RUDIMENTARY ASPECTS OF COMPLEX ANALYSIS

Simon Foldvik

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Abstract. We exhibit an isomorphism of $\mathbb{C}$ with a subalgebra of the matrix algebra $\mathcal{M}_2(\mathbb{R})$. From this we show $S^1 \simeq \mathrm{SO}(2)$ and $\mathbb{C} \setminus \{0\} \simeq (0, \infty) \times \mathrm{SO}(2)$ as groups.

Next, we discuss various senses in which a function $U \to Y$ on an open set $U \subseteq \mathbb{C}$ into a normed space Y can be differentiable. The interplay between the real and complex linear structures on $\mathbb{C}$ leads to the Cauchy–Riemann equations for complex differentiability. We conclude by showing that the space of holomorphic functions agrees with the kernel of the $\overline{\partial}$ operator.

Contents

1. The Complex Field

Consider the matrix algebra $\mathcal{M}_2(\mathbb{R})$ of real 2×2 matricies, and let I denote the identity. The matrix

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

generates a subalgebra $\mathcal{A} \subseteq \mathcal{M}_2(\mathbb{R})$, which we now describe.

Claim. $\mathcal{A} = \mathbb{R}I \oplus \mathbb{R}J$.

Proof. The subspaces $\mathbb{R}I$ and $\mathbb{R}J$ of $\mathcal{M}_2(\mathbb{R})$ are independent: $\mathbb{R}I \cap \mathbb{R}J = 0$.

Next, $J^2 = -I$, hence $I \in \mathcal{A}$. It follows that

$$\mathbb{R}I \oplus \mathbb{R}J \subseteq \mathcal{A}.$$

Thirdly, $\mathbb{R}I \oplus \mathbb{R}J$ is a linear subspace of $\mathcal{M}_2(\mathbb{R})$. One checks it is a subalgebra:

$$\begin{aligned}(\alpha I + \beta J)(\alpha' I + \beta' J) &= (\alpha\alpha')I + (\alpha\beta')J + (\alpha'\beta)J + (\beta\beta')J^2 \\ &= (\alpha\alpha' - \beta\beta')I + (\alpha\beta' + \alpha'\beta)J \in \mathbb{R}I \oplus \mathbb{R}J.\end{aligned}$$

Since $\mathbb{R}I \oplus \mathbb{R}J$ is a subalgebra containing J, it must contain all of $\mathcal{A}$. ■

Remark. Thus

$$\mathcal{A} = \mathbb{R}I \oplus \mathbb{R}J = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

We now define a map $M\colon \mathbb{C} \to \mathcal{A}$ by

$$M(z) := \begin{bmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{bmatrix} \qquad (z \in \mathbb{C}).$$

Considering $\mathbb{C}$ as an $\mathbb{R}$-algebra, we have:

Claim. *M is a unital $*$-isomorphism $\mathbb{C} \to \mathcal{A}$. Hence $\mathbb{C} \simeq \mathcal{A}$ as fields.*

Proof. Routine. The relevant algebraic properties are thus:

$$\begin{aligned} M(z+w) &= M(z) + M(w), \\ M(zw) &= M(z)M(w), \\ M(tz) &= tM(z), \\ M(1) &= I, \\ M(\overline{z}) &= M(z)^T, \end{aligned}$$

where $z, w \in \mathbb{C}$ and $t \in \mathbb{R}$. ■

Remark. Observe also $M(i) = J$. Hence the representation $z = a + bi$ becomes $M(z) = a + bJ$ in matrix form.

Claim. $\det M(z) = |z|^2$.

Proof. Immediate: $\det M(z) = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = |z|^2$. ■

Claim. *$M(z) \in \mathrm{GL}_2(\mathbb{R})$ if and only if $z \neq 0$, in which case*

$$M(z)^{-1} = M(1/z) = \frac{1}{\det M(z)} M(z)^T.$$

Proof. One has $\det M(z) = |z|^2 > 0$ if and only if $z \neq 0$, in which case

$$I = M(1) = M(z)M(1/z).$$

This proves $M(z)^{-1} = M(1/z)$. The second identity follows from

$$(\det M(z))I = M\big(\det M(z)\big) = M(z\overline{z}) = M(z)M(z)^T. \qquad ■$$

Remark. It follows that M provides an isomorphism of the multiplicative group $\mathbb{C} \setminus \{0\}$ with the subgroup $\mathcal{A} \cap \mathrm{GL}_2(\mathbb{R})$ of $\mathrm{GL}_2(\mathbb{R})$. In fact, this is the group $(0, \infty) \times \mathrm{SO}(2)$; we will have more to say about this soon.

