geq \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}). \tag{2}$$

If $(\boldsymbol{q},(\boldsymbol{y}^i))$ is not a competitive equilibrium, we have $\boldsymbol{y} \leq \boldsymbol{x}$ with strict inequality $y_j < x_j$ for at least one good j. Thus, $\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y}) > 0$, a contradiction. □

[7] The term 'constrained efficiency' is motivated by the fact that the allocations we consider must be supported by a price to form a feasible outcome (cf. Definition 1), and feasibility is constrained by budgets.

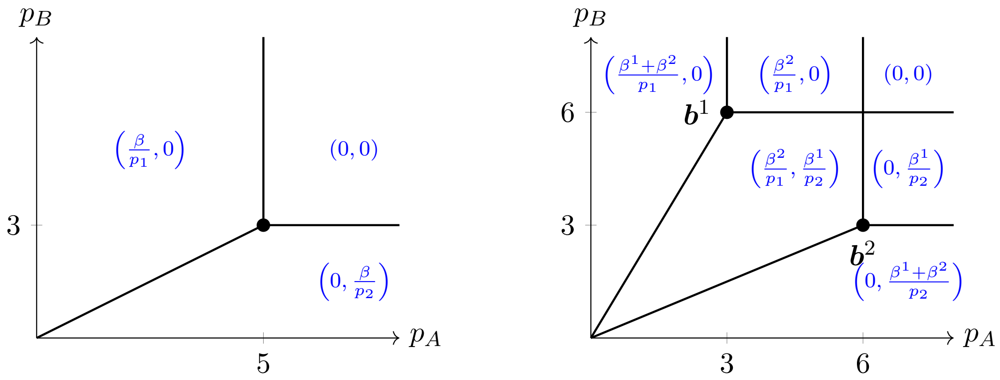

Figure 2: The demand of buyers with quasi-linear utilities, linear valuations and budget constraints, which divides price space into convex regions. In each region, we specify the bundle demanded, which depends on prices $\boldsymbol{p}$. **Left:** The demand of a single buyer with linear values $\boldsymbol{v} = (5, 3)$ and budget β leads to three regions. **Right:** The aggregate demand of two buyers, one with values $\boldsymbol{v}^1 = (3, 6)$ and budget β^1, and the other with values $\boldsymbol{v}^2 = (6, 3)$ and budget β^2.

5 The revenue-welfare coincidence

We now assume that each agent has a linear valuation. In this linear market setting, we show that revenue and social welfare are both maximised over feasible outcomes at the unique competitive equilibrium prices. By Theorem 1, these prices thus also uniquely support a constrained optimal outcome.

The linear valuation $v^i(\boldsymbol{x}) = \boldsymbol{v}^i \cdot \boldsymbol{x}$ of each agent i is expressed by a *valuation vector* $\boldsymbol{v}^i$ representing her linear per-unit values $v^i_j \geq 0$ for each good j. The per-unit value of money is $v^j_0 = 1$ for each buyer. A buyer's utility function is then $u^i(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{v}^i \cdot \boldsymbol{x} + \beta^i - \boldsymbol{p} \cdot \boldsymbol{x}$, and her *demand correspondence* is $D^i(\boldsymbol{p}) = \arg\max_{\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i} (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$. Fig. 2 illustrates the demand that arises from linear valuations.

Theorem 2. *In our market with linear valuations, competitive equilibrium prices are unique. Moreover, all competitive equilibrium outcomes maximise revenue and are uniquely constrained efficient.*

Theorem 1 thus implies that social welfare is maximised only at these unique competitive equilibrium prices. In contrast, revenue may also be maximised at higher prices, at which the seller only sells a subset of her supply.

In order to prove Theorem 2, we first consider the feasible region, which consists of the set of all feasible prices (cf. Definition 2). A key property of the feasible region is that it forms a lower semi-lattice. That is, there exists an elementwise-minimal price vector $\boldsymbol{p}^*$ that is dominated by all other feasible prices. This is illustrated in Fig. 1. We develop these geometric insights in Section 5.1. We then prove, in Section 5.2, that revenue is maximised, and the market is cleared, at these prices $\boldsymbol{p}^*$.

