

Optical Fiber Geophysics HW 1

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1 Ampere's Law

Start with Ampere's Law and constitutive relations as given in lecture:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}; \quad \vec{J} = \sigma \vec{E}; \quad \vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon \vec{E}; \quad \epsilon = \epsilon_0(1 + \chi_e) \quad (1)$$

Use constitutive relations to rewrite in terms of \vec{E} .

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad (2)$$

Stokes' Theorem allows us to integrate a curl of \vec{H} as a surface integral of \vec{H} . We can thus write Ampere's law as:

$$\oint (\vec{\nabla} \times \vec{H}) \cdot d\vec{l} = \iint \left(\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S} \quad (3)$$

Some re-arranging...

$$\oint (\vec{\nabla} \times \vec{H}) \cdot d\vec{l} = \iint (\sigma \vec{E}) \cdot d\vec{S} + \iint \left(\epsilon \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{S} \quad (4)$$

$$= \sigma \iint \vec{E} \cdot d\vec{S} + \epsilon \frac{\partial}{\partial t} \iint \vec{E} \cdot d\vec{S} \quad (5)$$

$$= \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \iint \vec{E} \cdot d\vec{S} \quad (6)$$

When the loop is parallel to the boundary surface the magnetic density field is reliant purely on \vec{E} and, as far as we know, \vec{E} and \vec{H} are continuous. So we can say $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$.

2 Snell's Law

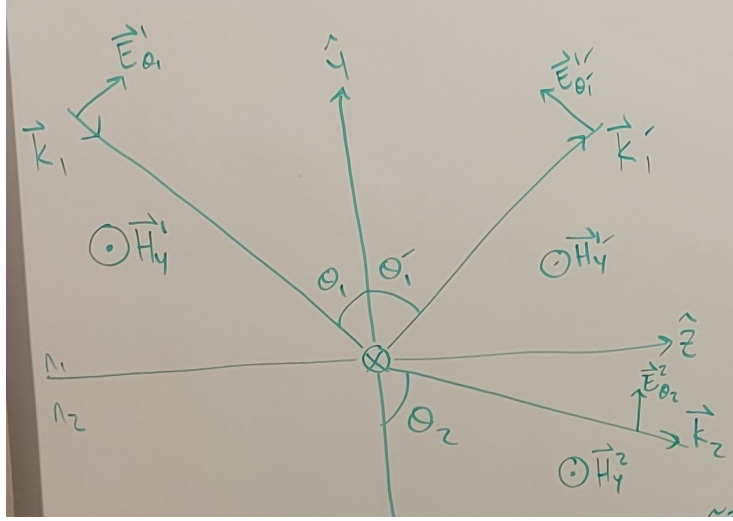


Figure 1: Transverse Magnetic Geometry

Start with the general solutions:

$$\vec{H}_y^1 = \tilde{H}_y^1 e^{(-i\vec{k}_1 \cdot \vec{r}_1)} e^{(-i\omega t)}; \quad \vec{k}_1 = n_1 \frac{\omega}{c_1} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}; \quad \vec{r}_1 = \begin{pmatrix} -x \\ z \end{pmatrix} \quad (7)$$

$$\vec{H}_y^{1'} = \tilde{H}_y^{1'} e^{(-i\vec{k}_1' \cdot \vec{r}_1')} e^{(-i\omega t)}; \quad \vec{k}_1' = n_1 \frac{\omega}{c_1} \begin{pmatrix} -\cos\theta_1' \\ \sin\theta_1' \end{pmatrix}; \quad \vec{r}_1' = \begin{pmatrix} -x \\ z \end{pmatrix} \quad (8)$$

$$\vec{H}_y^2 = \tilde{H}_y^2 e^{(-i\vec{k}_2 \cdot \vec{r}_2)} e^{(-i\omega t)}; \quad \vec{k}_2 = n_2 \frac{\omega}{c_2} \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}; \quad \vec{r}_2 = \begin{pmatrix} -x \\ z \end{pmatrix} \quad (9)$$

Our boundary condition states that components \vec{B} and \vec{E} parallel to the interface are continuous. $\therefore \vec{H}_y^2 = \vec{H}_y^1 + \vec{H}_y^{1'}$ at $x = 0$.

$e^{(-i\omega t)}$ appear in each term and can be divided out of the equation, allowing us to write:

$$\tilde{H}_y^2 e^{(-i\vec{k}_2 \cdot \vec{r}_2)} = \tilde{H}_y^1 e^{(-i\vec{k}_1 \cdot \vec{r}_1)} + \tilde{H}_y^{1'} e^{(-i\vec{k}_1' \cdot \vec{r}_1')} \quad (10)$$

$$\tilde{H}_y^2 e^{\left(-in_2 \frac{\omega}{c_2} \cdot \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \cdot \begin{pmatrix} -x \\ z \end{pmatrix}\right)} = \tilde{H}_y^1 e^{\left(-in_1 \frac{\omega}{c_1} \cdot \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \cdot \begin{pmatrix} -x \\ z \end{pmatrix}\right)} + \tilde{H}_y^{1'} e^{\left(-in_1 \cdot \begin{pmatrix} -\cos\theta_1' \\ \sin\theta_1' \end{pmatrix} \cdot \begin{pmatrix} -x \\ z \end{pmatrix}\right)} \quad (11)$$

Realize that $x = 0$, and $\theta'_1 = \theta_1$ and we can write:

$$\tilde{H}_y^2 e^{\left(-in_2 \frac{\omega}{c_2} z \sin \theta_2\right)} = \left(\tilde{H}_y^1 + \tilde{H}_y^{1'}\right) e^{\left(-in_1 \frac{\omega}{c_1} z \sin \theta_1\right)} \quad (12)$$

Divide both sides by \tilde{H}_2 and take the natural logarithm:

$$\left(-in_2 \frac{\omega}{c_2} z \sin \theta_2\right) = \left(-in_1 \frac{\omega}{c_1} z \sin \theta_1\right) \quad (13)$$

Now divide through by $-i\omega z$ and we have our solution:

$$\frac{n_2}{c_2} \sin \theta_2 = \frac{n_1}{c_1} \sin \theta_1 \quad (14)$$