Lecture 10

Saturday, October 15, 2016 4:37 PM

MATH 223 Linear Algebra

$\boxed{\text{Oct } 6, 2016}$

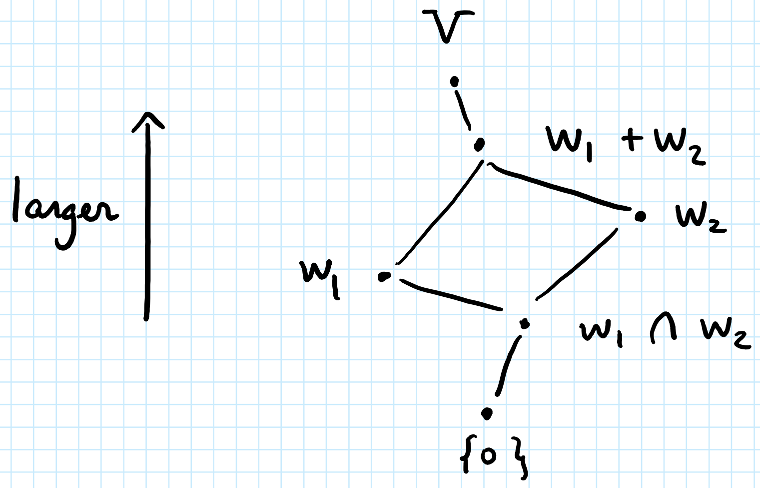
W_1, W_2 subspaces of V

$$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \}$$

Theorem:

(i) $W_1 + W_2$ subspaces containg W_1, W_2

(ii) $W_1 + W_2$ smallest subspace containing W_1, W_2



ex. $V = M_n(\mathbb{R})$

$W_1 = \{\text{upper triangular}\}$

$W_2 = \{\text{lower triangular}\}$

$W_1 + W_2 = V$

Linear Combinations

V vector space over K

$u_1, \dots, u_m$ vectors (elements of V)

$$v = \underbrace{a_1 u_1 + \dots + a_m u_m}_{\text{linear combination of vectors } \{u_1, \dots, u_m\}} \quad (a_i \in K)$$

3 important ideas of Linear Combination

- spanning set = "sufficiency"
- independent set = "no redundancy"
- basis (spanning & independent)

Def^n: $\{u_1, \dots, u_m\}$ spanning set

if each $v \in V$ is a linear combination

of $\{u_1, \dots, u_m\}$

ex. $V = \mathbb{R}^3$

$\{(1,0,0), (0,1,0), (0,0,1)\}$ spanning

$\{ (1,0,0), (0,1,0), (1,1,0) \}$ non-spanning

$$\because \quad a(1,0,0) + b(0,1,0) + c(1,1,0)$$
$$= (a+c, \ b+c, \ 0)$$

<u>counter</u>: vectors such as $(0,0,1) \in \mathbb{R}^3$ are not linear combinations of this set

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<u>ex.</u> $\{ (1,2), (2,1), (2,2) \}$ in $\mathbb{R}^2$

$$x(1,2) + y(2,1) + z(2,2)$$
$$= (x + 2y + 2z, \ 2x + y + 2z)$$

given $(a,b) \in \mathbb{R}^2$ solve for x, y, z (some sol^n)

let
$$a = x + 2y + 2z$$
$$b = 2x + y + 2z$$

Take $z=0$

$$a+b = 3(x+y) \Rightarrow y = a - \frac{a+b}{3} = \frac{2a-b}{3}$$
$$x = b - \frac{a+b}{3} = \frac{2b-a}{3}$$

<u>ex.</u> $\{ u_1, \dots, u_m, w \}$ independent

$\Rightarrow \{ u_1, \dots, u_m \}$ independent

<u>why?</u> $\sum a_i u_i = 0$

$$\sum a_i u_i + 0 \cdot w_i = 0$$
$$\Rightarrow \quad a_1 = a_2 = \dots = a_m = 0$$

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ex. $\{u_1, \dots, u_m\}$ independent

and w <u>not</u> a linear combination of $\{u_1, \dots, u_m\}$

$\Rightarrow \{u_1, \dots, u_m, w\}$ independent

Why? $\sum a_i u_i + a w = 0$

if $a \neq 0$, $w = -\sum \frac{a_i}{a} u_i \leftarrow$ linear combination of $\{u_1, \dots, u_m\}$ $\lightning$

