COMMENTS ON URYSOHN'S LEMMA

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1. Introduction

We modularise the standard proof of Urysohn's lemma, as for instance found in [2, 1], making it more transparent what is required to make the proof go through.

There are no new mathematical ideas contained herein. The only novelties may be said to be proof-technical, in the sense that we in Section 3 spell out in full detail exactly what is required to give a rigorous proof of the belowmentioned fundamental lemma, something which the referenced texts remain vague about.

Assume X is a normal topological space, meaning one-point sets are closed and that disjoint closed sets have disjoint neighbourhoods. It follows that X is Hausdorff.

Consider disjoint closed sets $E, F \subseteq X$. We want to exhibit a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on E and $f = 1$ on F. Such an f is known as a *Urysohn function* for the pair E, F.

Assume without loss of generality that E, F are non-empty. The construction of a Urysohn function for E and F relies on the following technical device, to be proved in Section 3.

Fundamental Lemma. *We assume there exists a countable dense subset Q of $[0,1]$ containing 0 and 1, along with a family $(A_q)_{q\in Q}$ of neighbourhoods of E such that $A_1 = X \setminus F$ and*

$$\overline{A_p} \subseteq A_q \quad \text{whenever} \quad p < q \text{ in } Q.$$

Thus A_0 encapsulates E and $A_1 = X \setminus F$ is the maximal neighbourhood of E not intersecting F. Intermediate between A_0 and A_1 there are stably increasing neighbourhoods A_q of E, one for each $q \in Q$. The stability requirement $\overline{A_p} \subseteq A_q$ for $p < q$ and the maximality of A_1 provides for enough regularity in the family (A_q) that the A_q flow from A_0 to $X \setminus F$ in such a way that the indicies Q provide for a time-like development of the neighbourhoods. The topological neighbourhoods of E are thus translated back into an increasing progression of a dense sequence of time steps from 0 to 1.

In case the topology of X is compatible with a metric d, the proof of Urysohn's lemma goes through easily as follows. One first verifies that for a non-empty subset $A \subseteq X$, the function

$$d(x, A) := \inf_{a\in A} d(x, a) \quad (x \in X)$$

is continuous with null set $\overline{A}$, and defining $f\colon X \to [0,1]$ by

$$f(x) := d(x, E)/\big(d(x, E) + d(x, F)\big) \quad (x \in X) \tag{1.1}$$

provides the desired Urysohn function for E and F in the form of a normalised distance from E to F.

In light of the above discussion, it is this normalised distance function (1.1) that is allowed to be replaced by definition (2.1) due to the time-like development of the neighbourhoods A_q of E.

2. Constructing the Urysohn Function

Assuming the fundamental lemma, we construct the desired Urysohn function f.

Claim. *$\bigcup A_q = A_1 = X \setminus F$.*

Proof. Clearly $X \setminus F = A_1 \subseteq \bigcup A_q$.

If instead $x \in A_q$ for some $q \in Q$, then either $q = 1$ and we are done, or $q < 1$ and $x \in \overline{A_q} \subseteq A_1 = X \setminus F$. ■

Definition. Define $f \colon X \to [0,1]$ by $f(x) \coloneqq 1$ for $x \in F$ and

$$f(x) \coloneqq \inf\{q \in Q : x \in A_q\} \quad (x \in X \setminus F). \tag{2.1}$$

We claim f has the desired properties. For $f = 1$ on F by construction, and if $x \notin F$ we have $x \in A_q$ for some $q \in Q$ by the above claim, hence $0 \leq f(x) \leq q \leq 1$. If $x \in E \subseteq A_0$, then $f(x) \leq 0$, from which $f(x) = 0$, thus $f = 0$ on E.

We show next that f is continuous. Since $[0,1]$ is a convex subset of $\mathbb{R}$, the induced order and subspace topologies agree. Therefore $[0,1]$ has a subbasis consisting of sets $[0,t)$, $(s,1]$ for $s,t \in [0,1]$, and it suffices to show $f^{-1}(U)$ is open in X for each such set U.

