Structure of White Dwarf Stars

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Abstract

In this report the equations of state for a white dwarf star are derived, scaled and solved numerically using the 4th order Runge-Kutta method. An explanation of the mathematics behind Euler's method and the 2nd order Runge-Kutta is provided. Data collected via the computational model is used to explore the Mass/Radius relationship in Carbon and Iron core dominated White Dwarf Stars. The data is then used to identify the Stella composition of White Dwarfs and to accurately find the Chandrasekhar limit.

Introduction

It is an inescapable truth that one day, several billion years from now the evolution of our star will make planetary conditions on Earth impossible to sustain life. As the sun builds up a deposit of Helium at its core that reaches a critical density and temperature, our star will continue to fuse hydrogen nuclei in expanding concentric shells. The increased thermal energy due to gravitational contraction of the growing helium core will exert more radially outward pressure. In order for Hydrostatic equilibrium between the force of gravity and thermal pressure the star will begin to expand, during this sub giant phase the sun will become larger and brighter until our planet falls out of the CHZ (Circumstellar habitable zone) and its oceans are boiled away.

Although our planet will once again become molten the evolution of our star will be far from over. At the end of our suns Red Giant phase, the envelope of the star will be expelled forming a planetary nebula while leaving behind the Stella core destined to become a White Dwarf Star. No longer undergoing nuclear fusion, the matter in the remaining core will compress under gravity until all the electrons have been separated from the atomic nuclei. Forming a gas the electrons will become compressed until a new hydrostatic equilibrium is formed between gravity and the electron degeneracy pressure halting the White Dwarfs compression.

Electron degeneracy pressure comes about from the quantum mechanical effect known as the Pauli Exclusion Principle where fermions may not simultaneously occupy the same quantum state. During core compression free electrons are forced very close together and the exclusion principle forces them to occupy different energy states. Adding a further electron to such a small volume forces an electron to occupy a higher state which requires energy. This energy requirement infers the electron degeneracy pressure which may be modeled as the opposing force to gravity during stellar hydrostatic equilibrium.

A stars mass will determine its final evolutionary form. A more massive star core experiences a greater compressive force due to gravity, if this force is great enough it can overcome the electron degeneracy pressure and lead to further compression. When this happens the gravitational pressure combines protons and electrons culminating in a core comprised of neutrons where

$$p^{+} + e^{-} \rightarrow n + \nu_e$$

Thus forming a neutron star. Above another mass threshold neutron degeneracy pressure can be overcome and a black hole may be formed. From low to high mass stars the common possible outcomes are a brown dwarf, white dwarf, neutron star or black hole.

Subrahmanyan Chandrasekhar an Indian Astrophysicist born in 1930 is credited with deriving the dubbed Chandrasekhar limit. Over several papers during his time as a student at the University of Cambridge he deduced the mass limit stars must be under in order to become stable White Dwarfs.

Throughout this report the derivations of the equations of state for White Dwarf Stars are pursued yielding coupled Ordinary differential equations which may be numerically solved. By implementing a 4th order Runge-Kutta algorithm this coupled system may be solved enabling graphical plots of Stella Mass vs Radius and the arrival at the Chandrasekhar limit ourselves.

Methodology

In order to model a white dwarf star the involved forces governing the hydrostatic equilibrium of gravity and electron degeneracy pressure are considered and linked. The forces due to gravity and pressure are respectively identified as $dF_g \; and \; dF_p$. By assuming a perfectly spherical star an infinitesimal volume of matter $dV = Adr$ was considered at a radius r with mass $dm = \rho(r)dV$. Owing to the symmetry of the star the gravitational force on dm may be found while ignoring the mass beyond the stars radius r while treating the mass as a point mass at the stars centre. The component force of gravity is thus

$$dF_g = -\frac{Gm(r)}{r^2}\rho(r)dV \tag{0.1}$$

Where the stars mass is defined as $m(r)$

$$m(r) = 4\pi\int_0^r \rho(r')r'^2 dr' \tag{0.2}$$

Both sides of equation 0.2 are then differentiated with respect to r

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \tag{0.3}$$

The outward radial force on dm is due to the resultant pressure difference arising from the two competing forces of gravity (directed at the stars centre of mass, radially inward) and the radially outward force due to the “degeneracy pressure” of electrons in their quantum states.

$$dF_p = P(r)A - P(r+dr)A \tag{0.4}$$

Where A is the area the radial pressure affects.

