lecture 3 presentation

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Section 1

Statistik opsumering

Expected value

$$E(aX + b) = aE(X) + b$$
$$E(X + Y) = E(X)E(Y)$$

$$E(XY) = E(X)E(Y)$$

- Hvis X og Y er uafhængige
- Bevis for dette behøves i ik at kende

Variance

$$Var(X) = E[X - E(X)]^2 = E(XX) - E(X)E(X)$$

- Vi kan se hvis $E(X) = 0$ er $Var(X) = E(X^2)$
- Bevis ovenstående

$$Var(aX + b) = a^2 Var(X)$$

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

- Bevis ovenstående

$$Var(X + Y) = Var(X) + Var(Y)$$

- Hvis X og Y er uafhængige og dermed $Cov(X, Y) = 0$

Bevis for Var(X)

$$Var(X) = E[X - E(X)]^2 = E(XX) - E(X)E(X)$$

- Løs på tavlen

Bevis for Var(X+Y)

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Covariance

$$Cov(X,Y) = E[X - E(X)][Y - E(Y)] = E(XY) - E(X)E(Y)$$

- Bevis ovenstående

$$Cov(X,X) = Var(X)$$

$$Cov(aX + b, cY + d) = acCov(X,Y)$$

$$Cov(X + Y, Z) = Cov(X,Z) + Cov(Y,Z)$$

Bevis Cov(X,Y)

- Forsøg at bevis nedenstående:

$$Cov(X,Y) = E[X - E(X)][Y - E(Y)] = E(XY) - E(X)E(Y)$$

- Løs på tavlen

Section 2

Moving Average process opsumering:

Mean, varians og Covariance

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, \sigma^2)$$

- Calculate mean of Y_t

$$E[Y_t] = E[\mu]E[\varepsilon_t] + E[\alpha\varepsilon_{t-1}]$$

- Vi ved $E[\varepsilon_t] = E[\varepsilon_{t-1}] = 0$

$$E[Y_t] = \mu$$

Mean, varians og Covariance

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, \sigma^2)$$

- Calculate the variance

$$\begin{aligned} Var[Y_t] &= E[(Y_t - \mu)^2] \\ &= E[(\varepsilon_t + \alpha\varepsilon_{t-1})^2] \\ &= E[\varepsilon_t^2 + 2\alpha\varepsilon_t\varepsilon_{t-1} + \alpha^2\varepsilon_{t-1}^2] \\ &= E[\varepsilon_t^2] + 2\alpha E[\varepsilon_t\varepsilon_{t-1}] + \alpha^2 E[\varepsilon_{t-1}^2] \end{aligned}$$

- Hvorfor ved vi $2\alpha E[\varepsilon_t\varepsilon_{t-1}] = 0$?
- Hvad sker der med $E[\varepsilon_t^2]$ og $E[\varepsilon_{t-1}^2]$, og hvorfor?

$$= E[\varepsilon_t^2] + \alpha^2 E[\varepsilon_{t-1}^2]$$
$$(1 + \alpha^2)\sigma^2$$

Mean, varians og Covariance

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, \sigma^2)$$

- Calculate autocovariance between Y_t and Y_{t-1}

$$Cov[Y_t, Y_{t-1}] = E[(Y_t - \mu) * (Y_{t-1} - \mu)]$$
$$E[(\varepsilon_t + \alpha\varepsilon_{t-1})(\varepsilon_{t-1} + \alpha\varepsilon_{t-2})]$$

- Forsøg selv at tage de sidste steps!

