SOLUTIONS TO CHAPTER 4

Problem 4.1

(a) From equation (4.11) in the text, output per person on the balanced growth path with the assumption that $G(E) = e^{\phi E}$ is given by

$$(1)\quad \left(\frac{Y}{N}\right)^{bgp} = y^* A(t) e^{\phi E} \frac{e^{-nE} - e^{-nT}}{1 - e^{-nT}},$$

where $y^* = f(k^*)$ which is output per unit of effective labor services on the balanced growth path. We can maximize the natural log of $(Y/N)^{bgp}$ with respect to E, noting that y^* and $A(t)$ are not functions of E. The log of output per person on the balanced growth path is

$$(2)\quad \ln\left(\frac{Y}{N}\right)^{bgp} = \ln y^* + \ln A(t) + \phi E + \ln\left[e^{-nE} - e^{-nT}\right] - \ln\left[1 - e^{-nT}\right],$$

and so the first-order condition is given by

$$(3)\quad \frac{\partial \ln(Y/N)^{bgp}}{\partial E} = \phi + \frac{1}{e^{-nE} - e^{-nT}} e^{-nE}(-n) = 0,$$

or

$$(4)\quad \phi\left(e^{-nE} - e^{-nT}\right) = ne^{-nE}.$$

Collecting the terms in e^{-nE} gives us

(5) $(\phi - n)e^{-nE} = \phi e^{-nT}$,

or simply

$$(6)\quad e^{-nE} = \frac{\phi}{\phi - n} e^{-nT}.$$

Taking the natural log of both sides of equation (6) yields

(7) $- nE = [\ln\phi - \ln(\phi - n)] - nT$.

Multiplying both sides of (7) by $- 1/n$ gives us the following golden-rule level of education:

$$(8)\quad E^* = T - \frac{1}{n}\ln\left[\frac{\phi}{\phi - n}\right].$$

(b) (i) Taking the derivative of E* with respect to T gives us

$$(9)\quad \frac{\partial E^*}{\partial T} = 1.$$

So a rise in T – an increase in lifespan – raises the golden-rule level of education one for one.

(b) (ii) Showing that a fall in n increases the golden-rule level of education is somewhat complicated. From equation (6), we can write

$$(10)\quad e^{-n(T-E^*)} = \frac{\phi - n}{\phi},$$

or

$$(11)\quad 1 - e^{-n(T-E^*)} = \frac{n}{\phi}.$$

Multiplying both sides of equation (11) by ϕ/n gives us

$$(12)\quad \frac{\phi}{n}[1 - e^{-n(T-E^*)}] = 1.$$

Now note that the left-hand side of equation (12) is equivalent to

(13) $V \equiv \phi \int_{s=0}^{T-E^*} e^{-ns} ds.$

Thus, totally differentiating equation (12) gives us

(14) $\frac{\partial V}{\partial n} dn + \frac{\partial V}{\partial E^*} dE^* = 0,$

and so

(15) $\frac{dE^*}{dn} = -\frac{\partial V/\partial n}{\partial V/\partial E^*}.$

Now note that

(16) $\frac{\partial V}{\partial n} = \phi \int_{s=0}^{T-E^*} -se^{-ns} ds < 0,$

and

(17) $\frac{\partial V}{\partial E^*} = -\phi e^{-n(T-E^*)} < 0.$

Thus $dE^*/dn < 0$ and so a fall in n raises the golden-rule level of education.

Problem 4.2

(a) In general, the present discounted value, at time zero, of the worker's lifetime earnings is

(1) $Y = \int_{t=E}^{T} e^{-\bar{r}t} w(t)L(t)dt.$

We can normalize L(t) to one and we are assuming that $w(t) = be^{gt}e^{\phi E}$. Thus (1) becomes

(2) $Y = \int_{t=E}^{T} e^{-\bar{r}t} be^{gt} e^{\phi E} dt = be^{\phi E} \int_{t=E}^{T} e^{-(\bar{r}-g)t} dt.$

Solving the integral in (2) gives us

(3) $Y = be^{\phi E}\left[\frac{-1}{(\bar{r}-g)} e^{-(\bar{r}-g)t} \Big|_{t=E}^{T} \right] = \frac{be^{\phi E}}{\bar{r}-g}\left[-e^{-(\bar{r}-g)T} + e^{-(\bar{r}-g)E} \right],$

which can be rewritten as

(4) $Y = \frac{b}{\bar{r}-g}\left[-e^{\phi E-(\bar{r}-g)T} + e^{[\phi-(\bar{r}-g)]E} \right].$

(b) The first-order condition for the choice of E is given by

(5) $\frac{\partial Y}{\partial E} = \frac{b}{\bar{r}-g}\left[-\phi e^{\phi E-(\bar{r}-g)T} + [\phi-(\bar{r}-g)]e^{[\phi-(\bar{r}-g)]E} \right] = 0.$

This can be rewritten as

(6) $[\phi-(\bar{r}-g)]e^{[\phi-(\bar{r}-g)]E} = \phi e^{\phi E-(\bar{r}-g)T}.$

Dividing both sides by $e^{\phi E}$ and rearranging yields

(7) $e^{-(\bar{r}-g)(E-T)} = \frac{\phi}{\phi-(\bar{r}-g)}.$

Taking the natural log of both sides of equation (7) gives us

(8) $-(\bar{r}-g)(E-T) = \ln\left[\frac{\phi}{\phi-(\bar{r}-g)} \right].$

Dividing both sides of (8) by $-(\bar{r}-g)$ and then adding T to both sides of the resulting expression gives us

(9) $$E^* = T - \frac{1}{\bar{r}-g}\ln\left[\frac{\phi}{\phi-(\bar{r}-g)}\right].$$

(c) (i) From equation (9),

(10) $$\frac{\partial E^*}{\partial T} = 1.$$

Thus an increase in lifespan increases the optimal amount of education. Intuitively, a longer lifespan provides a longer working period over which to receive the higher wages yielded by more education.

(c) (ii) & (iii) The interest rate, $\bar{r}$, and the growth rate, g, enter the optimal choice of education through their difference, $(\bar{r}-g)$. Intuitively, it should be clear that a rise in $\bar{r}$, and thus a rise in $(\bar{r}-g)$, will cause the individual to choose less education. Getting marginally more education foregoes current earnings for higher future earnings. A higher interest rate means that the higher future wages due to increased education will be worth less in present-value terms and hence the individual chooses less education.