By Taylors theorem

$$P(r+dr) = P(r) + \frac{dP(r)}{dr}dr + \cdots \tag{0.5}$$

It can be expressed via a 1st order expansion that

$$dF_p = P(r)A - \left[P(r) + \frac{dP(r)}{dr}dr\right]A = -\frac{dP(r)}{dr}dV \tag{0.6}$$

The condition for equilibrium is that the forces are balanced implying $dF_g + dF_p = 0$

Which yields

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r) \tag{0.7}$$

Equations 0.3 and 0.7 link $M(r)$, $\rho(r)$ and $P(r)$ to define the equilibrium state of the white dwarf.

But by using the chain rule

$$\frac{dP(r)}{dr} = \frac{d\rho}{dr} * \frac{dP}{d\rho} \tag{0.8}$$

The derivative of the density ρ may be found

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1} * \frac{dP(r)}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1} * \frac{GM(r)}{r^2}\rho(r) \tag{0.9}$$

Equations 0.3 and 0.9 give two coupled 1st order ODE's that may be solved using numerical methods. Before they may be solved however the term $\left(\frac{dP}{d\rho}\right)$ must be defined. This was achieved by approximating the equation of state for matter in a white dwarf to a relativistic free Fermi gas to describe the origin of the electron degeneracy pressure. The derivation that followed with complete steps may be referred to in the appendices of this report.

Firstly the kinematics were considered for the degenerate electron gas where

$$\epsilon_p^2 = m_e^2c^4 + p^2c^2 \tag{0.10}$$

Then a density of states was introduced in order to express the degeneracy pressure as an integral. After a change of variables this integral was evaluated to find the following equation.

$$P = \frac{\frac{hc}{4}\left(\frac{3}{8\pi}\right)^{\frac{1}{3}}\left(\frac{Y_e\rho}{m_H}\right)^{\frac{4}{3}} 3[x_F(1+x_F^2)^{\frac{1}{2}}\left(\frac{2}{3x_F^2}-1\right) + ln\left[x_F + (1+x_F^2)^{\frac{1}{2}}\right]]}{2x_F^4} \tag{0.11}$$

Where the dimensionless fermi momentum x_F is expressed as a function of density

$$x_F = \frac{\left(\frac{3Y_e}{8\pi m_H}\right)^{\frac{1}{3}} h}{m_e c}\rho^{\frac{1}{3}} \tag{0.12}$$

And the defined constants of

$$n_e = \frac{Y_e\rho}{m_H} = number\ of\ electrons\ per\ unit\ volume$$

$$Y_e = the\ number\ of\ electrons\ per\ nucleon$$

$$m_H = proton\ mass$$

By taking the radial derivative of equation 0.11 and performing a substantial amount of algebra (see appendices) it can be shown that

$$\frac{dP}{dr} = \frac{Y_e m_e c^2}{3m_H} * \frac{x^2}{(1+x^2)^{\frac{1}{2}}} * \frac{d\rho}{dr} \tag{0.13}$$

Where appropriate use of the chain rule

$$\frac{dP}{dr} = \frac{d\rho}{dr} * \frac{dP}{d\rho} \tag{0.14}$$

Reveals that

$$\frac{dP}{d\rho} = \frac{Y_e m_e c^2}{3m_H} * \frac{x^2}{(1+x^2)^{\frac{1}{2}}} \tag{0.15}$$

Using equation 0.12 and the following identity

$$x = \left(\frac{\rho}{\rho_o}\right)^{\frac{1}{3}} = \frac{\left(\frac{3Y_e}{8\pi m_H}\right)^{\frac{1}{3}} h}{m_e c}\rho^{\frac{1}{3}} \tag{0.16}$$

The constant ρ_o may be defined

$$p_o = \frac{m_p m_e^3 c^3}{3\pi^2 \hbar^3 Y_e} \tag{0.17}$$

$\frac{dP}{d\rho}$ is then simplified further to

$$\frac{dP}{d\rho} = \frac{Y_e m_e c^2}{m_H} * \frac{(\frac{\rho}{\rho_o})^{\frac{2}{3}}}{3\left(1+(\frac{\rho}{\rho_o})^{\frac{2}{3}}\right)^{\frac{1}{2}}} = \frac{Y_e m_e c^2}{m_H}\gamma\left(\frac{\rho}{\rho_o}\right) \tag{0.18}$$

Now that the quantity $\left(\frac{dP}{d\rho}\right)$ is identified it is substituted into equation 0.9 to give the new equations of state

$$\frac{d\rho}{dr} = -\frac{Gm(r)\rho(r)m_H}{r^2\gamma\left(\frac{\rho}{\rho_o}\right)Y_e m_e c^2} \tag{0.19}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{0.20}$$

Now that the two equations of state are identified, a numerical method of solving this coupled system is used to graphically illustrate the behavior of white dwarf stars.