$$\alpha E[\varepsilon_{t-1}]$$

$$\alpha\sigma^2$$

Mean, Variance og Covariance

- Calculate autocovariance between Y_t and Y_{t-2}

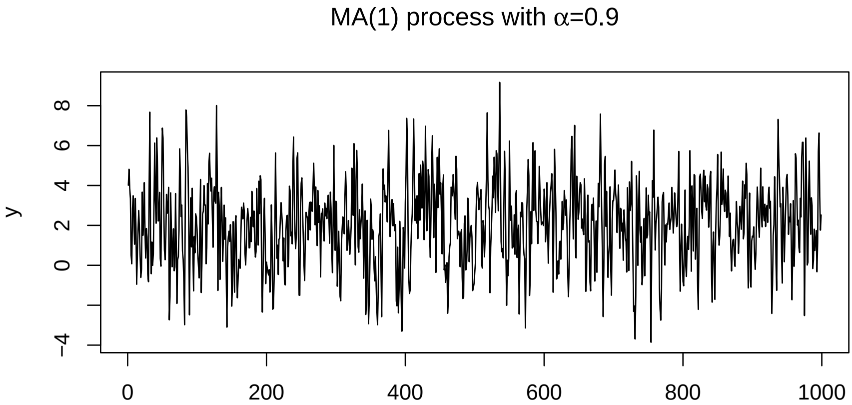
$$
\begin{aligned}
Cov[Y_t, Y_{t-2}] &= E[(Y_t - \mu) * (Y_{t-2} - \mu)] \\
&= E[(\varepsilon_t + \alpha\varepsilon_{t-1})(\varepsilon_{t-2} + \alpha\varepsilon_{t-3})]
\end{aligned}
$$

- Samme metode som vist på tavlen før:

$$= 0$$

MA(1) Simulation

$$Y_t = 2 + \varepsilon_t + 0.9\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, 1.5^2)$$



MA(1) Simulation

```
mean(y)# 2.064771
var(y) # 4.263845
cov(y[-length(y)],y[-1])# 2.073882
```

$$E(y_t) = \mu = 2$$

$$V(y_t) = (1 + \alpha^2)\sigma^2$$

$$V(y_t) = (1 + 0.9^2)1.5^2$$

$$V(y_t) = (1 + 0.9^2)1.5^2 = 4.0725$$

$$Cov = \alpha * \sigma^2$$

$$Cov = 0.9 * 1.5^2 = 2.025$$

Section 3

Auto regressive Processes opsumering

Auto regressive Processes opsumering

$$Y_t = \mu + \theta Y_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim IID(0, \sigma^2)$$

- Værktøjer vi skal bruge til properties af AR-modeller:

1. Geometriske serier.
2. Difference ligninger.

Geometriske serier review - Eksempler på serier

Eksempel 1:

$$\sum_{n=1}^{n} x^n = x + x^2 + x^3 + x^4 + \ldots + x^{n-1}$$

Eksempel 2:

$$x + \sqrt{x} + 1 + \frac{1}{\sqrt{x}} + \frac{1}{x} \ldots$$

Typer af geometriske serier

Note til senere:

- $a = \mu$
- $k = \theta$

Endelig serie

$\sum_{n=1}^{n} ak^n = a * \frac{1-k^n}{1-k}, \; k \neq 1$

Uendelig serie

$\sum_{n=1}^{n} ak^n = \frac{a}{1-k}, \; |k| < 1$
$\sum_{n=1}^{n} ak^n = na, \; |k| = 1$

Udregning

Endelig serie

$$S_n = \alpha + \alpha * k + \alpha * k^2 + \alpha * k^3 + ... + \alpha * k^{n-1}$$

$$k * S_n = \alpha * k + \alpha * k^2 + \alpha * k^3 + \alpha * k^4 + ... + \alpha * k^n$$

$$S_n - k * S_n = \alpha + (\alpha * k - \alpha * k) + (\alpha * k^2 - \alpha * k^2) + ... + (\alpha * k^{n-1} - \alpha * k^{n-1}) - \alpha * k^n$$

$$S_n - k * S_n = \alpha - \alpha * k^n$$

$$S_n(1 - k) = \alpha(1 - k^n)$$

$$S_n = \alpha * \frac{1 - k^n}{1 - k}$$

Udregning

Uendelig serie

Hvis $|k| < 1$ når $n \to \infty$ vil udtrykket gå mod:

$$S_n = \alpha * \frac{1}{1-k}$$

Dermed kan vi skrive:

$$\sum_{n=1}^{\infty} = \frac{\alpha}{1-k}$$

Difference equations: Math to econometrics

Looking at a 1. order difference equations

In 2. semester math seen like this:

$$x_t = a x_{t-1} + b$$

In time series econometrics you have seen it like this (AR(1) process):

$$y_t = \mu + \theta y_{t-1} + \varepsilon_t$$

What are the differences?

