MATH 223 Linear Algebra

$\boxed{\text{Oct 4, 2016}}$

Vector Spaces & Subspaces

Pol (K) : set of all polynomials with coefficients in a field K

$$\left\{ P(t) = a_0 + a_1 t + \dots + a_n t^n, \text{ where } a_1, \dots, a_n \in K, n \text{ arbitrary} \right\}$$

$\deg(P(t))$ = largest n s.t. $a_n \neq 0$

ex. $\deg(1 + t + 0 \cdot t^2 + t^3 + \cancel{0 t^4}) = 3$

$\deg(0)$ = DNE or $-\infty$ ← not important

addition
$(2 - t + t^2) + (2t - t^3) = 2 + t$

multiplication
$3(1 + t^2) = 3 + 3 + 3t^2$

Pol(K) vector space over K

Have to check axioms

Fun (X, K) : set of all functions from a (non-empty) set X to a field K } vector space over K

Have to check axioms

Fun $(\{1,2,3\}, \mathbb{R})$ is $\mathbb{R}^3$ (?)

when X finite, it is like tuples
X infinite, things get nice

Recall:

Theorem: let V vector space over a field K

Then a subset $W \subseteq V$ is a subspace IFF

(i) $0 \in W$

(ii) $u, v \in W \Rightarrow u + v \in W$

(iii) $k \in K, \ u \in W \Rightarrow ku \in W$

Ex. $Pol_n(K)$: set of all polynomials of degree $\leq n$

$\{ P(t) = a_0 + a_1 t + \dots + a_n t^n : a_i \in K \}$

Claim: Subspace of $Pol(K)$

① $0 \in Pol_n(K)$

② $P, Q \in Pol_n(K) \Rightarrow \deg(P+Q) \leq \underbrace{\max(\underbrace{\deg P}_{\leq n}, \underbrace{\deg Q}_{\leq n})}_{\leq n}$

③ $k \in K, \ P \in Pol_n(K)$

$\Rightarrow \deg(kP) = \deg(P) \quad (k \neq 0)$

Ex. subspaces of $Fun([a,b], \mathbb{R})$ — closed interval

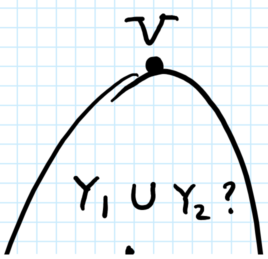
- continuous functions $[a,b] \to \mathbb{R}$
- differentiable functions $[a,b] \to \mathbb{R}$

$\vdots$

V : vector space over K

What are the subspaces ?

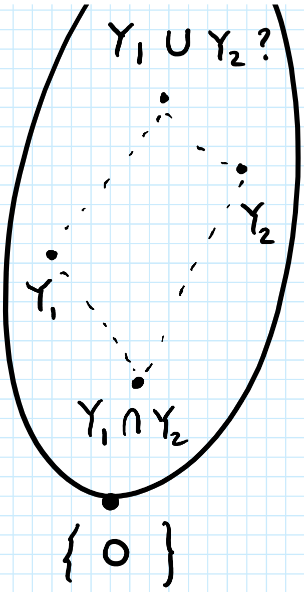
ex. $\{0\}$ & V



ex. $\{0\}$ & V

ex. $u \in V$ vector

$$\langle u \rangle = \{ ku : k \in K \}$$



Subspace

$$0 = 0 \cdot u$$

$$ku, lu = ku + lu = (k+l)u$$

$$k(lu) = (kl)u$$

Ex. W_1, W_2 subspaces $\Rightarrow$ $W_1 \cap W_2$ subspace

① $0 \in W_1 \cap W_2$ ✓ since $0 \in W_1$, $0 \in W_2$

② $u, v \in W_1 \cap W_2$ ✓ $u, v \in W_1 \Rightarrow u+v \in W_1$

$u, v \in W_2 \Rightarrow u+v \in W_2$

$\therefore u+v \in W_1 \cap W_2$

③ $k \in K$, $u \in W_1 \cap W_2$

$\Rightarrow ku \in W_1$, $ku \in W_2$

$\Rightarrow ku \in W_1 \cap W_2$

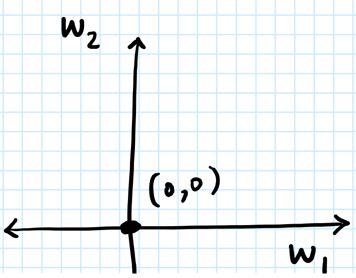
True / False: W_1, W_2 subspaces $\Rightarrow$ $W_1 \cup W_2$ subspace ?