Claim. *M provides a group isomorphism $S^1 \simeq \mathrm{SO}(2)$.*

Proof. Assuming $z \in S^1$, then $\det M(z) = |z|^2 = 1$ and

$$M(z)^T M(z) = M(\overline{z}z) = M(1) = I.$$

Similarly, $M(z)M(z)^T = I$. Hence M maps S^1 into $\mathrm{SO}(2)$.

We now prove M is onto. For consider $A \in \mathrm{SO}(2)$, say

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

The orthonormality of the columns of A dictates $(c,d) = \pm(-b,a)$. Now $\det A = 1$ forces $(c,d) = (-b,a)$. Hence

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Now $a + ib \in S^1$ and $M(a+ib) = A$ follows.

Being a surjective monomorphism, M is in fact an isomorphism $S^1 \xrightarrow{\sim} \mathrm{SO}(2)$. ∎

We now return to the complete description of the aforementioned group $\mathcal{A} \cap \mathrm{GL}_2(\mathbb{R})$.

Claim. *The map* $\Phi\colon \mathbb{C} \setminus \{0\} \to (0,\infty) \times \mathrm{SO}(2)$ *given by*

$$\Phi(z) := \big(|z|, M(z/|z|)\big) \quad (z \in \mathbb{C} \setminus \{0\})$$

is a group isomorphism. Hence

$$\mathcal{A} \cap \mathrm{GL}_2(\mathbb{R}) \simeq \mathbb{C} \setminus \{0\} \simeq (0,\infty) \times \mathrm{SO}(2).$$

Proof. Φ is multiplicative since M is.

Injectivity can be seen as follows. If $\Phi(z) = \Phi(w)$, then $|z| = |w|$, and M agrees at the two points $z/|z|$ and $w/|w|$ of S^1. Therefore, $z/|z| = w/|w|$, from which $z = w$ follows.

We now prove surjectivity. Given $r > 0$ and $A \in \mathrm{SO}(2)$, use that M provides an isomorphism $S^1 \xrightarrow{\sim} \mathrm{SO}(2)$ to find $w \in S^1$ such that $M(w) = A$. One then checks that $\Phi(rw) = (r, A)$. ∎

2. Differentiability

The complex field $\mathbb{C}$ carries the structure of a real Banach space of (real) dimension two and of a complex Banach space of (complex) dimension one. The interplay between these structures gives rise to the Cauchy–Riemann equations.

2.1. Complex linearity.

View $\mathbb{C}$ as a Banach space over $\mathbb{R}$ with canonical basis $\{1, i\}$. Given an $\mathbb{R}$-linear map $T\colon \mathbb{C} \to \mathbb{C}$, its representation $[T]$ with respect to the canonical basis is an arbitrary element of the matrix algebra $\mathcal{M}_2(\mathbb{R})$. We now set out to characterize when such a mapping is $\mathbb{C}$-linear.

Claim. *Let* $T\colon \mathbb{C} \to \mathbb{C}$ *be* $\mathbb{R}$*-linear. The following are equivalent:*

(1) T *is* $\mathbb{C}$*-linear;*
(2) $T(i) = iT(1)$*;*
(3) $T(iz) = iT(z)$ *for all* $z \in \mathbb{C}$*;*
(4) $[T]$ *commutes with* J*, that is,* $[T]J = J[T]$*;*
(5) $[T] \in \mathcal{A}$*, that is,* $[T]$ *takes the form*

$$[T] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad (a, b \in \mathbb{R}).$$

Proof. Routine and left to the reader. ∎

Let $L(\mathbb{C})$ denote the $\mathbb{C}$-algebra of $\mathbb{C}$-linear transformations $\mathbb{C} \to \mathbb{C}$.