Showing this formally is somewhat complicated, however. Taking the inverse of both sides of equation (7) gives us

(11) $$e^{-(\bar{r}-g)(T-E^*)} = \frac{\phi-(\bar{r}-g)}{\phi},$$

or

(12) $$1-e^{-(\bar{r}-g)(T-E^*)} = \frac{(\bar{r}-g)}{\phi}.$$

Multiplying both sides of equation (12) by $\phi/(\bar{r}-g)$ gives us

(13) $$\frac{\phi}{(\bar{r}-g)}[1-e^{-(\bar{r}-g)(T-E^*)}] = 1.$$

Now note that the left-hand side of equation (13) is equivalent to

(14) $$V \equiv \phi\int_{s=0}^{T-E^*} e^{-(\bar{r}-g)s}ds.$$

Thus, totally differentiating equation (13) gives us

(15) $$\frac{\partial V}{\partial(\bar{r}-g)}d(\bar{r}-g) + \frac{\partial V}{\partial E^*}dE^* = 0,$$

and so

(16) $$\frac{dE^*}{d(\bar{r}-g)} = -\frac{\partial V/\partial(\bar{r}-g)}{\partial V/\partial E^*}.$$

Now note that

(17) $$\frac{\partial V}{\partial(\bar{r}-g)} = \phi\int_{s=0}^{T-E^*} -se^{-(\bar{r}-g)s}ds < 0,$$

and

(18) $$\frac{\partial V}{\partial E^*} = -\phi e^{-(\bar{r}-g)(T-E^*)} < 0.$$

Thus $dE^*/d(\bar{r}-g) < 0$. So a rise in $\bar{r}$ decreases the optimal choice of education; a rise in g increases the optimal choice of education.

Problem 4.3

(a) The country's output is described by $Y_i = A_i Q_i e^{\phi E_i} L_i$ and the quality of education is described by $Q_i = B_i (Y_i/L_i)^{\gamma}$. Solving for output per worker and taking logs, we have

(1) $\ln(Y_i/L_i) = \ln A_i + \ln B_i + \gamma \ln(Y_i/L_i) + \phi E_i$.

Thus, the difference in log output per worker between two countries would be

(2) $\ln(Y_2/L_2) - \ln(Y_1/L_1) = \ln A_2 - \ln A_1 + \ln B_2 - \ln B_1 + \gamma(\ln(Y_2/L_2) - \ln(Y_1/L_1)) + \phi(E_2 - E_1)$.

Intuitively, attributing amount $\phi(E_2 - E_1)$ of the difference in log output per worker to education would capture only the direct effect of a higher level of education on output per worker. It would miss the fact that a higher level of education leads to higher output per worker, which results in a higher quality of education that in turn increases output per worker even more.

(b) We can solve equation (2) for the difference in log output per worker, giving us

(3) $\ln(Y_2/L_2) - \ln(Y_1/L_1) = \dfrac{1}{1-\gamma}(\ln A_2 - \ln A_1) + \dfrac{1}{1-\gamma}(\ln B_2 - \ln B_1) + \dfrac{\phi}{1-\gamma}(E_2 - E_1)$.

For a more accurate measure, we should attribute $\dfrac{\phi}{1-\gamma}(E_2 - E_1)$ of $\ln(Y_2/L_2) - \ln(Y_1/L_1)$ to education. Note that the larger is γ—the greater is the effect of output per worker on the quality of education—the larger is φ/(1-γ) and the more we would understate the true effect of education by simply using $\phi(E_2 - E_1)$.

Problem 4.4

(a) We know the production function is given by $Y = K^{\alpha}(e^{\phi E} L)^{1-\alpha}$ and that the marginal product of capital is the first derivative of output with respect to capital. Therefore, we get

(1) $\partial Y/\partial K = \alpha K^{\alpha-1}(e^{\phi E} L)^{1-\alpha}$.

(b) There is perfect capital mobility so the marginal product of capital equals the world rate of return. That is, $\partial Y/\partial K = r^*$. Setting the marginal product of capital equal to r* in equation (1) and solving for K, we have

(2) $K = (r^*/\alpha)^{1/\alpha - 1} L e^{\phi E}$.

(c) We would like to find an expression for $\partial \ln Y/\partial E$ so substitute the expression for K in equation (2) into our production function and take the natural log of both sides. Simplifying we get

(3) $\ln Y = \dfrac{\alpha}{1-\alpha} \ln(r^*/\alpha) + \ln L + \phi E$.

Taking the partial derivative with respect to E results in

(4) $\partial \ln Y/\partial E = \phi$.

(d) From our result in equation (4) we can see that capital mobility increases the impact of the change in E on output. With perfect capital mobility, more education increases the marginal product of capital as we can see in equation (1). That implies that for the marginal product to remain equal to the world rate of return, the capital stock must increase to offset this. This is reflected by the fact that K is increasing in E in equation (2). The increase in the level of the capital stock causes output to rise more. Note that without the response of the capital stock, $\partial \ln Y/\partial E = (1-\alpha)\phi$, which is less than φ.

<u>Problem 4.5</u>

(a) Substituting the expression for output given by equation (1),

(1) $Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}$,

into the definition of output per unit of effective labor, $y(t) \equiv [Y(t)/A(t)L(t)]$, gives us

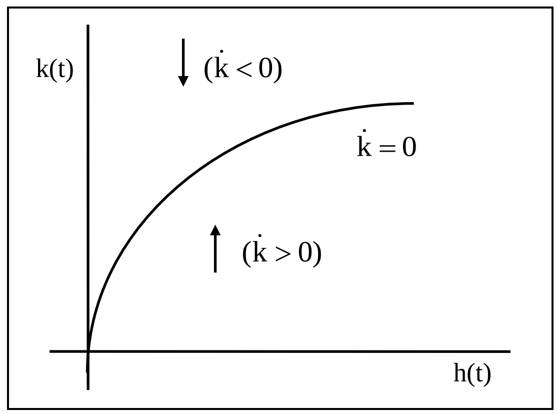
(2) $$y(t) = \frac{K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}}{A(t)L(t)}.$$

Now using the definitions of physical and human capital per unit of effective labor, $k(t) \equiv [K(t)/A(t)L(t)]$ and $h(t) \equiv [H(t)/A(t)L(t)]$, we can rewrite equation (2) as

(3) $$y(t) = \frac{(A(t)L(t)k(t))^{\alpha} (A(t)L(t)(h(t))^{\beta} (A(t)L(t))^{1-\alpha-\beta}}{A(t)L(t)}.$$

Noting that we have $A(t)L(t)$ in both the numerator and denominator, this simplifies to

(4) $y(t) = k(t)^{\alpha} h(t)^{\beta}$.