Computational Implementation

Numerical solutions to ordinary differential equations are often required when the solution is impractical or otherwise impossible to find analytically. Under such circumstances one may look towards Euler's Method to approach such a problem. However after a brief derivation of the method it will be briefly shown how it became the foundation for a more accurate algorithm by several orders of magnitude.

Assuming an ODE is of the form

$$\frac{dx}{dt} = f(x,t) \tag{0.21}$$

Some initial condition fixes x for some value of t. The value of x may be found after a short interval h by using a Taylor expansion.

$$x(t+h) = x(t) + h\frac{dx}{dt} + \frac{1}{2}h^2\frac{d^2x}{dt^2} = x(t) + hf(x,t) + O(h^2) \tag{0.22}$$

Where $O(h^2)$ represents the terms that go to h^2 and higher. If the interval h is small then the terms related to h^2 may be ignored and we're left with

$$x(t+h) = x(t) + hf(x,t) \tag{0.23}$$

With the value of x known at a time t another value of x may be found using this equation. And with repeated use one can find successive values for x at evenly spaced points. This method however yields only approximate solutions, this is due to neglecting the h^2 and higher order terms. For the specific step of h^2 the error is

$$\frac{1}{2}h^2\frac{d^2x}{dt^2} \tag{0.24}$$

Although this error decreases as h gets smaller the cumulative error should be considered. If for this example a solution was needed between $t = a$ and $t = b$ using a step size of h the total number of steps is

$$N = \left(\frac{b-a}{h}\right) \tag{0.25}$$

By writing the values of t at which the steps fall by $t_k = a + kh$ with the corresponding values of x by x_k then the total error over all steps is

$$\begin{aligned}\sum_{k=0}^{N-1}\frac{1}{2}h^2\left(\frac{d^2x}{dt^2}\right)_{\substack{x=x_k\\t=t_k}} &= \frac{1}{2}h\sum_{k=0}^{N-1}h\left(\frac{df}{dt}\right)_{\substack{x=x_k\\t=t_k}} = \frac{1}{2}h[f(x(b),b) - f(x(a),a] \\ &\cong \frac{1}{2}h\int_a^b \frac{df}{dt}dt\end{aligned} \tag{0.26}$$

This method is also known as the 1st order Runge-Kutta. The method essentially takes the slope $\frac{dx}{dt}$ at time t and extrapolates it for use at a future time equal to $t+h$. A better approximation is to perform such an extrapolation using the slope at time $t+\frac{1}{2}\ h$, this leads to a significantly better estimate of $x(t+h)$ and is the basis for the 2nd order Runge-Kutta method.

Once again a Taylor expansion is required around $t+\frac{1}{2}h$ to get a value for $x(t+h)$.

$$x(t+h)=x\left(t+\frac{1}{2}h\right)+\frac{1}{2}h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h}+\frac{1}{8}h^2\left(\frac{d^2x}{dt^2}\right)_{t+\frac{1}{2}h}+O(h^3) \qquad (0.27)$$

And is also used to find $x(t)$

$$x(t)=x\left(t+\frac{1}{2}h\right)-\frac{1}{2}h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h}+\frac{1}{8}h^2\left(\frac{d^2x}{dt^2}\right)_{t+\frac{1}{2}h}+O(h^3) \qquad (0.28)$$

Subtracting equations 0.27 and 0.28 yields equation 0.29 after rearranging

$$\begin{aligned} x(t+h) &= x(t)+h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h}+O(h^3) \\ &= x(t)+hf\left(x\left(t+\frac{1}{2}h\right),t+\frac{1}{2}h\right)+O(h^3) \end{aligned} \qquad (0.29)$$

From equation 0.29 it may be observed that the term of h^2 is now gone and our new error term has become $O(h^3)$. This implies that the 2nd order Runge-Kutta is a factor of h more accurate than Euler's method.

The method used to solve the coupled equations of state for a white dwarf in this report is the 4th order Runge-Kutta. This method is accurate to terms of order h^4 and only carries an error of order h^5 where in the methods steps the approximation involves ignoring the terms of $O(h^5)$ and above. The equations used for this method can be adapted for a coupled system of equations.

For the following system of coupled ODE's

$$\dot{x}(t)=f(x,y,t) \quad \dot{y}(t)=g(x,y,t) \qquad (0.30)$$

The Runge-Kutta equations are the following

$$k_1=hf(x_n,y_n,t_n) \qquad (0.31)$$

$$l_1=hg(x_n,y_n,t_n) \qquad (0.32)$$

$$k_2=hf(x_n+\frac{k_1}{2},y_n+\frac{l_1}{2},t_{n+\frac{1}{2}}) \qquad (0.33)$$

$$l_2 = hg(x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2}, t_{n+\frac{1}{2}}) \tag{0.34}$$

$$k_3 = hf(x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2}, t_{n+\frac{1}{2}}) \tag{0.35}$$