For these proofs, we also make use of an alternative characterisation of the demand correspondence of an agent with linear valuation. At any given prices $\boldsymbol{p}$, let $J^i(\boldsymbol{p}) := \arg\max_{j \in [n]_0} \frac{v^i_j}{p_j}$ be the set of goods that maximise *bang-per-buck*. (Note that $J^i(\boldsymbol{p})$ contains the money good 0 if $\max_{j \in [n]_0} \frac{v^i_j}{p_j} = 1$, as $v^i_0 = p_0 = 1$ by definition.) Lemma 1 makes the observation that demanded bundles only contain quantities of goods that maximise bang-per-buck; in other words, if $\boldsymbol{x}$ is a bundle demanded by buyer i at $\boldsymbol{p}$, then $x_j > 0$ implies $j \in J^i(\boldsymbol{p})$. Moreover, any demanded bundle is the convex combination of the 'extremal' bundles that arise when the entire budget is spent on a single demanded good in $J^i(\boldsymbol{p})$. The convex hull of S, i.e., the set of all convex combinations of elements in S, is written as $\operatorname{conv} S$.

Lemma 1. *For any buyer i with linear valuation v^i and budget β^i, we have $D^i(\boldsymbol{p}) = \operatorname{conv}\{\frac{\beta^i}{p_j}\boldsymbol{e}^j \mid j \in J^i(\boldsymbol{p})\}$.*

Proof. Recall that the buyer's demand is $D^i(\boldsymbol{p}) = \arg\max_{\boldsymbol{x}\in\mathbb{R}^n_{\geq 0}, \boldsymbol{x}\cdot\boldsymbol{p}\leq\beta^i}(\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{x}$. The space $\{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} \mid \boldsymbol{x} \cdot \boldsymbol{p} \leq \beta^i\}$ of all bundles that the agent can afford at prices $\boldsymbol{p}$ is a closed polyhedron spanned by vertices $\boldsymbol{y}^0 := \boldsymbol{0}$ and $\boldsymbol{y}^j := \frac{\beta^i}{p_j}\boldsymbol{e}^j$ for each good $j \in [n]$. The fundamental theorem of linear algebra tells us that any bundle $\boldsymbol{x} \in D^i(\boldsymbol{p})$ demanded at $\boldsymbol{p}$ is the convex combination of the vertices $\boldsymbol{y}^j$ that maximise $f(\boldsymbol{y}) := (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \boldsymbol{y}$. As $f(\boldsymbol{y}^j) = (\boldsymbol{v}^i - \boldsymbol{p}) \cdot \frac{\beta^i}{p_j}\boldsymbol{e}^j = \beta^i(\frac{v^i_j}{p_j} - 1)$ for each $j \in [n]_0$, we see that $\boldsymbol{y}^j$ maximises $f(\boldsymbol{y})$ iff $j \in J^i(\boldsymbol{p}) = \arg\max_{j\in[n]} \frac{v^i_j}{p_j}$. □

In Section 6, we will also see that the arctic product-mix auction introduces a bidding language which starts from this definition of demand to characterise a more general class of preferences.

5.1 Elementwise-minimal feasible prices

Recall from Definition 2 that prices $\boldsymbol{p}$ are feasible if they can be extended with an allocation $(\boldsymbol{x}^i)$ to a feasible outcome $(\boldsymbol{p}, (\boldsymbol{x}^i))$. We now show that the set of feasible prices form a lower semi-lattice. In particular, there exists a special price vector $\boldsymbol{p}^*$ that is elementwise smaller than all other feasible prices (so that $\boldsymbol{p}^* \leq \boldsymbol{p}$ for all feasible $\boldsymbol{p}$).[8]

Proposition 4. *The feasible region has a unique elementwise-minimal price vector $\boldsymbol{p}^*$.*

In order to develop the proof of Proposition 4, we first define the elementwise minimum of two prices. For any two price vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, let $\boldsymbol{p} \wedge \boldsymbol{q}$ denote their *element-wise minimum* defined as $(\boldsymbol{p} \wedge \boldsymbol{q})_i = \min\{p_i, q_i\}$. The following lemma is central to our proof of Proposition 4.

