SOLUTIONS TO CHAPTER 3

Problem 3.1

The production functions for output and new knowledge are given by

(1) $Y(t) = A(t)(1 - a_L)L(t)$,

and

(2) $\dot{A}(t) = Ba_L^{\gamma} L(t)^{\gamma} A(t)^{\theta}, \qquad \theta < 1.$

(a) On a balanced growth path,

(3) $\dot{A}(t)/A(t) = g_A{}^* = \gamma n/(1 - \theta)$.

Dividing both sides of equation (2) by A(t) yields

(4) $\dot{A}(t)/A(t) = Ba_L^{\gamma} L(t)^{\gamma} A(t)^{\theta-1}$.

Equating (3) and (4) yields

(5) $Ba_L^{\gamma} L(t)^{\gamma} A(t)^{\theta-1} = \gamma n/(1-\theta) \quad \Rightarrow \quad A(t)^{\theta-1} = \gamma n \big/ (1-\theta)Ba_L^{\gamma} L(t)^{\gamma}$.

Simplifying and solving for A(t) yields

(6) $A(t) = \left[(1-\theta)Ba_L^{\gamma} L(t)^{\gamma} \big/ \gamma n\right]^{1/(1-\theta)}$.

(b) Substitute equation (6) into equation (1):

$$Y(t) = \left[(1-\theta)Ba_L^{\gamma} L(t)^{\gamma} \big/ \gamma n\right]^{1/(1-\theta)} (1-a_L)L(t) = \left[(1-\theta)B/\gamma n\right]^{1/(1-\theta)} a_L^{\gamma/(1-\theta)} (1-a_L)L(t)^{[\gamma/(1-\theta)]+1}.$$

We can maximize the log of output with respect to a_L or maximize

(7) $\ln Y(t) = [1/(1-\theta)] \ln[(1-\theta)B/\gamma n] + [\gamma/(1-\theta)] \ln a_L + \ln(1-a_L) + [(\gamma/(1-\theta)) + 1] \ln L(t)$.

The first-order condition is given by

(8) $\dfrac{\partial \ln Y(t)}{\partial a_L} = \dfrac{\gamma}{(1-\theta)} \dfrac{1}{a_L} - \dfrac{1}{1-a_L} = 0.$

Solving for an expression for $a_L{}^*$ yields

(9) $a_L{}^* = \dfrac{\gamma}{(1-\theta)+\gamma}.$

The higher is θ, the importance of knowledge in the production of new knowledge, and the higher is γ, the importance of labor in the production of new knowledge, the more of the labor force that should be employed in the knowledge sector.

Problem 3.2

Substituting the production function, $Y_i(t) = K_i(t)^{\theta}$, into the capital-accumulation equation, $\dot{K}_i(t) = s_i Y_i(t)$, yields

(1) $\dot{K}_i(t) = s_i K_i(t)^{\theta}, \quad \theta > 1.$

Dividing both sides of equation (1) by $K_i(t)$ gives an expression for the growth rate of the capital stock, $g_{K,i}$:

(2) $g_{K,i}(t) \equiv \dot{K}_i(t)/K_i(t) = s_i K_i(t)^{\theta-1}$.

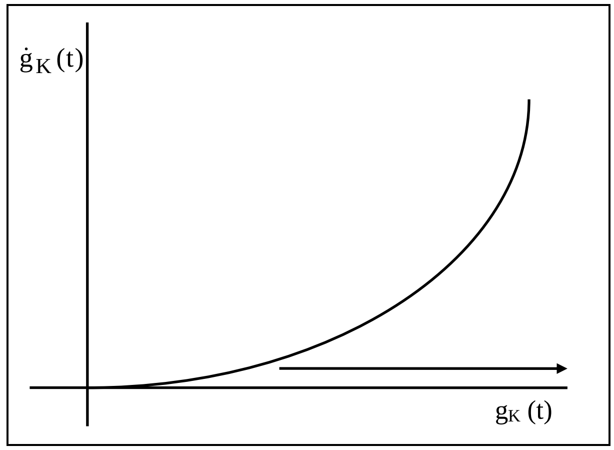
Taking the time derivative of the log of equation (2) yields an expression for the growth rate of the growth rate of capital:

(3) $\dot{g}_{K,i}(t)/g_{K,i}(t) = (\theta-1)g_{K,i}(t)$,

and thus

(4) $\dot{g}_{K,i}(t) = (\theta-1)g_{K,i}(t)^2$.

Equation (4) is plotted at right. With $\theta > 1$, $g_{K,i}$ will be always increasing. The initial value of $g_{K,i}$ is determined by the initial capital stock and the saving rate; see equation (2).



Since both economies have the same K(0) but one has a higher saving rate, then from equation (2), the economy with the higher s will have the higher initial $g_{K,i}(0)$.

From equation (3), the growth rate of $g_{K,i}$ is increasing in $g_{K,i}$. Thus the growth rate of the capital stock in the high-saving economy will always exceed the growth rate of the capital stock in the low-saving economy. That is, we have $g_{K,1}(t) > g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two growth rates will be increasing over time.

More formally, using the production function, we can write the ratio of output in the high-saving country, country 1, to output in the low-saving country, country 2, as

(5) $Y_1(t)/Y_2(t) = \left[K_1(t)/K_2(t)\right]^{\theta}$.

Taking the time derivative of the log of equation (5) yields an expression for the growth rate of the ratio of output in the high-saving economy to output in the low-saving economy:

$$(6)\quad \frac{\left[Y_1(t)/Y_2(t)\right]^{\bullet}}{\left[Y_1(t)/Y_2(t)\right]} = \theta\left[\frac{\dot{K}_1(t)}{K_1(t)} - \frac{\dot{K}_2(t)}{K_2(t)}\right] = \theta\left[g_{K,1}(t) - g_{K,2}(t)\right] > 0.$$

As explained above, $g_{K,1}(t)$ will exceed $g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two will be increasing over time. Thus the growth rate of the output ratio will be positive and increasing over time. That is, the ratio of output in the high-saving economy to output in the low-saving economy will be continually rising, and rising at an increasing rate.

Problem 3.3

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines are given by

(1) $\dot{g}_K = 0 \quad \Rightarrow \quad g_K = g_A + n,$

and

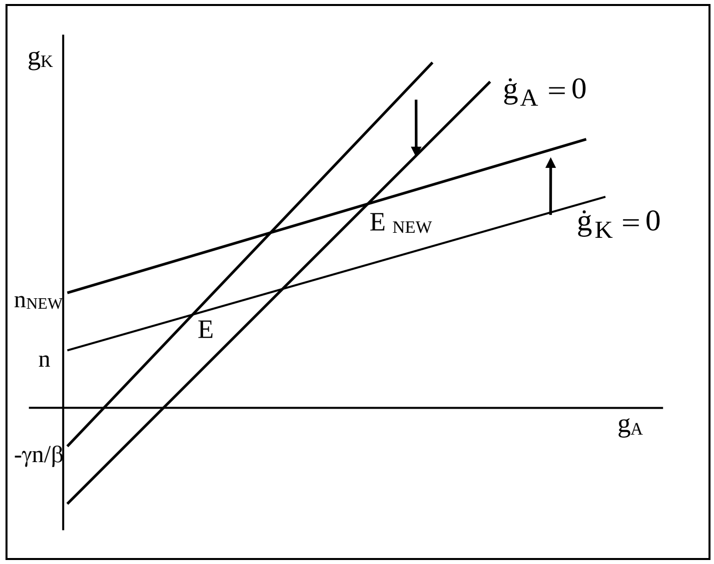
(2) $\dot{g}_A = 0 \quad \Rightarrow \quad g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}.$

The expressions for the growth rates of capital and knowledge are

(3) $g_K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha} \qquad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1} \qquad c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}.$

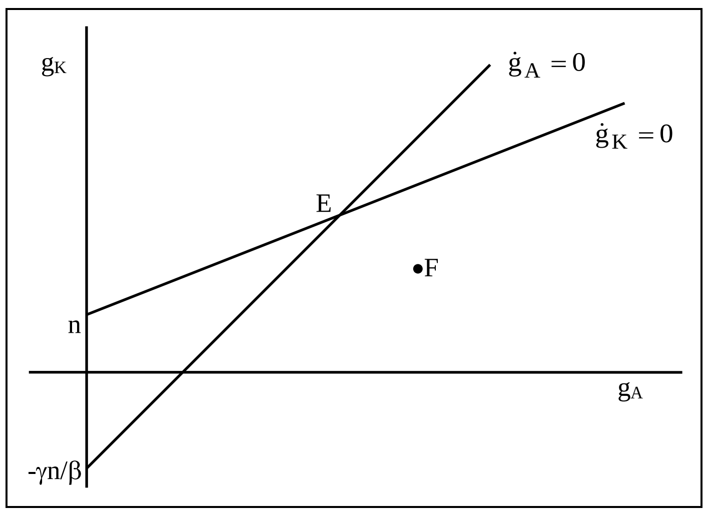
(a) From equation (1), for a given g_A, the value of g_K that satisfies $\dot{g}_K = 0$ is now higher as a result of the rise in population growth from n to n_{NEW}. Thus the $\dot{g}_K = 0$ locus shifts up. From equation (2), for a given g_A, the value of g_K that satisfies $\dot{g}_A = 0$ is now lower. Thus the $\dot{g}_A = 0$ locus shifts down.



Since n does not appear in equation (3), there is no jump in the value of g_K at the moment of the increase in population growth.

