Eksamensæt 2

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Endogenous growth theory

Assume production function of final goods sector is given by:

$$Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1-\alpha}$$

where $1 - \alpha_L$ and $1 - \alpha_K$ is proportion of labour and capital allocated for final production

Capital stock is given by:

$$\dot{K}(t) = sY(t)$$

Production functions of R&D sector is given by:

$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}$$

where $B > 0$ shows efficiency of research, $\gamma \in (0, 1)$ is the output elasticity of labour allocated in R&D, and $\theta \leq 1$ is a parameter describing the elasticity of existing knowledge (A) for the production of increases in the stock of knowledge.

Population growth is exogenous

$$\frac{\dot{L}(t)}{L(t)} = n \quad or \quad \dot{L}(t) = nL(t)$$

Derive an expression for the growth rate of capital $g_K(t)$ and growth rate of technology $g_A(t)$

$g_K(t)$

Vi starter med at opskrive produktionsfunktionen for final goods sector:

$$Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1-\alpha}$$

hvor det gælder, at $s(1 - a_K)^{\alpha} * (1 - a_L)^{1-\alpha} = C_K$, så:

$$\frac{\dot{k}}{k} = C_K * A(t)^{1-\alpha} * K(t)^{\alpha-1} * L(t)^{1-\alpha}$$

$$\frac{\dot{K}}{K} = C_K * \frac{A(t)^{1-\alpha} * L(t)^{1-\alpha}}{K(t)^{1-\alpha}} = C_K * [\frac{A(t) * L(t)}{K(t)}]^{1-\alpha} = g_k$$

Taking logs and differentiating with respect to time yields:

$$\frac{\dot{g}_k}{g_k} = (1-\alpha)(g_A(t) + n - g_K(t))$$

Sætter vi $= 0$ og fjerner $(1-\alpha)$ ved at dividere med den, så får vi

$$g_A^* + n - g_K^*$$

og isoleres g_K^* fåes

$$g_K^* = g_A^* + n$$

$g_A(t)$

Again we divide the find the change by dividing by $A(t)$ on both sides of the equation $\dot{A}(t) = B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta$:

$$\frac{\dot{A}(t)}{A(t)} = \frac{B[\alpha_K K(t)]^\beta [\alpha_L L(t)]^\gamma A(t)^\theta}{A(t)}$$

Then we simplify by getting rid of the brackets on the right hand side

$$g_A(t) = \frac{B * \alpha_K^\beta * K(t)^\beta * \alpha_L^\gamma * L(t)^\gamma * A(t)^\theta}{A(t)}$$

moving the other $A(t)$ down to the denominator by making the exponent negative and then simplifying $A(t)^1 * A(t)^{-\theta}$ to yield the final result:

$$g_A(t) = \frac{B * \alpha_K^\beta * K(t)^\beta * \alpha_L^\gamma * L(t)^\gamma}{A(t)^{1-\theta}}$$

Using the fact that $g_A(t)$ is constant overtime (i.e., $\frac{\partial g_A(t)}{\partial t} = 0$), derive an expression for the growth rate of technology g_A^* along a balanced growth path (steady state), for the case when $\theta + \beta < 1$? Intuitively, explain the equation

We start by taking logs to get rid of all the exponents from the above equation:

$$ln(g_A) = ln(B) + \beta * ln(\alpha_K) + \beta * ln(K) + \gamma * ln(\alpha_L) + \gamma * ln(L) + (\theta - 1) * ln(A)$$

Then we take the derivative with respect to time, where our constants disapear so we get:

$$\frac{\partial ln(g_A)}{\partial t} = \beta \frac{\partial ln(K)}{\partial t} + \gamma \frac{\partial ln(L)}{\partial t} + (\theta - 1) \frac{\partial ln(A)}{\partial t}$$

We know that $\frac{\partial ln(X)}{\partial t} = \frac{\dot{X}}{X}$ so we now get:

$$\frac{\dot{g}_A}{g_A} = \beta \frac{\dot{K}}{K} + \gamma \frac{\dot{L}}{L} + (\theta - 1) \frac{\dot{A}}{A}$$

all these are the growth rates so we get:

$$\frac{\dot{g}_A}{g_A} = \beta g_K + \gamma n + (\theta - 1) g_A$$

Knowing from earlier that $g_K * = g_A * + n$ we insert that to the above an setting equal to 0 as $\frac{\dot{g}_A}{g_A} = 0$ because it is constant over time we also use stars to denote that and get:

$$\beta g_A^* (\beta + \gamma) n + (\theta - 1) g_A^* = 0$$

which is the same as:

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n$$

From above, g_K^* is simply $g_A^* + n$ so from the production function we get, that when A and K are growing at these rates, output is growing at rate g_K^* and output per effective worker is therefor growing at rate g_A^*.

