Eksamensopgave 1

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Opgave 1

1. Explain crucial differences between the Solow model and the Ramsey, Cass and Koopman model. Does it affect the overall conclusions?

Solow antager konstant opsparingsrate (lille s), så kapitalakkumuleringen kan skrives som: $\dot{K}(t) = sY(t) - \delta K(t)$, da $I = S$

Den største forskel mellem Solow-modellen og Ramsey-Kass-Koopman modellen er, at sidstnævnte implementerer opsparing som en endogen faktor i modellen.

Solow-modellen holder opsparingsraten konstant, og således eksogent bestemt i modellen.

I Ramsey-modellen vælger husholdningerne den optimale opsparingsrate og denne er således endogeniseret. Her introduceres en utålmodighedsrate, ρ, og en rate der "smoother" forbruget, *theta*. Således vil opsparingsraten ikke længere være konstant udenfor steady state.

Derudover ligger der en forskel i de 2 modellers "balanced growth paths": Det er nemlig ikke muligt for Ramsey-Cass-Koopman at have en balanced growth path med kapitalniveau over golden rule niveauet (golden rule kapitalniveauet er det højeste niveau af forbrug som kan vedligeholdes. Det er givet ved $f'(k) = n + g$). Dette skyldes, at opsparingsraten er udledt af husholdningerne ud fra nyttemaksimering.

Udover ovenstående faktorer, er Ramsey- og Solow-modellen ens og der er ikke forskel på de overordnede konklusioner.

2. Piketty (2014) argues that a fall in the growth rate of the economy is likely to an increase in the difference between the real interest rate and the growth rate. This problem asks you to investigate this issue in the context of the Ramsey Cass Koopmans model. Specially, consider a Ramsey Cass Koopmans economy that is on its balanced growth path, and suppose there is a permanent fall in g.

2.1 How, if at all, does this affect the $\dot{k} = 0$ curve?

Det vides fra opgaven, at der sker et permanent fald i vækstraten g Her skal vi kigge på følgende ligning:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

.

Effekten skal udledes fra sidste led af ligningen:

$$-(n + g)k(t)$$

.

Da g falder bliver tallet større (mindre negativt). Vi ser ved denne omskrivning (hvor $\dot{k} = 0$),

$$c = f(k) - (n+g)k$$

at faldet i g medfører en stigning i $c(t)$ for at opretholde niveauet i $\dot{k}(t)$. Denne stigning i c skubber $\dot{k}$ kurven op og gør den dermed højere.

Derudover ved vi, at $f'(k) = n + g$ (ved at sætte $\dot{k} = 0$ og isolere f(k) og differentiere). Her ser vi, at et fald i g, får marginalproduktet af kapital ($f'(k)$) til at falde, hvilket får k til at stige, hvilket ligeledes forskyder kurven opad og udad.

(Inada betingelsen)

2.2 How, if at all, does this affect the $\dot{c} = 0$ curve?

Vi starter med Eulerligningen:

$$\frac{\dot{c}}{c} = \frac{r(t) - \rho - \theta g}{\theta}$$

Her ved vi, at $r(t)$ (realrenten) er lig med $f'(k(t))$ (marginalproduktet af kapital), da der ingen depreciering antages, hvorved vi også kan få:

$$\frac{\dot{c}}{c} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

Her sætter vi $\dot{c} = 0$ og isolerer for $f'(k(t))$:

$$f'(k(t)) = \rho + \theta g$$

Her ser vi, at et fald i g må betyde et tilsvarende fald i $f'(k(t))$. Dermed må det betyde at det k som er nødvendigt for at opretholde $\dot{c} = 0$ stiger, og således bliver kurven skubbet til højre, som det ses på billedet.