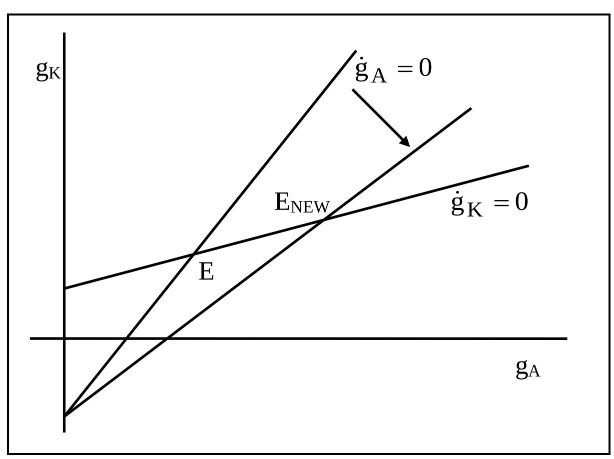
Similarly, since n does not appear in equation (4), there is no jump in the value of g_A at the moment of the rise in population growth.

(b) Note that a_K does not appear in equation (1), the $\dot{g}_K = 0$ line, or in equation (2), the $\dot{g}_A = 0$ line. Thus neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ line shifts as a result of the increase in the fraction of the capital stock used in the knowledge sector from a_K to a_K^{NEW}.



From equation (3), the rise in a_K causes the growth rate of capital, g_K, to jump down. From equation (4), the growth rate of knowledge, g_A, jumps up at the instant of the rise in a_K. Thus the economy moves to a point such as F in the figure.

(c) Since θ does not appear in equation (1), there is no shift of the $\dot{g}_K = 0$ locus as a result of the rise in θ, the coefficient on knowledge in the knowledge production function. From equation (2), the $\dot{g}_A = 0$ locus has slope (1 - θ)/β and therefore becomes flatter after the rise in θ. See the figure.



Since θ does not appear in equation (3), the growth rate of capital, g_K, does not jump at the time of the rise in θ. θ does appear in equation (4) and thus we need to determine the effect that the rise in θ has on the growth rate of knowledge. It turns out that g_A may jump up, jump down or stay the same at the instant of the change in θ. Taking the log of both sides of equation (4) gives us

(5) $\ln g_A(t) = \ln c_A + \beta \ln K(t) + \gamma \ln L(t) + (\theta - 1)\ln A(t)$.

Taking the derivative of both sides of equation (5) with respect to θ yields

(6) $\partial \ln g_A(t)/\partial\theta = \ln A(t)$.

So if A(t) is less than one, so that $\ln A(t) < 0$, the growth rate of knowledge jumps down at the instant of the rise in θ. However, if A(t) is greater than one, so that $\ln A(t) > 0$, the growth rate of knowledge jumps up at the instant of the rise in θ. Finally, if A(t) is equal to one at the time of the change in θ, there is no initial jump in g_A. This means the dynamics of the adjustment to E_{NEW} may differ depending on the value of g_A at the time of the change in θ, but the end result is the same.

Problem 3.4

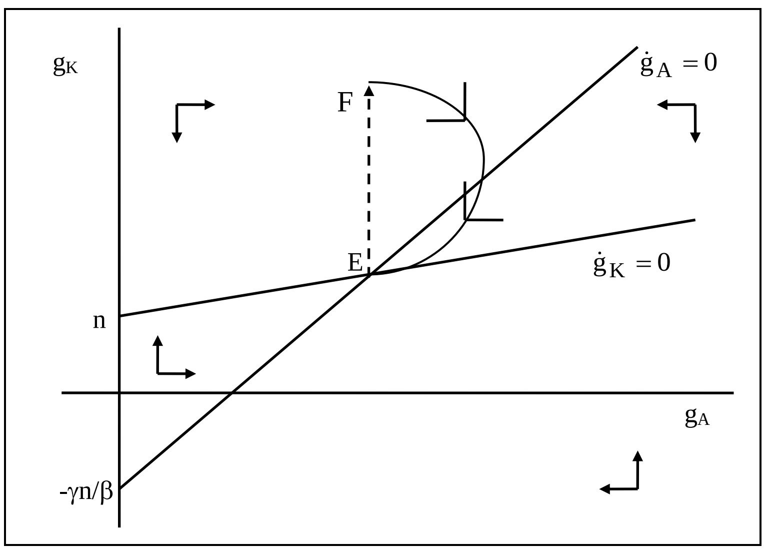
The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci are

(1) $\dot{g}_K = 0 \;\Rightarrow\; g_K = g_A + n$, and (2) $\dot{g}_A = 0 \;\Rightarrow\; g_K = \dfrac{(1-\theta)g_A - \gamma n}{\beta}$.

The equations defining the growth rates of capital and knowledge at any point in time are

(3) $g_K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$ $\qquad c_K \equiv s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$

(4) $g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$ $\qquad c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}$.



(a) Since the saving rate, s, does not appear in equations (1) or (2), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ locus shifts when s increases. From equation (4), the growth rate of knowledge, g_A, does not change at the moment that s increases. However, from equation (3), a rise in s causes an upward jump in the growth rate of capital, g_K. In the figure, the economy jumps from its balanced growth path at E to a point such as F at the moment that s increases.

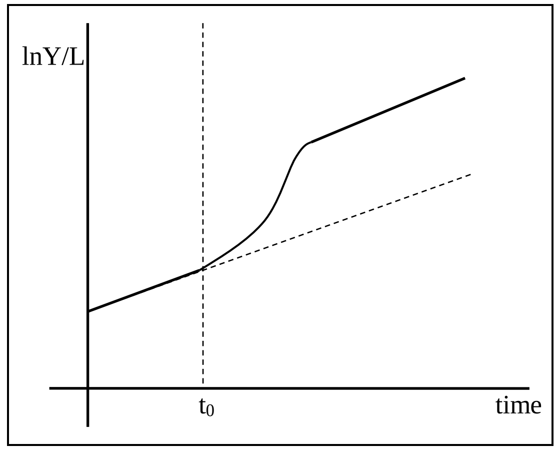
(b) At point F, the economy is above the $\dot{g}_A = 0$ locus and thus g_A is rising. Due to the increase in s, the growth rate of capital is higher than it would have been – the amount of capital going into the production of knowledge is higher than it would have been – and so the growth rate of knowledge begins to rise above what it would have been. Also at point F, the economy is above the $\dot{g}_K = 0$ locus and so g_K is falling. The economy drifts to the southeast and eventually crosses the $\dot{g}_A = 0$ locus at which point g_A begins to fall as well. Since there are decreasing returns to capital and knowledge in the production of new knowledge – $\theta + \beta < 1$ – the increase in s does not have a permanent effect on the growth rates of K and A. The economy eventually returns to point E.

The production function is given by

(5) $Y(t) = \left[(1-a_K)K(t)\right]^{\alpha}\left[A(t)(1-a_L)L(t)\right]^{1-\alpha}$.

Taking the time derivative of the log of equation (5) will yield the growth rate of total output:

(6) $\dfrac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)\left[g_A(t) + n\right].$



On the initial balanced growth path, from equation (1), $g_K{}^* = g_A{}^* + n$. From equation (6), this means that total output is also growing at rate $g_K{}^* = g_A{}^* + n$ on the initial balanced growth path. Thus output per person, Y(t)/L(t), is initially growing at rate $g_A{}^*$. During the transition period, both g_K and g_A are higher than on the balanced growth path and so output per worker must be growing at a rate greater than its balanced-growth-path value of $g_A{}^*$. Whether the growth rate of output per worker is rising or falling will depend, among other things, on the value of α since there is a period of time when g_K is falling and g_A is rising. The figure shows the growth rate of output per worker initially rising and then falling, but the important point is that during the entire transition, the growth rate itself is higher than its balanced-growth-path value of $g_A{}^*$. In the end, once the economy returns to point E, output per worker is again growing at rate $g_A{}^*$, which has not changed.

(c) Note that the effects of an increase in s in this model are qualitatively similar to the effects in the Solow model. Since there are net decreasing returns to the produced factors of production here – $\theta + \beta < 1$ – the increase in s has only a level effect on output per worker. The path of output per worker lies above the path it would have taken but there is no permanent effect on the growth rate of output per worker, which on the balanced growth path is equal to the growth rate of knowledge. This is the same effect that a rise in s has in the Solow model in which there are diminishing returns to the produced factor, capital. Quantitatively, the effect is larger than in the Solow model (for a given set of parameters). This is because, here, A rises above the path it would have taken whereas that is not true in the Solow model.

<u>Problem 3.5</u>

(a) From equations (3.14) and (3.16) in the text, the growth rates of capital and knowledge are given by

(1) $g_K(t) \equiv \dot{K}(t)/K(t) = c_K\left[A(t)L(t)/K(t)\right]^{1-\alpha}$, where $c_K \equiv s[1 - a_K]^{\alpha}[1 - a_L]^{1-\alpha}$, and

(2) $g_A(t) \equiv \dot{A}(t)/A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$, where $c_A \equiv B a_K{}^{\beta} a_L{}^{\gamma}$.

With the assumptions of $\beta + \theta = 1$ and $n = 0$, these equations simplify to

(3) $g_K(t) = [c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha}$, and (4) $g_A(t) = [c_A L^{\gamma}][K(t)/A(t)]^{\beta}$.

Thus given the parameters of the model and the population (which is constant), the ratio A/K determines both growth rates. The two growth rates, g_K and g_A, will be equal when

(5) $[c_K L^{1-\alpha}][A(t)/K(t)]^{1-\alpha} = [c_A L^{\gamma}][K(t)/A(t)]^{\beta}$,

or when

(6) $[A(t)/K(t)]^{1-\alpha+\beta} = [c_A/c_K]L^{\gamma-(1-\alpha)}$.

Thus the value of A/K that yields equal growth rates of capital and knowledge is given by

(7) $A(t)/K(t) = \left[(c_A/c_K)L^{\gamma-(1-\alpha)}\right]^{1/(1-\alpha+\beta)}$.