So $a = 0$ $\quad \sum a_i u_i = 0 \Rightarrow a_1 = \dots = a_m = 0$

$\because \{u_1, \dots, u_m\}$ independent

Def^n: $\{u_1, \dots, u_m\}$ <u>basis</u> for V

if both spanning and independent

Theorem: $\{u_1, \dots, u_m\}$ basis iff

every $v \in V$ $\underbrace{\text{can be written}}_{\text{spanning}}$ $\underbrace{\text{uniquely}}_{\text{independence}}$

as a linear combination of $\{u_1, \dots, u_m\}$

Proof: exercise

ex. $\mathbb{R}^n$ $(1, 0, \dots, 0)$

$(0, 1, \dots, 0)$

$\vdots$

$(0, 0, \dots, 1)$

$e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$

ith position

$\{ e_i : i = 1, \dots, n \}$

standard basis

$M_{m \times n}(K)$

e_{ij} = $m \times n$ matrix whose entries are all 0, except for the ijth entry which is $\underline{1}$

$$\sum_{i,j} a_{ij} e_{ij} = 0_{m \times n}$$

Linearly independent

$$(a_{ij}) = 0_{m \times n}$$

$$\{ e_{ij} : 1 \le i \le m, \ 1 \le j \le n \}$$

basis

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ex. upper triangular matrices

$$\{ e_{ij} : 1 \le i, j \le n \}$$

ex: $Pol_n(K)$

$\{ 1, x, x^2, \dots, x^n \} \leftarrow$ basis

$a_0 + a_1 x + \dots + a_n x^n$ linear combination

How to Get a Basis

① **Bottom-Up**

saturate an independent set

- pick an independent set
- check if it spans: if yes ✓
  - if No, take a vector which is not a linear combination of the set, and add it to the set
- repeat

② **Top-Down**

trim a spanning set

- pick a spanning set
- check if independent: if yes ✓
- if No, pick a vector which is a linear combination and drop it.
- repeat

Read: Sections 4.4, 4.5, 4.7, 4.9 – 4.11

Exercises: 4.6, 4.7, 4.9, 4.11, 4.15 – 4.24

Remarks: S spanning set

A superset of S : spanning

A subset of S : may or may not span

ex: $\{u_1, \dots, u_m\}$ spanning

$\Rightarrow \{u_1, \dots, u_m, w\}$ spanning

add a new vector

Why? Given $v \in V$

$$v = \sum a_i u_i + 0 \cdot w_i$$

ex. $\{ u_1, \dots, u_m, w \}$ spanning

and w linear combination of $\{u_1, \dots, u_m\}$

$\Rightarrow$ $\{ u_1, \dots, u_m \}$ spanning

Why? $v \in V$, $v = \sum a_i u_i + a w$

$$w = \sum c_i u_i$$

$$\Rightarrow v = \sum a_i u_i + a \left(\sum c_i u_i \right)$$
$$= \sum (a_i + a c_i) u_i$$

Def^n: $\{ u_1, \dots, u_m \}$ linearly independent if

there exists scalars $a_1, \dots, a_m$ not all 0

s.t. $a_1 u_1 + \dots + a_m u_m = 0$;

and linearly independent otherwise

i.e. if $a_1 u_1 + \dots + a_m u_m = 0$, then $a_1 = \dots = a_m = 0$

ex. $V = \mathbb{R}^3$

$$\{ (1,0,0), (0,1,0), (0,0,1) \}$$

$$a(1,0,0) + b(0,1,0) + c(0,0,1) = (0,0,0)$$

then $(a,b,c) = (0,0,0)$

$a = b = c = 0 \Rightarrow$ linearly independent

<u>ex.</u> $\{ (1,0,0), (0,1,0), (1,1,0) \}$

$$(1,0,0) + (0,1,0) + -1(1,1,0) = (0,0,0)$$

$\Rightarrow$ linearly dependent

<u>ex.</u> $\{ 0, v_1, \ldots, v_k \} \Rightarrow$ linearly dependent

$$1 \cdot 0 + 0 \cdot v_1 + \ldots + 0 \cdot v_k = 0$$

<u>ex.</u> $V = \mathbb{R}^3$

$$\{ (1,0,1), (1,1,0), (0,1,1) \}$$

$$a(1,0,1) + b(1,1,0) + c(0,1,1)$$

$$= (a+b, b+c, a+c)$$

Set $\Rightarrow$ $a + b = 0 \Rightarrow a = -b \Rightarrow$ linearly dependent

$b + c = 0$ $\quad \boxed{b = -c}$

$a + c = 0$ $\quad a = -c$

<u>Remark</u> : S independent

A superset of S : may or may not be independent

A subset of S : independent