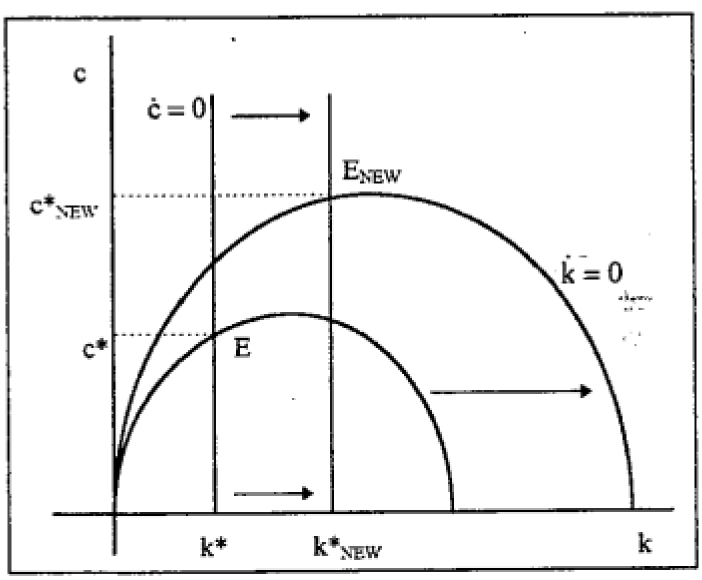


Figure 1: dotkc

2.3 At the time of the change, does c rise, fall, or stay the same, or is it not possible to tell?

At the time of change in g, the value of k, the stock of capital per unit of effective labor, is given by the history of the economy, and it cannot change discontinuosly. It remains equal to the k^* on the old balanced growth path.

In contrast, c, the rate at which households are consuming in units of effective labor, can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.

However, we cannot tell whether the new saddle path passes above or below the original point E. Thus we cannot tell whether c jumps up or down and in fact, if the new saddle path passes right through point E, c might even remain the same at the instant the g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

2.4 At the time of the change, does r-g rise, fall, or stay the same, or is it not possible to tell?

If we assume that the shock is unexpected then the amount of capital is historically build up and will not imidially change when the shock is made to g. Therefor there will be no changes in $r(t)$ because there will be no changes in k therefor no changes in $f(k)$ and last no change in $f'(k)$ and we know $f'(k) = r(t)$. Therefor the distance will be larger.

2.5 In the long run, does r-g rise, fall, or stay the same, or is it not possible to tell?

Vi starter ud fra følgende ligning

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho - \theta g}{\theta}$$

Man sætter $\dot{c} = 0$ og isolerer $f(k)$ og differencierer og får:

$$f'(k) = \rho + \theta g$$

Vi ved også at $f'(k) = r$, så på lang sigt gælder det, at

$$r = \rho + \theta g$$

Her trækkes g fra på begge sider, så vi får

$$r - g = \rho + \theta g - g$$

Der faktoreres for g på højresiden

$$r - g = \rho + g(\theta - 1)$$

Til sidst differentierer vi for g og får:

$$\frac{\partial r - g}{\partial g} = \theta - 1$$

Dermed kan følgende konkluderes om θ (villighed til at ændre forbrug på tværs af perioder) og g:

Hvis $\theta > 1$ vil g falde. Hvis $\theta = 1$ vil g forblive uændret. Hvis $\theta < 1$ vil g vokse.

2.6 Find an expression for the impact of a marginal change in g on the fraction of output that is saved on the balanced growth path. Can one tell whether this expression is positive or negative?

We can start by defining the fraction of output that is saved on the balanced growth path called s $s = \frac{(f(k^*)-c^*)}{f(k^*)}$

since k is constant on the balanced growth path we can write the function $\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$ as $f(k^*) - c^* = (n+g)k^*$

We can now rewrite the fraction of output saved on the balanced growth path. $s = \frac{(n+g)k^*}{f(k^*)}$