(b) To find the growth rate of A and K when $g_K = g_A \equiv g^*$, substitute equation (7) into (3):

(8) $g^* = \left[c_K L^{1-\alpha}\right]\left[(c_A/c_K)\, L^{\gamma-(1-\alpha)}\right]^{(1-\alpha)/(1-\alpha+\beta)}$.

Simplifying the exponents yields

(9) $g^* = \left[c_K^{(1-\alpha+\beta)-(1-\alpha)} c_A^{1-\alpha} L^{(1-\alpha)+\gamma(1-\alpha)-(1-\alpha)^2}\right]^{1/(1-\alpha+\beta)}$,

or simply

(10) $g^* = \left[c_K^{\beta} c_A^{1-\alpha} L^{(1-\alpha)(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}$.

(c) To see the way in which an increase in s affects the long-run growth rate of the economy, substitute the definitions of c_K and c_A into equation (10):

(11) $g^* = \left[s^{\beta}(1-a_K)^{\alpha\beta}(1-a_L)^{(1-\alpha)\beta} B^{1-\alpha} a_K^{\beta(1-\alpha)} a_L^{\gamma(1-\alpha)} L^{(1-\alpha)\cdot(\gamma+\alpha)}\right]^{1/(1-\alpha+\beta)}$.

Taking the natural logarithm of both sides of equation (11) gives us

(12) $\ln g^* = \left[1/(1-\alpha+\beta)\right]\{\beta \ln s + (1-\alpha)(\gamma+\alpha)\ln L + (1-\alpha)\ln B + \beta[\alpha\ln(1-a_K) + (1-\alpha)\ln a_K] +$
$(1-\alpha)[\beta\ln(1-a_L) + \gamma\ln a_L]\}$.

Using equation (12), the elasticity of the long-run growth rate of the economy with respect to the saving rate is

(13) $\partial \ln g^*/\partial \ln s = \beta/(1 - \alpha + \beta) > 0$.

Thus an increase in the saving rate increases the long-run growth rate of the economy. This is essentially because it increases the resources devoted to physical capital accumulation and in this model, we have constant returns to the produced factors of production.

(d) We can maximize $\ln g^*$ with respect to a_K to determine the fraction of the capital stock that should be employed in the R&D sector to maximize the long-run growth rate of the economy.

The first-order condition is

(14) $\dfrac{\partial \ln g^*}{\partial a_K} = \dfrac{\beta}{(1-\alpha+\beta)}\left[\dfrac{-\alpha}{(1-a_K)} + \dfrac{(1-\alpha)}{a_K}\right] = 0.$

Solving for the optimal a_K^* yields

(15) $\alpha/(1 - a_K) = (1 - \alpha)/a_K$.

Simplifying gives us

(16) $\alpha a_K = 1 - a_K + \alpha a_K - \alpha$,

and thus

(17) $a_K^* = (1 - \alpha)$.

Thus the optimal fraction of the capital stock to employ in the R&D sector is equal to effective labor's share in the production of output. Note that β, capital's share in the production function for new knowledge, does not affect the optimal allocation of capital to the R&D sector. The reason for this is that an increase in β has two effects. It makes capital more important in the R&D sector, thereby tending to raise the a_K that maximizes g^*. A rise in β also makes the production of new capital more valuable, and new capital is produced when there is more output to be saved and invested. This tends to lower the a_K that maximizes g^* since it implies that more resources should be devoted to the production of output rather than knowledge. In the case we are considering, these two effects exactly cancel each other out.

Problem 3.6

(a) From equation (3.15) in the text, the growth rate of the growth rate of capital is given by

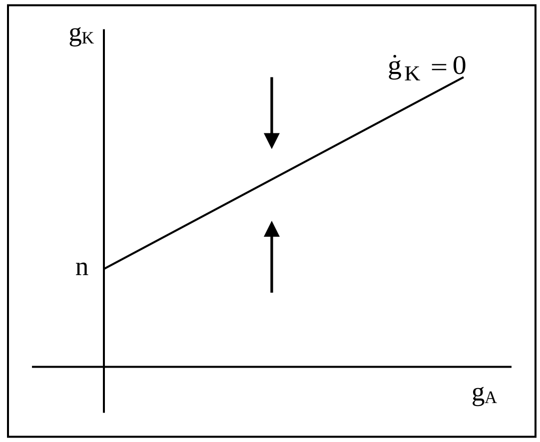
(1) $\dfrac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$.

Thus the equation of the $\dot{g}_K = 0$ locus is

(2) $g_K = g_A + n$.

The slope of the $\dot{g}_K = 0$ locus in (g_A, g_K) space is therefore equal to 1 and the vertical intercept is equal to n.

In addition, g_K is rising when $g_A + n - g_K > 0$ or when $g_K < g_A + n$ (below the $\dot{g}_K = 0$ line) and g_K is falling when $g_A + n - g_K < 0$ or when $g_K > g_A + n$ (above the $\dot{g}_K = 0$ line).



This information is summarized in the figure at right.

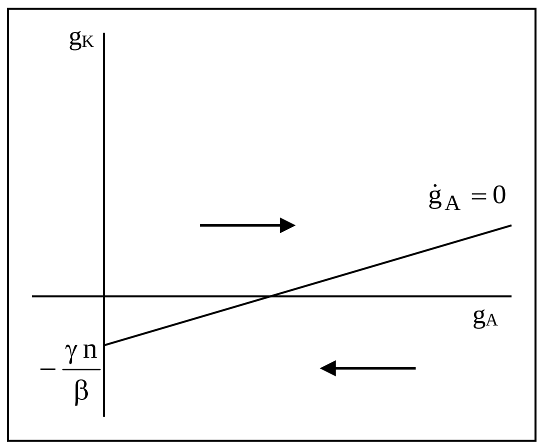
From equation (3.17) in the text, the growth rate of the growth rate of knowledge is given by

(3) $\dfrac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$.

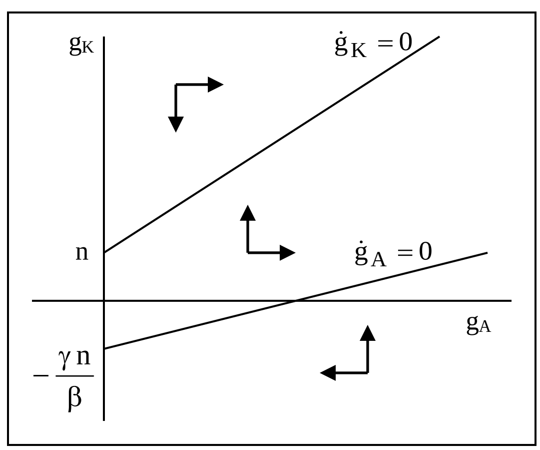
Thus the equation of the $\dot{g}_A = 0$ locus is

(4) $g_K = \dfrac{(1-\theta)}{\beta} g_A - \dfrac{\gamma n}{\beta}$.

The slope of the $\dot{g}_A = 0$ locus is therefore $(1 - \theta)/\beta$ and the vertical intercept is $-\gamma n/\beta$. In addition, g_A is rising when $\beta g_K + \gamma n + (\theta - 1)g_A$ is positive or when $g_K > [(1 - \theta)g_A - \gamma n]/\beta$ (above the $\dot{g}_A = 0$ line). Similarly, g_A is falling when $\beta g_K + \gamma n + (\theta - 1)g_A$ is negative or when $g_K < [(1 - \theta)g_A - \gamma n]/\beta$ (below the $\dot{g}_A = 0$ line).



Putting the two loci together gives us the phase diagram depicted in the figure at right. With $\theta + \beta$ greater than 1, or $\beta > (1 - \theta)$, the slope of the $\dot{g}_A = 0$ locus is less than the slope of the $\dot{g}_K = 0$ locus, which is 1. Thus the two lines diverge as shown in the figure.



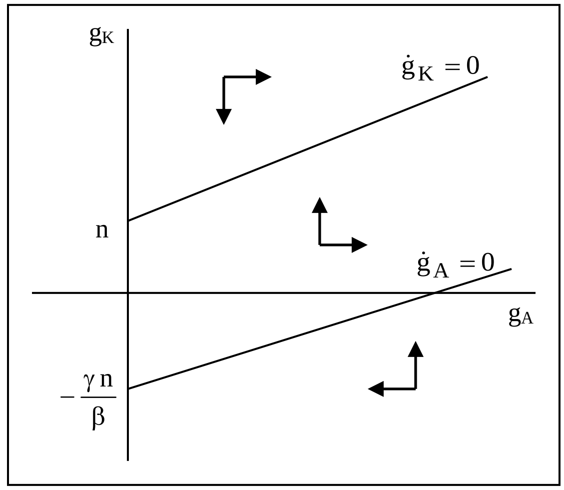
(b) The initial values of g_A and g_K are determined by the parameters of the model and by the initial values of A, K, and L. As the phase diagram from part (a) shows, regardless of where the economy starts, it eventually enters the region between the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines. Once this occurs, we can see that the growth rates of both A and K increase continually. Since output is given by

(5) $Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)]^{1 - \alpha}$,
the growth rate of output is

(6) $\dfrac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)\left[g_A(t) + n\right].$

Thus when g_A and g_K increase continually, so does output.

(c) In this case, $(1 - \theta)/\beta$ equals 1, and thus the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci have the same slope. Since $n > 0$, the $\dot{g}_K = 0$ line lies above the $\dot{g}_A = 0$ line. The dynamics of the economy are similar to the case where $\beta + \theta > 1$; see the phase diagram at right.



Regardless of where the economy starts, it eventually enters the region between the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines. Once this occurs, the growth rates of capital, knowledge, and output increase continually.

Problem 3.7

The relevant equations are

(1) $Y(t) = K(t)^{\alpha}A(t)^{1-\alpha}$, (2) $\dot{K}(t) = sY(t)$, and (3) $\dot{A}(t) = BY(t)$.