Claim. $L(\mathbb{C}) \simeq \mathbb{C}$ *as* $\mathbb{C}$*-algebras.*

Proof. Every $\mathbb{C}$-linear transformation of the complex plane is multiplication by a complex number. ■

The coordinate isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$ induced by the canonical $\mathbb{R}$-basis $\{1, i\}$ for $\mathbb{C}$ is the map $\operatorname{Re} \oplus \operatorname{Im} \colon z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$. Therefore, representing a $\mathbb{C}$-linear transformation $T \colon \mathbb{C} \to \mathbb{C}$ with respect to this basis yields an element of $\mathcal{A}$ equal to

$$[T] = \begin{bmatrix} \operatorname{Re} T(1) & -\operatorname{Im} T(1) \\ \operatorname{Im} T(1) & \operatorname{Re} T(1) \end{bmatrix}. \tag{2.1}$$

This transformation corresponds to multiplication by the complex number $z := T(1)$, hence (2.1) shows $[T] = M(z)$, from which we conclude that $\det[T] = \det M(z) = |z|^2 \geq 0$. In particular, $T \in \mathrm{GL}(\mathbb{C})$ and $[T] \in \mathrm{GL}_2(\mathbb{R})$ if and only if $z \neq 0$.

2.2. **Complex differentiability.** Consider now $\mathbb{C}$ as a complex Banach space of dimension one. Let an open set $U \subseteq \mathbb{C}$, a function $f \colon U \to \mathbb{C}$, and a point $p \in U$ be given. We consider two senses in which f can be differentiable at p, which in the end turn out to coalesce.

Definition. f is said to be *complex differentiable* at p if the difference quotient $\big(f(p+w) - f(p)\big)/w$ converges in the topology of $\mathbb{C}$ as $w \to 0$, in which case we denote this limit by $f'(p)$:

$$f'(p) = \lim_{\substack{w \to 0 \\ w \in \mathbb{C} \setminus \{0\}}} \frac{f(p+w) - f(p)}{w}.$$

The other sense in which f can be differentiable at p is in the sense of normed spaces. Thus, f is complex Fréchet differentiable at p if there is a $\mathbb{C}$-linear map $T \colon \mathbb{C} \to \mathbb{C}$ (automatically continuous) such that

$$f(p+w) = f(p) + T(w) + |w|\epsilon(w) \tag{2.2}$$

holds for small $w \in \mathbb{C}$, where $\epsilon(w) \to 0$ as $w \to 0$. If this is the case, the map T is unique and called the Fréchet derivative of f at p.

Claim. *f is complex differentiable at p if and only if it is Fréchet differentiable at p when $\mathbb{C}$ is viewed as a complex Banach space. If $T \colon \mathbb{C} \to \mathbb{C}$ denotes the Fréchet derivative of f at p, then $f'(p) = T(1)$, and*

$$T(w) = wf'(p) \quad (w \in \mathbb{C}).$$

Hence T is multiplication by $f'(p)$.

Proof. If f has Fréchet derivative T at p, estimate

$$\left| \frac{f(p+w) - f(p)}{w} - T(1) \right| = \frac{|f(p+w) - f(p) - T(w)|}{|w|} = |\epsilon(w)| \to 0.$$

Conversely, if f is complex differentiable at p, let $T \colon \mathbb{C} \to \mathbb{C}$ be multiplication by $f'(p)$ and employ elementary estimates as above to conclude f is Fréchet differentiable at p with derivative T. ■

A function $U \to \mathbb{C}$ complex differentiable on all of U is said to be *holomorphic*. It is elementary to show that the collection $H(U)$ of holomorphic functions on U is a unital and commutative $\mathbb{C}$-algebra under pointwise operations, and that the derivative $f \mapsto f'$ satisfies the usual formal properties.[1] Moreover, the composition of holomorphic functions is holomorphic.

2.3. **Real differentiability.** Consider $\mathbb{C}$ a Banach space over $\mathbb{R}$ and let Y be an arbitrary real normed vector space. Let an open set $U \subseteq \mathbb{C}$, a mapping $f\colon U \to Y$, and a point $p \in U$ be given.

Definition. The real Fréchet derivative of f at p, if it exists, is the unique $\mathbb{R}$-linear map $Df(p)\colon \mathbb{C} \to Y$ satisfying

$$f(p+w) = f(p) + Df(p)w + |w|\epsilon(w)$$

for small $w \in \mathbb{C}$, where $\epsilon(w) \to 0$ in Y as $w \to 0$ in the topology of $\mathbb{C}$.