Claim. *$f^{-1}([0,t)) = \bigcup_{q<t} A_q$ for all $0 < t \leq 1$.*

Proof. Consider such a t.

If $x \in X$ and $f(x) < t$, then $x \notin F$ and there exists by definition $q \in Q$ such that $q < t$ and $x \in A_q$.

If $q < t$ and $x \in A_q$, then $f(x) \leq q < t$. ■

Claim. *$f^{-1}([0,s]) = \bigcap_{r>s} \bigcup_{p<r} A_p$ for all $0 \leq s < 1$.*

Proof. Fix such an s and assume first that $x \in X$ is such that $f(x) \leq s$, and consider $r > s$. Then $f(x) < r$, hence $x \notin F$ and there exists by definition $p < r$ in Q such that $x \in A_p$, showing $f^{-1}([0,s]) \subseteq \bigcap_{r>s} \bigcup_{p<r} A_p$.

To prove the reverse inclusion, suppose $x \in X$ and that for every $r > s$ there exists $p < r$ such that $x \in A_p$; we show $f(x) \leq s$. For considering $r > s$ and picking $p < r$ with $x \in A_p$, we find $f(x) \leq p < r$, hence $f(x) < r$ for all $r > s$. If instead of $f(x) \leq s$ we had $s < f(x)$, the density of Q in $[0,1]$ would yield $r \in Q$ satisfying $s < r < f(x)$. This r would also satisfy $f(x) < r$ since $r > s$, a contradiction. Hence $f(x) \leq s$, and the claim follows. ■

Claim. *In the above situation, $\bigcap_{r>s} \bigcup_{p<r} A_p = \bigcap_{q>s} \overline{A_q}$.*

Proof. Assume first $x \in X$ has the property that there for every $r > s$ exists $p < r$ such that $x \in A_p$, and let $q > s$. By assumption there exists $p < q$ such that $x \in A_p \subseteq \overline{A_p} \subseteq A_q$, hence $x \in \overline{A_q}$, and we have inclusion to the right.

Assume next that $x \in \bigcap_{q>s} \overline{A_q}$, and let $r > s$. By density of Q in $[0,1]$, we find $p, q \in Q$ such that $s < q < p < r$, hence $x \in \overline{A_q} \subseteq A_p$, and we have inclusion to the left. ■

Claim. *f is continuous $X \to [0,1]$.*

Proof. By a previous claim, the sets $f^{-1}([0,t))$ for $t \in [0,1]$ are all open in X, and the same is true for sets of the form $f^{-1}((s,1]) = X \setminus f^{-1}([0,s])$ with $s \in [0,1]$. Since the collection of all such half-open intervals forms a subbasis for the topology of $[0,1]$, it follows that f is continuous. ■

3. Proof of the Fundamental Lemma

We now turn our attention to proving the fundamental lemma stated at the beginning of the note, making heavy use of the following property of normal spaces.

Lemma 3.1. *If E and U are respectively closed and open subsets of X such that $E \subseteq U$, there exists an open set V in X such that*

$$E \subseteq V \subseteq \overline{V} \subseteq U.$$

Proof. Apply normality to the disjoint closed sets E and $X \setminus U$. ■

For our index set Q we take the set $\mathbb{Q} \cap [0,1]$ of rational numbers in the interval $[0,1]$; for it is a countable dense subset containing both endspoints.

Let $(q_n)_{n \in \mathbb{N}}$ be a bijective enumeration of Q such that $q_0 = 1$ and $q_1 = 0$, and define

$$Q_n \coloneqq \{q_k : 0 \leq k \leq n\} \quad (n \in \mathbb{N}).$$

We observe that $Q_0 \subset Q_1 \subset \cdots$ and $\bigcup Q_n = Q$, hence the Q_n increase to Q.

Denoting the topology of X by $\mathscr{T}$, a map $A\colon Q_n \to \mathscr{T}$ for some $n \in \mathbb{N}$ will be called a *partial map*. It will be called *stable* when $\overline{A(p)} \subseteq A(q)$ whenever $p < q$ belong to Q_n.