(b) Differentiating both sides of the definition of $k(t) \equiv K(t)/A(t)L(t)$ with respect to time yields

(5) $$\dot{k}(t) = \frac{\dot{K}(t)A(t)L(t) - K(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}.$$

Using the definition of $k(t) \equiv K(t)/A(t)L(t)$, equation (5) can be rewritten as

(6) $$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right]k(t).$$

Substituting the capital-accumulation equation, $\dot{K}(t) = s_k Y(t) - \delta K(t)$ $\dot{K}(t) = sY(t) - \delta_K K(t)$, as well as the constant growth rates of knowledge and labor into equation (6) gives us

(7) $$\dot{k}(t) = \frac{s_k Y(t) - \delta K(t)}{A(t)L(t)} - (n+g)k(t),$$

which simplifies to

(8) $\dot{k}(t) = s_k y(t) - (n+g+\delta)k(t)$.

Substituting our expression for output per unit of effective labor given by equation (4) into (8) yields

(9) $\dot{k}(t) = s_k k(t)^{\alpha} h(t)^{\beta} - (n+g+\delta)k(t)$.

To derive the $\dot{k} = 0$ locus, set the right-hand-side of equation (9) to 0 to obtain

(10) $s_k k(t)^{\alpha} h(t)^{\beta} = (n+g+\delta)k(t)$.

Solving for k(t) gives us

(11) $$k(t)^{\alpha-1} = \frac{n+g+\delta}{s_k}\frac{1}{h(t)^{\beta}},$$

and thus finally

(12) $$k(t) = \left(\frac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} h(t)^{\frac{\beta}{1-\alpha}}.$$

To describe the $\dot{k} = 0$ locus, we can take the following derivatives:

(16) $dk(t)/dh(t)\big|_{\dot{k}=0} = c_k \dfrac{\beta}{1-\alpha} h(t)^{\frac{\alpha+\beta-1}{1-\alpha}} > 0$, and

(17) $d^2k(t)/dh(t)^2\big|_{\dot{k}=0} = c_k \dfrac{\beta}{1-\alpha}\dfrac{\alpha+\beta-1}{1-\alpha} h(t)^{\frac{\alpha+\beta-1}{1-\alpha}-1} < 0$,

where we have defined $c_k \equiv \left(\dfrac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} > 0$. The second derivative is negative because $\alpha + \beta < 1$. Thus, the $\dot{k} = 0$ locus is upward sloping and concave in (k, h) space. From equation (9) we can see that $\dot{k}$ is increasing in h(t) and so to the right of the $\dot{k} = 0$ locus, $\dot{k}$ is positive and so k is rising To the left of the $\dot{k} = 0$ locus, $\dot{k}$ is negative and so k is falling. See the figure above.

(c) Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields

(18) $\dot{h}(t) = \dfrac{\dot{H}(t)A(t)L(t) - H(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Equation (18) can be simplified to

(19) $\dot{h}(t) = \dfrac{\dot{H}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]h(t)$.

Substituting $\dot{H}(t) = s_h Y(t) - \delta H(t)$, the human-capital-accumulation equation, as well as the constant growth rates of knowledge and labor into equation (19) gives us

(20) $\dot{h}(t) = \dfrac{s_h Y(t) - \delta H(t)}{A(t)L(t)} - (n+g)h(t)$,

which simplifies to

(21) $\dot{h}(t) = s_h y(t) - (n+g+\delta)h(t)$.

Substituting our expression for output per unit of effective labor given by equation (4) into (21) yields

(22) $\dot{h}(t) = s_h k(t)^{\alpha} h(t)^{\beta} - (n+g+\delta)h(t)$.

To derive the $\dot{h} = 0$ locus, set the right-hand-side of equation (22) to 0 to obtain

(23) $s_h k(t)^{\alpha} h(t)^{\beta} = (n+g+\delta)h(t)$.

Solving for k(t) gives us

(24) $k(t)^{\alpha} = \dfrac{n+g+\delta}{s_h} h(t)^{1-\beta}$,

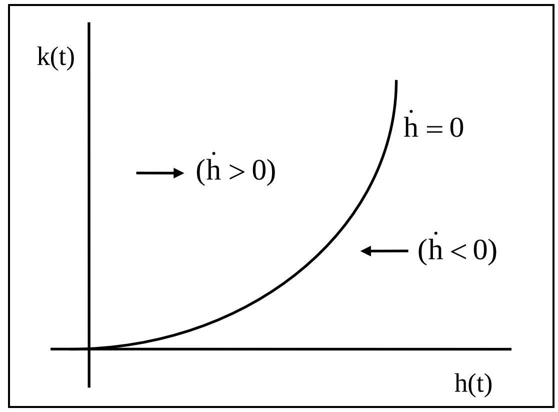
and thus finally

(25) $k(t) = \left(\dfrac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} h(t)^{\frac{1-\beta}{\alpha}}$.

To describe the $\dot{h} = 0$ locus, we can take the following derivatives:

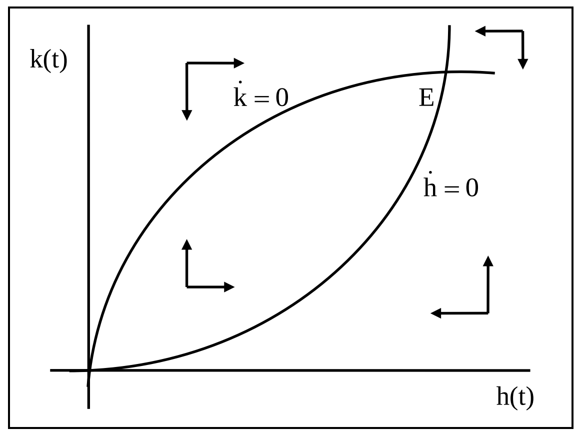
$$\text{(26)}\quad dk(t)/dh(t)\big|_{\dot h=0} = c_h \frac{1-\beta}{\alpha} h(t)^{\frac{1-\alpha-\beta}{\alpha}} > 0, \text{ and}$$

$$\text{(27)}\quad d^2k(t)\big/dh(t)^2\Big|_{\dot h=0} = c_h \frac{1-\beta}{\alpha}\frac{1-\alpha-\beta}{\alpha} h(t)^{\frac{1-\alpha-\beta}{\alpha}-1} > 0,$$



where we have defined $c_h \equiv \left(\frac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} > 0$. The second derivative is positive because α + β < 1. Thus, the $\dot h = 0$ locus is upward sloping and convex in (k, h) space. From equation (9) we can see that $\dot h$ is increasing in k(t) and so above the $\dot h = 0$ locus, $\dot h$ is positive and so h is rising Below the $\dot h = 0$ locus, $\dot h$ is negative and so h is falling. See the figure at right.

(d) Putting the $\dot k = 0$ and $\dot h = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (if we ignore the origin with k = h = 0). From the figure, physical capital per unit of effective labor, k(t) ≡ K(t)/A(t)L(t), is constant on a balanced growth path. Thus physical capital per person, K(t)/L(t) ≡ k(t)A(t), must grow at the same rate as knowledge, which is g. Similarly, human capital per unit of effective labor, h(t) ≡ H(t)/A(t)L(t), is constant on the balanced growth path. Thus human capital per person, H(t)/L(t) ≡ H(t)A(t), must also grow at the same rate as knowledge, which is g.



