TOPOLOGICAL BASES

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Throughout this document we consider a fixed set X.

Proposition 1. *If $\mathcal{C}$ is a collection of subsets of X, there exists a smallest topology $\mathscr{T}_{\mathcal{C}}$ on X containing $\mathcal{C}$.*

Proof. The collection $\mathcal{T}$ of topologies on X containing $\mathcal{C}$ is non-empty, and

$$\mathscr{T}_{\mathcal{C}} \coloneqq \bigcap_{\mathscr{T} \in \mathcal{T}} \mathscr{T}$$

is the desired topology. ∎

Definition 2. We say a collection $\mathscr{D}$ of subsets of X is *descending* if it has the property that whenever D_1 and D_2 are elements of $\mathscr{D}$ and $x \in D_1 \cap D_2$, there exists $D \in \mathscr{D}$ such that

$$x \in D \subseteq D_1 \cap D_2.$$

Definition 3. Let $\mathscr{T}$ be a topology on X.

(1) By a *basis* for the topological space $(X, \mathscr{T})$, we mean a descending open cover $\mathscr{B}$ of X such that

$$\mathscr{T}_{\mathscr{B}} = \mathscr{T}.$$

(2) By a *subbasis* for $(X, \mathscr{T})$, we mean an open cover $\mathscr{S}$ of X such that

$$\mathscr{T}_{\mathscr{S}} = \mathscr{T}.$$

Remark. We see that a basis for a topological space is nothing but a descending subbasis. In particular, every basis is a subbasis, but not conversely.

It is of interest to describe the topology on a set generated by a cover, descending or not, which in effect is to describe a topology in terms of a basis or a subbasis.

Lemma 4. *Let $\mathscr{S} \subseteq \mathscr{P}(X)$ be a collection of sets covering X. Then the collection $\mathscr{B}$ of sets of the form $\bigcap_{S \in \mathscr{S}_0} S$, where $\mathscr{S}_0 \subseteq \mathscr{S}$ is finite and non-empty, is a descending cover of X generating the same topology as $\mathscr{S}$:*

$$\mathscr{T}_{\mathscr{B}} = \mathscr{T}_{\mathscr{S}}.$$

Proof. Suppose without loss of generality that $\mathscr{S}$ is non-empty. That $\mathscr{B}$ is descending follows from the fact that whenever two members of $\mathscr{B}$ have non-empty intersection, their intersection in fact belongs to $\mathscr{B}$.

Next, since $\mathscr{S} \subseteq \mathscr{B} \subseteq \mathscr{T}_{\mathscr{B}}$, it follows that

$$\mathscr{T}_{\mathscr{S}} \subseteq \mathscr{T}_{\mathscr{B}}.$$

To prove the reverse inclusion, it suffices to prove that $\mathscr{B} \subseteq \mathscr{T}_{\mathscr{S}}$. But this follows from the fact that the members of $\mathscr{B}$ are finite intersections of sets belonging to $\mathscr{S} \subseteq \mathscr{T}_{\mathscr{S}}$. ∎

Proposition 5. *Suppose $\mathscr{B}$ is a descending cover of X. Then the topology $\mathscr{T}_{\mathscr{B}}$ generated by $\mathscr{B}$ may be described in the following ways:*

(1) *It equals the collection of all sets $V \subseteq X$ with the property that there for every $u \in V$ exists $B \in \mathscr{B}$ such that*

$$u \in B \subseteq V. \tag{1}$$

(2) *It equals the collection of all unions over $\mathscr{B}$.*

Proof. First let $\mathscr{T}$ be the collection of all subsets of X having the property indicated in (1); we show that $\mathscr{T} = \mathscr{T}_{\mathscr{B}}$. Since $\mathscr{B} \subseteq \mathscr{T}_{\mathscr{B}}$, it follows that $\mathscr{T} \subseteq \mathscr{T}_{\mathscr{B}}$, and we set out to prove the reverse inclusion. To this end we first observe that $\mathscr{B} \subseteq \mathscr{T}$, so it suffices to prove that $\mathscr{T}$ is a topology on X. Since it is clear that both $\emptyset$ and X belong to $\mathscr{T}$ (the collection $\mathscr{B}$ covers X), we prove that $\mathscr{T}$ is closed under arbitrary unions and finite intersections.