Lemma 2. *If $\boldsymbol{p}$ and $\boldsymbol{p}'$ are feasible, then so is their element-wise minimum $\boldsymbol{p} \wedge \boldsymbol{p}'$.*

Fix feasible prices $\boldsymbol{p}$ and $\boldsymbol{q}$ with element-wise minimum $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{q}$, and let $(\boldsymbol{x}^i)$ and $(\boldsymbol{y}^i)$ respectively denote allocations that extend $\boldsymbol{p}$ and $\boldsymbol{q}$ to feasible outcomes $(\boldsymbol{p}, (\boldsymbol{x}^i))$ and $(\boldsymbol{q}, (\boldsymbol{y}^i))$. In order to prove Lemma 2, we construct a third allocation $(\boldsymbol{z}^i)$ and show that $(\boldsymbol{r}, (\boldsymbol{z}^i))$ is a feasible outcome. We first define the set of goods A in which $\boldsymbol{p}$ is strictly dominated by $\boldsymbol{q}$, and its complement B, so $A = \{j \in [n] \mid p_j < q_j\}$ and $B = \{j \in [n] \mid p_j \geq q_j\}$. Then our allocation $(\boldsymbol{z}^i)$ is given by

$$\boldsymbol{z}^i = \begin{cases} \boldsymbol{y}^i & \text{if buyer } i \text{ demands some good } j \in B \text{ at } \boldsymbol{r}, \\ \boldsymbol{x}^i & \text{otherwise.} \end{cases} \tag{3}$$

In order to prove that $(\boldsymbol{r}, (\boldsymbol{z}^i))$ is feasible, we first state a technical lemma that establishes the connection between a buyer's demand at $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$.

[8] Efficient inner-point methods (e.g., Chen et al. (2007)) can be used to find $\boldsymbol{p}^*$.

Lemma 3. *Suppose buyer i demands good $j \in A$ at $\boldsymbol{r}$. Then she also demands j at $\boldsymbol{p}$ and $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$. Similarly, suppose buyer i demands $j \in B$ at $\boldsymbol{r}$. Then she also demands j at $\boldsymbol{q}$, and $J^i(\boldsymbol{q}) \subseteq J^i(\boldsymbol{r})$. Moreover, we have $J^i(\boldsymbol{q}) \subseteq B$.*

Proof. Fix a buyer i who demands good $j \in A$ at $\boldsymbol{r}$. As $p_j = r_j$, this implies $\frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_k^i}{r_k} \geq \frac{v_k^i}{p_k}$ for all goods $k \in [n]_0$. The first inequality holds due to the definition of demand, and the second inequality follows from $r_j \leq p_j, \forall j \in [n]_0$. Hence, the buyer demands good j at $\boldsymbol{p}$. For the second claim that $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r})$, fix a good $k \in J^i(\boldsymbol{p})$. Then we have $\frac{v_k^i}{r_k} \geq \frac{v_k^i}{p_k} \geq \frac{v_j^i}{p_j} = \frac{v_j^i}{r_j} \geq \frac{v_l^i}{r_l}$ for all goods $l \in [n]_0$. The first inequality holds due to $r_k \leq p_k$, and the second and third inequalities follow from the fact that the buyer i demands k at $\boldsymbol{p}$ and j at $\boldsymbol{r}$. Hence, if the buyer demands good k at $\boldsymbol{p}$, then they demand k at $\boldsymbol{r}$.

Now suppose that the buyer demands $j \in B$ at $\boldsymbol{r}$. The proof of the first claim is identical to the case $j \in A$. We prove the last claim that $J^i(\boldsymbol{q}) \subseteq B$. Suppose, for contradiction, that i demands a good $k \in A$ at $\boldsymbol{q}$, and good $j \in B$ at $\boldsymbol{r}$. This implies $\frac{v_k^i}{q_k} < \frac{v_k^i}{p_k} = \frac{v_k^i}{r_k} \leq \frac{v_j^i}{r_j} = \frac{v_j^i}{q_j}$, in contradiction to the fact that k is demanded at $\boldsymbol{q}$. $\square$

We can now prove Lemma 2.

Proof of Lemma 2. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two feasible prices and $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{q}$ denote their element-wise minimum. As above, $(\boldsymbol{x}^i)$ and $(\boldsymbol{y}^i)$ are allocations that extend $\boldsymbol{p}$ and $\boldsymbol{q}$ to feasible outcomes, and $(\boldsymbol{z}^i)$ is defined as in (3). We now prove that $(\boldsymbol{r}, (\boldsymbol{z}^i))$ is a feasible outcome, and so that it satisfies the two criteria in Definition 1. We can partition buyers into two sets: the set $\mathcal{B} \subseteq [m]$ of buyers that demand a good in B at $\boldsymbol{r}$, and the set $\mathcal{A} = [m] \setminus \mathcal{B}$ of buyers that do not. Note that, by Lemma 3, the buyers in $\mathcal{B}$ demand only goods in B at $\boldsymbol{q}$ and, by definition, the buyers in $\mathcal{A}$ demand only goods in A at $\boldsymbol{p}$. We thus observe: in outcome $(\boldsymbol{p}, (\boldsymbol{x}^i))$, each buyer $i \in \mathcal{A}$ is only allocated quantities of goods in A (so $\sum_{i \in \mathcal{A}} x_j^i > 0$ only if $j \in A$); similarly, in outcome $(\boldsymbol{q}, (\boldsymbol{y}^i))$, every buyer $i \in \mathcal{B}$ only receives quantities of goods in B (so $\sum_{i \in \mathcal{B}} y_j^i > 0$ only if $j \in B$).