Dividing the production function, equation (1) by L(t) gives us an expression for output per person:

$$\text{(28)}\quad \frac{Y(t)}{L(t)} = \left(\frac{K(t)}{L(t)}\right)^{\alpha}\left(\frac{H(t)}{L(t)}\right)^{\beta} A(t)^{1-\alpha-\beta}$$

Since K(t)/L(t), H(t)/L(t), and A(t) all grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

__**Problem 4.6**__

(a) To solve for the balanced-growth-path values of k* and h*, we can use equations (12) and (25) to write

$$\text{(1)}\quad \left(\frac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} h^{*\frac{\beta}{1-\alpha}} = \left(\frac{n+g+\delta}{s_h}\right)^{\frac{1}{\alpha}} h^{*\frac{1-\beta}{\alpha}}.$$

Solving for h* gives us

$$\text{(2)}\quad h^{*\frac{1-\beta}{\alpha}-\frac{\beta}{1-\alpha}} = \left(\frac{s_k}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}\left(\frac{s_h}{n+g+\delta}\right)^{\frac{1}{\alpha}}.$$

The exponent on h* simplifies to $(1-\alpha-\beta)/[\alpha(1-\alpha)]$. Taking both sides of (2) to the inverse of that exponent then gives us

$$(3)\quad h^* = s_K^{\frac{\alpha}{1-\alpha-\beta}} s_H^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}}.$$

To find k*, we can then substitute the expression for h* into equation (12) to obtain

$$(4)\quad k^* = s_k^{\frac{1}{1-\alpha}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \left[s_K^{\frac{\alpha}{1-\alpha-\beta}} s_H^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}} \right]^{\frac{\beta}{1-\alpha}}.$$

Simplifying gives us

$$(5)\quad k^* = s_k^{\frac{1}{1-\alpha}+\frac{\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha}+\frac{\beta}{(1-\alpha)(1-\alpha-\beta)}}.$$

The exponents then simplify, leaving us with

$$(6)\quad k^* = s_k^{\frac{1-\alpha-\beta+\alpha\beta}{(1-\alpha)(1-\alpha-\beta)}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1-\alpha-\beta+\beta}{(1-\alpha)(1-\alpha-\beta)}},$$

and thus, finally

$$(7)\quad k^* = s_k^{\frac{1-\beta}{1-\alpha-\beta}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{1}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}},$$

where we have used the fact that $(1-\alpha-\beta+\alpha\beta) = (1-\alpha)(1-\beta)$ to simplify the exponent on s_k.

(b) Since technology is the same in both countries, we can compare output per unit of effective labor. From equation (4) in the solution to problem 4.5, the ratio of output on a balanced growth path in country A to that in country B is given by

$$(8)\quad \frac{y_A^*}{y_B^*} = \left(\frac{k_A^*}{k_B^*}\right)^{\alpha} \left(\frac{h_A^*}{h_B^*}\right)^{\beta}.$$

We know that $\alpha = 1/3$ and $\beta = 1/2$. Substituting these values into equations (7) and (3) yields

$$(9)\quad k^* = s_k^3 s_h^3 \left(\frac{1}{n+g+\delta}\right)^6,$$

and

$$(10)\quad h^* = s_K^2 s_H^4 \left(\frac{1}{n+g+\delta}\right)^6.$$

Substituting equations (9) and (10) into equation (8) gives us

$$(11)\quad \frac{y_A^*}{y_B^*} = \left(\frac{s_{k,A}^3 s_{h,A}^3}{s_{k,B}^3 s_{h,B}^3}\right)^{\frac{1}{3}} \left(\frac{s_{k,A}^2 s_{h,A}^4}{s_{k,B}^2 s_{h,B}^4}\right)^{\frac{1}{2}} = \left(\frac{s_{k,A} s_{h,A}}{s_{k,B} s_{h,B}}\right) \left(\frac{s_{k,A} s_{h,A}^2}{s_{k,B} s_{h,B}^2}\right).$$

We also know that $s_{k,A} = 2s_{k,B}$ and $s_{h,A} = 2s_{h,B}$. Everything else is identical between Countries A and B. Thus, equation (11) becomes

(12) $\dfrac{y_A^*}{y_B^*} = 2 \cdot 2 \cdot 2 \cdot 2^2 = 32.$

Thus, output per worker in country A will be 32 times higher than in country B.

(c) We can compare the amount of skills per unit of effective labor since technology is the same across the two countries. Using equation (10), the ratio of skills in country A to those in country B is given by

(13) $\dfrac{h_A^*}{h_B^*} = \left(\dfrac{s_{k,A}^2 s_{h,A}^4}{s_{k,B}^2 s_{h,B}^4}\right).$

Substituting $s_{k,A} = 2s_{k,B}$ and $s_{h,A} = 2s_{h,B}$ into equation (13) yields

(14) $h_A^* / h_B^* = 2^2 \cdot 2^4 = 64.$

The level of skills per worker on the balanced growth path will be 64 times higher in country A than in country B.

Problem 4.7

(a) Since G(E) does not enter the expression for the evolution of the capital stock, $\dot{K}$, the balanced-growth-path values of capital and output per unit of effective labor services, k and y, are not affected by changes in E. We can write output per worker as

(1) Y/L = AG(E)y.

Thus, the growth rate of output per worker is given by

(2) $\dfrac{\dot{\overline{Y/L}}}{Y/L} = \dfrac{\dot{A}}{A} + \dfrac{G'(E)\dot{E}}{G(E)} + \dfrac{\dot{y}}{y}.$

On a balanced growth path, output per unit of effective labor services is constant so the last term is zero. Technology grows at rate g. Since $G(E) = e^{\phi E}$, where $\dot{E} = m$, we can therefore rewrite equation (2) as

(3) $\dfrac{\dot{\overline{Y/L}}}{Y/L} = g + \dfrac{e^{\phi E}\phi\dot{E}}{e^{\phi E}} = g + \phi m.$

(b) We are given that $\phi = 0.1$, m = 1/15, and that growth of output per worker is 0.02. Therefore, from equation (3), the ratio of education's contribution to the overall growth rate of output per worker is given by [(0.1)(1/15)]/(0.02), which equals one-third.

(c) The lifespan of an individual is fixed at T years, so years of education must be less than or equal to T. Therefore, $\dot{E}(t)$ cannot equal m forever since years of schooling will eventually be longer than one's lifespan. However, that is not to say that $\dot{E}(t)$ cannot equal m ever. For periods of time that $\dot{E}(t) = m$, output per worker growth would be larger, allowing for convergence for poorer economies.