Here the long run growth rate of the economy is endogenous, and again log-run growth is an increasing function of population growth and is zero if population growth is zero, as can be seen by n in the above expression. The fraction of the labor force and the capital stock engaged in R&D, α_L and α_K, do not affect long-run growth; nor does the svaing rate, s. The reason that these parameters do not affect long run growth is the same as in the simple model, which is that they only have a level effect so they effect the level of g_A and g_K but not in $\dot{g}_A$ as a function of g_A

Real business cycle theory

Assume the following utility function:

$$u_t = ln(c_t) + \frac{b(1 - l_t)^{1-\gamma}}{1 - \gamma}$$

where c_t is consumption, $b > 0$ and $\gamma > 0$

Income of the individuals is equal to $w_t l_t$

where w_t is wage, and l_t is the labour supply

Define the budget constraint and answer the following: How, if at all, does labour supply depend on wage?

The budget constraint is just consumption equal to wage times labour supply, which means that the Lagrangian is:

$$\Lambda = ln(c_t) + \frac{b(1 - l_l)^{1-\gamma}}{1 - \gamma} + \lambda(w_t l_t - c_t)$$

Taking first order condition with respect to c and l gives:

$$\frac{1}{c_t} - \lambda = 0 \quad and \quad -\frac{b}{(1 - l_t)^{\gamma}} + \lambda w_t$$

Using the fact that $c_t = w_t l_t$ gives $\lambda = \frac{1}{w_t l_t}$ inserting that into the FOC of l_t yields:

$$-\frac{b}{(1 - l_t)^{\gamma}} + \frac{1}{l_t}$$

this can also be writen as:

$$\frac{(1 - l_t)^{\gamma}}{l_t} = b$$

We see on the first equation that the wage in fact does not enter into this equation, which means in this static example that labour supply is independent of the wage level. This might not be the case over two periods, which will be looked at in the next task.

doesnt depend on wage because he only live 1 period, he has to work no matter the wage else you cant consume.

We also see, that a higher b implies more leisure.

Reconsider the above problem for two periods:

$$u = ln(c_1) + \frac{b(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\left[ln(c_2) + \frac{b(1-l_2)^{1-\gamma}}{1-\gamma}\right]$$

period 1 and 2 consumption is:

$$c_1 = w_1 l_1 \quad and \quad c_2 = w_2 l_2 + s_1(1+r)$$

where $s = w_1 l_1 - c_1$

How does the relative demand for leisure $\frac{1-l_1}{1-l_2}$ depend on the relative wage $\frac{w_1}{w_2}$?

The budget constraint is found using the two above equations for consumption and the fact that $s = s = w_1 l_1 - c_1$ and replacing that with s_1 in c_2:

$$c_2 = w_2 l_2 + (1+r)(w_1 l_1 - c_1)$$

Then dividing both sides with $(1+r)$ and adding c_1 yields:

$$c_1 + c_2 \frac{1}{1+r} = w_1 l_1 + w_2 l_2 \frac{1}{1+r}$$

then we can maximise by using the lagrangian:

$$\Lambda = ln(c_1) + \frac{b(1-l_1)^{1-\gamma}}{1-\gamma} + e^{-\rho}\left[ln(c_2) + \frac{b(1-l_2)^{1-\gamma}}{1-\gamma}\right] + \lambda(w_1 l_1 + w_2 l_2 \frac{1}{1+r} - c_1 - \frac{c_2}{1+r}$$

then we take the first order conditions of l_1 and l_2 with yields:

$$\frac{-b}{(1-l_1)^{\gamma}} = \lambda w_1 \quad and \quad \frac{-e^{-\rho} b}{(1-l_2)^{\gamma}} = \frac{1}{1+r}\lambda w_2$$

dividing both sides of the first by w_1 and both sides of the second by $\frac{w_2}{1+r}$ we get:

$$\frac{-b}{(1-l_1)^{\gamma}}\frac{1}{w_1} = \lambda \quad and \quad \frac{-e^{-\rho} b}{(1-l_2)^{\gamma}}\frac{1+r}{w_2} = \lambda$$

equating the two equations yields:

$$\frac{-e^{-\rho} b}{(1-l_2)^{\gamma}}\frac{1+r}{w_2} = \frac{-b}{(1-l_1)^{\gamma}}\frac{1}{w_1}$$

which is also:

$$(\frac{1-l_1}{(1-l_2)})^{\gamma} = \frac{1}{e^{-\rho}(1+r)}\frac{w_2}{w_1}$$

By taking a look at the above equation it can be stated that a relative fall in wages $\frac{w_2}{w_1}$ meaning smaller w_1, will lead to a relative fall in leisure $\frac{1-l_1}{(1-l_2)^{\gamma}}$. The relative fall in leisure means more people will work in period 1.

gamma is the smoothness of the change, the smaller the gamma the stronger the reaction on the change in wages.

How does the relative demand for leisure $\frac{1-l_1}{1-l_2}$ depend on the interest rate r?

Looking at the same equation as above it can be stated that an increase in the interest rate r will lead to a relative fall in leisure $\frac{1-l_1}{(1-l_2)^{\gamma}}$