There is also the notion of *directional derivative.* Indeed, given $w \in \mathbb{C}$ one defines

$$\partial_w f(p) \coloneqq \lim_{\substack{t\to 0\\ t\in\mathbb{R}\setminus\{0\}}} t^{-1}\big(f(p+tw) - f(p)\big),$$

when it exists, called the directional derivative of f at p in direction w.

It is well-known that if f is Fréchet differentiable at p in the real sense, then all directionals $\partial_w f$ of f exist at p, and

$$\partial_w f(p) = Df(p)w \quad (w \in \mathbb{C}).$$

The directionals

$$\partial_1 f(p) = Df(p)1 \quad \text{and} \quad \partial_i f(p) = Df(p)i$$

are of particular interest.

As a converse to the previous statement, if all directionals $\partial_w f$ exist in a neighbourhood of p and are continuous at p, then f is real differentiable at p. In fact, it is sufficient to restrict the w to lie in an $\mathbb{R}$-basis for $\mathbb{C}$: see Section 2.5. We therefore have:

Claim. *If the partials $\partial_1 f$ and $\partial_i f$ exist near p and are continuous at p, then f is real differentiable at p.*

Specialise to the real Banach space $Y = \mathbb{C}$ and decompose f into real and imaginary parts, thus $u \coloneqq \operatorname{Re} f\colon U \to \mathbb{R}$ and $v \coloneqq \operatorname{Im} f\colon U \to \mathbb{R}$.

Claim. *f is real differentiable at p if and only if u and v are, in which case*

$$Df(p) = Du(p) + iDv(p).$$

Proof. Elementary estimates suffice. ■

By similar estimates:

Claim. *Given $w \in \mathbb{C}$, the directional $\partial_w f$ exists at p if and only if both $\partial_w u$ and $\partial_w v$ exist at p, in which case*

$$\partial_w f(p) = \partial_w u(p) + i\partial_w v(p).$$

In particular,

$$\partial_1 f = \partial_1 u + i\partial_1 v \quad \text{and} \quad \partial_i f = \partial_i u + i\partial_i v \tag{2.3}$$

[1] In fact, it is a $\mathbb{C}$-algebra derivation $H(U) \to H(U)$.

whenever one side of the respective equalities is defined. Consequently, with respect to the canonical $\mathbb{R}$-basis $\{1, i\}$ for $\mathbb{C}$, one has the matrix representation

$$\begin{aligned}[Df(p)] &= \begin{bmatrix}[Df(p)1]_{\{1,i\}} & [Df(p)i]_{\{1,i\}}\end{bmatrix} \\ &= \begin{bmatrix}\operatorname{Re}\partial_1 f(p) & \operatorname{Re}\partial_i f(p) \\ \operatorname{Im}\partial_1 f(p) & \operatorname{Im}\partial_i f(p)\end{bmatrix} = \begin{bmatrix}\partial_1 u(p) & \partial_i u(p) \\ \partial_1 v(p) & \partial_i v(p)\end{bmatrix}\end{aligned}$$

whenever f is real differentiable at p.

2.4. **The Cauchy–Riemann equations.** We now investigate when the real differential is $\mathbb{C}$-linear, leading to the Cauchy–Riemann condition for complex differentiability.

Let $f : U \to \mathbb{C}$ and $p \in U$ be as in Section 2.3, with real and imaginary parts u and v, respectively. Supposing f is real differentiable at p, one may consider the following conditions on f:

(1) f is complex differentiable at p;
(2) $Df(p)$ is $\mathbb{C}$-linear;
(3) $Df(p)i = iDf(p)1$;
(4) $[Df(p)]$ commutes with J, that is, $[Df(p)]J = J[Df(p)]$.
(5) $[Df(p)]$ lies in the subalgebra $\mathcal{A} \subset \mathcal{M}_2(\mathbb{R})$, that is, takes the form

$$[Df(p)] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

for some $a, b \in \mathbb{R}$.

(6) f satisfies the Cauchy–Riemann equation

$$\partial_i f(p) = i\partial_1 f(p);$$

(7) The real and imaginary parts of f satisfy the Cauchy–Riemann equations

$$\begin{cases} \partial_1 u(p) = \partial_i v(p) \\ \partial_i u(p) = -\partial_1 v(p). \end{cases}$$

Claim. *All of the above conditions are equivalent.*

Proof. The equivalence of (1)–(5) follows from the results of Section 2.1. In this case, the real derivative $Df(p)$ is also the complex derivative.