(a) Substituting equation (1) into equation (2) yields $\dot{K}(t) = sK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by K(t) allows us to obtain the following expression for the growth rate of capital, $g_K(t)$:

(4) $g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha}$.

Substituting equation (1) into (3) gives us $\dot{A}(t) = BK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by A(t) allows us to obtain the following expression for the growth rate of knowledge, $g_A(t)$:

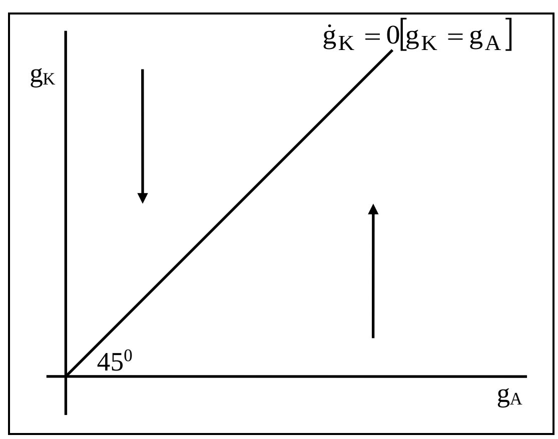
(5) $g_A(t) \equiv \dot{A}(t)/A(t) = BK(t)^{\alpha}A(t)^{-\alpha}$.

(b) Taking the time derivative of (4) yields the following growth rate of the growth rate of capital:

(6) $\dfrac{\dot{g}_K(t)}{g_K(t)} = (\alpha - 1)\dfrac{\dot{K}(t)}{K(t)} + (1-\alpha)\dfrac{\dot{A}(t)}{A(t)}$,

or

(7) $\dot{g}_K(t)/g_K(t) = (1-\alpha)\left[g_A(t) - g_K(t)\right].$



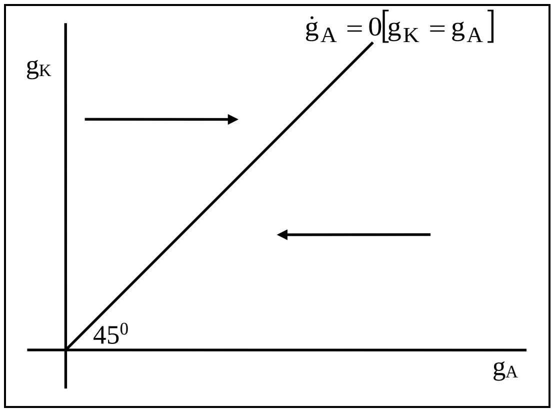
From equation (7), g_K will be constant when $g_A = g_K$. Thus the $\dot{g}_K = 0$ locus is a 45^0 line in (g_A, g_K) space. Also, g_K will be rising when $g_A > g_K$. Thus g_K is rising below the $\dot{g}_K = 0$ line. Lastly, g_K will fall when $g_A < g_K$. Thus g_K is falling above the $\dot{g}_K = 0$ line.

Taking the time derivative of the log of equation (5) yields the following growth rate of the growth rate of knowledge:

(8) $\dfrac{\dot{g}_A(t)}{g_A(t)} = \alpha\dfrac{\dot{K}(t)}{K(t)} - \alpha\dfrac{\dot{A}(t)}{A(t)}$,

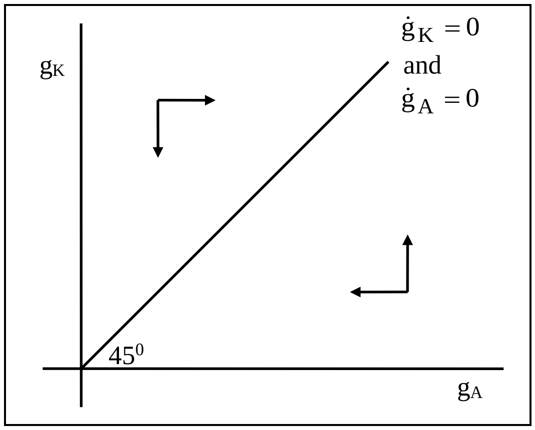
or

(9) $\dot{g}_A(t)/g_A(t) = \alpha\left[g_K(t) - g_A(t)\right]$.



From equation (9), g_A will be constant when $g_K = g_A$. Thus the $\dot{g}_A = 0$ locus is also a 45^0 line in (g_A ,g_K) space. Also, g_A will be rising when $g_K > g_A$. Thus above the $\dot{g}_A = 0$ line, g_A will be rising. Finally, g_A will be falling when $g_K < g_A$. Thus below the $\dot{g}_A = 0$ line, g_A will be falling.

(c) We can put the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci into one diagram.



Although we can see that the economy will eventually arrive at a situation where $g_K = g_A$ and they are constant, we still do not have enough information to determine the unique balanced growth path. Rewriting equations (4) and (5) gives us

(4) $g_K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha} = s\left[A(t)/K(t)\right]^{1-\alpha}$,

and

(5) $g_A(t) = BK(t)^{\alpha}A(t)^{-\alpha} = B\left[A(t)/K(t)\right]^{-\alpha}$.

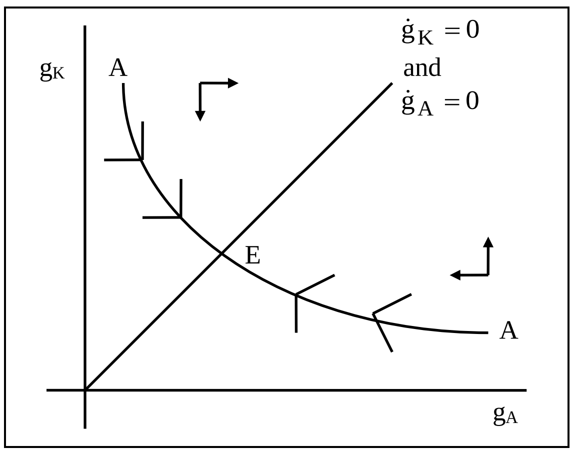
At any point in time, the growth rates of capital and knowledge are linked because they both depend on the ratio of knowledge to capital at that point in time. It is therefore possible to write one growth rate as a function of the other.

From equation (5), $\left[A(t)/K(t)\right]^{\alpha} = B/g_A(t)$ or simply

(10) $A(t)/K(t) = \left[B/g_A(t)\right]^{1/\alpha}$.

Substituting equation (10) into equation (4) gives us

(11) $g_K(t) = s\left[B/g_A(t)\right]^{(1-\alpha)/\alpha}$.



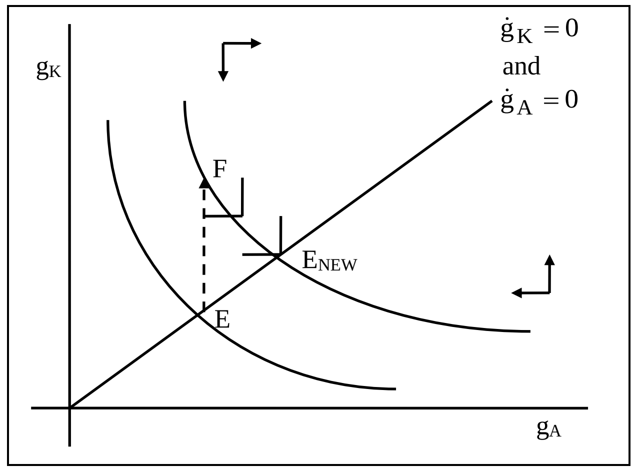
It must be the case that g_K and g_A lie on the locus satisfying equation (11), which is labeled AA in the figure. Regardless of the initial ratio of A/K the economy starts somewhere on this locus and then moves along it to point E. Thus the economy does converge to a unique balanced growth path at E.

To calculate the growth rates of capital and knowledge on the balanced growth path, note that at point E we are on the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci

where $g_K = g_A$. Letting g* denote this common growth rate, then from equation (11),
$g^* = s\left[B/g^*\right]^{(1-\alpha)/\alpha}$.
Rearranging to solve for g* yields
(12) $g^* = s^{\alpha}B^{1-\alpha}$.

Taking the time derivative of the log of the production function, equation (1), yields the growth rate of real output, $\dot{Y}(t)/Y(t) = \alpha g_K(t) + (1-\alpha)g_A(t)$. On the balanced growth path, $g_K = g_A \equiv g^*$, and thus
(13) $\dot{Y}(t)/Y(t) = \alpha g^* + (1-\alpha)g^* = g^* \equiv s^{\alpha}B^{1-\alpha}$.
On the balanced growth path, capital, knowledge and output all grow at rate g*.

(d) Clearly, from equation (12), a rise in the saving rate, s, raises g* and thus raises the long-run growth rates of capital, knowledge and output.



From equations (7) and (9), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ lines shift when s changes since s does not appear in either equation. From equation (4), a rise in s causes g_K to jump up. Also, the locus given by equation (11) shifts out. So at the moment that s rises, the economy moves from its balanced growth path at point E to a point such as F. It then moves down along the AA locus given by equation (11) until it reaches a new balanced growth path at point E_{NEW}.

<u>Problem 3.8</u>
Using the constant-relative-risk aversion equation (2.3) and (2.22) from the text, we have
$$U(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \quad \text{and} \quad \frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}.$$
Solving for r(t) gives us
(1) $r(t) = \rho + \theta\dfrac{\dot{C}(t)}{C(t)}$.
All output is consumed and all individuals are the same and thus choose the same consumption path so equilibrium in the goods market requires that $C(t)\bar{L} = Y(t)$ and so consumption grows at the same rate as output. Output grows at $[(1-\phi)/\phi]BL_A$ and so equation (1) becomes
(2) $r(t) = \rho + \theta\left(\dfrac{1-\phi}{\phi}\right)BL_A$,
which is analogous to equation (3.40).