$V = \mathbb{R}^2$ (vector space over $\mathbb{R}$)

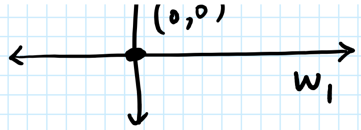
$W_1 = \{ (x, 0) : x \in \mathbb{R} \}$

$W_2 = \{ (0, y) : y \in \mathbb{R} \}$

$W_1 \cap W_2 = \{ (0,0) \}$



$W_1 \cap W_2 = \{(0,0)\}$



$W_1 \cup W_2 = ?$

① $(0,0) \in W_1 \cup W_2$

② $u, v \in W_1 \cup W_2 \not\Rightarrow u+v \in W_1 \cup W_2$

counter: $(1,0) \in W_1$, $(0,1) \in W_2$

but $(1,1) \notin W_1 \cup W_2$ $\lightning$

③ $k \in \mathbb{R}$, $u \in W_1 \cup W_2$

$\Rightarrow ku \in W_1 \cup W_2$

$\therefore W_1 \cup W_2$ not a subspace

Theorem: W_1, W_2 subspaces. Then TFAE:

(the following are equivalent)

(i) $W_1 \cup W_2$ subspace

(ii) either $W_1 \subseteq W_2$ OR $W_2 \subseteq W_1$

$\left.\right\}$ $i \Leftrightarrow ii$

Proof: $\boxed{i \rightarrow ii}$ by contradiction

$\boxed{ii \rightarrow i}$ easy

$\boxed{ii \rightarrow i}$ Say $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$

and W_2 is a subspace (by hypothesis)

$\therefore W_1 \cup W_2$ is a subspace.

$\boxed{i \rightarrow ii}$ Assume ii does not hold, i.e. $W_1 \nsubseteq W_2$

AND $W_2 \nsubseteq W_1$

Pick $w_1 \in W_1$, $w_1 \notin W_2$

$w_2 \in W_2$, $w_2 \notin W_1$

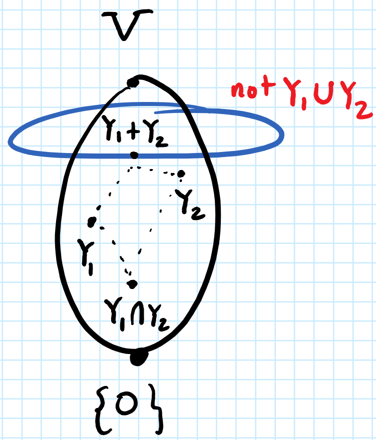
Look at $w_1 + w_2 \in W_1 \cup W_2$ by i

$$\left.\begin{array}{l} \text{Say } w_1 + w_2 \in W_1 \\ \text{but } w_1 \in W_1 \end{array}\right\} \Rightarrow w_2 \in W_1 \ \text{↯ contradiction}$$

W_1 , W_2 subspaces

Define $\underbrace{W_1 + W_2}_{\text{sum}} = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \}$

Theorem:

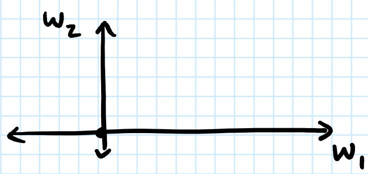


(i) $W_1 + W_2$ subspace

(ii) $W_1 + W_2$ contains W_1 and W_2

(iii) $W_1 + W$ smallest subspace containing W_1 and W_2

back to



$$W_1 + W_2 = \{ \underbrace{(x,0) + (0,y)}_{(x,y)} : x, y \in \mathbb{R} \}$$

$$= V$$

Proof:

① $0 \in W_1 + W_2 \rightarrow 0 = 0 + 0$

② $\left.\begin{array}{l} \omega_1 + \omega_2 \in W_1 + W_2 \\ \omega_1' + \omega_2' \in W_1 + W_2 \end{array}\right\} \rightarrow (\omega_1 + \omega_2) + (\omega_1' + \omega_2')$

$$= \underbrace{(\omega_1 + \omega_1')}_{\in W_1} + \underbrace{(\omega_2 + \omega_2')}_{\in W_2}$$

③ Scalar multiplication : Exercise

Given arbitrary $\omega_1 \in W_1$

$$\omega_1 = \omega_1 + \underbrace{0}_{\in W_2} \in W_1 + W_2$$