Problem 4.8

(a) Differentiating both sides of the definition of k(t) ≡ K(t)/A(t)L(t) with respect to time yields

(1) $\dot{k}(t) = \dfrac{\dot{K}(t)A(t)L(t) - K(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}.$

Using the definition of k(t) ≡ K(t)/A(t)L(t), equation (1) can be rewritten as

(2) $\dot{k}(t) = \dfrac{\dot{K}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]k(t).$

Substituting the capital-accumulation equation, $\dot{K}(t) = sY(t) - \delta_K K(t)$, as well as the constant growth rates of knowledge and labor into equation (2) gives us

(3) $\dot{k}(t) = \dfrac{sY(t) - \delta_K K(t)}{A(t)L(t)} - (n+g)\,k(t)$.

Substituting the production function, $Y(t) = [(1-a_K)K(t)]^{\alpha}[(1-a_H)H(t)]^{1-\alpha}$, into equation (3) yields

(4) $\dot{k}(t) = s\left[\dfrac{(1-a_K)K(t)}{A(t)L(t)}\right]^{\alpha}\left[\dfrac{(1-a_H)H(t)}{A(t)L(t)}\right]^{1-\alpha} - (n+g+\delta_K)k(t)$.

Finally, defining $c_K \equiv s(1 - a_K)^{\alpha}(1 - a_H)^{1-\alpha}$ and using $k(t) \equiv K(t)/A(t)L(t)$ as well as $h(t) \equiv H(t)/A(t)L(t)$, equation (4) can be rewritten as

(5) $\dot{k}(t) = c_K k(t)^{\alpha} h(t)^{1-\alpha} - (n+g+\delta_K)k(t)$.

Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields

(6) $\dot{h}(t) = \dfrac{\dot{H}(t)A(t)L(t) - H(t)\left[\dot{A}(t)L(t) + A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2}$.

Equation (6) can be simplified to

(7) $\dot{h}(t) = \dfrac{\dot{H}(t)}{A(t)L(t)} - \left[\dfrac{\dot{A}(t)}{A(t)} + \dfrac{\dot{L}(t)}{L(t)}\right]h(t)$.

Substituting $\dot{H}(t) = B[a_K K(t)]^{\gamma}[a_H H(t)]^{\phi}[A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t)$, the human-capital-accumulation equation, as well as the constant growth rates of knowledge and labor into equation (7) gives us

(8) $\dot{h}(t) = B\left[\dfrac{a_K K(t)}{A(t)L(t)}\right]^{\gamma}\left[\dfrac{a_H H(t)}{A(t)L(t)}\right]^{\phi}\left[\dfrac{A(t)L(t)}{A(t)L(t)}\right]^{1-\gamma-\phi} - (n+g+\delta_H)k(t)$.

Finally, defining $c_H \equiv B a_K^{\gamma} a_H^{\phi}$ allows us to rewrite equation (8) as

(9) $\dot{h}(t) = c_H k(t)^{\gamma} h(t)^{\phi} - (n+g+\delta_H)h(t)$.

(b) To find the combinations of h and k such that $\dot{k} = 0$, set the right-hand side of equation (5) equal to zero and solve for k as a function of h:

(10) $c_K\, k(t)^{\alpha} h(t)^{1-\alpha} = (n + g + \delta_K)k(t)$,

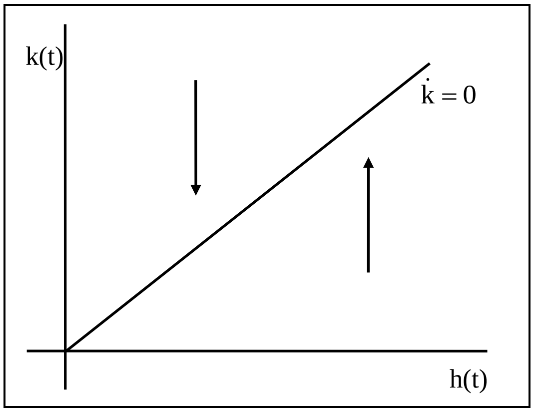
or

(11) $k(t)^{1-\alpha} = c_K\, h(t)^{1-\alpha}/(n + g + \delta_K)$,

and thus finally

(12) $k(t) = [c_K/(n + g + \delta_K)]^{1/(1-\alpha)}\, h(t)$.

The $\dot{k} = 0$ locus, as defined by equation (12), is a straight line with slope $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)} > 0$ that passes through the origin. See the figure at right. From equation (5), we can see that $\dot{k}(t)$ is increasing in h(t). Thus to the right of the $\dot{k} = 0$ locus, $\dot{k} > 0$ and so k(t) is rising. To the left of the $\dot{k} = 0$ locus, $\dot{k} < 0$ and so k(t) is falling.



To find the combinations of h and k such that $\dot{h} = 0$, set the right-hand side of equation (9) equal to zero and solve for k as a function of h:

(13) $c_H k(t)^{\gamma} h(t)^{\phi} = (n + g + \delta_H)h(t)$,

or

(14) $k(t)^{\gamma} = [(n + g + \delta_H)/c_H]h(t)^{1-\phi}$,

and thus finally

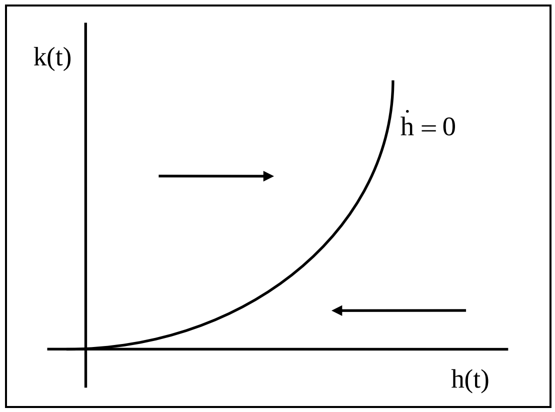
(15) $k(t) = [c_H/(n + g + \delta_H)]^{\gamma} h(t)^{(1-\phi)/\gamma}$.

The following derivatives will be useful:

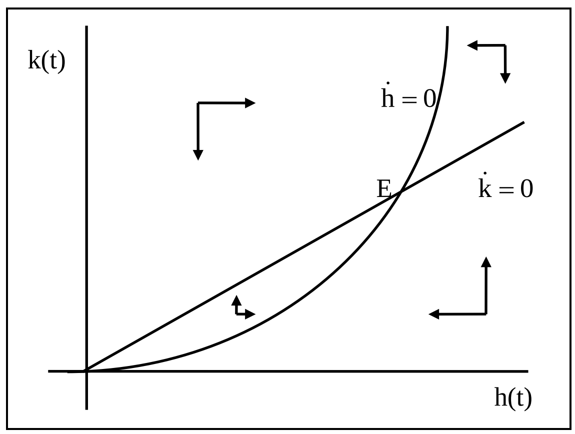
(16) $dk(t)/dh(t)\big|_{\dot{h}=0} = [(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma} h(t)^{(1-\phi-\gamma)/\gamma} > 0$, and

(17) $d^2k(t)\big/dh(t)^2\big|_{\dot{h}=0} = [(1-\phi-\gamma)/\gamma][(1-\phi)/\gamma][c_H/(n+g+\delta_H)]^{\gamma} h(t)^{(1-\phi-2\gamma)/\gamma} > 0$.