Suppose (V_j) is a family in $\mathscr{T}$, and let V be its union. If $u \in V$, we can pick j such that $u \in V_j$, hence a set $B \in \mathscr{B}$ with the property that

$$u \in B \subseteq V_j \subseteq V,$$

showing that V belongs to $\mathscr{T}$.

Consider now sets V and W belonging to $\mathscr{T}$, and suppose $u \in V \cap W$. Pick B_V and B_W in $\mathscr{B}$ such that

$$u \in B_V \subseteq V \quad \text{and} \quad u \in B_W \subseteq W.$$

Then $u \in B_V \cap B_W$, so we find $B \in \mathscr{B}$ with the property that

$$u \in B \subseteq B_V \cap B_W \subseteq V \cap W,$$

showing that $V \cap W$ belongs to $\mathscr{T}$.

The equality $\mathscr{T} = \mathscr{T}_{\mathscr{B}}$ follows.

To see that $\mathscr{T}_{\mathscr{B}}$ may also be described as the collection of unions over $\mathscr{B}$, first observe that every such union belongs to $\mathscr{T}_{\mathscr{B}}$ as $\mathscr{B} \subseteq \mathscr{T}_{\mathscr{B}}$. Secondly, if V belongs to $\mathscr{T}_{\mathscr{B}}$, we see from the above considerations that we for each $u \in V$ may pick $B_u \in \mathscr{B}$ such that $u \in B_u \subseteq V$, hence

$$V = \bigcup_{u \in V} B_u$$

is a union over $\mathscr{B}$. ∎

Corollary 6. *If $\mathscr{S} \subseteq \mathscr{P}(X)$ is a collection of sets covering X, then a set $V \subseteq X$ belongs to the topology $\mathscr{T}_{\mathscr{S}}$ generated by $\mathscr{S}$ if and only if it equals a union of sets of the form $\bigcap_{S \in \mathscr{S}_0} S$, where $\mathscr{S}_0 \subseteq \mathscr{S}$ is finite and non-empty.*

Proof. Combine Lemma 4 and Proposition 5. ∎

Proposition 7. *Let $\mathscr{T}$ be a topology on X. If $\mathscr{B}$ is a collection of open subsets of X such that there for every open set $V \in \mathscr{T}$, and every point $u \in V$, exists $B \in \mathscr{B}$ such that*

$$u \in B \subseteq V,$$

then $\mathscr{B}$ is a basis for $(X, \mathscr{T})$.

Proof. Firstly, since $\mathscr{B} \subseteq \mathscr{T}$ by assumption, it follows that $\mathscr{T}_{\mathscr{B}} \subseteq \mathscr{T}$. To prove the reverse inclusion, suppose $V \in \mathscr{T}$. We may for every $u \in V$ pick a set $B_u \in \mathscr{B}$ such that $u \in B_u \subseteq V$, hence

$$V = \bigcup_{u \in V} B_u \in \mathscr{T}_{\mathscr{B}}.$$

It remains to prove that $\mathscr{B}$ is a descending cover of X. Since $X \in \mathscr{T}$, we may for each point $x \in X$ find $B \in \mathscr{B}$ such that $x \in B \subseteq X$, hence $\mathscr{B}$ covers X. To see that $\mathscr{B}$ is descending, suppose $x \in B_1 \cap B_2$, where B_1 and B_2 are sets belonging to $\mathscr{B}$. Since $B_1 \cap B_2 \in \mathscr{T}$, we may by assumption find $B \in \mathscr{B}$ such that

$$x \in B \subseteq B_1 \cap B_2,$$

and the proof is complete. ■