$$l_3 = hg(x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2}, t_{n+\frac{1}{2}}) \tag{0.36}$$

$$k_4 = hf(x_n + k_3, y_n + l_3, t_{n+1}) \tag{0.37}$$

$$l_4 = hg(x_n + k_3, y_n + l_3, t_{n+1}) \tag{0.38}$$

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 3k_3 + k_4) \tag{0.39}$$

$$y_{n+1} = y_n + \frac{1}{6}(l_1 + 2l_2 + 3l_3 + l_4) \tag{0.40}$$

Before applying this method it is necessary to scale the equations of state. Using very large or small numbers will likely lead to a loss of precision during the programs steps leading to cumulative error. Recalling the equations of state (0.19 and 0.20) the following change of variables is introduced during the nondimensionalization of the equations of state.

$$\bar{r} = \frac{r}{R_0} \qquad \bar{p} = \frac{p}{p_0} \qquad \bar{m} = \frac{m}{M_0}$$

A substitution of the new variables is done and then the constants R_0 and M_0 are defined in order to set all constants in the equations of state equal to one.

$$\frac{dm}{dr} = 4\pi r^2 p = \frac{d\bar{m}M_0}{d\bar{r}R_0} = 4\pi \bar{r}^2 R_0^2 \bar{p} p_0 \tag{0.41}$$

$$\frac{d\bar{m}}{d\bar{r}} = \frac{4\pi \bar{r}^2 R_0^3 \bar{p} p_0}{M_0} \tag{0.42}$$

Set

$$M_0 = 4\pi R_0^3 p_0 = Y_e^2 * 5.656765603 * 10^{30} \tag{0.43}$$

$$\frac{dp}{dr} = -\frac{\left(\frac{dP}{dp}\right)^{-1} Gmp}{r^2} = -\frac{\left(\frac{Y_e m_e c^2}{m_p}\gamma\left(\frac{p}{p_0}\right)\right)^{-1} Gmp}{r^2} \tag{0.44}$$

$$\begin{aligned}\frac{d\bar{p}}{d\bar{r}} &= -\frac{G\bar{m}M_0\bar{p}p_0 m_p R_0}{\gamma(\bar{p})Y_e m_e c^2 \bar{r}^2 R_0^2 p_0} = -\frac{\bar{m}\bar{p}G(4\pi R_0^3 p_0)p_0 m_p R_0}{\gamma(\bar{p})\bar{r}^2 Y_e m_e c^2 R_0^2 p_0} \\ &= -\frac{\bar{m}\bar{p}G4\pi R_0^2 p_0 m_p}{\gamma(\bar{p})\bar{r}^2 Y_e m_e c^2}\end{aligned} \tag{0.45}$$

Set

$$R_0 = \left(\frac{Y_e m_e c^2}{G 4\pi p_0 m_p}\right)^{0.5} = Y_e * 7713064.722 \tag{0.46}$$

$$p_0 = \frac{m_p m_e^3 c^3}{3\pi^2 \bar{h}^3 Y_e} = 0.9810189408 * \frac{10^9}{Y_e} \tag{0.47}$$

Where

$$\gamma(\bar{\rho}) = \frac{\bar{\rho}^{\frac{2}{3}}}{3\left(1+\bar{\rho}^{\frac{2}{3}}\right)^{\frac{1}{2}}} \tag{0.48}$$

Substituting in the new constants the newly simplified and dimensionless equations of state (equations 0.49 and 0.50) are now suitable to solve using the 4th order Runge-Kutta method. After the values of $\bar{r}$ and $\bar{m}$ are returned by the program they are simply multiplied by their respective constants R_0 and M_0 then converted into units of Solar Radii and Solar Mass.

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2 \bar{p} \tag{0.49}$$

$$\frac{d\bar{p}}{d\bar{r}} = -\frac{\bar{m}\bar{p}}{\gamma(\bar{p})\bar{r}^2} \tag{0.50}$$

Results

The equations were planned to be integrated radially outwards from $r = 0$ with boundary conditions $m = 0$ and $\rho = \rho_c$ but due to computational issues with this they were integrated out from $r = 100^{-10}$ instead.

The number of electrons per nucleon Y_e was set to 0.5 and 0.464 to model a carbon C^{12} and an iron Fe^{56} composed star core respectively. The resultant graphs for these two star types displaying the relationship between the Mass and Radius expressed in terms of solar masses and solar radii are illustrated below.

