BAIRE SPACES

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1. BAIRE SPACES

Consider a topological space X. We employ the following terminology:

(1) A set $A \subseteq X$ is *nowhere dense* if $\overline{A}$ has empty interior in X.
Since a subset of X has empty interior if and only if its complement is dense, it follows that A is nowhere dense if and only if $X \setminus \overline{A}$ is dense; in fact, if and only if $X \setminus \overline{A}$ is dense and open. A closed set $A \subseteq X$ is nowhere dense if and only if its complement $X \setminus A$ is dense (and open).

(2) A subset of X is of *first category* or *meager* if it is a countable union of nowhere dense sets. This holds if and only if it is contained in the union of a sequence of closed sets with empty interiors.

(3) A complement of a meager set is *comeager* or *residual*. Some authors, e.g., [2] also use the term *generic*, but we shall not follow this convention.

(4) A subset which is not of the first category is said to be of the *second category*.

(5) We call a sequence of dense open sets an *open Baire sequence*. We call a sequence of closed sets with empty interiors a *closed Baire sequence*. This terminology is convenient, but non-standard.

(6) X is a *Baire space* if $\bigcap G_k$ is dense in X for every open Baire sequence $(G_k)_{k\in\mathbb{N}}$. Equivalently, X is Baire iff $\bigcup F_k$ has empty interior for every closed Baire sequence $(F_k)_{k\in\mathbb{N}}$.

(7) A (Baire) *category theorem* is a statement about when a topological space is Baire.

Claim. *A non-empty Baire space is not a countable union of nowhere dense sets.*

Proof. If the non-empty Baire space X was a countable union of nowhere dense sets A_k, then $X = \bigcup \overline{A_k}$ would yield $\emptyset = \bigcap (X \setminus \overline{A_k})$, contradicting the fact that the sets $X \setminus \overline{A_k}$ consitute an open Baire sequence. ■

Claim. *An open subspace of a Baire space is Baire.*

Proof. We follow [1, p. 297] and use the closed Baire sequence formulation.

Suppose Y is an open subspace of the Baire space X. Let (A_k) be a sequence of closed subsets of Y with empty interiors in Y. Then $\bigcup A_k$ has empty interior in Y.

Indeed, let $\overline{A_k}$ be the closure of A_k in X. Then $Y \cap \overline{A_k} = A_k$. One checks that each $\overline{A_k}$ has empty interior in X, hence so does $\bigcup \overline{A_k}$. One checks further that if $\bigcup A_k$ did not have empty interior in Y, then $\bigcup \overline{A_k}$ would not have empty interior in X, a contradiction. ■

2. Complete Metric Spaces are Baire

First a characterization of completeness.

Lemma 2.1. *A non-empty metric space (X, d) is complete if and only if $\bigcap E_n \neq \emptyset$ for every descending sequence (E_n) of non-empty, closed sets such that $\operatorname{diam} E_n \to 0$. In fact, $\bigcap E_n$ will in this case consist of a unique point.*

Proof. Assume first that X is complete and pick for each n a point $x_n \in E_n$. Since $\operatorname{diam} E_n \to 0$, the sequence (x_n) is Cauchy. Letting $x := \lim x_n$, we see that $x \in E$ since the tail $(x_k)_{k \geq n}$ belongs to the closed set E_n for all n. Hence $E \neq \emptyset$. If E contained two distinct points, we would have $\operatorname{diam} E > 0$, contradicting the fact that $\operatorname{diam} E \leq \operatorname{diam} E_n \to 0$.

Assume now that the closed set formulation holds, and let (x_n) be a Cauchy sequence. Consider the tails

$$T_N := \{x_n : n \geq N\} \quad (N \in \mathbb{N}).$$

Then $\overline{T_0} \supseteq \overline{T_1} \supseteq \cdots$ is a descending sequence of non-empty, closed sets such that $\operatorname{diam} \overline{T_N} = \operatorname{diam} T_N \to 0$ since (x_n) is Cauchy. Picking $x \in \bigcap \overline{T_N}$, one has $x_n, x \in \overline{T_n}$, hence $d(x_n, x) \leq \operatorname{diam} \overline{T_n} \to 0$, so that $x_n \to x$, and the proof is complete. ■

We henceforth let (X, d) be a non-empty, complete metric space and prove it is Baire. Indeed, let (G_n) be an open Baire sequence in X; we prove $\bigcap G_n$ is dense. To this end, let $W \subseteq X$ be a non-empty open set. We prove

$$W \cap \left(\bigcap G_n\right) \neq \emptyset. \tag{2.1}$$

The strategy will be to locate a point in the intersection (2.1) by means of a recursively constructed sequence of closely contained subsets, and then appeal to Lemma 2.1.

Lemma 2.2. *Given a non-empty, open subset $U \subseteq X$ and $\epsilon > 0$, there exists a non-empty open ball $B \subseteq X$ such that $\operatorname{diam} \overline{B} < \epsilon$ and $\overline{B} \subseteq U$.*

Proof. U contains arbitrarily small balls. ■

We now construct the aforementioned sequence.

Considering G_0 is open and dense, the intersection $G_0 \cap W$ is a non-empty open subset of X. By Lemma 2.2 we find a non-empty open ball B_0 in X such that $\operatorname{diam} B_0 < 2^{-0}$ and $\overline{B_0} \subseteq G_0 \cap W$.

