Dynamic correlations
Univariate time series: AR and MA models

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Time Series Econometrics - Økomometri II

Outline

Dynamic relationships

- Assume a time series:

$$X_t = seasonal_t + trend_t + irregular_t$$

- Once, we remove trend and seasonality, what component are we left with?
- Today, we use look at irregular component of an economic time series to explore some interesting correlations in time (dynamic correlations)
- The irregular component of an economic time series exhibit a cyclical pattern (the correlations are also called cyclical correlations)
- Even though the cyclical pattern contains some noise (irregular fluctuation) it does follow a pattern that is recognisable (cycles are usually associated with the phases of the business cycle)
- We will also look at a measure of volatility

Application of dynamic correlations

- The analysis of dynamic relationship is of great interest to macroeconomists.
- It is used to study the dynamics of business cycles.
- Below is the the set of Stylized Business Cycle Facts for the US

Variable x	Volatility of x	Cross-Correlations of output(t) with		
		x_{t-1}	x_t	x_{t+1}
output	2.13	0.51	1.00	0.51
consumption	1.85	0.73	0.90	0.34
investment	7.91	0.49	0.86	0.15
hours	2.09	0.24	0.91	0.68
productivity	0.93	0.64	0.29	-.43
capital	0.67	-.41	0.07	0.80

Table: The cyclical behavior of the US economy

- All variables are highly procyclical, except the capital stock which is uncorrelated with contemporaneous output.
- Investment is much more volatile than output and hours worked, which are, in turn, more volatile than consumption and the capital stock.
- Productivity leads the business cycle and the capital stock lags the business cycle.
- All other real variables peak at the same time as output.

Calculating dynamic correlations in R

Calculating dynamic correlations in R

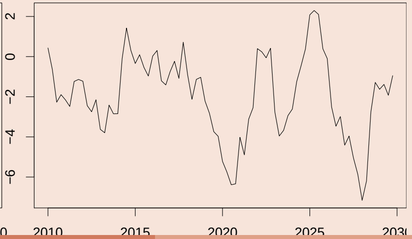
- Dynamic correlations are also called cross-correlation in time series data
- Let us assume two time series variable x(t) and y(t) in R:

```
set.seed(524)
a <- rnorm(80);    x <- cumsum(a)
x <- ts(x, start = c(2010, 1),    frequency = 4)

set.seed(525)
b <- rnorm(80);    y <- cumsum(b);
y <- ts(y, start = c(2010, 1),    frequency = 4)
```

Assume, our variables do not contain any trend and seasonality.

```
plot(x)
plot(y)
```



Calculating dynamic correlations

- **Contemporaneous correlation:** It is the correlation between two variables in time period (t)
- To calculate the contemporaneous correlation between $x(t)$ and $y(t)$ write: **cor(x,y)**
- To calculate a series of dynamic or cross-correlations between $x(t)$ and lags and leads of $y(t)$ we use the cross-correlation function **ccf**
- What are lags and leads?

Calculating dynamic correlations in R

- Let us explore the cross-correlations between our two variables

```
x <- as.numeric(x)  # this converts the variable from ts() to a normal vector (the ccf function below wor
y <- as.numeric(y)
cc <- ccf(x, y, 1, pl = F)
cc

##
## Autocorrelations of series 'X', by lag
##
##     -1      0      1
## -0.249 -0.241 -0.223
```

Correlation of y(t) with		
x(t-1)	x(t)	x(t+1)
-0.249	-0.241	-0.223

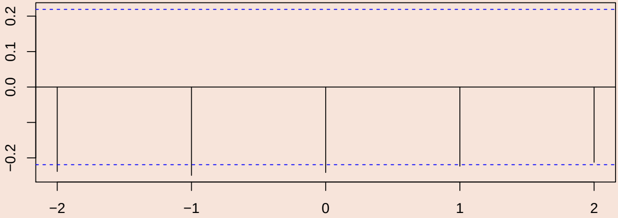
Note: the variable that comes the second in ccf command is the reference variable - (y in this case). Try switching the order of variables in ccf!

```
ccf(y, x, 1, pl = F)
```

Calculating dynamic correlations in R

- We can also visualise the relationship (note: this time I use 2 lags and leads)

```
cc=ccf(x, y, 2, pl=F)
plot(cc, ci=0.95, ylab="", xlab="", main="") # ci=0.95 means 95 percent confidence interval.
```



Note: reference variable is y(t) - you can see how lags and leads of x are correlated with $y(t)$.