Let $\mathcal{A}$ denote the collection of all stable partial maps. If $A\colon Q_n \to \mathscr{T}$ is one such, a stable partial map $\tilde{A}\colon Q_{n+1} \to \mathscr{T}$ such that $\tilde{A}|Q_n = A$ will be called a *stable extension* of A.

Claim. *Every stable partial map admits a stable extension.*

Proof. Let $A\colon Q_n \to \mathscr{T}$ be a stable partial map, and assume without loss of generality that $n \geq 1$. Since $Q_{n+1} = Q_n \cup \{q_{n+1}\}$, in order to define a stable extension $\tilde{A}$ of A, it is sufficient to define $\tilde{A}$ at q_{n+1}, considering $\tilde{A}(q) = A(q)$ otherwise.

Since $n \geq 1$, both 0 and 1 belong to Q_{n+1}, hence $0 < q_{n+1} < 1$. Moreover, $Q_{n+1} \subset [0,1]$ is finite, hence q_{n+1} has an immediate predecessor r and an immediate successor s in Q_{n+1}. In other words, $r, s \in Q_{n+1}$, and $r < q_{n+1} < s$, and if $q \in Q_{n+1}$ is different from q_{n+1}, either $q \leq r$ or $s \leq q$.

Now $r, s \neq q_{n+1}$, so $r, s \in Q_n$ and A is stable, so $\overline{A(r)} \subseteq A(s)$. By Lemma 3.1, we find an open set $U \in \mathscr{T}$ such that $\overline{A(r)} \subseteq U$ and $\overline{U} \subseteq A(s)$. We let $\tilde{A}(q_{n+1}) \coloneqq U$ and claim $\tilde{A}$ is stable.

For suppose $p < q$ belong to Q_{n+1}. If neither p nor q equals q_{n+1}, the inclusion $\overline{\tilde{A}(p)} \subseteq \tilde{A}(q)$ follows from the stability of A.

If $p = q_{n+1}$, then $p = q_{n+1} < s \leq q$, hence

$$\overline{\tilde{A}(p)} \subseteq \tilde{A}(s) \subseteq \tilde{A}(q).$$

If $q = q_{n+1}$, then $p \leq r < q_{n+1} = q$, hence

$$\overline{\tilde{A}(p)} \subseteq \overline{\tilde{A}(r)} \subseteq \tilde{A}(q). \qquad \blacksquare$$

For $A \in \mathcal{A}$ let $\mathcal{E}_A$ denote the non-empty collection of stable extensions of A. By the choice axiom we obtain an extension map $\mathcal{E}\colon \mathcal{A} \to \mathcal{A}$ such that $\mathcal{E}(A) \in \mathcal{E}_A$ is a stable extension of A for all $A \in \mathcal{A}$.

Noting that $Q_1 = \{1, 0\}$, define the stable partial map $A_0\colon Q_1 \to \mathscr{T}$ by letting $A_0(1) \coloneqq X \setminus F$ and use Lemma 3.1 to let $A_0(0)$ be a neighbourhood of E such that $\overline{A_0(0)} \subseteq A_0(1)$. Applying the recursion principle, we obtain a sequence $(A_n)_{n\in\mathbb{N}}$ of stable partial maps such that $A_{n+1} = \mathcal{E}(A_n)$ is a stable extension of A_n for all n. Since the sets Q_n increase to Q, the A_n patch uniquely to a map $A\colon Q \to \mathscr{T}$ such that $A|Q_n = A_n$ for all n.

Claim. *A is stable.*

Proof. Suppose $p < q$ in Q and pick $n \in \mathbb{N}$ such that $p, q \in Q_n$. Then $\overline{A(p)} \subseteq A(q)$ since $A|Q_n = A_n$ and A_n is stable. $\blacksquare$

This completes the proof of the fundamental lemma.

References

[1] James R. Munkres. *Topology.* Second Edition. Upper Saddle River: Prentice Hall, 2000.

[2] Gert K. Pedersen. *Analysis Now.* Revised Printing. Graduate Texts in Mathematics. New York: Springer, 1995.