Lets first see what they got in common:

- Both equations got a constant μ and b
- Both equations got a coefficient θ and a
- Both equations got a variable that changes over time (discrete) y_t and x_t

Difference equations: Math to econometrics

The difference is the error term ε_t with the diffenition:

$$\varepsilon_t \sim IID(0, \sigma^2)$$

Identical, Independent Distributed med $mean = 0$ og $Var = \sigma^2$

Later we take a look at how this changes things!

Differens equations (Math)

Lets look at the difference equation again:

$$x_t = ax_{t-1} + b_t$$

We can start from a given point x_0

$$x_1 = ax_0 + b_1$$
$$x_2 = ax_1 + b_2 = a(ax_0 + b_1) + b_2 = a^2x_0 + ab_1 + b_2$$
$$x_3 = ax_2 + b_3 = a(a^2x_0 + ab_1 + b_2) + b_3 = a^3x_0 + a^2b_1 + ab_2 + b_3$$

Differens equations (Math)

We can already see the pattern:

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k$$

We now assume the case when $b_k = b$ so we now got a constant as in the AR(1) model (fixed over time).

So we can now write the last term as:

$$\sum_{k=1}^{t} a^{t-k} b$$

Which is a geometric series! that we just covered!

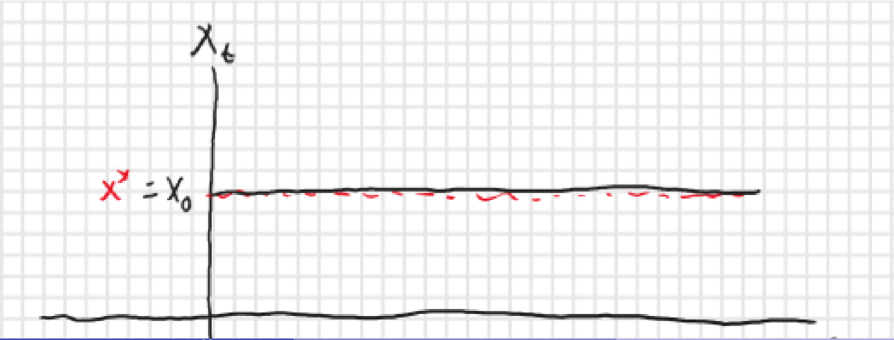
$$\sum_{k=1}^{t} a^{t-k} b = b(a^{t-1} + a^{t-2} + a^{t-3} + \ldots + a + 1) = \frac{(b - ba^t)}{(1-a)}$$

Differens equations (Math)

Therefor we can now write:

$$x_t = a^t(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$$

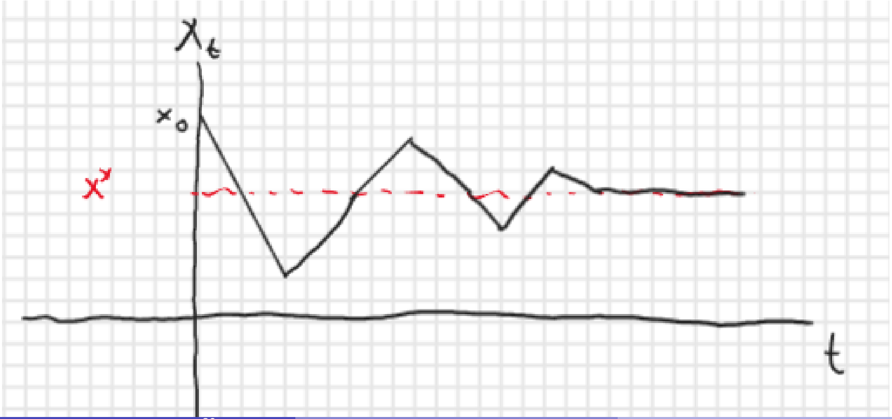
We can see that if $x_0 = \frac{b}{1-a}$ we get that $x_t = \frac{b}{1-a}$ which I have illustrated down below:



Differens equations (Math)