From the textbook, we know that the present value of profits is now given by

$$(3)\quad \pi(t)=\frac{\frac{1-\phi}{\phi}(\bar{L}-L_A)\frac{w(t)}{A(t)}}{\rho+\theta\left(\frac{1-\phi}{\phi}\right)BL_A-\left(\frac{1-2\phi}{\phi}\right)BL_A}=\frac{1-\phi}{\phi}\frac{w(t)}{A(t)}\left(\frac{\bar{L}-L_A}{\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)}\right),$$

and that the cost of an invention is w(t)/BA(t). At equilibrium the present value of profits must equal the costs of the invention, or

$$(4)\quad \frac{w(t)}{BA(t)}=\frac{1-\phi}{\phi}\frac{w(t)}{A(t)}\left(\frac{\bar{L}-L_A}{\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)}\right).$$

Solving equation (4) we can find the optimal level of labor in the R&D sector, L_A:

$$(5)\quad \bar{L}-L_A=\frac{\phi}{1-\phi}\frac{1}{B}\left(\rho+BL_A\left(\frac{\theta(1-\phi)}{\phi}-\frac{1-2\phi}{\phi}\right)\right),$$

which simplifies to

$$(6)\quad \bar{L}-\frac{\phi\rho}{(1-\phi)B}=L_A\left(\theta-\left(\frac{1-2\phi}{1-\phi}\right)+1\right).$$

Solving for L_A gives us

$$(7)\quad L_A=\left(\bar{L}-\frac{\phi\rho}{(1-\phi)B}\right)\left(\frac{1-\phi}{\phi+\theta-\phi\theta}\right),$$

or simply

$$(8)\quad L_A=\frac{1}{\phi+\theta-\phi\theta}\left((1-\phi)\bar{L}-\frac{\phi\rho}{B}\right).$$

The resulting optimal level of L_A is similar to the result in equation (3.43). However, the level of labor cannot be less than 0, so we rewrite equation (8) as

$$(9)\quad L_A=\max\left\{\frac{1}{\phi+\theta-\phi\theta}\left((1-\phi)\bar{L}-\frac{\phi\rho}{B}\right),0\right\}.$$

Problem 3.9

(a) The patent price is now $\delta w(t)/\phi$, where δ satisfies $\phi\le\delta\le 1$. Each patent-holder's profits at time t are now given by

$$(1)\quad \pi(t)=\frac{\bar{L}-L_A}{A(t)}\left[\frac{\delta w(t)}{\phi}-w(t)\right],$$

or simply

$$(2)\quad \pi(t)=\frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{A(t)}w(t).$$

Comparing equation (2) with equation (3.39) in the text, we can see that the growth rate of profits from a given invention will be unchanged from the standard model and equal to $[(1-2\varphi)/\varphi]BL_A$. In addition, the real interest rate remains equal to $\rho+[(1-\varphi)/\varphi]BL_A$. The present value of the profits earned from the discovery of a new idea at time t is therefore

(3) $$\pi(t)=\frac{\dfrac{\delta-\phi}{\phi}\dfrac{\bar{L}-L_A}{A(t)}w(t)}{\rho+\dfrac{1-\phi}{\phi}BL_A-\dfrac{1-2\phi}{\phi}BL_A},$$

which simplifies to

(4) $$\pi(t)=\frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{\rho+BL_A}\frac{w(t)}{A(t)}.$$

If the amount of R&D is strictly positive, equilibrium requires the present value of profits from an invention to equal the costs of an invention, w(t)/[BA(t)]. Thus, equilibrium requires

(5) $$\frac{\delta-\phi}{\phi}\frac{\bar{L}-L_A}{\rho+BL_A}\frac{w(t)}{A(t)}=\frac{w(t)}{BA(t)}.$$

We must now solve equation (5) for the equilibrium value of L_A. We can rewrite (5) as

(6) $$\bar{L}-L_A=\frac{1}{B}\frac{\phi}{\delta-\phi}(\rho+BL_A).$$

Isolating the terms in L_A yields

(7) $$L_A+\frac{\phi}{\delta-\phi}L_A=\bar{L}-\frac{\phi}{\delta-\phi}\frac{\rho}{B},$$

or simply

(8) $$\frac{\delta}{\delta-\phi}L_A=\bar{L}-\frac{\phi}{\delta-\phi}\frac{\rho}{B}.$$

Thus, L_A is given by

(9) $$L_A=\frac{\delta-\phi}{\delta}\bar{L}-\frac{\phi}{\delta}\frac{\rho}{B}.$$

To allow for the possibility of a corner solution, we need to modify (9) to

(10) $$L_A=\max\left\{\frac{\delta-\phi}{\delta}\bar{L}-\frac{\phi}{\delta}\frac{\rho}{B},0\right\}.$$

Since the growth rate of output is [(1-φ)/φ]BL_A, we have

(11) $$\frac{\dot{Y}(t)}{Y(t)}=\max\left\{\frac{\delta-\phi}{\delta}\frac{1-\phi}{\phi}B\bar{L}-\frac{1-\phi}{\delta}\rho,0\right\}.$$

To see the effects of a change in δ on the equilibrium growth rate, take the derivative of $\dot{Y}(t)/Y(t)$ with respect to δ in the case where output growth is strictly positive:

(12) $$\frac{\partial\dot{Y}(t)/Y(t)}{\partial\delta}=\frac{1-\phi}{\phi}B\bar{L}\left[\frac{\delta-(\delta-\phi)}{\delta^2}\right]+\frac{1-\phi}{\delta^2}\rho,$$

or simply

(13) $$\frac{\partial\dot{Y}(t)/Y(t)}{\partial\delta}=\frac{\phi}{\delta^2}\frac{1-\phi}{\phi}B\bar{L}+\frac{1-\phi}{\delta^2}\rho>0.$$

Thus, in the situation where output growth is positive, a decrease in the value of δ reduces the equilibrium growth rate of output.

(b) It is true that setting δ = φ would eliminate the monopoly distortion: if patent-holders were forced to charge marginal cost, the present value of the monopolist's profits would be zero. But the profits from an invention represent the incentive to innovate in this model. From the production function for new ideas, it takes 1/[BA(t)] units of labor to produce a new idea and so it costs w(t)/[BA(t)] to produce a new idea.

Marginal-cost pricing would imply that innovators would not be able to recoup this cost of producing an idea and there would be no incentive to innovate in this model. This would result in no R&D, which is not socially optimal.

Problem 3.10

(a) Equation (3.47) from the textbook is

$$(3.47)\quad U = \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} e^{\frac{1-\phi}{\phi} BL_A} e^{t} \right) dt\,,$$

which can be rewritten as

$$(1)\quad U = \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt + \int_{t=0}^{\infty} t e^{-\rho t} \frac{1-\phi}{\phi} BL_A \, dt\,.$$

Let's look at the first integral. We can write

$$(2)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \lim_{z\to\infty} \int_{t=0}^{z} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt\,,$$

which simplifies to

$$(3)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \lim_{z\to\infty} -\frac{1}{\rho} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) \Bigg|_{t=0}^{t=z}.$$

Thus, the first integral is given by

$$(4)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \frac{1}{\rho} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right),$$

or simply

$$(5)\quad \int_{t=0}^{\infty} e^{-\rho t} \ln\left(\frac{\bar{L} - L_A}{\bar{L}} A(0)^{\frac{1-\phi}{\phi}} \right) dt = \frac{1}{\rho} \left(\ln \frac{\bar{L} - L_A}{\bar{L}} + \frac{1-\phi}{\phi} \ln A(0) \right).$$

Now let's look at the second integral. We can write

$$(6)\quad \int_{t=0}^{\infty} t e^{-\rho t} \frac{1-\phi}{\phi} BL_A \, dt = \lim_{z\to\infty} \int_{t=0}^{z} t e^{-\rho t} \frac{1-\phi}{\phi} BL_A \, dt = \frac{1-\phi}{\phi} BL_A \lim_{z\to\infty} \int_{t=0}^{z} t e^{-\rho t} dt\,.$$

Integration by parts states that $\int u dv = uv - \int v du$. Here, let $u = t$ and $dv = e^{-\rho t}$. Then, $\frac{du}{dt} = 1$ and $v = -\frac{1}{\rho} e^{-\rho t}$ so we can rewrite equation (6) as

$$(7)\quad \int_{t=0}^{\infty} t e^{-\rho t} \frac{1-\phi}{\phi} BL_A \, dt = \frac{1-\phi}{\phi} BL_A \lim_{z\to\infty} \left[-\frac{t}{\rho} e^{-\rho t} + \int_{t=0}^{z} \frac{1}{\rho} e^{-\rho t} dt \right].$$

Solving the integral gives us

$$(8)\quad \int_{t=0}^{\infty} t e^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}BL_A \lim_{z\to\infty}\left[-\frac{t}{\rho}e^{-\rho t} - \frac{1}{\rho^2}e^{-\rho t}\Big|_{t=0}^{t=z}\right],$$

or simply

$$(9)\quad \int_{t=0}^{\infty} t e^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}BL_A \lim_{z\to\infty}\left[-\frac{z}{\rho}e^{-\rho z} - \frac{1}{\rho^2}e^{-\rho z} + \frac{1}{\rho^2}\right].$$

We cannot solve the first term on the right-hand-side of equation (9) because of its indeterminant form. Using L'Hôpital's Rule, we get:

$$(10)\quad \lim_{z\to\infty} -\frac{z}{\rho e^{\rho z}} = \lim_{z\to\infty} -\frac{1}{\rho^2 e^{\rho z}} = 0.$$

Therefore, the limit of the expression on the right-hand-side of (9) simplifies to $1/\rho^2$ and so we have

$$(11)\quad \int_{t=0}^{\infty} t e^{-\rho t}\frac{1-\phi}{\phi}BL_A dt = \frac{1-\phi}{\phi}\frac{1}{\rho^2}BL_A.$$

Therefore, using equations (5) and (11), equation (1) becomes

$$(12)\quad U = \frac{1}{\rho}\left(\ln\frac{\bar{L}-L_A}{\bar{L}} + \frac{1-\phi}{\phi}\ln A(0)\right) + \frac{1-\phi}{\phi}\frac{1}{\rho^2}BL_A,$$

which is equivalent to equation (3.48) in the text.