We can now differentiate both sides with respect to g. We use the rules for differentiating a fraction. We should also use the chainrule as g is in the function for f(k). (maybe lige skrive hvordan)

$$\frac{\delta s}{\delta g} = \frac{f(k^*)[(n+g)(\frac{\delta k^*}{\delta g}) + k^*] - (n+g)k^* f'(k^*)(\frac{\delta k^*}{\delta g})}{[f(k^*)]^2}$$

Vi ganger $f(k^*)$ ind i parentesen og setter derefter $(n+g)(\frac{\delta k^*}{\delta g})$ udenfor parentes (setter den bare on hver side af parentesen)

$$\frac{\delta s}{\delta g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)](\frac{\delta k^*}{\delta g}) + f(k^*)k^*}{[f(k^*)]^2}$$

We know that k^* is defined when $f'(k^*) = \rho + \theta g$ we can now differentiate both sites with respect to g, and we get $f''(k^*)(\frac{\delta k^*}{\delta g}) = \theta$ Solving for $\frac{\delta k^*}{\delta g}$ we get.

$$\frac{\delta k^*}{\delta g} = \frac{\theta}{f''(k^*)}$$

We can now substitute this into the above equation:

$$\frac{\delta s}{\delta g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)](\frac{\theta}{f''(k^*)}) + f(k^*)k^*}{[f(k^*)]^2}$$

We then multiply by $f''(k^*)$ above and under the fraction (above: As the term is multiplied on the brackets it just disapears and is multiplied to the second term)

$$\frac{\delta s}{\delta g} = \frac{(n+g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f''(k^*)}{[f(k^*)]^2 f''(k^*)}$$

- First term should be positive
- as we know $f''(k^*)$ is negative the last term in the nummerator is negative.
- the denominator will be negative for the same reason.

Da første del i tælleren er positiv og anden del er negativ ved vi ikke om faldet i g har en positiv eller negativ effekt på s (delen af output som opspares på den balanced growth path)

2.7 For the case where the production function is Cobb-Douglas, $f(x) = k^\alpha$, rewrite your answer to part (2.6) in terms of ρ, n, g, θ and α (Hint: Use the fact that $f'(k^*) = \rho + \theta g$

We know when $f(k) = k^\alpha$, then $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha-1)k^{a-2}$.

if we substitute this into the answer from the question above:

$$\frac{\delta s}{\delta g} = \frac{(n+g)[k^{*\alpha} - k^{*}\alpha k^{*\alpha-1}]\theta + k^{*\alpha}k^{*}\alpha(\alpha-1)k^{*\alpha-2}}{k^{*\alpha}k^{*\alpha}\alpha(\alpha-1)k^{*\alpha-2}}$$

We can now do some reduction:

$$\frac{\delta s}{\delta g} = \frac{(n+g)k^{*\alpha}(1-\alpha)\theta - k^{*a}\alpha(1-\alpha)k^{*\alpha-1}}{\frac{-(1-\alpha)k^{*\alpha}\alpha k^{*\alpha-1}\alpha k^{*\alpha-1}}{\alpha}}$$

And when we use $f'(k^{*}) = \rho + \theta g)$ we get that:

$$\frac{\delta s}{\delta g} = -\alpha\frac{[(n+g)\theta - (\rho+\theta g)]}{(\rho+\theta g)^2}$$

Multipling θ into the brackets:

$$\frac{\delta s}{\delta g} = -\alpha\frac{n\theta - \rho}{(\rho+\theta g)^2}$$

And we can make the expretion psoitive by:

$$\frac{\delta s}{\delta g} = \alpha\frac{(\rho - n\theta)}{(\rho+\theta g)^2}$$

3. Analyze the effect of a public procurement, including a thorough presentation of the dynamics in Figures 2.8 and 2.9

Firstly the implementation of the government sector makes our k dot function change to this:

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$$

where the government buys output at a rate of $G(t)$ per unit of effective labour per unit time. A higher G value shifts the $\dot{k} = 0$ down, because the more goods that are purchased by the government the fewer that can be purchased privately if k is to be held constant.