Assuming that f is complex differentiable at p, the Cauchy–Riemann equation for f is immediate:

$$\partial_i f(p) = Df(p)i = iDf(p)1 = i\partial_1 f(p). \tag{2.4}$$

Conversely, (2.4) shows f is complex differentiable if it satisfies the Cauchy–Riemann equation (6).

We now include (7) in the equivalence. Indeed, if f is complex differentiable at p, then $[Df(p)]$ belongs to $\mathcal{A}$, hence

$$[Df(p)] = \begin{bmatrix}\partial_1 u(p) & \partial_i u(p) \\ \partial_1 v(p) & \partial_i v(p)\end{bmatrix} = \begin{bmatrix}\partial_1 u(p) & -\partial_1 v(p) \\ \partial_1 v(p) & \partial_1 u(p)\end{bmatrix}, \tag{2.5}$$

which is (7). Conversely, if (7) holds, so does (2.5), from which $[Df(p)] \in \mathcal{A}$ and complex differentiability of f at p follows. ■

Incidentally, if f is differentiable at p in the complex sense, the matrix representation of $Df(p)$ with respect to the $\mathbb{C}$-basis $\{1\}$ is the 1×1-matrix $[f'(p)]$.

More interestingly, under the same circumstances, the matrix representation of $Df(p)$ with respect to the canonical $\mathbb{R}$-basis $\{1, i\}$ is an element of the subalgebra $\mathcal{A}$ of $\mathcal{M}_2(\mathbb{R})$. Being multiplication by $f'(p)$, it equals $M(f'(p))$, as seen in Section 2.1. In particular,

$$\det[Df(p)] = \det M(f'(p)) = |f'(p)|^2 \geq 0,$$

so that $Df(p)$ belongs to $\mathrm{GL}(\mathbb{C})$, and $[Df(p)]$ to $\mathrm{GL}_2(\mathbb{R})$, if and only if $f'(p) \neq 0$.

We also record the formulas

$$\begin{aligned} \partial_1 f(p) &= Df(p)1 = f'(p), \\ \partial_w f(p) &= wf'(p) \quad (w \in \mathbb{C}) \end{aligned}$$

in the presence of complex differentiability.

2.5. **Continuous differentiability.** We continue to view $\mathbb{C}$ as a Banach space over $\mathbb{R}$ and let Y be a real normed vector space.

Consider an open set $U \subseteq \mathbb{C}$. A (real) differentiable mapping $f\colon U \to Y$ induces a map $Df\colon U \to L(\mathbb{C}, Y)$ of U into the set $L(\mathbb{C}, Y)$ of $\mathbb{R}$-linear transformations $\mathbb{C} \to Y$, namely $Df\colon p \mapsto Df(p)$.

Definition. A differentiable mapping $f\colon U \to Y$ is said to be *continuously differentiable* if the indcued map $Df\colon U \to L(\mathbb{C}, Y)$ is continuous. The collection of continuously differentiable maps $U \to Y$ is denoted $C^1(U, Y)$.

Remark. Since $\mathbb{C}$ is finite-dimensional, every $\mathbb{R}$-linear transformation $\mathbb{C} \to Y$ is continuous, and the space $L(\mathbb{C}, Y)$ is a real normed space under the operator norm:

$$\|T\| \coloneqq \sup_{|z| \leq 1} \|T(z)\| \quad (T \in L(\mathbb{C}, Y)).$$

Continuous differentiability of $f\colon U \to Y$ may also be detected in terms of continuity of directionals or partials of f.

Claim. *The following are equivalent:*

(1) *f is continuously differentiable on U.*
(2) *$\partial_w f$ exists and is continuous on U for all $w \in \mathbb{C}$.*
(3) *$\partial_w f$ exists and is continuous on U for all w belonging to some $\mathbb{R}$-basis for $\mathbb{C}$.*

Remark. In particular, f is continuously differentiable if and only if $\partial_1 f$ and $\partial_i f$ exist and are continuous on U.

Proof. This is a standard result. ■

One also has the following variant of the above claim, whose proof is left to the reader.