In fact if just x_s at any point hits $\frac{b}{1-a}$ we wont get away from it as:

$$x_{s+1} = a\frac{b}{1-a} + b = \frac{b}{1-a}$$



Difference equations (stability)

Case 1

$$|a| < 1$$

We then see that a^t goes towards 0 as $t \to \infty$ in:

$$x_t = a^t(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$$

And we will end up with

$$x_t = \frac{b}{1-a}$$

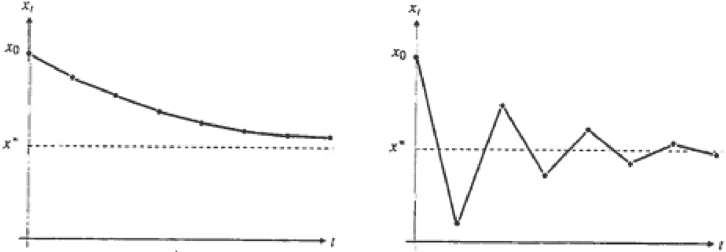
Difference equations (stability)

Case 2

$$|a| > 1$$

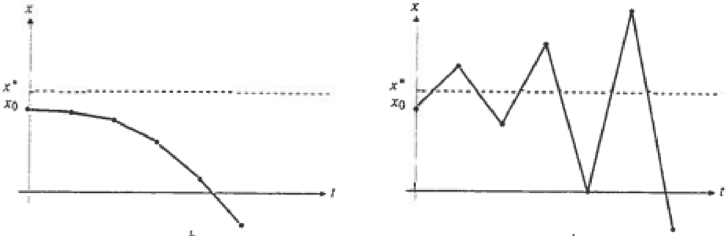
- We then see that a^t goes towards ∞ as $t \to \infty$ in and will explode.
- Lets look at the different scenarios

Difference equations (stability)



(a) $x_0 > x^* = \frac{b}{1-a}$, $0 < a < 1$

(b) $x_0 > x^* = \frac{b}{1-a}$, $-1 < a < 0$



AR(1) model Econometrics

Lets look at the AR(1) process again:

$$y_t = \mu + \theta y_{t-1} + \varepsilon_t$$

Where the only difference was the error term: ε_t lets see some examples and what the difference is

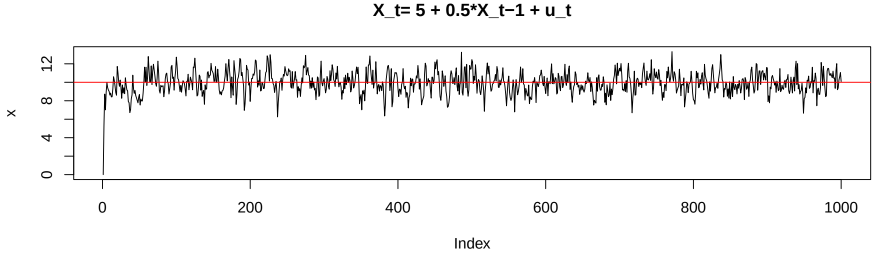
AR(1) model Exonometrics

To give an example we use the model:

$$y_t = 5 + 0.5y_{t-1} + \varepsilon_t$$

With start value $y_0 = 0$

Simulering

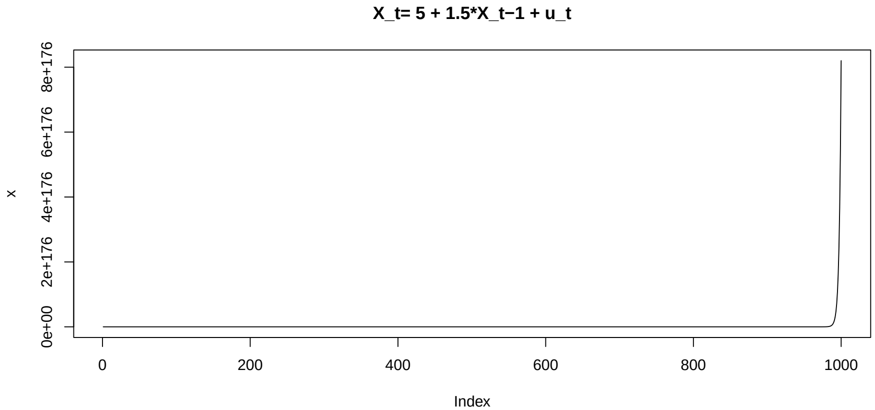


We see the shocks from ε_t does so we never stay in $\frac{b}{1-a}$ as we did with the difference equations before