- The dotted blue line shows 95 percent confidence intervals - in this case we can see that all correlations are statistically significant. If you write ci=99 in the code, it will set ci to 99 percent.

Calculating cyclical volatility

Cyclical volatility

- The purpose of this exercise is to determine the volatility of variable x (i.e. the size, or, magnitude of fluctuations in x) that are due to the business cycle behavior of x.
- Absolute (cyclical) Volatility:
  - Absolute, or, "own" volatility in (as shown in Table) is measured as the standard deviation of percentual fluctuations around the long-run trend

$$volatility\ x = s.d. \left[\left(\frac{x_t - trend(x)_t}{trend(x)_t} \right) * 100 \right].$$

- Relative (cyclical) Volatility.
  - Some authors prefer to report the volatility of a variable x relative to the volatility of output, y, e.g.

$$volatility\ x/y = \frac{s.d.(\ln x)}{s.d.(\ln y)}.$$

- Note that there are a number of other measures of volatility. So, look carefully at the definition used in each study you read. Be sure to do define your own definition of volatility carefully in each study you write.

Cyclical volatility

- Let us create a variables in $\mathbb{R}$ and calculate there cyclical volatility:
- Create a variable x (containing a trend):

```
set.seed(525)
a <- rnorm(80, mean = 1.5, sd=3)

x<- cumsum(a)

plot(x, t="l")
```

Since the variable has a trend, we can find the trend and then calculate the volatility around the trend

Cyclical volatility

- Fit an hp-filter and isolate the trend

```
library(mFilter)
lambda <- 1600
hp = hpfilter(x, lambda)
trend_x <- hp$trend
```

- We can now calculate the deviations from the trend

```
v_x <- ((x - trend_x) * 100/trend_x)
```

- Now take the standard deviation of v_x

```
sd(v_x)

## [1] 73.1
```

Exercise

- Can you find some business cycle facts for the Danish economy? Get the data from moodle!

Variable x	Volatility of x	Cross-Correlations of output(t) with		
		x_{t-1}	x_t	x_{t+1}
output	-	-	-	-
house prices	-	-	-	-
credit	-	-	-	-
stock prices	-	-	-	-

Table: Fill the table!

- Please answer the following:
  - Which variables are procyclical and which ones are counter-cyclical?
  - Which variable is the most volatile?
  - Note: Remember to work with seasonally adjusted and detrended data!
  - Note: If the correlation is significant at 99 percent level, denote it as, e.g., 0.63^{***} {use ** for 95 percent, and * for 90 percent}.

Outline

Univariate time series

This section will address univariate time series modeling

- What is univariate analysis? Discussion!
- Imagine we have the following, general univariate time series model

$$Y_t = f\left(Y_{t-1}, Y_{t-2}, ..., u_t\right).$$

- To make this model operational we must specify three things;
  1. the functional form of $f\left(\right)$,
  2. the number of lags,
  3. and the structure of the disturbance term, u_t.

Univariate time series

Why we use univariate time series analysis?

1. Purely statistical (atheoretical) models can oftentimes be extremely useful for summarizing information about a time series and for making reliable short-term forecasts.
2. Look at the individual data series BEFORE running your regressions! They may be able to tell you a lot.
   - Seasonal patterns
   - Long-run trends
   - Structural breaks and/or unusual historical events
3. Time series models are important for theoretical, analytical and numerical simulation methods. Not just for econometrics.

Why we use univariate time series analysis?