First we show that $(\boldsymbol{z}^i)$ satisfies condition (i) of Definition 1, i.e., $\sum_{i \in [m]} z_j^i \leq s_j$ for all goods $j \in [n]$. Fix some good $j \in A$. Then by definition of $(\boldsymbol{z}^i)$, we have $z_j^i = \sum_{i \in \mathcal{A}} x_j^i + \sum_{i \in \mathcal{B}} y_j^i$. Recalling that $\sum_{i \in \mathcal{B}} y_j^i = 0$ and that $(\boldsymbol{p}, (\boldsymbol{x}^i))$ is feasible, we get $\sum_{i \in [m]} z_j^i \leq \sum_{i \in [m]} x_j^i \leq s_j$. This implies that $(\boldsymbol{r}, (\boldsymbol{z}^i))$ does not over-allocate any goods $j \in A$. Analogously, we can show that $(\boldsymbol{z}^i)$ does not over-allocate any goods $j \in B$ by recalling that $\sum_{i \in \mathcal{A}} x_j^i = 0$ for any good $j \in B$. As $A \cup B = [n]$, we have shown that $(\boldsymbol{r}, \boldsymbol{z}^i)$ satisfies condition (i) in Definition 1.

Next we argue that $(\boldsymbol{r}, (\boldsymbol{z}^i))$ satisfies condition (ii) of Definition 1. Consider first a buyer $i \in \mathcal{A}$. By definition of $\mathcal{A}$, this buyer demands only goods in A at $\boldsymbol{r}$, and so by Lemma 3 we have $J^i(\boldsymbol{p}) \subseteq J^i(\boldsymbol{r}) \subseteq A$. Moreover, by Eq. (3), each buyer $i \in \mathcal{A}$ is allocated bundle $\boldsymbol{z}^i = \boldsymbol{x}^i$, and $x_j^i > 0$ implies $j \in A$. The prices of goods in A are the same at $\boldsymbol{p}$ and $\boldsymbol{r}$, by construction of $\boldsymbol{r}$, so each buyer $i \in A$ spends the same in outcome $(\boldsymbol{r}, (\boldsymbol{z}^i))$ as they do as they do in outcome $(\boldsymbol{p}, (\boldsymbol{x}^i))$. As the latter outcome is feasible, we have $\boldsymbol{z}^i \in D^i(\boldsymbol{r})$. Similarly, as the buyers in $\mathcal{B}$ only demand goods in B, we apply the same argument to see that $\boldsymbol{z}^i = \boldsymbol{y}^i \in D^i(\boldsymbol{r})$ for every $i \in \mathcal{B}$. $\square$

Proof of Proposition 4. Suppose there exists no elementwise-minimal price vector. This means that for all feasible $\boldsymbol{p}$, there exists some feasible $\boldsymbol{q}$ with $q_j < p_j$ for at least one good $j \in [n]$.

Fix some feasible prices $\boldsymbol{p}$ with the property that $\boldsymbol{p}$ cannot be reduced any further in any direction without breaking feasibility. Such a point must exist, as the feasible region is closed and restricted to $\mathbb{R}^n_+$. By assumption, there exists a feasible price vector $\boldsymbol{q}$ with $q_j < p_j$ for some $j \in [n]$. Now consider $\boldsymbol{r} = \boldsymbol{p} \wedge \boldsymbol{q}$. By Lemma 2, $\boldsymbol{r}$ is feasible. But as $\boldsymbol{r} \leq \boldsymbol{p}$ with $r_j < p_j$, this contradicts our assumption that $\boldsymbol{p}$ cannot be reduced further. □

5.2 Maximising revenue and welfare

In Section 5.1, we established that the set of feasible prices contains a unique elementwise-minimal price vector $\boldsymbol{p}^*$. We now show that revenue is maximised at these prices, and that the market is cleared at, and only at, $\boldsymbol{p}^*$. Note that we do not assume $\boldsymbol{p}^*$ to be the only prices at which revenue is maximised; indeed, there can be many revenue-maximising prices. However, a revenue-maximising outcome at $\boldsymbol{p}^*$ clears the market, and is thus optimal for buyers among all revenue-maximising outcomes.

