

# Eksamensæt 2

Andreas Methling

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## Endogenous growth theory

Assume production function of final goods sector is given by:

$$Y(t) = [(1 + \alpha_K)K(t)^\alpha][A(t)(1 - \alpha_L)L(t)]^{1-\alpha}$$

where  $1 - \alpha_L$  and  $1 - \alpha_K$  is proportion of labour and capital allocated for final production

Capital stock is given by:

$$\dot{K}(t) = sY(t)$$

Production functions of R&D sector is given by:

$$\dot{A}(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta$$

where  $B > 0$  shows efficiency of research,  $\gamma \in (0, 1)$  is the output elasticity of labour allocated in R&D, and  $\theta \leq 1$  is a parameter describing the elasticity of existing knowledge ( $A$ ) for the production of increases in the stock of knowledge.

Population growth is exogenous

$$\frac{\dot{L}(t)}{L(t)} = n \quad \text{or} \quad \dot{L}(t) = nL(t)$$

**Derive an expression for the growth rate of capital  $g_K(t)$  and growth rate of technology  $g_A(t)$**

$g_K(t)$

We know the growth rate is found by finding the change in the capital stock so:

$$\frac{\dot{K}(t)}{K(t)} = \frac{sY(t)}{K(t)}$$

which means that the growth rate of capital is equal to the fraction between the amount saved and the capital stock or just the change in the capital stock as above:

$$g_K(t) = \frac{sY(t)}{K(t)} \quad \text{and} \quad g_K(t) = \frac{\dot{K}(t)}{K(t)}$$

$g_A(t)$

Again we divide the find the change by dividing by  $A(t)$  on both sides of the equation  $\dot{A}(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta$ :

$$\frac{\dot{A}(t)}{A(t)} = \frac{B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta}{A(t)}$$

Then we simplify by getting rid of the brackets on the right hand side

$$g_A(t) = \frac{B * \alpha_K^\beta * K(t)^\beta * \alpha_L^\gamma * L(t)^\gamma * A(t)^\theta}{A(t)}$$

moving the other  $A(t)$  down to the denominator by making the exponent negative and then simplifying  $A(t)^1 * A(t)^{-\theta}$  to yield the final result:

$$g_A(t) = \frac{B * \alpha_K^\beta * K(t)^\beta * \alpha_L^\gamma * L(t)^\gamma}{A(t)^{1-\theta}}$$

**Using the fact that  $g_A(t)$  is constant overtime (i.e.,  $\frac{\partial g_A(t)}{\partial t} = 0$ ), derive an expression for the growth rate of technology  $g_A^*$  along a balanced growth path (steady state), for the case when  $\theta + \beta < 1$ ? Intuitively, explain the equation**

We start by taking logs to get rid of all the exponents from the above equation:

$$\ln(g_A) = \ln(B) + \beta * \ln(\alpha_K) + \beta * \ln(K) + \gamma * \ln(\alpha_L) + \gamma * \ln(L) + (\theta - 1) * \ln(A)$$

Then we take the derivative with respect to time, where our constants disappear so we get:

$$\frac{\partial \ln(g_A)}{\partial t} = \beta \frac{\partial \ln(K)}{\partial t} + \gamma \frac{\partial \ln(L)}{\partial t} + (\theta - 1) \frac{\partial \ln(A)}{\partial t}$$

We know that  $\frac{\partial \ln(X)}{\partial t} = \frac{\dot{X}}{X}$  so we now get:

$$\frac{\dot{g}_A}{g_A} = \beta \frac{\dot{K}}{K} + \gamma \frac{\dot{L}}{L} + (\theta - 1) \frac{\dot{A}}{A}$$

all these are the growth rates so we get:

$$\frac{\dot{g}_A}{g_A} = \beta g_K + \gamma n + (\theta - 1) g_A$$

which yields an equation for steady state as follows:

$$\dot{g}_A = (\beta g_K + \gamma n + (\theta - 1) g_A^*) g_A^*$$

Assuming a case where  $\theta + \beta < 1$  we either need  $g_A^*$  to be zero or other terms inside the bracket to be 0 to achieve a steady state. Given  $\dot{g}_A = (\beta g_K + \gamma n + (\theta - 1) g_A^*) g_A^* = 0$  we can find the growth rate where we can write:

$$(\beta g_K + \gamma n + (\theta - 1) g_A^*) g_A^* = 0$$

We then get ride of the brackets:

$$(\theta - 1) g_A^* * g_A^* = -\beta g_K g_A^* - \gamma n g_A^*$$

then dividing by  $g_A^*$  on both sides to get:

$$(\theta - 1)g_A^* = -\beta g_K - \gamma n$$

and lastly dividing by  $(\theta - 1)$  to isolated  $g_A^*$  and get the long run growth of technology:

$$g_A^* = \frac{-\beta g_K - \gamma n}{(\theta - 1)}$$

which is the same as:

$$g_A^* = \frac{\beta g_K + \gamma n}{(1 - \theta)}$$

So the long run growth rate of technology positively depends on five factors:  $\beta$  the output elasticity of capital involved in research,  $\gamma$  the output elasticity of labour involved in research,  $g_K$  the capital growth rate,  $n$  the population growth rate and  $\theta$  the elasticity of existing knowledge ( $A$ ) for the production of increases in the stock of knowledge

