Optical Fiber Geophysics HW 1

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1 Ampere's Law

Start with Ampere's Law and constitutive relations as given in lecture:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}; \qquad \vec{J} = \sigma \vec{E}; \qquad \vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon \vec{E}; \quad \epsilon = \epsilon_0 (1 + \chi_e) \tag{1}$$

Use constitutive relations to rewrite in terms of $\vec{E}$.

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial}{\partial t} \vec{E} \tag{2}$$

Stokes' Theorem allows us to integrate a curl of $\vec{H}$ as a surface integral of $\vec{H}$. We can thus write Ampere's law as:

$$\oint \left(\vec{\nabla} \times \vec{H} \right) \cdot dl = \iint \left(\sigma \vec{E} + \epsilon \frac{\partial}{\partial t} \vec{E} \right) \cdot dS \tag{3}$$

Some re-arranging...

$$\begin{aligned} \oint \left(\vec{\nabla} \times \vec{H} \right) \cdot dl &= \iint \left(\sigma \vec{E} \right) \cdot dS + \iint \left(\epsilon \frac{\partial}{\partial t} \vec{E} \right) \cdot dS && (4) \\ &= \sigma \iint \vec{E} \cdot dS + \epsilon \frac{\partial}{\partial t} \iint \vec{E} \cdot dS && (5) \\ &= \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \iint \vec{E} \cdot dS && (6) \end{aligned}$$

When the loop is parallel to the boundary surface the magnetic density field is reliant purely on $\vec{E}$ and, as far as we know, $\vec{E}$ and $\vec{H}$ are continuous. So we can say $\vec{E}_1^{||} = \vec{E}_2^{||}$.

2 Snell's Law

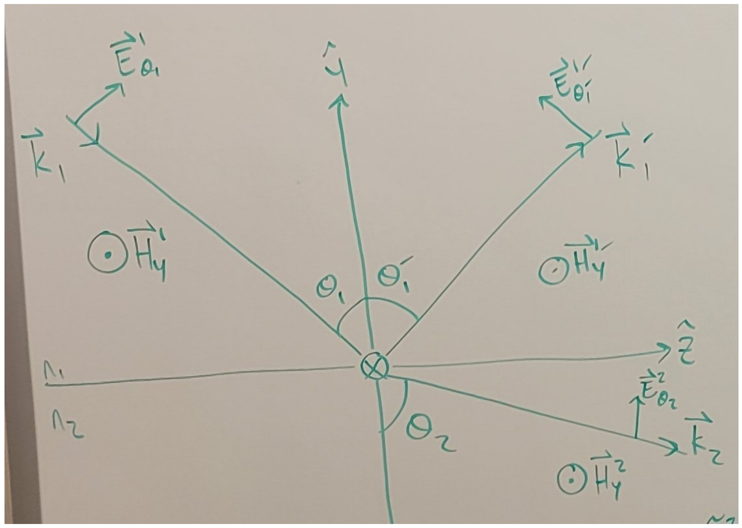

Figure 1: Transverse Magnetic Geometry

Start with the general solutions:

$$\vec{H}^1_y = \tilde{H}^1_y e^{\left(-i\vec{k}_1\cdot\vec{r}_1\right)} e^{(-i\omega t)}; \quad \vec{k}_1 = n_1 \frac{\omega}{c_1}\begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix}; \quad \vec{r}_1 = \begin{pmatrix} x \\ z \end{pmatrix} \tag{7}$$

$$\vec{H}^{1'}_y = \tilde{H}^{1'}_y e^{\left(-i\vec{k}_1'\cdot\vec{r}_1'\right)} e^{(-i\omega t)}; \quad \vec{k}_1' = n_1 \frac{\omega}{c_1}\begin{pmatrix} -cos\theta_1' \\ sin\theta_1' \end{pmatrix}; \quad \vec{r}_1' = \begin{pmatrix} x \\ z \end{pmatrix} \tag{8}$$

$$\vec{H}^2_y = \tilde{H}^2_y e^{\left(-i\vec{k}_2\cdot\vec{r}_2\right)} e^{(-i\omega t)}; \quad \vec{k}_2 = n_2 \frac{\omega}{c_2}\begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix}; \quad \vec{r}_2 = \begin{pmatrix} x \\ z \end{pmatrix} \tag{9}$$

Our boundary condition states that components $\vec{B}$ and $\vec{E}$ parallel to the interface are continuous. $\therefore \vec{H}^2_y = \vec{H}^1_y + \vec{H}^{1'}_y$ at $x = 0$.

$e^{(-i\omega t)}$ appear in each term and can be divided out of the equation, allowing us to write:

$$\tilde{H}^2_y e^{\left(-i\vec{k}_2\cdot\vec{r}_2\right)} = \tilde{H}^1_y e^{\left(-i\vec{k}_1\cdot\vec{r}_1\right)} + \tilde{H}^{1'}_y e^{\left(-i\vec{k}_1'\cdot\vec{r}_1'\right)} \tag{10}$$

$$\tilde{H}^2_y e^{\left(-in_2\frac{\omega}{c_2}\cdot\begin{pmatrix} cos\theta_2 \\ sin\theta_2 \end{pmatrix}\cdot\begin{pmatrix} x \\ z \end{pmatrix}\right)} = \tilde{H}^1_y e^{\left(-in_1\frac{\omega}{c_1}\cdot\begin{pmatrix} cos\theta_1 \\ sin\theta_1 \end{pmatrix}\cdot\begin{pmatrix} x \\ z \end{pmatrix}\right)} + \tilde{H}^{1'}_y e^{\left(-in_1\cdot\begin{pmatrix} -cos\theta_1' \\ sin\theta_1' \end{pmatrix}\cdot\begin{pmatrix} x \\ z \end{pmatrix}\right)} \tag{11}$$

Realize that $x = 0$, and $\theta_1^{'} = \theta_1$ and we can write:

$$\tilde{H}_y^2 e^{\left(-in_2 \frac{\omega}{c_2} z \sin\theta_2\right)} = \left(\tilde{H}_y^1 + \tilde{H}_y^{1'}\right) e^{\left(-in_1 \frac{\omega}{c_1} z \sin\theta_1\right)} \tag{12}$$

Divide both sides by $\tilde{H}_2$ and take the natural logarithm:

$$\left(-in_2 \frac{\omega}{c_2} z \sin\theta_2\right) = \left(-in_1 \frac{\omega}{c_1} z \sin\theta_1\right) \tag{13}$$

Now divide through by $-i\omega z$ and we have our solution:

$$\frac{n_2}{c_2} \sin\theta_2 = \frac{n_1}{c_1} \sin\theta_1 \tag{14}$$