(b) To arrive at the socially optimal level of L_A given by equation (3.49), we must maximize the expression in equation (3.48) for lifetime utility with respect to L_A. We can rewrite equation (3.48) as

$$(1)\quad \begin{aligned} U &= \frac{1}{\rho}\left(\ln\frac{\bar{L}-L_A}{\bar{L}} + \frac{1-\phi}{\phi}\ln A(0) + \frac{1-\phi}{\phi}\frac{BL_A}{\rho}\right) \\ &= \frac{1}{\rho}\ln(\bar{L}-L_A) - \frac{1}{\rho}\ln\bar{L} + \frac{1-\phi}{\phi}\frac{1}{\rho}\ln A(0) + \frac{1}{\rho^2}\frac{1-\phi}{\phi}BL_A \end{aligned}.$$

The first order-condition is given by

$$(2)\quad \frac{\partial U}{\partial L_A} = \frac{\partial}{\partial L_A}\left\{\frac{1}{\rho}\ln(\bar{L}-L_A) - \frac{1}{\rho}\ln\bar{L} + \frac{1-\phi}{\phi}\frac{1}{\rho}\ln A(0) + \frac{1}{\rho^2}\frac{1-\phi}{\phi}BL_A\right\} = 0.$$

Taking the partial derivative yields

$$(3)\quad -\frac{1}{\rho}\frac{1}{\bar{L}-L_A} + \frac{B}{\rho^2}\frac{1-\phi}{\phi} = 0.$$

Solving for L_A gives us

$$(4)\quad \frac{1}{\rho}\frac{1}{\bar{L}-L_A} = \frac{B}{\rho^2}\frac{1-\phi}{\phi},$$

or simply

$$(5)\quad \bar{L}-L_A = \frac{\phi}{1-\phi}\frac{\rho}{B},$$

and thus finally the optimal choice of L_A is given by

$$(6)\quad L_A^{OPT} = \bar{L} - \frac{\phi}{1-\phi}\frac{\rho}{B}.$$

However, the optimal level of labor in R&D cannot be negative, so we rewrite equation (6) as

(7) $L_A^{OPT} = \max\left\{\bar{L} - \frac{\phi}{1-\phi}\frac{\rho}{B}, 0\right\}.$

Now we need to check that L_A^{OPT} is indeed the maximum. By taking the second derivative, we get

(8) $\frac{\partial^2 U}{\partial L_A^2} = -\frac{1}{\rho}\frac{1}{(\bar{L} - L_A)^2} < 0.$

The second derivative is negative, so L_A^{OPT} is a maximum. Therefore, equation (7) provides the socially optimal level of L_A and is equivalent to equation (3.49) in the textbook.

Problem 3.11

(a) Equation (3.29) in the text, $Y = A^{(1-\phi)/\phi} L_Y$, continues to hold. Taking logs of both sides and then differentiating with respect to time gives us the following expression for the growth rate of Y:

(1) $\frac{\dot{Y}}{Y} = \frac{1-\phi}{\phi}\frac{\dot{A}}{A} + \frac{\dot{L}_Y}{L_Y}.$

On a balanced growth path, a_L is constant and so L_Y grows at rate n. Denoting $g_Y \equiv \dot{Y}/Y$ and $g_A \equiv \dot{A}/A$, we have

(2) $g_Y = \frac{1-\phi}{\phi} g_A + n.$

The growth rate of output per person, $g_{Y/L}$, is therefore

(3) $g_{Y/L} = g_Y - g_L = \frac{1-\phi}{\phi} g_A.$

The Euler equation for a CRRA utility function is given by equation (2.22) in the text, $g_{C/L} = (r - \rho)/\theta$, where $g_{C/L}$ denotes the growth rate of consumption per person. In the case of log utility, $\theta = 1$, and so the Euler equation becomes

(4) $g_{C/L} = r - \rho.$

On the balanced growth path, $g_{C/L} = g_{Y/L}$. Substituting equation (3) into (4) and solving for r gives us

(5) $r = \rho + \frac{1-\phi}{\phi} g_A.$

(b) Equation (3.39) in the text, which gives profits at a point in time, continues to hold although the labor force is no longer constant. Replacing $\bar{L} - L_A$ in (3.39) with $(1 - a_L)L$ gives us an analogous expression:

(6) $\pi(t) = \frac{1-\phi}{\phi}\frac{(1-a_L)L(t)}{A(t)} w(t).$

Taking logs of both sides of (6) and then differentiating with respect to time tells us that the growth rate of profits equals $g_L + g_w - g_A$. Since L grows at rate n, and wages grow at the same rate as Y/L on the balanced growth path, we have

(7) $g_\pi = n + \frac{1-\phi}{\phi} g_A - g_A,$

where we have used equation (3) to substitute for $g_w = gY_{/L}$. These profits are discounted at rate $r = \rho + [(1-\phi)/\phi] g_A$. Thus, on the balanced growth path, the present value of profits from the discovery of a new idea at time t is given by

(8) $$R(t) = \frac{\frac{1-\phi}{\phi}\frac{(1-a_L)L(t)}{A(t)}w(t)}{\rho + \frac{1-\phi}{\phi}g_A - n - \frac{1-\phi}{\phi}g_A + g_A} = \frac{\frac{1-\phi}{\phi}(1-a_L)\frac{L(t)w(t)}{A(t)}}{\rho + g_A - n}.$$

(c) The knowledge accumulation function is given by

(9) $\dot{A}(t) = BL_A(t)A(t)^{\theta}$.

Dividing both sides of equation (9) by A(t) gives the following expression for the growth rate of knowledge:

(10) $$\frac{\dot{A}(t)}{A(t)} = \frac{BL_A(t)}{A(t)^{1-\theta}}.$$

On the balanced growth path, $\dot{A}(t)/A(t)$ is constant. Thus the numerator and denominator of the term on the right-hand-side of equation (10) must grow at the same rate. On a balanced growth path, a_L is constant and so the numerator, $BL_A(t)$, grows at rate n. The denominator grows at rate $(1 - \theta)g_A$. Thus on a balanced growth path, we must have

(11) $n = (1 - \theta)g_A$.

Solving for the balanced-growth-path value of g_A gives us

(12) $$g_A = \frac{n}{1-\theta}.$$

Substituting the fact that $L_A(t) = a_L(t)L(t)$ into equation (9), and dividing both sides of the resulting expression by A(t) gives us

(13) $$\frac{\dot{A}(t)}{A(t)} = Ba_L(t)L(t)A(t)^{\theta-1}.$$

On a balanced growth path, $\dot{A}(t)/A(t) = n/(1 - \theta)$ and a_L is constant. Thus equation (13) becomes

(14) $$\frac{n}{1-\theta} = Ba_L L(t)A(t)^{\theta-1}.$$

Solving for $L(t)A(t)^{\theta-1}$ gives us

(15) $$L(t)A(t)^{\theta-1} = \frac{n}{(1-\theta)Ba_L}.$$

(d) The present value of profits from an invention, R(t), must equal the costs of the invention. Since one worker can produce $BA(t)^{\theta}$ ideas per unit time, the cost of an invention is $w(t)/[BA(t)^{\theta}]$. Thus equilibrium requires

(16) $$R(t) = \frac{w(t)}{BA(t)^{\theta}}.$$

Using equation (8) to substitute for R(t) yields

(17) $$\frac{\frac{1-\phi}{\phi}(1-a_L)\frac{L(t)w(t)}{A(t)}}{\rho + g_A - n} = \frac{w(t)}{BA(t)^{\theta}}.$$

We now need to solve for a_L. Cross-multiplying gives us

(18) $$\frac{1-\phi}{\phi}(1-a_L)BL(t)A(t)^{\theta-1} = \rho + g_A - n.$$

Using equations (12) and (15) to substitute for g_A and $L(t)A(t)^{\theta-1}$ in (18) yields

(19) $\dfrac{1-\phi}{\phi}(1-a_L)\dfrac{n}{(1-\theta)a_L} = \rho + \dfrac{n}{1-\theta} - n$.

Multiplying both sides by $(1 - \theta)a_L$ and obtaining a common denominator of $(1 - \theta)$ on the right-hand-side of (19) yields

(20) $\dfrac{1-\phi}{\phi}(1-a_L)n = (1-\theta)a_L\left[\dfrac{(1-\theta)\rho + n - (1-\theta)n}{1-\theta}\right]$.

Simplifying the right-hand-side of equation (20) leaves us with

(21) $\dfrac{1-\phi}{\phi}(1-a_L)n = a_L\left[(1-\theta)\rho + \theta n\right]$.

Rewriting the left-hand-side of equation (21) gives us

(22) $\dfrac{1-\phi}{\phi}n - \dfrac{1-\phi}{\phi}na_L = a_L\left[(1-\theta)\rho + \theta n\right]$.

Collecting the terms in a_L yields

(23) $a_L\left[\dfrac{1-\phi}{\phi}n + (1-\theta)\rho + \theta n\right] = \dfrac{1-\phi}{\phi}n$,

and so a_L is given by

(24) $a_L = \dfrac{\dfrac{1-\phi}{\phi}n}{\left[\dfrac{1-\phi}{\phi}n + (1-\theta)\rho + \theta n\right]}$.

Dividing the top and bottom of the right-hand-side of equation (24) by $[(1 - \phi)/\phi]n$ gives us the following expression for the share of the labor force employed in R&D on the balanced growth path:

(25) $a_L = \dfrac{1}{1+\Psi}$, where $\Psi \equiv \dfrac{\phi}{1-\phi}\left[\dfrac{\rho(1-\theta)}{n} + \theta\right]$.

