Variational and PDE Approaches to Image Denoising

Jun Hao (Simon) Hu *

February 27, 2019

Abstract

Image processing has long since been studied, typically by Electrical Engineers, for its various applications. Image processing is a fundamental tool used to understand the geometry and patterns of our three-dimensional world from a two-dimensional image. As such, it is vital that we develop a firm mathematical foundation that meets the demands of the modern era: computational efficiency, well-posedness and robustness. In this paper, I paint a broad picture of mathematical image processing through a PDE based approach which, within the past century, has proven to be a promising approach. I then study the problem of image denoising, which has important applications in many disciplines.

1 Introduction.

In today's modern world, digital images play a fundamental role in our daily lives. From taking photos to medical imaging, digital images provide us with insight into the complicated, yet beautiful, geometric world around us from a two-dimensional point of view. In the field of virology, electron microscopes are used to capture still images of viruses, which allows us to gain insight into their biological structure so that we can develop vaccines. Even with the advanced technology available to us, digital images suffer from noise, blur, incomplete information and other imperfections. As an example, images of the brain captured through medical resonance imaging (MRI) are inherently noisy due to the software implemented in the device. Given the importance of digital images in our daily lives, image restoration methods must be developed to remove imperfections.

To understand the role PDEs play in image processing, we have to first understand what an image really is. A digital image, captured through an image acquisition device, is a discrete representation of a scene in the continuous universe. Through sampling and quantization processes the device is able to lay out a grid and assign a value, such as the light intensity, to each grid element. If the grid is fine enough, the image appears to be continuous despite that it is a discrete object. In the mathematical image processing literature these grid elements are called *pixels* in two-dimensional space and *voxels* in three-dimensional space. Mathematically, we represent images as mappings defined on a continuous space. By casting images into mappings we make significant headway: we have taken a discrete object and cast it into a continuous object with rich mathematical structure. In essense, we cast an image $u = u(\mathbf{x}) : U \to \mathbb{R}^d$ as a real-valued function on a Lipschitz domain with Lipschitz boundary. In the case $d = 1$ we consider grayscale images and in the cases $d = 2, 3$ we consider color images[1]. This paper will only focus on the case where $d = 1$, corresponding to grayscale images. The methods presented here do work for $d > 1$, but the analysis and numerical methods become significantly more complicated.

Unfortunately, even with today's advanced technology, image acquisition devices cannot perfectly capture images without getting corrupted by noise. The task of removing noise from an image has become a fundamental task in image processing. Our main goal is to remove the noise from an image u_0 using variational and PDE methods. Though various models for images have been proposed (see [reference here] for discussion), we will use the simple model

$$u_0 = Au + \epsilon \tag{1.1}$$

where $A \in \mathbf{L}(\mathbb{R}^d, \mathbb{R}^d)$ is a linear bounded operator, and ϵ is some model of the corruption. In essense, we assume that the corrupted image is a linear combination of the true image u and some corruption. The corruption process depends

*Department of Electrical and Computer Engineering, University of California San Diego. Please send any corrections to simonhu@ieee.org

[1]More generally, we call these images multi-channel images.

on the operator A. For example, in image denoising problems we take $A = I_{d\times d}$ and in image deblurring problems we take A to be some convolution operator.

It is appropriate for us to assume that $\epsilon \sim \mathcal{N}(\mu, \mathbf{\Sigma})$ is Gaussian with mean μ and covariance $\mathbf{\Sigma} = \sigma^2 \mathbf{I}_{m\times m}$. This is not the *only* model for the noise—there are better models out there—but this one is chosen for convenience. In theory, solving the inverse problem gives us the original, noiseless, image u but this is a rather daunting task. Nowhere have we made the assumption that the operator A has an inverse. Even if an inverse were to exist, there is no guarantee on its numerical stability (i.e. the inverse is ill-conditioned so that the inverse operation may be numerically unstable). Furthermore, even if the operator were invertible and well-conditioned, we cannot account for the error between our corruption model and the actual corruption. The point is this: solving the inverse problem is not a viable option for recoving the original image, and thus our approach will have to avoid using the inverse. To avoid inverses, we cast the problem into solving a PDE through the use of tools from the calculus of variations. Our hope is that by casting the problem into solving a PDE, we can utilize the plethora of tools, developed in the field, to provide insight into the denoising problem.