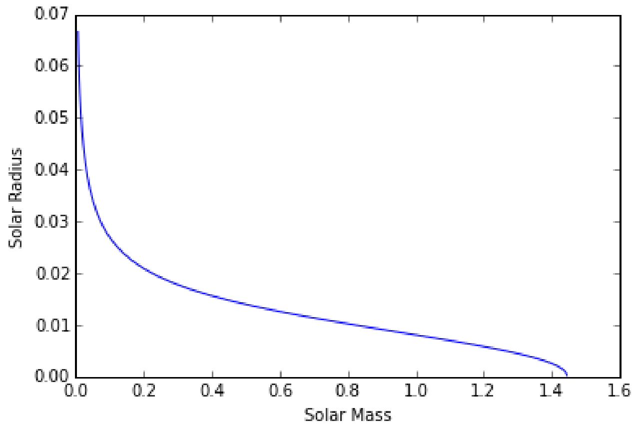


Figure 1:

Solar mass vs solar radius for a star with $Y_e = 0.5$
Graphed straight from Python

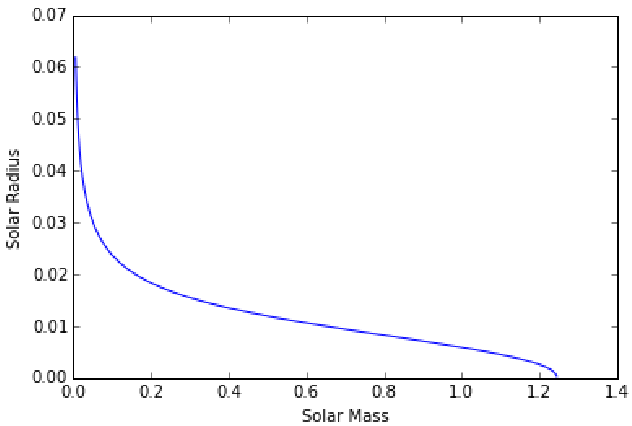


Figure 2:

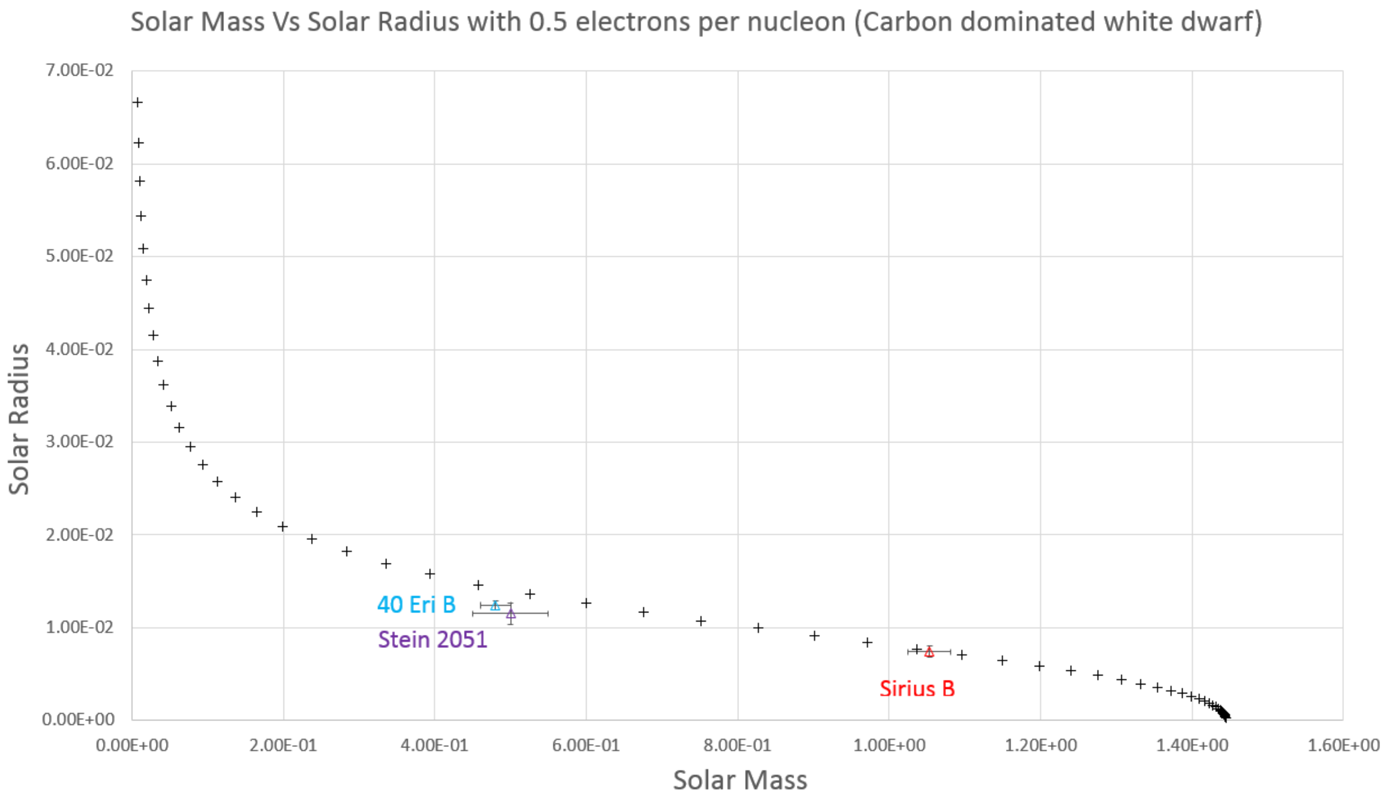
Solar mass vs solar radius for a star with $Y_e = 0.464$
Graphed straight from Python