AR(1) model Exonometrics

$$y_t = 5 + 1.5y_{t-1} + \varepsilon_t$$

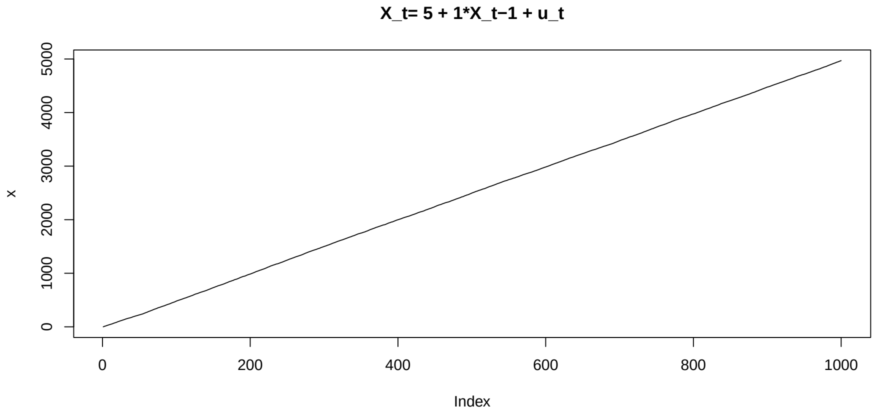
Simulering



AR(1) model Exonometrics

$$y_t = 5 + 1y_{t-1} + \varepsilon_t$$

Simulering



AR(1) model Exonometrics (mean)

We saw before that we never stay the value $\frac{b}{1-a}$ in the AR(1) model, but what if we calculate the mean?

$$
\begin{aligned}
E[y_t] &= E[\mu + \theta y_{t-1} + \varepsilon_t] \\
&= \mu + \theta E[y_{t-1}] + E[\varepsilon] \\
&= \mu + \theta E[\mu + \theta y_{t-2} + \varepsilon_{t-1}] \\
&= \mu + \mu\theta + \theta^2 E[y_{t-2}] \\
&= \mu + \mu\theta + \theta^2 E[\mu + \theta y_{t-3} + \varepsilon_{t-2}] \\
&= \mu(1 + \theta + \theta^2 + \theta^3 + \ldots + \theta^\infty)
\end{aligned}
$$

So back to the geometric series if $|\theta| < 1$ we get $\frac{\mu}{1-\theta}$

AR(1) model Exonometrics (mean)

Lets try calculating the mean using the example from before

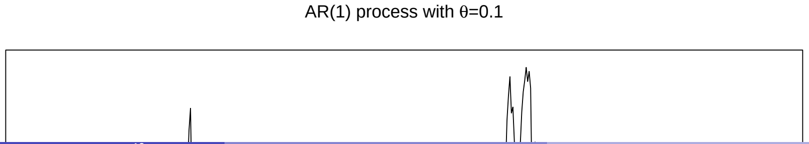
$$y_t = 5 + 0.5y_{t-1} + \varepsilon_t$$
$$E[y_t] = \frac{5}{1 - 0.5} = 10$$

Lets look at the plot again!

AR(1) model Exonometrics (Variance)

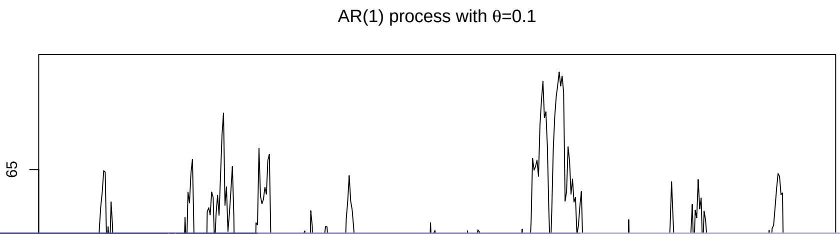
Da μ er en konstant vil denne ikke påvirke variancen og vi kan fjerne denne fra start.

```
set.seed(213)
n=500
e <- rnorm(n, mean = 0, sd = 1.5)
Y <- 2/(1-0.9)
for (t in 2:n) {
Y[t] <- 2+ 0.9*Y[t-1]+e[t]
}
ts.plot(Y, main=expression(paste("AR(1) process with ", theta,
```