Starting with figure 2.8 suppose the economy is on a balanced growth path with $G(t)$ constant at some level G_L, and that there is an unexpected, permanent increase in G to G_H. From the above equation we know, that this menas that $\dot{k} = 0$ has to shift downwards by the amount of increase in G. Since the purchase do not affect the euler equation, the $\dot{c} = 0$ is unaffected. This is shown below.

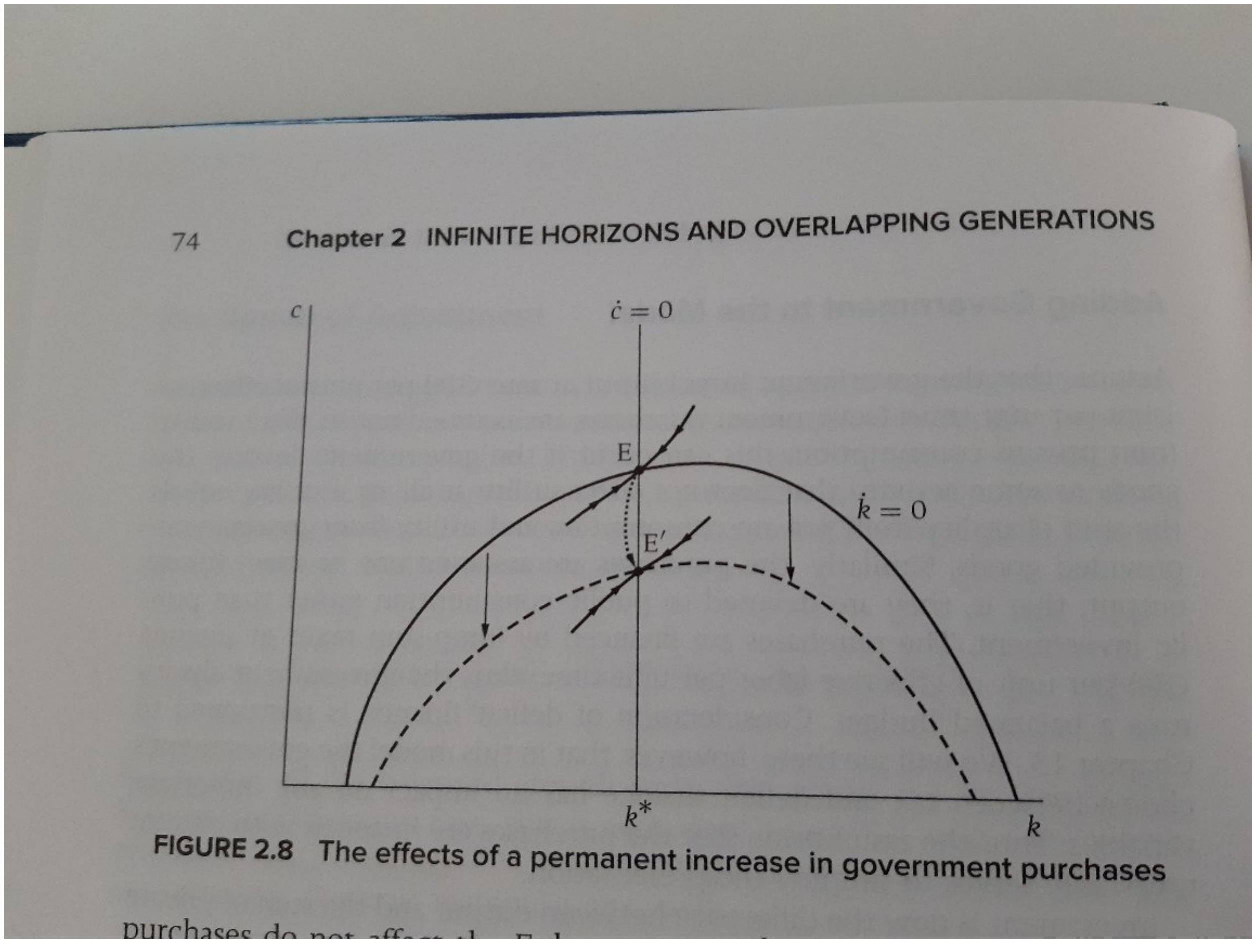


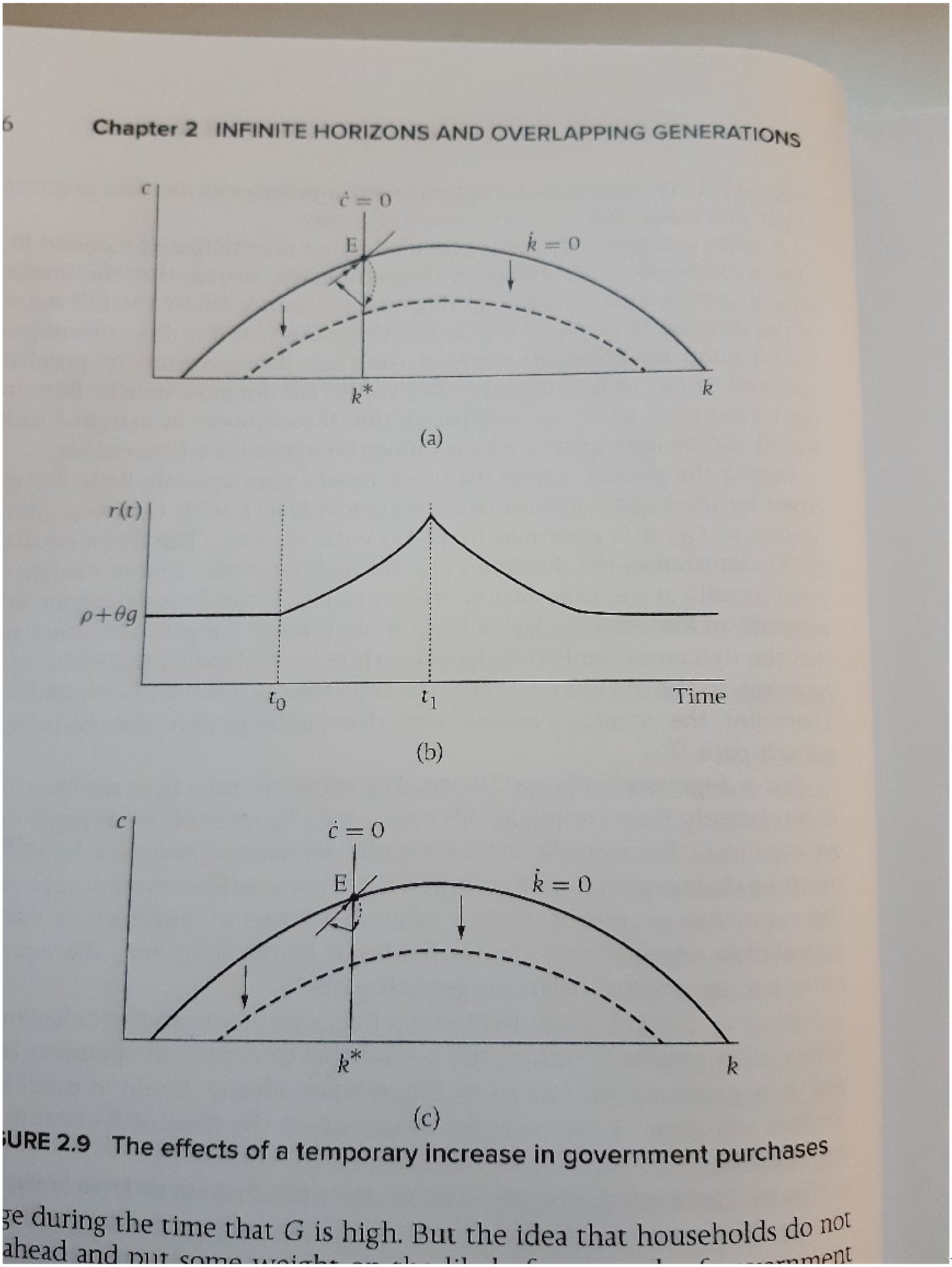
FIGURE 2.8 The effects of a permanent increase in government purchases

