MATH 223 Linear Algebra | Oct 18, 2016

Linear Mappings (continued)

Def^n: A map $F : V \to W$ between 2 linear spaces (over the same field K) is said to be linear if

$$F(v_1 + v_2) = F(v_1) + F(v_2), \quad \forall v_1, v_2 \in V$$

$$F(kv) = k\,F(v), \quad \forall k \in K, v \in V$$

example: $\text{trace} : M_n(K) \to K$

given $A \in M_{m \times n}(K)$ $\quad F_A : K^n \to K^m$

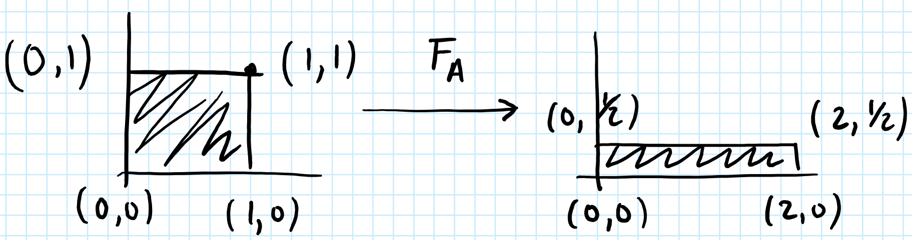
$F_A(v) = Av$

column vectors

$A \in M_2(\mathbb{R})$, $F_A : \mathbb{R}^2 \to \mathbb{R}^2$, where $F_A\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$

① Rescaling

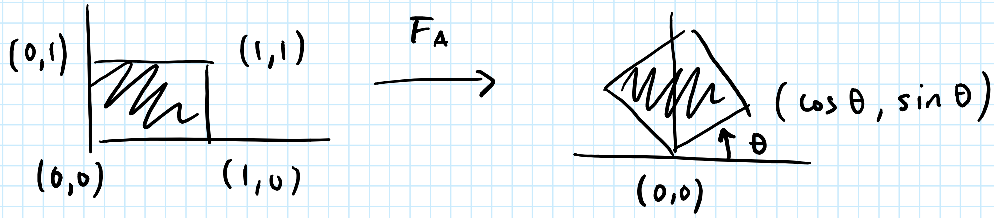
$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad F_A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{y}{2} \end{pmatrix}$$



$$F_A = \begin{pmatrix} 2 \cdot 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

② Rotation

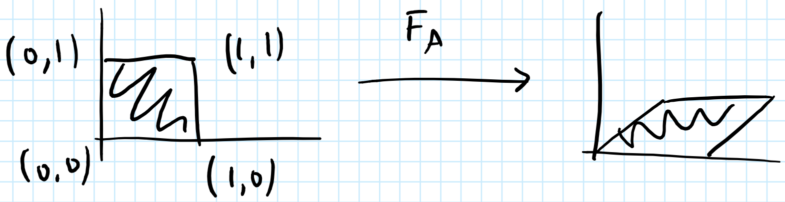
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$F_A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$F_A = \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

③ $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

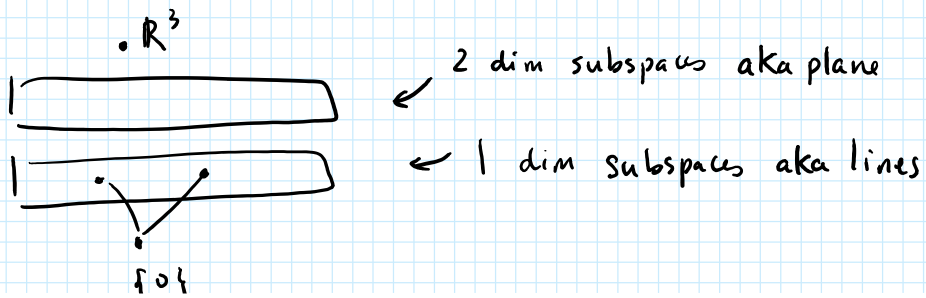


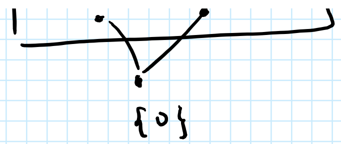
$$F_A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$F_A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