Proposition 5. *The elementwise-minimal feasible prices $\boldsymbol{p}^*$ maximise revenue.*

Proof. We show that, for any $\boldsymbol{p} \leq \boldsymbol{q}$, the maximum obtainable revenue obtainable at $\boldsymbol{p}$ is weakly greater than the revenue obtainable at $\boldsymbol{q}$. It then follows immediately that revenue is maximised at $\boldsymbol{p}^*$.

Let $(\boldsymbol{x}^i)$ and $(\boldsymbol{y}^i)$ be allocations that revenue-maximally extend $\boldsymbol{p}$ and $\boldsymbol{q}$ to feasible outcomes $(\boldsymbol{p}, (\boldsymbol{x}^i))$ and $(\boldsymbol{q}, (\boldsymbol{y}^i))$. Our goal is to determine an allocation $(\boldsymbol{z}^i)$ so that $(\boldsymbol{p}, (\boldsymbol{z}^i))$ is a feasible outcome with a revenue that is weakly greater than the revenue of $(\boldsymbol{q}, (\boldsymbol{y}^i))$. As the revenue of $(\boldsymbol{p}, (\boldsymbol{x}^i))$ is weakly greater than the revenue of $(\boldsymbol{p}, (\boldsymbol{z}^i))$, the result then follows by transitivity.

If $\boldsymbol{p} = \boldsymbol{q}$, there is nothing to prove. Hence we assume that $S := \{j \in [n] \mid p_j < q_j\}$, the set of goods which are priced strictly lower at $\boldsymbol{p}$ than at $\boldsymbol{q}$, is non-empty. Fix a buyer $i \in [m]$. In order to define the new allocation $\boldsymbol{z}^i$ to i at $\boldsymbol{p}$, we distinguish between the two cases that $J^i(\boldsymbol{p})$ is, and is not, a subset of S.

Case 1: Suppose buyer i demands a subset of S at $\boldsymbol{p}$, so $J^i(\boldsymbol{p}) \subseteq S$. In this case, we set $\boldsymbol{z}^i := \boldsymbol{x}^i$. As $(\boldsymbol{p}, (\boldsymbol{x}^i))$ is a feasible outcome, we see that $\boldsymbol{z}^i \in D^i(\boldsymbol{p})$. Moreover, as $0 \notin S$ by definition of S, the buyer spends her entire budget on $\boldsymbol{x}^i$ at prices $\boldsymbol{p}$.

Case 2: Suppose $J^i(\boldsymbol{p}) \not\subseteq S$. We note that $J^i(\boldsymbol{q}) \cap S = \emptyset$ and buyer i still demands all goods in $J^i(\boldsymbol{q})$, so $J^i(\boldsymbol{q}) \subseteq J^i(\boldsymbol{p})$. In this case, we set $\boldsymbol{z}^i := \boldsymbol{y}^i$. As the buyer is only allocated goods not in S, and $p_j = q_j$ for all goods $j \in [n]_0 \setminus S$, it follows that the buyer spends the same at both prices, and so $\boldsymbol{z}^i \in D^i(\boldsymbol{p})$.

Note that, in both cases, the buyer is only allocated goods that they demand. To summarise, we define $(\boldsymbol{z}^i)$ as

$$\boldsymbol{z}^i := \begin{cases} \boldsymbol{x}^i & \text{if buyer } i \text{ demands a subset of } S \text{ at } \boldsymbol{p}, \\ \boldsymbol{y}^i & \text{otherwise.} \end{cases}$$

We now prove that $(\boldsymbol{p}, (\boldsymbol{z}^i))$ is a feasible outcome. We have already argued above that condition (ii) of Definition 1 is satisfied. It remains to show that aggregate demand does not exceed supply s_j for any goods $j \in [n]_0$. Note that the outcome $(\boldsymbol{p}, (\boldsymbol{z}^i))$ allocates a positive quantity of a good $j \in S$ to a buyer if and only if the buyer satisfies Case 1 above. Indeed, in this case we set $z^i_j = x^i_j$. Hence, for any $j \in S$, we have $\sum_{i\in[m]} z^i_j \leq \sum_{i\in[m]} x^i_j \leq s_j$. Similarly, for any $j \notin S$ the buyer will satisfy Case 2, and we get $\sum_{i\in[m]} z^i_j \leq \sum_{i\in[m]} y^i_j \leq s_j$.