The $\dot{h} = 0$ locus, as defined by equation (15), is upward-sloping with a positive second derivative. See the figure at right. From equation (9), we can see that $\dot{h}(t)$ is increasing in k(t). Therefore, above the $\dot{h} = 0$ locus, $\dot{h} > 0$ and so h(t) is increasing. Below the $\dot{h} = 0$ locus, $\dot{h} < 0$ and so h(t) is falling.



(c) Putting the $\dot{k} = 0$ and $\dot{h} = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (if we ignore the origin with k = h = 0). From the figure, physical capital per unit of effective labor, $k(t) \equiv K(t)/A(t)L(t)$, is constant on a balanced growth path. Thus physical capital per person, $K(t)/L(t) \equiv k(t)A(t)$, must grow at the same rate as knowledge, which is g. Similarly, human capital per unit of effective labor, $h(t) \equiv H(t)/A(t)L(t)$, is constant on the balanced growth path. Thus human capital per person, $H(t)/L(t) \equiv H(t)A(t)$, must also grow at the same rate as knowledge, which is g.



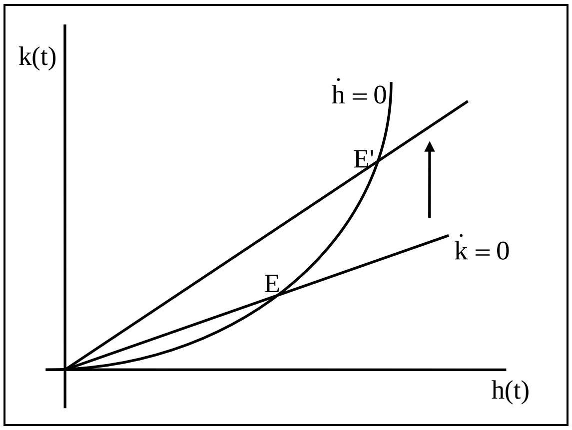
Dividing the production function by L(t) gives us an expression for output per person:

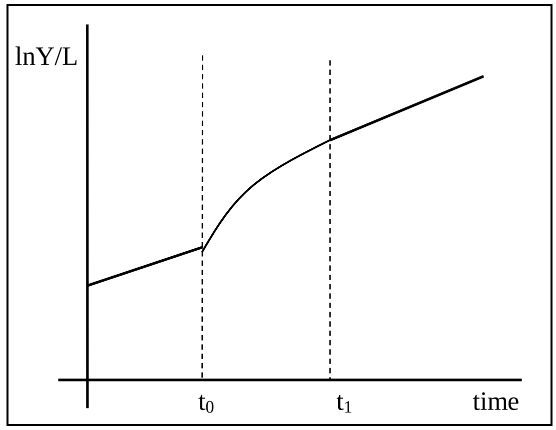
(18) $Y(t)/L(t) = [(1 - a_K)K(t)/L(t)]^{\alpha} [(1 - a_H)H(t)/L(t)]^{1-\alpha}$.

Since K(t)/L(t) and H(t)/L(t) both grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

(d) From equation (12), the slope of the $\dot{k} = 0$ locus is $[c_K/(n + g + \delta_K)]^{1/(1-\alpha)}$, where we have defined

$c_K \equiv s(1 - a_K)^{\alpha} (1 - a_H)^{1-\alpha}$. Thus a rise in s will make the $\dot{k} = 0$ locus steeper. Since s does not appear in equation (15), the $\dot{h} = 0$ locus is unaffected. See the figure on the left. The economy will move from its old balanced growth path at E to a new balanced growth path at E '.





Output per person grows at rate g until the time that s rises (denoted time t_0 in the figure on the right). During the transition from E to E ', both h(t) and k(t) are rising. Thus human capital per person and physical capital per person grow at a rate greater than g during the transition. From equation (18), this means that output per person grows at a rate greater than g during the transition as well. Once the economy reaches the new balanced growth path (at time t_1 in the diagram), h(t) and k(t) are constant again. Thus human and physical capital per person grow at rate g again. Thus output per person grows at rate g again on the new balanced growth path. A permanent rise in the saving rate has only a level effect on output per person, not a permanent growth rate effect.

Problem 4.9

The relevant equations are

(1) $Y(t) = K(t)^{\alpha}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$, (2) $\dot{H}(t) = Ba_H H(t)$, and (3) $\dot{K}(t) = sY(t)$,

where $0 < \alpha < 1$, $0 < \beta < 1$, and $\alpha + \beta > 1$.

(a) To get the growth rate of human capital – which turns out to be constant – divide equation (2) by H(t):

(4) $g_H \equiv \dot{H}(t)/H(t) = Ba_H$.

(b) Substitute the production function, equation (1), into the expression for the evolution of the physical capital stock, equation (3), to obtain

(5) $\dot{K}(t) = sK(t)^{\alpha}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$.

To get the growth rate of physical capital, divide equation (5) by K(t):

(6) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}\left[\left(1 - a_H\right) H(t)\right]^{\beta}$.

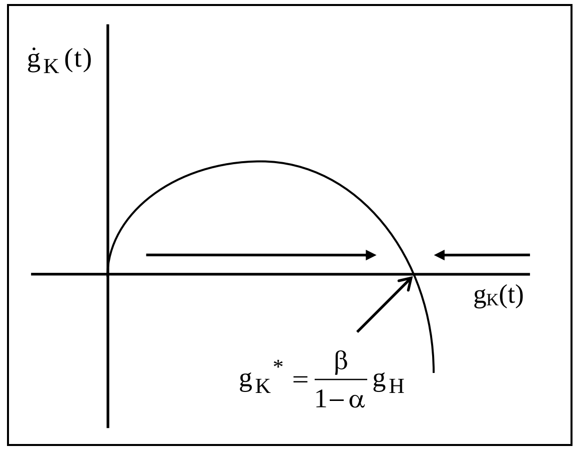
We need to examine the dynamics of the growth rate of physical capital. Taking the time derivative of the log of equation (6) yields the following growth rate of the growth rate of physical capital:

(7) $\dot{g}_K(t)/g_K(t) = (\alpha - 1)\, \dot{K}(t)/K(t) + \beta\, \dot{H}(t)/H(t) = (\alpha - 1)g_K(t) + \beta g_H$.