AR(1) model Exonometrics (Variance)

```
set.seed(213)
n=500
e <- rnorm(n, mean = 0, sd = 1.5)
Y <- 6/(1-0.9)
for (t in 2:n) {
Y[t] <- 6+ 0.9*Y[t-1]+e[t]
}
ts.plot(Y, main=expression(paste("AR(1) process with ", theta,
```



AR(1) model Exonometrics (Variance)

$$y_t = \mu + \theta y_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim IID(0, \sigma^2)$$

Vi antager derfor $\mu = 0$

$$V(y_t) = E[(y_t - E[y_t])^2]$$

$$V(y_t) = E[(\theta y_{t-1} + \varepsilon_t)^2]$$
$$V(y_t) = E[\varepsilon_t^2] + \theta^2 E[y_{t-1}^2]$$
$$V(y_t) = \sigma^2 + \theta^2 E[y_{t-1}^2]$$
$$V(y_t) = \sigma^2 + \theta^2 E[(\theta y_{t-2} + \varepsilon_{t-1})^2]$$

- Vi kan nu indsætte y_{t-2} og gøre nøjagtigt de samme steps:

AR(1) model Exonometrics (Auto covariance)

$$cov(Y_t, Y_{t-1}) = cov(\mu + \theta Y_{t-1} + \varepsilon_t, Y_{t-1})$$

- fra statistik ved vi at $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$

$$cov(\mu, Y_{t-1}) + \theta cov(Y_{t-1}, Y_{t-1}) + cov(\varepsilon_t, Y_{t-1})$$

- vi ved at $cov(\mu, Y_{t-1}) = cov(\varepsilon_t, Y_{t-1}) = 0$
- Og hvad er det nu $cov(Y_{t-1}, Y_{t-1})$ er?

$$cov(Y_t, Y_{t-1}) = \theta \frac{\sigma^2}{1 - \theta^2}$$

- Og antagelsen fra variance skal nu bruges: $|\theta| < 1$

AR(1) model Exonometrics (Auto covariance)

$$cov(Y_t, Y_{t-2}) = cov(\mu + \theta Y_{t-1} + \varepsilon_t, Y_{t-2})$$

$$cov(\mu, Y_{t-2}) + \theta cov(Y_{t-1}, Y_{t-2}) + cov(\varepsilon_t, Y_{t-1})$$

- vi ved at $cov(\mu, Y_{t-1}) = cov(\varepsilon_t, Y_{t-1}) = 0$
- Og vi kender $cov(Y_{t-1}, Y_{t-2})$ som vi fandt på sidste slide.

$$cov(Y_t, Y_{t-2}) = \theta^2 \frac{\sigma^2}{1-\theta^2}$$

- Da vi bruger covariancen med antagelsen, gælder den stadig: $|\theta| < 1$
- Derfor ACF aftager over tid når i plotter en AR-model.

Section 4

Done

Cheatsheet

Name	AR(1) $\lvert\theta\rvert < 1$	MA(1)	AR(1) (RW1) $\lvert\theta\rvert = 1$	AR(1) (RW2) $\lvert\theta\rvert = 1$	AR(1) (RW3) $\lvert\theta\rvert = 1$
Mean	$\frac{\mu}{1-\theta}$	μ	Y_0	$Y_0 + T\mu$	$Y_0 + T\mu + t$
Var	$\frac{\sigma^2}{1-\theta^2}$	$(1+\alpha^2)\sigma^2$	$T\sigma^2$	$T\sigma^2$	$T\sigma^2$
Cov	$\theta\frac{\sigma^2}{1-\theta^2}$	$\alpha\sigma^2$	$T\sigma^2$	$T\sigma^2$	$T\sigma^2$
Stationary?	YES!	YES!	NO!	NO!	NO!