Finally, we see that $(\boldsymbol{p}, (\boldsymbol{z}^i))$ achieves weakly greater revenue than $(\boldsymbol{q}, (\boldsymbol{y}^i))$. Each buyer satisfying Case 1 spends its entire budget and thus contributes a weakly greater amount to overall revenue in outcome $(\boldsymbol{p}, (\boldsymbol{z}^i))$ than in outcome $(\boldsymbol{q}, (\boldsymbol{y}^i))$. A buyer satisfying Case 2 contributes the same amount in both outcomes. □

Proposition 6. *The elementwise-minimal feasible prices $\boldsymbol{p}^*$ uniquely clear the market.*

Proof. We know that a competitive equilibrium $(\boldsymbol{q}, (\boldsymbol{y}^i))$ exists, as our market can be considered as a special case of the Arrow-Debreu model (Arrow and Debreu, 1954; Chen et al., 2007). As this equilibrium is a feasible outcome, and $\boldsymbol{p}^*$ is the elementwise-minimal feasible price vector, we have $\boldsymbol{q} \geq \boldsymbol{p}^*$. Suppose, for the sake of contradiction, that $\boldsymbol{q} \gneq \boldsymbol{p}^*$. We let $(\boldsymbol{x}^i)$ be an allocation that revenue-maximally extends $\boldsymbol{p}^*$ to a feasible outcome. For convenience, let $\boldsymbol{x} := \sum_{i\in[m]} \boldsymbol{x}^i$ be the aggregate demand of this outcome. As the entire supply is demanded at $\boldsymbol{q}$, the respective revenues achieved at prices $\boldsymbol{q}$ and $\boldsymbol{p}^*$ are $\sum_{j\in[n]} q_j s_j$ and $\sum_{j\in[n]} p^*_j x_j$. As $\boldsymbol{x} \leq \boldsymbol{s}$, and revenue is maximised at $\boldsymbol{p}^*$ by Proposition 5, we get the contradiction

$$\sum_{j\in[n]} q_j s_j \leq \sum_{j\in[n]} p^*_j x_j \leq \sum_{j\in[n]} p^*_j s_j < \sum_{j\in[n]} q_j s_j.$$

So $\boldsymbol{p}^*$ are the unique market-clearing prices. □

Theorem 2 now follows immediately from Propositions 5 and 6 and Theorem 1.

6 Arctic auctions

The market we consider can be interpreted as an important special case of the arctic product-mix auction market first proposed by Klemperer (2018) for the government of Iceland.[9] In this market, the seller has a fixed supply $\boldsymbol{s} \in \mathbb{R}^n_+$ of n divisible goods and wishes to find a feasible allocation of some subset of this supply among a finite set of buyers with the goal of maximising revenue.

The buyers express their demand preferences by submitting a collection of 'arctic bids', each of which consisting of an n-dimensional vector $\boldsymbol{b} \in \mathbb{R}^n_+$ and a monetary budget $\beta(\boldsymbol{b})$. Each bid is associated with a demand correspondence, and the demands of a buyer's bids are then aggregated to yield the buyer's demand.

At a market price *above the stated bid price*, an arctic bid *rejects the good*. At market prices below the stated bid prices, the bid spends its budget on goods that yield the highest 'bang-per-buck', i.e., the highest ratio of value to price $\frac{b_j}{p_j}$. When multiple goods maximise 'bang-per-buck', the bid can arbitrarily divide its expenditure between these goods. Moreover, if the maximal bang-per-buck is 1, then the bid may choose not to spend part of its budget (which we interpret

[9] The arctic product-mix auction is a variant of the original product-mix auction developed for the Bank of England by Paul Klemperer Klemperer (2008, 2010, 2018). In the general arctic product-mix auction, the seller can additionally choose cost functions to fine-tune its preferences, while in our model, we assume the seller's costs to be zero. See (Fichtl, 2022) for some discussion of the general case.

as ‘spending’ on the money good). Hence, Lemma 1 implies that each arctic bid $\boldsymbol{b}$, interpreted in isolation, induces a quasi-linear demand with linear valuation $v(\boldsymbol{x}) = \boldsymbol{b} \cdot \boldsymbol{x}$ and budget $\beta(\boldsymbol{b})$ as defined in Section 3. We denote the demand correspondence of each bid $\boldsymbol{b}$ by $D_{\boldsymbol{b}}$.