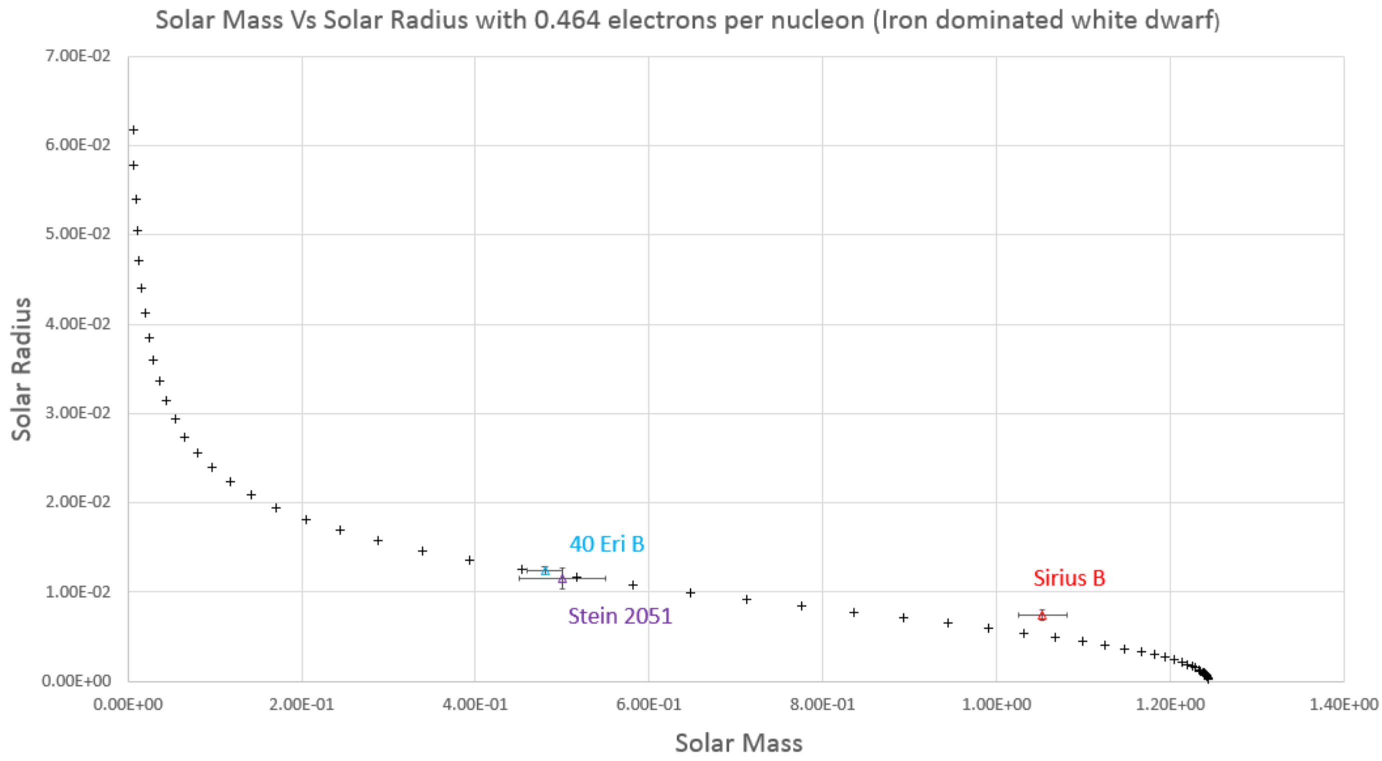
The first impressions from the data tell us that the stars volume decreases with increased mass, this fits the theory of hydrostatic equilibrium. The force of gravity increases with the mass causing the compression of the star. The cutoff point for the stars mass when the radius reaches zero represents the maximum mass and therefore force of gravity that the electron degeneracy pressure can resist, this limit is smaller for stars with fewer electrons per nucleon in this model.