Now we can plot the change in the growth rate of capital, $\dot{g}_K(t)$, as a function of the growth rate of capital itself, $g_K(t)$. Multiplying both sides of equation (7) by $g_K(t)$ gives us

(8) $\dot{g}_K(t) = (\alpha - 1)\, g_K(t)^2 + \beta g_H g_K(t)$.

Note that we are assuming that $\alpha < 1$ which means that there are decreasing returns to physical capital alone. The phase diagram implied by equation (8) is depicted in the figure at right. Note that $g_K(t)$ is constant when $\dot{g}_K(t) = 0$ or when $(\alpha - 1)g_K(t) + \beta g_H = 0$. Solving this expression for $g_K(t)$ yields $g_K^* = [\beta/(1 - \alpha)]g_H$.



Note that $g_K^* > g_H$ since $\alpha + \beta > 1$ or $\beta > 1 - \alpha$. To the left of g_K^*, from the phase diagram, $\dot{g}_K(t) > 0$ and so $g_K(t)$ rises toward g_K^*.

Similarly, to the right of g_K^*, $\dot{g}_K(t) < 0$ and so $g_K(t)$ falls toward g_K^*. Thus the growth rate of capital converges to a constant value of g_K^* and a balanced growth path exists.

Taking the time derivative of the log of equation (1) yields the growth rate of output:

(9) $\dot{Y}(t)/Y(t) = \alpha\, \dot{K}(t)/K(t) + \beta\, \dot{H}(t)/H(t) = \alpha g_K(t) + \beta g_H$.

On the balanced growth path, $g_K(t) = g_K^* = [\beta/(1 - \alpha)]g_H$ and so

(10) $$\frac{\dot{Y}(t)}{Y(t)} = \frac{\alpha\beta}{(1-\alpha)} g_H + \beta g_H = \frac{\alpha\beta + \beta - \alpha\beta}{(1-\alpha)} g_H = \frac{\beta}{(1-\alpha)} g_H \equiv g_K^*.$$

On the balanced growth path, output grows at the same rate as physical capital, which in turn is greater than the constant growth rate of human capital, $g_H = Ba_H$.

Problem 4.10

(a) The true relationship between social infrastructure and log income per person is expressed by $y_i(t) = a + bSI_i + e_i$. However, of the two different components of social infrastructure, we only have data for SI_i^A. Therefore, we substitute for SI_i and the equation becomes

(1) $y_i = a + bSI_i^A + bSI_i^B + e_i$.

(b) We would like to run an OLS regression on (1) but we cannot observe SI_i^B, so we run an OLS regression on

(2) $y_i = \alpha + \beta SI_i^A + e_i$.

We know that $\beta = \dfrac{\operatorname{cov}(SI_i^A, y_i)}{\operatorname{var}(SI_i^A)}$, where y_i represents the true log income per person, shown in equation (1). Substituting in the true log income per person and simplifying, we get

(3) $$\beta = \frac{\operatorname{cov}(SI_i^A, a + bSI_i^A + bSI_i^B + e_i)}{\operatorname{var}(SI_i^A)} = b + \frac{b\operatorname{cov}(SI_i^A, SI_i^B)}{\operatorname{var}(SI_i^A)} = b\left(1 + \frac{\operatorname{cov}(SI_i^A, SI_i^B)}{\operatorname{var}(SI_i^A)}\right).$$

Remember that SI_i^A is uncorrelated with e_i so $\operatorname{cov}(SI_i^A, e_i) = 0$.

(i) If SI_i^A and SI_i^B are uncorrelated, then $\text{cov}(SI_i^A, SI_i^B) = 0$. Equation (3) shows that $\beta = b$, such that the OLS regression for equation (2) would not produce a bias. Therefore, we would see the true impact of social infrastructure on log income per person.

(ii) If SI_i^A and SI_i^B are positively correlated, then $\text{cov}(SI_i^A, SI_i^B) > 0$. Equation (3) shows that $\beta > b$, such that the OLS regression for equation (2) would produce an upward bias. Therefore, we would be overestimating the impact of the component of social infrastructure that we observe. Intuitively, since the unobserved component of social infrastructure varies positively with the observed component, we would be incorrectly attributing some of the impact of the unobserved component to the observed component.

Problem 4.11

(a) False. Using this OLS method could result in bias. Assume we would like to run an OLS regression on $y_i(t) = a + bSI_i + e_i$, where y_i is income per person, SI_i is social infrastructure, and e_i is an error term that captures everything other than social infrastructure that has an impact on income per person. For an unbiased estimate, social infrastructure cannot be correlated with the error term; otherwise, the estimate will have an upward or downward bias (see problem 4.10). As discussed in the text, it is in fact likely that social infrastructure is correlated with other factors affecting income per person such as cultural factors or geography.

(b) False. Instrumenting a variable requires the variable to be uncorrelated with the error. Since we can only guarantee that the instrumental variable is uncorrelated with social infrastructure, the condition could be violated. Then, the result would be biased, and we would not measure the true effect of social infrastructure on output per person.

(c) False. The coefficient of determination measures the correlation between variables, so a high R^2 would show a good "fit" for the data. However, a high R^2 does not help determine whether the model reflects reality (that is, that the model captures the true effect of social infrastructure on output per person) or whether we have accounted for common problems like omitted variable bias.

Problem 4.12

(a) (i) We have

(1) $\dfrac{dy_i(t)}{dt} = -\lambda[y_i(t) - y^*].$

Since y^* is a constant, the derivative of $y_i(t)$ with respect to time is the same as the derivative of $y_i(t) - y^*$ with respect to time and so equation (1) is equivalent to

(2) $\dfrac{d[y_i(t) - y^*]}{dt} = -\lambda[y_i(t) - y^*],$

which implies that $y_i(t) - y^*$ grows at rate $-\lambda$. Thus

(3) $y_i(t) - y^* = e^{-\lambda t}[y_i(0) - y^*].$

Rearranging equation (3) to solve for $y_i(t)$ gives us

(4) $y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0).$

(a) (ii) Adding a mean-zero, random disturbance to $y_i(t)$ gives us

(5) $y_i(t) = (1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0) + u_i(t).$

Consider the cross-country growth regression given by

(6) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i.$

Using the hint in the question, the coefficient on $y_i(0)$ in this regression equals the covariance of $y_i(t) - y_i(0)$ and $y_i(0)$ divided by the variance of $y_i(0)$. Thus the estimate of β is given by

(7) $$\beta = \frac{\text{cov}[y_i(t) - y_i(0), y_i(0)]}{\text{var}[y_i(0)]}.$$

(If the sample size is large enough, we can treat sample parameters as equivalent to their population counterparts.) Now, use the fact that for any two random variables, X and Y, cov[(X - Y), Y] = cov[X, Y] - var[Y] and so