The demand correspondence of a collection $[m]$ of bids is defined by the Minkowski sum of demands $D_{\mathcal{B}}(\boldsymbol{p}) := \left\{\sum_{\boldsymbol{b}\in[m]} \boldsymbol{x}^{\boldsymbol{b}} \mid \boldsymbol{x}^{\boldsymbol{b}} \in D_{\boldsymbol{b}}(\boldsymbol{p})\right\}$. Equivalently, $D_{\mathcal{B}}$ can be understood as the aggregate demand of $|\mathcal{B}|$ buyers with quasi-linear demand, linear valuations $\boldsymbol{b} \in \mathcal{B}$ and budgets $\beta(\boldsymbol{b})$. By submitting multiple arctic bids, buyers can express richer preferences. Nevertheless, it is straightforward to see from the definition of $D_{\mathcal{B}}$ that, for the purposes of solving the auction (for welfare or for revenue), the seller can treat each bid independently, and we can assume without loss of generality that each buyer submits a single bid. Our results for quasi-Fisher markets on the coincidence of welfare and envy-free revenue thus hold also for the arctic auction we describe above.

7 Conclusion

In this article, we explore whether price-only mechanisms – common in the digital economy and financial asset exchanges – can achieve a form of efficiency that is attainable under budget constraints. We find that a unique set of prices is constrained socially optimal, respecting agents’ budgets, and simultaneously revenue-optimal for the seller. This coincidence of revenue optimality and constrained efficiency makes our market compelling in theory and highly attractive to sellers, buyers, and market platforms in practice. Our results contribute to the policy debate on the regulation of digital monopolies. They suggest that a regulatory framework that requires monopolies to set the market-clearing price vector might be justifiable (subject to the assumptions of our market setup), promoting fair competition and constrained social welfare in digital and financial markets.

Our approach is based on a novel geometric understanding of the structure underlying feasible market prices, which may be of independent interest. Future work includes addressing the open question of whether the revenue-welfare equivalence holds for other classes of preferences and markets with multiple buyers and sellers.

Omitted Proofs and Examples

Proof of Proposition 1. Recall that at any price p, the buyer demands the bundle x that maximises $v(x) - px$ subject to his budget constraint $px \leq \beta$. Given valuation $v(x)$, revenue is maximised at $(x, p) \in \arg\max_{x,p} px$ such that $v'(x) = p$ and $x \leq s$. Thus, maximal revenue given v and s is $v'(x)x$ for some $x \leq s$. Social welfare is maximised at $(x, p) \in \arg\max_{x,p} v(x) - px$ such that $x \leq s$, i.e. revenue at the social optimum is $v'(s)s$. We show that there exists x and s with $x < s$ and $v'(x)x > v'(s)s$. Recall that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is strongly concave if it satisfies $|f'(x) - f'(y)| \geq m\|x - y\|$ for all distinct $x, y \in \mathbb{R}$ (f does not need to be twice differentiable).

First, note that there exists $\overline{x} < \infty$ such that $v'(x) < mx$ for all $x \geq \overline{x}$, due to strict concavity of v. Fix some supply $s \geq \overline{x}$ and let $\varepsilon = \frac{1}{2}(ms - v'(s))$. As v is strongly concave, we also have $v'(s-\varepsilon) \geq v'(s) + m\varepsilon$ for any ε. Hence, $v'(s-\epsilon)(s-\epsilon) \geq (v'(s)+m\epsilon)(s-\epsilon) = v'(s)s + \epsilon(ms - v'(s) - m\epsilon) > v'(s)s$. □

Proof of Proposition 2. If there exists supply $s \in X$ with $v'(s) < ms$, then the result follows analogously to the proof of Proposition 1. We now prove the second part of the statement. The budget constraint is given by $px \leq \beta$. At the demanded quantity, $v'(x) = p$ holds. Thus, for all feasible x, it must hold that $v'(x) \leq \beta/x$, so $x \leq \tilde{x}$. Now we demonstrate that $v'(\tilde{x}-\epsilon) \leq m(\tilde{x}-\epsilon)$ for some small ϵ. Then the result follows from Proposition 2. For some $\delta > 0$, we have

$$m(\tilde{x}-\epsilon) \geq \left(\frac{v'(\tilde{x})}{\tilde{x}} + \delta\right)(\tilde{x}-\epsilon) = \frac{(v'(\tilde{x}) + \delta\tilde{x})(\tilde{x}-\epsilon)}{\tilde{x}} \geq v'(\tilde{x}-\epsilon).$$