Note that a higher balanced-growth-path value of the growth rate of Y/L – a higher value of $[(1 - \phi)/\phi][n/(1 - \theta)]$ – is associated with a larger share of labor employed in R&D.

(e) To see the effect on a_L, we can examine the effect of these parameters on Ψ since $\partial a_L/\partial[\cdot] = -\partial\Psi/\partial[\cdot]$. We have

(26) $\dfrac{\partial\Psi}{\partial\rho} = \dfrac{\phi}{1-\phi}\dfrac{1-\theta}{n} > 0.$

Thus an increase in ρ, which means that individuals become less patient, implies a fall in the balanced-growth-path share of labor engaged in R&D. This makes sense since R&D is a form of investment in this model. This effect is in the same direction as in the model of Section 3.5. From equation (3.43) in the text, ignoring the case where R&D is not strictly positive, we have

(27) $\dfrac{\partial L_A}{\partial\rho} = -\dfrac{\phi}{B} < 0.$

For the productivity parameter B, we have

(28) $\dfrac{\partial\Psi}{\partial B} = 0.$

Thus a change in B has no impact on the balanced-growth-path share of labor engaged in R&D. This is in contrast to the model in Section 3.5 where an increase in B drew additional workers into the R&D sector since

(29) $\dfrac{\partial L_A}{\partial B} = \dfrac{\phi\rho}{B^2} > 0.$

For ϕ, which measures the substitutability among inputs, we have

(30) $\dfrac{\partial \Psi}{\partial \phi} = \dfrac{K}{(1-\phi)^2} > 0$, where $K \equiv \left[\dfrac{\rho(1-\theta)}{n} + \theta\right]$.

Thus an increase in substitutability among inputs reduces the balanced-growth-path share of labor engaged in R&D. This effect is in the same direction as in the model of Section 3.5. There, we have

(31) $\dfrac{\partial L_A}{\partial \phi} = -\bar{L} - \dfrac{\rho}{B} < 0.$

For the labor force growth rate we have

(32) $\dfrac{\partial \Psi}{\partial n} = -\dfrac{\frac{\phi}{1-\phi}\rho(1-\theta)}{n^2} < 0.$

Thus an increase in the growth rate of the labor force increases the balanced-growth-path share of labor engaged in R&D.

For θ, the importance of the existing stock of knowledge in producing new knowledge, we have

(33) $\dfrac{\partial \Psi}{\partial \theta} = -\dfrac{\phi}{1-\phi}\dfrac{\rho}{n} + \dfrac{\phi}{1-\phi} = -\dfrac{\phi}{1-\phi}\left[\dfrac{\rho - n}{n}\right] < 0.$

The expression in (33) is negative because we require $\rho - n > 0$ so that lifetime utility does not diverge. So an increase in the importance of the existing stock of knowledge in producing new knowledge implies an increase in the balanced-growth-path share of labor engaged in R&D.

<u>**Problem 3.12**</u>

(a) (i) Substituting the assumption that $A(t) = BK(t)$ into the expression for firm i's output, $Y_i(t) = K_i(t)^{\alpha}(A(t)L_i(t))^{1-\alpha}$, we get $Y_i(t) = K_i(t)^{\alpha}(BK(t)L_i(t))^{1-\alpha}$.

To find the private marginal products of capital and labor, we take the first derivative of output with respect to the firm's choice of capital and labor assuming that the firm takes the aggregate capital stock, K, as given. The private marginal product of capital is therefore

(1) $\partial Y_i(t)/\partial K_i(t) = \alpha K_i(t)^{\alpha-1}(BK(t)L_i(t))^{1-\alpha}$

or simply

(2) $\partial Y_i(t)/\partial K_i(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^{-(1-\alpha)}$.

The private marginal product of labor is given by

(3) $\partial Y_i(t)/\partial L_i(t) = (1-\alpha)L_i(t)^{-\alpha}K_i(t)^{\alpha}(BK(t))^{1-\alpha}$,

or simply

(4) $\partial Y_i(t)/\partial L_i(t) = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}(K_i(t)/L_i(t))^{\alpha}$.

(ii) Because factor markets are competitive, at equilibrium the private marginal product of capital and labor cannot differ across firms. We can see from equations (2) and (4) that this implies the capital-labor ratio will be the same for all firms. Therefore,

(5) $K_i(t)/L_i(t) = K(t)/L(t)$,

for all firms.

(iii) With no depreciation, the real interest rate must equal the private marginal product of capital. From equation (2), this implies

(6) $r(t) = \partial Y_i(t)/\partial K_i(t) = \alpha B^{1-\alpha} K(t)^{1-\alpha} (K_i(t)/L_i(t))^{-(1-\alpha)}$.

Using the fact that the capital-labor ratio is the same across firms, we can substitute equation (5) into equation (6) to obtain

(6) $r(t) = \alpha B^{1-\alpha} K(t)^{1-\alpha} (K(t)/L)^{-(1-\alpha)}$,

which simplifies to

(7) $r(t) = \alpha B^{1-\alpha} L^{1-\alpha} = \alpha b$,

where $b \equiv B^{1-\alpha} L^{1-\alpha}$.

With no population growth, L is constant, and thus so is the real interest rate.

The real wage must equal the marginal product of labor. From equation (4) this implies

(8) $w(t) = \partial Y_i(t)/\partial L_i(t) = (1-\alpha) B^{1-\alpha} K(t)^{1-\alpha} (K_i(t)/L_i(t))^{\alpha}$.

Again, using the fact that the capital-labor ratio is the same across firms, we can substitute equation (5) into equation (8) to obtain

(9) $w(t) = (1-\alpha) B^{1-\alpha} K(t)^{1-\alpha} (K(t)/L)^{\alpha}$,

which simplifies to

(10) $w(t) = (1-\alpha) B^{1-\alpha} K(t) L^{-\alpha} = (1-\alpha) B^{1-\alpha} L^{1-\alpha} (K(t)/L) = (1-\alpha) b (K(t)/L)$,

or simply

(11) $w(t) = (1-\alpha) b (K(t)/L)$.

(b) Using the hint, since utility of the representative household takes the constant-relative-risk-aversion-form, consumption growth in equilibrium will be

(12) $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{r(t) - \rho}{\theta}$.

Substituting equation (7) for the real interest rate into equation (12) yields

(13) $\dfrac{\dot{C}(t)}{C(t)} = \dfrac{\alpha b - \rho}{\theta}$,

where $b \equiv B^{1-\alpha} L^{1-\alpha}$. Note that with no population growth so that L is constant, consumption growth is constant as well.

Using the zero-profit condition, we can write output as

(14) $Y(t) = r(t)K(t) + w(t)L$.

Substituting equations (7) and (11) into equation (14) gives us

(15) $Y(t) = \alpha b K(t) + (1-\alpha) b (K(t)/L) L$,

which simplifies to

(16) $Y(t) = bK(t)$.

Since b is a constant then output and capital grow at the same rate. Capital accumulation is then given by

(17) $\dot{K}(t) = sbK(t)$,

where s is the saving rate. Thus, the growth rate of the capital stock is given by

(18) $\dfrac{\dot{K}(t)}{K(t)} = sb$,

and so the growth rate of output also equals sb. Since C = (1 – s)Y, we can write the saving rate as

(19) $s = 1 - \dfrac{C}{Y}$.

Thus, we can write the growth rate of output as

(20) $\dfrac{\dot{Y}(t)}{Y(t)} = b\left(1 - \dfrac{C(t)}{Y(t)}\right)$.

If output growth were less than consumption growth, C/Y would rise over time. Output growth and capital growth would turn negative, which is not an allowable path. If output growth were greater than consumption growth, C/Y would fall to 0 over time. Output growth and capital growth would approach b. This implies that growth would eventually exceed the real interest rate, which is αb, and so this is also not an allowable path. Thus, the equilibrium growth rates of output and consumption must be equal.

(c) (i) We can take the derivative of the growth rate of output (which equals the growth rate of consumption) with respect to B to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial B} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}}{\theta} - \rho\right]}{\partial B} = \frac{\alpha(1-\alpha)L^{1-\alpha}}{\theta B^{\alpha}} > 0\,.$$

Thus an increase in B increases long-run growth.

(ii) We can take the derivative of the growth rate of output with respect to ρ to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial \rho} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}}{\theta} - \rho\right]}{\partial \rho} = -\frac{1}{\theta} < 0\,.$$

Thus an increase in ρ decreases long-run growth.

(iii) We can take the derivative of the growth rate of output with respect to L to obtain

$$\frac{\partial\,[\dot{Y}(t)/Y(t)]}{\partial L} = \frac{\partial\left[\dfrac{\alpha B^{1-\alpha}L^{1-\alpha}}{\theta} - \rho\right]}{\partial L} = \frac{\alpha(1-\alpha)B^{1-\alpha}}{\theta L^{\alpha}} > 0\,.$$

Thus an increase in L increases long-run growth.

(d) The equilibrium growth rate is less than the socially optimal growth rate. A social planner would internalize the knowledge spillovers and would set the growth rate of consumption dependent on the social return to capital, not the private return. We know that the private marginal product of capital is αb and the social marginal product is b (returns to capital are constant at the social level). Therefore, unless $\alpha = 1$, the growth rate set by the social planner would be greater than the decentralized equilibrium growth rate.

<u>Problem 3.13</u>

The production functions, after the normalization of T = 1, are given by

(1) $C(t) = K_C(t)^{\alpha}$, and (2) $\dot{K}(t) = BK_K(t)$.