- Theoretical models with lagged dependent variables can many time be reduced to meaningful univariate time series models.
- **Example: A simple macroeconomic model**

$$c_t = \beta_0 + \beta_1 y_t + \beta_2 c_{t-1} + \varepsilon_t \tag{1}$$

$$y_t = c_t + i_t \tag{2}$$

$$i_t = s y_t \tag{3}$$

where $0 < s < 1$ is a constant saving rate, i_t is investment, c_t is consumption, y_t is income

- We can replace investment in the GDP eq.

$$y_t = c_t + s y_t = \frac{c_t}{1-s} \tag{4}$$

Substituting y_t from Equation 4 into Equation 1 gives us:

$$c_t = \beta_0 + \frac{\beta_1 c_t}{1-s} + \beta_2 c_{t-1} + \varepsilon_t$$

$$c_t - \frac{\beta_1 c_t}{1-s} = \beta_0 + \beta_2 c_{t-1} + \varepsilon_t$$

$$\left(1 - \frac{\beta_1}{1-s}\right) c_t = \beta_0 + \beta_2 c_{t-1} + \varepsilon_t$$

$$c_t = \alpha_0 + \alpha_1 c_{t-1} + u_t$$

A quick revision

When deriving properties of time series processes, we shall frequently use rules of variances, covariances and mean

- Expected Value:

$$E(aX + b) = aE(X) + b$$

$$E(X + Y) = E(X) + E(Y)$$

$$E(XY) = E(X)E(Y), \text{ if } X \text{ and } Y \text{ are indep.}$$

- Variance:

$$Var(X) = E[X - E(X)]^2 = E(XX) - E(X)E(X)$$

Note: when $E(X) = 0$, in that case $Var(X) = E(X^2)$

$$Var(aX + b) = a^2 Var(X)$$

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

$$Var(X + Y) = Var(X) + Var(Y), \text{ if X and Y are indept.}$$

A quick revision

- Covariance:

$$Cov(X,Y) = E[X - E(X)][Y - E(Y)] = E(XY) - E(X)E(Y)$$

$$Cov(X,X) = Var(X)$$

$$Cov(aX + b, cY + d) = acCov(X,Y)$$

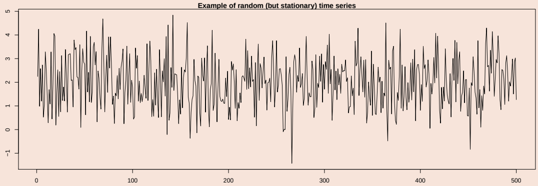
$$Var(X+Y) = Var(X) + Var(Y), \text{ if X and Y are indept.}$$

$$Cov(X+Y,Z) = Cov(X,Z) + Cov(Y,Z)$$

Some Basic Concepts:

- Stochastic Process:
  - A stochastic process is a collection of random variables ordered in time.
  - Let annual Danish GDP from 1950 to 2007 $\equiv \{Y_t\}_{1950}^{2007} = \{Y_{1950}, Y_{1951}, ..., Y_{2007}\}$. Each Y_t is a random variable (i.e. a single realization drawn from an infinite number of alternative realities).
  - The stochastic process $\{Y_t\}_{1950}^{2007}$ helps us to describe and draw inferences concerning the development of the Danish GDP over time.
- Lets create a random time series in R: $X_t = \mu + \varepsilon$, μ is a constant and ε is some random process:

```
eps <- rnorm(500, mean = 0, sd = 1) # purely random process with mean 0 and sd 1.5
mu <- 2 # the constant mean
X_t <- mu + eps # The process
ts.plot(X_t, main = "Example of random (but stationary) time series")
```

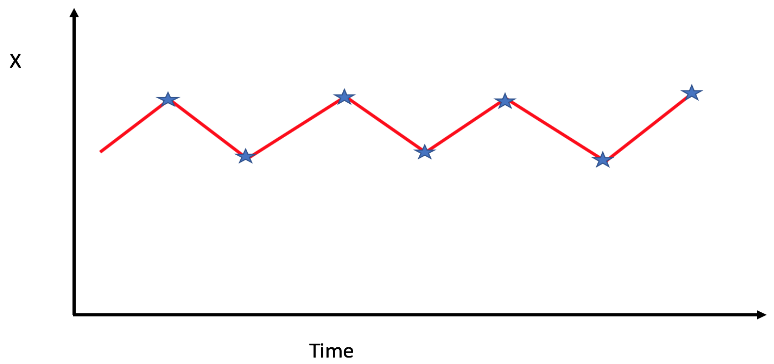


Stationarity:

- A stationary stochastic process is one whose ensemble statistics are the same for any value of time.
- A time series is said to be stationary if there is no systematic change in mean (no trend), if there is no systematic change in variance. In other words, the mean and variance are time invariant (also called weak stationarity).
- The concept of stationarity is further broken down in two sub-concepts
  - Strict stationarity
  - Weak stationarity

Some Basic Concepts:

- Strict Stationarity:
  - Strict Stationarity implies that the joint probability distribution of a stochastic variable is invariant over time.



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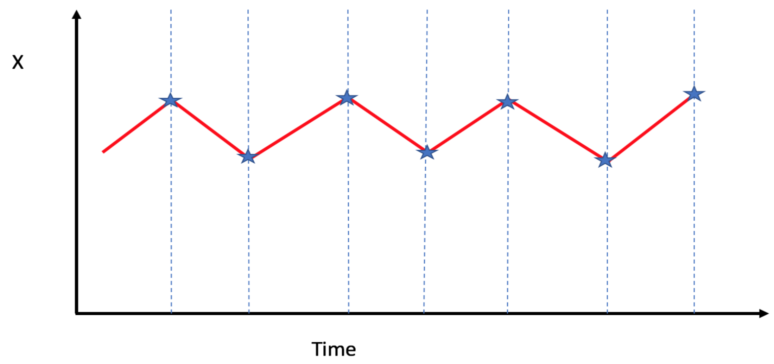

Some Basic Concepts:

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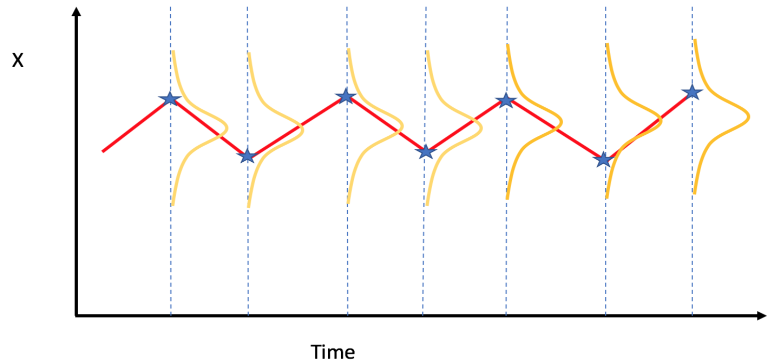


Some Basic Concepts:

- Weak Stationarity
  - Weak Stationarity implies that the first two moments of the joint probability distribution of a stochastic variable (i.e. the mean and variance-covariance matrix) are invariant over time, that is

$$E\{Y_t\} = \mu < \infty$$

$$V\{Y_t\} = E\{(Y_t - \mu)^2\} = \gamma_0 < \infty$$

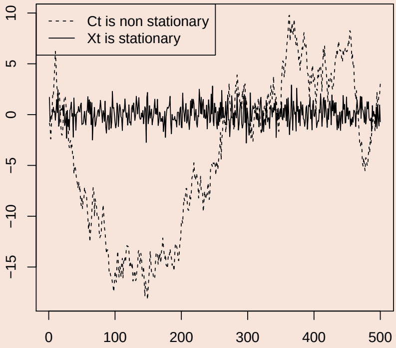
$$\begin{aligned} cov\{Y_t, Y_{t-k}\} &= E\{(Y_t - \mu)(Y_{t-k} - \mu)\} \\ &= \gamma_k,\ k = 1, 2, 3... \end{aligned}$$

- When are weak- and strict stationarity equivalent?
- For now, we need only be concerned with the concept of weak stationarity.
- Weak stationarity (henceforth stationarity) suffices for most of our needs (i.e. it allows us to proceed with our standard estimation, prediction and inference tools intact).