The last inequality holds for $\epsilon \to 0$ and some $\delta > 0$. □

Example 4. Consider an auction with a single good available in $s = 3$ units that has a single buyer. The buyer has valuation $v : \mathbb{R} \to \mathbb{R}$ given by $v(x) = \frac{4}{\log 2}(1 - 2^{-x})$ and budget 2. Then revenue is not maximised at market-clearing prices. Indeed, note that the utility of the buyer for quantity q at price p is $u(x,p) = v(x) - px$, so the buyer's demand $D(p)$ at p is found by solving $v'(x) = p$, which yields $x = -\log_2(\frac{p}{4})$. At $p = 0.5$, we have demand $x = 3$, so p clears the market. Revenue at p is $px = 1.5$. At price $q = 1$, we have demand $y = 2$, so q does not clear the market, but revenue is $qy = 2$, which is greater. Revenue is maximised at $p = \frac{4}{e}$ with a demanded quantity of $\frac{1}{\log(2)}$ and a revenue of $\frac{4}{e\log(2)}$.

References

Arrow, Kenneth J. and Gerard Debreu (1954), "Existence of an Equilibrium for a Competitive Economy." *Econometrica*, 22(3), 265–290.

Baldwin, Elizabeth, Paul Klemperer, and Edwin Lock (2024), "Implementing Walrasian Equilibrium: the Languages of Product-mix Auctions." Available at SSRN: https://ssrn.com/abstract=4931623.

Brainard, William C. and Herbert E. Scarf (2005), "How to Compute Equilibrium Prices in 1891." *American Journal of Economics and Sociology*, 64(1), 57–83.

Che, Yeon-Koo and Ian Gale (2000), "The optimal mechanism for selling to a budget-constrained buyer." *Journal of Economic Theory*, 92(2), 198–233.

Che, Yeon-Koo, Ian Gale, and Jinwoo Kim (2013), "Assigning resources to budget-constrained agents." *The Review of Economic Studies*, 80(1), 73–107.

Chen, Lihua, Yinyu Ye, and Jiawei Zhang (2007), "A note on equilibrium pricing as convex optimization." In *Internet and Network Economics* (Xiaotie Deng and Fan Chung Graham, eds.), 7–16, Springer, Berlin, Heidelberg.

Conitzer, Vincent, Christian Kroer, Debmalya Panigrahi, Okke Schrijvers, Nicolas E. Stier-Moses, Eric Sodomka, and Christopher A. Wilkens (2022), "Pacing equilibrium in first price auction markets." *Management Science*, 68(12), 8515–8535.

Conitzer, Vincent, Christian Kroer, Eric Sodomka, and Nicolas E Stier-Moses (2021), "Multiplicative pacing equilibria in auction markets." *Operations Research*.

Dobzinski, Shahar, Ron Lavi, and Noam Nisan (2012), "Multi-unit auctions with budget limits." *Games and Economic Behavior*, 74(2), 486–503.

Fichtl, Maximilian (2022), "Computing candidate prices in budget-constrained product-mix auctions." arXiv:2208.02633v1 [cs.GT], https://arxiv.org/abs/2208.02633.

Klemperer, Paul (2008), "A New Auction for Substitutes: Central Bank Liquidity Auctions, the U.S. TARP, and Variable Product-Mix Auctions." Working Paper.

Klemperer, Paul (2010), "The Product-Mix Auction: A new auction design for differentiated goods." *Journal of the European Economic Association*, 8(2-3), 526–536.

Klemperer, Paul (2018), "Product-Mix Auctions." Nuffield College Working Paper 2018-W07, http://www.nuffield.ox.ac.uk/users/klemperer/productmix.pdf.

Laffont, Jean-Jacques and Jacques Robert (1996), "Optimal auction with financially constrained buyers." *Economics Letters*, 52(2), 181–186.

Maskin, Eric S. (2000), "Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers." *European Economic Review*, 44(4), 667–681.

Murray, Riley, Christian Kroer, Alex Peysakhovich, and Parikshit Shah (2020), "Robust market equilibria with uncertain preferences." In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, 2192–2199.

Pai, Mallesh M. and Rakesh Vohra (2014), "Optimal auctions with financially constrained buyers." *Journal of Economic Theory*, 150, 383–425.

Richter, Michael (2019), "Mechanism design with budget constraints and a population of agents." *Games and Economic Behavior*, 115, 30–47.