Lecture 5

Sunday, September 25, 2016 3:11 PM

MATH 223 Linear Algebra

$\boxed{\text{Sept } 20, 2016}$

2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A) = ad - bc$$

$$\operatorname{tr}(A) = a + d$$

$$A' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad AA' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}$$

$$= A'A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Case 1 $\det A \neq 0$

A non-singular, with inverse $A^{-1} = \frac{1}{\det A}\overbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}^{A}$

Case 2 $\det A = 0$ $\quad \cancel{A^2} = 0$

$AA^{-1} = 0 \rightarrow A$ singular

Why? assume A non-singular, some A^{-1} exists

Then

$$\underbrace{\underbrace{(A^{-1}A)}_{I_2} A'}_{A'} = \underbrace{A^{-1}(0)}_{O_2}$$

$$\Rightarrow A' = 0$$

$$\Rightarrow a = b = c = d = 0$$

$$\Rightarrow A = 0 \text{, contradiction}$$

Principle

$AX = 0$, $X \neq 0$

$\Rightarrow A$ singular

Theorem : Cayley - Hamilton

Let A 2×2 matrix.

Then $\underbrace{A^2 - \operatorname{tr}(A)A + \det(A)I_2}_{\text{polynomial expression}} = O_2$

Proof:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}$$

$$-\operatorname{tr}(A)\, A = \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix}$$

$$A^2 - \operatorname{tr}(A)\, A = \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix}$$

$$= -\det(A)\, I_2$$

$\square$

<u>Remark</u>: if $\det A \neq 0$, then

$$\det(A)\, I_2 = \operatorname{tr}(A)\, A - A^2$$

$$= A\left(\operatorname{tr}(A)\, I_2 - A \right)$$

$$= \left(\operatorname{tr}(A)\, I_2 - A \right) A$$

where $\operatorname{tr}(A)\, I_2 - A$

$$= \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>Examples</u>: inverses

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{1-0} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{0-1}\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{Rotation}}$$

Diagonal Matices

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} + \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1+b_1 & & \\ & \ddots & \\ & & a_n+b_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix} = \begin{pmatrix} a_1 b_1 & & \\ & \ddots & \\ & & a_n b_n \end{pmatrix}$$

$$k\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} ka_1 & & \\ & \ddots & \\ & & ka_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} 1/a_1 & & \\ & \ddots & \\ & & 1/a_n \end{pmatrix}$$

inverse of diagonal matix

A, B diagonal $\Rightarrow$ AB diagonal

Let $A = (a_{ij})$ $\quad \underbrace{a_{ij} = 0 \text{ for } i \neq j}_{\text{all off-diagonal entries vanish}}$

$B = (b_{ij})$ $\quad b_{ij} = 0$ for $i \neq j$

ij-entry of AB for $i \neq j$:

$$\sum_k a_{ik} b_{kj}$$

$$= a_{ii} b_{ij} = 0$$

$\square$