Some Basic Concepts:

- Nonstationary Process:
  - A nonstationary process will have a time-varying mean and/or a time-varying variance. This prevents us from making inferences outside of the sample period.

```
TT <- 500; mean=0; sd=1
Xt <- ts(rnorm(TT, mean, sd), start=1, freq=1)
Ct <- ts(cumsum(rnorm(TT)), start=1, freq=1)
ts.plot(Ct, Xt, lty=2:1)
legend("topleft", c("Ct is non stationary", "Xt is stationary"), lty=2:1)
```



Some Basic Concepts:

- Purely Random Process:
  - A stochastic process which has a zero mean, constant variance, and is serially uncorrelated is called a purely random process, or, "white noise".
  - We will quite frequently assume that the shocks in our econometric models are purely random processes (and then test whether this is a good approximate or not), i.e., we will assume

$$\varepsilon_t \sim IID(0, \sigma^2).$$

Moving Average Process:

An $MA(1)$ Process:

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, \sigma^2).$$

- Calculate the mean of Y_t

$$\begin{aligned} E\{Y_t\} &= E\{\mu + \varepsilon_t + \alpha\varepsilon_{t-1}\} \\ &= E\{\mu\} + E\{\varepsilon_t\} + E\{\alpha\varepsilon_{t-1}\} \\ &= \mu \end{aligned}$$

- Calculate the variance of Y_t

$$\begin{aligned} V\{Y_t\} &= E\{(Y_t - \mu)^2\} \\ &= E\{(\varepsilon_t + \alpha\varepsilon_{t-1})^2\} \\ &= E\left\{\varepsilon_t^2 + 2\alpha\varepsilon_t\varepsilon_{t-1} + \alpha^2\varepsilon_{t-1}^2\right\} \\ &= E\left\{\varepsilon_t^2\right\} + 2\alpha E\{\varepsilon_t\varepsilon_{t-1}\} + \alpha^2 E\left\{\varepsilon_{t-1}^2\right\} \\ &= E\left\{\varepsilon_t^2\right\} + \alpha^2 E\left\{\varepsilon_{t-1}^2\right\} \\ &= (1 + \alpha^2)\sigma^2. \end{aligned}$$

Moving Average Process:

An $MA(1)$ Process:

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$
$$\varepsilon_t \sim IID(0, \sigma^2).$$

- Calculate the autocovariance between Y_t and Y_{t-1}

$$\begin{aligned} cov\{Y_t, Y_{t-1}\} &= E\{(Y_t - \mu)(Y_{t-1} - \mu)\} \\ &= E\{(\varepsilon_t + \alpha\varepsilon_{t-1})(\varepsilon_{t-1} + \alpha\varepsilon_{t-2})\} \\ &= \alpha E\left\{\varepsilon_{t-1}^2\right\} \\ &= \alpha\sigma^2. \end{aligned}$$

- Calculate the autocovariance between Y_t and Y_{t-2}

$$\begin{aligned} cov\{Y_t, Y_{t-2}\} &= E\{(Y_t - \mu)(Y_{t-2} - \mu)\} \\ &= E\{(\varepsilon_t + \alpha\varepsilon_{t-1})(\varepsilon_{t-2} + \alpha\varepsilon_{t-3})\} \\ &= 0. \end{aligned}$$

In general, $cov\{Y_t, Y_{t-k}\} = 0$ for $k = 2, 3, 4, ...$

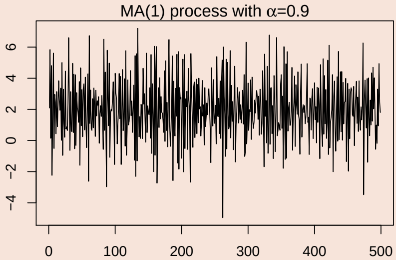
Moving Average Process:

- Moving Average MA(1) Process in R:

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$

- We simulate and MA(1) process in R:

```
n=500 # set time period to 100 obs
e <- rnorm(n, mean=0, sd=1.5) # purely random process with mean 0 and sd = 1.5
mu=2 #  we set mu= 2
y <- c()
for (t in 2:n) {
    y[t] <- mu + e[t] - 0.9*e[t-1]   # Note: we set alpha = 0.9$
    y <- y[!is.na(y)]
}
ts.plot(y, main=expression(paste("MA(1) process with ", alpha, "=0.9")), xlab="")
```



Moving Average Process:

- An example of a $MA(q)$ process

$$Y_t = \mu + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + ... + \alpha_q \varepsilon_{t-q}$$

$$\varepsilon_t \sim IID(0, \sigma^2).$$

- We can also simulate MA(q) process in R using arima.sim function
- For example, we simulate the following MA(2) process:

$$Y_t = \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \alpha_2 \varepsilon_{t-2}$$

where $\alpha_1 = 0.6$ and $\alpha_2 = -0.3$

```
ma.sim <- arima.sim(model = list(ma = c(0.6, -0.3)), n = 100)
plot(ma.sim)
```

Autoregressive Processes:

- An example of an $AR(1)$ process

$$Y_t = \mu + \theta Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim IID(0, \sigma^2).$$

- Calculate the mean of Y_t

$$\begin{aligned}
E\{Y_t\} &= E\{\mu + \theta Y_{t-1} + \varepsilon_t\} \\
&= \mu + E\{\theta Y_{t-1}\} + E\{\varepsilon_t\} \\
&= \mu + \theta E\{Y_{t-1}\} + E\{\varepsilon_t\} \\
&= \mu + \theta E(\mu + \theta Y_{t-2} + \varepsilon_{t-1}) \\
&= \mu + \theta\mu + \theta^2 E\{Y_{t-2}\} + \theta E\{\varepsilon_{t-1}\} \\
&= \mu + \theta\mu + \theta^2 E(\mu + \theta Y_{t-3} + \varepsilon_{t-2}) \\
&= \mu\left(1 + \theta + \theta^2 + ... + \theta^\infty\right) \\
&= \frac{\mu}{1-\theta} \text{ if } |\theta| < 1.
\end{aligned}$$

Autoregressive Processes:

- Geometric Series Theorem:
  - If $|r| < 1$, the geometric series $a + ar + ar^2 + ... + ar^{n-1} + ...$ converges to $a/(1 - r)$ as $n \to \infty$.
  - If $|r| \geq 1$, the series diverges unless $a = 0$.
  - If $a = 0$, the series converges to 0.

Autoregressive Processes:

- Calculate the variance of Y_t.
  - First, define $y_t \equiv Y_t$ - μ.

$$
\begin{aligned}
V\{y_t\} &= E\{(y_t - E\{y_t\})^2\} \\
&= E\{(\theta y_{t-1} + \varepsilon_t)^2\} \\
&= E\left\{\varepsilon_t^2\right\} + \theta^2 E\left\{y_{t-1}^2\right\} \\
&= \sigma^2 + \theta^2 E\left\{y_{t-1}^2\right\} \\
&= \sigma^2 + \theta^2 E\{(\theta y_{t-2} + \varepsilon_{t-1})^2\} \\
&= etc... \\
&= \sigma^2\left(1 + \theta^2 + \theta^4 + ... + \theta^\infty\right) \\
&= \frac{\sigma^2}{1-\theta^2} \text{ if } |\theta| < 1.
\end{aligned}
$$

Autoregressive Processes:

- Calculate the variance of Y_t.
  - A simpler derivation

$$\begin{aligned} V\{Y_t\} &= V\{\mu + \theta Y_{t-1} + \varepsilon_t\} \\ V\{Y_t\} &= V\{\mu\} + \theta^2 V\{Y_{t-1}\} + V\{\varepsilon_t\} \\ V\{Y_t\} &= \theta^2 V\{Y_{t-1}\} + \sigma^2 \\ V\{Y_t\} &- \theta^2 V\{Y_{t-1}\} = \sigma^2 \end{aligned}$$