## Real business cycle theory

Assume the following utility function:

$$u_t = \ln(c_t) + \frac{b(1 - l_t)^{1-\gamma}}{1 - \gamma}$$

where  $c_t$  is consumption,  $b > 0$  and  $\gamma > 0$

Income of the individuals is equal to  $w_t l_t$

where  $w_t$  is wage, and  $l_t$  is the labour supply

**Define the budget constraint and answer the following: How, if at all, does labour supply depend on wage?**

The budget constraint is just consumption equal to wage times labour supply, which means that the Lagrangian is:

$$\Lambda = \ln(c_t) + \frac{b(1 - l_t)^{1-\gamma}}{1 - \gamma} + \lambda(w_t l_t - c_t)$$

Taking first order condition with respect to  $c$  and  $l$  gives:

$$\frac{1}{c_t} - \lambda = 0 \quad \text{and} \quad -\frac{b}{(1 - l_t)^\gamma} + \lambda w_t$$

Using the fact that  $c_t = w_t l_t$  gives  $\lambda = \frac{1}{w_t l_t}$  inserting that into the FOC of  $l_t$  yields:

$$-\frac{b}{(1 - l_t)^\gamma} + \frac{1}{l_t}$$

this can also be written as:

$$\frac{(1 - l_t)^\gamma}{l_t} = b$$

We see on the first equation that the wage in fact does not enter into this equation, which means in this static example that labour supply is independent of the wage level. This might not be the case over two periods, which will be looked at in the next task.

doesn't depend on wage because he only live 1 period, he has to work no matter the wage else you can't consume.

We also see, that a higher  $b$  implies more leisure.

### Reconsider the above problem for two periods:

$$u = \ln(c_1) + \frac{b(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho} \left[ \ln(c_2) + \frac{b(1-l_2)^{1-\gamma}}{1-\gamma} \right]$$

period 1 and 2 consumption is:

$$c_1 = w_1 l_1 \quad \text{and} \quad c_2 = w_2 l_2 + s_1(1+r)$$

where  $s = w_1 l_1 - c_1$

**How does the relative demand for leisure  $\frac{1-l_1}{1-l_2}$  depend on the relative wage  $\frac{w_1}{w_2}$ ?**

The budget constraint is found using the two above equations for consumption and the fact that  $s = w_1 l_1 - c_1$  and replacing that with  $s_1$  in  $c_2$ :

$$c_2 = w_2 l_2 + (1+r)(w_1 l_1 - c_1)$$

Then dividing both sides with  $(1+r)$  and adding  $c_1$  yields:

$$c_1 + c_2 \frac{1}{1+r} = w_1 l_1 + w_2 l_2 \frac{1}{1+r}$$

then we can maximise by using the lagrangian:

$$\Lambda = \ln(c_1) + \frac{b(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho} \left[ \ln(c_2) + \frac{b(1-l_2)^{1-\gamma}}{1-\gamma} \right] + \lambda (w_1 l_1 + w_2 l_2 \frac{1}{1+r} - c_1 - \frac{c_2}{1+r})$$

then we take the first order conditions of  $l_1$  and  $l_2$  with yields:

$$\frac{-b}{(1-l_1)^\gamma} = \lambda w_1 \quad \text{and} \quad \frac{-e^{-\rho} b}{(1-l_2)^\gamma} = \frac{1}{1+r} \lambda w_2$$

dividing both sides of the first by  $w_1$  and both sides of the second by  $\frac{w_2}{1+r}$  we get:

$$\frac{-b}{(1-l_1)^\gamma} \frac{1}{w_1} = \lambda \quad \text{and} \quad \frac{-e^{-\rho} b}{(1-l_2)^\gamma} \frac{1+r}{w_2} = \lambda$$

equating the two equations yields:

$$\frac{-e^{-\rho} b}{(1-l_2)^\gamma} \frac{1+r}{w_2} = \frac{-b}{(1-l_1)^\gamma} \frac{1}{w_1}$$

which is also:

$$\left( \frac{1-l_1}{1-l_2} \right)^\gamma = \frac{1}{e^{-\rho}(1+r)} \frac{w_2}{w_1}$$

By taking a look at the above equation it can be stated that a relative fall in wages  $\frac{w_2}{w_1}$  meaning smaller  $w_1$ , will lead to a relative fall in leisure  $\frac{1-l_1}{(1-l_2)^\gamma}$ . The relative fall in leisure means more people will work in period 1.

*gamma* is the smoothness of the change, the smaller the gamma the stronger the reaction on the change in wages.

**How does the relative demand for leisure  $\frac{1-l_1}{1-l_2}$  depend on the interest rate  $r$ ?**

Looking at the same equation as above it can be stated that an increase in the interest rate  $r$  will lead to a relative fall in leisure  $\frac{1-l_1}{(1-l_2)^\gamma}$