Next, since G_1 is open and dense, the intersection $G_1 \cap B_0$ is open and non-empty. Appealing to Lemma 2.2 again, we find a non-empty open ball B_1 in X with $\operatorname{diam} B_1 < 2^{-1}$ and $\overline{B_1} \subseteq G_1 \cap B_0$.

In general, having chosen $B_0, \ldots, B_n$, using that G_{n+1} is open and dense in X, we see that $G_{n+1} \cap B_n$ is a non-empty open subset of X. By Lemma 2.2 we find a non-empty open ball B_{n+1} in X such that $\operatorname{diam} B_{n+1} < 2^{-(n+1)}$ and $\overline{B_{n+1}} \subseteq G_{n+1} \cap B_n$.

We thus obtain a sequence (B_n) of non-empty, open balls with properties that follow.

Claim. $\operatorname{diam} \overline{B_n} = \operatorname{diam} B_n < 2^{-n}$ *for all $n \in \mathbb{N}$.*

Proof. Evident. ∎

Claim. *$\overline{B_0} \subseteq W$.*

Proof. Evident. ∎

Claim. *$\overline{B_{n+1}} \subseteq B_n$ for all $n \in \mathbb{N}$.*

Proof. For each n we chose B_{n+1} such that

$$\overline{B_{n+1}} \subseteq G_{n+1} \cap B_n \subseteq B_n.$$ ∎

Claim. *$\overline{B_n} \subseteq G_n$ for all $n \in \mathbb{N}$.*

Proof. This holds by construction for $n = 0$. For $n \geq 1$, we have by construction

$$\overline{B_n} \subseteq G_n \cap B_{n-1} \subseteq G_n.$$ ∎

All in all, we see that $\overline{B_0} \supseteq \overline{B_1} \supseteq \cdots$ is a descending sequence of non-empty, closed subsets of X such that $\operatorname{diam} \overline{B_n} \to 0$. By Lemma 2.1, let $w \in \bigcap \overline{B_n}$. By the above claims, $w \in \overline{B_n} \subseteq G_n$ for all n, and $w \in \overline{B_0} \subseteq W$, hence $w \in W \cap \left(\bigcap G_n\right)$ and (2.1) follows.

3. Locally Compact Hausdorff Spaces are Baire

It suffices to prove that compact Hausdorff spaces are Baire. For assuming this and using that a locally compact Hausdorff space may be embedded as an open subspace of a compact Hausdorff space (its one-point compactification), the general case follows.

So we henceforth assume X is a non-empty compact Hausdorff space and proceed to show it is Baire. It is well-known that compact Hausdorff spaces are normal, from which the following lemma is direct.

Lemma 3.1. *Given a non-empty, open set $U \subseteq X$, there is a non-empty open set V such that $\overline{V} \subseteq U$.*

Proof. Pick $u \in U$. Then u does not lie in the closed set $X \setminus U$, hence we find by regularity disjoint open sets V, W such that $u \in V$ and $X \setminus U \subseteq W$. We claim $\overline{V} \subseteq U$.

Firstly, $V \subseteq X \setminus W$, hence $\overline{V} \subseteq X \setminus W$, since W is open.

Secondly, we have $X \setminus W \subseteq U$ since $X \setminus U \subseteq W$, and the claim follows. ∎

We mimick the proof that complete metric spaces are Baire, relying on Lemma 3.1 instead of Lemma 2.2.

We also recall that a non-empty collection $\mathscr{F}$ of subsets of X is said to have the *finite intersection property* if the intersection of every non-empty, finite subcollection of $\mathscr{F}$ has non-empty intersection.

Since X is compact, it follows that every non-empty collection of closed subsets of X with the finite intersection property has non-empty intersection; in fact, this criterion characterises compactness completely.

We now prove that X is Baire, and to this end we consider an open Baire sequence (G_n) in X.

Claim. *$\bigcap G_n$ is dense.*

Proof. Let $W \subseteq X$ be open and non-empty; we prove that (2.1) holds also in this case.

Since G_0 is open and dense, we find a non-empty open set V_0 in X such that $\overline{V_0} \subseteq G_0 \cap W$.

Having chosen $V_0, \ldots, V_n$, the set $G_{n+1} \cap V_n$ is open and non-empty, hence we obtain a non-empty open set V_{n+1} such that $\overline{V_{n+1}} \subseteq G_{n+1} \cap V_n$.

We thus obtain a descending sequence $\overline{V_0} \supseteq \overline{V_1} \supseteq \cdots$ of non-empty, closed sets automatically satisfying the finite intersection property. Let $w \in \bigcap \overline{V_n}$ by compactness of X. We claim $w \in W \cap (\bigcap G_n)$.

Firstly, $w \in \overline{V_0} \subseteq W$, so that $w \in W$.

Secondly, one shows as in Section 2 that $w \in \overline{V_n} \subseteq G_n$ for all n, and the claim follows. ∎

References

[1] James R. Munkres. *Topology.* Second Edition. Upper Saddle River: Prentice Hall, 2000.

[2] Elias M. Stein and Rami Shakarchi. *Functional Analysis: Introduction to Further Topics in Analysis.* Vol. 4. Princeton Lectures in Analysis. Princeton, N.J.: Princeton University Press, 2011.