  - If $|\theta| < 1$, we have a stationarity process that implies finite variance, so we can also write:

$$V\{Y_t\} - \theta^2 V\{Y_t\} = \sigma^2$$

$$V\{Y_t\}(1 - \theta^2) = \sigma^2$$

$$V\{Y_t\} = \frac{\sigma^2}{1 - \theta^2}$$

Autoregressive Processes:

- Calculate the autocovariance between Y_t and Y_{t-1}

$$\begin{aligned} cov\{Y_t, Y_{t-1}\} &= E\{(Y_t - \mu)(Y_{t-1} - \mu)\} \\ &= \theta \frac{\sigma^2}{1-\theta^2} \text{ if } |\theta| < 1. \end{aligned}$$

- In general, the autocovariance between Y_s and Y_t can be written as

$$cov\{Y_s, Y_t\} = \theta^{|s-t|} \frac{\sigma^2}{1-\theta^2} \text{ if } |\theta| < 1.$$

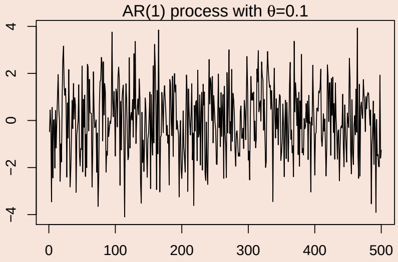
Autoregressive Process:

- Autoregressive Process AR(1) Process in R:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- We simulate an AR(1) process in R:

```
set.seed(213)
n=500
e <- rnorm(n, mean = 0, sd = 1.5)
Y <- rnorm(1)
for (t in 2:n) {
    Y[t] <- 0.1*Y[t-1]+e[t]
}
ts.plot(Y, main=expression(paste("AR(1) process with ", theta, "=0.1")), xlab="")
```



Autoregressive Process:

- Autoregressive Process AR(1) Process in R:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- Set n = 10000
- Set the parameter of AR(1) less than 1
- Set the parameter of AR(1) equal to 1
- Set the parameter of AR(1) greater than 1
- Compare the three plots
- What do you see?

Autoregressive Process:

- An example of a $AR(q)$ process

$$Y_t = \mu + \theta_1 Y_{t-1} + \ldots + \theta_q Y_{t-q} + \varepsilon_t$$

$$\varepsilon_t \sim IID(0, \sigma^2).$$

- We can also simulate AR(q) process in R using arima.sim function
- For example, we simulate the following AR(2) process:

$$Y_t = \mu + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$$

where $\theta_1 = 0.6$, $\theta_2 = -0.3$, and $\mu = 5$

```r
ar.sim <- arima.sim(model = list(ar = c(0.6, -0.3)), n = 100) + 5
plot(ar.sim)
```

The Lag Operator

- See pages 39-42 in Enders.
- Some basic rules for using lag operators:
  1. $L\alpha = \alpha$
  2. $L(x_t) = x_{t-1}$
  3. $L^2(x_t) = L\left[L(x_t)\right] = L(x_{t-1}) = x_{t-2}$
  4. $L^n(x_t) = x_{t-n}$
  5. $(1-L)\,x_t = x_t - L(x_t) = x_t - x_{t-1} = \Delta x_t$
  6. $L\,(1-L)\,x_t = (1-L)x_{t-1} = x_{t-1} - Lx_{t-1} = x_{t-1} - x_{t-2} = \Delta x_{t-1}$

Derivation using the Lag Operator

- 1. Now again assume MA(1) process:

$$Y_t = \mu + \alpha\varepsilon_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim IID(0, \sigma^2).$$

- Derive the variance of Y_t using lag operator rules

- 2. Assume AR(1) process:

$$Y_t = \mu + \alpha Y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim IID(0, \sigma^2).$$

- Derive the variance of Y_t using lag operator rules

Learning Outcome

- Our learning goals for today: We can:
  - Cyclical correlations and cyclical volatility ✓
  - Stationarity (strict stationarity, weak stationarity) ✓
  - Moving average process and its properties (mean, variance, autocovariance) ✓
  - Autoregressive process and its properties (mean, variance, autocovariance) ✓