Note: area is same! Why? Det = 1 ⇝ area-preserving

MATH RANT - Motivation





Theorem:

Let $F : V \to W$ linear map. Then:

- $\mathrm{Ker}(F)$ subspace of V and
  "all the things that send to 0"
- $\mathrm{Range}(F)$ subspace of W
  "Image"

Proof:

① $\mathrm{Ker}(F)$: need to check

$\boxed{\text{Ker}(F) \text{ sends all to } 0 \text{ by def}^n}$

- $0 \in \mathrm{Ker}(F)$ ← clearly, $F(0) = 0$
- $v_1, v_2 \in \mathrm{Ker}(F)$

  $\Rightarrow v_1 + v_2 \in \mathrm{Ker}(F)$

  $\because F(v_1) = 0,\ F(v_2) = 0$

  $F(v_1 + v_2) = F(v_1) + F(v_2) = 0$
- $k \in K,\ v \in \mathrm{Ker}(F)$

  $\Rightarrow kv \in \mathrm{Ker}(F)$

  $\because$ we know $F(v) = 0$, so $F(kv) = k \underbrace{F(v)}_{=0} = 0$

② $\mathrm{Range}(F)$:

- $0 \in \mathrm{Range}(F)$ ← $F(0) = 0$
  ↖ image of 0
- $w_1, w_2 \in \mathrm{Range}(F)$

  $\Rightarrow w_1 + w_2 \in \mathrm{Range}(F)$

for some v_1, v_2 : $F(v_1) = w_1$, $F(v_2) = w_2$

$$F(v_1 + v_2) = F(v_1) + F(v_2) = w_1 + w_2$$

- $k \in K$, $w \in \text{Range}(F)$

$\Rightarrow kw \in \text{Range}(F)$

for some v, $F(v) = w \Rightarrow Kw = kF(v) = F(kv)$

Theorem: Rank - Nullity

Let $F: V \to W$ linear map, assume V, W finite - dimensional

Then, $\underbrace{\dim \text{Ker}(F)}_{\text{nullity of } F} + \underbrace{\dim \text{Range}(F)}_{\text{Rank of } F} = \dim V$

(important that V is finite - dimensional)

Remark:

let $F: V \to W$ linear map

- F is surjective (or onto) means $\text{Range}(F) = W$
- F is injective (or one-to-one) $\Leftrightarrow \text{Ker}(F) = \{0\}$

$\boxed{\Rightarrow}$ take $v \in \text{Ker}(F)$.

Then, $F(v) = 0 = F(0) \Rightarrow v = 0$

$\boxed{\Leftarrow}$ let $F(v_1) = F(v_2)$

So, $F(v_1) - F(v_2) = 0$

$$\Rightarrow \quad F(v_1 - v_2) = 0 \qquad \because \ F(0) = 0$$

$$\Rightarrow \quad v_1 - v_2 = 0$$

$$\Rightarrow \quad v_1 = v_2$$

<u>Intuition</u>

Range (F) : "measures" how surjective F is.

Kernel (F) : "measures" how injective F is.

<u>Defn</u>: A bijective map $F: V \to W$ is said to be a <u>linear isomorphism</u>.
("same shaped")

Two linear spaces are isomorphic if $\exists$ a <u>linear isomorphism</u> $F: V \to W$

<u>Corollary</u>:

V, W isomorphic $\Rightarrow \dim V = \dim W$

<u>Proof</u>: let $F: V \to W$ linear isomorphism

$$\dim V = \underbrace{\dim \operatorname{Ker}(F)}_{\substack{=0 \\ \because F \text{ is injective}}} + \underbrace{\dim \operatorname{Ran}(F)}_{\substack{=\dim W \\ \because F \text{ is surjective}}}$$

$$\Rightarrow \dim V = 0 + \dim W$$
$$= \dim W$$

$\blacksquare$