In order to verify the achieved data is valid, three stars of known Radius and Mass with their respective error were plotted alongside both graphs. As well as determining the success of the model, the star core content may be identified as dominantly carbon or iron depending upon which graph they fit best with.

The known results for the three stars used are tabulated below

	$Mass\ (solar\ masses)$	$Radius\ (solar\ radii)$
$Sirius\ B$	1.053 ± 0.028	0.0074 ± 0.0006
$40\ Eri\ B$	0.48 ± 0.02	0.0124 ± 0.0005
$Stein\ 2051$	0.50 ± 0.05	0.0115 ± 0.0012





<u>Figures 3 and 4:</u>

Displays the data for three stars alongside the re-graphed program data.
The above graphs show the trend for a Carbon and an Iron dominated White Dwarf Star

Conclusion

From figures 3 and 4 it can be deduced that while Sirius B is mainly composed of carbon, 40 Eri B and Stein 2051 contain iron and clearly fit the trend of the computed points well within the error bars confirming the validity of the simulated data. By examining the data responsible (tabulated below) for the mass limit of both graphs the threshold mass that determines whether or not a star will become a white dwarf or not was found.

$Star\ Type$	$Threshold\ Mass\ (solar\ masses)$	$Radius\ (solar\ radii)$
$Y_e = 0.5\ (carbon)$	1.44443934	0.000277197986
$Y_e = 0.464\ (iron)$	1.24392805	0.000257239731

Where the result for a carbon modeled white dwarf reproduces the famous Chandrasekhar limit of 1.44 solar masses. By deriving the equations of state and evaluating them via a numerical method the Chandrasekhar limit was accurately found along with a means to discern iron and carbon based white dwarf stars. The match of three known stars to the generated graph indicating the validity of the model.

Improvements to this method of calculation could involve the use of a more sophisticated numerical method with less cumulative error, perhaps involving a higher order Runge-Kutta program. The reader may find it prudent to focus on refining early assumptions adding to the complexity of the model derived, such as the non-rotating star assumption or using a non-relativistic free fermi gas as a model. A possible extension further afield than this report could involve using similar methods to obtain the equations of state for neutron stars and to find the mass limit of such a star. Hydrostatic equilibrium in such cases would involve the forces of gravity and quantum originating degeneracy pressure from neutrons.

References

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Appendices

Derivation of the result

$$\frac{dP}{d\rho} \tag{0.51}$$

Express the kinematics of the degenerate electron gas as

$$\epsilon_p^2 = m_e^2 c^4 + p^2 c^2 \tag{0.52}$$

Introduce the density of states $g(p) = \frac{V}{h^3 4\pi p^2}$ and express pressure

$$P = \frac{1}{3V}\int_0^{P_F} \frac{p^2 c^2 g(p)}{\epsilon_p} \tag{0.53}$$

$$P = \frac{8\pi}{3h^3}\int_0^{P_F} p^4 c^2 (m_e^2 c^4 + p^2 c^2)^{-0.5} dp \tag{0.54}$$

Using change of variables

$$x = \frac{P}{m_e c} \; and \; dx = \frac{dp}{m_e c}$$

Pressure of the electron (degenerate) gas

$$P = \frac{8\pi m_e^4 c^5}{3h^3}\int_0^{x_F} \frac{x^4}{(1+x^2)^{0.5}} dx \tag{0.55}$$

Integrating yields

$$P = \frac{\frac{hc}{4}\left(\frac{3}{8\pi}\right)^{\frac{1}{3}}\left(\frac{Y_e\rho}{m_H}\right)^{\frac{4}{3}} 3[x_F(1+x_F^2)^{\frac{1}{2}}\left(\frac{2}{3x_F^2}-1\right) + ln\left[x_F + (1+x_F^2)^{\frac{1}{2}}\right]]}{2x_F^4} \tag{0.56}$$

Which may be simplified by the following definitions

$$K_1 = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{\frac{1}{3}} = constant$$

$$n_e = \frac{Y_e\rho}{m_H} = number\ of\ electrons\ per\ unit\ volume$$

$$I(x) = 3\left[x(1+x^2)^{\frac{1}{2}}\left(\frac{2}{3x^2}-1\right) + ln\left[x + (1+x^2)^{\frac{1}{2}}\right]\right]/2x^4$$

Where Y_e is the number of electrons per nucleon.