We know that in response to such a change c must jump so that the economy is on its new saddle path. If not, then as before, either capital would become negative at some point or households would accumulate infinite wealth. In this case, the adjustment takes a simple form: c falls by the amount of the increase in G, and the economy is immediately on its new balanced growth path. Intuitively, the permanent increase in government purchases and taxes reduce households' lifetime wealth. And because the increase in purchases and taxes are permanent, there is no scope for households to raise their utility by adjusting the time pattern of their consumption. Thus the size of the immediate fall in consumption is equal to the full amount of the increase in government purchases, and the capital stock and the real interest rate are unaffected. An older approach to modeling consumption behavior assumes that consumption depends only on current disposable income and that it moves less than one-for-one with disposable income. Recall for example, that the Solow model assumes that consumption is simply a fraction $1 - s$ of current disposable income. With that approach, consumption falls by less than the amount of the increase in government purchases. As a result, the rise in government purchases crowds out investment, and so the capital stock starts to fall and the real interest rate starts to rise. Our analysis shows that those results rest critically on the asumption that households follow mechanical rules: with full intertemporal optimization, a permanent increase in government purchases does not cause crowding out.

Moving on to 2.9 Panal (a) shows a case where the increase in G is relatively long-lasting. In this case c falls by most of the amount of the increase in G. Because the increase is not permanent, however, households decrease their capital holdings somewhat. c rises as the economy approaches the time that G returns to G_L. After that time, c continues to rise and households rebuild their capital holdings. IN the long run, the economy returns to its original balanced growth path.

Panel (b): since $r = f'(k)$, we can deduce the behavior of r from the behavior of k. Thus r rises gradually during the period that government spending is high and then gradually returns ti its initial level. This is shown in panel (b): t_0 denotes the time of the increase in G and t_1 the time of its return to its initial value.

Finally panel (c) shows the case of a short-lived rise in G. Here households change their consumption relatively little, choosing instead to pay for most of the temporarily higher taxes out of their savings. Because government purchases are high for only a short period, the effect on the capital stock and the real interest rate are similar. Note that once again allowing for forward-looking behavior yields insights we would not get from the older approach of assuming that consumption depends only on current disposable income. With that approach, the duration of the change in government purchases is irrelevant to the impact of thechange during the time that G is high. But the idea that households do not look ahead and put some weight on the likely future path of government purchases and taxes is implausable.



URE 2.9 The effects of a temporary increase in government purchases

ge during the time that G is high. But the idea that households do not
ahead and put

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Opgave 2 Diamond

Consider the following overlapping generations framework. Welfare is equal to:

$$U_t = u(C_{1t}) + \beta u(C_{2t+1})$$

with $\beta = \frac{1}{1+\rho} \leq$ the discount factor.

Assume the utility function is logarithmic:

$$u(C_{jt}) = lnC_{jt}$$

We will assume that the government implements a pension scheme, more specifically, a pay-as-you-go social security scheme, where the government taxes each young individual by an amount T , and uses that amount to pay benefits to old individuals. Hence, the budget constraints is given by:

$$C_{1t} + S_t = w - T$$

whereas period 2 consumption is givien by:

$$C_{2t+1} = S_t(1+r) + (1+n)T$$

Note that n - $L_t = (1+n)L_{t-1}$ represents population growth. Take the wage w and interest rate r as exogenous and solve the following:

Set up the intertemporal maximazation problem and derive the Euler equation:

Vores nyttefunktion er givet ved:

$$U_t = lnC_{1t} + [\frac{1}{1+\rho}]lnC_{2t+1}$$

Vi udleder nu budgetrestriktionen ved at omarrangere funktionen for forbrug for ældre (C_{2t+1}) for periode $t+1$

$$C_{2t+1} = S_t(1+r) + (1+n)T$$

Vi isolerer S_t ved at trække $(1+n)T$ fra på begge sider og derefter dividere med $(1+r)$ på begge sider:

$$S_t = \frac{C_{2t+1}}{(1+r)} - \frac{(1+n)}{(1+r)}T$$

Ved at sætte ind i forbrugsfunktionen i første periode får vi:

$$C_{1t} + \frac{C_{2t+1}}{(1+r)} - \frac{(1+n)}{(1+r)}T = w - T$$

Nu flytter vi $\frac{(1+n)}{(1+r)}T$ over på den anden side af lighedstegnet og får dermed budgetrestriktionen:

$$C_{1t} + \frac{C_{2t+1}}{(1+r)} = w - T + \frac{(1+n)}{(1+r)}T$$

Vi kan nu sætte Lagrangian op:

$$L = lnC_{1t} + [\frac{1}{1+\rho}]lnC_{2t+1} + \lambda\left[w - T + \frac{(1+n)}{(1+r)}T - \left(C_{1t} + \frac{C_{2t+1}}{(1+r)}\right)\right]$$

Tager vi nu First Order Conditions (FOC) mht. C_{1t} og C_{2t+1} får vi:

$$\frac{1}{C_{1t}} = \lambda$$

og

$$\frac{1}{C_{2t+1}(1+\rho)} = \frac{\lambda}{(1+r)}$$

Nu kan den første sættes ind i den anden, så vi får:

$$\frac{1}{C_{2t+1}(1+\rho)} = \frac{1}{C_{1t}(1+r)}$$

Til sidst får vi Euler ligningen ved at rearrangere:

$$\frac{C_{2t+1}}{C_{1t}} = \frac{(1+r)}{(1+\rho)}$$

Konkluderende kan det siges, at en større diskonteringsrate (ρ) vil betyde mere forbrug i periode 1. Samtidig kan det siges, at en højere rentesats (r) vil resultere i højere forbrug i periode 2.

Derive S_t as a function of r, w and T. How does an increase in T affect savings (show mathematically)? Discuss the result.

Hertil bruger vi Euler ligningen fra før til at udtrykke S_t i form af labor income, realrentesatsen og skat ved at bruge det faktum, at $\beta = \frac{1}{1+\rho}$. Dermed får vi:

$$\frac{C_2}{C_1} = \beta(1+r)$$

Nu indsætter vi funktionerne for C_1 og C_2

$$\frac{S_t(1+r) + T(1+n)}{w - T - S_t} = \beta(1+r)$$

Nu dividerer vi først igennem med $(1+r)$:

$$\frac{S_t}{(w - T - S_t)} + \frac{T(1+n)}{(w - T - S_t)(1+r)} = \beta$$

Dernæst ganger vi med $(w - T - S_t)$ på begge sider:

$$S_t + \frac{T(1+n)}{(1+r)} = \beta(w - T - S_t)$$

Vi tager S_t ud af parantesen og flytter den over på venstre side af lighedstegnet, og flytter $\frac{T(1+n)}{(1+r)}$ over på højre side af lighedstegnet, så vi får:

$$S_t + \beta S_t = \beta(w - T) - \frac{T(1+n)}{(1+r)}$$

Til sidst faktorerer vi med S_t og dividerer med $(1+\beta)$, så vi ender med:

$$S = \frac{\beta(w - T)}{(1+\beta)} - \frac{T(1+n)}{(1+r)(1+\beta)}$$

Vi ser, at hvis T stiger, vil første led blive mindre, mens andet led bliver større. Da der er minus imellem trækker begge effekter i retningen af, at **en stigning i T forårsager et fald i S**.