(8) $$\beta = \frac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \frac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$$

Using equation (5),

(9) $$\text{cov}[y_i(t), y_i(0)] = \text{cov}[(1 - e^{-\lambda t})y^* + e^{-\lambda t}y_i(0) + u_i(t), y_i(0)].$$

Since y* is a constant, and $u_i(t)$ and $y_i(0)$ are assumed to be uncorrelated, we have

(10) $$\text{cov}[y_i(t), y_i(0)] = e^{-\lambda t}\,\text{var}[y_i(0)].$$

Substituting equation (10) into equation (8) gives us

(11) $$\beta = \frac{e^{-\lambda t}\,\text{var}[y_i(0)]}{\text{var}[y_i(0)]} - 1,$$

or

(12) $$e^{-\lambda t} = 1 + \beta.$$

Taking the natural log of both sides of equation (12) and solving for λ gives us

(13) $$\lambda = -\frac{\ln(1+\beta)}{t}.$$

Thus, given an estimate of β, equation (13) could be used to calculate an estimate of the rate of convergence, λ.

(a) (iii) From equation (5), the variance of $y_i(t)$ is given by

(14) $$\text{var}[y_i(t)] = e^{-2\lambda t}\,\text{var}[y_i(0)] + \text{var}[u_i(t)].$$

From equation (13), if β < 0 then λ > 0. This does not, however, ensure that var[$y_i(t)$] < var[$y_i(0)$], so that the variance of cross-country income is falling. The is due to the variance of the random shocks to output, represented by the var[$u_i(t)$] term in equation (14). Thus the effect of β < 0 or λ > 0, which tends to reduce the dispersion of income, can be offset by the random shocks to output, which tend to raise income dispersion.

If β > 0 then λ < 0. From equation (14), we can see that this means var[$y_i(t)$] will be greater than var[$y_i(0)$]. In this case, the effect of β < 0 or λ > 0 is to increase income dispersion, and thus this works in the same direction as the random shocks which also tend to increase income dispersion.

(b) (i) Since y_i^* is time-invariant, analysis equivalent to that in part (a) (i) would yield

(15) $$y_i(t) = (1 - e^{-\lambda t})(a + bX_i) + e^{-\lambda t}y_i(0),$$

where we have used the fact that $y_i^* = a + bX_i$.

(b) (ii) We will determine the value of λ implied by an estimate of β in this model and compare it to the value implied by using the formula from part (a) (ii). In the cross-country growth regression given by

(16) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \varepsilon_i$,

again we have

(17) $\beta = \dfrac{\text{cov}[y_i(t), y_i(0)] - \text{var}[y_i(0)]}{\text{var}[y_i(0)]} = \dfrac{\text{cov}[y_i(t), y_i(0)]}{\text{var}[y_i(0)]} - 1.$

Then, since

(18) $y_i(t) = (1 - e^{-\lambda t})y_i^* + e^{-\lambda t}y_i(0) + e_i,$

we have

(19) $\text{cov}[y_i(t), y_i(0)] = (1 - e^{-\lambda t})\,\text{cov}[y_i^*, y_i(0)] + e^{-\lambda t}\,\text{var}[y_i(0)].$

Since

(20) $y_i(0) = y_i^* + u_i = a + bX_i + u_i$,

we have

(21) $\text{var}[y_i(0)] = b^2\,\text{var}[X_i] + \text{var}[u_i]$,

and

(22) $\text{cov}[y_i^*, y_i(0)] = \text{cov}[a + bX_i, a + bX_i + u_i] = b^2\,\text{var}[X_i]$,

since X_i and u_i are assumed to be uncorrelated. Substituting equations (21) and (22) into equation (19) gives us

(23) $\text{cov}[y_i(t), y_i(0)] = (1 - e^{-\lambda t})b^2\,\text{var}[X_i] + b^2 e^{-\lambda t}\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i],$

or simply

(24) $\text{cov}[y_i(t), y_i(0)] = b^2\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i].$

Substituting equations (21) and (24) into (17) gives us

(25) $\beta = \dfrac{b^2\,\text{var}[X_i] + e^{-\lambda t}\,\text{var}[u_i]}{b^2\,\text{var}[X_i] + \text{var}[u_i]} - 1 = \dfrac{-(1 - e^{-\lambda t})\,\text{var}[u_i]}{b^2\,\text{var}[X_i] + \text{var}[u_i]}.$

We can now solve for the value of λ implied by equation (25) and compare it to the one we would calculate if we used equation (13). Equation (25) implies

(26) $e^{-\lambda t} = 1 + \dfrac{b^2\,\text{var}[X_i] + \text{var}[u_i]}{\text{var}[u_i]}\beta.$

Taking the natural log of both sides of (26) and solving for λ gives us

(27) $\lambda = \dfrac{-\ln\left[1 + \dfrac{b^2\,\text{var}[X_i] + \text{var}[u_i]}{\text{var}[u_i]}\beta\right]}{t}.$

Since $(b^2\,\text{var}[X_i] + \text{var}[u_i])/\text{var}[u_i] > 1$, using the formula given by equation (13) would lead us to calculate an estimate for λ that is too small in absolute value. That is, if $\lambda > 0$, using the method of part (a) (ii) would yield an underestimate of the rate of convergence.

(b) (iii) Subtracting $y_i(0)$ from both sides of equation (18) gives us

(28) $y_i(t) - y_i(0) = (1 - e^{-\lambda t})y_i^* - (1 - e^{-\lambda t})y_i(0) + e_i.$

Substituting equation (20) into (28) yields

(29) $y_i(t) - y_i(0) = (1 - e^{-\lambda t})y_i^* - (1 - e^{-\lambda t})[y_i^* + u_i] + e_i,$

which simplifies to

(30) $y_i(t) - y_i(0) = (e^{-\lambda t} - 1)u_i + e_i.$

Defining $Q \equiv (e^{-\lambda t} - 1)$, we can see that the regression given by

(31) $y_i(t) - y_i(0) = \alpha + \beta y_i(0) + \gamma X_i + \varepsilon_i$

is equivalent to projecting $Qu_i + e_i$ on a constant, $y_i(0)$, and X_i, where e_i is simply a mean-zero, random error that is uncorrelated with the right-hand side variables. Rearranging $y_i(0) = a + bX_i + u_i$ to solve for u_i gives us

(32) $u_i = -a + y_i(0) - bX_i$,
and so
(33) $Qu_i = -Qa + Qy_i(0) - QbX_i$.
Thus, in the regression given by (31), an estimate of β provides an estimate of Q and an estimate of γ provides an estimate of -Qb. Thus, we can construct an estimate of b by taking the negative of the estimate of γ, divided by the estimate of β, or

(34) $-\frac{\gamma}{\beta} = -\frac{-Qb}{Q} = b.$