Simplified

$$P = K_1 n_e^{\frac{4}{3}} I\left(x_f\right) \tag{0.57}$$

Defining the dimensionless fermi momentum and then expressing it as a function of density

$$x_F = \frac{P_F}{m_e c} \tag{0.58}$$

$$x_F = \frac{\left(\frac{3n_e}{8\pi}\right)^{\frac{1}{3}} h}{m_e c} \tag{0.59}$$

$$x_F = \frac{\left(\frac{3Y_e}{8\pi m_H}\right)^{\frac{1}{3}} h}{m_e c} \rho^{\frac{1}{3}} \tag{0.60}$$

Take the radial derivative of …

$$\frac{dP}{dr} = \frac{d}{dr} K_1 n_e^{\frac{4}{3}} I(x_F) = \frac{d}{dr} K_1 \left(\frac{Y_e \rho(r)}{m_H}\right)^{\frac{4}{3}} I(x_F) = K_1 * \left(\frac{Y_e}{m_H}\right)^{\frac{4}{3}} \left(\frac{4}{3} * \rho^{\frac{1}{3}} * \frac{d\rho}{dr} I(x) + \rho^{\frac{4}{3}} \left(\frac{dI}{dx}\right) \Big|_{x_F} \left(\frac{dx_F}{dr}\right)\right) \tag{0.61}$$

Where

$$\frac{dI}{dx} = \frac{4x^2 + 6}{2x^4 (1 + x^2)^{\frac{1}{2}}} - \frac{6ln\left[(1 + x^2)^{\frac{1}{2}} + x\right]}{x^5} \tag{0.62}$$

And

$$\frac{dx_F}{dr} = \frac{1}{3K_2 \rho^{-\frac{2}{3}} * \frac{d\rho}{dr}} \tag{0.63}$$

$$\frac{dP}{dr} = K_1 \left(\frac{Y_e}{m_H}\right)^{\frac{4}{3}} \left(\frac{4\rho^{\frac{1}{3}}}{3} * \frac{d\rho}{dr} * I(x) + \rho^{\frac{2}{3}} * \frac{K_2}{3} * \frac{d\rho}{dr} * \frac{dI}{dx}\right) \tag{0.64}$$

Where

$$K_2 = \frac{\left(3 * \frac{Y_e}{8\pi m_H}\right)^{\frac{1}{3}} h}{m_e c} \tag{0.65}$$

Substituting remaining K_2 and ρ for x_F

$$\frac{dP}{dr} = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{\frac{1}{3}}\left(\frac{Y_e}{m_H}\right)^{\frac{4}{3}}\rho^{\frac{1}{3}} * \frac{4x}{3(1+x^2)^{\frac{1}{2}}} * \frac{d\rho}{dr} \tag{0.66}$$

$$\frac{dP}{dr} = \frac{Y_e m_e c^2}{3m_H} * \frac{x^2}{(1+x^2)^{\frac{1}{2}}} * \frac{d\rho}{dr} \tag{0.67}$$

From chain rule

$$\frac{dP}{dr} = \frac{d\rho}{dr} * \frac{dP}{d\rho} \tag{0.68}$$

Therefore

$$\frac{dP}{d\rho} = \frac{Y_e m_e c^2}{3m_H} * \frac{x^2}{(1+x^2)^{\frac{1}{2}}} \tag{0.69}$$

$$x = \left(\frac{\rho}{\rho_o}\right)^{\frac{1}{3}} = \frac{\left(\frac{3Y_e}{8\pi m_H}\right)^{\frac{1}{3}} h}{m_e c}\rho^{\frac{1}{3}} \tag{0.70}$$

Therefore

$$p_o = \frac{m_p m_e^3 c^3}{3\pi^2 \hbar^3 Y_e} \tag{0.71}$$

Where

$$\hbar = h/2\pi$$

Combining … with … while subbing in equation …

$$\frac{dP(r)}{dr} = -\frac{GM(r)}{r^2}\rho(r) = \frac{Y_e m_e c^2}{3m_H} * \frac{(\frac{\rho}{\rho_o})^{\frac{2}{3}}}{\left(1+(\frac{\rho}{\rho_o})^{\frac{2}{3}}\right)^{\frac{1}{2}}} * \frac{d\rho}{dr} \tag{0.72}$$

Yields

$$\frac{d\rho}{dr} = -\frac{GM(r)\rho(r)m_H}{r^2\gamma\left(\frac{\rho}{\rho_o}\right)Y_e m_e c^2} \tag{0.73}$$

Where

$$\gamma(y) = \frac{y^{2/3}}{3\left(1 + y^{\frac{2}{3}}\right)^{\frac{1}{2}}} \tag{0.74}$$

List of constants used

$$M_{\odot} = 1.989 * 10^{30}$$

$$R_{\odot} = 696342000$$

$$m_h = m_p = 1.6726219 \times 10^{-27}$$

$$m_e = 9.10938356 \times 10^{-31}$$

$$G = 6.67408 \times 10^{-11}$$

$$c = 299\,792\,458$$

$$h = 6.62607004 \times 10^{-34}$$