(a) The return to employing an additional unit of capital in the capital-producing sector is given by $\partial \dot{K}(t)/\partial K_K(t) = B$. This has value $P_K(t)B$ in units of consumption goods. The return from employing an additional unit of capital in the consumption-producing sector is $\partial C(t)/\partial [K_C(t)] = \alpha [K_C(t)]^{\alpha-1}$. Equating these returns gives us

(3) $P_K(t)B = \alpha [K_C(t)]^{\alpha-1}$.

Taking the time derivative of the log of equation (3) yields the growth rate of the price of capital goods relative to consumption goods,

(4) $\dfrac{\dot{P}_K(t)}{P_K(t)} + \dfrac{\dot{B}}{B} = \dfrac{\dot{\alpha}}{\alpha} + (\alpha - 1)\left[\dfrac{\dot{K}_C(t)}{K_C(t)}\right]$.

Using the fact that B and α are constants leaves us with

(5) $\dfrac{\dot{P}_K(t)}{P_K(t)} = (\alpha - 1)\dfrac{\dot{K}_C(t)}{K_C(t)}$

Now since $K_C(t)$ is growing at rate $g_K(t)$ and denoting the growth rate of $P_K(t)$ as $g_P(t)$, we have

(6) $g_P(t) = (\alpha - 1)g_K(t)$.

(b) (i) The growth rate of consumption is given by

(7) $g_C(t) \equiv \dot{C}(t)/C(t) = [r(t) - \rho]/\sigma = [B + g_p(t) - \rho]/\sigma = [B + (\alpha - 1)g_K(t) - \rho]/\sigma$,

where we have used equation (6) to substitute for $g_P(t)$.

(b) (ii) Taking the time derivative of the log of the consumption production function, equation(1), yields

(8) $g_C(t) \equiv \dot{C}(t)/C(t) = \alpha[\dot{K}_C(t)/K_C(t)] = \alpha g_K(t)$.

Equating the two expressions for the growth rate of consumption, equations (7) and (8), yields

(9) $\alpha g_K(t) = [B + (\alpha - 1)g_K(t) - \rho]/\sigma \quad \Rightarrow \quad \alpha\sigma g_K(t) + (1-\alpha)g_K(t) = B - \rho$.

Thus in order for C to be growing at rate $g_C(t)$, $K_C(t)$ must be growing at the following rate:

(10) $g_K(t) = (B - \rho)/[\alpha\sigma + (1-\alpha)]$.

(b) (iii) We have already solved for $g_K(t)$ in terms of the underlying parameters. To solve for $g_C(t)$, substitute equation (10) into equation (8):

(11) $g_C(t) = \alpha(B - \rho)/[\alpha\sigma + (1-\alpha)]$.

(c) The real interest rate is now $(1 - \tau)(B + g_P)$. Thus equation (7) becomes

(12) $g_C(t) = \dfrac{(1-\tau)[B + g_p(t)] - \rho}{\sigma} = \dfrac{(1-\tau)[B + (\alpha - 1)g_K(t)] - \rho}{\sigma}$,

where we have used equation (6) – which is unaffected by the imposition of the tax – to substitute for $g_P(t)$. Equating the two expressions for the growth rate of consumption, equations (12) and (8), yields

(13) $\alpha g_K(t) = \dfrac{(1-\tau)[B + (\alpha - 1)g_K(t)] - \rho}{\sigma}$,

which implies

(14) $\alpha\sigma g_K(t) + (1-\tau)(1-\alpha)g_K(t) = (1-\tau)B - \rho$,

and thus

(15) $g_K(t) = \dfrac{(1-\tau)B - \rho}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]}$.

Substituting equation (15) into equation (8) yields the following expression for the growth rate of consumption as a function of the underlying parameters of the model:

(16) $g_C(t) = \alpha\left[\dfrac{(1-\tau)B - \rho}{\alpha\sigma + (1-\tau)(1-\alpha)}\right]$.

To see the effects of the tax, take the derivative of $g_C(t)$ with respect to τ:

(17) $\dfrac{\partial g_C(t)}{\partial \tau} = -\alpha\left\{\dfrac{B\left[\alpha\sigma + (1-\tau)(1-\alpha)\right] - \left[(1-\tau)B - \rho\right](1-\alpha)}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]^2}\right\} = -\alpha\left\{\dfrac{B\alpha\sigma + \rho(1-\alpha)}{\left[\alpha\sigma + (1-\tau)(1-\alpha)\right]^2}\right\} < 0$

Thus an increase in the tax rate τ causes the growth rate of consumption to fall.

Problem 3.14

(a) We need to find a value of τ such that $[Y_N(t)/L_N]/[Y_S(t)/L_S]$, the ratio of output per worker in the north to that in the south, is equal to 10. From the northern production function,

(1) $Y_N(t)/L_N = A_N(t)(1 - a_L)$.

Taking the time derivative of the natural log of equation (1) yields an expression for the growth rate of northern output per worker:

(2) $\dfrac{\overset{\bullet}{\left[Y_N(t)/L_N\right]}}{Y_N(t)/L_N} = \dfrac{\dot{A}_N(t)}{A_N(t)} = 0.03,$

where we have used the information given in the problem that the growth rate of northern output per worker, and thus of northern knowledge, is 3% per year. Since $\dot{A}_N(t)/A_N(t) = 0.03$ then

(3) $A_N(t) = e^{0.03\tau}A_N(t - \tau)$.

From the southern production function,

(4) $Y_S(t)/L_S = A_S(t)$.

Dividing equation (3) by equation (4) yields an expression for the ratio of output per worker in the north to that in the south:

(5) $\dfrac{Y_N(t)/L_N}{Y_S(t)/L_S} = \dfrac{A_N(t)(1-a_L)}{A_S(t)} \approx \dfrac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau},$

where we have used the fact that $a_L \approx 0$, that $A_S(t) = A_N(t - \tau)$, and equation (3).

For output per person in the north to exceed that in the south by a factor of 10, we need a τ such that $e^{0.03\tau} = 10$, or $0.03\tau = \ln(10)$, which implies that τ must be approximately 76.8 years. Thus, attributing realistic cross-country differences in income per person to slow transmission of knowledge to poor countries requires the transmission to be very slow. Poor countries would currently need to be using technology that the rich countries developed in the early 1940s to explain a 10-fold difference in income per person.

(b) (i) Recall that in the Solow model, the balanced-growth-path value of $k \equiv K/AL$ is defined implicitly by the condition that actual investment, $sf(k^*)$, equal break-even investment, $(n + g + \delta)k^*$. Thus for the north, k_N^* is implicitly defined by

(6) $sf(k_N^*) = (n + g + \delta)k_N^*$,

where $g = \dot{A}_N(t)/A_N(t)$.

We are told that s, n, δ and the function $f(\bullet)$ are the same for the north and the south. The only possible source of difference is the growth rate of southern knowledge. However, it is straightforward to show that $\dot{A}_S(t)/A_S(t) = g$ also.

We are told that the knowledge used in the south at time t is the knowledge that was used in the north at time t - τ. That is,
(7) $A_S(t) = A_N(t - \tau)$.
Taking the time derivative of equation (7) yields
(8) $\dot{A}_S(t) = \dot{A}_N(t - \tau)$.
Dividing equation (8) by equation (7) yields
(9) $$\frac{\dot{A}_S(t)}{A_S(t)} = \frac{\dot{A}_N(t-\tau)}{A_N(t-\tau)}.$$
The growth rate of northern knowledge is constant and equal to g at all points in time and thus
(10) $\dot{A}_S(t)/A_S(t) = g$.
Therefore, for the south, k_S^* is implicitly defined by
(11) $sf(k_S^*) = (n + g + \delta)k_S^*$.
Since k_N^* and k_S^* are implicitly defined by the same equation, they must be equal.

(b) (ii) Introducing capital will not change the answer to part (a). Since $k_N^* = k_S^*$, output per unit of effective labor on the balanced growth path will also be equal in the north and the south. That is, $y_N^* = y_S^*$ where $y_i^* = [Y_i / A_iL_i]^*$. We can write the balanced-growth-path value of output per worker in the north as
(12) $Y_N(t)/L_N(t) \equiv A_N(t)y_N^*$.
Similarly, the balanced-growth-path value of output per worker in the south is
(13) $Y_S(t)/L_S(t) \equiv A_S(t)y_S^*$.
Dividing equation (12) by equation (13) yields
(14) $$\frac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \frac{A_N(t)y_N^*}{A_S(t)y_S^*} = \frac{A_N(t)}{A_S(t)} = \frac{A_N(t)}{A_N(t-\tau)}.$$
The second-to-last step uses the fact that $y_N^* = y_S^*$. The last step uses $A_S(t) = A_N(t - \tau)$. Using equation (3), we again have
(15) $$\frac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \frac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau}.$$
The same calculation as in part (a) would yield a value of τ = 76.8 years for the ratio $[Y_N(t)/L_N]/[Y_S(t)/L_S]$ to equal 10.

Problem 3.15

The best evidence that differences in saving rates are not important to cross-country income differences would be answer (2), a point estimate of the elasticity of long-run output with respect to the saving rate of 0.1 with a standard error of 0.01. A two-standard-error confidence interval in this case is (0.098,0.102). So it is true that at any reasonable significance level, we would reject the null hypothesis that the elasticity is zero. But that simply means the elasticity is statistically significant, not that it is economically important. Given a point estimate of 0.1 and an associated standard error of just 0.01, we have a precise estimate and therefore strong evidence that the true elasticity is very small.

Answer (3), a point estimate of 0.001 with a standard error of 5 might be tempting but it shows the problem of focusing on t-statistics. The two-standard-error confidence interval in this case is

(-9.999,10.001). With 0 safely in that interval, it is true that we would fail to reject the null hypothesis that the elasticity is zero. But since our estimate is so imprecise, and thus the confidence interval so large, we would also fail to reject the null hypothesis that the true elasticity is 10.0. And a true elasticity of 10 would in fact mean that saving rates have an important economic effect on cross-country income differences.