Lecture 11

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MATH 223 Linear Algebra $\boxed{\text{Oct 11, 2016}}$

Bases

V vector space over a field K

$S \subseteq V$ subset $S = \{ s_1, \dots, s_n \}$

Def^n: S basis for V if S independent & spanning

S basis $\Rightarrow$ every vector $v \in V$ can be written uniquely as a linear combination of $s_1, \dots, s_n$

$$v = a_1 s_1 + \dots + a_n s_n \quad , \; a_i \in K$$

n-Tuple $(a_1, \dots, a_n)$: coordinates of $v \in V$ with respect to basis S

! Theorem :

All bases of V have the same number of elements.

This allows for

Def^n: The Dimension of V, denoted $\dim(V)$, is the number of elements in any basis.

ex. $\dim \mathbb{R}^n = n$

$\dim M_n(\mathbb{R}) = n \cdot n = n^2$

$\dim \{ \text{upper triangular} \} = \dfrac{n(n+1)}{2}$

$\dim \{ \text{lower triangular} \} = \dfrac{n(n+1)}{2}$

$\dim \mathrm{Pol}_n(K) = n+1$ (basis : all monomials
$\{1, t, t^2, \dots t^n\}$)

$\dim \{0\} = 0$ (by convention)

$\dim \mathrm{Pol}(K) = \infty$ (there is no finite basis)
a basis: all monomials $\{1, t, t^2, \dots\}$

$\mathbb{R}$ as an $\mathbb{R}$-vector space has dim 1
$\mathbb{Q}$ ← cannot write basis

Hey Professor,
I was wondering if an infinite dimensional vector space can have a basis, and if so how?

! <u>Theorem</u>
There exists a basis for V (not a given)

Two Algorithms (for finite cases)

- top-down : trim a spanning set
- bottom-up : enrich an independent set

Does this <u>stop</u>?

<u>Theorem</u>: All bases of V have the same number of elements.

using <u>Lemma</u>: if $\{f_1, \dots, f_m\}$ independent and $\{s_1, \dots, s_n\}$ spanning, then $m \leq n$

<u>Proof</u>:

$\{u_1, \dots, u_m\}$ basis: independent & spanning
$\{v_1, \dots, v_n\}$ basis: independent & spanning

$\Rightarrow$ $m \leq n$, $n \leq m$

$\{ v_1, \dots, v_n \}$ basis : independent & spanning | $\underbrace{n \leq m}_{\substack{\text{Lemma} \\ \text{twice}}}$

$\Rightarrow m = n$

Proof (Lemma):

Idea: exchange f_1 for some s

exchange f_2 for some other s

$\vdots$

f_m

Write $f_1 = a_1 s_1 + \dots + a_n s_n$

at least one of $a_1, \dots, a_n$ is $\neq 0$

WLOG, assume $a_1 \neq 0$

$\Rightarrow$ write s_1 = linear combination of $f_1, s_2, \dots, s_n$

$\Rightarrow \{ f_1, s_2, \dots, s_n \}$ spanning

Write $f_2 = b_1 f_1 + b_2 s_2 + \dots + b_n s_n$

at least one of $b_2, \dots, b_n$ is $\neq 0$

WLOG, assume $b_2 \neq 0$

$\Rightarrow$ write s_2 = linear combination of $f_1, f_2, s_3, \dots, s_n$

$\Rightarrow \{ f_1, f_2, s_3, \dots, s_n \}$ spanning

and so on. In m steps,

get $\{ f_1, f_2, \dots, f_m, \underbrace{\dots}_{s} \}$ spanning

Theorem: Assume $\dim V = n$. Then:

① Any collection of more than n vectors

$S \subseteq V$

① S independent

$\Rightarrow |S| \leq n$

① Any collection of more than $\underline{n}$ vectors is not independent.

② Any collection of less than $\underline{n}$ vectors is not spanning.

① S independent $\Rightarrow |S| \leq n$

② S spanning $\Rightarrow |S| \geq n$

$|\ | = \#$ of elements

Proof: Let B be a basis, $|B| = n$

① S independent $\Rightarrow |S| \leq |B| = n$ (B spanning)

② S spanning $\Rightarrow |S| \geq |B| = n$ (B independent)

> Lemma

Theorem: Assume $\dim V = n$. Let $S \subseteq V$, $|S| = n$

Then S independent $\Leftrightarrow$ S spanning $\Leftrightarrow$ S basis

ex: $Pol_n(K)$. We know $\{1, t, t^2, \dots, t^n\}$ is a basis

Claim: given any scalar $c \in K$,

$$\{1, t-c, (t-c)^2, \dots, (t-c)^n\} \text{ basis}$$

spanning : a bit harder ($\rightarrow$ Taylor Expansion)

independent : a bit easier

Linear combination $= 0$

$$\underbrace{a_n (t-c)^n + a_{n-1}(t-c)^{n-1} + \dots + a_1 (t-c)}_{\text{polynomial of } a_n t^n + \text{ lower order term}} + a_0 = 0$$

polynomial of $a_n t^n$ + lower order term $= 0$

$\Rightarrow a_n = 0$

S independent $\Rightarrow S'$ spanning

assume S' not spanning, let v not a linear combination of elements from S'

$S' \cup \{v\}$ independent as well

$|S' \cup \{v\}| = n+1$ ↯

S' spanning $\Rightarrow$ S' independent

assume S' not independent, let $s \in S'$ expressed as a linear combination of $S' \setminus \{s\}$

$S' \setminus \{s\} \Rightarrow S' \setminus \{s\}$ spanning as well

$|S' \setminus \{s\}| = n-1$ ↯