Claim. *Let $f\colon U \to Y$ and suppose $\{w, w'\}$ is an $\mathbb{R}$-basis for $\mathbb{C}$. If both $\partial_w f$ and $\partial_{w'} f$ exist and are* bounded *on U, then f is continuous, and there exists $M \geq 0$ such that there for all $p \in U$ exists a radius $r > 0$ such that*

$$\|f(z) - f(p)\| \leq M|z - p|$$

whenever $|z - p| < r$.

We conclude this section with an example showing that boundedness of directionals does not imply differentiability.

Example 2.1. Define $u\colon \mathbb{C} \to \mathbb{C}$ by $u(0) = 0$ and

$$u(z) = \frac{(\operatorname{Re} z)(\operatorname{Im} z)^2}{|z|^2} \quad (z \neq 0).$$

One computes directly that

$$\partial_w u(0) = u(w) \quad \text{for all} \quad w \in \mathbb{C}.$$

Moreover, one can show that $u|(\mathbb{C} \setminus \{0\})$ is continuously differentiable, and

$$|\partial_w u| \leq 4|w| \quad \text{on } \mathbb{C}$$

for all $w \in \mathbb{C}$. Since all directionals of u exist and are bounded on all of $\mathbb{C}$, it follows that u is continuous. However, u is *not* differentiable at 0. Indeed, let $w := 1 + i$. Then

$$\partial_w u(0) = u(1 + i) = \frac{1}{2}.$$

On the other hand, if u was differentiable at 0, one would have

$$\partial_w u(0) = Du(0)w = \partial_1 u(0) + \partial_i u(0) = u(1) + u(i) = 0,$$

a contradiction.

2.6. **The $\overline{\partial}$ operator.** Let $U \subseteq \mathbb{C}$ be open and consider the $\mathbb{C}$-algebra $C^1(U)$ of continuously differentiable functions $U \to \mathbb{C}$. Each $w \in \mathbb{C}$ induces a linear partial differential operator

$$\partial_w\colon C^1(U) \to C(U),$$

namely $\partial_w\colon f \mapsto \partial_w f$.

Definition. We define linear differential operators $\partial, \overline{\partial}\colon C^1(U) \to C(U)$ by

$$\partial := \frac{1}{2}(\partial_1 - i\partial_i) \quad \text{and} \quad \overline{\partial} := \frac{1}{2}(\partial_1 + i\partial_i).$$

Split $f \in C^1(U)$ into real and imaginary parts, thus $u := \operatorname{Re} f \in C^1(U, \mathbb{R})$ and $v := \operatorname{Im} f \in C^1(U, \mathbb{R})$. Then f is real differentiable on U, and by the results of Section 2.4, f is holomorphic if and only if it satisfies the Cauchy–Riemann equation

$$\partial_i f = i\partial_1 f$$

on U. Equivalently, f is holomorphic if and only if its real and imaginary parts satisfy the Cauchy–Riemann equations on U:

$$\begin{cases} \partial_1 u = \partial_i v \\ \partial_i u = -\partial_1 v. \end{cases}$$

Recall that $H(U)$ is the collection of functions holomorphic on U.

Claim. *$H(U) \cap C^1(U) = \ker \overline{\partial}$.*

Remark. One can in fact show that a holomorphic function is automatically infinitely differentiable, hence $H(U) \cap C^1(U) = H(U)$ and

$$H(U) = \ker \overline{\partial}.$$

Proof. This is just a reformulation of the above considerations concerning the Cauchy–Riemann equations.

Supposing first that $f \in H(U) \cap C^1(U)$, one finds

$$\overline{\partial} f = \frac{1}{2}(\partial_1 f + i\partial_i f) = \frac{1}{2}(\partial_1 f + i^2 \partial_1 f) = 0.$$

On the other hand, if $\overline{\partial} f = 0$, then $\partial_1 f = -i\partial_i f$, from which $\partial_i f = i\partial_1 f$ and $f \in H(U)$ follows. ∎

Claim. *$\partial f = f'$ for all $f \in H(U) \cap C^1(U)$.*

Remark. As remarked above, $H(U) \cap C^1(U) = H(U)$, hence

$$\partial f = f' \quad \text{for all} \quad f \in H(U).$$

Proof. Assuming $f \in H(U) \cap C^1(U)$,

$$\partial f = \frac{1}{2}(\partial_1 f - i\partial_i f) = \frac{1}{2}(\partial_1 f - i^2 \partial_1 f) = \partial_1 f = f'.$$ ∎